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Algebra objects and algebra families for finite limit theories

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Introduction

Diers [1] has introduced the notion of locally representable functors. Let $\mathscr{C}$ be a category. A functor $F : \mathscr{C} \to \mathbf{Set}$ is locally representable if it is isomorphic to a sum of representable functors $\mathscr{C} \to \mathbf{Set}$. Equivalently, such a functor is locally representable if its unique extension to the free product completion of $\mathscr{C}$ is representable [1, Proposition 4.1.1]. The free product completion of $\mathscr{C}$ is the category $(\mathrm{Fam}\ \mathscr{C}^{\mathrm{op}})^{\mathrm{op}}$ and a representing object is then a family of objects of $\mathscr{C}$. For this reason we will here call locally representable functors *familially representable*.

In his Theorem 3.0.4 Diers [1] provides a characterization of familially representable functors in the case when $\mathscr{C}$ has all small connected limits: A functor $F : \mathscr{C} \to \mathbf{Set}$ is familially representable if and only if it preserves connected limits and satisfies the solution set condition (for which see [6]). In many such cases the representing family is easy to predict and has an interesting structure. The purpose of this paper is to describe the structure required on representing families for certain important functors. Rather than analysing when functors are representable we aim to construct functors by finding families of objects which bear the correct structure.

Algebraic structures on objects in categories have been treated in great detail. One can, for example, study topological groups as group objects in the category

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of topological spaces. Similarly category objects in a category have been deeply analysed. The structures that we investigate here will be algebraic (in the sense of finite limit theories) but the structure is distributed over a family of objects. By analogy with the term 'algebra object' (group object, category object) we will refer to them as 'algebra families' (group families, category families, etc.).

Our principal applications are to categorical constructions needed in computer science, but we will develop the theory beside an example chosen from homotopy theory: The Moore construction of a category of paths in a topological space which has a number of forms. One variant is as follows.

Let I_n denote the real interval $[0, n]$ of length n with the usual Euclidean topology. For each n let $\lceil 0 \rceil$ be the continuous map $I_0 \to I_n$ given by $\lceil 0 \rceil(0) = 0$ and let $\lceil n \rceil$ be the continuous map $I_0 \to I_n$ given by $\lceil n \rceil(0) = n$. Let X be any topological space and let $M_X = \sum_{n\in\omega} \mathbf{Top}(I_n, X)$ denote the set of continuous maps into X each of which has some I_n as its domain. Then M_X has a category structure given by: Suppose $f \in M_X$, say $f : I_j \to X$, then let $s(f)$ be the map $f\lceil 0 \rceil : I_0 \to X$ and $t(f)$ the map $f\lceil j \rceil : I_0 \to X$, and if $f : I_j \to X$, $g : I_k \to X$ are such that $s(g) = t(f)$, then we define $g * f : I_{j+k} \to X$ by

$$g * f(a) = \begin{cases} f(a) & \text{if } 0 \le a \le j\,, \\ g(a-j) & \text{if } j \le a \le j+k\,. \end{cases}$$

The verification that this definition satisfies the associative and identity laws (with identities given by maps $I_0 \to X$) is routine.

The Moore construction is functorial: Given any other topological space X' and a continuous map $h : X \to X'$, then composition with h defines a functor $M_h : M_X \to M_{X'}$. Arguments due to Eckmann and Hilton (see e.g. [2]) show that if such an algebraic structure is borne by a single hom set $\mathscr{C}(D, X)$, then the parametrizing object D is a coalgebra in $\mathscr{C}$. However, the 'multi-sorted' case where D is replaced by many objects (I_n), and $\mathscr{C}(D, X)$ by a coproduct of many hom sets $(\sum_{n\in\omega} \mathbf{Top}(I_n, X))$ has not been treated.

In Section 1 we define algebra families in a category $\mathscr{C}$ by shifting our attention to a category of families of objects of $\mathscr{C}$. Section 2 returns the problem to the category $\mathscr{C}$ and gives an alternative description of algebra families as algebras in $\mathscr{C}$ for certain theories. Section 3 explores some of the relationships between algebra objects and algebra families and their respective theories. Finally Section 4 includes examples of algebra families demonstrating that several well-known constructions are familially representable although they are not representable in the ordinary sense.

