Cartesian closed categories

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A **cartesian closed category** $\mathbb{C}$ can be defined as a category having a right adjoint of the unique functor to the terminal category $*\colon \mathbb{C} \to 1$, a right adjoint of the diagonal functor $\Delta\colon \mathbb{C} \to \mathbb{C} \times \mathbb{C}$, and a right adjoint of $- \times A\colon \mathbb{C} \to \mathbb{C}$ for each $A \in \mathbb{C}$. These three adjoints correspond to the existence of a *terminal* object, binary *products* and *exponentials*.

$$\frac{* \longrightarrow *}{C \xrightarrow{!} 1} \qquad \frac{C, C \xrightarrow{f,g} A, B}{C \xrightarrow{\langle f,g \rangle} A \times B} \qquad \frac{C \times A \xrightarrow{f} B}{C \xrightarrow{\widetilde{f}} B^A}$$

These three rules match the three introduction rules for the simply typed lambda calculus. We interpret C as a context Γ and each morphism $a\colon C \to A$ as a term $\Gamma \vdash a : A$.

$$\frac{}{\Gamma \vdash * : 1} \qquad \frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : B}{\Gamma \vdash a, b : A \times B} \qquad \frac{\Gamma, a : A \vdash b : B}{\Gamma \vdash (\lambda a.b) : A \to B}$$

Now, we should discuss if β-equivalence corresponds to the equality between morphisms.