Pasting pullbacks

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This is an exercise on Samson Abramsky's notes on the course Categories, Proofs and Processes. [AT10]

Proposition 1. *In the following commutative diagram, if $ABCD$ and $BEDF$ are pullback squares, so is $AECF$.*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & E \arrow[d] \\
C \arrow[r] & D \arrow[r] & F
\end{tikzcd}$$

Proof. Given some $u\colon Z \to E$ and $v\colon Z \to C$ such that $tu = dcv$, we can use the pullback to construct some $Z \to B$ and the other pullback to construct some $Z \to A$. This proves existence. Given two morphisms z and z' such that $baz = u = baz'$ and $rz = v = rz'$, as in the following diagram, we will prove they are equal.

$$\begin{tikzcd}
Z \arrow[drrr, bend left, "u"] \arrow[drr, dashed, bend left, "w"] \arrow[dr, dashed, bend left, "z"] \arrow[dr, dashed, bend right, "z'"'] \arrow[ddr, bend right, "v"'] & & & \\
& A \arrow[r, "a"] \arrow[d, "r"] & B \arrow[r, "b"] \arrow[d, "s"] & E \arrow[d, "t"] \\
& C \arrow[r, "c"] & D \arrow[r, "d"] & F
\end{tikzcd}$$

By the universal property of the pullback $BEDF$ there is a unique morphism $w\colon Z \to B$ such that $bw = u$ and $sw = cv$, but $baz = baz' = u$ and $saz = crz = cv = crz' = saz'$, so $az = az'$. Then using the pullback $ABCD$, there would be a unique morphism $y\colon Z \to A$ such that $ay = az$ and $v = ry$, but both z and z' satisfy this condition and thus they are equal. $\square$

Proposition 2. *In the following commutative diagram, if $BEDF$ and $AECF$ are pullback squares, so is $ABCD$.*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & E \arrow[d] \\
C \arrow[r] & D \arrow[r] & F
\end{tikzcd}$$

Proof. Given some $u\colon Z \to B$ and $v\colon Z \to C$ such that $su = cv$, we can apply the universal property of the pullback $AECF$ to the maps bu and v, which satisfy $tbu = dsu = dcv$, to construct some $z\colon Z \to A$ such that $baz = bu$ and $v = rz$.

By the universal property of $BEDF$, there exists a unique y such that $by = bu$ and $sy = cv$, but both az and u satisfy that, and thus they are equal. This proves existence.

$$\begin{tikzcd}
Z \arrow[drr, bend left, "u"] \arrow[dr, dashed, bend left, "z"] \arrow[dr, dashed, bend right, "z'"'] \arrow[ddr, bend right, "v"'] & & & \\
& A \arrow[r, "a"] \arrow[d, "r"] & B \arrow[r, "b"] \arrow[d, "s"] & E \arrow[d, "t"] \\
& C \arrow[r, "c"] & D \arrow[r, "d"] & F
\end{tikzcd}$$

Given two morphisms $z, z'\colon Z \to A$ such that $az = u = az'$ and $rz = rz' = v$; we know by the universal property of the pullback $AECF$ applied to bu and v that there is a unique morphism $x\colon Z \to A$ such that $bax = bu$ and $v = rx$; and thus they must be equal. $\square$

References

[AT10] Samson Abramsky and Nikos Tzevelekos. Introduction to categories and categorical logic. In *New structures for physics*, pages 3–94. Springer, 2010.