NOTES ON PROFUNCTOR OPTICS

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Contents

1. Preliminaries

Note: *This section should follow* [Lor15] *and* [PS08].

1.1. Dinaturality and ends.

Definition 1. A **dinatural transformation** $\alpha\colon P \xrightarrow{..} Q$ between profunctors $P, Q\colon \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{D}$ is a family $\alpha_c\colon P(c,c) \to Q(c,c)$ such that for any $f\colon c \to c'$ the following hexagon

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commutes.

$$\begin{tikzcd}
& P(c,c) \arrow[dr, "\alpha_c"] & \\
P(c',c) \arrow[ur, "{P(f,c)}"] \arrow[d, "{P(c',f)}"] & & Q(c,c) \arrow[d, "{Q(c,f)}"] \\
P(c',c') \arrow[dr, "\alpha_{c'}"'] & & Q(c,c') \\
& Q(c',c') \arrow[ur, "{Q(f,c')}"'] &
\end{tikzcd}$$

Definition 2. The **end** of a profunctor $F\colon \mathbf{C} \times \mathbf{C}^{op} \to \mathbf{D}$ is an object giving a terminal wedge $\omega\colon \mathrm{end}(F) \mathrel{\ddot{\to}} F$.

Explicitly, this means that, if we have any other wedge $\beta\colon d \mathrel{\ddot{\to}} F$, and any function $f\colon c \to c'$, there is a unique $h\colon d \to \mathrm{end}(F)$ making the diagram commute.

$$\begin{tikzcd}
& d \arrow[d, dashed, "h"] \arrow[ddl, bend right, "\beta_c"'] \arrow[ddr, bend left, "\beta_{c'}"] & \\
& \mathrm{end}(F) \arrow[dl, "\omega_{c'}"'] \arrow[dr, "\omega_c"] & \\
F(c',c') \arrow[dr, "{F(f,1)}"'] & & F(c,c) \arrow[dl, "{F(1,f)}"] \\
& F(c,c') &
\end{tikzcd}$$

1.2. Ninja Yoneda lemma.

Proposition 3. *For any functor $K\colon \mathbf{C}^{op} \to \mathbf{Sets}$, we have*

$$K \cong \int^{c\in\mathbf{C}} Kc \times \mathbf{C}(\Box, c), \qquad K \cong \int_{c\in\mathbf{C}} \mathbf{Set}(\mathbf{C}(c,\Box), Kc) \cong \int_{c\in\mathbf{C}} Kc^{\mathbf{C}(c,\Box)}.$$

For any functor $H\colon \mathbf{C} \to \mathbf{Sets}$, we have

$$H \cong \int^{c\in\mathbf{C}} Hc \times \mathbf{C}(c, \Box), \qquad H \cong \int_{c\in\mathbf{C}} Hc^{\mathbf{C}(\Box,c)}.$$

1.3. Day convolution.

Definition 4. Let $\mathbf{A}$ be a V-enriched monoidal closed category. The V-category of V-functors $[\mathbf{A}, V]$ is monoidal with the tensor product given by **Day convolution** and the unit given by the states.

$$(F * G)(a) = \int^{x,y\in\mathbf{A}} \mathbf{A}(x \otimes y, a) \otimes Fx \otimes Gy, \qquad Ja = \mathbf{A}(i, a).$$

Note that Day convolution can be equivalently written as follows using the adjunction and the Dirac Yoneda lemma when we have closedness.

$$(F * G)(a) = \int^{y\in\mathbf{A}} F({}^{y}a) \otimes Gy = \int^{x\in\mathbf{A}} Fx \otimes Ga^{x}.$$

Which also coincide with the two equivalent ways of writting convolution, once we have substraction.

1.4. Tambara modules.

Definition 5. Let $\mathbf{A}$ be a monoidal V-category. A **left Tambara module** on $\mathbf{A}$ is a V-profunctor $T\colon \mathbf{A}^{op} \otimes \mathbf{A} \to V$ with a family of morphisms

$$\alpha(a)_{x,y}\colon T(x,y) \to T(a \otimes x, a \otimes y)$$

V-natural in the three arguments and such that $\alpha(i)_{x,y} = \mathrm{id}$ and $\alpha(a)_{a'\otimes x, a'\otimes y} \circ \alpha(a')_{x,y} = \alpha(a \otimes a')_{x,y}$.

