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THE CALCULUS OF ENDS OVER A BASE TOPOS*

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The paper develops a calculus of ends for categories enriched in the bicategory Span($\mathscr{E}$) (when $\mathscr{E}$ is a topos), which is suitable for the notion of functor category with domain a category internal to $\mathscr{E}$. The basic property of the bicategory Span($\mathscr{E}$) which is appropriate to this calculus is 'interchanging of variables', which takes the form of a compact closed monoidal structure. In fact, in the presence of compact closedness, the calculus of ends becomes exactly the calculus of indexed limits. The leading example of locally internal categories over $\mathscr{E}$ is followed throughout the paper.

1. Introduction

This paper is a sequel to [4], to which we refer for general terminology. In [4], and earlier in [3] and in [14], it was pointed out that a locally-internal category over an elementary topos $\mathscr{E}$ can be usefully regarded as a category enriched in Span($\mathscr{E}$).

The objects of Span($\mathscr{E}$) are thought of as indexing types. To consider several variables at a time it is necessary to consider the product in $\mathscr{E}$, which induces a tensor product in Span($\mathscr{E}$). The analogy with symmetric monoidal closed categories (underlined in [3] and [4]) then becomes even more evident. Indeed, most of the classical theory of categories enriched over a symmetric monoidal closed category extends in a natural way.

In this paper we investigate further completeness of categories enriched in Span($\mathscr{E}$), introducing *ends* and *functor categories* with domain an internal category. The structure on the base bicategory which is appropriate to this study, as abstracted from Span($\mathscr{E}$), is a compact closed monoidal structure (developed in [7], extending the notion of Kelly [10]). In fact the calculus of ends [9] is exactly the calculus of indexed limits in the presence of compact closedness. We apply this calculus to the important special case of locally internal categories, the study of which can be largely seen as an axiomatization of families of objects (see Lawvere's Perugia notes [11],

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and also [1, 2, 12, 13]. Hence in this paper, as previously in [3], we pay particular attention to the example of Fam(**C**), where **C** is an ordinary locally small category, regarded as a Span(Sets)-category.

2. Tensor product

2.1. Let $\mathscr{E}$ be a category with finite limits. There is a coherently associative, symmetric and unitary *tensor product* which is a homomorphism of bicategories

$$\otimes : \mathrm{Span}(\mathscr{E}) \times \mathrm{Span}(\mathscr{E}) \to \mathrm{Span}(\mathscr{E})$$

given on objects by the product in $\mathscr{E}$. On arrows $\alpha : u \nrightarrow v$ and $\beta : u' \nrightarrow v'$ the product is defined as follows:

$$\alpha \otimes \beta : u \times u' \nrightarrow v \times v'$$

is given by the matrix $\alpha_{i,j} \times \beta_{k,l}$ $(i \in u,\ j \in v,\ k \in u',\ l \in v')$. On 2-cells the product is defined termwise.

Observe further that the process of taking the transpose of a matrix provides an involutory homomorphism

$$(\ )^\circ : \mathrm{Span}(\mathscr{E})^{\mathrm{op}} \to \mathrm{Span}(\mathscr{E}),$$

which is the identity on objects. If $\mathscr{X}$ is a Span($\mathscr{E}$)-category, we then define the *opposite category* $\mathscr{X}^{\mathrm{op}}$ as the category with the same objects as $\mathscr{X}$, the same underlying object function $e : \mathrm{ob}\,\mathscr{X} \to \mathrm{ob}\,\mathscr{E}$, and the hom given by

$$\mathscr{X}^{\mathrm{op}}(x, y) = \mathscr{X}(y, x)^\circ.$$

To fix notation, we shall denote categories enriched in Span($\mathscr{E}$) by $\mathscr{X}, \mathscr{Y}, \ldots,$ but we reserve the notation $A, B, \ldots,$ for one-object Span($\mathscr{E}$)-categories (= categories internal to $\mathscr{E}$).

