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The compact closed bicategory of left adjoints

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Abstract

We show that, for any braided compact closed bicategory $\mathscr{B}$, the bicategory **Ladj** ($\mathscr{B}$) of left adjoints in $\mathscr{B}$ also admits a braided compact closed structure.

1. *Introduction*

Over the last few years monoidal bicategories have been studied in connection with physics and geometry ([**1**, **2**, **4**, **7**, **8**]) and of particular interest to these applications have been the notions of a braiding for a tensor and a dual for an object. In this paper, we prove a theorem which demonstrates how an old construction (the bicategory of squares [**5**]) can be used to define braided compact closed bicategories which arise in the study of topological quantum field theories.

Specifically, the main theorem of this paper states the following. Suppose **Sq** ($\mathscr{B}$) is the monoidal bicategory of squares in a monoidal bicategory $\mathscr{B}$. Then an object $f{:}\,A \to B$ of **Sq** ($\mathscr{B}$) has a right bidual if and only if (i) both A and B have right biduals in $\mathscr{B}$ and (ii) the arrow f has a right adjoint in $\mathscr{B}$.

A compact closed bicategory is defined to be a braided monoidal bicategory in which every object has a right bidual. A consequence of the above mentioned theorem is that the full sub-bicategory **Ladj** ($\mathscr{B}$) of **Sq** ($\mathscr{B}$) comprising left adjoints is compact closed if $\mathscr{B}$ is compact closed.

Instances of the construction of a bicategory **Ladj** ($\mathscr{B}$) of left adjoints can be found in work on the categorical structures (first identified in [**11**]) which arise in the study of topological quantum field theories. For example, in [**2**] the notion of a *cobordism category* is defined and a topological quantum field theory is taken to be a structure-preserving morphism from a topological cobordism category to an algebraic one. (In this paper we do not take into account the 'vertical' composition of squares that is considered in [**2**].)

A typical candidate for a topological cobordism category is **Ladj** (Cobord), where Cobord is a (higher dimensional) category of cobordisms of compact, oriented smooth manifolds; and typical candidates for algebraic cobordism categories can be obtained by iterating the construction **Ladj** on the category Hilb of finite dimensional Hilbert spaces.

The original motivation for this work arose out of a general program to use monoidal bicategories with feedback for studying concurrency. The reader is referred to [**9**] for an overview of this work.

Most of the technical definitions in this paper are taken from [**3**] and, as is done there, we first deal with Gray monoids (which are strict monoidal bicategories), using the coherence theorem of [**6**] to transfer our definitions and results.

2. *Preliminaries*

In order to facilitate calculations – in particular, the passage between algebraic and diagrammatic representations of expressions – horizontal and vertical composition will be written in diagrammatic order: if $f: A \to B$ and $f': B \to C$ are arrows then $f \cdot f': A \to C$ denotes their composite (we also use the symbol $\cdot$ to denote horizontal composition of 2-cells); and, if $\alpha: f \Rightarrow g$ and $\beta: g \Rightarrow h$ are 2-cells, then $\alpha \bullet \beta: f \Rightarrow h$ denotes their vertical composite.

2·1. *Gray monoids*

To begin with, we recall the definition of a *Gray monoid* $\mathscr{M}$ from [**3**]. It is a 2-category equipped with the following data:

(a) an object I;

(b) for all objects A, two 2-functors $L_A, R_A: \mathscr{M} \to \mathscr{M}$ satisfying conditions
$$L_A(B) = R_B(A) \quad \text{(and define } A \otimes B = L_A(B)),$$
$$L_I = R_I = 1_{\mathscr{M}},$$
$$L_{A\otimes B} = L_B \cdot L_A, \quad R_{A\otimes B} = R_A \cdot R_B, \quad L_A \cdot R_B = R_B \cdot L_A,$$
for all objects A, B (remember that we are writing composition in diagrammatic order); and

(c) for all arrows $f: A \to A'$, $g: B \to B'$, an invertible 2-cell

$$\begin{tikzcd}
A \otimes B \arrow[r, "R_B(f)"] \arrow[d, "L_A(g)"'] \arrow[dr, phantom, "\Downarrow c_{f,g}"] & A' \otimes B \arrow[d, "L_{A'}(g)"] \\
A \otimes B' \arrow[r, "R_{B'}(f)"'] & A' \otimes B'
\end{tikzcd} .$$

The structure 2-cells $c_{f,g}$ are subject to the following four axioms:

(i) if both f and g are identity arrows then $c_{f,g}$ is an identity 2-cell;

(ii) for all arrows $f: A \to A'$, $g: B \to B'$ and $h: C \to C'$, there are equalities
$$L_A(c_{g,h}) = c_{L_A(g),h}, \quad c_{f,L_B(h)} = c_{R_B(f),h}, \quad R_C(c_{f,g}) = c_{f,R_C(g)};$$

