AN ALTERNATIVE APPROACH TO UNIVERSAL ALGEBRA

by

R. F. C. Walters
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The method of triples for defining ranked varieties of algebras over Sets (see [4] pp. 20-21) uses information about all free algebras. With the following related construction we need information about only two of the free algebras in defining the variety.

1. Definition of the construction.

Let $\underline{A}$ be a category. A device D over $\underline{A}$ consists of three things: X, η and E. X is a subclass of the objects of $\underline{A}$; η assigns to each $x \in X$ a morphism $\eta_x : x \to Tx$ of $\underline{A}$ (and Tx will always denote the codomain of η_x); $E = \{E_{x,y};\ x, y \in X\}$ is a family of sets of morphisms where all the morphisms in $E_{x,y}$ have domain Tx and codomain Ty. We require the following additional properties:

$$(1) \qquad E_{y,z}E_{x,y} \subseteq E_{x,z} \qquad (x, y, z \in X),$$

$$(2) \qquad 1_{Tx} \in E_{x,x} \qquad (x \in X),$$

and (3) to each $\mu : x \to Ty$ $(x, y \in X)$ there is a unique $\varepsilon \in E_{x,y}$ such that $\mu = \varepsilon\eta_x$. (For our theorem we shall be interested in devices over Sets with X of the form $\{x, \emptyset\}$.) We next define a category $\underline{A}^D$ of "D-objects and D-morphisms".

D-objects are pairs (a, F) where a is an object of $\underline{A}$ and $F = \{F_x;\ x \in X\}$ is a family of sets of morphisms of $\underline{A}$ where all morphisms in F_x have domain Tx and codomain a. We require further that

$$(1)\quad F_y E_{x,y} \subseteq F_x \quad (x,\ y \in X),$$

and (2) for each $x \in X$ and each $\mu : x \rightarrow a$ there is a unique $\varphi \in F_x$ such that $\mu = \varphi\eta_x$. A D-morphism from (a, F) to (b, G) is a morphism α: a $\rightarrow$ b such that $\alpha F_x \subseteq G_x$ $(x \in X)$. D-objects and D-morphisms make up the category $\underline{A}^D$ and there is an obvious forgetful functor $U^D: \underline{A}^D \rightarrow \underline{\text{Sets}}$ defined by:

$$U^D: (a,\ F) \rightsquigarrow a$$

and

$$U^D: (\alpha : (a,\ F) \rightarrow (b,\ G)) \rightsquigarrow (\alpha : a \rightarrow b).$$

Every triple (T, η, μ) yields a device with X the objects of A, $\eta_x: x \rightarrow Tx$ the value of the natural transformation η at x, and $E_{x,y} = \{\mu(y)T(\alpha);\ \alpha : x \rightarrow Ty\}$. Further, all devices with X equal to the objects of $\underline{A}$ can be obtained from triples in this way and the construction of $\underline{A}^D$ is equivalent to the Eilenberg-Moore construction ([2]).

2. <u>The classical definition of variety</u> (for details see [1] or [5]).

To define algebras we need a set Ω of "formal operations" with a set $n(\omega)$ assigned to each $\omega \in \Omega$ called the "arity" of ω.

(In [1] $n(\omega)$ is always a finite ordinal and in [5] an ordinal.) Then an Ω-algebra $\underline{a}$ is a set a and to each $\omega \in \Omega$ an operation $\omega_{\underline{a}}: a^{n(\omega)} \to a$. If $\alpha: n(\omega) \to a$ we denote the image of α under $\omega_{\underline{a}}$ by $\omega_{\underline{a}}[\alpha]$. A homomorphism from $\underline{a}$ to $\underline{b}$ is a map $\lambda: a \to b$ such that for all $\omega \in \Omega$ and all $\alpha: n(\omega) \to a$ we have

$$\lambda \omega_{\underline{a}}[\alpha] = \omega_{\underline{b}}[\lambda \alpha].$$

All Ω-algebras and all homomorphisms constitute a category $\underline{\Omega\text{-Alg}}$. There is a clear forgetful functor $U_\Omega: \underline{\Omega\text{-Alg}} \to \underline{\text{Sets}}$ which has a left adjoint W_Ω.