Definition 6. Let $\mathbf{A}$ be a monoidal V-category. A **Tambara module** is a V-profunctor $T\colon \mathbf{A}^{op} \otimes \mathbf{A} \to V$ with both α_l and α_r, left and right Tambara structures respectively. They must satisfy a bimodule compatiblity rule.

$$\alpha_r(b) \circ \alpha_l(a) = \alpha_l(a) \circ \alpha_r(b)$$

1.5. The category Prof.

Definition 7. The bicategory **Prof** has categories as 0-cells, profunctors as 1-cells, and natural transformations as 2-cells. Profunctor composition has the hom endoprofunctor as its identity and it is associative, both only up to isomorphism. Natural transformations can be *vertically* composed and the *horizontal* composition of two natural transformations $\alpha\colon P \to R$ and $\beta\colon Q \to S$ is the natural transformation $(\alpha \circ \beta)\colon PQ \to RS$ that is defined in the component (a,c) by a morphism

$$\left(\int_b P(a,b) \otimes Q(b,c)\right) \to \left(\int_b R(a,b) \otimes S(b,c)\right),$$

that is constructed applying the universal property of the coends to the following morphism

$$(\alpha \otimes \beta)\colon P(a,b) \otimes Q(b,c) \to R(a,b) \otimes S(b,c).$$

A **promonad** is a monoid in **Prof**. Fixing a category $\mathbf{C}$, a promonad $P\colon \mathbf{C} \nrightarrow \mathbf{C}$ is given by some unit $\eta_{a,b}\colon \mathbf{C}(a,b) \to P(a,b)$ and some multiplication $\mu_{a,c}\colon \left(\int^b P(a,b) \otimes P(b,c)\right) \to P(a,c)$. Using continuity of the hom-functor and closedness of the Benabou cosmos, we can rewrite multiplication as an end.

$$\mu\colon \int_b P(a,b) \to P(b,c) \to P(a,c)$$

Proposition 8. *A promonad determines an identity-on-objects functor. An identity-on-objects functor determines a promonad.*

Proof. We can take a category with the same objects but using the hom-sets given by the profunctor and the multiplication as composition. The action of the functor on morphisms is given by η. This is functorial because of the monoid laws.

On the other direction, given an identity-on-objects functor $J\colon \mathbf{C} \to \mathbf{C}'$, we can construct the representable endoprofunctor $P(c,d) = \mathbf{C}'(Jc, Jd)$. (?) $\square$