(iii) for all arrows $f, h: A \to A'$, $g, k: B \to B'$ and 2-cells $\alpha: f \Rightarrow h$, $\beta: g \Rightarrow k$,

$$\begin{tikzcd}
A \otimes B \arrow[rr, "R_B(f)"] \arrow[d, bend right=50, "L_A(k)"'] \arrow[d, bend left=50, "L_A(g)"] \arrow[d, phantom, "\overset{L_A(\beta)}{\Leftarrow}"] \arrow[drr, phantom, "\Downarrow c_{f,g}"] & & A' \otimes B \arrow[d, "L_{A'}(g)"] \\
A \otimes B' \arrow[rr, bend left, "R_{B'}(f)"] \arrow[rr, bend right, "R_{B'}(h)"'] \arrow[rr, phantom, "\Downarrow R_{B'}(\alpha)"] & & A' \otimes B'
\end{tikzcd}
=
\begin{tikzcd}
A \otimes B \arrow[rr, bend left, "R_B(f)"] \arrow[rr, bend right, "R_B(h)"'] \arrow[rr, phantom, "\Downarrow R_B(\alpha)"] \arrow[d, "L_A(k)"'] \arrow[drr, phantom, "\Downarrow c_{h,k}"] & & A' \otimes B \arrow[d, bend right=50, "L_{A'}(k)"'] \arrow[d, bend left=50, "L_{A'}(g)"] \arrow[d, phantom, "\overset{L_{A'}(\beta)}{\Leftarrow}"] \\
A \otimes B' \arrow[rr, "R_{B'}(h)"'] & & A' \otimes B'
\end{tikzcd} ;$$

(iv) for all arrows $f\colon A \to A'$, $g\colon B \to B'$, $f'\colon A' \to A''$, $g'\colon B' \to B''$,

$$
\begin{tikzcd}
A \otimes B \arrow[r, "R_B(f)"] \arrow[d, "L_A(g)"'] \arrow[dr, phantom, "\Downarrow c_{f,g}"] & A' \otimes B \arrow[r, "R_B(f')"] \arrow[d, "L_{A'}(g)"] \arrow[dr, phantom, "\Downarrow c_{f',g}"] & A'' \otimes B \arrow[d, "L_{A''}(g)"] \\
A \otimes B' \arrow[r, "R_{B'}(f)"] \arrow[d, "L_A(g')"'] \arrow[dr, phantom, "\Downarrow c_{f,g'}"] & A' \otimes B' \arrow[r, "R_{B'}(f')"] \arrow[d, "L_{A'}(g')"] \arrow[dr, phantom, "\Downarrow c_{f',g'}"] & A'' \otimes B' \arrow[d, "L_{A''}(g')"] \\
A \otimes B'' \arrow[r, "R_{B''}(f)"'] & A' \otimes B'' \arrow[r, "R_{B''}(f')"'] & A'' \otimes B''
\end{tikzcd}
\;=\;
\begin{tikzcd}
A \otimes B \arrow[r, "R_B(f\cdot f')"] \arrow[d, "L_A(g\cdot g')"'] \arrow[dr, phantom, "\Downarrow c_{f\cdot f',g\cdot g'}"] & A'' \otimes B \arrow[d, "L_{A''}(g\cdot g')"] \\
A \otimes B'' \arrow[r, "R_{B''}(f\cdot f')"'] & A'' \otimes B''
\end{tikzcd}.
$$

Perhaps not apparent from these axioms is the fact that if either f or g is an identity arrow then $c_{f,g}$ is an identity 2-cell. Henceforth, the symbol $\mathscr{M}$ will denote a Gray monoid.

Following the conventions of [**3**], we put $L_A(g) = A \otimes g$, $L_A(\beta) = A \otimes \beta$, $R_B(f) = f \otimes B$, $R_B(\alpha) = \alpha \otimes B$, $L_A(g) \cdot R_{B'}(f) = f \otimes g$ and $L_A(\beta) \cdot R_{B'}(\alpha) = \alpha \otimes \beta$. (Note that $R_B(f) \cdot L_{A'}(g)$ could be another choice for what we denote by $f \otimes g$; but the structure 2-cell $c_{f,g}$ provides as isomorphism between these two choices.) In order to avoid the proliferation of symbols, we will sometimes suppress the symbol $\otimes$, for example we may write fg for $f \otimes g$; and, when the context is clear, we will omit the subscripts of the structure 2-cells $c_{f,g}$.

2·2. *Biduals*

Recall that a *right bidual* for an object A of $\mathscr{M}$ comprises an object B, an arrow $e\colon A \otimes B \to I$ and, for all objects C, D, adjoint pseudo-inverses to the functor

$$e^{\#}\colon \mathscr{M}(C, B \otimes D) \to \mathscr{M}(A \otimes C, D),$$

given by $e^{\#}(f) = (A \otimes f) \cdot (e \otimes D)$. In such a case, the object B is often referred to as a right bidual for A (and A is called a left bidual for B). As right biduals are determined up to equivalence, we write A° for B. The following proposition [**10**] gives an alternative definition for biduals. Its proof is relatively straightforward, but involves non-trivial manipulations of the structure 2-cells c of $\mathscr{M}$.

