NOTES ON PROFUNCTOR OPTICS

ACT2019 SCHOOL GROUP 6

Contents

1. Preliminaries

Note: *This section should follow* [Lor15] *and* [PS08].

1.1. Dinaturality and ends.

Definition 1. A **dinatural transformation** $\alpha\colon P \xrightarrow{\cdot\cdot} Q$ between profunctors $P, Q\colon \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{D}$ is a family $\alpha_c\colon P(c,c) \to Q(c,c)$ such that for any $f\colon c \to c'$ the following hexagon

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commutes.

$$\begin{tikzcd}
& P(c,c) \arrow[dr, "\alpha_c"] & \\
P(c',c) \arrow[ur, "P(f,c)"] \arrow[d, "P(c',f)"] & & Q(c,c) \arrow[d, "Q(c,f)"] \\
P(c',c') \arrow[dr, "\alpha_{c'}"'] & & Q(c,c') \\
& Q(c',c') \arrow[ur, "Q(f,c')"'] &
\end{tikzcd}$$

Definition 2. The **end** of a profunctor $F\colon \mathbf{C} \times \mathbf{C}^{op} \to \mathbf{D}$ is an object giving a terminal wedge $\omega\colon \mathrm{end}(F) \stackrel{..}{\to} F$.

Explicitly, this means that, if we have any other wedge $\beta\colon d \stackrel{..}{\to} F$, and any function $f\colon c \to c'$, there is a unique $h\colon d \to \mathrm{end}(F)$ making the diagram commute.

$$\begin{tikzcd}
& d \arrow[d, dashed, "h"] \arrow[ddl, bend right, "\beta_c"'] \arrow[ddr, bend left, "\beta_{c'}"] & \\
& \mathrm{end}(F) \arrow[dl, "\omega_{c'}"'] \arrow[dr, "\omega_c"] & \\
F(c',c') \arrow[dr, "F(f,1)"'] & & F(c,c) \arrow[dl, "F(1,f)"] \\
& F(c,c') &
\end{tikzcd}$$

1.2. **Ninja Yoneda lemma.**

Proposition 3. *For any functor $K\colon \mathbf{C}^{op} \to \mathbf{Sets}$, we have*

$$K \cong \int^{c\in\mathbf{C}} Kc \times \mathbf{C}(\Box, c), \qquad K \cong \int_{c\in\mathbf{C}} \mathbf{Set}(\mathbf{C}(c,\Box), Kc) \cong \int_{c\in\mathbf{C}} Kc^{\mathbf{C}(c,\Box)}.$$

For any functor $H\colon \mathbf{C} \to \mathbf{Sets}$, we have

$$H \cong \int^{c\in\mathbf{C}} Hc \times \mathbf{C}(c, \Box), \qquad H \cong \int_{c\in\mathbf{C}} Hc^{\mathbf{C}(\Box,c)}.$$

1.3. **Day convolution.**

Definition 4. Let $\mathbf{A}$ be a V-enriched monoidal closed category. The V-category of V-functors $[\mathbf{A}, V]$ is monoidal with the tensor product given by **Day convolution** and the unit given by the states.

$$(F * G)(a) = \int^{x,y\in\mathbf{A}} \mathbf{A}(x \otimes y, a) \otimes Fx \otimes Gy, \qquad Ja = \mathbf{A}(i, a).$$

Note that Day convolution can be equivalently written as follows using the adjunction and the Dirac Yoneda lemma when we have closedness.

$$(F * G)(a) = \int^{y\in\mathbf{A}} F({}^{y}a) \otimes Gy = \int^{x\in\mathbf{A}} Fx \otimes Ga^{x}.$$

Which also coincide with the two equivalent ways of writting convolution, once we have substraction.

1.4. **Tambara modules.**

Definition 5. Let $\mathbf{A}$ be a monoidal V-category. A **left Tambara module** on $\mathbf{A}$ is a V-profunctor $T\colon \mathbf{A}^{op} \otimes \mathbf{A} \to V$ with a family of morphisms

$$\alpha(a)_{x,y} \colon T(x,y) \to T(a \otimes x, a \otimes y)$$

V-natural in the three arguments and such that $\alpha(i)_{x,y} = \mathrm{id}$ and $\alpha(a)_{a'\otimes x, a'\otimes y} \circ \alpha(a')_{x,y} = \alpha(a \otimes a')_{x,y}$.