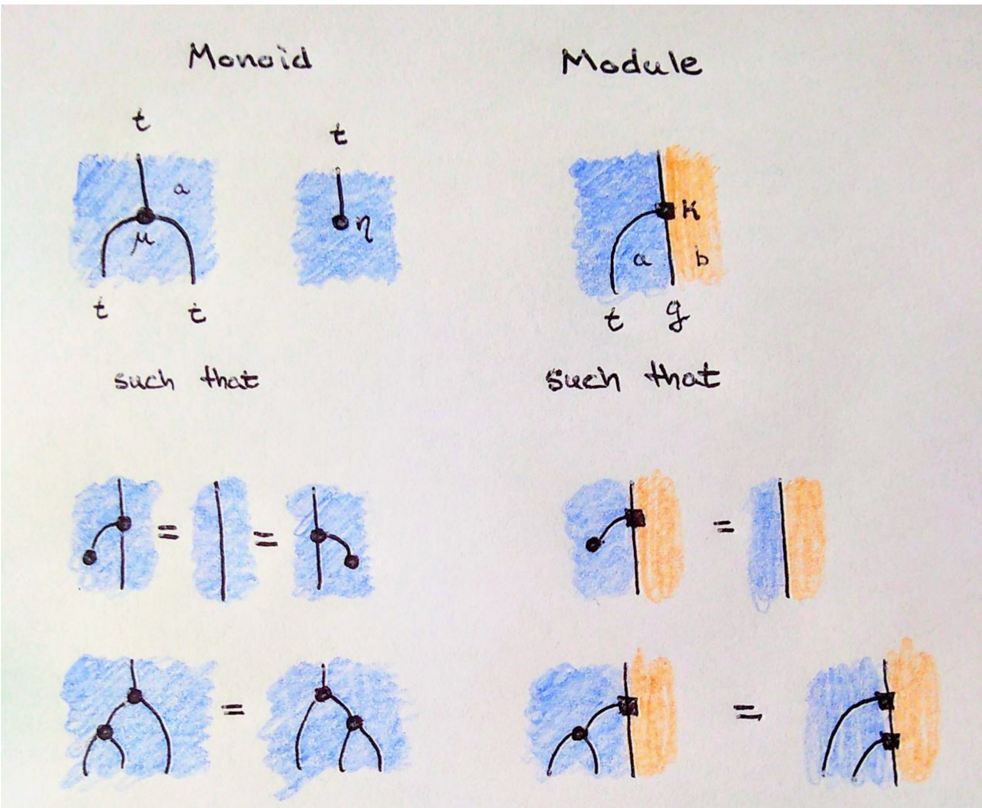

FIGURE 1. Monoids and modules over a monoid.

1.6. **Kleisli object.** Recall that we can consider monoids (a, t, μ, η) and right monoid modules (b, g, κ) in a bicategory.

Definition 9. A **Kleisli object**, $(a_t, f_t \colon a \to a_t, \lambda \colon f_t \circ t \Rightarrow f_t)$ is the universal right module, in the sense that any right module (b, g, κ) can be written uniquely as this module in parallel with some $u \colon a_t \to b$.

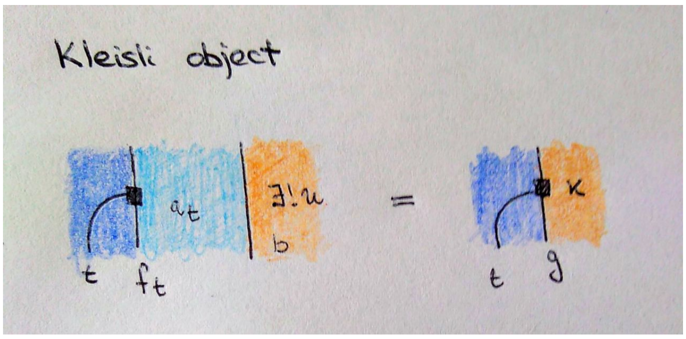

FIGURE 2. A Kleisli object

2. PROFUNCTOR REPRESENTATION OF LENSES AND PRISMS

2.1. **The Θ comonad.**

Definition 10. For any category $\mathbf{C}$, we define a functor $\Theta \colon \mathbf{Prof}(\mathbf{C}) \to \mathbf{Prof}(\mathbf{C})$ that acts in objects as follows.

$$\Theta(p) = \int_{c \in \mathbf{C}} p(c \otimes x, c \otimes y)$$

It acts sending a morphism $\alpha \colon p \Rightarrow q$ to the natural transformation given by the arrows making the following diagram commute for each $d \in \mathbf{C}$, which exists and is unique by the universal property of the end.

$$\begin{tikzcd}
\int_{c \in \mathbf{C}} p(c \otimes x, c \otimes y) \arrow[r, dashed, "\exists!"] \arrow[d, "\pi"] & \int_{c \in \mathbf{C}} q(c \otimes x, c \otimes y) \arrow[d, "\pi"] \\
p(d \otimes x, d \otimes y) \arrow[r, "\alpha_{d \otimes x, d \otimes y}"] & q(d \otimes x, d \otimes y)
\end{tikzcd}$$

Lemma 11. *This functor Θ is a comonad.*

> **Note:** *This proof was written by Fosco on a separate file.*

2.2. **Coalgebras for Θ are Tambara modules.**

Lemma 12. *There is an equivalence* $\mathbf{Coalg}(\Theta) \simeq \mathbf{Tamb}$.

Proof. Given a coalgebra, we will construct a Tambara module. The coalgebra gives us the following family of functions, natural in a and b.

$$\alpha_{a,b} \colon p(a,b) \to \int_{c \in \mathbf{C}} p(c \otimes a, c \otimes b)$$

Composing with all the possible projections for the end, we get the a family $\alpha_{a,b,c} \colon p(a,b) \to p(c \otimes a, c \otimes b)$. This family is extranatural in c because of the wedge condition of the end. This is the data for the Tambara module. Note that every Tambara module can be written in this way.