Proposition 1. *To give a right bidual for A is to give an object A°, two arrows $e\colon A \otimes A^{\circ} \to I$ and $n\colon I \to A^{\circ} \otimes A$ as well as two invertible 2-cells $\epsilon\colon (A \otimes n) \cdot (e \otimes A) \Rightarrow 1_A$ and $\eta\colon 1_{A^{\circ}} \Rightarrow (n \otimes A^{\circ}) \cdot (A^{\circ} \otimes e)$ such that the following two equations hold.*

$$
\begin{tikzcd}
 & AA^{\circ} \arrow[r, "e"] \arrow[dr, phantom, "\Downarrow c^{-1}"] & I \\
AA^{\circ} \arrow[ur, "1"] \arrow[r, "AnA^{\circ}"'] \arrow[rr, bend right, "1"'] \arrow[rr, phantom, bend right=20, "\Downarrow \epsilon A^{\circ}"] \arrow[ur, phantom, "\Downarrow A\eta", shift right=3] & AA^{\circ}AA^{\circ} \arrow[u, "AA^{\circ}e"] \arrow[r, "eAA^{\circ}"'] & AA^{\circ} \arrow[u, "e"']
\end{tikzcd}
\;=\;
\begin{tikzcd}
AA^{\circ} \arrow[r, bend left, "e"] \arrow[r, bend right, "e"'] \arrow[r, phantom, "\Downarrow 1"] & I
\end{tikzcd},
$$

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$$
\begin{tikzcd}
& A^\circ A \arrow[dl, "1"'] \arrow[d, "nA^\circ A"] \arrow[dl, phantom, shift right=3, "\Downarrow\eta A"] \arrow[dr, phantom, "\Downarrow c^{-1}"] & I \arrow[l, "n"'] \arrow[d, "n"] \\
A^\circ A & A^\circ A A^\circ A \arrow[l, "A^\circ eA"] \arrow[d, phantom, "\Downarrow A^\circ\epsilon"] & AA^\circ \arrow[l, "A^\circ An"] \arrow[ll, bend left, "1"] \\
& {} &
\end{tikzcd}
=
\begin{tikzcd}
A^\circ A \arrow[r, phantom, "\Downarrow 1"] & I \arrow[l, bend right, "n"'] \arrow[l, bend left, "n"]
\end{tikzcd} .
$$

If A has a right bidual as above we call $e\colon A \otimes A^\circ \to I$ a pairing and $n\colon I \to A^\circ \otimes A$ a copairing for A. Also, ϵ and η are respectively called the counit and unit isomorphisms for A. Of course, when we say $e\colon A \otimes A^\circ \to I$ is a pairing for A we mean there exists a right bidual for A of the form $(A^\circ, e, n, \epsilon, \eta)$ (this quintuple being interpreted in the sense of the above proposition).

We end this section, by using the structure of objects with biduals to establish a correspondence which will be fundamental to the proof of the main theorem (Section 3).

First we establish some notation. Suppose that A and B are objects of $\mathscr{M}$ with right biduals $(A^\circ, e_A, n_A, \epsilon_A, \eta_A)$ and $(B^\circ, e_B, n_B, \epsilon_B, \eta_B)$. Let $(-)^\circ\colon \mathscr{M}(B, A) \to \mathscr{M}(A^\circ, B^\circ)$ be the functor mapping $g\colon B \to A$ to

$$g^\circ = (n_B A^\circ) \cdot (B^\circ g A^\circ) \cdot (B^\circ e_A)\colon A^\circ \to B^\circ.$$

We also define a functor $(-)^*\colon \mathscr{M}(A^\circ, B^\circ) \to \mathscr{M}(B, A)$ as follows: given an arrow $h\colon A^\circ \to B^\circ$, define the arrow $h^*\colon B \to A$ to be the composite $(Bn_A) \cdot (BhA) \cdot (e_B A)$.

Let $\phi\colon (g^\circ)^* \Rightarrow g$ be the invertible 2-cell

$$[(Bn_B n_A) \cdot c^{-1}] \bullet [c^{-1} \cdot e_A A] \bullet [\epsilon_B \cdot g \cdot \epsilon_A].$$

Let $\psi\colon (An_A) \cdot (fg^\circ A) \cdot (e_B A) \Rightarrow f \cdot g$ be the invertible 2-cell defined by

$$[c^{-1} \cdot (e_B A)] \bullet [f \cdot \phi].$$

Let $\theta\colon g \cdot f \Rightarrow (Bn_A) \cdot (Bg^\circ f) \cdot (e_B B)$ be the invertible 2-cell defined by

$$(\phi^{-1} \cdot f) \bullet ((Bn_A) \cdot c).$$

Given a 2-cell of the form $\alpha\colon e_A \Rightarrow (fg^\circ) \cdot e_B$, define the 2-cell $\tilde{\alpha}\colon 1_A \Rightarrow f \cdot g$ to be

$$
\begin{tikzcd}
A \arrow[r, "An_A"] \arrow[rrr, bend left=40, "1_A"] \arrow[rrr, bend right=70, "f \cdot g"'] & AA^\circ A \arrow[rr, "e_A A"] \arrow[dr, "fg^\circ A"'] \arrow[rr, phantom, bend left=30, "\Downarrow\epsilon_A^{-1}"] \arrow[d, phantom, shift right=2, "\Downarrow\psi"] & \arrow[d, phantom, "\Downarrow\alpha A"] & A . \\
& {} & BB^\circ A \arrow[ur, "e_B A"'] &
\end{tikzcd}
$$

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This construction, of course, has a dual: given a 2-cell of the form $\beta\colon n_A\cdot(g^\circ f) \Rightarrow n_B$, define the 2-cell $\vec{\beta}\colon g\cdot f \Rightarrow 1_B$ to be

$$\begin{tikzcd}
& BA^\circ A \arrow[dr, "Bg^\circ f"] \arrow[d, phantom, "\Downarrow B\beta"] & \\
B \arrow[ur, "Bn_A"] \arrow[rr, "Bn_B"'] \arrow[rrr, bend left=60, "g\cdot f"] \arrow[rrr, bend right=40, "1_B"'] & {} & BB^\circ B \arrow[r, "e_B B"] & B.
\end{tikzcd}$$

The following proposition follows from the basic properties of biduals.

