Coend Calculus via String Diagrams

The Coend approach to Incomplete Diagrams

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Abstract

If we interpret the morphisms in a monoidal category as black boxes that can be composed sequentially and in parallel, we are faced with the problem of interpreting what non-square boxes ought to represent. We propose a description in terms of coends and the cartesian bicategory of profunctors, with features of what could be considered a graphical calculus for these coends.

1 Introduction

1.1 Incomplete Diagrams

Morphisms in monoidal categories are interpreted as processes with inputs and outputs and generally represented by boxes. However, that raises the question of what to do if the process does not consume all the inputs at the same time or produces all the outputs at the same time. Consider for instance a process that consumes an input, produces an output, then consumes a second input and ends up producing an final output. Graphically, we have a clear idea of how this process should be represented, even if it is not a morphism in the category.

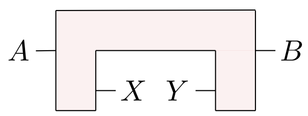


Figure 1: A process with a non-standard shape. The input A is taken at the beginning, then the output X is produced, strictly after that, the input Y is taken; finally, the output B is produced.

Reasoning graphically, it is obvious, for instance, that we should be able to *plug* a morphism connecting the first output to the second input inside this process and get back an actual morphism of the category.

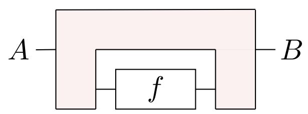


Figure 2: It is possible to plug a morphism $f\colon X \to Y$ inside the previous process (Figure 1), and, importantly, get back a morphism $A \to B$.

The particular shape depicted above has been extensively studied by [Ril18] under the name of (monoidal) *optic*; and it has many applications in bidirectional data accessing [PGW17, Kme18]. It has appeared also in the work of Kissinger and Uijlen as a notational convention [KU17]. It can be shown that *boxes* of that shape should correspond to elements of a suitable *coend* (Figure 5, for an introduction to coend calculus see [Lor15]). The intuition under a coend representation is that one should consider a tuple of morphisms and then quotient out by an equivalence relation generated by all the wires that are connected between these morphisms.

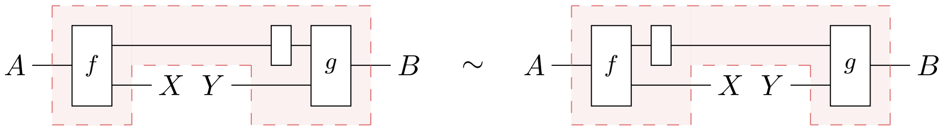


Figure 3: A box of this shape is meant to represent a pair of morphisms in a monoidal category quotiented out by "sliding a morphism" over the upper wire.

It had remained unclear, however, how this process should be carried in full generality and if it was in solid ground. Are we being formal when we use *incomplete* diagrams? What happens with all the other possible shapes that one would want to consider in a monoidal category?[1] They are interesting on their own; for instance, the last one of these has three inputs and two outputs, but the first input cannot affect the last output; and the last input cannot affect the first one.

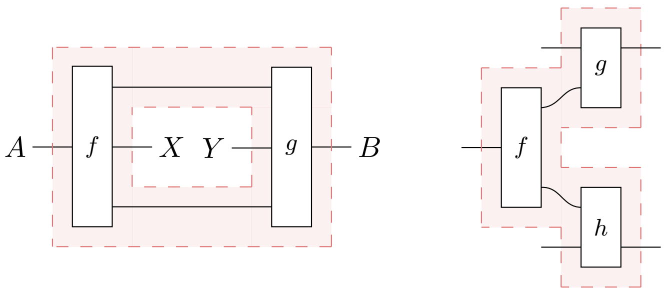


Figure 4: Some other shapes for boxes in a monoidal category.

This note starts from the idea that incomplete diagrams should be interpreted in the cartesian bicategory of profunctors. Compositions of incomplete diagrams corresponds to reductions that employ the cartesian bicategory structure.

