Optics form a category, graphically

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Abstract

This is a short note that slightly extends the graphical calculus Mitchell Riley [Ril18] introduced for monoidal optics to the general case of arbitrary optics. This graphical calculus is then used to prove that optics for a strong monoidal action form a category. Compare these simple proofs with unintuitive formal derivations of this same author [Rom19, §3.1].

1 Optics

In functional programming, **optics** are a compositional representation of bidirectional data accessors, provided by libraries such as [Kme18]. Optics are divided into various *families*; each one of them encapsulating some data accessing pattern. For instance, *lenses* access subfields, *prisms* pattern match, and *traversals* iterate over containers.

Definition 1.1. Let $\oslash\colon \mathcal{M} \times \mathcal{C} \to \mathcal{C}$ be a functor and let $A, B, S, T \in \mathcal{C}$. An **optic** from (A, B) to (S, T) is an element of the set

$$\mathit{Optic}_{\oslash}((S,T),(A,B)) \coloneqq \int^{M \in \mathcal{M}} \mathcal{C}(S, M \oslash A) \times \mathcal{C}(M \oslash B, T).$$

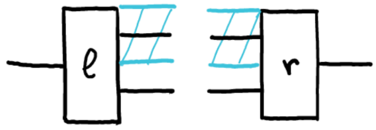
In other words, optics are pairs $\langle l \mid r \rangle$, where $l \in \mathcal{C}(S, M \oslash A)$ and $r \in \mathcal{C}(M \oslash B, T)$, quotiented by the equivalence relation generated by $\langle (\alpha \oslash \mathrm{id}) \circ l \mid r \rangle \sim \langle l \mid r \circ (\alpha \oslash \mathrm{id}) \rangle$ for every $\alpha \in \mathcal{M}(M, N)$.

2 The category of optics

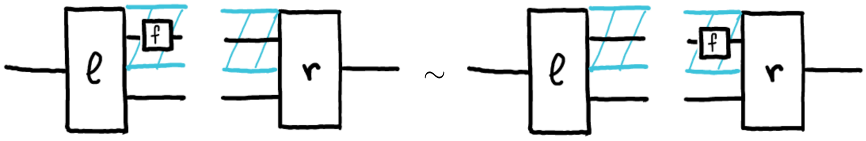
In order to depict optics, we shall employ the graphical calculus for bicategories [Mar14] and specifically for the bicategory of categories Cat, in which 0-cells are categories, 1-cells are functors and 2-cells are natural transformations. We shall also make use of monoidal functor boxes [Mel06] on the monoidal category $\mathit{Cat}(\mathcal{C}, \mathcal{C})$ for a fixed category $\mathcal{C}$.

Let us describe the specific elements that come into play when representing optics. The action $(\oslash)\colon \mathcal{M} \times \mathcal{C} \to \mathcal{C}$ will be seen as a functor $\mathcal{M} \to [\mathcal{C}, \mathcal{C}]$ represented by a functor box. Objects on the category $\mathcal{C}$ will be represented as functors (wires) from $\mathbf{1}$, the terminal category. The different categories we are using are usually represented by coloring the regions. However, given that ambiguity will not be a problem, we prefer to avoid coloring the regions in order to make diagrams clearer. After these considerations, an optic $\langle l \mid r \rangle \in \mathit{Optic}_{\oslash}((A, B), (S, T))$ can be depicted as the pair of functions $l\colon S \to M \oslash A$

and $r\colon M \oslash B \to T$,



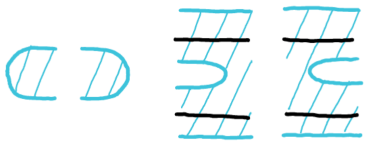
where the naturality condition quotients optics by the equivalence relation given by any natural transformation $f\colon M \to N$ travelling through the upper wire.



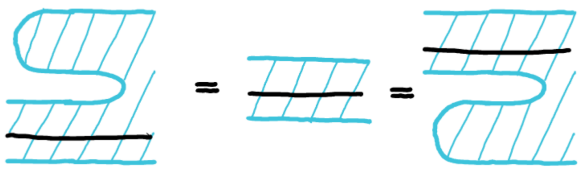
Remark 2.1. There exist some other graphical calculi for optics on the literature [Hed17, Boi19]. This proposal is not to be understood as a graphical calculus for categories of optics but a way of understanding what optics are in the already existing language of bicategories.

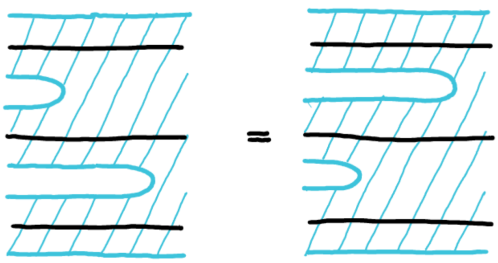
We have defined optics in full generality, allowing $\oslash$ to be an arbitrary functor from an arbitrary category $\mathcal{M}$. However, the most interesting case, and the one commonly studied when one talks about optics, is the one where $\mathcal{M}$ is a monoidal category and the functor is a *monoidal action*. In that case, we can endow optics over an action with category structure.