Now an Ω-law in variables x is a pair of elements of $U_\Omega W_\Omega x$. An Ω-algebra $\underline{a}$ satisfies the law (t_1, t_2) if $\alpha t_1 = \alpha t_2$ for every homomorphism $\alpha: W_\Omega x \to \underline{a}$. Given L a set of Ω-laws in variables x, $\underline{V} = \underline{\text{Var (L)}}$ is the category of all Ω-algebras satisfying these laws (together with all homomorphisms). Again the natural forgetful functor $U_V: \underline{V} \to \underline{\text{Sets}}$ has a left adjoint. (These categories of algebras $\underline{\text{Var (L)}}$ are called varieties.) Finally, a varietal functor U is a functor from some category $\underline{A}$ to $\underline{\text{Sets}}$ such that there exists a category $\underline{V} = \underline{\text{Var (L)}}$ and an isomorphism $K: \underline{V} \to \underline{A}$ such that $UK = U_V$.

I am assuming that it is these varietal functors one studies in universal algebra rather than the particular way of constructing them. Certainly one can retrieve from a functor the "theory" of the corresponding variety (see [3]). However, we do not distinguish between two varieties with the same theory.

3. Theorem.

If $D = (X, \eta, E)$ is a device over Sets with $X = \{x, \emptyset\}$, then U^D: $\underline{\text{Sets}}^D \longrightarrow$ Sets is a varietal functor. Further, to each varietal functor U: $\underline{A} \longrightarrow$ Sets there is a device D of this sort and an isomorphism K: $\underline{A} \longrightarrow \underline{\text{Sets}}^D$ such that $U^D K = U$.

Proof.

(i) Given $D = (X, \eta, E)$ with $X = \{x, \emptyset\}$;

to construct the required variety we need to select an operation set Ω, an arity function n, a morphism $\xi: \Omega \longrightarrow Tx$ and to each $\omega \in \Omega$ an injection $\iota_\omega : n(\omega) \longrightarrow x$. The following is a suitable selection: $\Omega = Tx$; $n(\omega) = \emptyset$ if $\omega \in$ image $(\varepsilon_{\emptyset,x})$ ($\varepsilon_{\emptyset,x}$ is the single morphism in $E_{\emptyset,x}$) and $n(\omega) = x$ for all other $\omega \in \Omega$; $\xi = 1_{Tx}$; $\iota_\omega = \emptyset \longrightarrow x$ if $n(\omega) = \emptyset$ and $\iota_\omega = 1_x$ if $n(\omega) = x$. More generally, any choice which satisfies the following properties will suffice:

(1) $n(\omega) = \emptyset$ only if $\xi\omega \in$ image $(\varepsilon_{\emptyset,x})$,

(2) $n(\omega) = y \neq \emptyset$ implies that for any α_1, α_2: $x \longrightarrow Tx$

$$\alpha_1 \iota_\omega = \alpha_2 \iota_\omega \Rightarrow \varepsilon_1 \xi \omega = \varepsilon_2 \xi \omega$$

where $\alpha_i = \varepsilon_i \eta_x \quad (\varepsilon_i \in E_{x,x};\ i = 1, 2)$,

and (3) if $\underline{Tx} = \text{alg}\ (Tx, \{E_{x,x}, E_{\emptyset,x}\})$ and

$\underline{T\emptyset} = \text{alg}\ (T\emptyset, \{E_{x,\emptyset}, E_{\emptyset,\emptyset}\})$ are defined as in the next paragraph, (the images of) η_x: $x \longrightarrow Tx$ and $\eta_\emptyset$: $\emptyset \longrightarrow T\emptyset$ generate (in the algebraic sense) $\underline{Tx}$ and $\underline{T\emptyset}$ respectively.