1.2 Coend Calculus

Definition 1.1. The **coend** $\int^{X \in \mathbf{C}} P(X,X)$ of a profunctor $P \colon \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{Set}$ is the universal object endowed with morphisms

$$i_A \colon P(A,A) \to \left(\int^{X \in \mathbf{C}} P(X,X) \right)$$

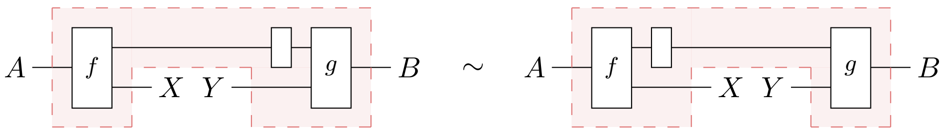
[1] For the last of these shapes, this author thanks Nathaniel Virgo, who posed it as a question on `categorytheory.zulipchat.com`.

for every $A \in \mathbf{C}$ such that, for any morphism $f: B \to A$ in $\mathbf{C}$, they satisfy $i_B \circ P(f, \mathrm{id}) = i_A \circ P(\mathrm{id}, f)$. It is universal in the sense that any other object D endowed with morphisms $j_A: P(A, A) \to D$ satisfying the same condition factors uniquely through it.

In other words, the coend is the coequalizer of the action on morphisms on both arguments of the functor. In other words, an element of the coend is an equivalence class of pairs $[X, x \in P(X, X)]$ quotiented by the equivalence relation generated by $[X, P(f, -)(u)] \sim [Y, P(-, f)(u)]$.

$$\int^{X \in \mathbf{C}} P(X, X) \cong \mathrm{coeq}\left(\bigsqcup_{f: B \to A} P(A, B) \rightrightarrows \bigsqcup_{X \in \mathbf{C}} P(X, X) \right).$$

Our plan is to use these quotient relations to deal with the naturality arising in non-square monoidal shapes.



$$\int^{M} \mathbf{C}(A, M \otimes X) \times \mathbf{C}(M \otimes Y, B).$$

Figure 5: We go back to the previous example to check how it coincides with the quotienting arising from the dinaturality of a coend.

2 The Cartesian Bicategory of Profunctors

Definition 2.1. There exists a cartesian bicategory **Prof** of profunctors having as objects the (small) categories; as 1-cells from $\mathbf{A}$ to $\mathbf{B}$ the profunctors $\mathbf{A}^{op} \times \mathbf{B} \to \mathbf{Set}$; as 2-cells, the natural transformations; and as tensor product, the cartesian product of categories. Composition of two profunctors $P: \mathbf{A}^{op} \times \mathbf{B} \to \mathbf{Set}$ and $Q: \mathbf{B}^{op} \times \mathbf{C} \to \mathbf{Set}$ returns a profunctor $(P \diamond Q): \mathbf{A}^{op} \times \mathbf{C} \to \mathbf{Set}$ defined by

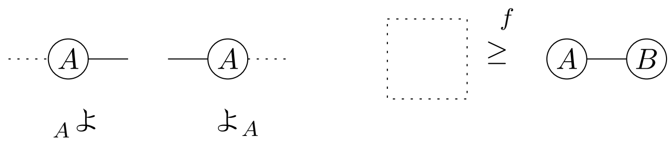
$$(P \diamond Q)(A, C) \coloneqq \int^{B \in \mathbf{B}} P(A, B) \times Q(B, C).$$

The monoidal product of two profunctors $P_1: \mathbf{A}_1^{op} \times \mathbf{B}_1 \to \mathbf{Set}$ and $P_2: \mathbf{A}_2^{op} \times \mathbf{B}_2 \to \mathbf{Set}$ is the profunctor $(P_1 \otimes P_2): (\mathbf{A}_1 \times \mathbf{A}_2)^{op} \times (\mathbf{B}_1 \times \mathbf{B}_2) \to \mathbf{Set}$ defined by