1. Algebra and coalgebra families

The Moore construction and several others (see below) depend upon homming out of a collection of objects $(D_i)_{i\in I}$ of a category $\mathscr{C}$ and obtaining a structure on the collection of morphisms $\sum_{i\in I}\mathscr{C}(D_i, X)$. In this section we interpret constructions of this sort in the category Fam $\mathscr{C}^{\mathrm{op}}$.

The section begins by showing that the collection of morphisms $\sum_{i\in I}\mathscr{C}(D_i, X)$ is a single hom set in Fam $\mathscr{C}^{\mathrm{op}}$. We then define algebra families, show that Moore's domains $(I_n)_{n\in\omega}$ are a cocategory family in **Top**, and note that this yields a 'classical' explanation of the Moore construction.

Suppose $\mathscr{C}$ is a category. The category of families of objects of $\mathscr{C}$, Fam $\mathscr{C}$, is defined as follows. An object of Fam $\mathscr{C}$ is a small set I and an I-indexed family $(A_i)_{i\in I}$ of objects of $\mathscr{C}$. An arrow of Fam $\mathscr{C}$ from $(A_i)_{i\in I}$ to $(B_j)_{j\in J}$ consists of a function $\phi: I\to J$ and a family of arrows of $\mathscr{C}$, $f_i: A_i\to B_{\phi(i)}$. Fam $\mathscr{C}$ is fibred over **Set** with the projection $p:\mathrm{Fam}\,\mathscr{C}\to\mathbf{Set}$ given by $(A_i)_{i\in I}\mapsto I$ and if $(\phi, f_i):(A_i)_{i\in I}\to(B_j)_{j\in J}$, then $p(\phi, f_i)=\phi: I\to J$.

Notice that $\mathscr{C}$ is a full subcategory of (Fam $\mathscr{C}^{\mathrm{op}})^{\mathrm{op}}$. The inclusion J is given by sending C to the singleton family (C), and $f: C\to C'$ to the opposite of the map $(C')\to(C)$ whose single component is $f^{\mathrm{op}}: C'\to C$ in $\mathscr{C}^{\mathrm{op}}$.

Lemma 1. *Suppose $\mathscr{C}$ is a category and $(D_i)_{i\in I}$ a collection of objects of $\mathscr{C}$; then for any $X\in\mathscr{C}$,*

$$\sum_{i\in I}\mathscr{C}(D_i, X)\cong \mathrm{Fam}\,\mathscr{C}^{\mathrm{op}}(J^{\mathrm{op}}X, (D_i)_{i\in I})\,.$$

Proof. The bijection is clear if we note that a morphism in the right-hand side is a function ϕ from the indexing set of the family $J^{\mathrm{op}}X$, i.e. from a one-point set $\{*\}$, into the set I, together with a $\mathscr{C}^{\mathrm{op}}$ morphism $X\to D_{\phi(*)}$. In other words, an element of the right-hand side is given by an element $i\in I$, and a $\mathscr{C}$ morphism $D_i\to X$. $\square$

Recall that a theory T is a small, finitely complete category. An algebra in a category $\mathscr{C}$ for the theory T is a finite limit preserving functor $\mathsf{T}\to\mathscr{C}$ and is called a T-algebra in $\mathscr{C}$. A T-algebra in **Set** is often referred to as just a T-algebra.

Now suppose T is a theory and $\mathscr{C}$ a category.

Definition 2. An *algebra family for* T *in* $\mathscr{C}$ is an (ordinary) algebra for T in Fam $\mathscr{C}$ and a *co-algebra family for* T *in* $\mathscr{C}$ is a T-algebra in Fam $\mathscr{C}^{\mathrm{op}}$.

Example 3. The collection $(I_n)_{n\in\omega}$ is a cocategory family in **Top**. To see this we verify that $(I_n)_{n\in\omega}$ has a category structure in Fam $\mathbf{Top}^{\mathrm{op}}$.

Firstly define $s:(I_n)_{n\in\omega}\to(I_n)_{n\in\omega}$ by $s=(\phi,(s_n)_{n\in\omega})$ with $\phi:\omega\to\omega$ given by $\phi(n)=0$, and $s_n: I_n\to I_{\phi(n)}$ in $\mathbf{Top}^{\mathrm{op}}$ given by the map $\lceil 0\rceil: I_0\to I_n$ in **Top**. Similarly $t=(\phi,(t_n)_{n\in\omega})$ with $t_n: I_n\to I_{\phi(n)}$ determined by the map $\lceil n\rceil: I_0\to I_n$. It is easy to see that s and t satisfy the equations required for source and target maps in a category.