Definition 2.2. An action $\oslash\colon \mathcal{M} \times \mathcal{C} \to \mathcal{C}$ from a monoidal category $\mathcal{M}$ is a (strong) monoidal action when the associated functor $\mathcal{M} \to [\mathcal{C}, \mathcal{C}]$ is strong monoidal. In other words, it comes equipped with natural isomorphisms $\varepsilon_A\colon A \to I \oslash A$ and $\mu_{M,N,A}\colon M \oslash N \oslash A \to M \otimes N \oslash A$.



These isomorphisms must satisfy the usual unitality and associativity requirements, which can be translated into the graphical calculus as saying that the following equalities between diagrams hold.



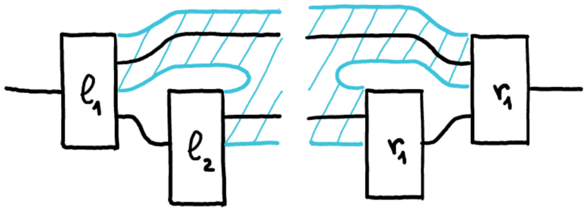


Proposition 2.3. *Let* $(\oslash)\colon \mathcal{M} \times \mathcal{C} \to \mathcal{C}$ *be a monoidal action.* $\mathit{Optic}_{\oslash}$ *can be given category structure.*

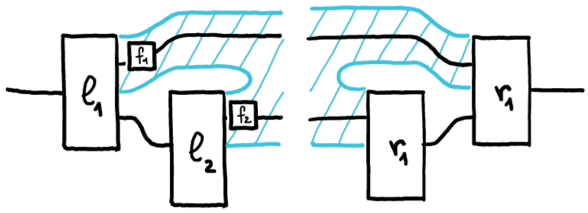
Proof. We start by proving that $\mathit{Optic}_{\oslash}$ defines a category. Let

$$\langle l_1 \mid r_1 \rangle \in \mathit{Optic}_{\oslash}((A,B),(S,T)), \qquad \text{and} \qquad \langle l_2 \mid r_2 \rangle \in \mathit{Optic}_{\oslash}((X,Y),(A,B)),$$

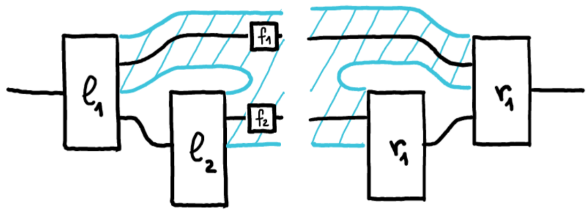
we define their composition in $\mathit{Optic}((X,Y),(S,T))$ to be the following.



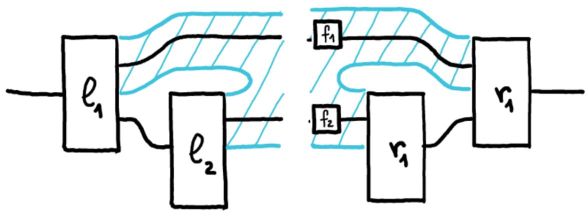
This is well defined, as it preserves the equivalence relation.



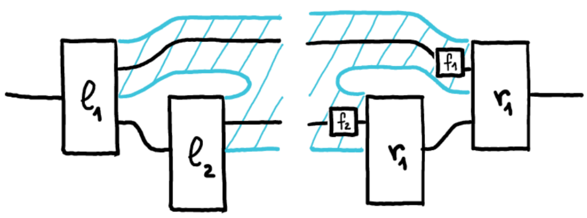
= (Naturality)



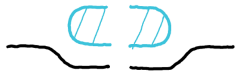
= (Equivalence relation)



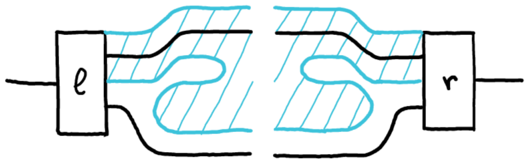
= (Naturality)



The identity, on the other hand, is defined as follows.



Composing with the identity leaves the optic unchanged on both sides.



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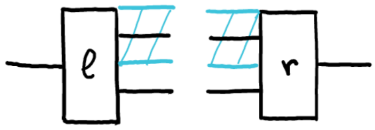

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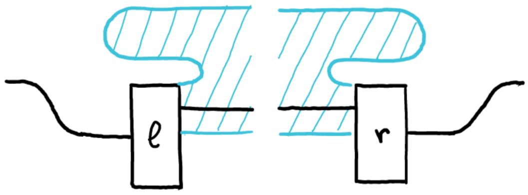
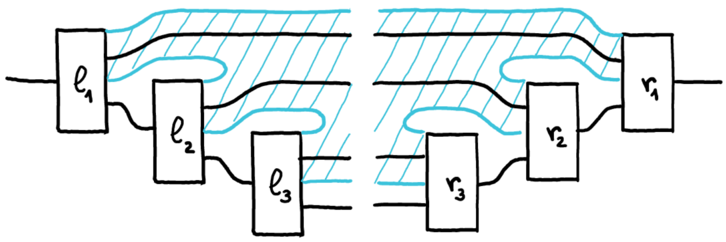


We also prove associativity of composition.