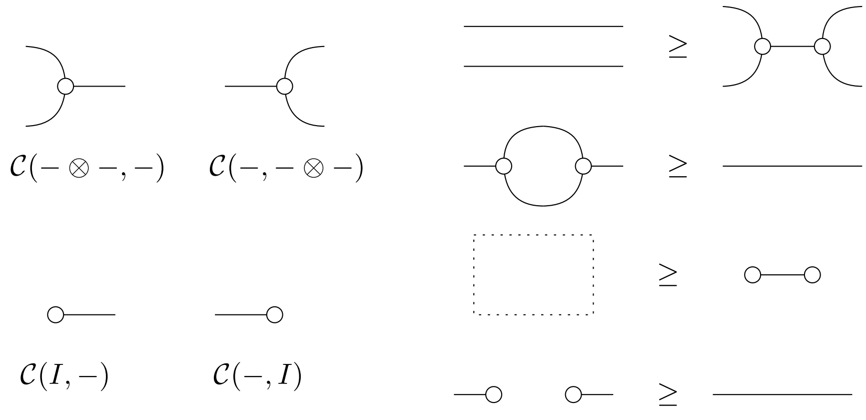
$$(P_1 \otimes P_2)(A_1, A_2, B_1, B_2) \coloneqq P_1(A_1, B_1) \times P_2(A_2, B_2).$$

We will employ the graphical calculus of monoidal categories for the cartesian structure of **Prof**; and, at the same time, we will use inequalities between diagrams to denote the 2-cells of the bicategorical structure, which are natural transformations. This is in analogy with the graphical calculus for categories of relations.

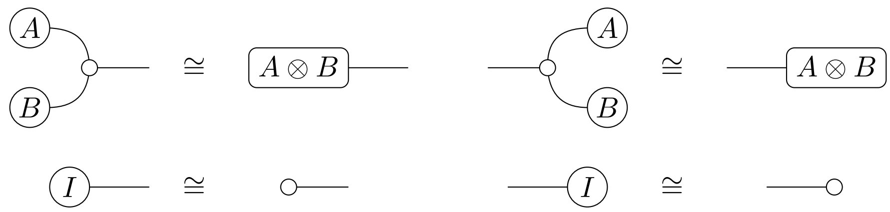
Definition 2.2 (Yoneda embeddings)**.** Every object $A \in \mathbf{C}$ determines two profunctors $_{A}\text{よ} \coloneqq \mathbf{C}(A, -): \mathbf{1}^{op} \times \mathbf{C} \to \mathbf{Set}$ and $\text{よ}_{A} \coloneqq \mathbf{C}(-, A): \mathbf{C}^{op} \times \mathbf{1} \to \mathbf{Set}$ called their contravariant and covariant Yoneda embeddings. Every morphism $f \in \mathbf{C}(A, B)$ can be seen as a natural transformation $f\colon \mathbf{1} \to {}_{A}\text{よ} \diamond \text{よ}_{B}$.



Definition 2.3. A **promonoidal category** is a pseudomonoid in the category **Prof**. A **procomonoidal category** is a pseudocomonoid in the category **Prof**. Every monoidal category is both promonoidal and procomonoidal. Moreover, the promonoidal structure is right adjoint to the procomonoidal structure.



This monoidal structure interacts nicely with Yoneda embeddings in the following sense.



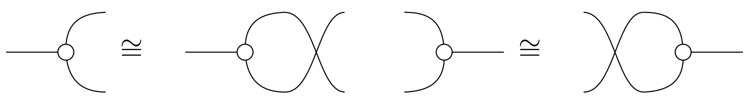
Remark 2.4. Scalars on this monoidal category are sets. We will be defining sets by using this graphical calculus.

2.1 Cartesian and cocartesian categories

Proposition 2.5. *Note that every object is a monoid and a comonoid in a canonical way in* **Prof***. A monoidal category is* ***cartesian*** *if and only if its promonoidal structure coincides with its canonical comonoid structure; a monoidal category is* ***cocartesian*** *if and only if its procomonoidal structure coincides with its canonical monoid structure.*

