

Tannakian reconstruction of Tambara modules

Or, why that "optics" formula?

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1 Definition of Tambara module

Let \mathbf{M} be a monoidal category acting both on two arbitrary categories \mathbf{C} and \mathbf{D} . We write \underline{M} for the image of $M \in \mathbf{M}$ both in $[\mathbf{C}, \mathbf{C}]$ and $[\mathbf{D}, \mathbf{D}]$.

Definition 1. A **Tambara module** consists of a profunctor $P: \mathbf{C}^{op} \times \mathbf{D} \rightarrow \mathbf{Sets}$ endowed with a family of morphisms $\alpha_M: P(A, B) \rightarrow P(\underline{M}A, \underline{M}B)$ natural in both $A \in \mathbf{C}$ and $B \in \mathbf{D}$, and dinatural in $M \in \mathbf{M}$; which additionally makes the following diagrams commute.

$$\begin{array}{ccc} P(A, B) & \xrightarrow{\alpha_I} & P(\underline{I}A, \underline{I}B) \\ & \searrow \text{id} & \downarrow \cong \\ & & P(A, B) \end{array} \quad \begin{array}{ccc} P(\underline{N}A, \underline{N}B) & \xrightarrow{\alpha_M} & P(\underline{MNA}, \underline{MNB}) \\ \alpha_N \uparrow & & \downarrow \cong \\ P(A, B) & \xrightarrow{\alpha_{N \otimes M}} & P(\underline{M \otimes N}A, \underline{M \otimes N}B) \end{array}$$

Remark 2. The original definition of Tambara module [T⁺06] deals only with actions that arise from a monoidal product $\otimes: \mathbf{C} \rightarrow [\mathbf{C}, \mathbf{C}]$. We use the term *Tambara module* also for the more general concept, allowing for arbitrary monoidal actions.

We can extend Pastro and Street [PS08] construction of free Tambara module over a profunctor $P: \mathbf{C}^{op} \times \mathbf{D} \rightarrow \mathbf{Sets}$ to the case of general monoidal actions. Tambara modules are equivalently algebras for a monad Ψ defined by

$$\Psi P(S, T) = \int^{M, X, Y} \mathbf{C}(S, \underline{M}X) \times \mathbf{D}(\underline{M}Y, T) \times P(X, Y).$$

We know how to construct free Tambara modules. What is the free Tambara module over a representable functor $\text{hom}((A, B), -)$? We call it **Optic** $((A, B), -)$, and it can be written as

$$\mathbf{Optic}((A, B), -) \cong \int^M \mathbf{C}(S, \underline{M}A) \times \mathbf{D}(\underline{M}B, T).$$

That is, the formula for optics is given by the free Tambara module on a representable functor.

2 Tannakian reconstruction

Milewski [Mil17], and then Boisseau and Gibbons [BG18], proved a unified profunctor representation theorem for optics, that is widely used in programming libraries such as Kmett's *lens* [Kme18].

Theorem 3. Let $\mathcal{U}_{(A,B)}: \mathcal{T} \rightarrow \mathbf{Sets}$ the functor that evaluates a Tambara module on the object (A, B) . There exists an isomorphism

$$[\mathcal{T}, \mathbf{Sets}] (\mathcal{U}_{(A,B)}, \mathcal{U}_{(S,T)}) \cong \mathbf{Optic}((A, B), (S, T)),$$

natural on both (A, B) and (S, T) .

Proof. The claim is that this theorem is precisely Tannakian reconstruction for Tambara modules. We first note that, by definition, the functor $\mathcal{U}_{(A,B)}$ is represented by $\mathbf{Optic}((A, B), -)$, the free Tambara module over the hom-profunctor. In fact, for any Tambara module $P: \mathbf{C}^{\text{op}} \times \mathbf{C} \rightarrow \mathbf{Sets}$,

$$\mathcal{U}_{(A,B)} P \cong \text{Nat}(\text{hom}((A, B), -), P) \cong \mathcal{T}(\mathbf{Optic}((A, B), -), P).$$

Then, by Tannakian reconstruction, $[\mathcal{T}, \mathbf{Sets}] (\mathcal{U}_{(A,B)}, \mathcal{U}_{(S,T)}) \cong \mathbf{Optic}((A, B), (S, T))$. \square

References

- [BG18] Guillaume Boisseau and Jeremy Gibbons. What you needa know about Yoneda: Profunctor optics and the Yoneda Lemma (functional pearl). *PACMPL*, 2(ICFP):84:1–84:27, 2018.
- [Kme18] Edward Kmett. lens library, version 4.16. Hackage <https://hackage.haskell.org/package/lens-4.16>, 2012–2018.
- [Mil17] Bartosz Milewski. Profunctor optics: the categorical view. <https://bartoszmilewski.com/2017/07/07/profunctor-optics-the-categorical-view/>, 2017.
- [PS08] Craig Pastro and Ross Street. Doubles for monoidal categories. *Theory and applications of categories*, 21(4):61–75, 2008.
- [T⁺06] Daisuke Tambara et al. Distributors on a tensor category. *Hokkaido mathematical journal*, 35(2):379–425, 2006.