The compact closed bicategory of left adjoints

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Abstract

We show that, for any braided compact closed bicategory \mathcal{B} , the bicategory Ladj(\mathcal{B}) of left adjoints in \mathcal{B} also admits a braided compact closed structure.

1. Introduction

Over the last few years monoidal bicategories have been studied in connection with physics and geometry ([1, 2, 4, 7, 8]) and of particular interest to these applications have been the notions of a braiding for a tensor and a dual for an object. In this paper, we prove a theorem which demonstrates how an old construction (the bicategory of squares [5]) can be used to define braided compact closed bicategories which arise in the study of topological quantum field theories.

Specifically, the main theorem of this paper states the following. Suppose $\mathbf{Sq}(\mathcal{B})$ is the monoidal bicategory of squares in a monoidal bicategory \mathcal{B} . Then an object $f: A \to B$ of $\mathbf{Sq}(\mathcal{B})$ has a right bidual if and only if (i) both A and B have right biduals in \mathcal{B} and (ii) the arrow f has a right adjoint in \mathcal{B} .

A compact closed bicategory is defined to be a braided monoidal bicategory in which every object has a right bidual. A consequence of the above mentioned theorem is that the full sub-bicategory $\mathbf{Ladj}(\mathcal{B})$ of $\mathbf{Sq}(\mathcal{B})$ comprising left adjoints is compact closed if \mathcal{B} is compact closed.

Instances of the construction of a bicategory $Ladj(\mathcal{B})$ of left adjoints can be found in work on the categorical structures (first identified in [11]) which arise in the study of topological quantum field theories. For example, in [2] the notion of a *cobordism* category is defined and a topological quantum field theory is taken to be a structure-preserving morphism from a topological cobordism category to an algebraic one. (In this paper we do not take into account the 'vertical' composition of squares that is considered in [2].)

A typical candidate for a topological cobordism category is **Ladj** (Cobord), where Cobord is a (higher dimensional) category of cobordisms of compact, oriented smooth manifolds; and typical candidates for algebraic cobordism categories can be obtained by iterating the construction **Ladj** on the category Hilb of finite dimensional Hilbert spaces.

The original motivation for this work arose out of a general program to use monoidal bicategories with feedback for studying concurrency. The reader is referred to [9] for an overview of this work.

Most of the technical definitions in this paper are taken from [3] and, as is done there, we first deal with Gray monoids (which are strict monoidal bicategories), using the coherence theorem of [6] to transfer our definitions and results.

2. Preliminaries

In order to facilitate calculations – in particular, the passage between algebraic and diagrammatic representations of expressions – horizontal and vertical composition will be written in diagrammatic order: if $f: A \to B$ and $f': B \to C$ are arrows then $f \cdot f' \colon A \to C$ denotes their composite (we also use the symbol \cdot to denote horizontal composition of 2-cells); and, if $\alpha: f \Rightarrow g$ and $\beta: g \Rightarrow h$ are 2-cells, then $\alpha \bullet \beta: f \Rightarrow h$ denotes their vertical composite.

2.1. Gray monoids

To begin with, we recall the definition of a Gray monoid M from [3]. It is a 2category equipped with the following data:

- (a) an object I:
- (b) for all objects A, two 2-functors $L_A, R_A: \mathcal{M} \to \mathcal{M}$ satisfying conditions $L_A(B) = R_B(A)$ (and define $A \otimes B = L_A(B)$),

$$L_I = R_I = 1_{\mathcal{M}},$$

 $L_{A\otimes B}=L_B\cdot L_A,\quad R_{A\otimes B}=R_A\cdot R_B,\quad L_A\cdot R_B=R_B\cdot L_A,$ for all objects $A,\,B$ (remember that we are writing composition in diagrammatic order); and