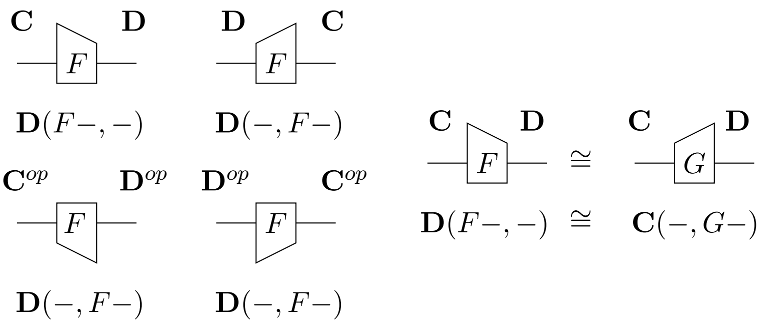
2.2 Symmetric categories

Proposition 2.6. *A category* **C** *is symmetric exactly when its probimonoidal structure is commutative.*



2.3 Adjoints

Definition 2.7. Let $F\colon \mathbf{C} \to \mathbf{D}$ be a functor. It can be embedded into profunctors in two ways, as a profunctor $\mathbf{D}(F-,-)\colon \mathbf{C}^{op} \times \mathbf{D} \to \mathbf{Set}$ or as a profunctor $\mathbf{D}(-,F-)\colon \mathbf{D}^{op} \times \mathbf{C} \to \mathbf{Set}$. We denote them with the following graphical notation and we note the particular case of what it means to be an adjoint functor.



The suggestive shape of the boxes is actually matched by their semantics. Every category has a dual (namely, its opposite category) and functors circulate as expected through the cups and the caps that represent dualities.

3 Examples of Open Diagrams

We have all the ingredients to start considering open diagrams. Let us consider the following examples. On the left we have a shape, on the right we have its description as an open shape. Lastly, as they are scalars, their interpretation is a set described as a coend.

3.1 Motivating examples

Consider the examples that motivated the introduction (in Figure 4). By writing them again with the graphical calculus of the cartesian bicategory **Prof**, we can obtain formulaic descriptions of the shapes, but also compose them in different ways.

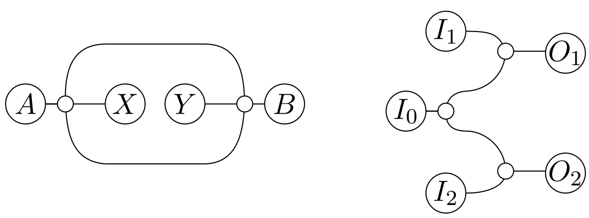


Figure 6: The shapes of Figure 4, interpreted as coends.

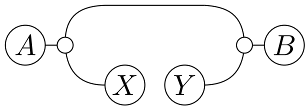
The corresponding coend descriptions are as follows.

$$\int^{M,N} \mathbf{C}(A, M \otimes X \otimes N) \times \mathbf{C}(M \otimes Y \otimes N, B),$$

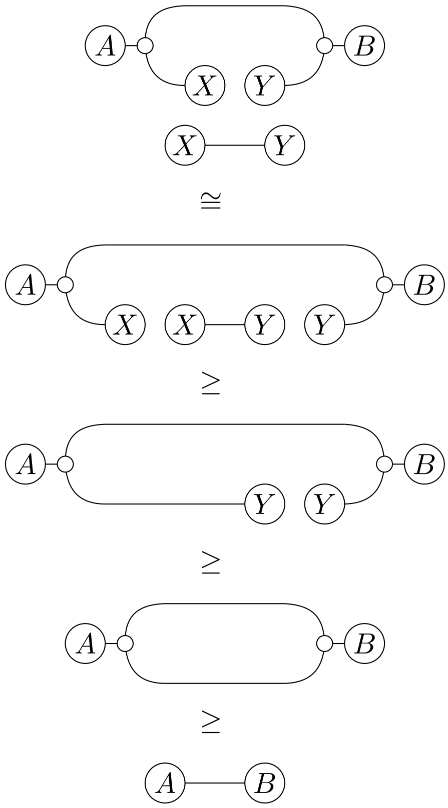
$$\int^{M,N} \mathbf{C}(I_0, M \otimes N) \times \mathbf{C}(I_1 \otimes M, O_1) \times \mathbf{C}(N \otimes I_2, O_2).$$

3.2 Lenses and optics

Definition 3.1. A monoidal lens [Ril18, “Optic” in Definition 2.0.1], from $A, B \in \mathbf{C}$ to $X, Y \in \mathbf{C}$ is an element of the following set.