The 'composable maps' in the category (I_n) are given by the pullback

$$\begin{tikzcd} M \arrow[r] \arrow[d] & (I_n)_{n\in\omega} \arrow[d, "s"] \\ (I_n)_{n\in\omega} \arrow[r, "t"'] & (I_n)_{n\in\omega} \end{tikzcd}$$

which is easily calculated in Fam $\mathbf{Top}^{\mathrm{op}}$: M is the family of *pushouts* (calculated in **Top**)

$$\begin{tikzcd} M_{m,n} & I_n \arrow[l, "b"'] \\ I_m \arrow[u, "a"] & I_0 \arrow[l, "t_m"] \arrow[u, "s_n"'] \end{tikzcd}$$

indexed by $\omega \times \omega$. Notice that $M_{m,n}$ may be chosen to be I_{m+n} with the injections a,b given by $a(x) = x$, $x \in [0, m]$ and $b(y) = m + y$, $y \in [0, n]$ (and of course any other choice of pushout is canonically isomorphic to this).

The 'composition' in the category is a Fam $\mathbf{Top}^{\mathrm{op}}$ morphism $c : (M_{m,n})_{m,n\in\omega\times\omega} \to (I_n)_{n\in\omega}$. To give such a morphism is to give, for each $m,n \in \omega \times \omega$, a natural number $\phi(m, n)$ and a continuous map $c_{m,n} : I_{\phi(m,n)} \to M_{m,n}$. In this case ϕ is given by $\phi(m, n) = m + n$ and $c_{m,n}$ is the identity $I_{m+n} \to M_{m,n}$ (or the canonical isomorphism if some other choice of pushout is made above).

That c is associative is easy to see. Identities for c come from the object of objects (the equalizer of s and the identity $(I_n) \to (I_n)$ or equivalently of t and the identity) which is the singleton family (I_0). 'Composable maps', the first of which is an identity, are given by the pullback

$$\begin{tikzcd} (I_n)_{n\in\omega} \arrow[r] \arrow[d] & (I_n)_{n\in\omega} \arrow[d, "s"] \\ (I_0) \arrow[r, "t'"'] & (I_n)_{n\in\omega} \end{tikzcd}$$

(where t' is the composition of t with the equalizer) and then c is equal to the projection of the pullback onto the second factor. Similarly for left identities.

Thus, by the classical argument of Eckmann and Hilton, homming into (I_n) in Fam $\mathbf{Top}^{\mathrm{op}}$, which by Lemma 1 is the same as homming out of each of the I_n in **Top**, yields an ordinary category—the Moore category.

2. Theories for algebra families

The fact that $\mathscr{C}(X, D)$ inherits an algebra structure if D is an algebra object in $\mathscr{C}$ has long been understood. The work of Eckmann and Hilton demonstrated that

$\mathscr{C}(D, X)$ inherits an algebra structure if D is an algebra object, *not* in $\mathscr{C}$, but in $\mathscr{C}^{\mathrm{op}}$. The preceding section demonstrates that $\sum_{i \in I} \mathscr{C}(D_i, X)$ inherits an algebra structure if (D_i) is an algebra object in Fam $\mathscr{C}^{\mathrm{op}}$ and that this is the basis of the Moore construction.

Although a coalgebra D in $\mathscr{C}$ is defined as an algebra in $\mathscr{C}^{\mathrm{op}}$, the structure that D bears in $\mathscr{C}$ is easy to describe. In this section we describe the structure borne in $\mathscr{C}$ by the collection of objects D_i which corresponds to an algebra structure on the family (D_i) in Fam $\mathscr{C}^{\mathrm{op}}$.

Let T be a theory, $\mathscr{C}$ a category and F a T-coalgebra family in $\mathscr{C}$ (i.e. $F : \mathsf{T} \to \mathrm{Fam}\ \mathscr{C}^{\mathrm{op}}$ preserves finite limits). Recall that $p : \mathrm{Fam}\ \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$ given by $(C_i)_{i \in I} \mapsto I$ and $(\phi, (f_i)_{i \in I}) \mapsto \phi$ preserves finite limits. The composite $G = pF : \mathsf{T} \to \mathbf{Set}$ is a T-algebra and will be called *the algebra of sorts* of F.