(c) for all arrows $f: A \to A'$, $g: B \to B'$, an invertible 2-cell

$$A \otimes B \xrightarrow{R_B(f)} A' \otimes B$$

$$\downarrow^{L_{A(g)}} \qquad \downarrow^{L_{A'}(g)} .$$

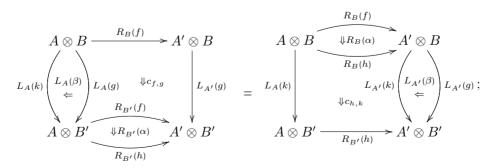
$$A \otimes B' \xrightarrow{R_{B'}(f)} A' \otimes B'$$

The structure 2-cells $c_{f,q}$ are subject to the following four axioms:

- (i) if both f and g are identity arrows then c_{f,g} is an identity 2-cell;
 (ii) for all arrows f: A → A', g: B → B' and h: C → C', there are equalities

$$L_A(c_{g,h}) = c_{L_A(g),h}, \quad c_{f,L_B(h)} = c_{R_B(f),h}, \quad R_C(c_{f,g}) = c_{f,R_C(g)};$$

(iii) for all arrows $f, h: A \to A', g, k: B \to B'$ and 2-cells $\alpha: f \Rightarrow h, \beta: g \Rightarrow k$,



(iv) for all arrows $f: A \to A'$, $g: B \to B'$, $f': A' \to A''$, $g': B' \to B''$,

$$A \otimes B \xrightarrow{R_B(f)} A' \otimes B \xrightarrow{R_B(f')} A'' \otimes B$$

$$\downarrow_{L_A(g)} \downarrow \qquad \downarrow_{c_{f,g}} \qquad \downarrow_{L_{A'}(g)} \qquad \downarrow_{c_{f',g}} \downarrow_{L_{A''}(g)} \qquad A \otimes B \xrightarrow{R_B(f \cdot f')} A'' \otimes B$$

$$A \otimes B' \xrightarrow{R_{B'}(f)} A' \otimes B' \xrightarrow{R_{B'}(f')} A'' \otimes B' \qquad = \qquad L_A(g \cdot g') \downarrow \qquad \downarrow_{c_{f \cdot f',g \cdot g'}} \downarrow_{L_{A''}(g \cdot g')} L_{A''}(g \cdot g') \cdot A \otimes B'' \xrightarrow{R_{B''}(f)} A'' \otimes B''$$

$$A \otimes B'' \xrightarrow{R_{B''}(f)} A' \otimes B'' \xrightarrow{R_{B''}(f')} A'' \otimes B''$$

Perhaps not apparent from these axioms is the fact that if either f or g is an identity arrow then $c_{f,g}$ is an identity 2-cell. Henceforth, the symbol \mathcal{M} will denote a Gray monoid.

Following the conventions of [3], we put $L_A(g) = A \otimes g$, $L_A(\beta) = A \otimes \beta$, $R_B(f) = f \otimes B$, $R_B(\alpha) = \alpha \otimes B$, $L_A(g) \cdot R_{B'}(f) = f \otimes g$ and $L_A(\beta) \cdot R_{B'}(\alpha) = \alpha \otimes \beta$. (Note that $R_B(f) \cdot L_{A'}(g)$ could be another choice for what we denote by $f \otimes g$; but the structure 2-cell $c_{f,g}$ provides as isomorphism between these two choices.) In order to avoid the proliferation of symbols, we will sometimes suppress the symbol \otimes , for example we may write fg for $f \otimes g$; and, when the context is clear, we will omit the subscripts of the structure 2-cells $c_{f,g}$.

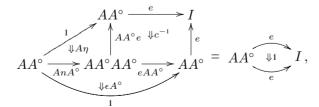
2.2. Biduals

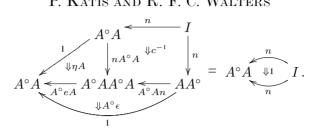
Recall that a *right bidual* for an object A of \mathcal{M} comprises an object B, an arrow $e: A \otimes B \to I$ and, for all objects C, D, adjoint pseudo-inverses to the functor

$$e^{\#}: \mathcal{M}(C, B \otimes D) \to \mathcal{M}(A \otimes C, D),$$

given by $e^{\#}(f) = (A \otimes f) \cdot (e \otimes D)$. In such a case, the object B is often referred to as a right bidual for A (and A is called a left bidual for B). As right biduals are determined up to equivalence, we write A° for B. The following proposition [10] gives an alternative definition for biduals. Its proof is relatively straightforward, but involves non-trivial manipulations of the structure 2-cells c of \mathcal{M} .

