

Optics form a category, graphically

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Abstract

This is a short note that slightly extends the graphical calculus Mitchell Riley [Ril18] introduced for monoidal optics to the general case of arbitrary optics. This graphical calculus is then used to prove that optics for a strong monoidal action form a category. Compare these simple proofs with unintuitive formal derivations of this same author [Rom19, §3.1].

1 Optics

In functional programming, **optics** are a compositional representation of bidirectional data accessors, provided by libraries such as [Kme18]. Optics are divided into various *families*; each one of them encapsulating some data accessing pattern. For instance, *lenses* access subfields, *prisms* pattern match, and *traversals* iterate over containers.

Definition 1.1. Let $\odot: \mathcal{M} \times \mathcal{C} \rightarrow \mathcal{C}$ be a strong monoidal action and let $A, B, S, T \in \mathcal{C}$. An **optic** from (A, B) to (S, T) is an element of the set

$$\text{Optic}_{\odot}((S, T), (A, B)) := \int^{M \in \mathcal{M}} \mathcal{C}(S, M \odot A) \times \mathcal{C}(M \odot B, T).$$

In other words, optics are pairs $\langle l \mid r \rangle$, where $l \in \mathcal{C}(S, M \odot A)$ and $r \in \mathcal{C}(M \odot B, T)$, quotiented by the equivalence relation generated by $\langle (\alpha \odot \text{id}) \circ l \mid r \rangle \sim \langle l \mid r \circ (\alpha \odot \text{id}) \rangle$ for every $\alpha \in \mathcal{M}(M, N)$.

2 The category of optics

In order to depict optics, we shall employ the graphical calculus for bicategories [Mar14] and specifically for the bicategory of categories Cat , in which 0-cells are categories, 1-cells are functors and 2-cells are natural transformations. We shall also make use of monoidal functor boxes [Mel06] on the monoidal category $Cat(\mathcal{C}, \mathcal{C})$ for a fixed category \mathcal{C} .

Let us describe the specific elements that come into play when representing optics. The action $(\odot): \mathcal{M} \times \mathcal{C} \rightarrow \mathcal{C}$ will be seen as a functor $\mathcal{M} \rightarrow [\mathcal{C}, \mathcal{C}]$ represented by a functor box. Objects on the category \mathcal{C} will be represented as functors (wires) from **1**, the terminal category. The different categories we are using are usually represented by coloring the regions. However, given that ambiguity will not be a problem, we prefer to avoid coloring the regions in order to make diagrams clearer. After these considerations, an optic $\langle l \mid r \rangle \in \text{Optic}_{\odot}((A, B), (S, T))$ can be depicted as the pair of morphisms $l: S \rightarrow M \odot A$

and $r: M \otimes B \rightarrow T$,



where the naturality condition quotients optics by the equivalence relation given by any natural transformation $f: M \rightarrow N$ travelling through the upper wire.



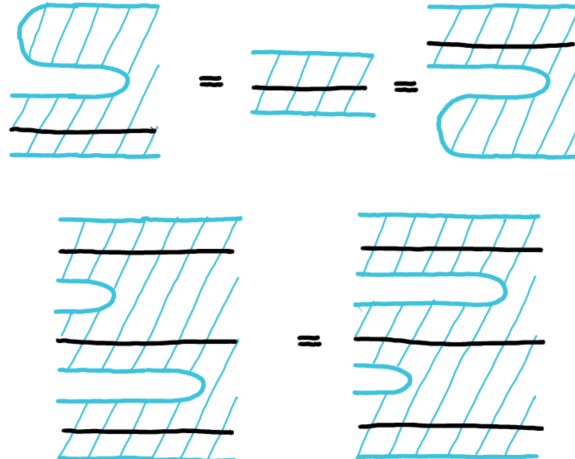
Remark 2.1. There exist some other graphical calculi for optics on the literature [Hed17, Boi19]. This proposal is not to be understood as a graphical calculus for categories of optics but a way of understanding what optics are in the already existing language of bicategories.

We have defined optics in full generality, allowing \otimes to be an arbitrary functor from an arbitrary category \mathcal{M} . However, the most interesting case, and the one commonly studied when one talks about optics, is the one where \mathcal{M} is a monoidal category and the functor is a *monoidal action*. In that case, we can endow optics over an action with category structure.

Definition 2.2. An action $\otimes: \mathcal{M} \times \mathcal{C} \rightarrow \mathcal{C}$ from a monoidal category \mathcal{M} is a (strong) monoidal action when the associated functor $\mathcal{M} \rightarrow [\mathcal{C}, \mathcal{C}]$ is strong monoidal. In other words, it comes equipped with natural isomorphisms $\varepsilon_A: A \rightarrow I \otimes A$ and $\mu_{M,N,A}: M \otimes (N \otimes A) \rightarrow (M \otimes N) \otimes A$.



These isomorphisms must satisfy the usual unitality and associativity requirements, which can be translated into the graphical calculus as saying that the following equalities between diagrams hold.