If $\phi : A \nrightarrow B$ is a module of Span($\mathscr{E}$)-categories, then $\phi^\circ : B^{\mathrm{op}} \nrightarrow A^{\mathrm{op}}$ denotes the module whose components are $\phi^\circ(b, a) = \phi(a, b)^\circ$.

Definition. If $\mathscr{X}$ and $\mathscr{Y}$ are Span($\mathscr{E}$)-categories, the *tensor product category* $\mathscr{X} \otimes \mathscr{Y}$ is defined as follows: the objects are the pairs (x, y) with x in $\mathscr{X}$ and y in $\mathscr{Y}$, the underlying object of (x, y) is $e_{(x,y)} = e_x \otimes e_y$, and the hom is given by

$$(\mathscr{X} \otimes \mathscr{Y})((x, y), (x', y')) = \mathscr{X}(x, x') \otimes \mathscr{Y}(y, y').$$

2.2. Observe that Span($\mathscr{E}$), with respect to the tensor product, has a *compact closed structure*, in the sense that there are isomorphisms of categories

$$\mathrm{Span}(\mathscr{E})(u \otimes v, w) \cong \mathrm{Span}(\mathscr{E})(u, v^\circ \otimes w), \tag{1}$$

pseudo-natural in u, v and w, (see [7] in the locally partially ordered case, [10] for monoidal categories).

Moreover a calculation shows that this compact closed structure lifts to modules, in the sense that for modules between Span($\mathscr{E}$)-categories there are the isomorphisms

$$\mathbf{Mod}(A \otimes B, C) \cong \mathbf{Mod}(A, B^{\mathrm{op}} \otimes C), \tag{2}$$

pseudo-natural in each variable, provided it varies over small categories. Observe that these isomorphisms include those of (1), as their restriction to discrete one-object Span($\mathscr{E}$)-categories.

We denote objects corresponding under the isomorphisms (2) as follows: $\phi : A \otimes B \mathrel{+\!\!\!\to} C$ corresponds to $\phi : A \mathrel{+\!\!\!\to} B^{\mathrm{op}} \otimes C$. Furthermore, for brevity, we make an abuse of language treating the isomorphisms in (2) as if they were in fact natural. It is worth writing explicitly the naturality laws, observing that the category which varies, in each case, is required to be one-object:

Rule 1:

$$\frac{\phi \cdot (\alpha \otimes 1) : A' \otimes B \mathrel{+\!\!\!\to} A \otimes B \mathrel{+\!\!\!\to} C}{\phi \cdot \alpha : A' \mathrel{+\!\!\!\to} A \mathrel{+\!\!\!\to} B^{\mathrm{op}} \otimes C},$$

Rule 2:

$$\frac{\gamma \cdot \phi : A \otimes B \mathrel{+\!\!\!\to} C \mathrel{+\!\!\!\to} C'}{(1 \otimes \gamma) \cdot \phi : A \mathrel{+\!\!\!\to} B^{\mathrm{op}} \otimes C \mathrel{+\!\!\!\to} B^{\mathrm{op}} \otimes C'},$$

Rule 3:

$$\frac{\phi \cdot (1 \otimes \beta) : A \otimes B' \mathrel{+\!\!\!\to} A \otimes B \mathrel{+\!\!\!\to} C}{(\beta^{\circ} \otimes 1) \cdot \phi : A \mathrel{+\!\!\!\to} B^{\mathrm{op}} \otimes C \mathrel{+\!\!\!\to} B'^{\mathrm{op}} \otimes C}.$$

Example. Recall from [3,4] that a Span($\mathscr{E}$)-category $\mathscr{X}$ is *locally internal* to $\mathscr{E}$ if it admits substitution along maps, i.e. if for every map $f : u \to v$ in $\mathscr{E}$, regarded as the span $(1, f) : u \mathrel{+\!\!\!\to} v$, and for every object x lying over v, there exist an object f^*x, lying over u, and isomorphisms

$$\mathscr{X}(f^*x, y) \cong \mathscr{X}(x, y) \cdot f, \tag{3}$$

natural in y. Recall that categories internal to $\mathscr{E}$ are just one-object Span($\mathscr{E}$)-categories.