PROPOSITION 1. To give a right bidual for A is to give an object A° , two arrows $e: A \otimes A^{\circ} \to I$ and $n: I \to A^{\circ} \otimes A$ as well as two invertible 2-cells $\epsilon: (A \otimes n) \cdot (e \otimes A) \Rightarrow 1_A$ and $\eta: 1_{A^{\circ}} \Rightarrow (n \otimes A^{\circ}) \cdot (A^{\circ} \otimes e)$ such that the following two equations hold.





If A has a right bidual as above we call $e: A \otimes A^{\circ} \to I$ a pairing and $n: I \to A^{\circ} \otimes A$ a copairing for A. Also, ϵ and η are respectively called the counit and unit isomorphisms for A. Of course, when we say $e: A \otimes A^{\circ} \to I$ is a pairing for A we mean there exists a right bidual for A of the form $(A^{\circ}, e, n, \epsilon, \eta)$ (this quintuple being interpreted in the sense of the above proposition).

We end this section, by using the structure of objects with biduals to establish a correspondence which will be fundamental to the proof of the main theorem (Section 3).

First we establish some notation. Suppose that A and B are objects of \mathcal{M} with right biduals $(A^{\circ}, e_A, n_A, \epsilon_A, \eta_A)$ and $(B^{\circ}, e_B, n_B, \epsilon_B, \eta_B)$. Let $(-)^{\circ}: \mathcal{M}(B, A) \to \mathcal{M}(A^{\circ}, B^{\circ})$ be the functor mapping $g: B \to A$ to

$$g^{\circ} = (n_B A^{\circ}) \cdot (B^{\circ} g A^{\circ}) \cdot (B^{\circ} e_A) : A^{\circ} \to B^{\circ}.$$

We also define a functor $(-)^*: \mathcal{M}(A^\circ, B^\circ) \to \mathcal{M}(B, A)$ as follows: given an arrow $h: A^{\circ} \to B^{\circ}$, define the arrow $h^*: B \to A$ to be the composite $(Bn_A) \cdot (BhA) \cdot (e_BA)$. Let $\phi: (g^{\circ})^* \Rightarrow g$ be the invertible 2-cell

$$[(Bn_Bn_A)\cdot c^{-1}]\bullet [c^{-1}\cdot e_AA]\bullet [\epsilon_B\cdot g\cdot \epsilon_A].$$

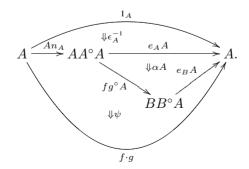
Let $\psi: (An_A) \cdot (fg^{\circ}A) \cdot (e_BA) \Rightarrow f \cdot g$ be the invertible 2-cell defined by

$$[c^{-1} \cdot (e_B A)] \bullet [f \cdot \phi].$$

Let $\theta: g \cdot f \Rightarrow (Bn_A) \cdot (Bg^{\circ}f) \cdot (e_BB)$ be the invertible 2-cell defined by

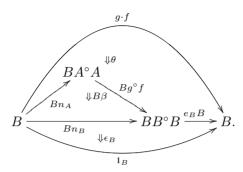
$$(\phi^{-1} \cdot f) \bullet ((Bn_A) \cdot c).$$

Given a 2-cell of the form $\alpha: e_A \Rightarrow (fg^{\circ}) \cdot e_B$, define the 2-cell $\tilde{\alpha}: 1_A \Rightarrow f \cdot g$ to be



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This construction, of course, has a dual: given a 2-cell of the form $\beta: n_A \cdot (g^{\circ} f) \Rightarrow n_B$, define the 2-cell $\vec{\beta}: g \cdot f \Rightarrow 1_B$ to be



The following proposition follows from the basic properties of biduals.

