COMPUTING QUOTIENTS OF ACTIONS OF A FREE CATEGORY

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In [1] we have described a procedure for computing left Kan extensions (Chapter 10 of [6]) of functors into **Set** which generalizes the traditional Todd-Coxeter procedure (described in [2],[4],[5]) for coset enumeration. In this paper we describe and prove the correctness of a key sub-procedure – the Quotient Procedure – which computes quotients of actions of a free category. The quotient procedure, being highly recursive, is the most subtle part of the left Kan extension procedure. In fact published proofs (as in [3],[7]) of the Todd-Coxeter procedure appear to lack a proof of the correctness of the quotient procedure for this special case.

A further subtlety arises from the fact that infinite mathematical structures must, for computational purposes, be handled in terms of finite presentations. More specifically, we consider a finite directed graph G and $\mathcal{F}G$, the free category on G, and define the notion of a partial description of a functor $\mathcal{F}G \to \mathbf{Set}$, which we call a *presentation*. Given a presentation we describe its *completion* to a functor $\overline{P}: \mathcal{F}G \to \mathbf{Set}$. We then consider a finite system of *equations* S between the elements of a presentation P and develop the notion of the *quotient* P/S which is the natural one to take so that $\overline{P}/S$ corresponds to the usual quotient of the action by the congruence generated by S. The key result justifying computing with finite presentations in this paper, given in THEOREM 1, is that $\overline{P}/S$ is isomorphic to $\overline{P/S}$. The typical situation then is that P is finite, whilst $\overline{P}$ is infinite, but in the procedure to compute a quotient of $\overline{P}$ we need only work with a quotient of P and hence a finite amount of data.

The ideas developed in this paper are very naturally motivated through graphical interpretations and to this end we include several diagrams by way of illustrative example.

In sections 1 and 2 we treat presentations of functors and quotients of presentations. The quotient procedure itself is described in section 3 and analysed in section 4. We finish in section 5 with a sketch of the application of this procedure to the calculation of left Kan extensions.

§1 Presentations of Functors

We first make precise the idea of a *presentation of a functor*, which we shall often refer to simply as a *presentation*.

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DEFINITION 1.1 A **presentation of a functor**, P, on a graph G consists of

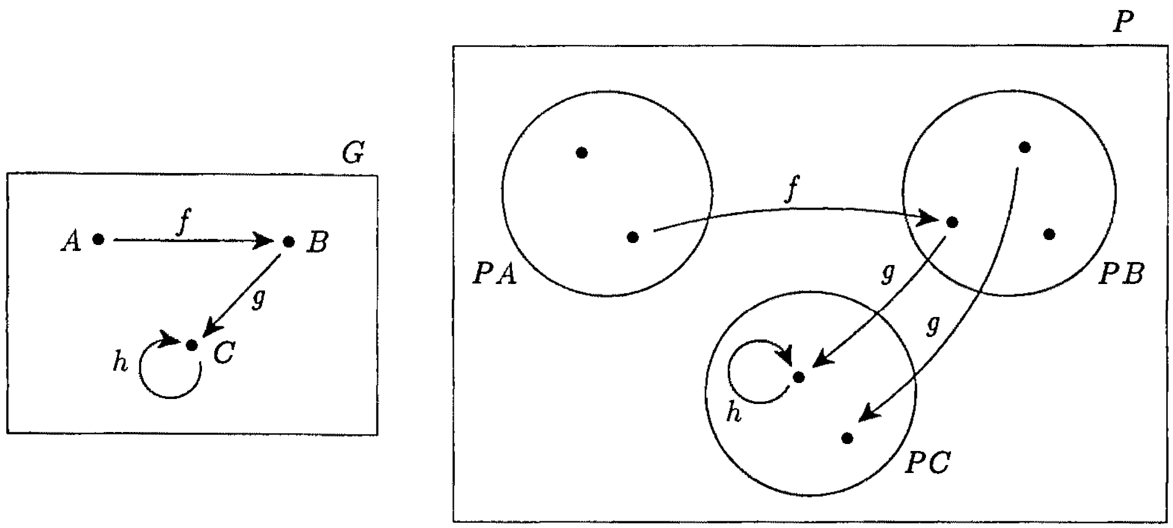
$$\text{sets} \quad PA \qquad \text{for all } A \in G$$

$$\text{and partial functions } Pg \subseteq PA \times PB \quad \text{for all } g\colon A \to B \text{ in } G.$$

Notation: If $(x, y) \in Pg$ we write $x \xrightarrow{g} y$ in P.

