Morphisms of open games for iterated games

Supervisors: Dr Julian Hedges, Dr Jamie Vicary

Elena Di Lavore

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Outline

- Definition of open games
- An example: the Prisoner's dilemma
- Two different definitions of morphisms of open games
- How to model subgame perfection and iterated games
- Attempts to generalise this construction

Introduction

- Game theory models decision-making processes involving more than one agent
- The classical approach does not allow scalability
- The categorical approach is compositional

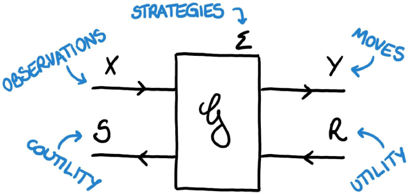
Open game

An open game

$$\mathcal{G} : \binom{X}{S} \xrightarrow{\Sigma}\!\!\!\!\!\!\!\!\not\;\;\;\; \binom{Y}{R}$$

is given by

- X, Y, R, S, Σ sets
- $\mathbb{P}_{\mathcal{G}} : \Sigma \times X \longrightarrow Y$ play function
- $\mathbb{C}_{\mathcal{G}} : \Sigma \times X \times R \longrightarrow S$ coplay function
- $\mathbb{B}_{\mathcal{G}} : X \times (Y \to R) \longrightarrow \mathcal{R}el(\Sigma)$ best response function



Ghani N., Hedges J., Winschel V., Zahn P., *Compositional game theory*, LICS '18

The category of open games

There is a symmetric monoidal category Game where open games are morphisms.

- objects are disets $\begin{pmatrix} X \\ S \end{pmatrix}$
- morphisms are open games $\mathcal{G} : \begin{pmatrix} X \\ S \end{pmatrix} \xrightarrow{\Sigma}\!\!\!\!\!\!\not\;\; \begin{pmatrix} Y \\ R \end{pmatrix}$

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A typical example: the prisoner's dilemma

Players: $P_i : \begin{pmatrix} 1 \\ 1 \end{pmatrix} \overset{\Sigma}{\nrightarrow} \begin{pmatrix} X \\ \mathbb{R} \end{pmatrix}$,

where $X = \{cooperate, defect\}$ and $\Sigma = 1 \to X$

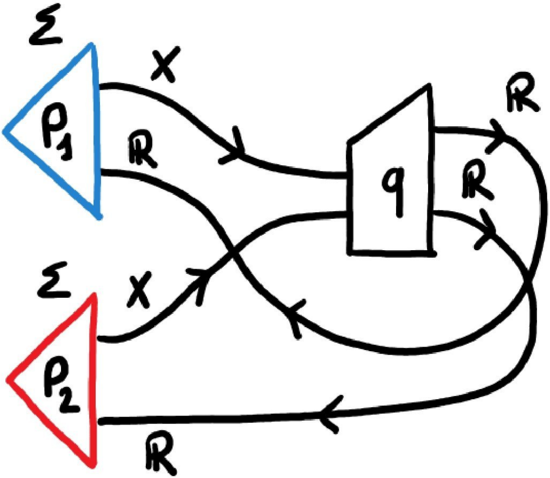
$\mathbb{B}_i(f) = \{(\sigma, \sigma') \in \Sigma^2 : f(\sigma) \leq f(\sigma')\}$

Utility function: $q : \begin{pmatrix} X \times X \\ 1 \end{pmatrix} \overset{1}{\nrightarrow} \begin{pmatrix} \mathbb{R} \times \mathbb{R} \\ 1 \end{pmatrix}$

$\mathbb{P}_q =$ *see table*

	C	D
C	22	30
D	03	11

A typical example: the prisoner's dilemma



Morphisms of open games - contravariant

A morphism of open games $\alpha : \mathcal{G} \longrightarrow \mathcal{G}'$ is given by

- a lens $s(\alpha) : \binom{X'}{S'} \longrightarrow \binom{X}{S}$
- a lens $t(\alpha) : \binom{Y'}{R'} \longrightarrow \binom{Y}{R}$
- a function $\Sigma(\alpha) : \Sigma \longrightarrow \Sigma'$

$$
\begin{tikzcd}
\binom{X}{S} \arrow[rr, "\mathcal{G}", "\Sigma"'] & {} \arrow[d, "\Sigma(\alpha)"] & \binom{Y}{R} \\
\binom{X'}{S'} \arrow[u, "s(\alpha)"] \arrow[rr, "\Sigma'", "\mathcal{G}'"'] & {} & \binom{Y'}{R'} \arrow[u, "t(\alpha)"']
\end{tikzcd}
$$

Hedges J., *Morphisms of open games*, arXiv:1711.07059v2 (2017)

The double category of open games

There is a symmetric monoidal pseudo double category 2Game $\uparrow$.

