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Some properties of the continued fraction expansion of $(m/n)\,e^{1/q}$

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Introduction. Continued fractions of the form

$$\left[b_1, \ldots, b_h, \overline{f_1(x), \ldots, f_k(x)}\right]_{x=0}^{\infty}$$

are called Hurwitzian if $b_1, \ldots, b_h$ are positive integers, $f_1(x), \ldots, f_k(x)$ are polynomials with rational coefficients which take positive integral values for $x = 0, 1, 2, \ldots$, and at least one of the polynomials is not constant. $f_1(x), \ldots, f_k(x)$ are said to form a quasi-period.

The expansions

$$e = \left[2, \overline{1, 2x+2, 1}\right]_{x=0}^{\infty} \quad \text{and} \quad e^{1/q} = \left[\overline{1, (2x+1)\,q-1, 1}\right]_{x=0}^{\infty},$$

where q is a positive integer, $q > 1$, are well-known examples. (See Perron (4), Davis (1), or Walters (5).)

If m and n are coprime positive integers, it follows from a theorem of A. Hurwitz (see Perron (4)) that $(m/n)\,e^{1/q}$ also has a Hurwitzian continued fraction with a quasi-period consisting of linear progressions, apart from constants.

In this paper we prove the existence of a quasi-period containing exactly mn linear progressions as above, and a necessary and sufficient condition for each of these progressions to have the form $2qx+b$ is also derived.

The proofs are based on matrix methods developed by Kolden (2) and Walters (5).

Notation. Let $\{A_r\}$ be a sequence of non-singular matrices with real number elements, and let

$$\prod_{r=0}^{N} A_r = \begin{pmatrix} p_N & r_N \\ q_N & s_N \end{pmatrix}.$$

Then we write $\xi \sim \prod_{r=0}^{\infty} A_r$ if each of the sequences p_N/q_N and r_N/s_N converges to ξ, as $N \to \infty$. Continued fractions and matrices are connected by the relation

$$[a_0, a_1, \ldots] \sim \prod_{r=0}^{\infty} U_{a_r}, \quad \text{where} \quad U_a = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}.$$

Any non-empty product $U_{b_1} U_{b_2} \ldots U_{b_r}$, $b_i > 0$, is denoted by P. Kolden proved that any matrix M with non-negative integer elements and determinant ± 1 (apart from U_0 and the identity matrix) has one of the forms P, U_0P, PU_0, U_0PU_0. The factorization is unique. (See (2) pages 159–161.)

The following facts are used frequently:

(i) $PU_0P = P$, $U_0PU_0P = U_0P$, $MP = P$ or U_0P, $MU_0P = P$ or U_0P. (These follow from $U_aU_0U_b = U_{a+b}$.)

(ii) If
$$P = \begin{pmatrix} p & r \\ q & s \end{pmatrix}$$
then $p \geqslant r$ and $q \geqslant s$.

Lemma 6 of Walters (5) is used twice. It implies, under conditions satisfied here, that
$$A_0 \prod_{r=1}^{\infty} A_r \sim \prod_{r=1}^{\infty} (A_0 A_r A_0^{-1}).$$

Our investigation is based on a result of Lehmer (3).

LEMMA 1.
$$\prod_{r=1}^{d} \begin{pmatrix} (2r-1)q+1 & (2r-1)q \\ (2r-1)q & (2r-1)q-1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & (-1)^d \end{pmatrix} \pmod{d}.$$

Proof. An easy induction shows the matrix product is
$$\begin{pmatrix} A_d(q) & B_d(q) \\ (-1)^d B_d(-q) & (-1)^d A_d(-q) \end{pmatrix},$$
where
$$A_d(q) = \sum_{k=0}^{d} \binom{d}{k} \frac{(d+k-1)!}{(d-1)!} q^k$$
and
$$B_d(q) = \sum_{k=0}^{d-1} \binom{d-1}{k} \frac{(d+k)!}{(d-1)!} q^{k+1}.$$
The lemma follows, since $(d+k-1)!/(d-1)!$ is divisible by d if $k \geqslant 1$, while
$$(d+k)!/(d-1)! \text{ is divisible by } d \text{ if } k \geqslant 0.$$

DEFINITION. *Polynomials in x are defined by*
$$\begin{pmatrix} F_d(x) & R_d(x) \\ G_d(x) & S_d(x) \end{pmatrix} = \prod_{r=1}^{d} \begin{pmatrix} \{2(dx+r)-1\}q+1 & \{2(dx+r)-1\}q \\ \{2(dx+r)-1\}q & \{2(dx+r)-1\}q-1 \end{pmatrix}.$$
Then by Lemma 1 the coefficients of $G_d(x)$ and $R_d(x)$ are divisible by d.