Given a presentation P on a graph G, if $x \xrightarrow{g} y$ in P for some y, we say $Pg(x)$ is **defined** and that $Pg(x) = y$; otherwise it is **undefined**.

The following diagram gives an example of a graph G (later illustrations will be based on the same graph) and a presentation P on G:



DEFINITION 1.2 A presentation P on a graph G is called **complete** if for each $g\colon A \to B$ in G, the relations Pg are in fact *functions.*

Note: In this case P can can be considered as a functor $\mathcal{F}G \to \mathbf{Set}$ and in speaking of isomorphisms of complete presentations, we will mean isomorphisms of the corresponding functors.

REMARK In the material that follows, given a presentation P, we will frequently be considering families of equivalence relations $\sim_A$ on PA indexed by $A \in G$. By a convenient abuse of terminology, we shall sometimes refer to $\sim$ as an equivalence relation on P, and by an abuse of notation, we shall often drop the subscript from $\sim_A$.

We now turn to the question of obtaining a functor (complete presentation) from any given presentation P. We take two approaches, which turn out to be equivalent. The first, $\overline{P}$, has an obvious connection with the coend formula for left Kan extensions, which will be discussed further in §5, while the second, $P + P^{\infty}$, suggests itself very naturally from the diagrammatic representations of presentations, and is also convenient to work with as it does not involve a quotient by an equivalence relation.

DEFINITION 1.3 Given a presentation P on a graph G, we define a *complete* presentation $\overline{P}$ — the completion of P — as follows:
for each $A \in G$, define

$$\overline{P}A = \Big(\sum_{B\in\mathcal{F}G} \mathcal{F}G(B,A) \times PB\Big)\Big/\!\sim_A$$

where $\sim$ is the smallest equivalence relation such that:

- for $g\colon A \to B$ in G, f, $f'\colon A \to B$ in $\mathcal{F}G$, $x, x' \in PA$, $y \in PB$,
  (i) $x \xrightarrow{g} y$ in $P \Rightarrow (g,x) \sim (1_B, y)$
  (ii) $(f,x) \sim (f',x') \Rightarrow (hf,x) \sim (hf',x')$ whenever $\operatorname{dom} h = B$ in G.

We denote the equivalence class of (f,x) by $[f,x]$.

- for each $g\colon A \to B$ in G, define

$$\overline{P}g\colon \overline{P}A \to \overline{P}B\colon [h,x] \longmapsto [gh,x]$$

which is well-defined by (ii).

DEFINITION 1.4 Given a presentation P on a graph G, we define a complete presentation $P + P^\infty$ as follows:

- for each $A \in G$,

$$P^\infty A = \left\{(g_n \dots g_1, x) \in \sum_{B\in\mathcal{F}G} \mathcal{F}G(B,A) \times PB \;\middle|\; g_i \text{ in } G,\ Pg_1(x) \text{ undefined}\right\}$$

We denote $(g_n \dots g_1, x)$, with $Pg_1(x)$ not defined, by $g_n \dots g_1 x$. Now define

$$(P + P^\infty)A = PA + P^\infty A.$$

- for each $h\colon A \to B$ in G, define

$$(P + P^\infty)h\colon (P + P^\infty)A \to (P + P^\infty)B$$

by

$$x \longmapsto \begin{cases} Ph(x) & \text{if defined} \\ hx & \text{otherwise} \end{cases}$$

$$g_n \dots g_1 x \longmapsto h g_n \dots g_1 x.$$

The following proposition makes the connection between these two completions:

PROPOSITION 1.1 Given a presentation P on a graph G,

$$\overline{P} \cong P + P^{\infty}$$

Proof For each $A \in G$, define $\phi_A: (P + P^{\infty})A \to \overline{P}A$ by

$$\begin{aligned} x &\longmapsto [1_A, x] \\ g_n \dots g_1 x &\longmapsto [g_n \dots g_1, x]. \end{aligned}$$

Now by (1.3) (i) and (ii) we can express any element y of $\overline{P}A$ in the form $[g_n \dots g_1, x]$ where either $Pg_1(x)$ is not defined or $g_n \dots g_1 = 1_A$. In the first case, $y = \phi_A([g_n \dots g_1 x])$ and in the second, $y = \phi_A(x)$. Thus ϕ_A is surjective.