When $\mathscr{X}$ is a locally internal category, $\mathscr{X}^{\mathrm{op}}$ is also locally internal. This follows from the fact that the substitution property (3) can be equivalently stated in the form

$$\mathscr{X}(x, f^*y) \cong f^{\circ} \cdot \mathscr{X}(x, y).$$

When $\mathscr{X}$ and $\mathscr{Y}$ are locally internal categories, $\mathscr{X} \otimes \mathscr{Y}$ is not necessarily locally internal. However, the following formula holds for general Span($\mathscr{E}$)-categories:

$$\mathscr{L}(\mathscr{X} \otimes \mathscr{Y}) \cong \mathscr{L}\mathscr{X} \times \mathscr{L}\mathscr{Y}$$

where $\mathscr{L}$ denotes the free locally internal category on a Span($\mathscr{E}$)-category (see [3]). Hence, if $\mathscr{X}$ and $\mathscr{Y}$ are locally internal, then $\mathscr{L}(\mathscr{X} \otimes \mathscr{Y}) \cong \mathscr{X} \times \mathscr{Y}$. Observe moreover that when A and B are internal categories, $A \otimes B$ is the usual product of internal categories.

2.3. Hom functor. Suppose now that $\mathscr{E}$ is an elementary topos. To a Span($\mathscr{E}$)-category A with one object, another Span($\mathscr{E}$)-category $\mathscr{P}A$ can be associated which represents modules out of A:

$$\frac{F:\mathscr{X}\to\mathscr{P}A}{\tilde{F}:A \nrightarrow \mathscr{X}}. \tag{4}$$

The objects of $\mathscr{P}A$ over u are modules $A \nrightarrow u$ and the hom is given by right Kan extension:

$$\mathscr{P}A(\alpha,\beta)\cong \hom^{A}(\alpha,\beta).$$

Note. We often use the same notation for a module and the functor corresponding to it under (4).

Let I denote the terminal object of $\mathscr{E}$ regarded as a (discrete) internal category. Then $\mathscr{P}I$ can be thought of as $\mathscr{E}$, regarded as a locally internal category over itself: objects over u are arrows $I \nrightarrow u$, i.e. maps in $\mathscr{E}$ with codomain u. If $f:w\to u$ and $g:w'\to v$ are two objects, then the formulae for right extensions in Span($\mathscr{E}$) (see for instance [3]) give:

$$\mathscr{P}I(f,g)\cong \prod_{f\times 1}(1\times g). \tag{5}$$

Definition. If $\mathscr{X}$ is a Span($\mathscr{E}$)-category, then

$$\mathrm{Hom}_{\mathscr{X}}:\mathscr{X}^{\mathrm{op}}\otimes\mathscr{X}\to\mathscr{P}I$$

is the Span($\mathscr{E}$)-functor which takes (x,x') to

$$\mathscr{X}(x,x'):I\nrightarrow e_x\otimes e_{x'}.$$

It is now easy to see that modules can be represented as functors with codomain $\mathscr{P}I$.

Proposition. *There is an isomorphism of categories* $\mathbf{Mod}(\mathscr{X},\mathscr{Y})\cong\mathbf{Cat}(\mathscr{X}^{\mathrm{op}}\otimes\mathscr{Y},\mathscr{P}I)$, *where* **Mod** *and* **Cat** *are the bicategories of* Span($\mathscr{E}$)*-modules and of* Span($\mathscr{E}$)*-categories.*

Proof.