Definition 6. Let $\mathbf{A}$ be a monoidal V-category. A **Tambara module** is a V-profunctor $T\colon \mathbf{A}^{op} \otimes \mathbf{A} \to V$ with both α_l and α_r, left and right Tambara structures respectively. They must satisfy a bimodule compatiblity rule.

$$\alpha_r(b) \circ \alpha_l(a) = \alpha_l(a) \circ \alpha_r(b)$$

1.5. **The category Prof.**

Definition 7. The bicategory **Prof** has categories as 0-cells, profunctors as 1-cells, and natural transformations as 2-cells. Profunctor composition has the hom endoprofunctor as its identity and it is associative, both only up to isomorphism. Natural transformations can be *vertically* composed and the *horizontal* composition of two natural transformations $\alpha\colon P \to R$ and $\beta\colon Q \to S$ is the natural transformation $(\alpha \circ \beta)\colon PQ \to RS$ that is defined in the component (a,c) by a morphism

$$\left(\int_b P(a,b) \otimes Q(b,c)\right) \to \left(\int_b R(a,b) \otimes S(b,c)\right),$$

that is constructed applying the universal property of the coends to the following morphism

$$(\alpha \otimes \beta)\colon P(a,b) \otimes Q(b,c) \to R(a,b) \otimes S(b,c).$$

A **promonad** is a monoid in **Prof**. Fixing a category $\mathbf{C}$, a promonad $P\colon \mathbf{C} \nrightarrow \mathbf{C}$ is given by some unit $\eta_{a,b}\colon \mathbf{C}(a,b) \to P(a,b)$ and some multiplication $\mu_{a,c}\colon \left(\int_b P(a,b) \otimes P(b,c)\right) \to P(a,c)$. Using continuity of the hom-functor and closedness of the Benabou cosmos, we can rewrite multiplication as an end.

$$\mu\colon \int^b P(a,b) \to P(b,c) \to P(a,c)$$

Proposition 8. *A promonad determines an identity-on-objects functor. An identity-on-objects functor determines a promonad.*

Proof. We can take a category with the same objects but using the hom-sets given by the profunctor and the multiplication as composition. The action of the functor on morphisms is given by η. This is functorial because of the monoid laws.

On the other direction, given an identity-on-objects functor $J\colon \mathbf{C} \to \mathbf{C}'$, we can construct the representable endoprofunctor $P(c,d) = \mathbf{C}'(Jc, Jd)$. (?) □

1.6. **Kleisli object.**

Recall that we can consider monoids (a,t,μ,η) and right monoid modules (b,g,κ) in a bicategory.

Definition 9. A **Kleisli object**, $(a_t, f_t\colon a \to a_t, \lambda\colon f_t \circ t \Rightarrow f_t)$ is the universal right module, in the sense that any right module (b,g,κ) can be written uniquely as this module in parallel with some $u\colon a_t \to b$.

