

# Pasting pullbacks

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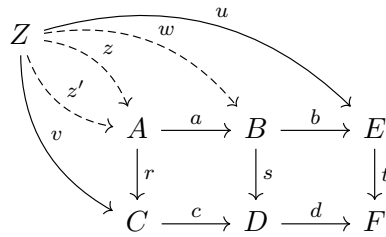
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This is an exercise on Samson Abramsky's notes on the course *Categories, Proofs and Processes*. [AT10]

**Proposition 1.** *In the following commutative diagram, if  $ABCD$  and  $BEDF$  are pullback squares, so is  $AECF$ .*

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow \\ C & \longrightarrow & D & \longrightarrow & F \end{array}$$

*Proof.* Given some  $u: Z \rightarrow E$  and  $v: Z \rightarrow C$  such that  $tu = dcv$ , we can use the pullback to construct some  $Z \rightarrow B$  and the other pullback to construct some  $Z \rightarrow A$ . This proves existence. Given two morphisms  $z$  and  $z'$  such that  $baz = u = baz'$  and  $rz = v = rz'$ , as in the following diagram, we will prove they are equal.



By the universal property of the pullback  $BEDF$  there is a unique morphism  $w: Z \rightarrow B$  such that  $bw = u$  and  $sw = cv$ , but  $baz = baz' = u$  and  $saz = crz = cv = crz' = saz'$ , so  $az = az'$ . Then using the pullback  $ABCD$ , there would be a unique morphism  $y: Z \rightarrow A$  such that  $ay = az$  and  $v = ry$ , but both  $z$  and  $z'$  satisfy this condition and thus they are equal.  $\square$

**Proposition 2.** *In the following commutative diagram, if  $BEDF$  and  $AECF$  are pullback squares, so is  $ABCD$ .*

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow \\ C & \longrightarrow & D & \longrightarrow & F \end{array}$$

*Proof.* Given some  $u: Z \rightarrow B$  and  $v: Z \rightarrow C$  such that  $su = cv$ , we can apply the universal property of the pullback  $AECF$  to the maps  $bu$  and  $v$ , which satisfy  $tbu = dsu = dcv$ , to construct some  $z: Z \rightarrow A$  such that  $baz = bu$  and  $v = rz$ .

By the universal property of  $BEDF$ , there exists a unique  $y$  such that  $by = bu$  and  $sy = cv$ , but both  $az$  and  $u$  satisfy that, and thus they are equal. This proves existence.

Given two morphisms  $z, z': Z \rightarrow A$  such that  $az = u = az'$  and  $rz = rz' = v$ ; we know by the universal property of the pullback  $AECF$  applied to  $bu$  and  $v$  that there is a unique morphism  $x: Z \rightarrow A$  such that  $ba x = bu$  and  $v = rx$ ; and thus they must be equal.  $\square$

## References

- [AT10] Samson Abramsky and Nikos Tzevelekos. Introduction to categories and categorical logic. In *New structures for physics*, pages 3–94. Springer, 2010.