Uniqueness of identity proofs

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> *Attribution: this is only a summary, the original ideas exposed here can be found on the reference articles.*

Equality in Martin-Löf type theory is a beautiful concept but it is notoriously tricky to understand. This is the second time I feel like I get it; and it probably won't be the last one. I have been reading the groupoid interpretation article, and at least the first sections are recommended lecture to anyone interested in the topic; it is really well-written and does not require any previous knowledge. [HS98]

Let's talk about **Uniqueness of identity proofs** (UIP). This is the principle that says that any two proofs of the same equality must be equal themselves. That is, for any type A, the following type is inhabited

$$\prod_{x,y:A} \prod_{p,q:x=y} p = q.$$

Idris and Agda allow the user to prove this proposition in general via pattern matching.

```
-- In Agda.
uip : {A : Set} -> {x y : A} -> (p q : x   y) -> p   q
uip refl refl = refl
```

```
-- In Idris
uip : (t : Type)
    -> (x : t) -> (y : t)
    -> (p : x = y) -> (q : x = y)
    -> p = q
uip t x x Refl Refl = Refl
```

However, UIP is not valid in general in Martin-Löf type theory. Hofmann and Streicher's groupoid interpretation provides a model where the UIP fails. In particular, this proves that pattern matching, in full generality, is not conservative over Martin-Löf type theory. [HS98] In Agda, we can disable this behaviour with the flag `--without-K`, this is necessary if we want to do Homotopy Type Theory, for example.

In general, the rule we *can* use in Intensional Type Theory is the J-eliminator. The usual properties of equality, like the Leibniz' indiscernibility of identicals, are consequences of it. It is subtle to notice how it works as it should while not implying UIP. For any type C depending two variables $x : A$ and $y : A$ we have the following rule.

$$\frac{\begin{array}{cc} \Gamma \vdash a : A & \Gamma \vdash b : A \\ \Gamma, x : A \vdash c : C(x,x) & \Gamma \vdash p : a = b \end{array}}{\Gamma \vdash \mathsf{J}_C(c,p) : C(a,b)}$$

There is also a nice justification of the rule from category theory and an adjoint characterization of equality due to Lawvere. More on this can be read on the Michael's Shulman article on homotopy type theory. [Shu17]

References

[HS98] Martin Hofmann and Thomas Streicher. The groupoid interpretation of type theory. *Twenty-five years of constructive type theory (Venice, 1995)*, 36:83–111, 1998.

[Shu17] Michael Shulman. Homotopy type theory: the logic of space. *arXiv preprint arXiv:1703.03007*, 2017.