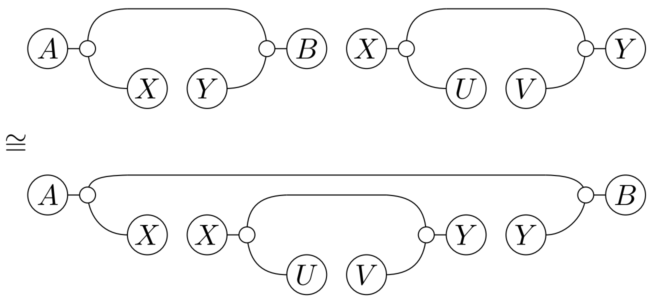


Example 3.2. As detailed in the introduction, a lens $(A, B) \to (X, Y)$ can be composed with a continuation $X \to Y$ to obtain a morphism $A \to B$. Let us illustrate this composition in the graphical calculus of **Prof**. It is also interpreted into the following chain of coend calculus, that describes that same composition.



$$
\begin{aligned}
&\left(\int^{M} \mathbf{C}(A, M \otimes X) \times \mathbf{C}(M \otimes Y, B)\right) \times \mathbf{C}(X, Y) \\
\to\ &\int^{M} \mathbf{C}(A, M \otimes X) \times \mathbf{C}(M \otimes X, M \otimes Y) \times \mathbf{C}(M \otimes Y, B) \\
\to\ &\int^{M,N} \mathbf{C}(A, M \otimes N) \times \mathbf{C}(M \otimes N, B) \\
\to\ &\mathbf{C}(A, B).
\end{aligned}
$$

Example 3.3. Two lenses of types $(A, B) \to (X, Y)$ and $(X, Y) \to (U, V)$ can be also composed with each other.



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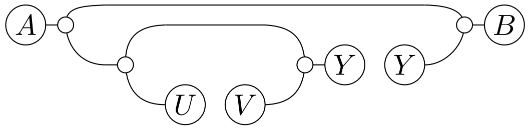

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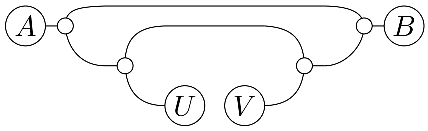

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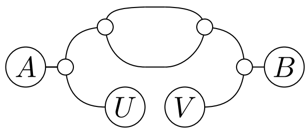

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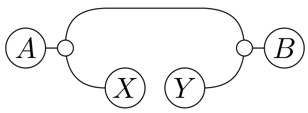
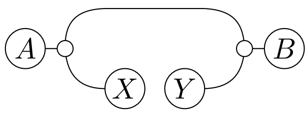


Example 3.4. There is, however, another way of composing two lenses. Let us write a lens $(A, B) \to (X, Y)$ as $(A, Y) \twoheadrightarrow (B, X)$. It can be composed with a lens $(B, X) \twoheadrightarrow (C, Z)$ into a lens $(A, B) \twoheadrightarrow (C, Z)$. As we will see, this is an option only when the category is symmetric.

3.2.1 Cartesian lenses

Proposition 3.5. *In a cartesian category* $\mathbf{C}$*, a lens* $(A, B) \to (X, Y)$ *is given by a pair of morphisms* $\mathbf{C}(A, X)$ *and* $\mathbf{C}(A \times Y, B)$*. In a cocartesian category, these are called* prisms *[Kme18] and they are given by a pair of morphisms* $\mathbf{C}(S, A + T)$ *and* $\mathbf{C}(B, T)$*.*

Proof. We write the proof for lenses, the proof for prisms is dual. The original proof can be found in [Mil17].