$$\frac{\dfrac{\mathscr{X}\nrightarrow\mathscr{Y}}{I\nrightarrow\mathscr{X}^{\mathrm{op}}\otimes\mathscr{Y}}\ \text{(by (2))}}{\mathscr{X}^{\mathrm{op}}\otimes\mathscr{Y}\to\mathscr{P}I.}\ \text{(by (4))} \qquad \square$$

2.4. The category $\mathscr{P}I$ admits a tensor product $\odot:\mathscr{P}I\otimes\mathscr{P}I\to\mathscr{P}I$, induced from the tensor product on Span($\mathscr{E}$) as follows: given $f:a\to u$ and $g:b\to v$, then

$$f\odot g=f\times g:a\times b\to u\times v.$$

With this tensor and the hom defined above, $\mathscr{P}I$ is a 'monoidal closed Span($\mathscr{E}$)-category', in the sense that:

Proposition. *For any f, g, h in $\mathscr{P}I$*

$$\mathscr{P}I(f \odot g, h) \cong \mathscr{P}I(f, \mathscr{P}I(g, h)).$$

Proof. From formula (5), observing that

$$\prod_{f\times g\times 1} (1\times h) \cong \prod_{f\times 1} \Big(1 \times \prod_{g\times 1} (1\times h)\Big). \qquad \square$$

3. Ends and the Fubini theorem

3.1. Suppose A is a category internal to $\mathscr{E}$, and consider the module

$$\mathrm{Hom}_A : I \rightarrow\!\!\!\!\!\!+\;\; A^{\mathrm{op}} \otimes A.$$

Definition. Given a functor $T : A^{\mathrm{op}} \otimes A \to \mathscr{X}$, the *end* of T, if it exists, is the limit $\{\mathrm{Hom}_A, T\}$ of T indexed by Hom_A.

As usual, we use the notation

$$\int_A T = \{\mathrm{Hom}_A, T\}$$

for the end of T.

Remarks. (1) Recall that the limit $\{\phi, F\}$ of the functor $F : C \to \mathscr{X}$ indexed by the module $\phi : u \rightarrow\!\!\!\!\!\!+\;\; A$, when it exists, is an object representing the right lifting $\mathrm{hom}_C(\phi, F^*)$ of $F^* = \mathscr{X}(1, F)$ through ϕ, where in general $\mathscr{X}(G, F)$ for functors

$$A \xrightarrow{F} \mathscr{X} \xleftarrow{G} B$$

denotes the module with components

$$\mathscr{X}(G, F)(b, a) = \mathscr{X}(Gb, Fa)$$

(see [6] for this module calculus).

The limit $\{\phi, F\}$ is characterized by the following universal property:

$$\frac{\gamma \to \mathscr{X}(G, \{\phi, F\})}{\phi \cdot \gamma \to \mathscr{X}(G, F)}$$

for any module $\gamma : B \rightarrow\!\!\!\!\!\!+\;\; u$.

When the enrichment is in a symmetric monoidal category, indexed limits coincide with those introduced in [5], and the notion of end coincides with the usual one for enriched categories (see [9]).

(2) We can also consider ends with extra variables: given a functor $T: D \otimes A^{\mathrm{op}} \otimes A \to \mathscr{X}$, we denote by $\int_A T: D \to \mathscr{X}$ the limit $\{1_D \otimes \mathrm{Hom}_A, T\}$.

(3) By [4], if $\mathscr{X}$ is a *complete* locally internal category, then it admits all ends.

3.2. The end formula. The end formula for limits of Day and Kelly [8] can be extended to our case.

Proposition. *Suppose A is an internal category. For any functor $F: A \to \mathscr{X}$ and any module $\phi: I \mathrel{+\!\!\!\to} A$,*

$$\{\phi, F\} \cong \int_A \{\phi^\circ \otimes 1_A, F\}$$

if the right-hand side exists.