Proposition 2. The above associations $\alpha \mapsto \tilde{\alpha}$ and $\beta \mapsto \vec{\beta}$ establish bijections

$$\mathcal{M}[AA^{\circ}, I](e_A, (fg^{\circ}) \cdot e_B) \cong \mathcal{M}[A, A](1_A, f \cdot g)$$

and

$$\mathcal{M}[I, B^{\circ}B](n_A \cdot (g^{\circ}f), n_B) \cong \mathcal{M}[B, B](g \cdot f, 1_B).$$

$2 \cdot 3$. Squares in \mathcal{M}

Let **2** be the category comprising the 'free living arrow'. For any 2-category **C**, define the 2-category $\mathbf{Sq}(\mathbf{C})$ of squares in **C** to be the 2-category of 2-functors, op-lax transformations and modifications from **2** to **C**. So an object of $\mathbf{Sq}(\mathbf{C})$ is an arrow $f: A \to A'$ in **C** and an arrow from $f: A \to A'$ to $g: B \to B'$ in $\mathbf{Sq}(\mathbf{C})$ is a pair of arrows $h: A \to B$, $i: A' \to B'$ in **C** together with a 2-cell of the form

$$\begin{array}{ccc}
A & \xrightarrow{h} & B \\
f \downarrow & \downarrow \alpha & \downarrow g \\
A' & \xrightarrow{i} & B'
\end{array}$$

The arrows of Sq(C) are called squares in C. There is an obvious pair of 2-functors Dom, Cod: $Sq(C) \rightarrow C$ which are locally fully faithful.

PROPOSITION 3. If \mathcal{M} is a Gray monoid then $\mathbf{Sq}(\mathcal{M})$ admits a Gray monoid structure such that the two 2-functors Dom, $\operatorname{Cod}: \mathbf{Sq}(\mathcal{M}) \to \mathcal{M}$ preserve the Gray monoid structure on the nose.

Proof. The unit for the Gray monoid structure of $\mathbf{Sq}(\mathcal{M})$ is the object $1_I: I \to I$. Consider the following two squares.

$$\begin{array}{ccc}
A & \xrightarrow{j} & C & & B & \xrightarrow{k} & D \\
f \downarrow & \downarrow \alpha & \downarrow h & & g \downarrow & \downarrow \beta & \downarrow i \\
A' & \xrightarrow{l} & C' & & B' & \xrightarrow{m} & D'
\end{array}$$

We define $L_f, R_i: \mathbf{Sq}(\mathcal{M}) \to \mathbf{Sq}(\mathcal{M})$ on objects by $L_f(i) = R_i(f) = f \otimes i$ and on

arrows by:

$$AB \xrightarrow{Ak} AD \qquad AD \xrightarrow{jD} CD$$

$$Ag \downarrow \quad \downarrow A\beta \quad \downarrow Ai \qquad \qquad Ai \downarrow \quad \downarrow c \quad \downarrow Ci$$

$$L_f(\beta) = AB' \xrightarrow{Am} AD' \quad , \qquad R_i(\alpha) = AD' \xrightarrow{jD'} CD' \quad .$$

$$fB' \downarrow \quad \downarrow c^{-1} \quad \downarrow fD' \qquad \qquad fD' \downarrow \quad \downarrow \alpha D' \quad \downarrow hD'$$

$$A'B' \xrightarrow{A'm} A'D' \qquad \qquad A'D' \xrightarrow{ID'} C'D'$$

There is a unique way to define L_f and R_i on 2-cells so that Dom and Cod preserve this structure. Similarly there is a unique choice for the structure 2-cells c of $\mathbf{Sq}(\mathcal{M})$. That this data is well defined is a straightforward exercise in the calculus of the structure 2-cells of a Gray monoid. That the structure 2-cells for $\mathbf{Sq}(\mathcal{M})$ satisfy the axioms for a Gray monoid is guaranteed by the fact that Dom and Cod are locally fully faithful.