- Lens horizontal category
- Game $\uparrow$ vertical category

Hedges J., *Morphisms of open games*, arXiv:1711.07059v2 (2017)

Morphisms of open games - covariant

A morphism of open games $\alpha : \mathcal{G} \longrightarrow \mathcal{G}'$ is given by

- a lens $s(\alpha) : \begin{pmatrix} X \\ S \end{pmatrix} \longrightarrow \begin{pmatrix} X' \\ S' \end{pmatrix}$
- a lens $t(\alpha) : \begin{pmatrix} Y \\ R \end{pmatrix} \longrightarrow \begin{pmatrix} Y' \\ R' \end{pmatrix}$
- a function $\Sigma(\alpha) : \Sigma \longrightarrow \Sigma'$

$$\begin{tikzcd}
\begin{pmatrix} X \\ S \end{pmatrix} \arrow[rr, "\mathcal{G}", "\Sigma"'] \arrow[d, "s(\alpha)"] \arrow[drr, phantom, "\Sigma(\alpha)"] & & \begin{pmatrix} Y \\ R \end{pmatrix} \arrow[d, "t(\alpha)"'] \\
\begin{pmatrix} X' \\ S' \end{pmatrix} \arrow[rr, "\Sigma'", "\mathcal{G}'"'] & & \begin{pmatrix} Y' \\ R' \end{pmatrix}
\end{tikzcd}$$

The category Open

There is a category Open where open games are objects.

- objects are games of the form $\mathcal{G} : \begin{pmatrix} 1 \\ R \end{pmatrix} \xrightarrow[\Sigma]{} \begin{pmatrix} Y \\ R \end{pmatrix}$
- morphisms $\alpha : \mathcal{G} \longrightarrow \mathcal{G}'$ are given by functions $\alpha_Y : Y \longrightarrow Y'$, $\alpha_\Sigma : \Sigma \longrightarrow \Sigma'$

$$\begin{tikzcd}
\begin{pmatrix} 1 \\ R \end{pmatrix} \arrow[rr, "\mathcal{G}", "\Sigma"'] \arrow[d, equal] \arrow[drr, phantom, "\Sigma \xrightarrow{\alpha_\Sigma} \Sigma'"] & & \begin{pmatrix} Y \\ R \end{pmatrix} \arrow[d, "\alpha_Y"'] \\
\begin{pmatrix} 1 \\ R \end{pmatrix} \arrow[rr, "\Sigma'", "\mathcal{G}'"'] & & \begin{pmatrix} Y' \\ R \end{pmatrix}
\end{tikzcd}$$

Ghani N., Kupke C., Lambert A., Nordvall Forsberg F., *A compositional treatment of iterated open games*, Theoretical Computer Science, 741, 48-57 (2018)

The subgame perfection functor

Let $\mathcal{G} : \binom{X}{S} \xrightarrow[\Sigma]{} \binom{Y}{R}$.

For any set A, define $A \to \mathcal{G} : \binom{A \times X}{S} \xrightarrow[A \to \Sigma]{} \binom{A \times Y}{R}$ with

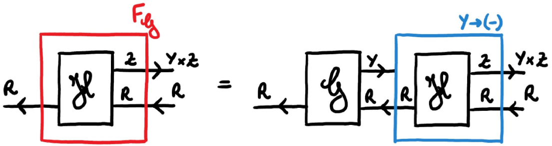
$$
\begin{aligned}
\mathbb{P}_{A\to\mathcal{G}}(f, a, x) &\coloneqq (a, \mathbb{P}_{\mathcal{G}}(f(a), x)) \\
\mathbb{C}_{A\to\mathcal{G}}(f, a, x, r) &\coloneqq \mathbb{C}_{\mathcal{G}}(f(a), x, r) \\
\mathbb{B}_{A\to\mathcal{G}}(a, x, k) &\coloneqq \{(f, f') \in (A \to \Sigma)^2 : \\
&\qquad \forall\ a' \in A\ (f(a'), f'(a')) \in \mathbb{B}_{\mathcal{G}}(x, k(a', -))\}
\end{aligned}
$$

$A \to (-)$ can be extended to a functor in the category Open.

