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SHEAVES ON SITES AS CAUCHY-COMPLETE CATEGORIES

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In [9] I showed that sheaves on a space H can be regarded as *symmetric Cauchy-complete B-categories* for a certain bicategory B formed from H. Here I extend this result to sheaves on a general site. More particularly, from a cite C, with pretopology P, I construct a bicategory of 'relations', $\mathbf{Rel}(C, P)$. Then the category of sheaves on C is biequivalent to the bicategory of symmetric Cauchy-complete $\mathbf{Rel}(C, P)$-categories.

There are several ways of looking at this result. Firstly it extends, to general sites, the idea [4, 3] (derived from Boolean-valued set theory) that sheaves on a space are sets with equality taking values in the open-set lattice of the space. Secondly, it places topos theory (and its logic) in the context of generalized logic [6], where equality exists as symmetric hom. Thirdly, it exhibits the sheaf condition as a further example of Cauchy-completeness for categories (which property was so named by Lawvere [6] since it includes the usual notion of Cauchy-completeness for metric spaces).

1. The bicategory of relations

Let C be a locally small (but not necessarily small) category with pullbacks, and P a pretopology on C (see, for example, p. 12 of [5]).

If u, v are objects in C then a *crible* R from u to v is a set of spans from u to v such that

$$\begin{tikzcd}[column sep=small] & p \arrow[dl] \arrow[dr] & \\ u & & v \end{tikzcd} \in R \quad \text{implies} \quad \begin{tikzcd}[column sep=small] & p' \arrow[d] & \\ & p \arrow[dl] \arrow[dr] & \\ u & & v \end{tikzcd} \in R$$

for any arrow $p' \to p$ with codomain p.

Of course any span

$$\begin{tikzcd} & p \arrow[dl] \arrow[dr] & \\ u & & v \end{tikzcd}$$

generates a *principal crible* denoted

$$\left\langle \begin{tikzcd} & p \arrow[dl] \arrow[dr] & \\ u & & v \end{tikzcd} \right\rangle.$$

Given a crible R from u to v, the *reverse* crible R^* from v to u is defined by

$$R^* = \{(\alpha, \beta);\ (\beta, \alpha) \in R\}.$$

We define a *closure operation* on the poset Cribles (u, v) (order is inclusion) as follows:

$$\bar{R} = \left\{ \begin{tikzcd} & p \arrow[dl] \arrow[dr] & \\ u & & v \end{tikzcd} ;\ \exists \text{ a cover } (p_i \to p)_i \text{ such that } \begin{tikzcd} & p_i \arrow[d] & \\ & p \arrow[dl] \arrow[dr] & \\ u & & v \end{tikzcd} \in R \text{ for all } i \right\}.$$

Call a closed crible a *relation*.

Examples. If C is a locale, or a Grothendieck topos (or even a lex-total category [8, 7]) and P is the canonical topology, then the relations are all principal – that is, generated as cribles by a single span. In fact, a relation R from u to v is generated by the union of those subobjects of $u \times v$ which are in R. Further a crible generated by a subobject of $u \times v$ is closed. Hence there is a bijection between relations from u to v and subobjects of $u \times v$.

Composition of relations. Given cribles R from u to v and S from v to w we define

$$S \circ R = \{(\alpha, \beta);\ \exists \gamma \text{ such that } (\alpha, \gamma) \in R,\ (\gamma, \beta) \in S\}.$$

It is straightforward to check the following properties:

(i) $T \circ (S \circ R) = (T \circ S) \circ R$.

(ii) $\langle (1_v, 1_v) \rangle \circ R = R = R \circ \langle (1_u, 1_u) \rangle$.

(iii) If $S \subseteq S'$ and $R \subseteq R'$ then

$$S \circ R \subseteq S' \circ R \quad \text{and} \quad S \circ R \subseteq S \circ R'.$$

(iv) $(\bigcup_i S_i) \circ R = \bigcup_i (S_i \circ R)$, $S \circ \bigcup_i R_i = \bigcup_i (S \circ R_i)$

(v) $(S \circ R)^* = R^* \circ S^*$.

(vi) $\overline{S \circ R} = \overline{\bar{S} \circ \bar{R}}$.

Definition of $B = \mathbf{Rel}(C, P)$. The objects of B are the objects of C. The arrows from u to v are the relations. The 2-cells are inclusions.

Composition of relations R from u to v, S from v to w is given by

$$S \cdot R = \overline{S \circ R}.$$

The identity of u is $\overline{\langle (1_u, 1_u) \rangle}$.