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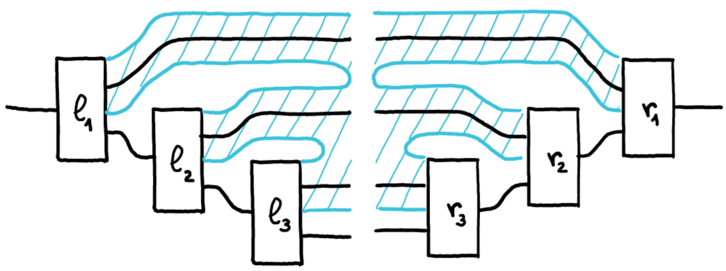


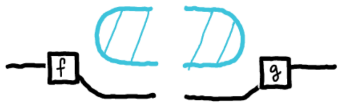
$\square$

3 Identity-on-objects embedding

An important, and possibly overlooked detail on the theory of optics, is the existence of an identity-on-objects functor that embeds the category $\mathcal{C} \times \mathcal{C}^{op}$ into $\mathit{Optic}_{\oslash}$. Monoids in the bicategory of profunctors, which we will call *promonads*, can be characterized to be equivalent to identity-on-objects functors. This is to say that the following result makes $\mathit{Optic}_{\oslash}$ a promonad.

Theorem 3.1. *There exists an identity-on-objects functor* $i \colon \mathcal{C} \times \mathcal{C}^{op} \to \mathit{Optic}_{\oslash}$.

Proof. The embedding of a morphism of $\mathcal{C} \times \mathcal{C}^{op}$ given by a pair of functions (f, g) is determined by the following diagram.



Using the graphical calculus, it is particularly easy to check that this defines in fact a functor. □

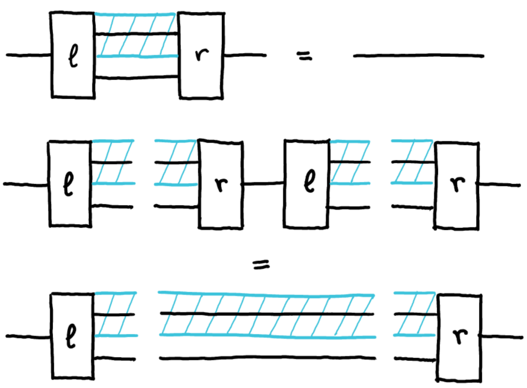
4 Lawful optics

Optics, and in particular lenses, were originally considered to be particularly well-behaved if they were to satisfy some extra axioms. In practice, these axioms are used to ensure that optics behave as the final user expects them to (the *lens* library [Kme18] examplifies this convention). An important contribution in the work of Riley [Ril18, §3] is to characterize the laws of optics as the axioms of a comonoid homomorphism. For completeness, we will depict them following the graphical calculus we just introduced.

Definition 4.1. [Ril18] Consider a type-invariant three-leg variant of optics where elements are the elements of a coend given as follows. Elements of this type can be written as triples quotiented by the equivalence relation of the coend, and depicted as triples of diagrams.

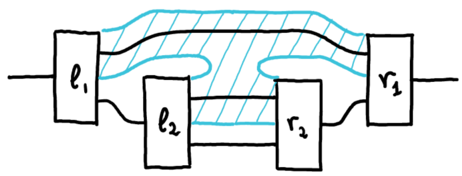
$$\mathit{Optic}^2_{\oslash}(S, A) := \int^{M_1, M_2 \in \mathcal{M}} \mathcal{C}(S, M_1 \oslash A) \times \mathcal{C}(M_1 \oslash A, M_2 \oslash A) \times \mathcal{C}(M_2 \oslash A, S).$$

Definition 4.2. An optic $\langle l \mid r \rangle$ is **lawful** when $r \circ l = \mathrm{id}$ and $\langle l \mid r \circ l \mid r \rangle = \langle l \mid \mathrm{id} \mid r \rangle$. That is to say that the following diagrammatic equations hold.



Proposition 4.3. *Assume* $(\oslash)$ *is a monoidal action. Lawful optics form a subcategory of the category of optics.*

Proof. We prove that the composition of two lawful optics is again a lawful optic.



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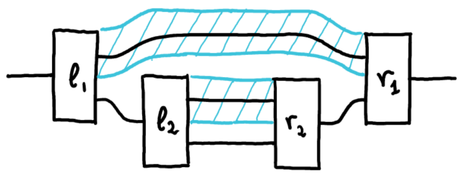

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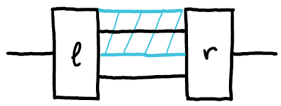

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