Example 4. Section 1 showed that the family $(I_n)_{n \in \omega}$ is a cocategory family. Its algebra of sorts is the category ω which has a single object and arrows corresponding to the natural numbers with composition given by addition. Explicitly: If we write the elements of the set ω as n (as usual), then source and target maps are given by $s(n) = 0 = t(n)$ for all $n \in \omega$; the composable pairs are given by the pullback of s along t which is $\omega \times \omega$ (the set indexing the composable maps of the category (I_n)); and composition is given by the image of the indexing map ϕ for the composition c of (I_n) which sends $(n, m) \mapsto n + m$.

Let $*$ be a one-point set. If $F : \mathsf{T} \to \mathbf{Set}$ recall that $e\ell\, F$, the category of elements of F, is the comma category $*/F$. The elements construction is functorial—considering the category **set** of small sets as an object of **Cat**, we have a functor $e\ell : \mathbf{Cat}/\mathbf{set} \to \mathbf{Cat}$. The Fam construction used above is also functorial with $\mathrm{Fam} : \mathbf{Cat} \to \mathbf{Cat}/\mathbf{set}$.

Proposition 5. *The functor $e\ell$ is left adjoint to the functor* Fam.

Proof. The isomorphism

$$\mathbf{Cat}(e\ell(G : \mathscr{A} \to \mathbf{set}), \mathscr{C}) \cong \mathbf{Cat}/\mathbf{set}(G : \mathscr{A} \to \mathbf{set}, p : \mathrm{Fam}\ \mathscr{C} \to \mathbf{set})$$

is easily seen: A functor in the left-hand side gives for each $A \in \mathscr{A}$ a collection of objects of $\mathscr{C}$, one for each element of GA, while functors in the right-hand side give a family of objects of C indexed by the set GA. □

Remark 6. Indeed the bijection is an isomorphism of categories, so we have a 2-adjoint, and it carries left exact functors to left exact functors.

Proposition 7. *Suppose* T *is small and finitely complete, and* $G : \mathsf{T} \to \mathbf{Set}$ *preserves finite limits; then* $e\ell\, G$ *is small and finitely complete.*

Proof. The projection $Q : e\ell\, G \to \mathsf{T}$ creates limits (see [6, p. 117]). □

Thus for any theory T and any T-algebra G the comma category $e\ell\, G$ is a theory. The remark says that the category of T-algebra families in $\mathscr{C}$ with algebra of sorts G is isomorphic to the category of $e\ell\, G$-algebras in $\mathscr{C}$. In other words to give a T-algebra structure on (D_i) in Fam $\mathscr{C}$ is to give an $e\ell\, G$-algebra structure on the objects D_i in $\mathscr{C}$ and dually (replacing Fam $\mathscr{C}$ by Fam $\mathscr{C}^{\mathrm{op}}$ and $e\ell\, G$-algebra by $e\ell\, G$-coalgebra). Thus $e\ell\, G$ is *the theory of* T*-algebra families with algebra of sorts* G.

In what follows we will move freely between descriptions of T-algebra families in $\mathscr{C}$ as T-algebras in Fam $\mathscr{C}$ and as $e\ell\, G$-algebras in $\mathscr{C}$ for an appropriate choice of G.

An explicit description of the structure of a monoid family in **Set** appears in Section 4.

3. Algebras and algebra families

This section records a few elementary results about the relationships between ordinary algebras and algebra families and their theories. Specifically, it demonstrates that every algebra is a degenerate algebra family, and every algebra family can be made into an ordinary algebra by 'joining up the sorts'.

Let T be a theory and let G be a T-algebra. Let $\mathsf{T}' = e\ell\, G$ be the theory of T-algebra families with algebra of sorts G. Since $\mathsf{T}' = e\ell\, G = */G$ (where $*$ is a one-point set and $G : \mathsf{T} \to \mathbf{Set}$), there is a projection Q of the comma category T' onto the theory T.

Lemma 8. *The projection* $Q : \mathsf{T}' \to \mathsf{T}$ *preserves finite limits.*

Proof. Immediate from [6, p. 113, Theorem 2] since Q creates limits. □

Corollary 9. *Let* $G : \mathsf{T} \to \mathbf{Set}$ *be any* T*-algebra. Then every* T*-algebra in* $\mathscr{C}$ *is a* T*-algebra family in* $\mathscr{C}$ *with algebra of sorts* G.

