Coend Calculus and Open Diagrams

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Abstract

Morphisms in a monoidal category are usually interpreted as *processes* or *black boxes* that can be composed sequentially and in parallel. In practice, we are faced with the problem of interpreting what non-square boxes ought to represent and, more importantly, how should they be composed. Examples of this situation include *lenses* or *learners*. We propose a description of these non-square boxes, which we call *open diagrams*, in terms of coends and the monoidal bicategory of profunctors, with features of what could be considered a graphical coend calculus. This graphical calculus allows us to describe possible compositions of these open diagrams but also to reason about their concrete descriptions. This is work in progress.

1 Introduction

1.1 Open Diagrams

Morphisms in monoidal categories are interpreted as processes with inputs and outputs and generally represented by square boxes. This interpretation, however, raises the question of how to represent a process that does not consume all the inputs at the same time or a process that does not produce all the outputs at the same time. Consider for instance a process that consumes an input, produces an output, then consumes a second input and ends producing an output. Graphically, we have a clear idea of how this process should be represented, even if it is not a morphism in the category.

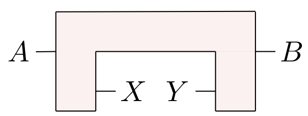


Figure 1: A process with a non-standard shape. The input A is taken at the beginning, then the output X is produced, strictly after that, the input Y is taken; finally, the output B is produced.

Reasoning graphically, it is obvious, for instance, that we should be able to *plug* a morphism connecting the first output to the second input inside this process and get back an actual morphism of the category.

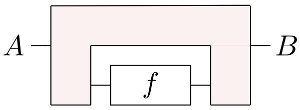


Figure 2: It is possible to plug a morphism $f: X \to Y$ inside the previous process (Figure 1), and, importantly, get back a morphism $A \to B$.

The particular shape depicted above has been extensively studied by [Ril18] under the name of (monoidal) *optic*; it can be also called a *monoidal lens*; and it has applications in bidirectional data accessing [PGW17, BG18, Kme18] or compositional game theory [GHWZ18]. A multi-legged generalization has appeared also in the work on causality of Kissinger and Uijlen as a notational convention [KU17, Rom20]. It can be shown that *boxes* of that shape should correspond to elements of a suitable *coend* (Figure 3, see also §1.2 and [Mil17]). The intuition for this representation is that one should consider a tuple of morphisms and then quotient out by an equivalence relation generated by all the wires that are connected between these morphisms.

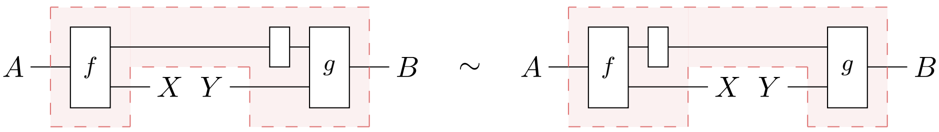


Figure 3: A box of this shape is meant to represent a pair of morphisms in a monoidal category quotiented out by "sliding a morphism" over the upper wire.

It has remained unclear, however, how this process should be carried in full generality and if it was in solid ground. Are we being formal when we use these *open* or *incomplete* diagrams? What happens with all the other possible shapes that one would want to consider in a monoidal category? They are different from the usual squares. For instance, the second one of the shapes in Figure 4 has three inputs and two outputs, but the first input cannot affect the last output; and the last input cannot affect the first one.[1]

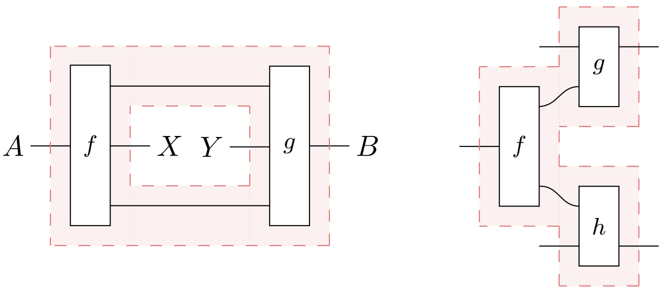


Figure 4: Some other shapes for boxes in a monoidal category.

This note presents the idea that incomplete diagrams should be interpreted in the monoidal bicategory of profunctors; compositions of incomplete diagrams correspond to reductions that employ the monoidal bicategory structure. At the same time, this gives a graphical presentation of *coend calculus*.

[1] This particular shape comes from a question by Nathaniel Virgo on `categorytheory.zulipchat.com`.

1.2 Coend Calculus

Coends are particular cases of colimits and *coend calculus* is a practical formalism that uses Yoneda reductions to compute isomorphisms between them. Their dual counterparts are *ends*, and formalisms for both interact nicely in a *(Co)End calculus* [Lor19].

Definition 1.1. The **coend** $\int^{X\in\mathbf{C}} P(X,X)$ of a profunctor $P\colon \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{Set}$ is the universal object endowed with morphisms

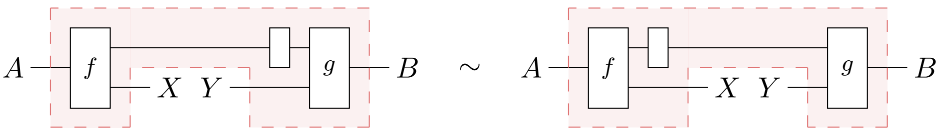
$$i_A\colon P(A,A) \to \left(\int^{X\in\mathbf{C}} P(X,X)\right)$$

for every $A \in \mathbf{C}$ such that, for any morphism $f\colon B \to A$ in $\mathbf{C}$, they satisfy $i_B \circ P(f,\mathrm{id}) = i_A \circ P(\mathrm{id}, f)$. It is universal in the sense that any other object D endowed with morphisms $j_A\colon P(A,A) \to D$ satisfying the same condition factors uniquely through it.

In other words, the coend is the coequalizer of the action on morphisms on both arguments of the profunctor. An element of the coend is an equivalence class of pairs $[X, x \in P(X,X)]$ quotiented by the equivalence relation generated by $[X, P(f,-)(u)] \sim [Y, P(-,f)(u)]$.

$$\int^{X\in\mathbf{C}} P(X,X) \cong \mathrm{coeq}\left(\textstyle\bigsqcup_{f\colon B\to A} P(A,B) \rightrightarrows \bigsqcup_{X\in\mathbf{C}} P(X,X) \right).$$

Our main idea is to use these quotient relations to deal with the naturality arising in non-square monoidal shapes.



$$\int^{M} \mathbf{C}(A, M \otimes X) \times \mathbf{C}(M \otimes Y, B).$$

Figure 5: We can go back to the previous example (Figure 3) to check how it coincides with the quotienting arising from the dinaturality of a coend.