Using the properties listed above, it is easy to prove that B is a bicategory, locally a cocomplete poset (with $\bigvee R_i = \overline{\bigcup R_i}$), and that composition (on either side) preserves suprema. Further $(\ )^* : B \to B$ is an involution which is the identity on objects.

2. *B*-categories and modules

I recall here briefly some of the theory of categories based on a bicategory B [1,2,9]. I assume that B satisfies the conditions of the last paragraph.

A *B-category* X is a set X with functions $e: X \to \operatorname{obj} B$ and $d: X \times X \to \operatorname{morph} B$ satisfying:

(i) $d(x_1, x_2): e(x_1) \to e(x_2)$,

(ii) $1_{e(x)} \leq d(x, x)$,

(iii) $d(x_2, x_3) \cdot d(x_1, x_2) \leq d(x_1, x_3)$.

We say that X is *small* if $e^{-1}(u)$ is small for all $u \in B$.

A B-functor, $f: X \to Y$, is a function $f: X \to Y$ satisfying

(i) $e(x) = e(f(x))$,

(ii) $d(x_1, x_2) \leq d(fx_1, fx_2)$.

A *B-transformation* exists from f to $g: X \to Y$ (and we write $f \leq g$) if $1_{e(x)} \leq d(fx, gx)$ all $x \in X$.

So B-categories form a bicategory B-**Cat**, and it makes sense to talk about B-equivalent B-categories. A B-category X is called *skeletal* if, whenever $e(x) = e(x') = u$ and $1_u \leq d(x, x')$, $1_u \leq d(x', x)$, then $x = x'$. As usual we have that every B-category is B-equivalent to a skeletal B-category.

A B-category X is *symmetric* if

$$d(x_1, x_2) = (d(x_2, x_1))^* \quad \text{all } x_1, x_2 \text{ in } X.$$

To each object u of B there is a special B-category denoted $\hat{u}$ defined by $\hat{u} = \{*\}$, $e(*) = u$, $d(*, *) = 1_u$.

Now to describe Cauchy-completeness I need to define the notion 'adjoint pair of modules' (see [6]) between B-categories. For brevity I will look only at the special case when one of the B-categories is of the form $\hat{u}$. An *adjoint pair of modules from* $\hat{u}$ *to* X is a pair of functions, $\phi, \psi: X \to \operatorname{morph} B$, satisfying

AM(i) $\phi(x): u \to e(x)$, $\psi(x): e(x) \to u$.

AM(ii) $d(x, x') \cdot \phi(x) \leq \phi(x')$, $\psi(x) \cdot d(x', x) \leq \psi(x')$.

AM(iii) $1_u \leq \bigvee_x \psi(x) \cdot \phi(x)$.

AM(iv) $\phi(x') \cdot \psi(x) \leq d(x, x')$.

A B-category X is *Cauchy-complete* if for each $u \in B$ and each adjoint pair of bimodules ϕ, ψ from $\hat{u}$ to X there is an element $x \in X$ such that $e(x) = u$ and

$$\phi(y) = d(x, y), \quad \psi(y) = d(y, x) \qquad (\text{all } y \in X).$$

3. Sheaves

We define a functor

$$L : \mathbf{Shv}(C, P) \to B\text{-}\mathbf{Cat} \quad (B = \mathbf{Rel}(C, P))$$

as follows: If F is a sheaf then

$$L(F) = \coprod_{u \in C} F(u).$$

Further, if $s \in F(u)$, $t \in F(v)$ then

$$e(s) = u, \qquad d(s,t) = \left\{ \begin{tikzcd}[column sep=small] & p \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ u & & v \end{tikzcd} ; F\alpha(s) = F\beta(t) \right\}.$$

It is straightforward to show that $d(s,t)$ is a relation, and in fact that $L(F)$ is a B-category. The extension of L to morphisms is immediate and it is easy to see that L is a fully-faithful functor. Notice that $d(t,s) = (d(s,t))^*$; and that if s and t have $e(s) = e(t) = u$, $1_u \le d(s,t)$ then $s = t$. Hence L lands on symmetric skeletal B-categories.