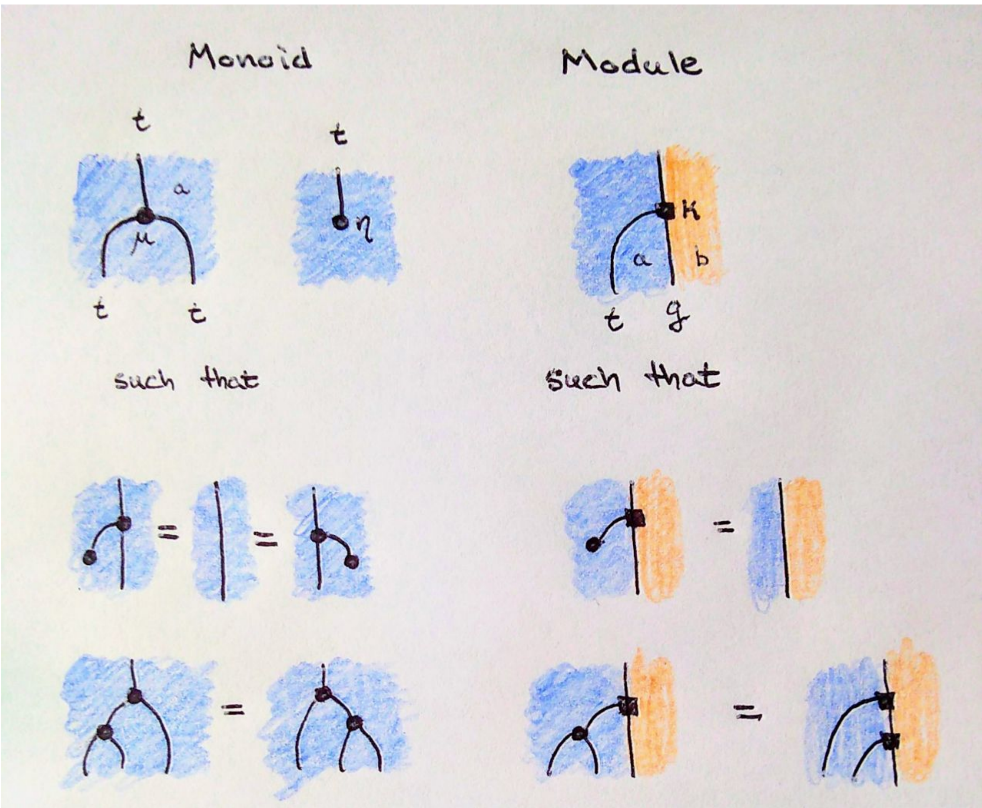

FIGURE 1. Monoids and modules over a monoid.

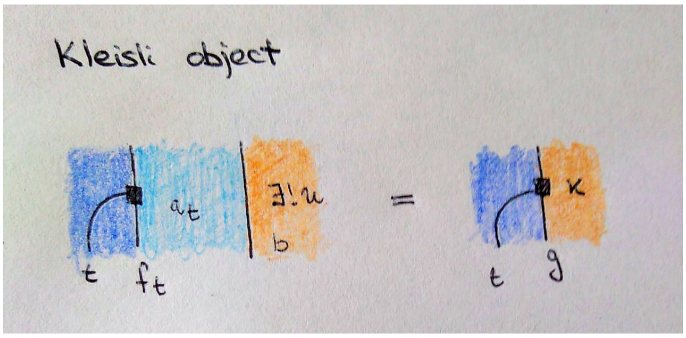

FIGURE 2. A Kleisli object

2. Profunctor representation of lenses and prisms

2.1. Θ is a comonad.

2.2. Coalgebras for Θ are Tambara modules.

Lemma 10. *There is an equivalence* $\mathbf{Coalg}(\Theta) \simeq \mathbf{Tamb}$.

Proof. Given a coalgebra, we will construct a Tambara module. The coalgebra gives us the following family of functions, natural in a and b.

$$\alpha_{a,b} \colon p(a,b) \to \int_{c \in \mathbf{C}} p(c \otimes a, c \otimes b)$$

Composing with all the possible projections for the end, we get the a family $\alpha_{a,b,c} \colon p(a,b) \to p(c \otimes a, c \otimes b)$. This family is extranatural in c because of the wedge condition of the end. This is the data for the Tambara module. Note that every Tambara module can be written in this way.

We check now that it satisfies the conditions on a Tambara module thanks to the coalgebra axioms. Because of the definition of the counit, $\alpha_{a,b,i} = \eta \circ \alpha_{a,b} = \mathrm{id}$; and because of the definition of the comultiplication, we have that $\alpha_{c' \otimes a, c' \otimes b, c} \circ \alpha_{a,b} = \mu \circ \alpha_{a,b}$. The following diagram proves this second equation: (1) the top square commutes because of the definition of the functor Θ, (2) the left one because of the coalgebra axioms, and (3) the right one because of the definition of σ.

$$\begin{tikzcd}
 & & p(ca,cb) \arrow[drr, "\alpha"] & & \\
\int_x p(xa,xb) \arrow[urr, "\pi_c"] \arrow[drr, "\Theta\alpha"] & & & & \int_x p(xca,xcb) \arrow[d, "\pi_d"] \\
p(a,b) \arrow[u, "\alpha"] \arrow[drr, "\alpha"] & & \int_y \int_x p(xya,xyb) \arrow[urr, "\pi_c"] & & p(dca,dcb) \\
 & & \int_x p(xa,xb) \arrow[u, "\sigma"] \arrow[urr, "\pi_{dc}"] & &
\end{tikzcd}$$

For morphisms, we check that, given a natural transformation between profunctors that determines a coalgebra homomorphism, we can see that it is a Tambara morphism. Here the upper square commutes because of the definition of Θ and the lower one because of the coalgebra axioms.