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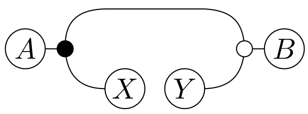

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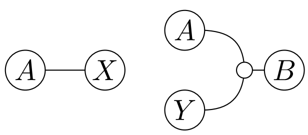
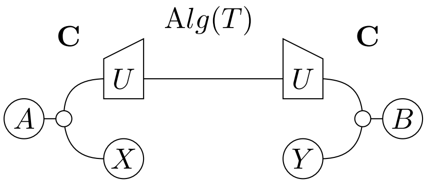


$\square$

3.2.2 Algebraic lenses

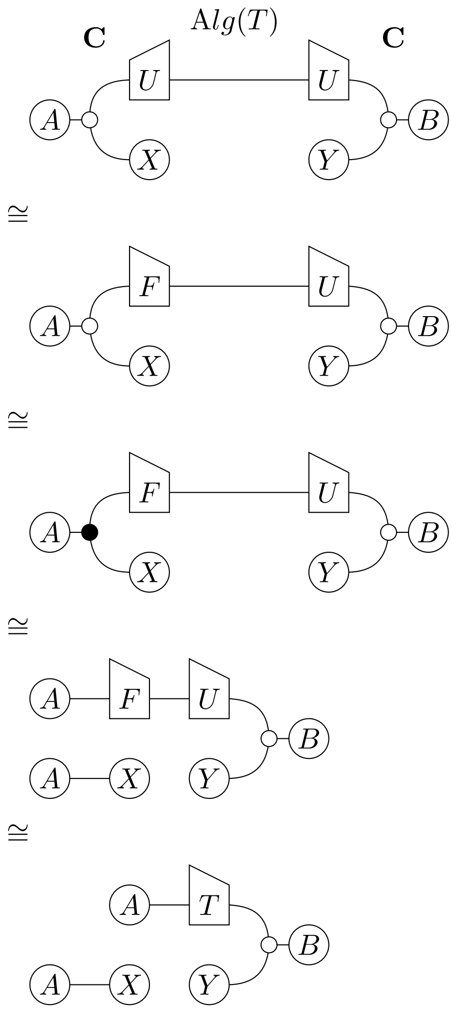
We have discussed the topic of monoidal lenses in the introduction, but there is a much more general notion of lenses, called optics. All of them can be nicely represented in the graphical calculus of **Prof** but, for simplicity, we will focus on a particular family of lenses that restrict the category over which the coend is taken. This showcases a technique that we consider important on its own: restricting the coend to slightly adjust a definition.

Definition 3.6. [CEG+20, Definition 3.9] Let $\mathbf{C}$ be a cartesian category and let $T\colon \mathbf{C} \to \mathbf{C}$ be a monad; let $F\colon \mathbf{C} \to \mathrm{Alg}(T)$ and $U\colon \mathrm{Alg}(T) \to \mathbf{C}$ be the free and forgetful functors to the Eilenberg-Moore category, respectively. An algebraic lens $(A, B) \to (X, Y)$ is an element of the following set.



Proposition 3.7. *In a cartesian category, an algebraic lens is equivalent to a pair of functions* $\mathbf{C}(A, X)$ *and* $\mathbf{C}(TA \times Y, B)$.

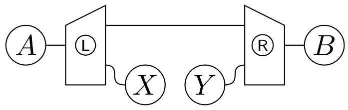
Proof.



□

3.2.3 Mixed optics

For the specialized reader, it may be of interest to finally consider the definition of a **mixed optic**, the generalized version of an optic. Let $\mathbf{M}$ a monoidal category acting with two monoidal actions $\textcircled{\scriptsize L}\colon \mathbf{M} \times \mathbf{C} \to \mathbf{C}$ and $\textcircled{\scriptsize R}\colon \mathbf{M} \times \mathbf{D} \to \mathbf{D}$. A mixed optic $(A, B) \to (X, Y)$ is an element of the following set.

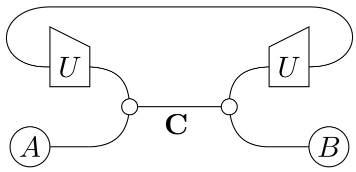


3.3 Circ construction

As one may expect, depicting diagrams in the category of profunctors gives us much more freedom than one would have drawing diagrams in the monoidal category. This extra freedom can be used to represent monoidal processes of different shapes, but in some sense it also allows us to identify any two wires. If we do not preserve the flow of time (understood as the order of sequential compositions) during this identification, we can introduce notions of feedback.

