

Pasting pullbacks

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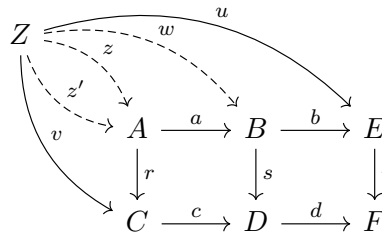
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This is an exercise on Samson Abramsky's notes on the course *Categories, Proofs and Processes*. [AT10]

Proposition 1. *In the following commutative diagram, if $ABCD$ and $BEDF$ are pullback squares, so is $AECF$.*

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow \\ C & \longrightarrow & D & \longrightarrow & F \end{array}$$

Proof. Given some $u: Z \rightarrow E$ and $v: Z \rightarrow C$ such that $tu = dc v$, we can use the pullback to construct some $Z \rightarrow B$ and the other pullback to construct some $Z \rightarrow A$. This proves existence. Given two morphisms z and z' such that $baz = u = baz'$ and $rz = v = rz'$, as in the following diagram, we will prove they are equal.



By the universal property of the pullback $BEDF$ there is a unique morphism $w: Z \rightarrow B$ such that $bw = u$ and $sw = cv$, but $baz = baz' = u$ and $saz = crz = cv = crz' = saz'$, so $az = az'$. Then using the pullback $ABCD$, there would be a unique morphism $y: Z \rightarrow A$ such that $ay = az$ and $v = ry$, but both z and z' satisfy this condition and thus they are equal. \square

Proposition 2. *In the following commutative diagram, if $BEDF$ and $AECF$ are pullback squares, so is $ABCD$.*

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow \\ C & \longrightarrow & D & \longrightarrow & F \end{array}$$

Proof. Given some $u: Z \rightarrow B$ and $v: Z \rightarrow C$ such that $su = cv$, we can apply the universal property of the pullback $AECF$ to the maps bu and v , which satisfy $tbu = dsu = dc v$, to construct some $z: Z \rightarrow A$ such that $baz = bu$ and $v = rz$.

By the universal property of $BEDF$, there exists a unique y such that $by = bu$ and $sy = cv$, but both az and u satisfy that, and thus they are equal. This proves existence.

$$\begin{array}{ccccccc} & & Z & & & & \\ & \swarrow & \downarrow & \searrow & & & \\ & & A & \xrightarrow{a} & B & \xrightarrow{b} & E \\ & \swarrow & \downarrow r & & \downarrow s & & \downarrow t \\ & & C & \xrightarrow{c} & D & \xrightarrow{d} & F \end{array}$$

(Note: In the original image, there are additional curved arrows from Z to A labeled z and z', and from Z to C labeled v, and a curved arrow from Z to B labeled u.)

Given two morphisms $z, z': Z \rightarrow A$ such that $az = u = az'$ and $rz = rz' = v$; we know by the universal property of the pullback $AECF$ applied to bu and v that there is a unique morphism $x: Z \rightarrow A$ such that $ba x = bu$ and $v = rx$; and thus they must be equal. \square

References

- [AT10] Samson Abramsky and Nikos Tzevelekos. Introduction to categories and categorical logic. In *New structures for physics*, pages 3–94. Springer, 2010.