$$\begin{tikzcd}
p(ca,cb) \arrow[r, "\eta"] & q(ca,cb) \\
\int_x p(xa,xb) \arrow[u, "\pi"] \arrow[r, "\Theta\eta"] & \int_x q(xa,xb) \arrow[u, "\pi"] \\
p(a,b) \arrow[u, "\alpha"] \arrow[r, "\eta"] \arrow[uu, bend left=60] & q(a,b) \arrow[u, "\alpha"] \arrow[uu, bend right=60]
\end{tikzcd}$$

Because of the universal property of the end, given a Tambara morphism making the external square commute, by definition of Θ, we have that $\Theta\eta$ makes also the lower square commute. We have a full and faithful translation between homomorphisms. This proves the equivalence. $\square$

2.3. **Representation of lenses and prisms.**

We want to show that the two representations of lenses are equivalent.

Theorem 11. *For any monoidal category* $\mathbf{C}$*, and any objects* $a, b, s, t \in \mathbf{C}$*, the following is a bijection.*

$$\int_{p \in \mathbf{Tamb}} \mathbf{Set}(p(a,b), p(s,t)) \cong \int^{c \in \mathbf{C}} \mathbf{C}(s, c \otimes a) \times \mathbf{C}(c \otimes b, t)$$

Proof. We define a functor $\Theta\colon \mathbf{Prof}(\mathbf{C},\mathbf{C}) \to \mathbf{Prof}(\mathbf{C},\mathbf{C})$, which is a comonad.

$$(\Theta p)(s,t) = \int_{c\in\mathbf{C}} p(c\otimes s, c\otimes t)$$

It has a left adjoint $\Phi \dashv \Theta$ that must be a monad.

$$(\Phi p)(s,t) = \int^{c,x,y\in\mathbf{C}} \mathbf{C}(s, c\otimes x) \times \mathbf{C}(c\otimes y, t) \times p(x,y)$$

We consider the category of coalgebras over this comonad, $\mathbf{Coalg}(\Theta)$, which is equivalent to the category of algebras over the monad $\mathbf{Alg}(\Phi)$. Both share the forgetful functor $U\colon \mathbf{Coalg}(\Theta) \to \mathbf{Prof}(\mathbf{C},\mathbf{C})$ up to this equivalence.

Note: *This is an adjoint triple.*

In general we have the following formula

$$\int_{p\in\mathbf{Coalg}(\Theta)} Up(a,b) \to Up(s,t) \cong \Phi(\mathbf{C}^{op}\times\mathbf{C}((a,b),-))(s,t)$$

and integrating over the coend we get the desired result.

$$\begin{aligned}&\Phi(\mathbf{C}^{op}\times\mathbf{C}((a,b),-))(s,t)\\ &\quad\cong \int^{c,x,y\in\mathbf{C}} \mathbf{C}(s,c\otimes x)\times\mathbf{C}(c\otimes y,t)\times(\mathbf{C}^{op}\times\mathbf{C})((a,b),(x,y))\\ &\quad\cong \int^{c} \mathbf{C}(s,c\otimes a)\times\mathbf{C}(c\otimes b,t). \quad \square\end{aligned}$$

A coalgebra for that comonad is the following natural transformation, which can be rewritten as the data for the profunctor to be a Tambara module using continuity of the hom-functor.

$$p(a,b) \to \int_{c\in\mathbf{C}} p(c\otimes a, c\otimes b) \cong \int_{c\in\mathbf{C}} p(a,b)\to p(c\otimes a, c\otimes b)$$

Corollary 12. *For any cartesian closed category $\mathbf{C}$, and any objects $a,b,s,t\in\mathbf{C}$, the following is a bijection.*

$$\int_{p\in\mathbf{Cartesian}} \mathbf{Set}(p(a,b),p(s,t)) \cong \mathbf{C}(s, a\times(b\to t))$$

Proof. We can apply Theorem 11. The cartesian closed structure gives $\mathbf{C}(c\times b,t)\cong\mathbf{C}(c,b\to t)$, and then by Yoneda lemma we have the following.

$$\int^{c\in\mathbf{C}} \mathbf{C}(s,c\times a)\times\mathbf{C}(c\times b,t) \cong \mathbf{C}(s,(b\to t)\times a) \qquad \square$$