1.3 Contributions

- A description of how to use the graphical calculus of the monoidal bicategory of profunctors to reason about *open diagrams* and *coend calculus*, paying special attention to how the monoidal structure of a category is represented with profunctors (§2 and §3.1).
- A study of the multiple ways of composing monoidal lenses, and a recast of some coend calculus constructions on the literature on optics in terms of monoidal categories (§3.2).
- A description of how some two constructions in the literature are instances of *open diagrams* and can be described in terms of coends. These are the Circ construction [KSW02] (§3.3) and *learners* [FJ19] (§3.4).

2 The Monoidal Bicategory of Profunctors

Definition 2.1. There exists a symmetric monoidal bicategory **Prof** having as objects the (small) categories; as 1-cells from $\mathbf{A}$ to $\mathbf{B}$ the profunctors $\mathbf{A}^{op} \times \mathbf{B} \to \mathbf{Set}$; as 2-cells, the natural transformations; and as tensor product, the cartesian product of categories [Lor19]. Composition of two profunctors $P\colon \mathbf{A}^{op} \times \mathbf{B} \to \mathbf{Set}$ and $Q\colon \mathbf{B}^{op} \times \mathbf{C} \to \mathbf{Set}$ returns a profunctor $(P \diamond Q)\colon \mathbf{A}^{op} \times \mathbf{C} \to \mathbf{Set}$ defined by

$$(P \diamond Q)(A, C) \coloneqq \int^{B \in \mathbf{B}} P(A, B) \times Q(B, C).$$

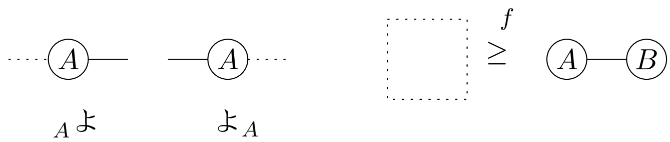
The monoidal product of two profunctors $P_1\colon \mathbf{A}_1^{op} \times \mathbf{B}_1 \to \mathbf{Set}$ and $P_2\colon \mathbf{A}_2^{op} \times \mathbf{B}_2 \to \mathbf{Set}$ is the profunctor $(P_1 \otimes P_2)\colon (\mathbf{A}_1 \times \mathbf{A}_2)^{op} \times (\mathbf{B}_1 \times \mathbf{B}_2) \to \mathbf{Set}$ defined by

$$(P_1 \otimes P_2)(A_1, A_2, B_1, B_2) \coloneqq P_1(A_1, B_1) \times P_2(A_2, B_2).$$

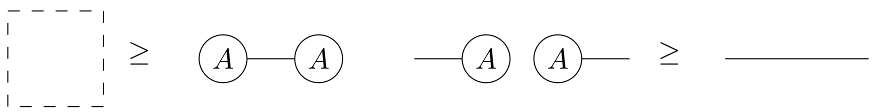
We will employ the graphical calculus of monoidal categories for the cartesian structure of **Prof**; and, at the same time, we will use inequalities between diagrams to denote the 2-cells of the bicategorical structure, which are natural transformations. This is in analogy with the graphical calculus for categories of relations, but note that the category is not posetal.

Remark 2.2. Scalars in this monoidal structure are sets. We will be defining sets by using this graphical calculus during the rest of the text.

Definition 2.3 (Yoneda embeddings)**.** Every object $A \in \mathbf{C}$ determines two profunctors ${}_A\text{よ} \coloneqq \mathbf{C}(A, -)\colon \mathbf{1}^{op} \times \mathbf{C} \to \mathbf{Set}$ and $\text{よ}_A \coloneqq \mathbf{C}(-, A)\colon \mathbf{C}^{op} \times \mathbf{1} \to \mathbf{Set}$ called their contravariant and covariant Yoneda embeddings. Every morphism $f \in \mathbf{C}(A, B)$ can be seen as a natural transformation $f\colon \mathbf{1} \to {}_A\text{よ} \diamond \text{よ}_B$.

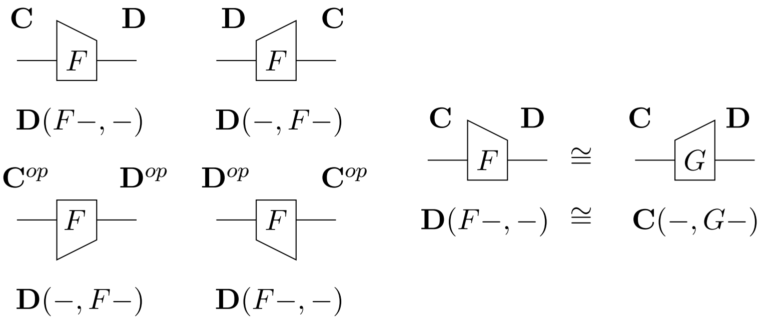


In particular, identities and composition give the following morphisms. Unitality of the identity under composition becomes the zig-zag equation making these two adjoints (see Proposition 2.5).

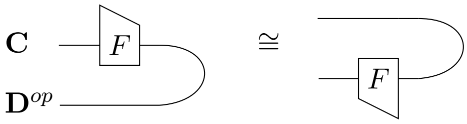


Definition 2.4 (Yoneda embedding of functors)**.** Let $F\colon \mathbf{C} \to \mathbf{D}$ be a functor. It can be embedded into profunctors in two ways, as a profunctor $\mathbf{D}(F-, -)\colon \mathbf{C}^{op} \times \mathbf{D} \to \mathbf{Set}$ or as a profunctor $\mathbf{D}(-, F-)\colon \mathbf{D}^{op} \times \mathbf{C} \to \mathbf{Set}$. We denote them with the following graphical

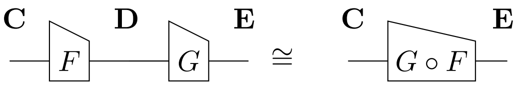
notation and we note the particular case of what it means to be an adjoint functor.



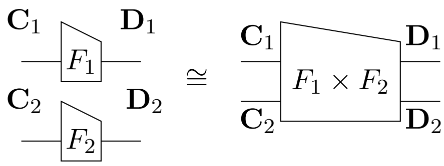
The suggestive shape of the boxes (taken from [CK17]) is actually matched by their semantics. Every category has a dual (namely, its opposite category) and functors circulate as expected through the cups and the caps that represent dualities.