We check now that it satisfies the conditions on a Tambara module thanks to the coalgebra axioms. Because of the definition of the counit, $\alpha_{a,b,i} = \eta \circ \alpha_{a,b} = \mathrm{id}$; and because of the definition of the comultiplication, we have that $\alpha_{c' \otimes a, c' \otimes b, c} \circ \alpha_{a,b} = \mu \circ \alpha_{a,b}$. The following diagram proves this second equation: (1) the top square commutes because of the definition of the functor Θ, (2) the left one because of the coalgebra axioms, and (3) the right one because of the definition of σ.

$$\begin{tikzcd}
 & & p(ca,cb) \arrow[drr, "\alpha"] & & \\
\int_x p(xa,xb) \arrow[urr, "\pi_c"] \arrow[drr, "\Theta\alpha"] & & & & \int_x p(xca,xcb) \arrow[d, "\pi_d"] \\
p(a,b) \arrow[u, "\alpha"] \arrow[drr, "\alpha"] & & \int_y \int_x p(xya,xyb) \arrow[urr, "\pi_c"] & & p(dca,dcb) \\
 & & \int_x p(xa,xb) \arrow[u, "\sigma"] \arrow[urr, "\pi_{dc}"] & &
\end{tikzcd}$$

For morphisms, we check that, given a natural transformation between profunctors that determines a coalgebra homomorphism, we can see that it is a Tambara morphism. Here the upper square commutes because of the definition of Θ and the lower one because of the coalgebra axioms.

$$\begin{tikzcd}
p(ca,cb) \arrow[r, "\eta"] & q(ca,cb) \\
\int_x p(xa,xb) \arrow[u, "\pi"] \arrow[r, "\Theta\eta"] & \int_x q(xa,xb) \arrow[u, "\pi"] \\
p(a,b) \arrow[u, "\alpha"] \arrow[r, "\eta"] \arrow[uu, bend left=60] & q(a,b) \arrow[u, "\alpha"] \arrow[uu, bend right=60]
\end{tikzcd}$$

Because of the universal property of the end, given a Tambara morphism making the external square commute, by definition of Θ, we have that $\Theta\eta$ makes also the lower square commute. We have a full and faithful translation between homomorphisms. This proves the equivalence. $\square$

2.3. **Representation of lenses and prisms.** We want to show that the two representations of lenses are equivalent.

Theorem 13. *For any monoidal category* $\mathbf{C}$*, and any objects* $a, b, s, t \in \mathbf{C}$*, the following is a bijection.*

$$\int_{p \in \mathbf{Tamb}} \mathbf{Set}(p(a,b), p(s,t)) \cong \int^{c \in \mathbf{C}} \mathbf{C}(s, c \otimes a) \times \mathbf{C}(c \otimes b, t)$$

Proof. We define a functor $\Theta \colon \mathbf{Prof}(\mathbf{C},\mathbf{C}) \to \mathbf{Prof}(\mathbf{C},\mathbf{C})$, which is a comonad.

$$(\Theta p)(s,t) = \int_{c \in \mathbf{C}} p(c \otimes s, c \otimes t)$$

It has a left adjoint $\Phi \dashv \Theta$ that must be a monad.

$$(\Phi p)(s,t) = \int^{c,x,y \in \mathbf{C}} \mathbf{C}(s, c \otimes x) \times \mathbf{C}(c \otimes y, t) \times p(x,y)$$

We consider the category of coalgebras over this comonad, $\mathbf{Coalg}(\Theta)$, which is equivalent to the category of algebras over the monad $\mathbf{Alg}(\Phi)$. Both share the forgetful functor $U \colon \mathbf{Coalg}(\Theta) \to \mathbf{Prof}(\mathbf{C},\mathbf{C})$ up to this equivalence.

Note: *This is an adjoint triple.*

In general we have the following formula

$$\int_{p \in \mathbf{Coalg}(\Theta)} Up(a,b) \to Up(s,t) \cong \Phi(\mathbf{C}^{op} \times \mathbf{C}((a,b),-))(s,t)$$

and integrating over the coend we get the desired result.

$$\begin{aligned}
&\Phi(\mathbf{C}^{op} \times \mathbf{C}((a,b),-))(s,t) \\
&\quad \cong \int^{c,x,y \in \mathbf{C}} \mathbf{C}(s, c \otimes x) \times \mathbf{C}(c \otimes y, t) \times (\mathbf{C}^{op} \times \mathbf{C})((a,b),(x,y)) \\
&\quad \cong \int^{c} \mathbf{C}(s, c \otimes a) \times \mathbf{C}(c \otimes b, t). \quad \square
\end{aligned}$$

A coalgebra for that comonad is the following natural transformation, which can be rewritten as the data for the profunctor to be a Tambara module using continuity of the hom-functor.