Now given a selection of Ω and n and associated morphisms ξ and ι_ω ($\omega \in \Omega$) satisfying (1) and (2) above, we define to each (a, F) $\in$ $\underline{\text{Sets}}^D$ an Ω-algebra alg(a, F) as follows. alg(a, F) has underlying set a and if $n(\omega) = y$ and α: $y \longrightarrow a$ then $\alpha = \beta\,\iota_\omega$ for some β and $\beta = \varphi\eta_x(\varphi \in F_x)$. (A suitable morphism β may not exist when $n(\omega) = \emptyset$ and $a = \emptyset$ but this situation cannot occur since the existence of a nullary operation ω implies that $T\emptyset \neq \emptyset$ and hence, since $F_\emptyset$ is non-empty, that $a \neq \emptyset$.) We put $\omega_{\underline{a}}[\alpha] = \varphi\,\xi\,\omega$. We have to check that $\omega_{\underline{a}}[\alpha]$ does not depend on the particular β chosen; that is, that $\beta_1\iota_\omega = \beta_2\iota_\omega$ implies that $\varphi_1\xi\omega = \varphi_2\xi\omega$ where $\varphi_i\eta_x = \beta_i$ (i = 1, 2) ($\varphi_i \in F_x$). When $n(\omega) = \emptyset$ we use the fact that $\varphi_i \cdot \varepsilon_{\emptyset,x} = \varphi_\emptyset$ ($\varepsilon_{\emptyset,x} \in E_{\emptyset,x}$; $\varphi_\emptyset \in F_\emptyset$). When $n(\omega) = y \neq \emptyset$ there certainly exists a map β : $x \longrightarrow a$ and a map γ: $x \longrightarrow x$ such that $\beta = \beta_i\gamma$ (i = 1, 2) and $\gamma\,\iota_\omega = \iota_\omega$. Let $\eta_x\gamma = \varepsilon\eta_x$ ($\varepsilon \in E_{x,x}$) and $\beta = \varphi\eta_x$ ($\varphi \in F_x$). Then $(\varphi_i\varepsilon)\eta_x = \varphi_i\eta_x\gamma = \beta$ and hence $\varphi_i\varepsilon = \varphi$. Now since $\eta_x\gamma\,\iota_\omega = \eta_x\iota_\omega$ we have $\varepsilon\,\xi\,\omega = \xi\,\omega$. (This is restriction (2) on the selection of Ω and n.) Hence $\varphi_1\xi\omega = \varphi_1\varepsilon\,\xi\,\omega = \varphi\,\xi\,\omega$ $= \varphi_2\varepsilon\,\xi\,\omega = \varphi_2\xi\omega$, which is what we wished to prove. Thus alg(a, F) is a well defined Ω-algebra.

(ii) Next we wish to show that λ: $a \longrightarrow b$ is a homomorphism from $\underline{a}$ = alg (a, F) to $\underline{b}$ = alg (b, G) if and only if it is a D-morphism from (a, F) to (b, G).

Suppose λ is a D-morphism, then $\lambda F_x \subseteq G_x$. For operations ω of arity y consider α: y $\longrightarrow$ a. We need that

$$\lambda \omega_{\underline{a}}[\alpha] = \omega_{\underline{b}}[\lambda\alpha].$$

Let $\alpha = \varphi\eta_x \iota_\omega (\varphi \in F_x)$. Then the left hand side is equal to $\lambda\varphi\xi\omega$. Further $\lambda\alpha = \lambda\varphi\eta_x\iota_\omega$ and since $\lambda\varphi \in G_x$ the right hand side is also $\lambda\varphi\xi\omega$.

Conversely let λ be a homomorphism from $\underline{a}$ to $\underline{b}$. Consider $\lambda\varphi$ where $\varphi \in F_x$. There exists a $\gamma \in G_x$ such that $\gamma\eta_x = \lambda\varphi\eta_x$. Now $\lambda\varphi$ and γ are both homomorphisms from $\underline{Tx}$ to $\underline{b}$ and they agree on the generators so they are equal. That is, $\lambda F_x \subseteq G_x$. It is clear that $\lambda F_\emptyset = G_\emptyset$ since for any $\varphi \in F_x$, $\varphi\varepsilon_{\emptyset,x}$ is the only map in $F_\emptyset$ and $\lambda\varphi\varepsilon_{\emptyset,x}$ is the only map in $G_\emptyset$.