In fact, each one of these embeddings is a monoidal pseudofunctor **Cat** $\to$ **Prof**. In particular, because of pseudofunctoriality, we have the following isomorphism and its counterpart.

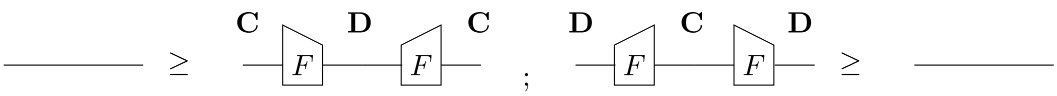


Because of monoidality, we have the following isomorphism and its counterpart.

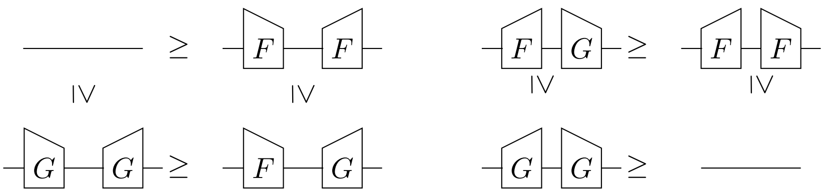


Moreover, this is a fully faithful embedding thanks again to the Yoneda lemma. The natural transformations between (co)representable profunctors correspond to natural transformations between the (co)represented functors themselves.

Proposition 2.5. *In the category of profunctors, functors are left adjoints, in the sense that there exist morphisms as follows and they verify the zig-zag identities [Bor94].*



Moreover, every natural transformation commutes with these dualities in the sense that the following are two commutative squares.[2]

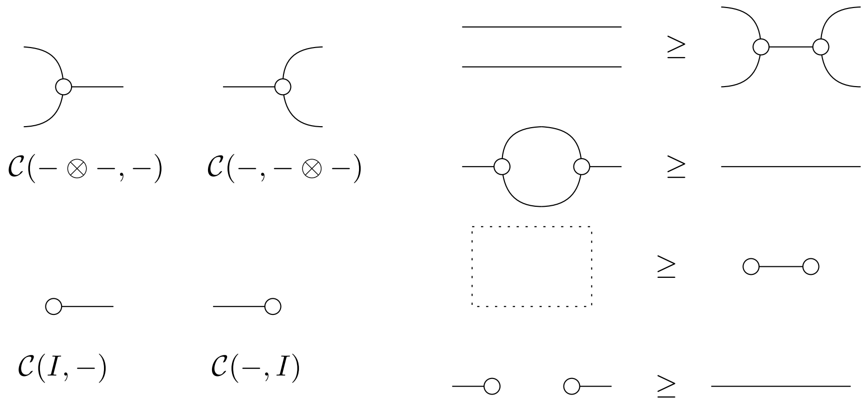


The converse holds for profunctors whose codomain is Cauchy complete, see [Bor94].

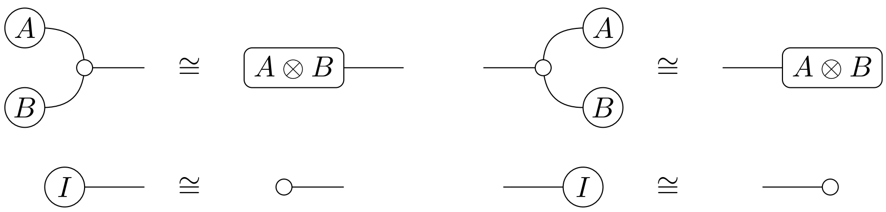
[2]It feels much more instructive to see these equalities from the graphical calculus of a bicategory. However, for the sake of simplicity, we are emphasizing only the monoidal structure.

2.1 Monoidal categories

Definition 2.6. A **promonoidal category** is a pseudomonoid in the monoidal bicategory **Prof**. A **procomonoidal category** is a pseudocomonoid in the category **Prof**. Every monoidal category is both promonoidal and procomonoidal. Moreover, because the monoidal tensor and unit are functors, the promonoidal structure is adjoint to the procomonoidal structure.

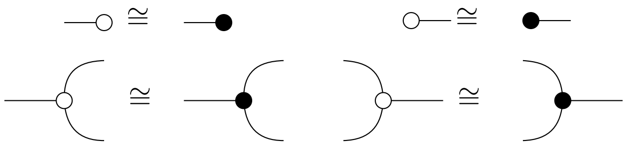


This monoidal structure interacts nicely with Yoneda embeddings precisely because it is represented by a functor.



2.2 Cartesian, cocartesian and symmetric monoidal categories

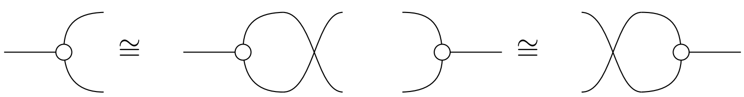
Proposition 2.7. *Note that every object is a monoid and a comonoid in a canonical way in* **Prof**. *A monoidal category is* ***cartesian*** *if and only if its procomonoidal structure coincides with its canonical one; a monoidal category is* ***cocartesian*** *if and only if its promonoidal structure coincides with its canonical one.*



(Cartesian) *(Cocartesian)*

During this text, we use the black dots for the comonoid structure inherited from **Cat** *that every representable functor preserves. We keep the white dots to represent monoidal structures of particular categories.*

Proposition 2.8. *A monoidal category* **C** *is* symmetric *exactly when the pseudomonoid giving its monoidal structure is commutative and the application of commutativity commutes with unitors and associators.*



3 Examples of Open Diagrams

We have all the ingredients to start considering open diagrams. We will start by the examples discussed on the introduction, focus then on the case of *lenses* [Ril18] and finally comment both on the construction of free categories with feedback (the *Circ* construction) [KSW02] and *learners* [FST19].

3.1 Motivating examples

Consider the examples that motivated the introduction (in Figure 4). By writing them again with the graphical calculus of the cartesian bicategory **Prof**, we can obtain formulaic descriptions of the shapes, but we can also compose them in different ways.