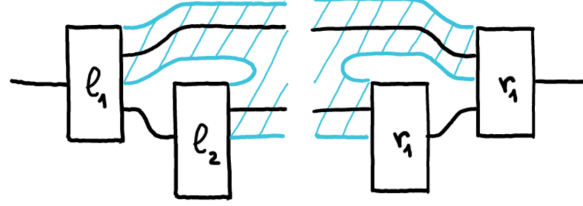


Proposition 2.3. Let $(\odot): \mathcal{M} \times \mathcal{C} \rightarrow \mathcal{C}$ be a monoidal action. $Optic_{\odot}$ can be given category structure.

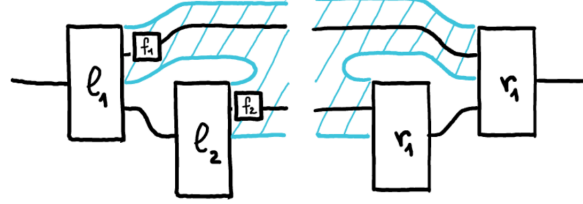
Proof. We start by proving that $Optic_{\odot}$ defines a category. Let

$$\langle l_1 \mid r_1 \rangle \in Optic_{\odot}((A, B), (S, T)), \quad \text{and} \quad \langle l_2 \mid r_2 \rangle \in Optic_{\odot}((X, Y), (A, B)),$$

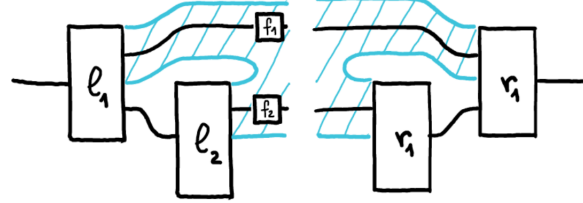
we define their composition in $Optic_{\odot}((X, Y), (S, T))$ to be the following.



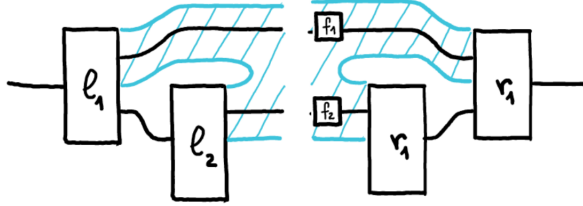
This is well defined, as it preserves the equivalence relation.



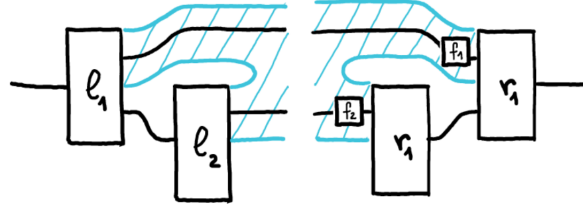
= (Naturality)



= (Equivalence relation)



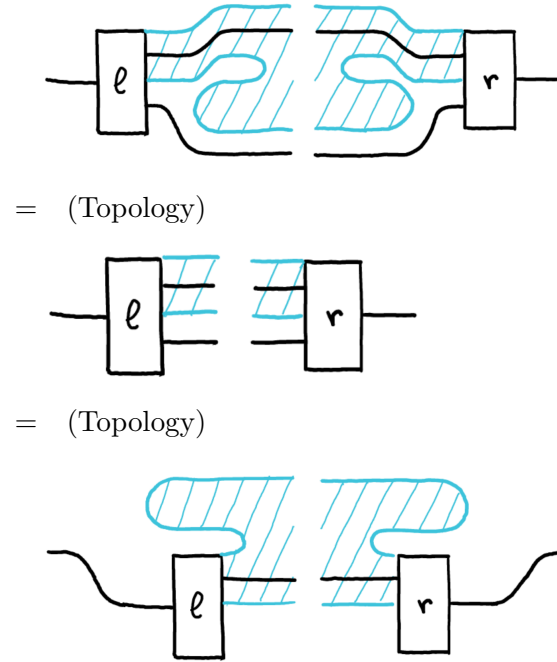
= (Naturality)



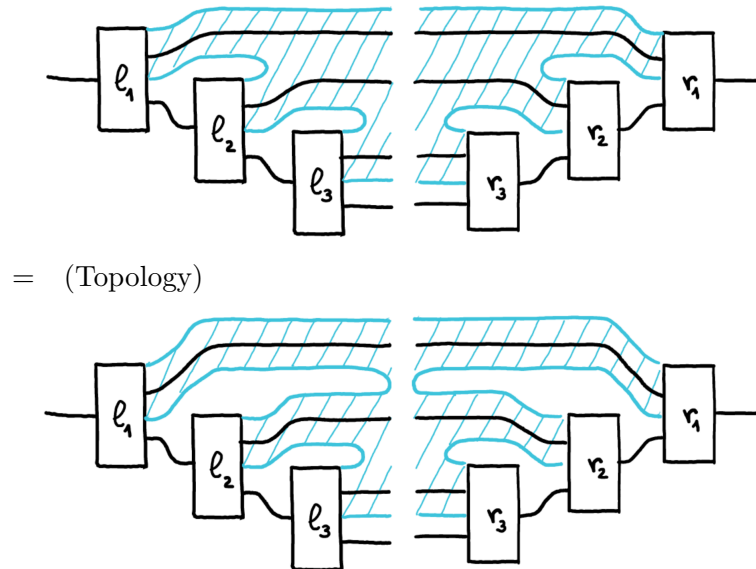
The identity, on the other hand, is defined as follows.



