Tannakian reconstruction of Tambara modules

Or, why that "optics" formula?

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1 Definition of Tambara module

Let $\mathbf{M}$ be a monoidal category acting both on two arbitrary categories $\mathbf{C}$ and $\mathbf{D}$. We write $\underline{M}$ for the image of $M \in \mathbf{M}$ both in $[\mathbf{C}, \mathbf{C}]$ and $[\mathbf{D}, \mathbf{D}]$.

Definition 1. A **Tambara module** consists of a profunctor $P \colon \mathbf{C}^{op} \times \mathbf{D} \to \mathbf{Sets}$ endowed with a family of morphisms $\alpha_M \colon P(A, B) \to P(\underline{M}A, \underline{M}B)$ natural in both $A \in \mathbf{C}$ and $B \in \mathbf{D}$, and dinatural in $M \in \mathbf{M}$; which additionally makes the following diagrams commute.

$$\begin{tikzcd} P(A,B) \arrow[r, "\alpha_I"] \arrow[dr, "\mathrm{id}"'] & P(\underline{I}A, \underline{I}B) \arrow[d, "\cong"] \\ & P(A,B) \end{tikzcd} \qquad \begin{tikzcd} P(\underline{N}A, \underline{N}B) \arrow[r, "\alpha_M"] & P(\underline{M}\underline{N}A, \underline{M}\underline{N}B) \arrow[d, "\cong"] \\ P(A,B) \arrow[u, "\alpha_N"] \arrow[r, "\alpha_{N\otimes M}"'] & P(\underline{M \otimes N}A, \underline{M \otimes N}B) \end{tikzcd}$$

Remark 2. The original definition of Tambara module [T+06] deals only with actions that arise from a monoidal product $\otimes \colon \mathbf{C} \to [\mathbf{C}, \mathbf{C}]$. We use the term *Tambara module* also for the more general concept, allowing for arbitrary monoidal actions.

We can extend Pastro and Street [PS08] construction of free Tambara module over a profunctor $P \colon \mathbf{C}^{op} \times \mathbf{D} \to \mathbf{Sets}$ to the case of general monoidal actions. Tambara modules are equivalently algebras for a monad Ψ defined by

$$\Psi P(S,T) = \int^{M,X,Y} \mathbf{C}(S, \underline{M}X) \times \mathbf{D}(\underline{M}Y, T) \times P(X,Y).$$

We know how to contruct free Tambara modules. What is the free Tambara module over a representable functor $\hom((A,B), -)$? We call it $\mathbf{Optic}((A,B),-)$, and it can be written as

$$\mathbf{Optic}((A,B),-) \cong \int^{M} \mathbf{C}(S, \underline{M}A) \times \mathbf{D}(\underline{M}B, T).$$

That is, the formula for optics is given by the free Tambara module on a representable functor.

2 Tannakian reconstruction

Milewski [Mil17], and then Boisseau and Gibbons [BG18], proved a unified profunctor representation theorem for optics, that is widely used in programming libraries such as

Kmett's *lens* [Kme18]. Milewski suggested to me that profunctor representation was surprisingly similar to Tannakian reconstruction; we will prove the theorem following the proof of Tannakian reconstruction (for, say, groups).

Theorem 3. *Let* $\mathcal{U}_{(A,B)} \colon た \to \mathbf{Sets}$ *the functor that evaluates a Tambara module on the object* (A, B)*. There exists an isomorphism*

$$[た, \mathbf{Sets}]\left(\mathcal{U}_{(A,B)}, \mathcal{U}_{(S,T)}\right) \cong \mathbf{Optic}((A, B), (S, T)),$$

natural on both (A, B) *and* (S, T)*.*

Proof. The claim is that this theorem is precisely Tannakian reconstruction for Tambara modules. We first note that, by definition, the functor $\mathcal{U}_{(A,B)}$ is represented by $\mathbf{Optic}((A, B), -)$, the free Tambara module over the hom-profunctor. In fact, for any Tambara module $P \colon \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{Sets}$,

$$\mathcal{U}_{(A,B)}P \cong \mathrm{Nat}(\hom((A, B), -), P) \cong た(\mathbf{Optic}((A, B), -), P).$$

Then, by Tannakian reconstruction, $[た, \mathbf{Sets}]\left(\mathcal{U}_{(A,B)}, \mathcal{U}_{(S,T)}\right) \cong \mathbf{Optic}((A, B), (S, T))$. $\square$

References

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