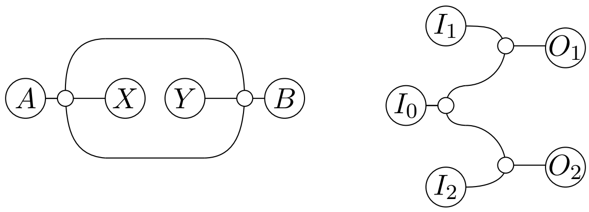


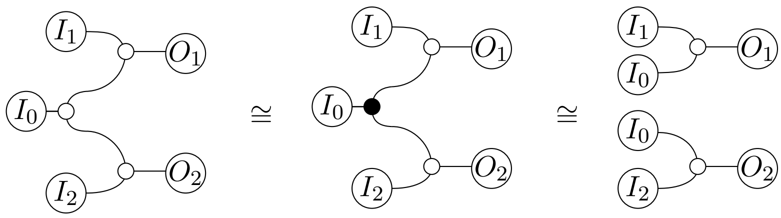
Figure 6: The shapes of Figure 4, interpreted as coends.

The corresponding coend descriptions are as follows.

$$\int^{M,N} \mathbf{C}(A, M \otimes X \otimes N) \times \mathbf{C}(M \otimes Y \otimes N, B),$$

$$\int^{M,N} \mathbf{C}(I_0, M \otimes N) \times \mathbf{C}(I_1 \otimes M, O_1) \times \mathbf{C}(N \otimes I_2, O_2).$$

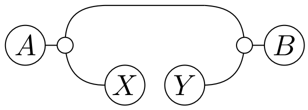
Remark 3.1. If $\mathbf{C}$ is cartesian monoidal, the second shape reduces to a pair of morphisms $\mathbf{C}(I_0 \times I_1, O_1)$ and $\mathbf{C}(I_0 \times I_2, O_2)$, which coincides with our previous intuition in Figure 4 that the input I_1 should not be able to affect O_2, while the input I_2 should not be able to affect O_1.



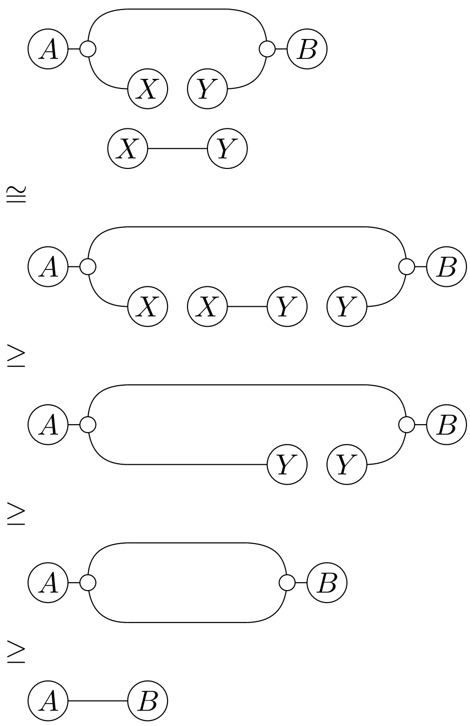
3.2 Lenses and optics

Profunctor optics have been extensively studied in functional programming [Kme18, Mil17, PGW17, BG18] for bidirectional data accessing. Categories of monoidal optics and the interpretation of optics as *diagrams with holes* were studied in depth by Riley [Ril18]. The theory of optics uses coend calculus both to describe how optics compose and how to reduce them in sufficiently well-behaved cases to tuples of morphisms. We are going to be studying optics from the perspective of the graphical calculus of **Prof**. This presents a new way of computing with coend calculus and describing reductions that also formalizes the intuition of optics as *diagrams with holes*.

Definition 3.2. A monoidal lens [Ril18, "Optic" in Definition 2.0.1], from $A, B \in \mathbf{C}$ to $X, Y \in \mathbf{C}$ is an element of the following set.

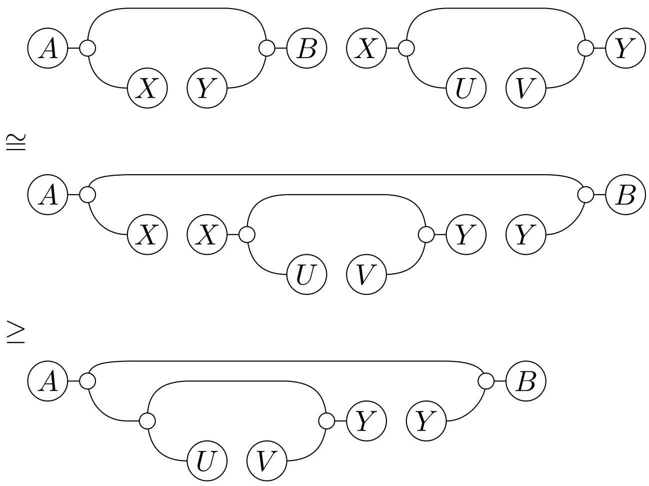


Example 3.3. As detailed in the introduction, a lens $(A, B) \to (X, Y)$ can be composed with a continuation $X \to Y$ to obtain a morphism $A \to B$. Let us illustrate this composition in the graphical calculus of **Prof**. It is also interpreted into the following chain of coend calculus, that describes that same composition.



$$\begin{aligned}
&\left(\int^{M} \mathbf{C}(A, M \otimes X) \times \mathbf{C}(M \otimes Y, B)\right) \times \mathbf{C}(X, Y) \\
\cong\ &\int^{M} \mathbf{C}(A, M \otimes X) \times \mathbf{C}(X, Y) \times \mathbf{C}(M \otimes Y, B) \\
\to\ &\int^{M} \mathbf{C}(A, M \otimes Y) \times \mathbf{C}(M \otimes Y, B) \\
\to\ &\int^{M,N} \mathbf{C}(A, M \otimes N) \times \mathbf{C}(M \otimes N, B) \\
\to\ &\mathbf{C}(A, B)
\end{aligned}$$

Example 3.4. Two lenses of types $(A, B) \to (X, Y)$ and $(X, Y) \to (U, V)$ can be also composed with each other. This is the composition that gives rise to categories of optics [Ril18].



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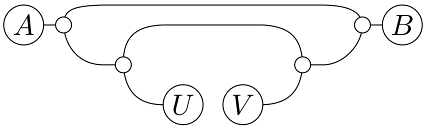

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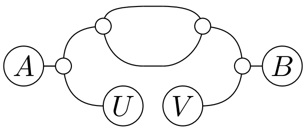

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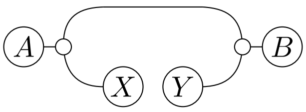
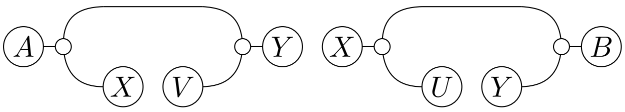
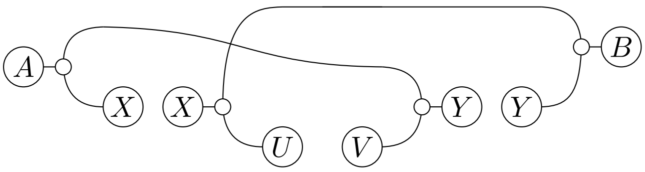


Example 3.5. There is, however, another way of composing two lenses when the category is symmetric. A lens $(A, Y) \to (X, V)$ can be composed with a lens $(X, B) \to (U, Y)$ into a lens $(A, B) \to (U, Y)$. As we will see during its construction, this is an option only when the category is symmetric.