LEMMA 2. *Let $d = mn$. Then*
$$\frac{m}{n} e^{1/q} \sim \prod_{x=0}^{\infty} \begin{pmatrix} F_d(x) & (m/n)R_d(x) \\ (n/m)G_d(x) & S_d(x) \end{pmatrix},$$
where the matrices are unimodular with integer elements, positive unless $m = n = q = 1$.

Proof. We have
$$e^{1/q} \sim \prod_{y=0}^{\infty} \begin{pmatrix} (2y+1)q+1 & (2y+1)q \\ (2y+1)q & (2y+1)q-1 \end{pmatrix},$$
by Theorem 1 and Lemma 2 of (5).

The regrouping Lemma 3 of (5) gives
$$e^{1/q} \sim \prod_{x=0}^{\infty} \begin{pmatrix} F_d(x) & R_d(x) \\ G_d(x) & S_d(x) \end{pmatrix}.$$

Consequently, by Lemma 1 of (5),

$$\frac{m}{n}e^{1/q} \sim \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix} \prod_{x=0}^{\infty} \begin{pmatrix} F_d(x) & R_d(x) \\ G_d(x) & S_d(x) \end{pmatrix}.$$

Our Lemma 2 then follows from the equation

$$\begin{pmatrix} F_d(x) & (m/n)\,R_d(x) \\ (n/m)\,G_d(x) & S_d(x) \end{pmatrix} = \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix} \begin{pmatrix} F_d(x) & R_d(x) \\ G_d(x) & S_d(x) \end{pmatrix} \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix}^{-1}, \dots \tag{1}$$

either by direct comparison of convergents or by using Lemma 6 of (5).

The factorization of the matrices occurring in Lemma 2 must now be examined. Our method is based essentially on the proof of Hurwitz's theorem presented in Perron (4), pages 110–123.

Lemma 3. *Let* $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ *have non-negative integer elements, with* $a_1 a_3 \neq 0$, $\Delta = a_1 a_4 - a_2 a_3 \neq 0$ *and either* $a_1 \geqslant a_2$ or $a_3 \geqslant a_4$. *Then A may be factorized uniquely as*

$$A = BC,$$

where B is of type P or $U_0 P$ and $\quad C = \begin{pmatrix} p & r \\ 0 & s \end{pmatrix},$

with p and s positive integers, and r an integer.

If $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$, then $\det B = \Delta/|\Delta|$, $p = (a_1, a_3)$, $b_1 = a_1/p$, $b_3 = a_3/p$, $ps = |\Delta|$ and $-s < r \leqslant p$. (See Perron (4) page 111 for the proof of a similar result.)

The matrix A_x defined by

$$A_x = \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix} \begin{pmatrix} F_d(x) & R_d(x) \\ G_d(x) & S_d(x) \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

is evidently of type A for integers $x \geqslant 0$, and we can obtain the following factorization:

Lemma 4. $$A_x = \begin{pmatrix} F_d(x) & (m/n)\,R_d(x) \\ (n/m)\,G_d(x) & S_d(x) \end{pmatrix} B_0 C_1,$$

where $B_0 C_1$ is the factorization of $\begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

Proof. $$A_x = \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix} \begin{pmatrix} F_d(x) & R_d(x) \\ G_d(x) & S_d(x) \end{pmatrix} \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix}^{-1} \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} F_d(x) & (m/n)\,R_d(x) \\ (n/m)\,G_d(x) & S_d(x) \end{pmatrix} \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{by equation (1).}$$

Lemma 5. *Matrices B_t and* $C_{t+1} = \begin{pmatrix} p_{t+1} & r_{t+1} \\ 0 & s_{t+1} \end{pmatrix}$ *are defined by the recurrence relation*

$$B_t C_{t+1} = Q_t^{-1} C_t U_{a-1} U_1 U_1 \quad (t = 1, 2, \dots, d),$$

where $a = \{2(dx+t)-1\}q$ *and* $Q_t = Q_t(x) = U_k$, *where* $k = p_t^2(2qx-1)$. *Here* $B_t C_{t+1}$ *is the factorization of*

$$\begin{pmatrix} 2s_t & s_t \\ p_t\{2d-1+(4t-2)q\}+2r_t & p_t\{d+(2t-1)q\}+r_t \end{pmatrix}$$

and C_1 *is defined in Lemma* 4. *Then the factorization*

$$A_x = B_0\left\{\prod_{t=1}^{d} Q_t B_t\right\} C_{d+1}$$

holds.