PROPOSITION 2. *The above associations $\alpha \mapsto \tilde{\alpha}$ and $\beta \mapsto \vec{\beta}$ establish bijections*

$$\mathscr{M}[AA^\circ, I](e_A, (fg^\circ)\cdot e_B) \cong \mathscr{M}[A,A](1_A, f\cdot g)$$

and

$$\mathscr{M}[I, B^\circ B](n_A\cdot(g^\circ f), n_B) \cong \mathscr{M}[B,B](g\cdot f, 1_B).$$

2·3. *Squares in $\mathscr{M}$*

Let $\mathbf{2}$ be the category comprising the 'free living arrow'. For any 2-category $\mathbf{C}$, define the 2-category $\mathbf{Sq}\,(\mathbf{C})$ of *squares* in $\mathbf{C}$ to be the 2-category of 2-functors, op-lax transformations and modifications from $\mathbf{2}$ to $\mathbf{C}$. So an object of $\mathbf{Sq}\,(\mathbf{C})$ is an arrow $f\colon A\to A'$ in $\mathbf{C}$ and an arrow from $f\colon A\to A'$ to $g\colon B\to B'$ in $\mathbf{Sq}\,(\mathbf{C})$ is a pair of arrows $h\colon A\to B$, $i\colon A'\to B'$ in $\mathbf{C}$ together with a 2-cell of the form

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "g"] \\
A' \arrow[r, "i"'] & B'
\end{tikzcd}.$$

The arrows of $\mathbf{Sq}\,(\mathbf{C})$ are called squares in $\mathbf{C}$. There is an obvious pair of 2-functors $\mathrm{Dom},\ \mathrm{Cod}\colon \mathbf{Sq}\,(\mathbf{C})\to\mathbf{C}$ which are locally fully faithful.

PROPOSITION 3. *If $\mathscr{M}$ is a Gray monoid then $\mathbf{Sq}\,(\mathscr{M})$ admits a Gray monoid structure such that the two 2-functors $\mathrm{Dom},\mathrm{Cod}\colon\mathbf{Sq}\,(\mathscr{M})\to\mathscr{M}$ preserve the Gray monoid structure on the nose.*

Proof. The unit for the Gray monoid structure of $\mathbf{Sq}\,(\mathscr{M})$ is the object $1_I\colon I\to I$. Consider the following two squares.

$$\begin{tikzcd}
A \arrow[r, "j"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & C \arrow[d, "h"] \\
A' \arrow[r, "l"'] & C'
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "k"] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\beta"] & D \arrow[d, "i"] \\
B' \arrow[r, "m"'] & D'
\end{tikzcd}.$$

We define $L_f, R_i\colon \mathbf{Sq}\,(\mathscr{M})\to\mathbf{Sq}\,(\mathscr{M})$ on objects by $L_f(i) = R_i(f) = f\otimes i$ and on

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arrows by:

$$L_f(\beta) \;=\; \begin{tikzcd} AB \arrow[r, "Ak"] \arrow[d, "Ag"'] \arrow[dr, phantom, "\Downarrow A\beta"] & AD \arrow[d, "Ai"] \\ AB' \arrow[r, "Am"'] \arrow[d, "fB'"'] \arrow[dr, phantom, "\Downarrow c^{-1}"] & AD' \arrow[d, "fD'"] \\ A'B' \arrow[r, "A'm"'] & A'D' \end{tikzcd}\;, \qquad R_i(\alpha) \;=\; \begin{tikzcd} AD \arrow[r, "jD"] \arrow[d, "Ai"'] \arrow[dr, phantom, "\Downarrow c"] & CD \arrow[d, "Ci"] \\ AD' \arrow[r, "jD'"'] \arrow[d, "fD'"'] \arrow[dr, phantom, "\Downarrow \alpha D'"] & CD' \arrow[d, "hD'"] \\ A'D' \arrow[r, "lD'"'] & C'D' \end{tikzcd}\;.$$

There is a unique way to define L_f and R_i on 2-cells so that Dom and Cod preserve this structure. Similarly there is a unique choice for the structure 2-cells c of **Sq** $(\mathscr{M})$. That this data is well defined is a straightforward exercise in the calculus of the structure 2-cells of a Gray monoid. That the structure 2-cells for **Sq** $(\mathscr{M})$ satisfy the axioms for a Gray monoid is guaranteed by the fact that Dom and Cod are locally fully faithful.