$\cong$



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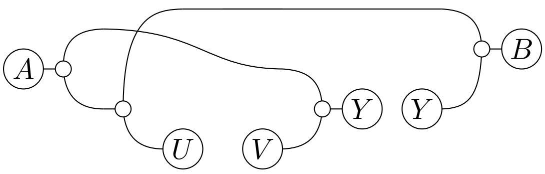

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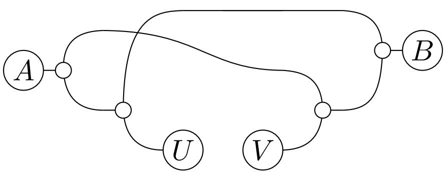

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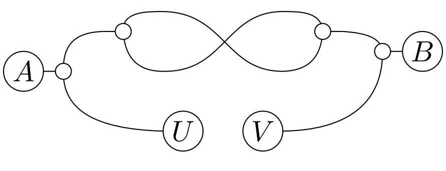
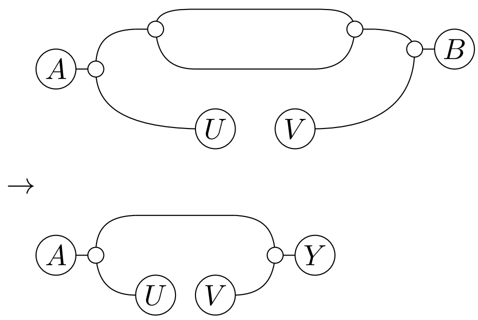


$\cong$ (This step requires symmetry)

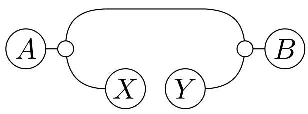


That is, even when the category **Prof** is symmetric; in order to produce a valid composition into a new lens, one explicitly needs symmetry on the base category $\mathbf{C}$.

3.2.1 Cartesian and linear lenses

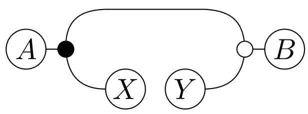
Proposition 3.6. *In a cartesian category* $\mathbf{C}$*, a lens* $(A, B) \to (X, Y)$ *is given by a pair of morphisms* $\mathbf{C}(A, X)$ *and* $\mathbf{C}(A \times Y, B)$*. In a cocartesian category, these are called* prisms *[Kme18] and they are given by a pair of morphisms* $\mathbf{C}(S, A + T)$ *and* $\mathbf{C}(B, T)$*.*

Proof. We write the proof for lenses, the proof for prisms is dual and can be obtained by mirroring the diagrams. The coend derivation can be found, for instance, in [Mil17].



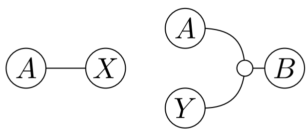
$$\int^{M} \mathbf{C}(A, M \times X) \times \mathbf{C}(M \times Y, B)$$

$\cong$



$$\int^{M} \mathbf{C}(A, M) \times \mathbf{C}(A, X) \times \mathbf{C}(M \times Y, B)$$

$\cong$

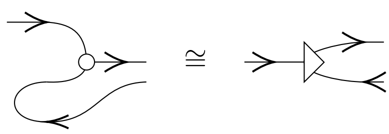


$$\int^{M} \mathbf{C}(A, X) \times \mathbf{C}(A \times Y, B)$$

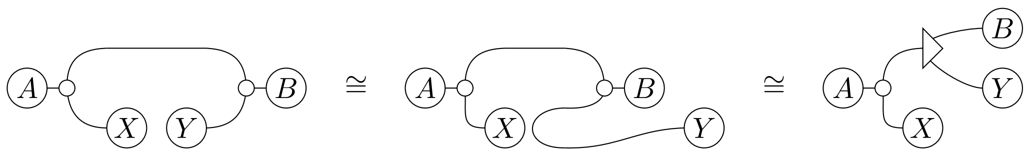
□

Proposition 3.7. *In a monoidal closed category* $(\mathbf{C}, \otimes, I, \multimap)$*, a lens* $(A, B) \to (X, Y)$ *is given by a single morphism* $\mathbf{C}(A, (Y \multimap B) \otimes X)$*.*

Proof. The adjunction that defines the exponential of the closed monoidal structure is $\mathbf{C}(X \otimes Y, Z) \cong \mathbf{C}(X, Y \multimap Z)$. This can be translated as saying that there is a functor $(\multimap)\colon \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{C}$, that we represent by a triangle, such that the following two diagrams are isomorphic.



We can then rewrite the lens as follows.

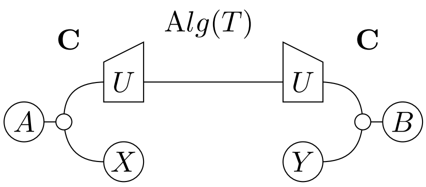


□

3.2.2 Algebraic lenses

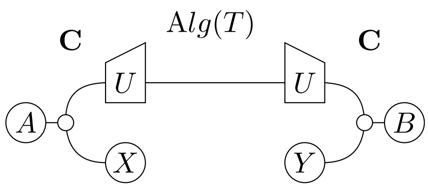
We have discussed so far the topic of monoidal lenses, but *optics* are a more general notion of lens. For simplicity, let us focus on a particular family of optics, similar to lenses, that restrict the category over which the coend is taken. This family can be used to construct lenses that deal with *creation* of missing entries (*achromatic lenses*, [Boi17]) and classification of the focus (*classifying lens*, [CEG+20, Remark 3.12]). This showcases a technique that we consider important on its own: restricting the coend to slightly adjust a definition.

Definition 3.8. [CEG+20, Definition 3.9] Let $\mathbf{C}$ be a cartesian category and let $T\colon \mathbf{C} \to \mathbf{C}$ be a monad; let $F\colon \mathbf{C} \to \mathrm{Alg}(T)$ and $U\colon \mathrm{Alg}(T) \to \mathbf{C}$ be the free and forgetful functors to the Eilenberg-Moore category, respectively. An algebraic lens $(A, B) \to (X, Y)$ is an element of the following set.