2.4. Doubles.

Proposition 13. *Left adjoint functors $[\mathbf{D}^{op},\mathbf{Set}]\to[\mathbf{C}^{op},\mathbf{Set}]$ are equivalent to profunctors $\mathbf{C}^{op}\times\mathbf{D}\to\mathbf{Set}$.*

Note: *I should prove that this is an equivalence, describing the action on morphisms and showing this is full and faithful.*

Proof. Given a functor $\Psi\colon[\mathbf{C}^{op},\mathbf{Set}]\to[\mathbf{D}^{op},\mathbf{Set}]$, we can define the profunctor $\Omega(x,y)=\Psi(\mathbf{C}(-,y))(x)$. Given any profunctor Ω, its preimage under this assignment must be such that $\Psi(\mathbf{C}(-,y))=\Omega(-,y)$; and using that Ψ is a left adjoint (cocontinuous) functor and that every presheaf is a colimit of representables, this determines the totality of Ψ. $\square$

In [PS08], the monad on endoprofunctors $\Psi\colon [\mathbf{C}^{op}\times\mathbf{C},\mathbf{Set}] \to [\mathbf{C}^{op}\times\mathbf{C},\mathbf{Set}]$, which is left adjoint to the Θ comonad, is seen as a promonad $\check{\Psi}\colon \mathbf{C}^{op}\times\mathbf{C} \nrightarrow \mathbf{C}^{op}\times\mathbf{C}$.

3. Tannakian reconstruction for endofunctors

Let (M,η,μ) be a monad on a category $\mathbf{C}$, complete and cartesian closed. Modules over the monad (that are endofunctors) form the category $\mathbf{Mmod}$, with natural transformations that preserve the monad action as morphisms. There is an obvious forgetful functor $U\colon \mathbf{Mmod} \to [\mathbf{C},\mathbf{C}]$.

The monad can be represented in itself, thanks to the monad axioms. We want to show that $U \cong \mathbf{Mmod}(M,-)$. Can we show that $\mathbf{Mmod}$ is enriched over $[\mathbf{C},\mathbf{C}]$? Note that $[\mathbf{C},\mathbf{C}]$ is cartesian closed.

This should be an application Theorem 2.3 in nlab https://ncatlab.org/nlab/show/Tannaka+duality.

4. Reading: optics.tex

4.1. General equation for optics.

```
-- We define some profunctor categories that are given by certain
-- classes. Cartesian (strong), cocartesian (choice), closed functors
-- are examples.
class Profunctor p => Cartesian p where
  first :: p a b -> p (a,c) (b,c)
  second :: p a b -> p (c,a) (c,b)

class Profunctor p => Choice p where
  first :: p a b -> p (Either a c) (Either b c)
  second :: p a b -> p (Either c a) (Either c b)

class Profunctor p => Closed p where
  closed :: p a b -> p (x -> a) (x -> b)

-- With these, we can get the basic profunctor optics for Haskell.
type Iso   s t a b = forall p . Profunctor  p => p a b -> p s t
type Lens  s t a b = forall p . Cartesian   p => p a b -> p s t
type Prism s t a b = forall p . Cocartesian p => p a b -> p s t
type Grate s t a b = forall p . Closed      p => p a b -> p s t
```

Isos, lenses, prisms and grates can be described by a single equation. For some $\mathbf{E}$ subcategory of profunctors.

$$O(s,t,a,b) = \int_{p\in\mathbf{E}} p(a,b) \to p(s,t)$$

Using the component functor, this can be rewritten as follows.

$$O(s,t,a,b) = [\mathbf{E},\mathbf{Set}]\left(V_{(a,b)}, V_{(s,t)}\right)$$

4.2. Component functor.

Definition 14. The **component functor** $V_a\colon [\mathbf{C}, \mathbf{D}] \to \mathbf{D}$ for an object $a \in \mathbf{C}$ is given by evaluation of functors and natural transformations on that object. This is a representable functor by the Yoneda lemma.

$$[\mathbf{C}, \mathbf{Set}](\mathbf{C}(a, -), f) \cong V_a f$$

Proposition 15. *The Day convolution of two component functors is the component functor of their tensor product,* $V_a \star V_b = V_{a\otimes b}$.

References

[Lor15] Fosco Loregian. This is the (co)end, my only (co)friend. *arXiv preprint arXiv:1501.02503*, 2015.

[PS08] Craig Pastro and Ross Street. Doubles for monoidal categories. *Theory and applications of categories*, 21(4):61–75, 2008.