In order to avoid confusing the Gray monoid structure of $\mathscr{M}$ with that of **Sq** $(\mathscr{M})$, we will denote the 'tensor product' of the latter by the symbol $\sharp$. For example, if f is an arrow of $\mathscr{M}$ and β is a square in $\mathscr{M}$ (as in the proof of the above proposition), then the square $L_f(\beta)$ defined by the Gray monoid structure on **Sq** $(\mathscr{M})$ is denoted $f\sharp\beta$.

3. *The main theorem*

In this section we prove the following theorem.

Theorem 1. *Let $f\colon A \to B$ be an arrow of a Gray monoid $\mathscr{M}$. The object f of* **Sq** $(\mathscr{M})$ *has a right bidual if and only if* (i) *both A and B have right biduals in $\mathscr{M}$ and* (ii) *the arrow f has a right adjoint in $\mathscr{M}$.*

In order to facilitate the proof, we break down the notions of both an adjoint as well as a bidual into two parts.

Suppose $f\colon A \to B$ and $g\colon B \to A$ are arrows and $\lambda\colon 1_A \Rightarrow f \cdot g$ and $\mu\colon g \cdot f \Rightarrow 1_B$ are 2-cells, in a 2-category. We call the triple (g, λ, μ) a right proto-adjoint for f if $1_f = (\lambda \cdot f) \bullet (f \cdot \mu)$; and we call it a right coproto-adjoint for f if $1_g = (g \cdot \lambda) \bullet (\mu \cdot g)$. Of course, if (g, λ, μ) is both a right proto-adjoint and a right coproto-adjoint for f, then g is a right adjoint for f.

Suppose that X and Y are objects, $e\colon XY \to I$ and $n\colon I \to YX$ are arrows and $\epsilon\colon (Xn)\cdot(eX) \Rightarrow 1_X$ and $\eta\colon 1_Y \Rightarrow (nY)\cdot(Ye)$ are invertible 2-cells in a Gray monoid. We call the quadruple (Y, e, n, ϵ) a right proto-bidual for X and we call the quadruple (Y, e, n, η) a right coproto-bidual for X. Of course, the data $(Y, e, n, \epsilon, \eta)$ defines a right bidual for X if the two equations of Proposition 1 hold.

The 2-functors Dom, Cod: **Sq** $(\mathscr{M}) \to \mathscr{M}$ preserve biduals. Explicitly, if an object $f\colon A \to B$ of **Sq** $(\mathscr{M})$ has a right bidual $(h\colon C \to D, e, n, \epsilon, \eta)$, then $(C, \mathrm{Dom}\,(e), \mathrm{Dom}\,(n), \mathrm{Dom}\,(\epsilon), \mathrm{Dom}\,(\eta))$ is a right bidual of A in $\mathscr{M}$ and $(D, \mathrm{Cod}\,(e), \mathrm{Cod}\,(n), \mathrm{Cod}\,(\epsilon), \mathrm{Cod}\,(\eta))$ is a right bidual of B in $\mathscr{M}$.

In this case, without loss of generality we may assume that the pairing e and copairing n for f will be squares of the form

$$
\begin{tikzcd}
AA^\circ \arrow[r, "e_A"] \arrow[d, "fg^\circ"'] \arrow[dr, phantom, "\Downarrow e"] & I \arrow[d, "1"] \\
BB^\circ \arrow[r, "e_B"'] & I,
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, "n_A"] \arrow[d, "1"'] \arrow[dr, phantom, "\Downarrow n"] & A^\circ A \arrow[d, "g^\circ f"] \\
I \arrow[r, "n_B"'] & B^\circ B,
\end{tikzcd}
$$

where $(A^\circ, e_A, n_A, \epsilon_A, \eta_A)$ is a right bidual for A, $(B^\circ, e_B, n_B, \epsilon_B, \eta_B)$ is a right bidual for B and $g\colon B \to A$ is an arrow in $\mathscr{M}$. Notice that, in this case, we can apply the constructions $e \mapsto \tilde{e}$ and $n \mapsto \vec{n}$ of Proposition 2.

LEMMA 1. *Suppose A and B have right biduals A° and B° in a Gray monoid $\mathscr{M}$. Let $f\colon A \to B$ and $g\colon B \to A$ be any arrows. Then the following two conditions are equivalent.*

(i) *In $\mathbf{Sq}(\mathscr{M})$, $(g^\circ, e, n, \epsilon)$ is a right proto-bidual for f such that* $\mathrm{Dom}(e) = e_A$, $\mathrm{Dom}(n) = n_A$, $\mathrm{Dom}(\epsilon) = \epsilon_A$, $\mathrm{Cod}(e) = e_B$, $\mathrm{Cod}(n) = n_B$ *and* $\mathrm{Cod}(\epsilon) = \epsilon_B$.

