The traversal is the optic for polynomial functors

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We present a derivation of the traversal as the optic associated with polynomial functors. This is still work in progress and I have not checked all the details, I might retract this if it does not work.

The idea here is that optics can be described as associated to the action of some pseudomonoid. This approach was previously taken by Riley but we make this work with Bartosz' Yoneda-with-adjunction trick applied to arbitrary actions. We can consider the same monad Pastro-Street use for Tambara modules but for an arbitrary pseudomonoid action.

Yoneda with adjunction

Lemma 1. *Let Φ be a monad in $\mathbf{Prof}$ and let $U\colon \Phi{-}\mathbf{alg} \to \mathbf{Prof}$ be the forgetful functor. We know the following isomorphism.*

$$\int_p (Up)(a,b) \to (Up)(s,t) \cong (\Phi(\mathbf{C}^{op} \times \mathbf{C})((a,b),-))(s,t)$$

Comonad for an action

Lemma 2. *Let $(\cdot)\colon \mathbf{M} \times \mathbf{C} \to \mathbf{C}$ be an action of a monoidal category on a category. The following should form a comonad where the unit and the multiplication are given by the unit and tensor of the monoidal category (or, in general, the pseudomonoid structure).*

$$(\Theta p)uv = \int_{m \in \mathbf{M}} p(m \cdot u, m \cdot v)$$

The laws should follow from the laws of the pseudomonoid.

Theorem 3. *The comonoid has a right adjoint given by*

$$(\Phi p)xy = \int^{u,v,m} \mathbf{C}(x, m \cdot u) \times \mathbf{C}(m \cdot v, y) \times p(u,v).$$

General formula

Applying Yoneda with adjunction to this monad yields

$$\left(\int_{p \in \Theta\text{-}\mathbf{coAlg}} p(a,b) \to p(s,t)\right) \cong \int^{m \in \mathbf{M}} \mathbf{C}(s, m \cdot a) \times \mathbf{C}(m \cdot b, t).$$

Where we are quantifying over the functors that have a coalgebra for the comonad. That is, the profunctors that preserve the the action.

$$p(a,b) \to \int_{m \in \mathbf{M}} p(m \cdot a, m \cdot b)$$

The idea is that it should generalize the results described (in Riley and) first in Bartosz' post on profunctor optics for Tambara modules to an arbitrary action. We need to check details.

Traversal

Polynomial functors $f \colon \mathbf{C} \to \mathbf{C}$ in a bicartesian closed category are these that can be written as $f(x) = \sum_n \lambda_n x^n$, where x^n is the n-fold cartesian product. We consider the category $\mathbf{Poly}$ of polynomial functors as a full subcategory of $[\mathbf{C}, \mathbf{C}]$. This category is monoidal with the monoidal product given by composition of endofunctors. There exists an action $\mathbf{Poly} \to \mathbf{Sets} \to \mathbf{Sets}$ given by evaluation.

Let $f \in \mathbf{Poly}$. It must be of the form $f(x) = \sum_n \lambda_n x^n$. We apply cocontinuity of hom, then we apply Yoneda n-times. Yoneda substitutes the coefficient λ_n inside f by $(b^n \to t)$.

$$\begin{aligned}
\int^{p \in \mathbf{Poly}} \mathbf{C}(s, f(a)) \times \mathbf{C}(f(b), t) \cong \\
\int^{p \in \mathbf{Poly}} \mathbf{C}(s, f(a)) \times \mathbf{C}(\sum_n \lambda_n b^n, t) \cong \\
\int^{p \in \mathbf{Poly}} \mathbf{C}(s, f(a)) \times \prod_n \mathbf{C}(\lambda_n b^n, t) \cong \\
\int^{p \in \mathbf{Poly}} \mathbf{C}(s, f(a)) \times \prod_n \mathbf{C}(\lambda_n, b^n \to t) \cong \\
\mathbf{C}\left(s, \sum_n a^n \times (b^n \to t)\right)
\end{aligned}$$

And thus the traversal is the associated optic to the action of polynomial functors on the category of sets. In other words,

$$(s \to \sum_n a^n \times (b^n \to t)) \cong \int_{p \in \mathsf{TambPoly}} p(a,b) \to p(s,t),$$

where $\mathsf{TambPoly}$ is the subcategory of profunctors that are Tambara modules for the action of polynomial functors. That is, that have families of morphisms as follows.

$$p(x,y) \to \int_{f \in \mathbf{Poly}} p(f(x), f(y))$$