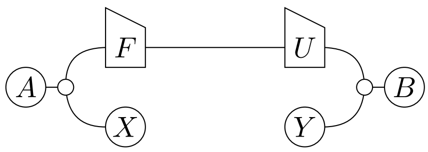


Proposition 3.9. *[CEG+20, Proposition 3.10] In a cartesian category, an algebraic lens is equivalent to a pair of functions* $\mathbf{C}(A, X)$ *and* $\mathbf{C}(TA \times Y, B)$.

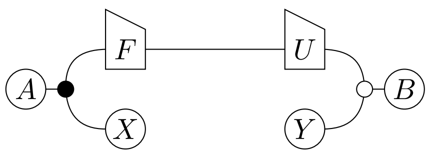
Proof.



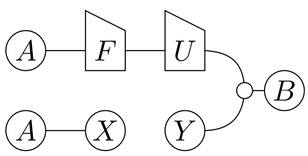
$\cong$



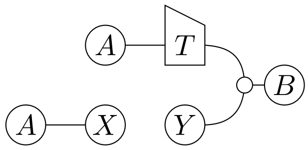
$\cong$



$\cong$



$\cong$

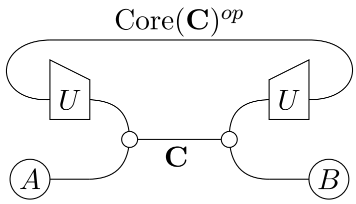


□

3.3 Circ construction

As one may expect, depicting diagrams in the category of profunctors gives us much more freedom than one would have drawing diagrams in the monoidal category. This extra freedom can be used to represent monoidal processes of more exotic shapes. If we do not preserve the flow of time (understood as the order of sequential compositions) during this identification, we can introduce notions of feedback in any monoidal $\mathbf{C}$.

Definition 3.10. [KSW02, Definition 2.4] Let $U\colon \mathrm{Core}(\mathbf{C}) \to \mathbf{C}$ be the obvious inclusion functor from its core subgroupoid. For any two objects $X, Y \in \mathbf{C}$, we define a morphism in the $\mathsf{Circ}_{\mathbf{C}}$ construction to be an element of the following set.



This is equivalently, an element of the following coend.

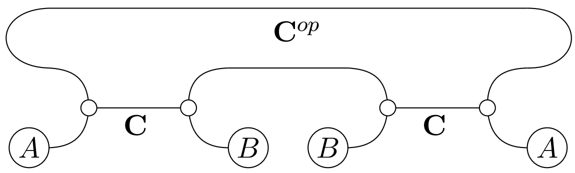
$$\mathsf{Circ}_{\mathbf{C}}(X, Y) \coloneqq \int^{M \in \mathrm{Core}(\mathbf{C})} \mathbf{C}(M \otimes X, M \otimes Y).$$

As we can see this, is a much compressed description of the free category with feedback. It is also arguable that it makes intuitive sense: we are literally adding the feedback as a wire, and the fact that only isomorphisms travel the feedback reflects in the choice of wire. From this description, is easy to see that the Circ construction can be easily generalized into multiple variants by just changing the category over which we take the coend, in the same style we did for algebraic lenses (§3.2.2).

3.4 Learners

The connections between the approach of Fong, Spivak and Tuyéras [FST19] to compositional supervised machine learning and lenses have generated recent interest [FJ19]. The approach is based on a category of learners where a morphism can be described as follows.

Definition 3.11. Let $(\mathbf{C}, \otimes, I)$ be a monoidal category. A (monoidal) **learner** taking inputs on an object A and producing outputs on an object B is an element of the following set.



This is a generalization to the monoidal case of an alternative definition proposed by Riley [Ril18]. It lets us view learners under a different light, even if it can be extremely different from the usual presentation.

$$\int^{P,Q\in\mathbf{C}} \mathbf{C}(P\otimes A, Q\otimes B)\times\mathbf{C}(Q\otimes B, P\otimes A)$$

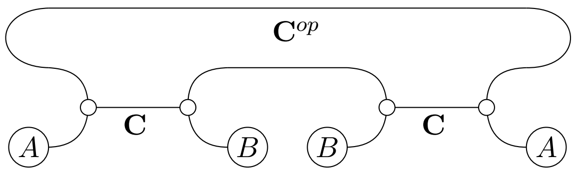
The original definition is conceptually clearer as it splits the learner into the functions is should implement. Let us show both are equivalent in the case of **Set**.

Proposition 3.12. *[FJ19, following Definition 4.1] Let* $\mathbf{C}$ *be a cartesian category. A* ***learner*** *taking inputs on an object* A *and producing outputs on an object* B *is given by*

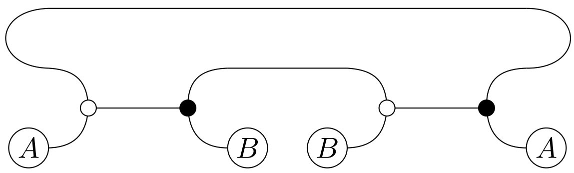
- *a set of* parameters P*,*
- *an* implementation *function* $i\colon P\times A\to B$*,*
- *an* update *function* $u\colon P\times A\times B\to P$*, and*
- *a* request *function* $r\colon P\times A\times B\to A$*.*

Under the right quotient, a learner is precisely a monoidal learner in a cartesian category.

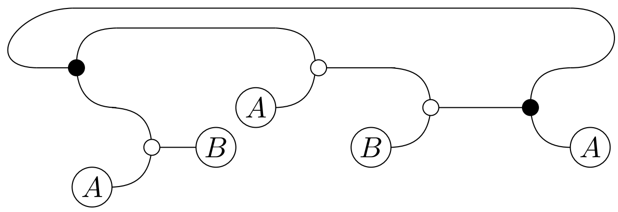
Proof. The following coend derivation appears, in the case of **Set**, in [Ril18, Definition 6.4.1]. It is included here for the sake of completeness and the slight generalization, but also to compare how it works under the graphical calculus.