Proof. First observe that, by applying rule 1 and rule 2 of 2.2, ϕ factors as

$$\begin{tikzcd} I \arrow[rr, "\phi", "+"{marking}] \arrow[dr, "\mathrm{Hom}_A"', "\times"{marking}] & & A \cong I \otimes A \\ & A^{\mathrm{op}} \otimes A \arrow[ur, "\phi^\circ \otimes 1_A"', "\times"{marking}] & \end{tikzcd}$$

Then use the fact that the limit $\{\phi \cdot \psi, F\}$ of a functor F indexed by a composite can be computed as $\{\psi, \{\phi, F\}\}$. So we have:

$$\{\phi, F\} \cong \{(\phi^\circ \otimes 1_A) \cdot \mathrm{Hom}_A, F\} \cong \int_A \{\phi^\circ \otimes 1_A, F\}. \qquad \square$$

Remark. Suppose F and ϕ are as in the proposition above, and $\mathscr{X}$ is complete. Then

$$\mathscr{X}(1, \{\phi^\circ \otimes 1_A, F\}) \cong \hom_{I \otimes A}(\phi^\circ \otimes 1_A, F^*) : \mathscr{X} \mathrel{+\!\!\!\to} A^{\mathrm{op}} \otimes A$$

corresponds to $\hom_I(\phi^\circ, F^*): A^{\mathrm{op}} \otimes \mathscr{X} \mathrel{+\!\!\!\to} A^{\mathrm{op}}$ (by applying rule 2) which does not involve the category structure of A.

3.3. Proposition. *Suppose $G: \mathscr{Y} \otimes A^{\mathrm{op}} \otimes A \to \mathscr{P}I$ and $H: \mathscr{Y} \to \mathscr{P}I$ are functors. Then there is a bijection of natural transformations*

$$\frac{H \to \int_A G}{H \odot \mathrm{Hom}_A \to G}$$

where $H \odot \mathrm{Hom}_A$ denotes the composite

$$\mathscr{Y} \otimes A^{\mathrm{op}} \otimes A \xrightarrow{H \otimes \mathrm{Hom}_A} \mathscr{P}I \otimes \mathscr{P}I \xrightarrow{\odot} \mathscr{P}I.$$

Proof. The result comes from the following sequence of bijections:

$$\begin{array}{c l}
H \to \int_A G & \\
\hline
1_{\mathscr{Y}} \to \mathscr{P}I(H, \int_A G) & \text{(def. of end)} \\
\hline
1_{\mathscr{Y}} \otimes \mathrm{Hom}_A \to \mathscr{P}I(H, G) & \text{(right lifting)} \\
\hline
1_{\mathscr{Y} \otimes A^{\mathrm{op}} \otimes A} \to \mathscr{P}I(\mathrm{Hom}_A, \mathscr{P}I(H, G)) & \text{(Prop. 2.4)} \\
\hline
1_{\mathscr{Y} \otimes A^{\mathrm{op}} \otimes A} \to \mathscr{P}I(H \odot \mathrm{Hom}_A, G) & \\
\hline
H \odot \mathrm{Hom}_A \to G &
\end{array}$$

□

3.4. Proposition (Fubini theorem). *Suppose $\mathscr{X}$ is complete. Given a functor*

$$T : A^{\mathrm{op}} \otimes A \otimes D^{\mathrm{op}} \otimes D \to \mathscr{X}$$

we have

$$\int_A \int_D T \cong \int_{A \otimes D} T \cong \int_D \int_A T.$$

Proof. Just observe that

$$\begin{aligned}
\mathrm{Hom}_{A \otimes D} &\cong (1_{A^{\mathrm{op}} \otimes A} \otimes \mathrm{Hom}_D) \cdot \mathrm{Hom}_A \\
&\cong (\mathrm{Hom}_A \otimes 1_{D^{\mathrm{op}} \otimes D}) \cdot \mathrm{Hom}_D
\end{aligned}$$

and apply the fact that limits can be computed iteratively. □

Example. Consider the category Fam(**C**), where **C** is an ordinary small-complete, locally small category. Let A be a small category, regarded as a one-object Span(Sets)-category, and consider a functor

$$T : A^{\mathrm{op}} \otimes A \to \mathrm{Fam}(\mathbf{C}).$$

We prove that $\int_A T$ exists and give a formula to compute it.