To show that ϕ_A is injective, note that two elements u, $v \in \overline{P}A$ are equivalent iff there is a chain

$$u = u_1 \sim u_2 \sim \dots \sim u_n = v$$

where each step of the chain is of the form

$$u_i = (g_n \dots g_1 g, x) \sim (g_n \dots g_1, y) = u_{i+1}$$

where $x \xrightarrow{g} y$ in P, or the reverse of this step.

We may assume that throughout such a chain, the length of the path in the first component of each term is either strictly increasing or strictly decreasing, since the only possible change from decreasing to increasing occurs as

$$(g_n \dots g_1 g, x) \sim (g_n \dots g_1, y) \sim (g_n \dots g_1 g, x)$$

where $x \xrightarrow{g} y$ in P since Pg is a partial function. Such a pair of steps may be removed from the chain. The same applies to changes from increasing to decreasing.

Thus $(1, x) \sim (1, y)$ implies $x = y$ and also if $(g_n \dots g_1, x) \sim (h_m \dots h_1, y)$ with $Pg_1(x)$ undefined then $n \le m$. Thus if $Pg_1(x)$ and $Ph_1(y)$ are both undefined, it clearly follows that $m = n$ and $g_i = h_i$ for $i = 1, \dots, n$ and $x = y$. Hence the ϕ_A are injective and thus isomorphisms.

We now check naturality of ϕ.

For each $h: A \to B$ in G,

$$\begin{aligned} \overline{P}h\phi_A(x) &= \overline{P}h[1_A, x] \\ &= [h, x], \end{aligned}$$

while

$$\begin{aligned} \phi_B(P + P^{\infty})h(x) &= \begin{cases} \phi_B Ph(x) & \text{if } Ph(x) \text{ defined} \\ \phi_B(hx) & \text{otherwise} \end{cases} \\ &= \begin{cases} [1_A, Ph(x)] = [h, x] & \text{if } Ph(x) \text{ defined} \\ [h, x] & \text{otherwise} \end{cases} \end{aligned}$$

and

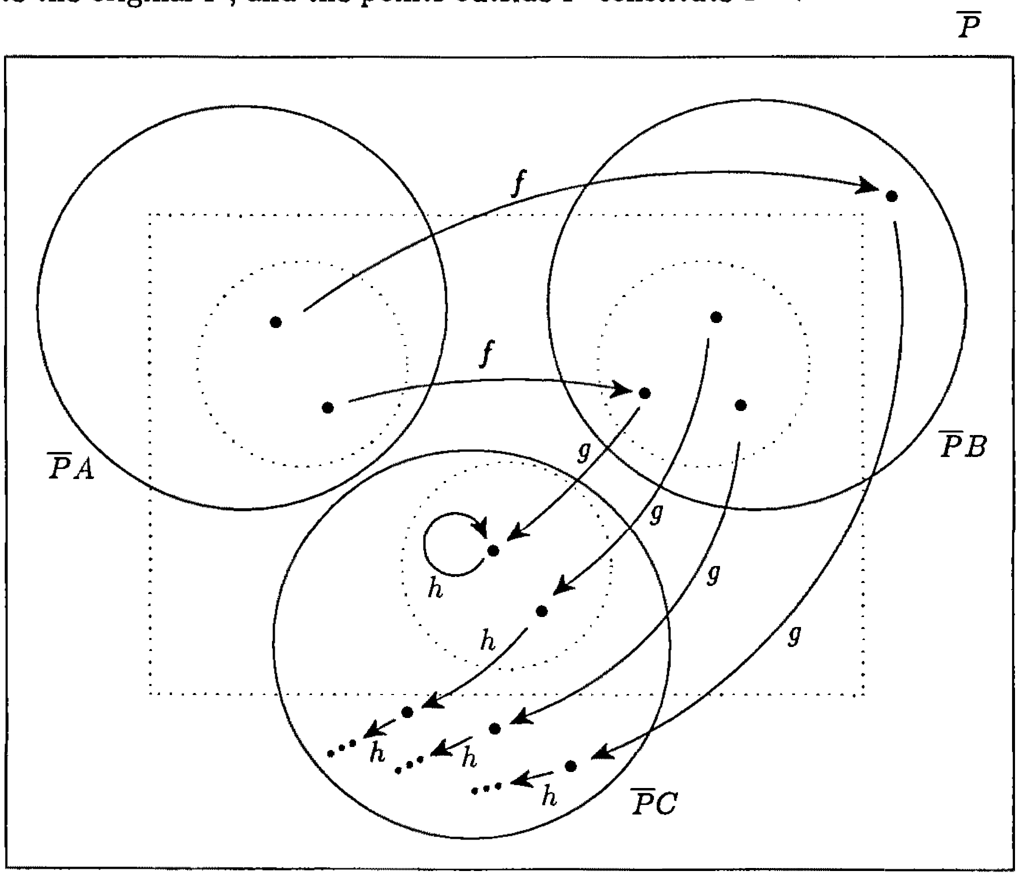
$$\begin{aligned} \overline{P}h\phi_A(g_n \dots g_1 x) &= \overline{P}h[g_n \dots g_1, x] \\ &= [hg_n \dots g_1, x] \end{aligned}$$