$$p(a,b) \to \int_{c \in \mathbf{C}} p(c \otimes a, c \otimes b) \cong \int_{c \in \mathbf{C}} p(a,b) \to p(c \otimes a, c \otimes b)$$

Corollary 14. *For any cartesian closed category* $\mathbf{C}$*, and any objects* $a, b, s, t \in \mathbf{C}$*, the following is a bijection.*

$$\int_{p \in \mathbf{Cartesian}} \mathbf{Set}(p(a,b), p(s,t)) \cong \mathbf{C}(s, a \times (b \to t))$$

Proof. We can apply Theorem 13. The cartesian closed structure gives $\mathbf{C}(c \times b, t) \cong \mathbf{C}(c, b \to t)$, and then by Yoneda lemma we have the following.

$$\int^{c \in \mathbf{C}} \mathbf{C}(s, c \times a) \times \mathbf{C}(c \times b, t) \cong \mathbf{C}(s, (b \to t) \times a) \qquad \square$$

2.4. Doubles.

Proposition 15. *Left adjoint functors* $[\mathbf{D}^{op}, \mathbf{Set}] \to [\mathbf{C}^{op}, \mathbf{Set}]$ *are equivalent to profunctors* $\mathbf{C}^{op} \times \mathbf{D} \to \mathbf{Set}$.

Note: *I should prove that this is an equivalence, describing the action on morphisms and showing this is full and faithful.*

Proof. Given a functor $\Psi\colon [\mathbf{C}^{op}, \mathbf{Set}] \to [\mathbf{D}^{op}, \mathbf{Set}]$, we can define the profunctor $\Omega(x, y) = \Psi(\mathbf{C}(-, y))(x)$. Given any profunctor Ω, its preimage under this assignment must be such that $\Psi(\mathbf{C}(-, y)) = \Omega(-, y)$; and using that Ψ is a left adjoint (cocontinuous) functor and that every presheaf is a colimit of representables, this determines the totality of Ψ. $\square$

Note: ***Derek:*** *Here's just some extra context. It asks for a functor to be a left adjoint arguably mainly for cocontinuity. It turns out (at least when V=Set) these are identical assumptions. A functor between locally presentable categories is a left adjoint if and only if it is cocontinuous.* `https://ncatlab.org/nlab/show/adjoint+functor+theorem#AdjFuncTheoremForLocallyPresentableCats` *Categories of (pre)sheaves are locally (finitely) presentable.* `https://ncatlab.org/nlab/show/locally+presentable+category#locally_finitely_presentable_categories` *In fact, a lot of categories are locally presentable so this is a handy general fact to have in one's pocket.*

In [PS08], the monad on endoprofunctors $\Psi\colon [\mathbf{C}^{op} \times \mathbf{C}, \mathbf{Set}] \to [\mathbf{C}^{op} \times \mathbf{C}, \mathbf{Set}]$, which is left adjoint to the Θ comonad, is seen as a promonad $\check{\Psi}\colon \mathbf{C}^{op} \times \mathbf{C} \nrightarrow \mathbf{C}^{op} \times \mathbf{C}$. We will now take the Kleisli construction over an arbitrary promonad and apply it to this particular case.

Proposition 16. *Promonads admit a Kleisli construction.*

Proof. Let $p\colon \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{Sets}$ be a promonad. We define the category $\mathbf{C}_p$ as a category having the same objects as $\mathbf{C}$ and morphisms given by $\mathbf{C}_p(x, y) = p(x, y)$. Composition in this category is given by promonad multiplication and the identity is the image of the identity under the unit.

We now define the profunctor $f_p\colon \mathbf{C}^{op} \times \mathbf{C_p} \to \mathbf{Sets}$. On objects, it acts as $f_p(c, d) = p(c, d)$. On morphisms, given a pair of $u \in \mathbf{C}(c', c)$ and $v \in p(d, d')$, we can precompose with u to get $p(c', d)$ and then use the monad multiplication to get an element in $p(c', d')$.