Let u be the set of objects of A regarded as a discrete category. First consider the two arrows in the base

$$\mathrm{Hom}_A, \mathrm{Hom}_u : I \rightrightarrows u^{\mathrm{op}} \otimes u.$$

We can compute the limits $\{\mathrm{Hom}_u, T\}$ and $\{\mathrm{Hom}_A, T\}$ separately.

The formulae for right liftings in Span(Sets) give:

$$\{\mathrm{Hom}_u, T\} \cong \prod_i T(i, i)$$

and

$$\{\mathrm{Hom}_A, T\} \cong \prod_{i,j} T(i, j)^{A(i,j)}$$

(such products exist because **C** is small-complete).

There are two arrows

$$\{\mathrm{Hom}_A, T\} \rightrightarrows \{\mathrm{Hom}_u, T\}$$

assigned by combining the effect of T on arrows, namely $T(1,f)$ and $T(f,1)$, and their equalizer in **C** is $\int_A T$.

As a consequence it is easy to show that Fam(**C**) is a complete Span(Sets)-category in the sense of limits indexed by modules $\phi : v \to A$, where A is one-object. By the end formula on limits (see 3.2) we have

$$\{\phi, F\} \cong \int_A \{\phi^\circ \otimes 1_A, F\}$$

and by the remark of 3.2, the limit $\{\phi^\circ \otimes 1_A, F\}$ can be obtained by a discrete limit. A direct calculation shows that discrete limits always exist in Fam(**C**) when **C** is small-complete. More precisely, when $x : A \to \mathrm{Fam}(\mathbf{C})$ is a functor and $\psi : u \nrightarrow v$ is a module (u and v discrete), the limit $\{\psi, x\}$ can be calculated by the formula

$$\{\psi, x\}_i \cong \prod_j x_j^{\phi_{ij}}$$

where $i \in u$ and $j \in v$.

4. Functor categories

4.1. Suppose $\mathscr{X}$ is a Span($\mathscr{E}$)-category, and A is internal to $\mathscr{E}$. The *functor category* $\mathscr{X}^A$ has objects over u the functors

$$F : u \otimes A \to \mathscr{X}.$$

The hom in $\mathscr{X}^A$ is defined by

$$\mathscr{X}^A(F,G) = \int_A \mathrm{Hom}_{\mathscr{X}}(F,G)$$

as in the following diagram:

$$\begin{tikzcd}
u^{\mathrm{op}} \otimes v \otimes A^{\mathrm{op}} \otimes A \arrow[r, "F\otimes G"] & \mathscr{X}^{\mathrm{op}} \otimes \mathscr{X} \arrow[r, "\mathrm{Hom}_{\mathscr{X}}"] & \mathscr{P}I \\
u^{\mathrm{op}} \otimes v \otimes I \arrow[u, "1_{u^{\mathrm{op}} \otimes v} \otimes \mathrm{Hom}_A", "|"'] \arrow[urr, "\mathscr{X}^A(F,G)"'] & &
\end{tikzcd}$$

Observe that $\int_A \mathrm{Hom}_{\mathscr{X}}(F,G)$ is an object of $\mathscr{P}I$ lying over $u^\circ \times v$, i.e. it corresponds to an arrow $u \nrightarrow v$ in Span($\mathscr{E}$).

To define composition in $\mathscr{X}^A$, we first recall from [3] how limits can be computed in a category of the form $\mathscr{P}A$. Recall from 2.3 that when B is an internal category, any functor $F: B \to \mathscr{P}A$ corresponds bijectively to a module $\tilde{F}: A \nrightarrow B$. Now the indexed limit $\{\phi, F\}$, as an object of $\mathscr{P}A$ is given by the right extension $\hom_B(\phi, F)$.