(iii) alg(a, F) = alg(b, G) $\Rightarrow$ (a, F) = (b, G).

Clearly the left hand side implies that a = b. Suppose that the right hand side is nevertheless false. Then either $F_\emptyset \neq G_\emptyset$ or $F_x \neq G_x$. In the first case it follows that $\varphi\varepsilon_{\emptyset,x} \neq \gamma\varepsilon_{\emptyset,x}$ and hence that $\varphi \neq \gamma$ for any $\varphi \in F_x$, $\gamma \in G_x$. Hence we may assume that $F_x \neq G_x$. Then there exists $\varphi \in F_x$ and $\gamma \in G_x$ such that $\varphi\eta_x = \gamma\eta_x$ but $\varphi \neq \gamma$. This however cannot occur since φ and γ are both homomorphisms from $\underline{Tx}$ to $\underline{a}$ (where $\underline{a}$ = alg(a, F) = alg(a, G)) and they agree on the generators so they are equal.

(iv) We next wish to identify the algebras alg(a, F) as belonging to a certain variety. Let $\underline{Wx}$ be the free Ω-algebra with underlying set Wx, freely generated by $x \xrightarrow{\tau} Wx$. Then there is a unique homomorphism ν from $\underline{Wx}$ to $\underline{Tx}$ such that $\nu\tau = \eta_x$. We define a set of laws L as follows:

$$L = \{(t_1, t_2);\ t_1, t_2 \in Wx \text{ and } \nu t_1 = \nu t_2\}.$$

All algebras $\underline{a}$ = alg(a, F) satisfy these laws. Clearly this would follow if $F_x\nu$ were all homomorphisms from $\underline{Wx}$ to $\underline{a}$. But to each $\mu : x \longrightarrow a$ there is a homomorphism $\varphi\nu$ from $\underline{Wx}$ to $\underline{a}$ belonging to $F_x\nu$ such that $\varphi\nu\tau = \mu$. Any homomorphism from $\underline{Wx}$ to $\underline{a}$ must agree with one of these on the generators of $\underline{Wx}$ and hence must be one of them.

(v) Any algebra $\underline{a}$ of $\underline{\text{Var}(L)}$ is of the form alg(a, F) for some $(a, F) \in \underline{\text{Sets}}^D$. Take F_x to be all homomorphisms from $\underline{Tx}$ to $\underline{a}$ and $F_\emptyset$ to be all homomorphisms from $\underline{T\emptyset}$ to $\underline{a}$. Property (1) for D-objects is then clearly true for (a, F). To check (2) consider any $\mu : x \longrightarrow a$. There exists a homomorphism $\lambda : \underline{Wx} \longrightarrow \underline{a}$ such that $\lambda\tau = \mu$. Now ν is an epimorphism since $\eta_x : x \longrightarrow Tx$ generates $\underline{Tx}$. Further whenever $\nu t_1 = \nu t_2$ $(t_1, t_2 \in Wx)$ then $(t_1, t_2) \in L$ so that $\lambda t_1 = \lambda t_2$. Under these conditions there exists a homomorphism $\kappa : \underline{Tx} \longrightarrow \underline{a}$ such that $\kappa\nu = \lambda$ and hence $\kappa\eta_x = \kappa\nu\tau = \lambda\tau = \mu$. Since η_x generates $\underline{Tx}$ there is at most one such homomorphism. To check (2) we also have to show that any $\emptyset \longrightarrow a$ is of the form $\varphi_\emptyset\eta_\emptyset$ for a unique $\varphi_\emptyset \in F_\emptyset$. This amounts to showing that $F_\emptyset$ contains precisely one element. It contains at most one since $\emptyset \longrightarrow T\emptyset$ generates $\underline{T\emptyset}$. If $T\emptyset = \emptyset$

there is the empty homomorphism from $\underline{T\emptyset}$ to $\underline{a}$. If $T\emptyset \neq \emptyset$ then there are nullary operations so that $\underline{a}$ cannot be the empty algebra and hence F_x is non-empty. Then $\varphi\,\varepsilon_{\emptyset,x} \in F_\emptyset$ for any $\varphi \in F_x$.