while

$$\begin{aligned}\phi_B(P + P^\infty)h(g_n \ldots g_1 x) &= \phi_B h g_n \ldots g_1 x \\ &= [hg_n \ldots g_1, x].\end{aligned}$$

□

We can now illustrate $P + P^\infty$ and hence $\overline{P}$ with an example. We think of P^∞ as adding necessary extra elements to P so as to ensure it is complete. The dotted lines indicate the original P, and the points outside P constitute P^∞.



The first simple property of the completion process is the following:

Proposition 1.2 Given a presentation P on a graph G,

$$P \text{ complete } \Rightarrow \overline{P} \cong P.$$

Proof

$$\begin{aligned} P \text{ complete} &\Rightarrow P^\infty A = \emptyset \qquad \text{for all } A \in G \\ &\Rightarrow P + P^\infty = P \end{aligned}$$

□

§2 Quotients of Presentations

DEFINITION 2.1 Given a presentation P on a graph G and a family of equivalence relations $\sim_A$ on PA for each $A \in G$, we say $\sim$ is closed under P if for all g in G and x, $x' \in PA$,

$$x \sim x' \Rightarrow Pg(x) \sim Pg(x') \quad \text{when both } Pg(x) \text{ and } Pg(x') \text{ are defined.}$$

DEFINITION 2.2 A presentation with **equations** consists of a presentation P on a graph G and for each $A \in G$ sets $SA \subseteq (PA)^2$. We refer to these sets collectively as the *equations* S.

Notation: If $(x, x') \in SA$, we write $x - x'$ in S.

DEFINITION 2.3 Given a presentation P on a graph G with equations S, we define the **quotient presentation** P/S as follows:

- for all $A \in G$, define

$$(P/S)A = PA/\sim_A$$

where $\sim$ is the smallest equivalence relations on P such that

$$x - x' \Rightarrow x \sim x'$$

and $\sim$ is closed under P.

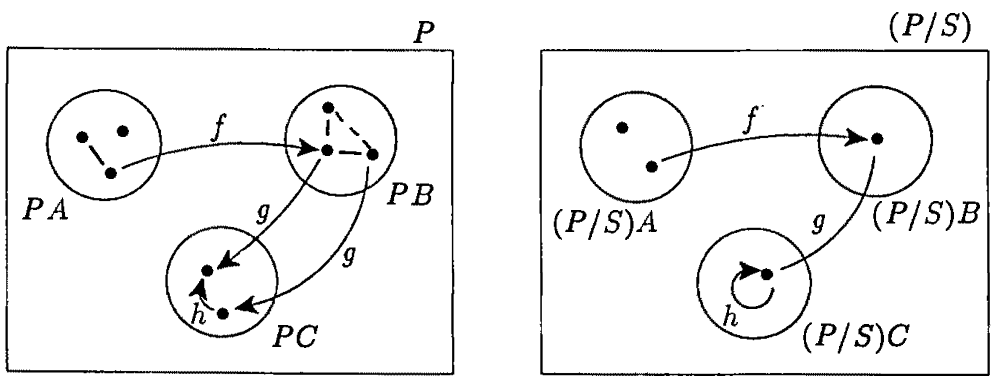
- for all $g: A \to B$ in G, define

$$(P/S)g = \{([x], [y]) \mid x \xrightarrow{g} y \text{ in } P\}$$

where $[\ ]$ denotes equivalence class under $\sim$.

We call $\sim$ the **quotient equivalence relation** induced by S on P.

We illustrate the quotient process diagrammatically. The equations S are indicated by a solid line, other induced equivalences by a dotted line.



PROPOSITION 2.1 Given a presentation P on a graph G and equations S, if P is complete then (P/S) is complete.