Proof. Suppose $F : \mathsf{T} \to \mathscr{C}$ is an algebra in $\mathscr{C}$ and $G : \mathsf{T} \to \mathbf{Set}$ the given T-algebra. Let T' be the theory of T-algebra families with algebra of sorts G, and let Q be the projection $\mathsf{T}' \to \mathsf{T}$. Since Q preserves finite limits the composite FQ is a T'-algebra. □

A T-algebra family $F' : \mathsf{T}' \to \mathscr{C}$ which factors as FQ where $F : \mathsf{T} \to \mathscr{C}$ is a T-algebra and $Q : \mathsf{T}' \to \mathsf{T}$ is the projection of the comma category T', is called a *degenerate* T-algebra family. In the Fam $\mathscr{C}$ description of a T-algebra family the degeneracy manifests itself through the image containing only families of the form $(C)_{i \in I}$ (that is, a copy of the same object C of $\mathscr{C}$ for each $i \in I$). Notice however that the algebra of sorts of a T-algebra family retains its structure even when the T-algebra family is degenerate.

Conversely every T-algebra family (in **Set**) yields, by Kan extension, an ordinary T-algebra.

Proposition 10. *Suppose $F' : \mathsf{T}' \to \mathbf{Set}$ is a T-algebra family and let $Q : \mathsf{T}' \to \mathsf{T}$ be the projection of the comma category T'; then the left Kan extension of F' along Q preserves finite limits.*

Proof. Notice first that the category $e\ell\, F'$ is a cofiltered category because T' has finite limits and F' preserves them. Now the left Kan extension of F' along Q can be computed as the (filtered) colimit in the first variable of the functor $H : (e\ell\, F')^{\mathrm{op}} \times \mathsf{T} \to \mathbf{Set}$ given by $((x, t'), t) \mapsto \mathsf{T}(Qt', t)$. The proposition follows since filtered colimits commute with finite limits in **Set**. □

Furthermore, the left Kan extension is easy to compute.

Proposition 11. *Suppose $F' : \mathsf{T}' \to \mathbf{Set}$ is a T-algebra family with algebra of sorts G and $Q : \mathsf{T}' \to \mathsf{T}$ is the projection of the comma category $e\ell\, G = \mathsf{T}'$ onto T. If $B \in \mathsf{T}$, then*

$$(\mathrm{Lan}_Q F')B = \sum_{B' \in GB} F'B'$$

Proof. Recall that the Kan extension may also be computed as a colimit over Q/B [6, p. 234]. In Q/B the full subcategory determined by the objects $QB' \xrightarrow{\mathrm{Id}} B$ is final and discrete so the colimit may be computed over this subcategory and becomes a coproduct. □

The preceding two results provide another explanation of the Moore construction of a category of paths in a topological space. In **Top** the collection of objects I_n, $n \in \omega$, forms a cocategory family (an $e\ell\, G$-coalgebra where G is the category ω described in Example 4). Thus, classically, homming out of the I_n yields a category family (an $e\ell\, G$-algebra in **Set**). The category family in turn yields an ordinary category—the Moore category—via Proposition 11. Indeed the coproduct which is computed there amounts to 'forgetting' the distinction between the different sorts which in this case means collecting all the paths together irrespective of the length of their domains $[0, n]$.

4. Examples

Example 12. The first example is the description of a monoid family (in **Set**) showing its structure both in Fam **Set** and as an algebra for a theory of the form $e\ell\, G$. This example may illuminate some of the preceding definitions.

A monoid family is an object in Fam **Set**, say $(M_i)_{i\in I}$, together with a multiplication $m:(M_i)_{i\in I}\times(M_i)_{i\in I}\to(M_i)_{i\in I}$ which is associative and has a left and right identity given by $e:(*)\to(M_i)_{i\in I}$ (where $(*)$ is a terminal object in Fam **Set**, say the singleton family containing a one-point set $*$). Of course this is strictly only a *presentation* of a monoid in Fam **Set**—the whole structure is given by a finite limit preserving functor from the theory T of monoids into Fam **Set**, but the relationships between such a presentation and a T-algebra are well understood and will not be entered into here.