It remains to be shown that $\underline{\underline{a}} = \underline{a}$ where $\underline{\underline{a}} = \mathrm{alg}(a, F)$. Consider $\omega \in \Omega$ and $\alpha: n(\omega) \to a$. Let $\alpha = \varphi\eta_x\iota_\omega \,(\varphi \in F_x)$. Then $\omega_{\underline{\underline{a}}}[\alpha] = \varphi\xi\omega = \varphi\omega_{\underline{Tx}}[\eta_x\iota_\omega] = \omega_{\underline{a}}[\varphi\eta_x\iota_\omega] = \omega_{\underline{a}}[\alpha]$.

Thus we have shown that alg: $\underline{\text{Sets}}^D \to \underline{\text{Var(L)}}$ defined by:

$$\mathrm{alg}: (a, F) \rightsquigarrow \mathrm{alg}(a, F)$$

and $$\mathrm{alg}: (\alpha : (a, F) \to (b, G)) \rightsquigarrow (\alpha : \mathrm{alg}(a, F) \to \mathrm{alg}(b, G))$$

is an isomorphism and it is clear that $U_V\,\mathrm{alg} = U^D$. Hence U^D is a varietal functor.

(vi) We shall now discuss the second part of the theorem. Let $\underline{V}$ be any variety with operation set Ω and arity function n. Let $\underline{Wx}$ be the free Ω-algebra freely generated by $\uparrow : x \to Wx$ and let $\underline{Tu}$ be the $\underline{V}$-free algebra freely generated by $\eta_u: u \to Tu$. Let ν be the homomorphism from $\underline{Wx}$ to $\underline{Tx}$ such that $\nu\uparrow = \eta_x$. Then it is a fact of universal algebra that for all sufficiently large x, $\underline{V} = \underline{\text{Var(L)}}$ where

$$L = \{(t_1, t_2);\ t_1, t_2 \in Wx \text{ and } \nu t_1 = \nu t_2\}.$$

Take such an x with $|x| > |n(\omega)|$ for all $\omega \in \Omega$. Then consider the device with $X = \{x, \emptyset\}$, η_x and $\eta_\emptyset$ as above, and $E_{u,v}$ all homomorphisms from $\underline{Tu}$ to $\underline{Tv}$ $(u, v \in X)$. Certainly $D = (X, \eta, E)$

is a device. We wish to consider the variety $\underline{V}'$ obtained from this device by the method given in the earlier parts of this theorem. Now Ω and n form a suitable operation set and arity function for $\underline{V}'$, if ι_ω is taken to be any injection from $n(\omega)$ to x and ξ is defined by:

$$\xi : \omega \rightsquigarrow \omega_{\underline{Tx}}[\eta_x \iota_\omega].$$

Now for any $\alpha : n(\omega) \longrightarrow Tx$ let $\alpha = \varepsilon \eta_x \iota_\omega$ $(\varepsilon \in E_{x,x})$. Then if $\underline{\underline{Tx}} = \mathrm{alg}(Tx, \{E_{x,x}, E_{\emptyset,x}\})$ we see that

$$\omega_{\underline{\underline{Tx}}}[\alpha] = \varepsilon \xi \omega = \varepsilon \omega_{\underline{Tx}}[\eta_x \iota_\omega] = \omega_{\underline{Tx}}[\alpha].$$

This means that $\underline{\underline{Tx}} = \underline{Tx}$. Now $\underline{V}'$ and $\underline{V}$ have the same operation set and arity function. Further the laws defining $\underline{V}'$ are obtained from η_x and $\underline{\underline{Tx}}$ in precisely the same way that the laws L of $\underline{V}$ are obtained from η_x and $\underline{Tx}$. Hence $\underline{V}' = \underline{V}$ and $U_{V'} = U_V$. So there exists an isomorphism $K: \underline{\mathrm{Sets}}^D \longrightarrow \underline{V}$ such that $U_V K = U^D$, and this is what we were required to prove.

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Australian National University,

Canberra.