Proof Since the quotient equivalence relation $\sim$ is closed under P,

$$x \sim x' \Rightarrow Pg(x) \sim Pg(x')$$

Hence for each $g: A \to B$ in G,

$$(P/S)g : (P/S)A \to (P/S)B : [x] \longmapsto [Pg(x)]$$

is a well-defined *function*. □

Our first main result justifies computing with finite presentations.

THEOREM 1 Given a presentation P on a graph G and equations S,

$$\overline{(P/S)} \cong (\overline{P}/S).$$

Proof We first construct a closed equivalence relation on $\overline{P}$.

Let [ ] denote equivalence class under the quotient equivalence relation induced on P by S. We now describe a subdivision of P^∞:

• for each $A \in G$, let

$$QA = \{g_n \dots g_1 x \in P^\infty A \mid (P/S)g_n \dots (P/S)g_1[x] \text{ defined}\}$$

$$Q^\infty A = \{g_n \dots g_1 x \in P^\infty A \mid (P/S)g_n \dots (P/S)g_1[x] \textit{ not } \text{defined}\}.$$

We now partition the sets $\overline{P}A$ as follows:

• for each $A \in G$, $y \in PA$, let

$$[y]^e = [y] \cup \{g_n \dots g_1 x \in QA \mid (P/S)g_n \dots (P/S)g_1[x] = [y]\}$$

which partitions $PA + QA$ and now we partition $Q^\infty A$:
Given $g_n \dots g_1 x \in Q^\infty A$, if

$$[y] = (P/S)g_i \dots (P/S)g_1[x] \text{ is defined,}$$

but

$$(P/S)g_{i+1}[y] \text{ is } \textit{not} \text{ defined}$$

we place $g_n \dots g_1 x$ in a set denoted

$$g_n \dots g_{i+1}[y]^e.$$

This clearly partitions each set $\overline{P}A$. We let $\approx$ denote the equivalence relation corresponding to this partition. We claim this equivalence relation is closed under $\overline{P}$. First we consider the effect of $\overline{P}g$ on elements of $[y]^e \subseteq \overline{P}A$ where $g: A \to B$. This breaks into two cases:

(i) $(P/S)g[y]$ defined, then if $z \in [y]$

(a) if $Pg(z)$ defined then

$$\begin{aligned}\overline{P}g(z) = Pg(z) &\in (P/S)g[y] \\ &\subseteq ((P/S)g[y])^{e}\end{aligned}$$

(b) if $Pg(z)$ not defined then $\overline{P}g(z) \in QA$ and

$$\overline{P}g(z) \in ((P/S)g[y])^{e},$$

whereas if $z \in [y]^{e} \setminus [y]$, then clearly

$$\overline{P}g(z) \in ((P/S)g[y])^{e}.$$

(ii) $(P/S)g[y]$ not defined, then clearly for any $z \in [y]^{e}$,

$$\overline{P}g(z) \in g[y]^{e}.$$

Now considering $g_n \dots g_1 x \in Q^{\infty}A$ in $g_n \dots g_{i+1}[y]^{e}$, clearly

$$\overline{P}g(g_n \dots g_1 x) = g g_n \dots g_1 x \in g g_n \dots g_{i+1}[y]^{e}.$$

Thus $\approx$ is indeed closed under $\overline{P}$. It is clear elements equivalent under $\approx$ are also equivalent under the closed equivalence relation induced by S on $\overline{P}$ and hence

$$(\overline{P}/\approx) \cong (\overline{P}/S).$$

There is, however an obvious isomorphism

$$\phi_A \colon (\overline{P}A/\approx) \to \overline{(P/S)}A$$

given by

$$g_n \dots g_1[y]^{e} \longmapsto g_n \dots g_1[y]$$

which is clearly natural. Thus

$$\overline{(P/S)} \cong (\overline{P}/S).$$

□

§3 The Quotient Procedure

We now describe a procedure to calculate quotients of the form $\overline{P}/S$. Given a presentation P on a graph G and equations S, the procedure specifies a way to modify both P and S which preserves $\overline{P/S}$ and hence $\overline{P}/S$ by THEOREM 1. We begin with an equation $u — v$ in S with $u \neq v$ and then the modification is carried out in two steps, producing P_1, S_1 and then P_2, S_2. The procedure terminates when every equation has the form $u — u$. When the graph G and the sets PA are finite, the procedure terminates after a finite number of steps. If we commence the procedure with P and S, terminating with Q and T, then clearly $\overline{Q/T} \cong \overline{Q}$. Further, since the procedure preserves $\overline{P/S}$ (proved in §4), it follows that $\overline{P/S} \cong \overline{Q/T}$. Therefore $\overline{P}/S \cong \overline{Q}$ and the desired $\overline{P}/S$ has effectively been calculated. We leave the analysis to section 4, and now simply describe the procedure.