Proof. With an obvious simplified notation, noting that

$$\begin{pmatrix} a+1 & a \\ a & a-1 \end{pmatrix} = U_1 U_{a-1} U_1,$$

we have
$$\begin{aligned} A_x &= B_0 C_1 U_1^{-1}\begin{pmatrix} F & R \\ G & S \end{pmatrix} U_1 = B_0 C_1 U_1^{-1}\left\{\Pi\begin{pmatrix} a+1 & a \\ a & a-1 \end{pmatrix}\right\} U_1 \\ &= B_0 C_1 U_1^{-1}\{\Pi U_1 U_{a-1} U_1\} U_1 = B_0 C_1 \Pi(U_{a-1} U_1 U_1) \\ &= B_0 C_1 \Pi(C_t^{-1} Q_t B_t C_{t+1}), \quad \text{by the recurrence relation;} \\ &= B_0\{\Pi Q_t B_t\} C_{d+1}. \end{aligned}$$

LEMMA 6.
$$\begin{pmatrix} F_d(x) & (m/n)R_d(x) \\ (n/m)G_d(x) & S_d(x) \end{pmatrix} = B_0\left\{\prod_{t=1}^{d} Q_t B_t\right\} B_0^{-1}.$$

Proof. Lemmas 4 and 5 give

$$A_x = \begin{pmatrix} F_d(x) & (m/n)R_d(x) \\ (n/m)G_d(x) & S_d(x) \end{pmatrix} B_0 C_1 = B_0\left\{\prod_{t=1}^{d} Q_t B_t\right\} C_{d+1}.$$

Using remarks (i) and (ii) of the introduction, it is not hard to see that the uniqueness conditions of Lemma 3 are satisfied if $x \geqslant 1$. Hence $C_1 = C_{d+1}$. Multiplication on the right by $C_1^{-1}B_0^{-1}$ then gives Lemma 6.

THEOREM 1.

(*a*)
$$\frac{m}{n}e^{1/q} \sim B_0 \prod_{x=0}^{\infty}\prod_{t=1}^{\infty} Q_t(x) B_t$$

(*b*)
$$\frac{m}{n}e^{1/q} \sim \begin{pmatrix} F_d(0) & (m/n)R_d(0) \\ (n/m)G_d(0) & S_d(0) \end{pmatrix} B_0 \prod_{x=1}^{\infty}\prod_{t=1}^{d} Q_t(x) B_t.$$

Proof. By Lemmas 2 and 6, and Lemma 6 of (5)

$$\frac{m}{n}e^{1/q} \sim \prod_{x=0}^{\infty}\left\{B_0\left(\prod_{t=1}^{d} Q_t(x) B_t\right) B_0^{-1}\right\} = B_0 \prod_{x=0}^{\infty}\prod_{t=1}^{d} Q_t(x) B_t.$$

The proof of (*b*) is similar.

Remarks. (i) The matrices of (*b*) are of type M and may be factorized, as mentioned in the introduction. It follows that the regular continued fraction for $(m/n)e^{1/q}$ has a quasi-period containing exactly mn linear progressions.

(ii) The expansion (*a*) does not give the regular continued fraction for $(m/n)e^{1/q}$ as the matrix $Q_t(0)$ has a negative element.

As a simple example we consider $3e$. It is easily verified that

$$p_1 = p_2 = p_3 = 1, \quad B_0 = U_3, B_1 = B_2 = U_0 U_1 U_5 U_1, \quad B_3 = U_0 U_1 U_1 U_5,$$

$$\begin{pmatrix} F_3(0) & 3R_3(0) \\ \frac{1}{3}G_3(0) & S_3(0) \end{pmatrix} = \begin{pmatrix} 106 & 261 \\ 13 & 32 \end{pmatrix} = U_8 U_6 U_2 U_2 U_0.$$

Theorem 1 (*b*) then gives

$$3e \sim U_8 U_6 U_2 U_2 U_0 U_3 \prod_{x=1}^{\infty} \{U_{2x-1} U_0 U_1 U_5 U_1\}\{U_{2x-1} U_0 U_1 U_5 U_1\}\{U_{2x-1} U_0 U_1 U_1 U_5\}$$

$$\sim U_8 U_6 U_2 U_5 \prod_{x=1}^{\infty} \{U_{2x} U_5 U_1 U_{2x} U_5 U_1 U_{2x} U_1 U_5\},$$

and the corresponding regular continued fraction is

$$3e = \left[8, 6, 2, 5, \overline{2x, 5, 1, 2x, 5, 1, 2x, 1, 5}\right]_{x=1}^{\infty}.$$

