# Hom is continuous and then all right adjoints are

Mario Román

*<2018-12-23 Sun 16:20>*

*This is an exercise on Samson Abramsky's notes on the course Categories, Proofs and Processes.* [AT10]

**Statement.** *Let $A$ be an object in the category $\mathcal{C}$. Show that the covariant* hom *functor $\mathcal{C}(A,-)$, as defined in Lecture III, preserves all limits.*

*Proof.* Let $\mathcal{I}$ be a small category, $F\colon \mathcal{I} \to \mathcal{C}$ a functor and let $L$ with the morphisms $(l_I)_{I\in\mathcal{I}}$ be the limit of that functor. We know that for every morphism $i\colon I \to J$ in $\mathcal{I}$, we have $l_J = Fi \circ l_I$.

Let $Z$ be a set with a family of morphisms $f_I\colon Z \to \mathcal{C}(A, FI)$ determining a cone to $\mathcal{C}(A, F-)\colon \mathcal{I} \to \mathbf{Sets}$; that is, such that for each $i\colon I \to J$ in $\mathcal{I}$, we have that $f_J = (Fi \circ -) \circ f_I$; or, in other words, for each $z \in Z$ we have $f_J(z) = Fi \circ f_I(z)$. Thus, fixing any $z \in Z$ gives as a family of morphisms $f_I(z) \in \mathcal{C}(A, FI)$ defining a cone. This implies that there exists some unique morphism $a(z)\colon A \to L$ such that $l_I \circ a(z) = f_I(z)$.

Repeating this for every $z \in Z$ we have obtained a function $a\colon Z \to \mathcal{C}(A, L)$ with the property $l_I \circ a = f_I$. This function must be the unique one with this property, because for any other $a'\colon Z \to \mathcal{C}(A, L)$, for each $z \in Z$ we would have $l_I \circ a'(z) = f_I(z)$ and that would imply that $a'(z) = a(z)$. □

**Proposition 1.** *Right adjoints preserve limits.*

*Proof.* (From [Awo10]) Let $L\colon \mathbb{C} \to \mathbb{D}$ and $R\colon \mathbb{D} \to \mathbb{C}$ be a pair of adjoints $L \dashv R$. Let $X_i$ be objects determining any diagram and $Y$ an arbitrary object. We have the following chain of natural transformations between hom-sets.

$$\begin{aligned}
\hom(Y, R\lim X_i) &\cong \hom(LY, \lim X_i) \\
&\cong \lim \hom(LY, X_i) \\
&\cong \lim \hom(Y, RX_i) \\
&\cong \hom(Y, \lim RX_i)
\end{aligned}$$

By Yoneda Lemma, $R\lim X_i \cong \lim RX_i$. □

## References

[AT10] Samson Abramsky and Nikos Tzevelekos. Introduction to categories and categorical logic. In *New structures for physics*, pages 3–94. Springer, 2010.

[Awo10] Steve Awodey. *Category theory*. Oxford University Press, 2010.