The Procedure

Begin with presentation P, equations S with $u - v \in PC$ and $u \neq v$.

Step 1

For all $A \in G$ and g in G, define

$$P_1 A = PA$$
$$P_1 g(x) = (Pg(x))^\circ \quad \text{if } Pg(x) \text{ defined}$$

where

$$(\ )^\circ : PA \to P_1 A : a \longmapsto \begin{cases} a & \text{if } a \neq v \\ u & \text{if } a = v. \end{cases}$$

Also

$$S_1 A = \begin{cases} S^\circ C \cup TC \cup \{(u,v)\} & \text{if } A = C \\ S^\circ A \cup TA & \text{otherwise} \end{cases}$$

where

$$S^\circ A = \{(a^\circ, b^\circ) \mid (a,b) \in SA\}$$

and

$$TA = \{(Pg(u)^\circ, Pg(v)^\circ) \mid \text{if both defined for } g: C \to A \text{ in } G\}.$$

Step 2

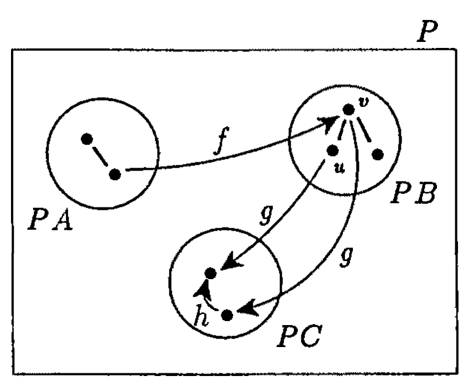
For all $A \in G$ and g in G, define

$$P_2 A = \begin{cases} P_1 C \setminus \{v\} & \text{if } A = C \\ P_1 A & \text{otherwise} \end{cases}$$
$$P_2 g = P_1 g|_{P_2 A}$$

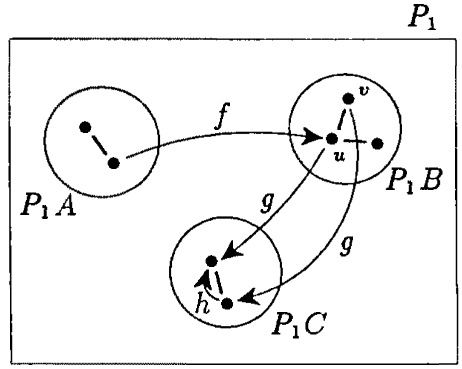
and

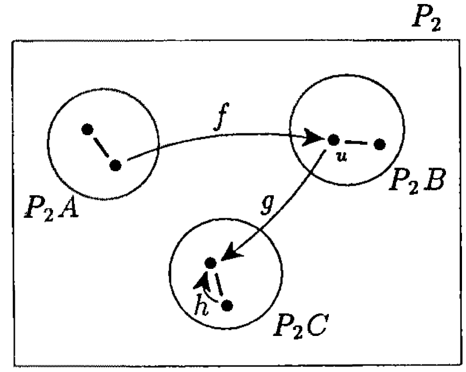
$$S_2 A = \begin{cases} S_1 C \setminus \{(u,v)\} & \text{if } A = C \\ S_1 A & \text{otherwise.} \end{cases}$$

We can illustrate this modification process graphically. If we begin with the presentation P and equations S as:



Then the modified presentations P_1 and P_2 can then be illustrated as:





§4 Analysis of the Procedure

The following two Lemmas show that both steps in the procedure preserve the quotient (P/S) at least in the case when P (and hence of course (P/S) by PROPOSITION 2.1) is complete.