(ii) *In $\mathscr{M}$, $(g, \tilde{e}, \vec{n})$ is a right proto-adjoint for f.*

Proof. Let $\tau\colon f \cdot g \cdot f \Rightarrow (An_A) \cdot (fg^\circ f) \cdot (e_B B)$ be the invertible 2-cell

$$
\begin{tikzcd}
 & A \arrow[dl, "An_A"'] \arrow[dr, "f \cdot g"] \arrow[d, phantom, "\Downarrow\psi^{-1}"] & \\
AA^\circ A \arrow[r, "fg^\circ A"'] \arrow[d, "AA^\circ f"'] & BB^\circ A \arrow[r, "e_B A"'] \arrow[d, phantom, "\Downarrow c"] & A \arrow[d, "f"] \\
AA^\circ B \arrow[r, "fg^\circ B"'] & BB^\circ B \arrow[r, "e_B B"'] & B
\end{tikzcd}
.
$$

A straightforward calculation shows

$$
\begin{tikzcd}
A \arrow[r, "An_A"] \arrow[rr, bend left=50, "1_A"] \arrow[drr, bend right=60, "f\cdot g\cdot f"'] \arrow[dr, phantom, "\Downarrow\tau^{-1}"] & AA^\circ A \arrow[r, "e_A A"] \arrow[d, "fg^\circ f"] \arrow[u, phantom, "\Downarrow\epsilon_A^{-1}"] \arrow[dr, phantom, "\Downarrow e\sharp f"] & A \arrow[d, "f"] \\
 & BB^\circ B \arrow[r, "e_B B"'] & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, bend left, "1_A"] \arrow[r, bend right, "f\cdot g"'] \arrow[r, phantom, "\Downarrow\tilde{e}"] & A \arrow[r, "f"] & B.
\end{tikzcd}
$$

We also have

$$
\begin{tikzcd}
A \arrow[r, "An_A"] \arrow[d, "f"'] \arrow[drr, bend left=60, "f\cdot g\cdot f"] \arrow[dr, phantom, "\Downarrow f\sharp n"] & AA^\circ A \arrow[d, "fg^\circ f"] \arrow[dr, phantom, "\Downarrow\tau"] & \\
B \arrow[r, "Bn_B"'] \arrow[rr, bend right=50, "1_B"'] & BB^\circ B \arrow[r, "e_B B"'] \arrow[d, phantom, "\Downarrow\epsilon_B"] & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[r, bend left, "g\cdot f"] \arrow[r, bend right, "1_B"'] \arrow[r, phantom, "\Downarrow\vec{n}"] & B.
\end{tikzcd}
$$

Combining these last two equalities, we get

$$\begin{tikzcd} A \arrow[r, "An_A"] \arrow[d, "f"'] \arrow[rr, bend left=50, "1"] & AA^\circ A \arrow[r, "e_A A"] \arrow[d, "fg^\circ f"] \arrow[u, phantom, "\Downarrow\epsilon_A^{-1}"] & A \arrow[d, "f"] \\ B \arrow[r, "Bn_B"'] \arrow[rr, bend right=50, "1"'] \arrow[ur, phantom, "\Downarrow f\sharp n"] & BB^\circ B \arrow[r, "e_B B"'] \arrow[ur, phantom, "\Downarrow e\sharp f"] \arrow[d, phantom, "\Downarrow\epsilon_B"] & B \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\tilde{e}"] & A \arrow[d, "f"] \\ B \arrow[r, "1_B"'] \arrow[ur, "g"] \arrow[ur, phantom, "\Downarrow\tilde{n}"'] & B \end{tikzcd} .$$

This last equality defines a 2-cell in $\mathbf{Sq}\,(\mathscr{M})$ whose domain is the square $(f\sharp n)\cdot(e\sharp f)$. The result is now immediate.

Of course, there is a 'co' version of this lemma obtained by replacing ϵ with η and replacing the words proto-adjoint and proto-bidual with coproto-adjoint and coproto-bidual, respectively. (This 'co' version is proved in a similar way.)

Note that if both condition (i) of the above lemma as well as the 'co' version of condition (i) hold, then the fact that the 2-functors Dom and Cod are locally fully faithful implies that $(g^\circ, e, n, \epsilon, \eta)$ is a right bidual for f in $\mathbf{Sq}\,(\mathscr{M})$.

Theorem 1 now follows from the above lemma and its 'co' version.

4. *Compact closed Gray monoids and* **Ladj**

We begin by recalling from [**3**] the notion of a braided Gray monoid (though, for our purposes, we find it more natural to consider structure 2-cell isomorphisms with the reverse orientation). A *braiding* for a Gray monoid $\mathscr{M}$ consists of arrows $\rho_{X,Y}: X\otimes Y \to Y\otimes X$ which are equivalences for all objects X, Y and invertible 2-cells

$$\begin{tikzcd} XY \arrow[r, "\rho_{X,Y}"] \arrow[d, "fg"'] \arrow[dr, phantom, "\Downarrow\rho_{f,g}"] & YX \arrow[d, "gf"] \\ X'Y' \arrow[r, "\rho_{X',Y'}"'] & Y'X' \end{tikzcd} \qquad \begin{tikzcd} WXYZ \arrow[r, "WX\rho_{Y,Z}"] \arrow[d, "\rho_{W,X}YZ"'] \arrow[dr, phantom, "\Downarrow\omega_{W,X,Y,Z}"] & WXZY \arrow[d, "\rho_{W,XZ}Y"] \\ XWYZ \arrow[r, "X\rho_{WY,Z}"'] & XZWY \end{tikzcd}$$

satisfying a number of conditions which we will not repeat here. We will call $\rho_{X,Y}: XY \to YX$ a braid arrow, $\rho_{f,g}$ a type-a-braid 2-cell and $\omega_{W,X,Y,Z}$ a type-b-braid 2-cell. A *braided Gray monoid* is a Gray monoid $\mathscr{M}$ together with a braiding for $\mathscr{M}$.