It is natural to ask when the numbers p_t of Theorem 1 satisfy $p_t = 1$ for $t = 1, 2, \ldots, d$. This is answered by Theorem 2. It is convenient to state, without proof, the following lemma:

Lemma 7. *If* $\begin{pmatrix} X & V \\ Y & W \end{pmatrix}$ *is a matrix of type P or $U_0 P$, with $XW - YV = 1$ and $X > 1$, then V is determined by the conditions $YV \equiv -1 \pmod X$ and $0 < V < X$.*

Theorem 2. *Each of the progressions $p_t^2(2qx-1)$ in Theorem 1 reduces to $2qx-1$ if, and only if, the integers K_t defined by the recurrence relation*

$$K_t = (4t-2)\,qK_{t-1} + K_{t-2} \quad \textit{for} \quad 2 \leqslant t \leqslant d,$$

with $K_0 = 1$, $K_1 = 2q - 1 + 2r_1$, satisfy $(K_t, d) = 1$.

Proof. (i) Assume $p_t = 1$ for $t = 1, 2, \ldots, d$. Then the definition of $B_t C_{t+1}$ in Lemma 5 gives

$$\begin{pmatrix} 2d & d \\ 2d - 1 + (4t-2)\,q + 2r_t & d + (2t-1)\,q + r_t \end{pmatrix} = B_t C_{t+1}.$$

Hence

$$B_t = \begin{pmatrix} 2d & v_t \\ 2d - 1 + (4t-2)\,q + 2r_t & w_t \end{pmatrix}, \quad C_{t+1} = \begin{pmatrix} 1 & r_{t+1} \\ 0 & d \end{pmatrix}.$$

By Lemma 7, v_t is determined by

$$\{2r_t + (4t-2)\,q - 1\}\,v_t \equiv -1 \pmod{2d} \tag{2}$$

and $0 < v_t < 2d$.

The matrix equation obtained by equating the two expressions for $B_t C_{t+1}$ gives

$$1 = 2r_{t+1} + v_t. \tag{3}$$

Congruence (2) and equation (3) imply

$$\{(4t-2)\,q - v_{t-1}\}\,v_t \equiv -1 \pmod{2d} \tag{4}$$

for $t = 2, \ldots, d$.

Residue classes k_t, each prime to $2d$, can now be defined by the recurrence relation

$$k_t \equiv k_{t-1}\{(4t-2)q - v_{t-1}\} \pmod{2d} \quad (2 \leqslant t \leqslant d), \tag{5}$$

with $k_1 \equiv 2q + 2r_1 - 1 \pmod{2d}$.
Congruence (4) may then be written as

$$k_t v_t \equiv -k_{t-1} \pmod{2d}.$$

This congruence is valid for $t = 1$ on taking $k_0 \equiv 1 \pmod{2d}$. Substituting for $k_{t-1}v_{t-1}$ in congruence (5) gives

$$k_t \equiv (4t-2)qk_{t-1} + k_{t-2} \pmod{2d},$$

completing the 'only if' part of the proof of Theorem 2.

(ii) Assume the integers K_t, defined in Theorem 2, satisfy $(K_t, d) = 1$ for $t = 1, 2, \ldots, d$. Since K_t is clearly odd, $(K_t, 2d) = 1$.

By definition, $B_t C_{t+1}$ is the factorization of

$$\begin{pmatrix} 2s_t & s_t \\ p_t\{2d-1+(4t-2)q\}+2r_t & p_t\{d+(2t-1)q\}+r_t \end{pmatrix}.$$

Also

$$B_t = \begin{pmatrix} X_t & V_t \\ Y_t & W_t \end{pmatrix},$$

where

$$X_t = 2s_t/p_{t+1},\ Y_t = [p_t\{2d-1+(4t-2)q\}+2r_t]/p_{t+1},$$

and

$$p_{t+1} = (2s_t, p_t\{2d-1+(4t-2)q\}+2r_t).$$

By Lemma 7, V_t is determined by

$$Y_t V_t \equiv -1 \pmod{X_t} \quad \text{and} \quad 0 < V_t < X_t.$$

Since $(K_t, 2d) = 1$ for $t = 0, 1, \ldots, d$, we can define integers v_t by

$$K_t v_t \equiv -K_{t-1} \pmod{2d} \quad \text{and} \quad 0 < v_t < 2d$$

for $t = 1, 2, \ldots, d$.

