

# Powerset-algebras are complete semilattices

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*This is a combination of two exercises from Samson Abramsky's notes on the course Categories, Proofs and Processes. [AT10]*

**Definition 1.** We take a **complete semilattice** to mean a poset  $(P, \leq)$  where every subset  $S \subseteq P$  has a least upper bound we write as  $\bigvee S$ . That is to say that it is a category with at most one morphism between any two objects that has all coproducts.

A morphism between complete semilattices is a map between posets that preserves all least upper bounds, that is  $h(\bigvee S) = \bigvee h(S)$ , where we write  $h(S) = \{h(x) \mid x \in S\}$ . This property implies monotonicity: note that from  $x \leq y$  we get  $h(y) = h(x \vee y) = h(x) \vee h(y)$ , and thus  $h(x) \leq h(y)$ . We have created a category of complete semilattices with morphisms between them that we call **SL**. There is a forgetful functor  $U: \mathbf{SL} \rightarrow \mathbf{Set}$  assigning the underlying set to each poset.

## Free-forgetful adjunction

**Proposition 2.** *The forgetful functor  $U: \mathbf{SL} \rightarrow \mathbf{Set}$  has a left adjoint.*

*Proof.* We will show that the left adjoint is the functor  $\mathcal{P}: \mathbf{Set} \rightarrow \mathbf{SL}$  sending each set to its powerset ordered by inclusion, which is a complete semilattice because the union of a family of sets is its least upper bound. Given two sets  $A$  and  $B$ , any function  $f: A \rightarrow B$  induces a  $f^*: \mathcal{P}A \rightarrow \mathcal{P}B$  sending a subset to its image under  $f$ ; this constitutes the action of the functor on morphisms. We can check that this is a morphism of complete semilattices because, for any family  $\mathcal{X} \subseteq \mathcal{P}(A)$ , we have

$$f^*\left(\bigcup_{Y \in \mathcal{X}} Y\right) = \left\{f(y) \mid y \in \bigcup_{Y \in \mathcal{X}} Y\right\} = \bigcup_{Y \in \mathcal{X}} \{f(y) \mid y \in Y\} = \bigcup_{Y \in \mathcal{X}} f^*(Y).$$

To construct the adjunction, we start by defining an isomorphism  $\mathbf{Set}(A, UB) \rightarrow \mathbf{SL}(\mathcal{P}A, B)$  that sends  $f: A \rightarrow B$  to the function that acts on some  $Y \subseteq A$  as

$$\bar{f}(Y) = \bigvee_{y \in Y} f(y).$$

This is a morphism of complete semilattices because for any family of subsets  $\mathcal{X} \subseteq \mathcal{P}(A)$  we have

$$\bar{f}\left(\bigcup_{Y \in \mathcal{X}} Y\right) = \bigvee_{y \in \bigcup_{Y \in \mathcal{X}} Y} f(y) = \bigvee_{Y \in \mathcal{X}} \bigvee_{y \in Y} f(y) = \bigvee_{Y \in \mathcal{X}} \bar{f}(Y)$$

This isomorphism has an inverse sending each morphism  $h: \mathcal{P}(A) \rightarrow B$  to the function  $A \rightarrow B$  defined as  $a \mapsto h(\{a\})$ . We can check that these are in fact inverses because for any  $f: A \rightarrow B$  and  $a \in A$  we have

$$\bar{f}(\{a\}) = \bigvee_{a \in \{a\}} f(a) = f(a)$$

and for any semilattice morphism  $h: \mathcal{P}(A) \rightarrow B$  we have for any  $X \subseteq A$ , using that it preserves least upper bounds, that

$$\overline{h(\{-\})}(X) = \bigvee_{x \in X} h(\{x\}) = h\left(\bigvee_{x \in X} \{x\}\right) = h(X).$$

We now show that the isomorphism is in fact natural. Given any  $f: A \rightarrow UB$ , any function  $a: A' \rightarrow A$  and any morphism of semilattices  $b: B \rightarrow B'$ , we have for a given  $Y \in \mathcal{P}(A)$  that

$$b\bar{f}a^*(Y) = b\left(\bigvee_{y \in Y} f(a(y))\right) = \bigvee_{y \in Y} bfa(y) = \overline{bfa}(Y),$$

because  $b$  preserves least upper bounds. This proves that  $b \circ \bar{f} \circ a^* = \overline{b \circ f \circ a}$  and the isomorphism is thus natural.  $\square$

## Powerset-algebras

Note that the powerset monad acts on objects as  $UP$ , simply taking the powerset and forgetting about its semilattice structure. The unit of the adjunction,  $a \mapsto \{a\}$  is precisely the unit of the monad, and the counit of the adjunction, the lattice homomorphism  $\bigvee: \mathcal{P}L \rightarrow L$  for any lattice  $L$ , is such that  $U \bigvee_{\mathcal{P}} = \bigcup: \mathcal{P}PA \rightarrow$

$\mathcal{P}A$  is precisely the multiplication of the monad for any set  $A$ .

We can now prove that  $\mathcal{P}$ -algebras are complete semilattices. Note that an algebra would be a function  $f: \mathcal{P}A \rightarrow A$  such that

$$f\{a\} = a, \quad f\left(\bigcup_{i \in I} A_i\right) = f\{f(A_i) \mid i \in I\},$$

for any element  $a \in A$  and any family of subsets  $A_i \subseteq A$ . We can define a partial order where for any two elements  $x, y \in A$ , we have that  $x \leq y$  when  $f\{x, y\} = y$ . This satisfies

- **reflexivity**, because  $f\{x, x\} = x$ ;
- **transitivity**, because if  $x \leq y \leq z$ , then we have  $f\{x, z\} = f\{f\{x\}, f\{y, z\}\} = f\{x, y, z\} = f\{f\{x, y\}, f\{z\}\} = z$ ;
- and **antisymmetry**, because  $x \leq y \leq x$  implies  $x = f\{x, y\} = y$ .

For this particular preorder, the function  $f$  is the lowest great bound  $\bigvee$ ; this can be proved checking that, for any subset  $S \subseteq A$ ,

- for each  $s \in S$ , we have  $f\{s, f(S)\} = f(S \cup \{s\}) = f(S)$ , so  $s \leq f(S)$ ;
- and given some  $x \in A$  such that  $s \leq x$  for all  $s \in S$ , we have  $f\{f(S), x\} = f(\bigcup_{s \in S} \{s, x\}) = f\{x\} = x$ , and thus  $f(S) \leq x$ .

Finally, we can check that taking the lowest great bound on a subset of a poset provides a valid algebra.

$$\bigvee\{a\} = a, \quad \bigvee\left(\bigcup_{i \in I} A_i\right) = \bigvee\left\{\bigvee A_i \mid i \in I\right\}.$$

Note that once we know that all algebras are of this form, a  $\mathcal{P}$ -algebra morphism is precisely a function between complete semilattices  $f: A \rightarrow B$  satisfying,  $f(\bigvee A) = \bigvee f(A)$ ; that is, a complete semilattice morphism.

## Monadicity theorem

This is also a consequence of Beck's monadicity theorem. The adjunction  $\mathcal{P} \dashv U$  is monadic and that implies that the comparison functor  $\mathbf{SL} \rightarrow \mathbf{Set}^{\mathcal{P}}$  between complete semilattices and powerset-algebras is an equivalence.

## References

- [AT10] Samson Abramsky and Nikos Tzevelekos. Introduction to categories and categorical logic. In *New structures for physics*, pages 3–94. Springer, 2010.