Powerset-algebras are complete semilattices

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<2018-12-23 Sun 13:10>

This is a combination of two exercises from Samson Abramsky's notes on the course Categories, Proofs and Processes.

Definition 1. We take a **complete semilattice** to mean a poset $(P, \leqslant)$ where every subset $S \subseteq P$ has a least upper bound we write as $\bigvee S$. That is to say that it is a category with at most one morphism between any two objects that has all coproducts.

A morphism between complete semilattices is a map between posets that preserves all least upper bounds, that is $h\left(\bigvee S\right) = \bigvee h(S)$, where we write $h(S) = \{h(x) \mid x \in S\}$. This property implies monotonicity: note that from $x \leqslant y$ we get $h(y) = h(x \vee y) = h(x) \vee h(y)$, and thus $h(x) \leqslant h(y)$. We have created a category of complete semilattices with morphisms between them that we call $\mathbf{SL}$. There is a forgetful functor $U \colon \mathbf{SL} \to \mathbf{Set}$ assigning the underlying set to each poset.

Free-forgetful adjunction

Proposition 2. *The forgetful functor* $U \colon \mathbf{SL} \to \mathbf{Set}$ *has a left adjoint.*

Proof. We will show that the left adjoint is the functor $\mathcal{P} \colon \mathbf{Set} \to \mathbf{SL}$ sending each set to its powerset ordered by inclusion, which is a complete semilattice because the union of a family of sets is its least upper bound. Given two sets A and B, any function $f \colon A \to B$ induces a $f^* \colon \mathcal{P}A \to \mathcal{P}B$ sending a subset to its image under f; this constitutes the action of the functor on morphisms. We can check that this is a morphism of complete semilattices because, for any family $\mathcal{X} \subseteq \mathcal{P}(A)$, we have

$$f^*\left(\bigcup_{Y \in \mathcal{X}} Y\right) = \left\{ f(y) \;\middle|\; y \in \bigcup_{Y \in \mathcal{X}} Y \right\} = \bigcup_{Y \in \mathcal{X}} \{f(y) \mid y \in Y\} = \bigcup_{Y \in \mathcal{X}} f^*(Y).$$

To construct the adjunction, we start by defining an isomorphism $\mathbf{Set}(A, UB) \to \mathbf{SL}(\mathcal{P}A, B)$ that sends $f \colon A \to B$ to the function that acts on some $Y \subseteq A$ as

$$\overline{f}(Y) = \bigvee_{y \in Y} f(y).$$

This is a morphism of complete semilattices because for any family of subsets $\mathcal{X} \subseteq \mathcal{P}(A)$ we have

$$\overline{f}\left(\bigcup_{Y \in \mathcal{X}} Y\right) = \bigvee_{y \in \bigcup_{Y \in \mathcal{X}} Y} f(y) = \bigvee_{Y \in \mathcal{X}} \bigvee_{y \in Y} f(y) = \bigvee_{Y \in \mathcal{X}} \overline{f}(Y)$$

This isomorphism has an inverse sending each morphism $h \colon \mathcal{P}(A) \to B$ to the function $A \to B$ defined as $a \mapsto h(\{a\})$. We can check that these are in fact inverses because for any $f \colon A \to B$ and $a \in A$ we have

$$\overline{f}(\{a\}) = \bigvee_{a \in \{a\}} f(a) = f(a)$$

and for any semilattice morphism $h \colon \mathcal{P}(A) \to B$ we have for any $X \subseteq A$, using that it preserves least upper bounds, that

$$\overline{h(\{-\})}(X) = \bigvee_{x \in X} h(\{x\}) = h\left(\bigvee_{x \in X} \{x\}\right) = h(X).$$

We now show that the isomorphism is in fact natural. Given any $f \colon A \to UB$, any function $a \colon A' \to A$ and any morphism of semilattices $b \colon B \to B'$, we have for a given $Y \in \mathcal{P}(A)$ that

$$b\overline{f}a^*(Y) = b\left(\bigvee_{y \in Y} f(a(y))\right) = \bigvee_{y \in Y} bfa(y) = \overline{bfa}(Y),$$

because b preserves least upper bounds. This proves that $b \circ \overline{f} \circ a^* = \overline{b \circ f \circ a}$ and the isomorphism is thus natural. □

Powerset-algebras

Note that the powerset monad acts on objects as $U\mathcal{P}$, simply taking the powerset and forgetting about its semilattice structure. The unit of the adjunction, $a \mapsto \{a\}$ is precisely the unit of the monad, and the counit of the adjunction, the lattice homomorphism $\bigvee \colon \mathcal{P}L \to L$ for any lattice L, is such that $U \bigvee_{\mathcal{P}} = \bigcup \colon \mathcal{P}\mathcal{P}A \to$

$\mathcal{P}A$ is precisely the multiplication of the monad for any set A.

We can now prove that $\mathcal{P}$-algebras are complete semilattices. Note that an algebra would be a function $f\colon \mathcal{P}A \to A$ such that

$$f\{a\} = a, \qquad f\left(\bigcup_{i\in I} A_i\right) = f\{f(A_i) \mid i \in I\},$$

for any element $a \in A$ and any family of subsets $A_i \subseteq A$. We can define a partial order where for any two elements $x, y \in A$, we have that $x \leqslant y$ when $f\{x, y\} = y$. This satisfies

- **reflexivity**, because $f\{x, x\} = x$;
- **transitivity**, because if $x \leqslant y \leqslant z$, then we have $f\{x, z\} = f\{f\{x\}, f\{y, z\}\} = f\{x, y, z\} = f\{f\{x, y\}, f\{z\}\} = z$;
- and **antisymmetry**, because $x \leqslant y \leqslant x$ implies $x = f\{x, y\} = y$.

For this particular preorder, the function f is the lowest great bound $\bigvee$; this can be proved checking that, for any subset $S \subseteq A$,

- for each $s \in S$, we have $f\{s, f(S)\} = f(S \cup \{s\}) = f(S)$, so $s \leqslant f(S)$;
- and given some $x \in A$ such that $s \leqslant x$ for all $s \in S$, we have $f\{f(S), x\} = f\left(\bigcup_{s\in S}\{s, x\}\right) = f\{x\} = x$, and thus $f(S) \leqslant x$.

Finally, we can check that taking the lowest great bound on a subset of a poset provides a valid algebra.

$$\bigvee\{a\} = a, \qquad \bigvee\left(\bigcup_{i\in I} A_i\right) = \bigvee\left\{\bigvee A_i \mid i \in I\right\}.$$

Note that once we know that all algebras are of this form, a $\mathcal{P}$-algebra morphism is precisely a function between complete semilattices $f\colon A \to B$ satisfying, $f\left(\bigvee A\right) = \bigvee f(A)$; that is, a complete semilattice morphism.

Monadicity theorem

This is also a consequence of Beck's monadicity theorem. The adjunction $\mathcal{P} \dashv U$ is monadic and that implies that the comparison functor $\mathbf{SL} \to \mathbf{Set}^{\mathcal{P}}$ between complete semilattices and powerset-algebras is an equivalence.