It is now shown that $v_t = V_t$ and $p_t = 1$, for $t = 1, 2, \ldots, d$. We use induction and assume

$$p_1 = 1, p_2 = 1, \ldots p_t = 1 \quad \text{and} \quad V_1 = v_1, \ldots, V_{t-1} = v_{t-1}$$

for some t with $2 \leqslant t < d$.

We first verify that the assumption is correct for $t = 2$.

By definition, $p_1 = (m, n) = 1$. Hence $s_1 = d$ and

$$B_1 C_2 = \begin{pmatrix} 2d & d \\ 2d-1+2q+2r_1 & d+q+r_1 \end{pmatrix}$$
$$= \begin{pmatrix} 2d & d \\ 2d+K_1 & d+q+r_1 \end{pmatrix}.$$

Consequently $p_2 = (2d, 2d+K_1) = (2d, K_1) = 1$,

and

$$B_1 = \begin{pmatrix} 2d & V_1 \\ 2d+K_1 & W_1 \end{pmatrix},$$

where $(2d+K_1)V_1 \equiv -1 \pmod{2d}$ and $0 < V_1 < 2d$. But $K_1 v_1 \equiv -1 \pmod{2d}$ and $0 < v_1 < 2d$, so that $v_1 = V_1$. This completes the verification for $t = 2$.

To complete the induction, we need the following result which is easily deduced:

$$\{(4t-2)q - v_{t-1}\} v_t \equiv -1 \pmod{2d} \tag{6}$$

for $t \geqslant 2$.

It is now shown that $p_{t+1} = 1$.

The induction hypothesis gives $p_t = 1$ and $s_t = d$, so that

$$B_t C_{t+1} = \begin{pmatrix} 2d & d \\ 2d-1+(4t-2)q+2r_t & d+(2t-1)q+r_t \end{pmatrix}$$

and consequently $p_{t+1} = (2d, (4t-2)q + 2r_t - 1)$.

Also since $p_{t-1} = 1$, $s_{t-1} = d$, and $v_{t-1} = V_{t-1}$, we have

$$B_{t-1}C_t = \begin{pmatrix} 2d & d \\ 2d-1+(4t-6)q+2r_{t-1} & d+(2t-3)q+r_{t-1} \end{pmatrix}$$
$$= \begin{pmatrix} 2d & v_{t-1} \\ 2d-1+(4t-6)q+2r_{t-1} & W_{t-1} \end{pmatrix} \begin{pmatrix} 1 & r_t \\ 0 & d \end{pmatrix}.$$

Hence $2r_t + v_{t-1} = 1$ and $\quad p_{t+1} = (2d, (4t-2)q - v_{t-1})$.

Congruence (6) then implies $p_{t+1} = 1$. Finally, we show that $V_t = v_t$.

By definition

$$B_t = \begin{pmatrix} 2d & V_t \\ 2d+(4t-2)q-v_{t-1} & W_t \end{pmatrix},$$

where by Lemma 7,

$$\{2d+(4t-2)q-v_{t-1}\}V_t \equiv -1 \pmod{2d} \quad \text{and} \quad 0 < V_t < 2d.$$

Congruence (6) together with $0 < v_t < 2d$, gives $V_t = v_t$ and the induction is complete.

The following theorem deals with $me^{1/q}$. Here $r_1 = 0$, $K_0 = 1$, and $K_1 = 2q-1$.

THEOREM 3. *Let q_i denote a prime with the property that $(K_t, q_i) = 1$ for $t = 1, 2, \ldots, q_i$. Also let S be the set of natural numbers m for which each of the progressions $p_i^2(2qx-1)$ reduces to $2qx-1$, in the expansions of $me^{1/q}$ given by Theorem 1. Then S is identical with T, the set of numbers which contain only $q_1, q_2, \ldots$ as prime factors.*

Proof. (i) Suppose m belongs to S. Then by Theorem 2, $(K_t, m) = 1$ for $t = 1, 2, \ldots, m$. Hence $(K_t, q) = 1$ for $t = 1, 2, \ldots, q$, for each prime q dividing m.

(ii) Suppose m belongs to T. Then to deduce that m belongs to S, it suffices, by Theorem 2, to prove that the condition $(K_t, q) = 1$ for $t = 1, 2, \ldots, q$ is equivalent to $(K_t, q) = 1$ for all t.

However, this follows from the congruence

$$K_{t+q} \equiv (-1)^q K_t \pmod{q} \quad \text{for} \quad t \geqslant 0.$$

This congruence is easily deduced from the formula

$$K_t = \sum_{r=0}^{t} \binom{t}{r} (-1)^{t-r} \frac{(t+r)!}{t!} q^r,$$

which may be verified by induction.

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