Definition 3.8. [KSW02, Definition 2.4] Let $\mathbf{C}$ be a monoidal category and let $U\colon \mathrm{Core}(\mathbf{C}) \to \mathbf{C}$ be the obvious inclusion functor from its core. For any two objects $X, Y \in \mathbf{C}$, we define a morphism in the $\mathsf{Circ}_{\mathbf{C}}$ construction to be an element of the following set.

$\mathrm{Core}(\mathbf{C})^{op}$



This is equivalently, an element of the following coend.

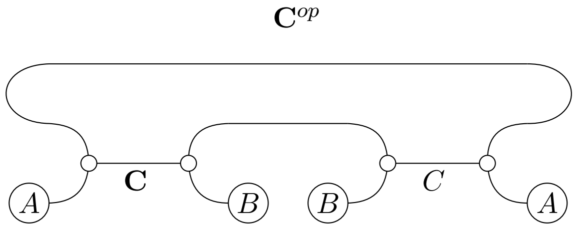
$$\mathsf{Circ}_{\mathbf{C}}(X, Y) \coloneqq \int^{M \in \mathrm{Core}(\mathbf{C})} \mathbf{C}(M \otimes X, M \otimes Y).$$

As we can see this is a much compressed description of the free category with feedback. It is also arguable that it makes intuitive sense: we are literally adding the feedback as a wire, and the fact that only isomorphisms travel the feedback reflects in the choice of wire. From this description, is easy to see that the Circ construction can be slightly modified into multiple variants by just changing the category over which we take the coend, in the same style we did for algebraic lenses.

3.4 Learners

On a recent article, Fong and Johnson [FJ19] have proposed an approach to machine learning based on category theory by constructing a monoidal category where morphisms represent supervised learning algorithms. In this category **Learn**, a morphism is given by the following data.

Definition 3.9. Let $(\mathbf{C}, \otimes, I)$ be a monoidal category. A (monoidal) **learner** taking inputs on an object A and producing outputs on an object B is an element of the following set.



This is a generalization to the monoidal case of an alternative definition proposed by Riley [Ril18]. It lets us view learners under a different light, even if it can be extremely different from the usual presentation.

$$\int^{P,Q\in\mathbf{C}} \mathbf{C}(P \otimes A, Q \otimes B) \times \mathbf{C}(Q \otimes B, P \otimes A)$$

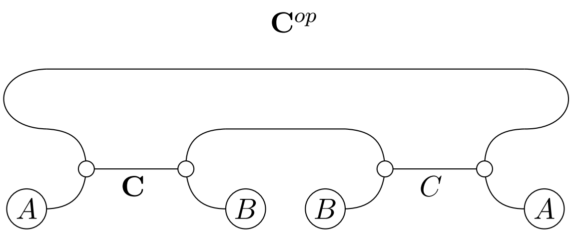
The original definition is conceptually clearer as it splits the learner into the functions is should implement. Let us show both are equivalent in the case of **Set**.

Proposition 3.10. *A **learner** taking inputs on a set A and producing outputs on a set B is given by*

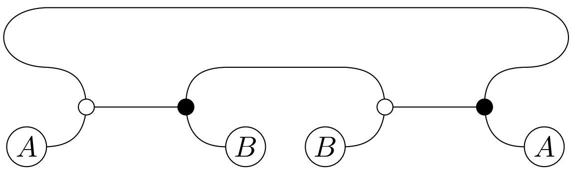
- *a set of* parameters *P,*
- *an* implementation *function $i\colon P \times A \to B$,*
- *an* update *function $u\colon P \times A \times B \to P$, and*
- *a* request *function $r\colon P \times A \times B \to A$.*

Let $(\mathbf{C}, \otimes, I)$ be a monoidal category and consider shapes of the following form. In the case of cartesian categories, this coincides with a learner.