In order to avoid confusing the Gray monoid structure of \mathcal{M} with that of $\mathbf{Sq}(\mathcal{M})$, we will denote the 'tensor product' of the latter by the symbol \sharp . For example, if f is an arrow of \mathcal{M} and β is a square in \mathcal{M} (as in the proof of the above proposition), then the square $L_f(\beta)$ defined by the Gray monoid structure on $\mathbf{Sq}(\mathcal{M})$ is denoted $f\sharp\beta$.

3. The main theorem

In this section we prove the following theorem.

THEOREM 1. Let $f: A \to B$ be an arrow of a Gray monoid \mathcal{M} . The object f of $\mathbf{Sq}(\mathcal{M})$ has a right bidual if and only if (i) both A and B have right biduals in \mathcal{M} and (ii) the arrow f has a right adjoint in \mathcal{M} .

In order to facilitate the proof, we break down the notions of both an adjoint as well as a bidual into two parts.

Suppose $f: A \to B$ and $g: B \to A$ are arrows and $\lambda: 1_A \Rightarrow f \cdot g$ and $\mu: g \cdot f \Rightarrow 1_B$ are 2-cells, in a 2-category. We call the triple (g, λ, μ) a right proto-adjoint for f if $1_f = (\lambda \cdot f) \bullet (f \cdot \mu)$; and we call it a right coproto-adjoint for f if $1_g = (g \cdot \lambda) \bullet (\mu \cdot g)$. Of course, if (g, λ, μ) is both a right proto-adjoint and a right coproto-adjoint for f, then g is a right adjoint for f.

Suppose that X and Y are objects, $e: XY \to I$ and $n: I \to YX$ are arrows and $\epsilon: (Xn) \cdot (eX) \Rightarrow 1_X$ and $\eta: 1_Y \Rightarrow (nY) \cdot (Ye)$ are invertible 2-cells in a Gray monoid. We call the quadruple (Y, e, n, ϵ) a right proto-bidual for X and we call the quadruple (Y, e, n, η) a right coproto-bidual for X. Of course, the data $(Y, e, n, \epsilon, \eta)$ defines a right bidual for X if the two equations of Proposition 1 hold.

The 2-functors Dom, Cod: $\mathbf{Sq}(\mathcal{M}) \to \mathcal{M}$ preserve biduals. Explicitly, if an object $f: A \to B$ of $\mathbf{Sq}(\mathcal{M})$ has a right bidual $(h: C \to D, e, n, \epsilon, \eta)$, then $(C, \mathrm{Dom}(e), \mathrm{Dom}(n), \mathrm{Dom}(\epsilon), \mathrm{Dom}(\eta))$ is a right bidual of A in \mathcal{M} and $(D, \mathrm{Cod}(e), \mathrm{Cod}(n), \mathrm{Cod}(\epsilon), \mathrm{Cod}(\eta))$ is a right bidual of B in \mathcal{M} .

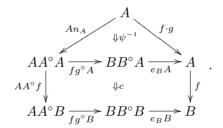
In this case, without loss of generality we may assume that the pairing e and copairing n for f will be squares of the form

where $(A^{\circ}, e_A, n_A, \epsilon_A, \eta_A)$ is a right bidual for A, $(B^{\circ}, e_B, n_B, \epsilon_B, \eta_B)$ is a right bidual for B and $g: B \to A$ is an arrow in \mathscr{M} . Notice that, in this case, we can apply the constructions $e \mapsto \tilde{e}$ and $n \mapsto \vec{n}$ of Proposition 2.

Lemma 1. Suppose A and B have right biduals A° and B° in a Gray monoid \mathcal{M} . Let $f: A \to B$ and $g: B \to A$ be any arrows. Then the following two conditions are equivalent.