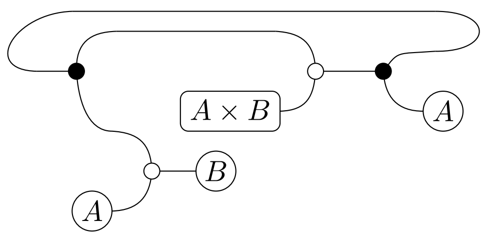
$\cong$ ($\mathbf{C}$ is cartesian)



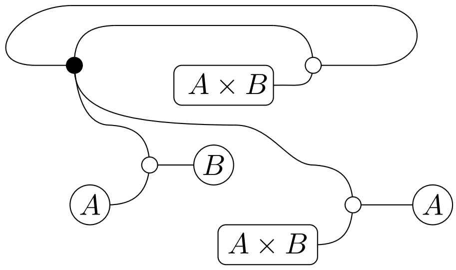
$\cong$ (Split)



$\cong$ (Monoidal product)



$\cong$ (Split)



This is the definition in terms of parameters, implementation, update and request. The looping wire represents the parameters quotiented by a coend. □

4 Related work

- Finite *combs*, as described by [CDP08], [KU17] and [Rom20], can be combined in many different ways with this technique. We are presenting a way of describing these possible compositions. The fact that they can be nicely structured on an operad has been claimed by Jules Hedges in personal communication. Note however, that present approach is strictly more general than considering combs, as *combs* assume the symmetric structure of the underlying monoidal category, whereas the present technique can express arbitrary shapes in non-symmetric monoidal categories. Moreover, symmetry can be additionally required when needed (see, for instance, the case of Remark 3.2).
- Graphical calculi for the cartesian bicategory of relations, as used in [BPS17], and graphical regular logic [FS18] are arguably presenting a decategorification of this same idea. Previous graphical calculi for lenses and optics [Hed17, Boi20] have elegantly captured some aspects of optics by working on the Kleisli or Eilenberg-Moore categories of the Pastro-Street monoidal monad [PS08]. The present approach diverges greatly in the formalism (diagrams using the cartesian structure of profunctors) and presents a more general application. In any case, it also allows us to reason about categories of optics themselves. We believe that it is closer to, and it provides a formal explanation to the *diagrams with holes* of [Ril18, Definition 2.0.1], which were missing from previous approaches.

5 Conclusions

We have presented a way to study and compose *processes* in monoidal categories that do not necessarily have the usual shape of a square box. This allows us to consider processes with non-trivial dependencies between inputs and outputs. The calculus of the category **Prof** seems particularly suited to accomodate descriptions of these open systems without losing the usual language of monoidal categories.

This technique is backed up at the same time with a practical formalism in terms of coend calculus [Lor19]. We can argue that the graphical representation of coend calculus is helpful to its understanding. Constrasting with usual presentations of coends that are usually centered around the Yoneda reductions; the graphical approach seems to put more weight in the non-reversible transformations and make most applications of Yoneda lemma transparent.

We have been working in the symmetric monoidal bicategory of profunctors for simplicity, but the same results extend to the symmetric monoidal bicategory of $\mathcal{V}$-profunctors for $\mathcal{V}$ a Bénabou cosmos. This is the setting chosen in [CEG+20] to study optics; and it is conceptually interesting to note how it does not seem to significantly alter the proofs. We can also consider other symmetric monoidal bicategories and even to drop the requirement for symmetry if we only need to work with monoidal shapes that do not involve symmetry in their compositions.

Finally, there is an important shortcoming to this approach that we leave as further work: the present graphical calculus is an extremely good tool for *coend calculus*, but it remains to see if it is so for *(co)end calculus*. In other words, *ends* enter the picture only as natural transformations. As it happens with diagrammatic presentations of regular logic, only the existential quantifier is recovered. Diagrammatic approaches to obtaining the universal quantifier in a situation like this go back to Peirce and are described by [HS20].

5.1 Three-dimensional calculus

The graphical calculus we are using is three-dimensional in nature. The work of Willerton [Wil08] is an example where the three-dimensional structure of the monoidal bicategory **Cat** is depicted. It is only because of simplicity and technical constraints that we choose to work in some form of sliced two-dimensional graphical calculus. Arguably, it should not to be this way and recent developments try to overcome these constraints. For instance, `Homotopy.io` is a proof assistant supporting n-dimensional diagrammatic reasoning that can render three-dimensional diagrams [HHV19].

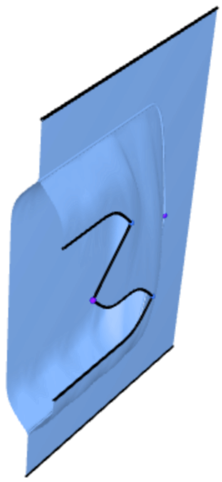


Figure 7: A monoidal lens is composed with a particular continuation. It is similar (except for the choice of continuation) to the construction in Example 3.3.

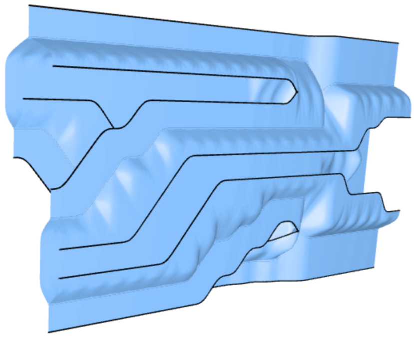


Figure 8: Two lenses composing. At the left end, we can see the figure of the two lenses. They get merged in the middle. At the right end, we can see a single lens.

6 Acknowledgements

Thanks go to whole Compositional Systems and Methods group at TalTech for discussions both on optics and cartesian bicategories (including a seminar on these by Jens Seeber, remarks on surface diagrams by Edward Morehouse and the account of coend calculus of Fosco Loregian [Lor19]). A question by Nathaniel Virgo motivates the last of the examples in Figure 4.

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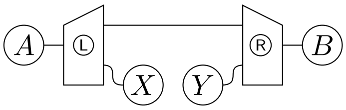
7 Appendix

7.1 A Diagrammatic Profunctor Representation of Optics

For the specialized reader, it may be of interest to finally consider the definition of a **mixed optic**, the generalized version of an optic. Let $\mathbf{M}$ a monoidal category acting with two monoidal actions ⓛ: $\mathbf{M} \times \mathbf{C} \to \mathbf{C}$ and ⓡ: $\mathbf{M} \times \mathbf{D} \to \mathbf{D}$. A mixed optic $(A, B) \to (X, Y)$ is an element of the following set.



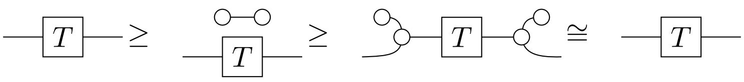
The theory of optics generalizes in the expected way to mixed optics [CEG+20]; the graphical calculus helps us explain why that should be the case, as all the derivations we need hold the same for monoidal actions and monoidal products.