In terms of sets a monoid family is a collection of sets M_i, one for each $i\in I$, together with a number of multiplications. The multiplication given in Fam **Set**

$$m:(M_i)_{i\in I}\times(M_i)_{i\in I}\cong(M_i\times M_j)_{(i,j)\in I\times I}\to(M_i)_{i\in I}$$

gives for each $(i,j)\in I\times I$ an index $\phi(i,j)$ and a function $m_{i,j}:M_i\times M_j\to M_{\phi(i,j)}$. Associativity amounts to asking that

$$\phi(\phi(i,j),k)=\phi(i,\phi(j,k)) \tag{1}$$

and that

$$m_{\phi(i,j),k}(m_{i,j}\times 1)=m_{i,\phi(j,k)}(1\times m_{j,k}):M_i\times M_j\times M_k\to M_{\phi(\phi(i,j),k)}\,.$$

The identity amounts to choosing a $\psi(*)\in I$ and an element of $M_{\psi(*)}$ which acts as a left and right identity in any multiplication $m_{\psi(*),j}$ or $m_{j,\psi(*)}$, $j\in I$, in which it is involved. (We are allowing $*$ to ambiguously denote either the one-point set or its single element.)

The elements $i\in I$, are called *sorts* and the monoid family is sometimes called a *multi-sorted monoid*—it is a monoid in the sense that any two elements can be multiplied together (associatively) and there exists a left and right identity, but the sorts give a 'partitioning' of the elements into sets which behave similarly in that the product of any two elements from the sets M_i and M_j is an element of the set $M_{\phi(i,j)}$.

Notice that the sorts themselves have a monoid structure (the *monoid of sorts* of the monoid family). Any two sorts may be multiplied with the product of i and j given by $\phi(i,j)$. This product is associative by (1) and $\psi(*)$ acts as a left and right identity. Thus, a monoid family is a graded monoid, graded by the monoid of sorts. The reader may wish to convince himself that if this monoid of sorts is thought of as a monoid $G:\mathsf{T}\to\mathbf{Set}$, then the category $e\ell\, G$ has all the structure required of a multi-sorted monoid and that the collection M_i is an $e\ell\, G$-algebra in **Set**.

Finally Proposition 11 says that the left Kan extension of this $e\ell\, G$-algebra along the projection $e\ell\, G\to\mathsf{T}$ (which by Proposition 10 gives a monoid) is the monoid obtained by forming the coproduct of the M_i—forgetting the 'partitioning' and viewing the whole collection of elements as a single ordinary monoid.

Dually a comonoid family is a monoid in $\mathrm{Fam}\,\mathbf{Set}^{\mathrm{op}}$ which amounts to a multi-sorted comonoid—a collection of sets M_i together with, for each $(i, j) \in I \times I$, a comultiplication $m_{i,j} : M_{\phi(i,j)} \to M_i + M_j$ which is coassociative etc. Again the set I has a monoid structure—the monoid of sorts of the comonoid family.

Example 13. Write $[k] = \{0, 1, \ldots, k-1\}$ and hence $[0] = \emptyset$. In **Set** the collection $[k]$ for $k = 0, 1, \ldots$ is a comonoid family. To verify this we show that $([k])_{k \in \omega}$ is a monoid in $\mathrm{Fam}\,\mathbf{Set}^{\mathrm{op}}$.

The multiplication $m : ([k])_{k\in\omega} \times ([k])_{k\in\omega} \cong ([j] + [k])_{(j,k)\in\omega\times\omega} \to ([k])_{k\in\omega}$ in $\mathrm{Fam}\,\mathbf{Set}^{\mathrm{op}}$ is given by a function $\phi : \omega \times \omega \to \omega$ which sends $(j, k) \mapsto j + k$ and functions $m_{j,k} : [\phi(j, k)] \to [j] + [k]$ in **Set**. Notice that $[j + k]$ is canonically isomorphic to $[j] + [k]$ and choose $m_{j,k}$ to be this isomorphism. Finally let $e : 1 \to ([k])_{k\in\omega}$ be given by (ψ, e_*), where ψ picks out the index 0 and e_* is the opposite of the unique map $[0] = \emptyset \to$ the one-point set. The verification that $(\phi, (m_{j,k}))$ and (ψ, e_*) satisfy the associative and identity laws is routine.

Since the collection $[k]$ is a comonoid family we know that homming out of it into any set X yields a monoid family. Furthermore, Proposition 11 says that forgetting the distinctions between the sorts gives an ordinary monoid. The ordinary monoid has as elements functions from $\{0, 1, \ldots, k-1\}$ into X which may be thought of as words of length k in elements of X. The identity for this ordinary monoid is the empty word, and the multiplication is given by concatenation of words.