Proposition 1. *$L(F)$ is Cauchy-complete.*

Proof. Consider an adjoint pair of modules ϕ, ψ from $\hat{u}$ to $L(F)$. Condition AM(iii) says that

$$(1_u, 1_u) \in \overline{\bigcup_{s \in L(F)} \psi(s) \circ \phi(s)}.$$

Hence there is a cover

$$(u_i \xrightarrow{\alpha_i} u)_{i \in I}$$

such that for each $i \in I$ there is an $s_i \in L(F)$ with

$$(\alpha_i, \alpha_i) \in \psi(s_i) \circ \phi(s_i).$$

This means that for these s_i there are arrows λ_i such that $(\alpha_i, \lambda_i) \in \phi(s_i)$ and $(\lambda_i, \alpha_i) \in \psi(s_i)$.

Make a choice of this data: the cover, the s_i and the λ_i. Define sections σ_i over u_i by

$$\sigma_i = F(\lambda_i)(s_i).$$

I claim that these σ_i are a compatible family of sections. First note that σ_i does not depend on the choice of λ_i. Suppose μ_i is another arrow with $(\alpha_i, \mu_i) \in \phi(s_i)$, $(\mu_i, \alpha_i) \in \psi(s_i)$. Then

$$(\lambda_i, \mu_i) \in \phi(s_i) \circ \psi(s_i) \le d(s_i, s_i) \quad \text{(by AM(iv))}.$$

Hence $F\lambda_i(s_i) = F\mu_i(s_i)$. Now to prove the compatibility of the σ_i consider the diagram

$$\begin{tikzcd} & & u_{ij} \arrow[dl, "\beta_i"'] \arrow[dr, "\beta_j"] & & \\ & u_i \arrow[dl, "\lambda_i"'] \arrow[dr, "\alpha_i"] & & u_j \arrow[dl, "\alpha_j"'] \arrow[dr, "\lambda_j"] & \\ e(s_i) & & u & & e(s_j) \end{tikzcd}$$

where the diamond commutes. We need to show that $F\beta_i(\sigma_i) = F\beta_j(\sigma_j)$. But clearly

$$(\lambda_i\beta_i, \lambda_j\beta_j) \in \phi(s_j) \circ \psi(s_i) \le d(s_i, s_j) \quad \text{(by AM(iv))}.$$

Hence

$$F\beta_i(\sigma_i) = F(\lambda_i\beta_i)(s_i) = F(\lambda_j\beta_j)(s_j) = F\beta_j(\sigma_j).$$

Since F is a sheaf we get the existence of a section σ over u such that $F\alpha_i(\sigma) = \sigma_i$.

I will now prove that

$$\phi(s) = d(\sigma, s) \quad \text{and} \quad \psi(s) = d(s, \sigma).$$

The first step is to show that

$$d(\sigma, s) = \overline{\bigcup_{i \in I} d(\sigma_i, s) \circ \langle(\alpha_i, 1_{u_i})\rangle}.$$

The inclusion of the right-hand-side in the left is easy. To see the opposite inclusion consider the diagram

$$\begin{tikzcd} & u_i \times_u w \arrow[dl] \arrow[dr] & & \\ u_i \arrow[dr] & & w \arrow[dl] \arrow[dr] & \\ & u & & v \end{tikzcd} \qquad \text{where} \qquad \begin{tikzcd} & w \arrow[dl] \arrow[dr] & \\ u & & v \end{tikzcd} \in d(\sigma, s).$$

Clearly,

$$\begin{tikzcd} & u_i \times_u w \arrow[dl] \arrow[dr] & & \\ u_i & & w \arrow[dr] & \\ & & & w \end{tikzcd}$$

is in $d(\sigma_i, s)$ and hence

$$\begin{tikzcd} u_i \times_u w \arrow[dr] & & \\ & w \arrow[dl] \arrow[dr] & \\ u & & v \end{tikzcd}$$

belongs to $d(\sigma_i,s)\circ\langle(\alpha_i,1_u)\rangle$. But $(u_i\times_u w\to w)_i$ is a cover and so

$$\begin{tikzcd} & w \arrow[dl] \arrow[dr] & \\ u & & v \end{tikzcd}$$

is in the right-hand-side.

Next, notice that $\langle(\alpha_i,1_{u_i})\rangle\le\phi(\sigma_i)$ (by AM(ii)) and so

$$d(\sigma,s)\le\overline{\bigcup d(\sigma_i,s)\cdot\phi(\sigma_i)}\le\phi(s)\quad\text{(by AM(ii))}.$$