- (i) In $\mathbf{Sq}(\mathcal{M})$, $(g^{\circ}, e, n, \epsilon)$ is a right proto-bidual for f such that $\mathrm{Dom}(e) = e_A$, $\mathrm{Dom}(n) = n_A$, $\mathrm{Dom}(\epsilon) = \epsilon_A$, $\mathrm{Cod}(e) = e_B$, $\mathrm{Cod}(n) = n_B$ and $\mathrm{Cod}(\epsilon) = \epsilon_B$.
- (ii) In \mathcal{M} , (g, \tilde{e}, \vec{n}) is a right proto-adjoint for f.

Proof. Let $\tau: f \cdot g \cdot f \Rightarrow (An_A) \cdot (fg^{\circ}f) \cdot (e_BB)$ be the invertible 2-cell



A straightforward calculation shows

$$A \xrightarrow{An_A} AA^{\circ}A \xrightarrow{e_A A} A$$

$$\downarrow \downarrow \tau^{-1} fg^{\circ}f \qquad \downarrow e\sharp f \qquad \downarrow f \qquad = \qquad A \xrightarrow{\downarrow \tilde{e}} A \xrightarrow{f \cdot g} B.$$

$$BB^{\circ}B \xrightarrow{e_B B} B$$

We also have

$$A \xrightarrow{An_A} AA^{\circ}A$$

$$f \downarrow \qquad \downarrow f \sharp n \qquad \downarrow f g^{\circ} f \qquad \qquad = \qquad A \xrightarrow{f} B \xrightarrow{\downarrow \vec{n}} B.$$

$$B \xrightarrow{Bn_B} BB^{\circ}B \xrightarrow{\downarrow \tau} B$$

$$\downarrow \epsilon_B \qquad \qquad \downarrow \epsilon_B$$

Combining these last two equalities, we get

$$A \xrightarrow{An_A} AA^{\circ}A \xrightarrow{e_A A} A \qquad A \xrightarrow{1_A} A$$

$$f \downarrow \qquad \downarrow f \sharp n \qquad f g^{\circ} f \qquad \downarrow e \sharp f \qquad \downarrow f \qquad = \qquad f \downarrow \psi \tilde{e} / \chi \qquad \downarrow f \qquad B$$

$$B \xrightarrow{Bn_B} BB^{\circ}B \xrightarrow{e_B B} B \qquad B \xrightarrow{1_B} B$$

$$\downarrow \iota \epsilon_B \qquad \downarrow \iota \epsilon_B \qquad \downarrow$$

This last equality defines a 2-cell in $\mathbf{Sq}(\mathcal{M})$ whose domain is the square $(f\sharp n)\cdot (e\sharp f)$. The result is now immediate.

Of course, there is a 'co' version of this lemma obtained by replacing ϵ with η and replacing the words proto-adjoint and proto-bidual with coproto-adjoint and coproto-bidual, respectively. (This 'co' version is proved in a similar way.)

Note that if both condition (i) of the above lemma as well as the 'co' version of condition (i) hold, then the fact that the 2-functors Dom and Cod are locally fully faithful implies that $(g^{\circ}, e, n, \epsilon, \eta)$ is a right bidual for f in $\mathbf{Sq}(\mathcal{M})$.

Theorem 1 now follows from the above lemma and its 'co' version.

4. Compact closed Gray monoids and Ladi

We begin by recalling from [3] the notion of a braided Gray monoid (though, for our purposes, we find it more natural to consider structure 2-cell isomorphisms with the reverse orientation). A *braiding* for a Gray monoid \mathcal{M} consists of arrows $\rho_{X,Y} \colon X \otimes Y \to Y \otimes X$ which are equivalences for all objects X, Y and invertible 2-cells

satisfying a number of conditions which we will not repeat here. We will call $\rho_{X,Y}: XY \to YX$ a braid arrow, $\rho_{f,g}$ a type-a-braid 2-cell and $\omega_{W,X,Y,Z}$ a type-braid 2-cell. A braided Gray monoid is a Gray monoid \mathcal{M} together with a braiding for \mathcal{M} .