Finally, we need a natural transformation $\lambda\colon f_p \circ p \to f_p$. We already have a family of morphisms $\mu\colon f_p \circ p \to f_p$. $\square$

Note: *We need to check naturality of the family* λ*. It looks like it should follow from* μ *being associative. Then we need to check that the composition works as expected; this is better explained in the comments by Alex Campbell and* `bikleisli.pdf`*.*

3. Tannakian reconstruction for endofunctors

Let (M, η, μ) be a monad on a category $\mathbf{C}$, complete and cartesian closed. Modules over the monad (that are endofunctors) form the category $\mathbf{Mmod}$, with natural transformations that preserve the monad action as morphisms. There is an obvious forgetful functor $U \colon \mathbf{Mmod} \to [\mathbf{C}, \mathbf{C}]$.

The monad can be represented in itself, thanks to the monad axioms. We want to show that $U \cong \mathbf{Mmod}(M, -)$. Can we show that $\mathbf{Mmod}$ is enriched over $[\mathbf{C}, \mathbf{C}]$? Note that $[\mathbf{C}, \mathbf{C}]$ is cartesian closed.

This should be an application Theorem 2.3 in nlab https://ncatlab.org/nlab/show/Tannaka+duality.

4. Reading: optics.tex

4.1. General equation for optics.

```
-- We define some profunctor categories that are given by certain
-- classes. Cartesian (strong), cocartesian (choice), closed functors
-- are examples.
class Profunctor p => Cartesian p where
  first :: p a b -> p (a,c) (b,c)
  second :: p a b -> p (c,a) (c,b)

class Profunctor p => Choice p where
  first :: p a b -> p (Either a c) (Either b c)
  second :: p a b -> p (Either c a) (Either c b)

class Profunctor p => Closed p where
  closed :: p a b -> p (x -> a) (x -> b)

-- With these, we can get the basic profunctor optics for Haskell.
type Iso   s t a b = forall p . Profunctor  p => p a b -> p s t
type Lens  s t a b = forall p . Cartesian   p => p a b -> p s t
type Prism s t a b = forall p . Cocartesian p => p a b -> p s t
type Grate s t a b = forall p . Closed      p => p a b -> p s t
```

Isos, lenses, prisms and grates can be described by a single equation. For some $\mathbf{E}$ subcategory of profunctors.

$$O(s,t,a,b) = \int_{p \in \mathbf{E}} p(a,b) \to p(s,t)$$

Using the component functor, this can be rewritten as follows.

$$O(s,t,a,b) = [\mathbf{E}, \mathbf{Set}] \left(V_{(a,b)}, V_{(s,t)}\right)$$

4.2. Component functor.

Definition 17. The **component functor** $V_a \colon [\mathbf{C}, \mathbf{D}] \to \mathbf{D}$ for an object $a \in \mathbf{C}$ is given by evaluation of functors and natural transformations on that object. This is a representable functor by the Yoneda lemma.

$$[\mathbf{C}, \mathbf{Set}](\mathbf{C}(a, -), f) \cong V_a f$$

Proposition 18. *The Day convolution of two component functors is the component functor of their tensor product,* $V_a \star V_b = V_{a \otimes b}$.

5. Notions of lens morphism

6. Optics for a monoidal action

Note: *The plan is to describe the whole construction at the end of Riley's chapter 2 in terms of modules in Cat. Then we would like to get the same construction as in Pastro-Street for the general case.*

6.1. Pseudomonoids and their modules.

Definition 19. Let $\mathbf{E}$ be a monoidal 2-category. A **pseudomonoid** [DS97] is given by an object $M \in \mathbf{E}$ with arrows $\eta \colon I \to A$ and $\mu \colon A \otimes A \to A$; and with invertible 2-cells witnessing associativity $\alpha \colon \mu \circ (\mu \otimes \mathrm{id}) \Rightarrow (\mathrm{id} \otimes \mu) \circ \mu$ and unitality on both sides, $l \colon \mu \circ (\eta \otimes \mathrm{id}) \Rightarrow \mathrm{id}$ and $r \colon \mu \circ (\mathrm{id} \otimes \eta) \Rightarrow \mathrm{id}$.

The following coherence equations must hold.

Note: *TODO. Write down coherence equations, how are they fulfilled in a monoidal category. There should exist a coherence theorem for pseudomonoids that allows you to have a nice definition.*

In the particular case of $\mathbf{Cat}$, a pseudomonoid is precisely a monoidal category. We want to consider actions of pseudomonoids on this category.