Similarly

$$d(s,\sigma)\le\psi(s).$$

Finally, notice that

$$\begin{aligned}\phi(s)&=\phi(s)\circ\langle(1_u,1_u)\rangle\\ &\le\phi(s)\circ\overline{\bigcup_i(d(\sigma_i,\sigma)\circ\langle(\alpha_i,1_{u_i})\rangle)}\\ &\le\overline{\bigcup_i\phi(s)\circ d(\sigma_i,\sigma)\circ\langle(\alpha_i,1_{u_i})\rangle}\\ &\le\overline{\bigcup_i\phi(s)\circ\psi(\sigma_i)\circ\langle(\alpha_i,1_{u_i})\rangle}\quad\text{(just proved)}\\ &\le\overline{\bigcup_i d(\sigma_i,s)\circ\langle(\alpha_i,1_{u_i})\rangle}\quad\text{(AM(iv))}\\ &=d(\sigma,s).\qquad\Box\end{aligned}$$

Proposition 2. *If X is a Cauchy-complete, skeletal, symmetric B-category then X is $L(F)$ for some sheaf F.*

Proof. It is clear how to define F on objects:

$$Fu=e^{-1}(u).$$

Given an arrow $\gamma:u\to v$ in C, and an element $x\in X$ over v we want to define $F\gamma(x)$ over u. To construct this element we consider the adjoint pair of bimodules from $\hat{u}$ to X

$$\phi(y)=\{(\alpha,\beta);\ (\gamma\alpha,\beta)\in d(x,y)\},\qquad\psi(y)=\phi^*(y).$$

Then $F\gamma(x)$ is the element guaranteed by the Cauchy-completeness of X. Since X is skeletal $F\gamma(x)$ is uniquely determined by

$$d(F\gamma(x),y)=\phi(y)\quad\text{all } y\in X.$$

Functoriality of F follows easily.

Notice that if $\delta:u\to w$ and x' lies over w then $F\gamma(x)=F\delta(x')$ iff $(\gamma,\delta)\in d(x,x')$; so $L(F)=X$.

Now let's check that F is a sheaf. Suppose $(x_i)_{i\in I}$ is a compatible family of sections (x_i lying over u_i), where

$$(u_i \xrightarrow{\alpha_i} u)_{i\in I}$$

is a cover. To produce a section over u consider the adjoint pair of modules from $\hat{u}$ to X:

$$\phi(y) = \overline{\bigcup d(x_i, y) \circ \langle(\alpha_i, 1_{u_i})\rangle}, \qquad \psi(y) = \phi^*(y).$$

It is not hard to check that the resulting section, x say, is unique with the property that $F\alpha_i(x) = x_i$ ($i \in I$). □

As a consequence of these two propositions we have:

Theorem. **Shv**(C, P) *is biequivalent to the bicategory of symmetric Cauchy-complete* **Rel**(C, P)*-categories.*

Proof. The functor L defined earlier is in fact a homorphism of bicategories. The fully-faithfulness of L, together with the scarcity of 2-cells involving skeletal B-categories, imply that L is always an equivalence of hom-categories. Since every B-category is B equivalent to a skeletal B-category, Propositions 1 and 2 imply that the B-categories of the form $L(F)$ are exactly those which are B-equivalent to symmetric Cauchy-complete B-categories. □

Corollary. *If E is a Grothendieck topos (or even a lex-total category) then E is bi-equivalent to the bicategory of small symmetric Cauchy-complete* **Rel** *E-categories.*

Proof. Complete the identification of **Rel**(E, P) (P the canonical topology), begun in Section 1, as the usual bicategory of relations **Rel** E. Then $E \simeq$ **Shv**(E, P) (see [7] for the lex-total case). □

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References

[1] J. Bénabou, Introduction to Bicategories, Lecture Notes in Mathematics No. 47 (Springer, Berlin–New York, 1967).

[2] R. Betti and R.F.C. Walters, Categories based on a bicategory, in preparation.

[3] M.P. Fourman and D.S. Scott, Sheaves and Logic, Lecture Notes in Mathematics No. 753 (Springer, Berlin–New York, 1979).

[4] Denis Higgs, A category approach to boolean-valued set theory, Lecture Notes, University of Waterloo (1973).

[5] P.T. Johnstone, Topos Theory (Academic Press, New York, 1977).

[6] F.W. Lawvere, Metric Spaces, generalized logic, and closed categories, Rend. Semi. Mat. Fis. Milano 43 (1974) 135–166.

[7] R.H. Street, Notions of topos, Bull. Austral. Math. Soc. 23 (1981) 199–208.

[8] R.H. Street and R.F.C. Walters, Yoneda Structures on 2-categories, J. Algebra 50 (1978) 350–379.

[9] R.F.C. Walters, Sheaves and Cauchy-complete categories, Cahiers Top. et Géom. Diff. 22 (1981) 283–286.