If \mathcal{M} is a braided Gray monoid then we can also define a braiding for $\mathbf{Sq}(\mathcal{M})$. (In this section, we will suppress the symbol \sharp and merely use juxtaposition to denote the 'tensor' on $\mathbf{Sq}(\mathcal{M})$.) Suppose $x: X \to X'$ and $y: Y \to Y'$ are objects of $\mathbf{Sq}(\mathcal{M})$. Then the braid arrow $\rho_{x,y}: xy \to yx$ in $\mathbf{Sq}(\mathcal{M})$ is the type-a-braid 2-cell $\rho_{x,y}$ of \mathcal{M} . Suppose

$$\begin{array}{cccc} X & \xrightarrow{f} & X' & & Y & \xrightarrow{g} & Y' \\ x & & \downarrow \phi & \downarrow x' & & y & \downarrow \psi \gamma & \downarrow y' \\ \overline{X} & \xrightarrow{\overline{f}} & \overline{X}' & & \overline{Y} & \xrightarrow{\overline{g}} & \overline{Y}' \end{array}$$

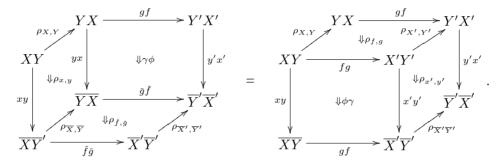
are squares in \mathcal{M} . The type-a-braid 2-cell

$$xy \xrightarrow{\rho_{x,y}} yx$$

$$\phi\gamma \downarrow \qquad \downarrow \rho_{\phi,\gamma} \qquad \downarrow \gamma\phi$$

$$x'y' \xrightarrow{\rho_{x',y'}} y'x'$$

in $\mathbf{Sq}(\mathcal{M})$ is the unique 2-cell $\rho_{\phi,\gamma}$ of $\mathbf{Sq}(\mathcal{M})$ satisfying $\mathrm{Dom}(\rho_{\phi,\gamma}) = \rho_{f,g}$ and $\mathrm{Cod}(\rho_{\phi,\gamma}) = \rho_{f,g}$. So the type-a-braid 2-cell $\rho_{\phi,\gamma}$ of $\mathbf{Sq}(\mathcal{M})$ naturally has the structure of a cube in \mathcal{M} :



Suppose $w: W \to \overline{W}$, $x: X \to \overline{X}$, $y: Y \to \overline{Y}$ and $z: Z \to \overline{Z}$ are objects of $\mathbf{Sq}(\mathcal{M})$. Then the type-b-braid 2-cell $\omega_{w,x,y,z}$ is defined to be the unique 2-cell of $\mathbf{Sq}(\mathcal{M})$ such that $\mathrm{Dom}(\omega_{w,x,y,z}) = \omega_{W,X,Y,Z}$ and $\mathrm{Cod}(\omega_{w,x,y,z}) = \omega_{\overline{W},\overline{X},\overline{Y},\overline{Z}}$.

That the structure 2-cells $\rho_{\phi,\gamma}$ and $\omega_{w,x,y,z}$ satisfy the conditions for a braiding follow from the fact that the 2-functors Dom and Cod are locally fully-faithful.

Recall from [3] that a *right autonomous* Gray monoid is a monoidal bicategory such that every object has a right bidual. We define a *compact closed* Gray monoid to be a braided right autonomous Gray monoid. A compact closed Gray monoid is autonomous in the sense of [3] (that is, every object has both a right and a left bidual).

Definition 1. If \mathscr{M} is a Gray monoid then $\operatorname{Ladj}(\mathscr{M})$ is defined to be the full and locally-full sub-bicategory of $\operatorname{Sq}(\mathscr{M})$ comprising objects $f \colon A \to B$ such that f is a left adjoint (that is, has a right adjoint) as an arrow of \mathscr{M} . We call $\operatorname{Ladj}(\mathscr{M})$ the Gray monoid of left adjoints in \mathscr{M} .