Definition 20. Given a monoidal category $\mathbf{M}$, a pseudomodule or **actegory** ([nLa18], [JK01]) is a category $\mathbf{C}$ with a functor $\oslash \colon \mathbf{M} \times \mathbf{C} \to \mathbf{C}$ and natural isomorphisms $\alpha_{m,n,a} \colon (m \otimes n) \oslash c \to m \oslash n \oslash c$ and $\lambda_a \colon i \oslash a \to a$ for any $m, n \in \mathbf{M}$ and $c \in \mathbf{C}$. We impose the commutativity of the following diagrams as coherence conditions that ensure they interplay nicely with the monoidal structure given by the associator a and the unitors l, r.

$$\begin{tikzcd}
((X \otimes Y) \otimes Z) \oslash A \arrow[r, "\alpha"] \arrow[d, "a \oslash \mathrm{id}"] & (X \otimes Y) \oslash Z \oslash A \arrow[r, "\alpha"] & X \oslash Y \oslash Z \oslash A \\
(X \otimes (Y \otimes Z)) \oslash A \arrow[rr, "\alpha"] & & X \otimes ((Y \otimes Z) \oslash A) \arrow[u, "1 \oslash \alpha"']
\end{tikzcd}$$

$$\begin{tikzcd}
(I \otimes X) \oslash A \arrow[rr, "\alpha"] \arrow[dr, "l \oslash \mathrm{id}"'] & & I \oslash (X \oslash A) \arrow[dl, "\lambda"] \\
& X \oslash A &
\end{tikzcd}$$

$$\begin{tikzcd}
(X \otimes I) \oslash A \arrow[rr, "\alpha"] \arrow[dr, "r \oslash \mathrm{id}"'] & & X \oslash (I \oslash A) \arrow[dl, "\mathrm{id} \oslash \lambda"] \\
& X \oslash A &
\end{tikzcd}$$

Note: *Note that the natural isomorphisms mimic the general definition of module, just up to isomorphism. The coherence conditions I feel are a bit more difficult to justify (a coherence theorem for actegories would be nice to have). A nice theory of how these things work can be found in* [JK01]. *`http://www.tac.mta.ca/tac/volumes/9/n4/n4.pdf` They also discuss what happens when* $f \colon \mathbf{M} \to [\mathbf{C}, \mathbf{C}]$ *has a right adjoint. Does this have any implication for the associated optic?*

Note: *I would like the definition to be something as follows. Are these two equivalent? Is a morphism of actions a module homomorphism? They are and this is described in the introduction to* [JK01]. *http://www.tac.mta.ca/tac/volumes/9/n4/n4.pdf*

Lemma 21. *An **action** of a monoidal category* $\mathbf{M}$ *on a category* $\mathbf{C}$ *is an actegory in* $\mathbf{Cat}$*. Equivalently, that is a monoidal functor* $\mathbf{M} \to [\mathbf{C}, \mathbf{C}]$*. Homomorphisms of actions can be defined as actegory homomorphisms.*

Proof. A strong monoidal functor $F\colon \mathbf{M} \to [\mathbf{C}, \mathbf{C}]$, corresponds to the functor $\oslash\colon \mathbf{M} \times \mathbf{C} \to \mathbf{C}$ where $F(X)(A) = X \oslash A$. We have natural isomorphisms $a\colon FI \cong \mathrm{id}$ and $l\colon F(X \otimes Y) \cong FX \circ FY$ that correspond to $a_{m,n,a}\colon (X \otimes Y) \oslash A \to X \oslash Y \oslash A$ and $\lambda_a\colon I \oslash A \to A$. The conditions on associativity and unitarlity for F correspond to the coherence conditions of the action.

A homomorphism of actions is thus some functor $H\colon \mathbf{C} \to \mathbf{D}$ with a natural isomorphism $\varphi_{X,A}\colon H(X \oslash A) \to X \oslash HA$, which is $H((FX)A) \to FX(HA)$ in terms of F. □

6.2. **Optic of an action.**

Definition 22. Given an action of $\mathbf{M}$ on $\mathbf{C}$, we define the optic associated to that action to be

$$\mathbf{Optic}_{\mathbf{M}}\left(\begin{pmatrix} s \\ t \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix}\right) = \int^{m \in \mathbf{M}} \mathbf{C}(s, m \cdot a) \times \mathbf{C}(m \cdot b, t).$$

We now introduce a notation for elements of $\mathbf{Optic}_{\mathbf{M}}((s,t),(a,b))$. Following [Ril18], we call $\langle l \mid r \rangle$ to the element of the coend given by the morphisms $l\colon s \to m \cdot a$ and $r\colon m \cdot b \to t$, for some $m \in \mathbf{M}$ we call the *reminder*.