Thus homming out of this comonoid family into a set X is a construction of *the free monoid on X*. The free monoid monad is familially representable and the collection $[k]$ is a representing family for the free monoid construction.

Example 14. Let $\mathscr{C}$ be a category. We will describe briefly what a cocategory family $\mathbf{X}$ in $\mathscr{C}$ is, with the next two examples in mind. We now take the two-sorted point of view that a category has an object of objects and an object of arrows.

The first piece of data is an (ordinary) category of sorts, with objects denoted $i, j, k, \ldots$, and arrows denoted $f : i \to, \ g : j \to k, \ldots$.

In addition to this there are families $(X_i, X_j, \ldots)$ and $(X_f, X_g, \ldots)$, of objects of $\mathscr{C}$ indexed by the objects and arrows, respectively, of the category of sorts. For each arrow $f : i \to j$ there are corresponding source and target arrows $s_f : X_i \to X_f$ and $t_f : X_j \to X_f$. For each composable pair $f : i \to j$, $g : j \to k$, there is a composition arrow $c_{f,g} : X_{g \cdot f} \to P_{f,g}$ where $P_{f,g}$ is the pushout of t_f, s_g. Given an object i and its identity arrow 1_i in the category of sorts there is an arrow $X_{1_i} \to X_i$.

The axioms to be satisfied by this data are parametrized, dual versions of the usual axioms of categories. As a guide to writing down the axioms, if the category of sorts has just one arrow, a cocategory family over it is just a cocategory object in $\mathscr{C}$.

Example 15. Let L be the category with two distinct objects and two parallel non-identity arrows, then $\mathbf{Set}^L$ is the category **Graph** of graphs and graph morphisms. We shall describe a cocategory family $\mathbf{X}$ in **Graph**.

Take the category of sorts to have one object, and the natural numbers as arrows, with addition as composition. Then take the object indexed by the one object in the category of sorts to be the graph with just one vertex, and no edges. The object indexed by the natural number k is the graph

$$[k] = 0 \to 1 \to \cdots \to k .$$

The source and target arrows pick out the first and last vertices, respectively, of the graph $[k]$. The composition arrows are identities.

Homming out of the $[k]$'s into some graph G gives a category family which becomes a category if we forget the distinctions between the sorts. The category so obtained is the free category on the graph G.

Example 16. We will now describe a cocategory family $\mathbf{X}$ in the category of small categories, which represents the free category-with-finite-products construction.

The category of sorts is the dual of the category of finite sets. We shall denote the finite sets by $I, J, \ldots$, and arrows between them by $f : I \to J$, $g : J \to K, \ldots$. Of course f here is a function from J to I.

The object is X_I is I regarded as a discrete category. The object X_f indexed by $f : I \to J$ is the category whose set of objects is the disjoint union of I and J and whose non-identity arrows are as follows: for each $j \in J$ there is an arrow from $f(j)$ to j.

The source and target arrows of X_f are the injections of the disjoint union of I and J. Given $f : I \to J$ and $g : J \to K$ the pushout of t_f and s_g has as set of objects the disjoint union of I, J, and K. The arrows of the pushout are those of X_f and X_g as well as formal composites of them of the form $f(g(k)) \to g(k) \to k$. The composition arrow from $X_{g \cdot f}$ to the pushout is the functor, which on objects is the injection of $I + K$ into $I + J + K$, and on arrows takes $f(g(k)) \to k$ to the formal composite $f(g(k)) \to g(k) \to k$.

It is more or less immediate that homming from this cocategory family into a category $\mathscr{C}$ yields the category $(\mathrm{Fin\,Fam}\,\mathscr{C}^{\mathrm{op}})^{\mathrm{op}}$.

The free category with products on a *multigraph*, rather than on a category, is also familially representable; some details are given in [5].

Example 17. In the category of simplicial sets the well-formed simplicial sets (defined in [4]) form a co-ω-category family. Thus homming out of the well-formed simplicial sets in the category of simplicial sets, or equivalently homming out of their realizations in **Top**, yields an ω-category. This provides a solution to the problem of generalizing the Moore construction to an ω-category of 'higher-dimensional paths' in a topological space. For details and applications see [3].

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