Composing with the identity leaves the optic unchanged on both sides.



We also prove associativity of composition.



□

3 Identity-on-objects embedding

An important, and possibly overlooked detail on the theory of optics, is the existence of an identity-on-objects functor that embeds the category $\mathcal{C} \times \mathcal{C}^{op}$ into $Optic_{\mathcal{O}}$. Monoids in the bicategory of profunctors, which we will call *promonads*, can be characterized to be equivalent to identity-on-objects functors. This is to say that the following result makes $Optic_{\mathcal{O}}$ a promonad.

Theorem 3.1. *There exists an identity-on-objects functor $i: \mathcal{C} \times \mathcal{C}^{op} \rightarrow Optic_{\mathcal{O}}$.*

Proof. The embedding of a morphism of $\mathcal{C} \times \mathcal{C}^{op}$ given by a pair of functions (f, g) is determined by the following diagram.



Using the graphical calculus, it is particularly easy to check that this defines in fact a functor. \square

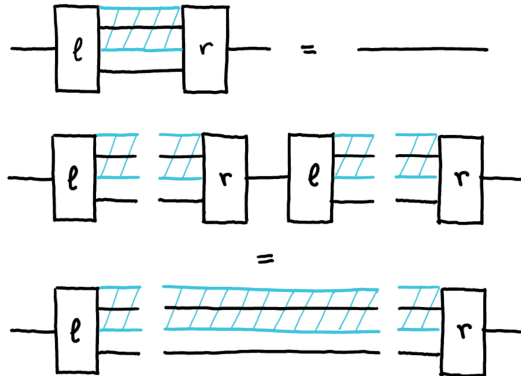
4 Lawful optics

Optics, and in particular lenses, were originally considered to be particularly well-behaved if they were to satisfy some extra axioms. In practice, these axioms are used to ensure that optics behave as the final user expects them to (the *lens* library [Kme18] exemplifies this convention). An important contribution in the work of Riley [Ril18, §3] is to characterize the laws of optics as the axioms of a comonoid homomorphism. For completeness, we will depict them following the graphical calculus we just introduced.

Definition 4.1. [Ril18] Consider a type-invariant three-leg variant of optics where elements are the elements of a coend given as follows. Elements of this type can be written as triples quotiented by the equivalence relation of the coend, and depicted as triples of diagrams.

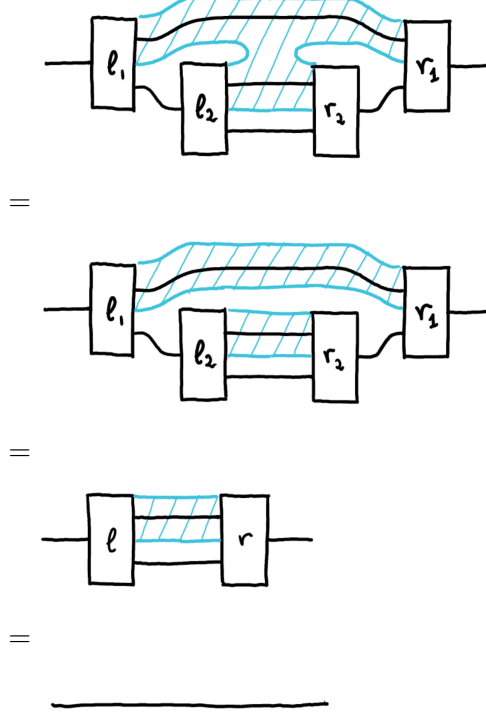
$$Optic_{\mathcal{O}}^2(S, A) := \int^{M_1, M_2 \in \mathcal{M}} \mathcal{C}(S, M_1 \otimes A) \times \mathcal{C}(M_1 \otimes A, M_2 \otimes A) \times \mathcal{C}(M_2 \otimes A, S).$$

Definition 4.2. An optic $\langle l \mid r \rangle$ is **lawful** when $r \circ l = \text{id}$ and $\langle l \mid r \circ l \mid r \rangle = \langle l \mid \text{id} \mid r \rangle$. That is to say that the following diagrammatic equations hold.



Proposition 4.3. Assume (\odot) is a monoidal action. Lawful optics form a subcategory of the category of optics.

Proof. We prove that the composition of two lawful optics is again a lawful optic.



□

References

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