Linear logic is quantity-sensitive

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This is an exercise on Samson Abramsky's notes on the course Categories, Proofs and Processes. [AT10]

The main idea here is that we can formalize the notion that linear logic is sensitive to the number of ocurrences of each variable using the model it has on groups, which are particular cases of symmetric monoidal closed categories.

Statement. *Can you construct proofs in Linear Logic of the following sequents?*

- $A \vdash A \otimes A$
- $\vdash (A \multimap (A \multimap B)) \multimap (A \multimap B)$
- $\vdash A \multimap (B \multimap A)$

Consider the discrete category of the integers with the monoidal structure given by addition, which is associative, and 0 as the unit. With this we can construct the structural isomorphisms as identities. This is a symmetric category because addition is commutative. It is monoidal closed with $(a \multimap b) = b - a$ because we have the following adjunction, where morphisms represent equalities.

$$\frac{a + b = c}{a = c - b}$$

Naturality of the structural isomorphisms and of the adjunction follow from the fact that we are in a discrete category.

Now, neither $a = a+a$, $0 = (b-a)-((b-a)-a) = a$, nor $0 = (a-b)-a = -b$ are true in general. It is impossible to create a morphism of these types in a symmetric monoidal closed category in general.

This shows that linear logic is resource sensitive. We could assume that each object has a weight and that morphisms have to be balanced. The interpretation on integers precisely reflects this 'weight'.

References

[AT10] Samson Abramsky and Nikos Tzevelekos. Introduction to categories and categorical logic. In *New structures for physics*, pages 3–94. Springer, 2010.