Lawvere's fixed point theorem

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<2019-02-28 Thu 10:23>

This is a note stating the main result from Diagonal arguments and Cartesian Closed Categories by William Lawvere.

Definition. A morphism s : X Y is *point-surjective* if for each y : 1 Y, there exists some x : 1 X such that sx = y.

Theorem (Lawvere, 1969). In any cartesian closed category, if there exists a point-surjective morphism d : A (A B), then each morphism f : B B has a fixed point, that is, some b : B such that f b = b.

Proof. As d is point-surjective, there exists x : A such that dx = a.f(daa), but then, dxx = (a.f(daa)) x = f(dxx) is a fixed point.