LEMMA 4.1 Let P be a complete presentation on a graph G, with equations S. Given $u — v$ in S where $u, v \in PC$, with $u \neq v$, if the complete presentation P' and equations S' are defined as in **Step 1** of the quotient procedure then

$$(P/S) \cong (P'/S').$$

Proof Let $\sim_A$ denote the quotient equivalence relation E_{SA} induced by S on PA and $\sim'_A$ the quotient equivalence relation $E_{S'A}$ induced by S' on $P'A$ for each $A \in G$. Note that $u — v$ in $S \Rightarrow$

$$\begin{aligned} \text{for all } a \quad & a^\circ \sim a \\ \text{and} \quad & a^\circ \sim' a. \end{aligned} \qquad (1)$$

Hence, using transitivity,

$$\begin{aligned} a \sim b &\iff a^\circ \sim b^\circ \\ a \sim' b &\iff a^\circ \sim' b^\circ. \end{aligned} \qquad (2)$$

Now

$$\begin{aligned} a — b \text{ in } S &\Rightarrow a \sim b \\ &\Rightarrow a^\circ \sim b^\circ. \end{aligned}$$

Hence $S^\circ A \subseteq E_{SA}$.
Further, for $a, b \in PA$,

$$\begin{aligned} a \sim b &\Rightarrow Pg(a) \sim Pg(b) && \text{whenever dom } g = A \text{ in } G \\ &\Rightarrow Pg(a)^\circ \sim Pg(b)^\circ && \text{by (2)} \\ &\Rightarrow P'g(a) \sim P'g(b). \end{aligned}$$

Thus $\sim$ is closed under P' and in particular, $P'g(u) \sim P'g(v)$ so $TA \subseteq E_{SA}$ for all $A \in G$.
Of course $(u, v) \in E_{SC}$.
Thus $S'A \subseteq E_{SA}$ for all $A \in G$.
Since $\sim$ is closed under P', but $\sim'$ is the smallest such equivalence relation,

$$E_{S'A} \subseteq E_{SA} \quad \text{for all } A \in G$$

Conversely,

$$\begin{aligned} a — b \text{ in } S &\Rightarrow a^\circ — b^\circ \text{ in } S^\circ \\ &\Rightarrow a^\circ — b^\circ \text{ in } S' \\ &\Rightarrow a^\circ \sim' b^\circ \\ &\Rightarrow a \sim' b \qquad \text{by (2)} \end{aligned}$$

Hence $SA \subseteq E_{S'A}$.
Further, for a, $b \in P'A$

$$\begin{aligned} a \sim' b &\Rightarrow P'g(a) \sim' P'g(b) \qquad \text{whenever } \operatorname{dom} g = A \text{ in } G \\ &\Rightarrow Pg(a)^\circ \sim' Pg(b)^\circ \\ &\Rightarrow Pg(a) \sim' Pg(b) \qquad \text{by (2)} \end{aligned}$$

so $\sim'$ is closed under P, thus $E_{SA} \subseteq E_{S'A}$ for all $A \in G$.
Hence

$$E_{SA} = E_{S'A} \quad \text{for all } A \in G.$$

Thus we have isomorphisms

$$\phi_A : (P/S)A \to (P'/S')A : [x] \longmapsto [x].$$

To ensure naturality of ϕ we require

$$(P/S)g[x] = (P'/S')g[x] \qquad \text{for all } g \text{ in } G$$

But

$$\begin{aligned} (P/S)g[x] &= [Pg(x)] \\ &= [(Pg(x))^\circ] \\ &= [P'g(x)] \\ &= (P'/S')g[x] \end{aligned}$$

$\square$

DEFINITION 4.1 Let P be a complete presentation on a graph G, with equations S such that $u — v$ in S, where u, $v \in PC$ say, and $u \neq v$. We say that v is removable via u if
(i) for all g in G with domain C,

$$Pg(u) — Pg(v) \text{ in S}$$

(ii) for all h in G with codomain C,

$$v \notin \operatorname{im} Ph$$

(iii) for all x — y in S with $(x,y) \neq (u,v)$,

$$x, y \neq v.$$

LEMMA 4.2 Given a presentation P on a graph G and coincidences S with v removable via u with u, $v \in PC$, if we define a complete presentation P' and coincidences S' as follows:
for all $A \in G$,

$$P'A = \begin{cases} PC \setminus \{v\} & \text{if } A = C \\ PA & \text{otherwise} \end{cases}$$

for all g in G,

$$P'g = Pg|_{P'A}$$

and

$$S'A = \begin{cases} SC \setminus \{(u,v)\} & \text{if } A = C \\ SA & \text{otherwise} \end{cases}$$

then

$$(P/S) \cong (P'/S')$$

Proof Note that hypothesis (iii) ensures that $S'A \subseteq (P'A)^2$ for all A in P'.