Note: *The identity is not given by* $\langle \mathrm{id} \mid \mathrm{id} \rangle$ *but by the iso witnessing that* $i \cdot a \cong a$*. What is a good notion (not too strict) of module in* $\mathbf{Cat}$*? It exists! and it is called "pseudomonoid". Happy to have an intuition confirmed.*

Let $s, t \in \mathbf{C}$. The **identity optic** $(s,t) \twoheadrightarrow (s,t)$ is given by $\langle \mathrm{id} \mid \mathrm{id} \rangle$. Given two optics, $\langle l, r \rangle : (s,t) \twoheadrightarrow (a,b)$ with residual m and $\langle l', r' \rangle : (a,b) \twoheadrightarrow (u,v)$ with residual n; we define the composite to be $\langle (m \cdot l') \circ l, r \circ (m \cdot r') \rangle$; note that the residual is the multiplication, mn, of the two residuals.

Proposition 23. *We have a category* $\mathbf{Optic}_{\mathbf{M}}$ *with a functor* $\iota\colon \mathbf{C} \times \mathbf{C}^{op} \to \mathbf{Optic}_{\mathbf{M}}$.

Proof. TODO □

Proposition 24. *A morphism of actions induces a functor between the associated categories of optics.*

Proof. Consider two actions of $\mathbf{M}$ on categories $\mathbf{C}$ and $\mathbf{D}$ and let $H\colon \mathbf{C} \to \mathbf{D}$ be a homomorphism for those actions. Given an object $(s,t) \in \mathbf{Optic}_{\mathbf{C}}$, we have $(Hs, Ht) \in \mathbf{Optic}_{\mathbf{D}}$. For morphisms, we can use the natural isomorphism $\varphi\colon H(m \oslash a) \to m \oslash Ha$ to induce the following morphism between the coends corresponding to $\mathbf{Optic}_{\mathbf{C}}((s,t),(a,b)) \to \mathbf{Optic}_{\mathbf{D}}((Hs, Ht),(Ha, Hb))$.

$$\begin{aligned} \int^{m \in \mathbf{M}} \mathbf{C}(s, m \oslash a) \times \mathbf{C}(m \oslash b, t) &\to \int^{m \in \mathbf{M}} \mathbf{C}(Hs, H(m \oslash a)) \times \mathbf{C}(H(m \oslash b), Ht) \\ &\to \int^{m \in \mathbf{M}} \mathbf{C}(Hs, m \oslash Ha) \times \mathbf{C}(m \oslash Hb, Ht) \quad \square \end{aligned}$$

In other words, this sends the morphism $\langle l \mid r \rangle$ to $\langle \varphi^{-1} \circ Hl, Hr \circ \varphi \rangle$.

We need to check that this is indeed a functor. Let $(s,t) \in \mathbf{Optic_C}$ have the identity $\langle \lambda_s^{-1}, \lambda_t \rangle$. **Note:** *I think I need something like the quotienting of lens morphisms. But that is precisely the end condition (!).*

Proposition 25. *This previous construction describes a functor from the category of actions to the subcategory of* **Cat** *given by optics.*

Proof. We need to check this preserves identities and compositions. □

Note: *It looks like it will do but we need to use conditions on many of our morphisms.*

Note: *Is this a functor from the category of modules to some category of categories of optics? I would like this to be functorial. Even better, this should go to the category of symmetric monoidal categories or something like that.*

What are optics associated to (pseudo?)Hopf or other classes of monoids? I would expect them to have some nice characterization.

6.3. Particular cases.

- $\mathbf{Lens} = \mathbf{Optic}_\times$
- $\mathbf{Prism} = \mathbf{Optic}_+$
- $\mathbf{Traversal} = \mathbf{Optic}_{\text{traversable}}$

Note: *Relation to coalgebraic lenses? Well-behavedness?*

7. Traversables

Note: *This section should follow [Ril18]. It is a way of getting profunctor traversals. The idea of getting Tambara modules over arbitrary module actions looks very interesting.*

```haskell
-- Profunctors that preserve the action of any traversable.
class TambaraTraversable p where
  action :: forall s t a b f . Traversable f => p a b -> p (f a) (f b)
data Traversal s t a b = forall p . TambaraTraversable p => p a b -> p s t
```

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