If $\mathscr{M}$ is a braided Gray monoid then we can also define a braiding for $\mathbf{Sq}\,(\mathscr{M})$. (In this section, we will suppress the symbol $\sharp$ and merely use juxtaposition to denote the 'tensor' on $\mathbf{Sq}\,(\mathscr{M})$.) Suppose $x: X \to X'$ and $y: Y \to Y'$ are objects of $\mathbf{Sq}\,(\mathscr{M})$. Then the braid arrow $\rho_{x,y}: xy \to yx$ in $\mathbf{Sq}\,(\mathscr{M})$ is the type-a-braid 2-cell $\rho_{x,y}$ of $\mathscr{M}$. Suppose

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "x"'] \arrow[dr, phantom, "\Downarrow\phi"] & X' \arrow[d, "x'"] \\ \overline{X} \arrow[r, "\bar{f}"'] & \overline{X}' \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "y"'] \arrow[dr, phantom, "\Downarrow\gamma"] & Y' \arrow[d, "y'"] \\ \overline{Y} \arrow[r, "\bar{g}"'] & \overline{Y}' \end{tikzcd}$$

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are squares in $\mathscr{M}$. The type-a-braid 2-cell

$$\begin{tikzcd}
xy \arrow[r, "\rho_{x,y}"] \arrow[d, "\phi\gamma"'] \arrow[dr, phantom, "\Downarrow\rho_{\phi,\gamma}"] & yx \arrow[d, "\gamma\phi"] \\
x'y' \arrow[r, "\rho_{x',y'}"'] & y'x'
\end{tikzcd}$$

in $\mathbf{Sq}(\mathscr{M})$ is the unique 2-cell $\rho_{\phi,\gamma}$ of $\mathbf{Sq}(\mathscr{M})$ satisfying $\mathrm{Dom}(\rho_{\phi,\gamma}) = \rho_{f,g}$ and $\mathrm{Cod}(\rho_{\phi,\gamma}) = \rho_{\bar{f},\bar{g}}$. So the type-a-braid 2-cell $\rho_{\phi,\gamma}$ of $\mathbf{Sq}(\mathscr{M})$ naturally has the structure of a cube in $\mathscr{M}$:

$$\begin{tikzcd}
 & YX \arrow[rr, "gf"] \arrow[dd, "yx"] \arrow[ddrr, phantom, "\Downarrow\gamma\phi"] & & Y'X' \arrow[dd, "y'x'"] \\
XY \arrow[ur, "\rho_{X,Y}"] \arrow[dd, "xy"'] \arrow[r, phantom, "\Downarrow\rho_{x,y}"] & & & \\
 & \overline{Y}\overline{X} \arrow[rr, "\bar{g}\bar{f}"] \arrow[dr, phantom, "\Downarrow\rho_{\bar{f},\bar{g}}"] & & \overline{Y}'\overline{X}' \\
\overline{XY}' \arrow[ur, "\rho_{\overline{X},\overline{Y}}"'] \arrow[rr, "\bar{f}\bar{g}"'] & & \overline{X}'\overline{Y}' \arrow[ur, "\rho_{\overline{X}',\overline{Y}'}"'] &
\end{tikzcd}
=
\begin{tikzcd}
 & YX \arrow[rr, "gf"] \arrow[d, phantom, "\Downarrow\rho_{f,g}"] & & Y'X' \arrow[dd, "y'x'"] \\
XY \arrow[ur, "\rho_{X,Y}"] \arrow[rr, "fg"'] \arrow[dd, "xy"'] \arrow[ddrr, phantom, "\Downarrow\phi\gamma"] & & X'Y' \arrow[ur, "\rho_{X',Y'}"'] \arrow[dd, "x'y'"] \arrow[r, phantom, "\Downarrow\rho_{x',y'}"] & \\
 & & & \overline{Y}'\overline{X}' \\
\overline{XY} \arrow[rr, "gf"'] & & \overline{X}'\overline{Y}' \arrow[ur, "\rho_{\overline{X}'\overline{Y}'}"'] &
\end{tikzcd} .$$

Suppose $w: W \to \overline{W}$, $x: X \to \overline{X}$, $y: Y \to \overline{Y}$ and $z: Z \to \overline{Z}$ are objects of $\mathbf{Sq}(\mathscr{M})$. Then the type-b-braid 2-cell $\omega_{w,x,y,z}$ is defined to be the unique 2-cell of $\mathbf{Sq}(\mathscr{M})$ such that $\mathrm{Dom}(\omega_{w,x,y,z}) = \omega_{W,X,Y,Z}$ and $\mathrm{Cod}(\omega_{w,x,y,z}) = \omega_{\overline{W},\overline{X},\overline{Y},\overline{Z}}$.

That the structure 2-cells $\rho_{\phi,\gamma}$ and $\omega_{w,x,y,z}$ satisfy the conditions for a braiding follow from the fact that the 2-functors Dom and Cod are locally fully-faithful.