During the following section we fix a pair of actions ⓛ: $\mathbf{M} \times \mathbf{C} \to \mathbf{C}$ and ⓡ: $\mathbf{M} \times \mathbf{D} \to \mathbf{D}$. They are both, together with the monoidal structure of $\mathbf{M}$, to be represented by a white dot. In most cases, they are precisely the monoidal structure on $\mathbf{C}$, and the added complexity of the notation would add nothing interesting to the proof.

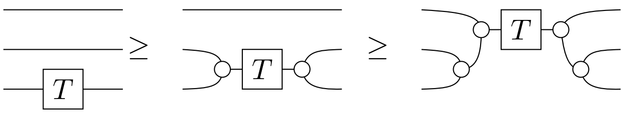
7.1.1 Tambara modules and the Pastro-Street monad

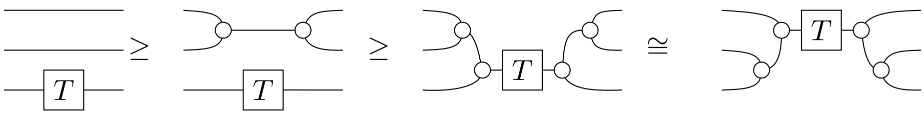
Definition 7.1 (Tambara module)**.** A **Tambara module** is a profunctor $T \colon \mathbf{C}^{op} \times \mathbf{D} \to \mathbf{Set}$ equipped with a natural transformation as follows.

Such that the following morphism is the identity.

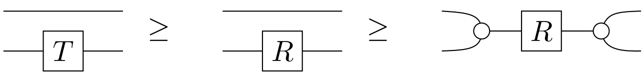


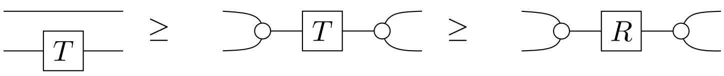
And such that the following two morphisms coincide





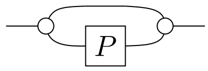
Definition 7.2 (Morphism of Tambara modules)**.** A morphism of Tambara modules $T \to R$ is a natural transformation between the profunctors such that the following two morphisms coincide.



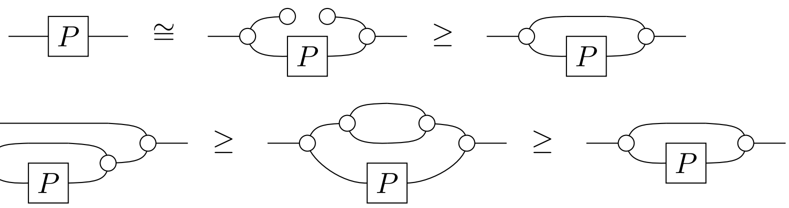


Tambara modules form a category. It can be checked that the composition of two morphisms of Tambara modules is a morphism of Tambara modules. We call た (hiragana for "ta") to the category of Tambara modules.

Definition 7.3 (Pastro-Street monad)**.** Consider the functorial assigment that maps a profunctor P to the following profunctor.

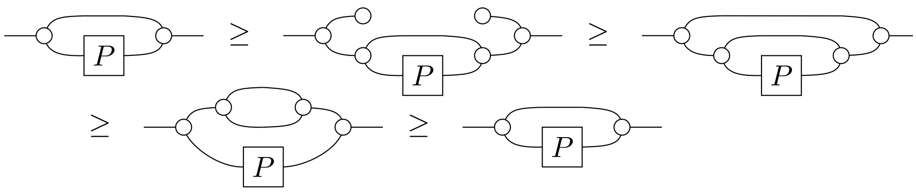


It is a monad Φ with the unit and multiplication given by the following morphisms.

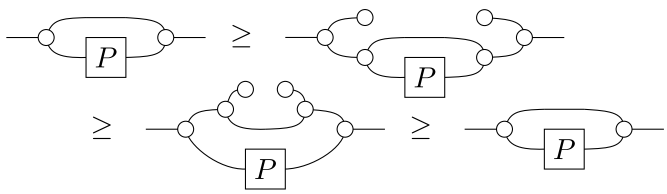


Proof. Let us prove left unitality, right unitality is analogous. The idea is to see that the following three derivations are homotopic.

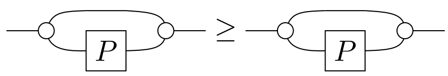
- Slice one.



- Slice two.



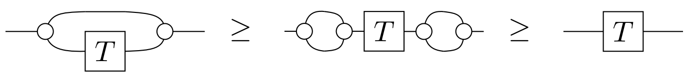
- Slice three.



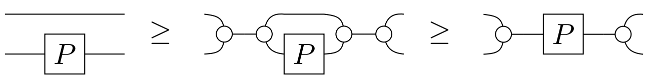
□

Theorem 7.4. *Tambara modules are the algebras of the Pastro-Street monad.*

Proof. Given a Tambara module, we construct an algebra as follows.



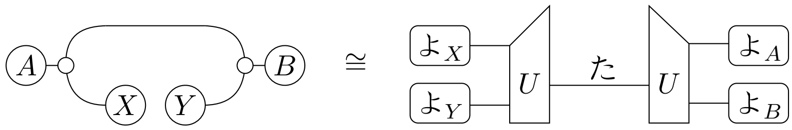
Given an algebra, we construct a Tambara module as follows.



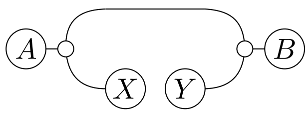
It remains to check that the two morphisms are inverse to each other, this follows from the axioms of monoidal categories. □

7.1.2 Profunctor representation

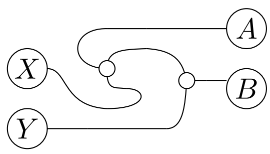
Theorem 7.5 (Profunctor Representation Theorem)**.** *There exists an isomorphism between elements of the following shape that moreover preserves composition.*



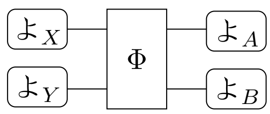
Proof. This proofs closely follows [Rom19, Theorem 5.3.1].



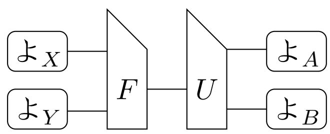
$\cong$



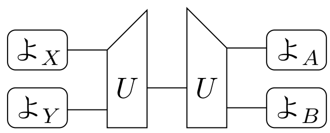
$\cong$ (Yoneda)



$\cong$ (Adjunction giving rise to Φ)



$\cong$



□