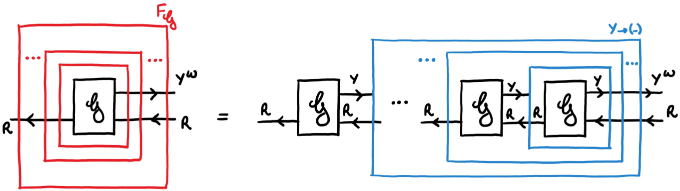
Ghani N., Kupke C., Lambert A., Nordvall Forsberg F., *A compositional treatment of iterated open games*, Theoretical Computer Science, 741, 48-57 (2018)

Iterated open games

Let $F_{\mathcal{G}}(\mathcal{H}) := (Y \to \mathcal{H}) \circ \mathcal{G}$. This is a functor on the category Open.



The final coalgebra of $F_{\mathcal{G}}$ models the infinite iteration of a game.



Ghani N., Kupke C., Lambert A., Nordvall Forsberg F., *A compositional treatment of iterated open games*, Theoretical Computer Science, 741, 48-57 (2018)

The subgame perfection NOT functor

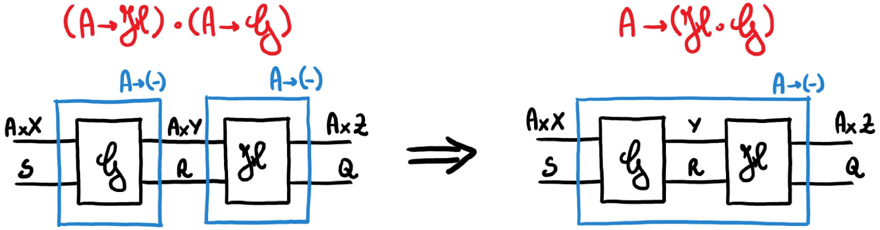
The definition of $A \to (-)$ can be extended to the double category $2\mathrm{Game}\uparrow$ by defining it on lenses as follows.

$$A \to \lambda : \begin{pmatrix} A \times X \\ S \end{pmatrix} \longrightarrow \begin{pmatrix} A \times Y \\ R \end{pmatrix}$$

- $v_{A\to\lambda}(a, x) = (a, v_\lambda(x))$ view function
- $u_{A\to\lambda}(a, x, r) = u_\lambda(x, r)$ update function

And it turns out this is NOT a (lax) double functor.

The problem



$(A \to \mathcal{H}) \circ (A \to \mathcal{G})$
$A \to (-)$
$A \to (-)$
$A \times X$
S
$\mathcal{G}$
$A \times Y$
R
$\mathcal{H}$
$A \times Z$
Q
$\Longrightarrow$
$A \to (\mathcal{H} \circ \mathcal{G})$
$A \to (-)$
$A \times X$
S
$\mathcal{G}$
Y
R
$\mathcal{H}$
$A \times Z$
Q

The subgame perfection functor - revisited

Let $\mathcal{G} : \binom{X}{S} \xrightarrow{\Sigma}\!\!\!\!\!\!\!\!\not\;\;\;\; \binom{Y}{R}$.

For any set A, define $A \to \mathcal{G} : \binom{A \times X^A}{S} \xrightarrow{A \to \Sigma}\!\!\!\!\!\!\!\!\!\!\!\!\not\;\;\;\;\;\;\;\; \binom{A \times Y^A}{R}$ with

$$
\begin{aligned}
\mathbb{P}_{A \to \mathcal{G}}(f, a, x) &:= (a, \mathbb{P}_{\mathcal{G}}(f(-), x(-))) \\
\mathbb{C}_{A \to \mathcal{G}}(f, a, x, r) &:= \mathbb{C}_{\mathcal{G}}(f(a), x(a), r) \\
\mathbb{B}_{A \to \mathcal{G}}(a, x, k) &:= \{(f, f') \in (A \to \Sigma)^2 : \\
&\qquad \forall\ a' \in A\ (f(a'), f'(a')) \in \mathbb{B}_{\mathcal{G}}(x(a'), \lambda y.ka'(\lambda t.y))\}
\end{aligned}
$$

The subgame perfection NOT functor - revisited

The definition of $A \to (-)$ can be extended to the double category $2\mathrm{Game}\uparrow$ by defining it on lenses as follows.

$$A \to \lambda : \begin{pmatrix} A \times X^A \\ S \end{pmatrix} \longrightarrow \begin{pmatrix} A \times Y^A \\ R \end{pmatrix}$$

- $v_{A\to\lambda}(a, x) = (a, v_\lambda \circ x)$ view function
- $u_{A\to\lambda}(a, x, r) = u_\lambda(x(a), r)$ update function

And it turns out even this is NOT a (lax) double functor.

Summary

- Definition of open games
- The Prisoner's dilemma
- Two different definitions of morphisms of open games
- The subgame perfection functor and how to model iterated games
- Attempts to generalise the functor to the double category