Recall from [**3**] that a *right autonomous* Gray monoid is a monoidal bicategory such that every object has a right bidual. We define a *compact closed* Gray monoid to be a braided right autonomous Gray monoid. A compact closed Gray monoid is autonomous in the sense of [**3**] (that is, every object has both a right and a left bidual).

Definition 1. If $\mathscr{M}$ is a Gray monoid then $\mathbf{Ladj}(\mathscr{M})$ is defined to be the full and locally-full sub-bicategory of $\mathbf{Sq}(\mathscr{M})$ comprising objects $f: A \to B$ such that f is a left adjoint (that is, has a right adjoint) as an arrow of $\mathscr{M}$. We call $\mathbf{Ladj}(\mathscr{M})$ the Gray monoid of left adjoints in $\mathscr{M}$.

Note that the tensor product of two left adjoint arrows is still left adjoint, and that if g is a right adjoint arrow then g° is a left adjoint. The following result now follows from Theorem 1.

Corollary 1. *If $\mathscr{M}$ is a right autonomous* (*resp. compact closed*) *Gray monoid then* $\mathbf{Ladj}(\mathscr{M})$ *is right autonomous* (*resp. compact closed*).

5. *The bicategory of left adjoints*

The higher dimensional algebraic structures that appear in nature tend to be 'weaker' than their lower dimensional counterparts. In particular, we find that monoidal bicategories – that is, one object tricategories – often arise in practice rather than Gray monoids. (Of course, a Gray monoid is a type of monoidal bicategory.) It is the case, however, that for any monoidal bicategory $\mathscr{B}$ there exists a Gray monoid

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$\mathscr{M}$ and strong monoidal biequivalence (that is, a triequivalence) $E: \mathscr{B} \to \mathscr{M}$. This follows from the coherence theorem for tricategories ([**6**]). In this section we extend the results of the previous section to the setting of monoidal bicategories.

Strong monoidal biequivalences between monoidal bicategories preserve adjoints and right biduals. If $\mathscr{B}$ is a monoidal bicategory then we can define the monoidal bicategory $\mathbf{Sq}(\mathscr{B})$ of squares in $\mathscr{B}$ (which coincides with the earlier definition in the case of Gray monoids). Moreover, given any strong monoidal homomorphism $F: \mathscr{B} \to \mathscr{C}$ we can define a strong monoidal homomorphism $\mathbf{Sq}(F): \mathbf{Sq}(\mathscr{B}) \to \mathbf{Sq}(\mathscr{C})$. If F is a biequivalence then so is $\mathbf{Sq}(F)$. Also, the notions of a right autonomous, braided and compact closed Gray monoid can be extended in the obvious way to the setting of monoidal bicategories.

For any $\mathscr{B}$, there exists an op-lax transformation

$$\begin{tikzcd} \mathbf{Sq}(\mathscr{B}) \arrow[r, bend left, "\mathrm{Dom}"] \arrow[r, bend right, "\mathrm{Cod}"'] \arrow[r, phantom, "\Downarrow\gamma"] & \mathscr{B}, \end{tikzcd}$$

where $\gamma_f = f$. For any homomorphism $F: \mathscr{B} \to \mathscr{C}$, we have that $\mathbf{Sq}(F) \cdot \mathrm{Dom}_{\mathscr{C}} = \mathrm{Dom}_{\mathscr{B}} \cdot F$, $\mathbf{Sq}(F) \cdot \mathrm{Cod}_{\mathscr{C}} = \mathrm{Cod}_{\mathscr{B}} \cdot F$ and $\mathbf{Sq}(F) \cdot \gamma_{\mathscr{C}} = \gamma_{\mathscr{B}} \cdot F$. The extension of Theorem 1 is now straightforward.

THEOREM 2. *Let $f: A \to B$ be an arrow of a monoidal bicategory $\mathscr{B}$. The object f of* $\mathbf{Sq}(\mathscr{B})$ *has a right bidual if and only if* (i) *both A and B have right biduals in $\mathscr{B}$ and* (ii) *the arrow f has a right adjoint in $\mathscr{B}$.*

Proof. Let $\mathscr{B}$ be a monoidal bicategory and let $E: \mathscr{B} \to \mathscr{M}$ be a strong monoidal biequivalence with codomain a Gray monoid. Let $f: A \to B$ be an arrow of $\mathscr{B}$ – that is, an object of $\mathbf{Sq}(\mathscr{B})$. Since E and $\mathbf{Sq}(E)$ are biequivalences, we have that the following four conditions are equivalent.

(a) The object f has a right bidual.
(b) The object $\mathbf{Sq}(E)(f)$ has a right bidual.
(c) $\mathrm{Dom}(\mathbf{Sq}(E)(f))$ and $\mathrm{Cod}(\mathbf{Sq}(E)(f))$ have right biduals and (ii) $\gamma_{\mathbf{Sq}(E)(f)}$ has a right adjoint.
(d) A and B have right biduals and (ii) f has a right adjoint.

COROLLARY 2. *If $\mathscr{B}$ is a right autonomous* (*resp. compact closed*) *bicategory then* $\mathbf{Ladj}(\mathscr{B})$ *is right autonomous* (*resp. compact closed*).

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