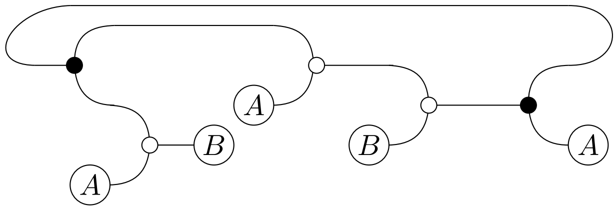
Proof. The following derivation appears, in the case of **Set**, in [Ril18, Definition 6.4.1]. It is included here for the sake of completeness, but also to show how it works under the graphical calculus.



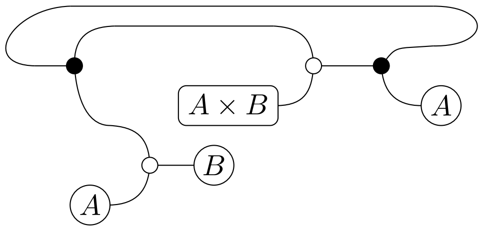
$\cong$ (Cartesianness)



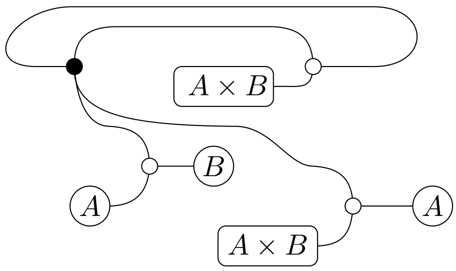
$\cong$ (Split)



$\cong$ (Monoidal product)



$\cong$ (Split)



$\square$

4 Related work

- The cartesian bicategory of relations, as used in [BPS17], and graphical regular logic [FS18] are arguably presenting a decategorification of this same idea.
- Finite *combs*, as described in [Rom20], can be combined in many different things with this technique. We are presenting a way of describing these possible compositions. The fact that they can be structured on an operad has been claimed by Jules Hedges[2]. Note however, that present approach is strictly more general than considering combs, as *combs* assume the symmetric structure of the underlying monoidal category, whereas we are dealing here with arbitrary shapes.

5 Conclusions

We have presented a way to study and compose *black boxes* in monoidal categories that do not necessarily have the usual squared box. This allows us to consider process with non-trivial dependencies between inputs and outputs. The calculus of the category **Prof** seems particularly practical to accomodate open systems; being backed up at the same time with a nice formalism in terms of coend calculus [Lor15].

There is an important shortcoming to this approach: this allows us to compute using coend calculus, but not (co)end calculus. In other words, *ends* are missing from the picture;

[2] Personal communication.

and this is expected. As it happens with regular logic, only the existential quantifier is recovered.

5.1 Three dimensional calculus

The graphical calculus we are using is three-dimensional in nature. It is only because of simplicity and technical constraints that we choose to work in some form of sliced two-dimensional graphical calculus. It does not need to be this way, `homotopy.io` is a proof assistant supporting n-dimensional diagrammatic calculus that can render three-dimensional diagrams [HHV19].

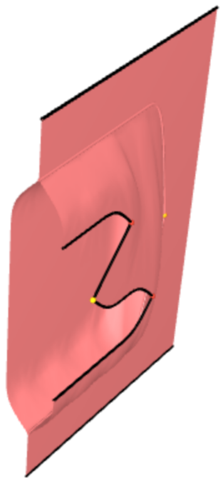


Figure 7: A monoidal lens is composed with a particular continuation. It is similar (except for the choice of continuation) to the construction in Example 3.2.

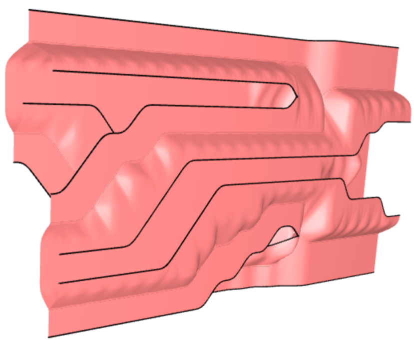


Figure 8: Two lenses composing. At the left end, we can see the figure of the two lenses. They get merged in the middle. At the right end, we can see a single lens.

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