Note that the tensor product of two left adjoint arrows is still left adjoint, and that if g is a right adjoint arrow then g° is a left adjoint. The following result now follows from Theorem 1.

COROLLARY 1. If \mathcal{M} is a right autonomous (resp. compact closed) Gray monoid then $Ladj(\mathcal{M})$ is right autonomous (resp. compact closed).

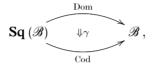
5. The bicategory of left adjoints

The higher dimensional algebraic structures that appear in nature tend to be 'weaker' than their lower dimensional counterparts. In particular, we find that monoidal bicategories – that is, one object tricategories – often arise in practice rather than Gray monoids. (Of course, a Gray monoid is a type of monoidal bicategory.) It is the case, however, that for any monoidal bicategory $\mathcal B$ there exists a Gray monoid

 \mathcal{M} and strong monoidal biequivalence (that is, a triequivalence) $E: \mathcal{B} \to \mathcal{M}$. This follows from the coherence theorem for tricategories ([6]). In this section we extend the results of the previous section to the setting of monoidal bicategories.

Strong monoidal biequivalences between monoidal bicategories preserve adjoints and right biduals. If \mathcal{B} is a monoidal bicategory then we can define the monoidal bicategory $\mathbf{Sq}(\mathcal{B})$ of squares in \mathcal{B} (which coincides with the earlier definition in the case of Gray monoids). Moreover, given any strong monoidal homomorphism $F: \mathcal{B} \to \mathcal{C}$ we can define a strong monoidal homomorphism $\mathbf{Sq}(F): \mathbf{Sq}(\mathcal{B}) \to \mathbf{Sq}(\mathcal{C})$. If F is a biequivalence then so is $\mathbf{Sq}(F)$. Also, the notions of a right autonomous, braided and compact closed Gray monoid can be extended in the obvious way to the setting of monoidal bicategories.

For any B, there exists an op-lax transformation



where $\gamma_f = f$. For any homomorphism $F: \mathcal{B} \to \mathcal{C}$, we have that $\mathbf{Sq}(F) \cdot \mathrm{Dom}_{\mathcal{C}} = \mathrm{Dom}_{\mathcal{B}} \cdot F$, $\mathbf{Sq}(F) \cdot \mathrm{Cod}_{\mathcal{C}} = \mathrm{Cod}_{\mathcal{B}} \cdot F$ and $\mathbf{Sq}(F) \cdot \gamma_{\mathcal{C}} = \gamma_{\mathcal{B}} \cdot F$. The extension of Theorem 1 is now straightforward.

THEOREM 2. Let $f: A \to B$ be an arrow of a monoidal bicategory \mathcal{B} . The object f of $\mathbf{Sq}(\mathcal{B})$ has a right bidual if and only if (i) both A and B have right biduals in \mathcal{B} and (ii) the arrow f has a right adjoint in \mathcal{B} .

Proof. Let \mathcal{B} be a monoidal bicategory and let $E:\mathcal{B}\to \mathcal{M}$ be a strong monoidal biequivalence with codomain a Gray monoid. Let $f:A\to B$ be an arrow of \mathcal{B} – that is, an object of $\mathbf{Sq}(\mathcal{B})$. Since E and $\mathbf{Sq}(E)$ are biequivalences, we have that the following four conditions are equivalent.

- (a) The object f has a right bidual.
- (b) The object $\mathbf{Sq}(E)(f)$ has a right bidual.
- (c) Dom ($\mathbf{Sq}(E)(f)$) and Cod ($\mathbf{Sq}(E)(f)$) have right biduals and (ii) $\gamma_{\mathbf{Sq}(E)(f)}$ has a right adjoint.
- (d) A and B have right biduals and (ii) f has a right adjoint.

Corollary 2. If \mathcal{B} is a right autonomous (resp. compact closed) bicategory then $Ladj(\mathcal{B})$ is right autonomous (resp. compact closed).

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