By rule 2, we now have that $\int_A \mathrm{Hom}_{\mathscr{X}}(F, G)$ can be calculated by means of a right lifting in Span($\mathscr{E}$):

$$\mathscr{X}^A(F,G) = \hom_{\upsilon \otimes A^{\mathrm{op}} \otimes A}(\upsilon \otimes \mathrm{Hom}_A, \mathrm{Hom}_{\mathscr{X}}(F,G))$$

which is a module $u \nrightarrow \upsilon$.

Now composition in $\mathscr{X}^A$ is defined using the universal property of right liftings.

Theorem. *If A is an internal category, there is an isomorphism of categories*

$$\frac{F: \mathscr{Y} \to \mathscr{X}^A}{\hat{F}: A \otimes \mathscr{Y} \to \mathscr{X}}.$$

Proof. Given $F: \mathscr{Y} \to \mathscr{X}^A$, the corresponding functor $\hat{F}$ is given on objects by $\hat{F}(a, y) = (Fy)(a)$, where a is the only object of A and y is an object of $\mathscr{Y}$.

The effect on arrows of $\hat{F}$ corresponds to that of F under the bijection of Proposition 3.3, taking

$$H(y') = \mathscr{Y}(y, y')$$

and

$$G(a, a', y') = \mathscr{X}((Fy)(a), (Fy')(a')). \qquad \square$$

Remark. It is easy to check that $\mathscr{X}^A$ is complete whenever $\mathscr{X}$ is complete. Moreover, limits can be computed *pointwise*, i.e. $\{\phi, F\}$ corresponds to $\{\phi \otimes 1_A, \hat{F}\}$ under the isomorphisms (2), where D is an internal category and $\hat{F}: D \otimes A \to \mathscr{X}$ corresponds to $F: D \to \mathscr{X}^A$ as in the previous theorem.

In particular, if $\mathscr{X}$ is a locally internal category, also $\mathscr{X}^A$ is locally internal: to see this it is enough to observe that the indexing module $f \otimes 1_A$ is a map if f is a map.

Corollary. *If A is an ordinary small category,* Fam(**C**) *and* Fam(**D**) *are categories of families, with* **C** *and* **D** *locally small, then there is an equivalence of categories*

$$\frac{\mathrm{Fam}(\mathbf{D}) \to \mathrm{Fam}(\mathbf{C})^A}{\mathscr{L}A \times \mathrm{Fam}(\mathbf{D}) \to \mathrm{Fam}(\mathbf{C})}.$$

Proof. By the fact that $\mathscr{L}(A \otimes \mathrm{Fam}\,\mathbf{D}) \cong \mathscr{L}A \times \mathscr{L}\mathrm{Fam}\,\mathbf{D}$ (see 2.2) and the universal property of $\mathscr{L}$. $\square$

4.2. Examples. (1) *Internal presheaves.* For any internal category A, we have

$$\mathscr{P}A = \mathscr{P}I^{A^{\mathrm{op}}}.$$

The correspondence on objects is the following: given an object $\phi : A \nrightarrow u$ over u in $\mathscr{P}A$, we get a module $I \nrightarrow u \otimes A^{\mathrm{op}}$ by interchanging the variables. Hence an object (over u):

$$u \otimes A^{\mathrm{op}} \nrightarrow \mathscr{P}I$$

in $\mathscr{P}I^{A^{\mathrm{op}}}$.

(2) *Families.* Consider a small set u. If $\mathbf{C}$ is an ordinary locally small category, then $\mathrm{Fam}(\mathbf{C})^u$ has objects over v the families indexed by $u \times v$. If x is over v and y is over w, by applying the formula for right liftings in Span(Sets), we have:

$$\mathrm{Fam}(\mathbf{C})^u(x, y)_{jk} = \prod_i \mathbf{C}(x_{ij}, y_{ik}).$$

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