Let $\sim_A$ denote the quotient equivalence relation E_{SA} induced by S on P and $\sim'_A$ the quotient equivalence relation $E_{S'A}$ induced by S' on P' for each $A \in G$.

We define operators e (extension) and c (contraction) between equivalence relations on P and equivalence relations on P' as follows:
Given equivalence relations E_A on PA and $E'A$ on $P'A$ for each $A \in G$,

$$E_A^c = E_A \setminus \{(a,b) \mid a \text{ or } b = v\}$$

$$E'^e_A = E'_A \cup \{(v,a),(a,v) \mid (a,u) \in E'_A\}$$

Note that E_A^c and $E'_A{}^e$ are clearly themselves equivalence relations, and that e and c both preserve inclusions.
Note that $E_A^{ce} = E_A$ and $E'_A{}^{ec} = E'_A$, hence in particular

$$E_{SA}^{ce} = E_{SA}$$
$$E_{S'A}^{ec} = E_{S'A}$$

Now $S'A \subseteq SA$ so $S'A \subseteq E_{SA}^c$.

Also, for all $g: A \to B$ in G,

$$\begin{aligned}(a,b) \in E^{c}_{SA} &\Rightarrow (a,b) \in E_{SA} \\ &\Rightarrow a \sim b \\ &\Rightarrow Pg(a) \sim Pg(b) \\ &\Rightarrow P'g(a) \sim P'g(b) \\ &\Rightarrow (P'g(a), P'g(b)) \in E^{c}_{SB} \quad \text{by property (ii).}\end{aligned}$$

Thus E^{c}_{SA} is closed under P' for each $A \in G$, hence

$$E_{S'A} \subseteq E^{c}_{SA} \text{ and so } E^{e}_{S'A} \subseteq E^{ce}_{SA} = E_{SA}$$

We now establish the reverse inclusions:
Firstly, $SA \subseteq E^{e}_{S'A}$ since $S'A \subseteq E^{e}_{S'A}$ and e adds (u,v) to $E_{S'C}$, as $(u,u) \in E_{S'C}$.
Now consider $(a,b) \in E^{e}_{S'A}$ and $g: A \to B$ in G:

If $a,b \neq v$ then $(a,b) \in E_{S'A}$
so $(Pg(a), Pg(b)) \in E_{S'B} \subseteq E^{e}_{S'B}$.

If $a = v$ but $b \neq v$, then by definition of e, we have $b \sim' u$.
and,

$$\begin{aligned}b \sim' u &\Rightarrow P'g(b) \sim' P'g(u) \\ &\Rightarrow Pg(b) \sim' Pg(u) \quad \text{by definition of } P'g.\end{aligned}$$

Also, it follows from (i) and (ii) that $Pg(u) — Pg(v)$ is in S' and hence

$$\begin{aligned}Pg(u) \sim' Pg(v) = Pg(a) &\Rightarrow Pg(a) \sim' Pg(b) \\ &\Rightarrow (Pg(a), Pg(b)) \in E^{e}_{S'B}.\end{aligned}$$

If $a = b = v$ then trivially $(Pg(a), Pg(b)) \in E^{e}_{S'B}$.
Thus $E^{e}_{S'}$ is closed under P, which means for all $A \in G$,

$$E_{SA} \subseteq E^{e}_{S'A} \text{ and so } E^{c}_{SA} \subseteq E^{ec}_{S'A} = E_{S'A}$$

Hence

$$E_{SA} = E^{e}_{S'A} \text{ and } E_{S'A} = E^{c}_{SA}$$

This allows us to consider the well-defined isomorphisms

$$\phi_A : (P'/S')A \to (P/S)A : [x]' \longmapsto [x]$$

where [ ] and [ ]$'$ represent equivalence class under $\sim$ and $\sim'$ respectively.

Then ϕ is natural since for all $g: A_1 \to A_2$ in G,

$$\begin{aligned}(P/S)g\phi_{A_1}[x]' &= (P/S)g[x] \\ &= [Pg(x)] \\ &= \phi_{A_2}[Pg(x)]' \\ &= \phi_{A_2}[P'g(x)]' \\ &= \phi_{A_2}(P'/S')g[x]'.\end{aligned}$$

□

LEMMA 4.3 Given a complete presentation P and coincidences S on the graph G, and producing a presentation P' and equations S' using the quotient procedure, it follows that

$$(P/S) \cong (P'/S').$$

Proof We adopt the notation of §3. Noting that the definitions of $^\circ$ and T in the procedure ensure that v is removable via u in P_1 and S_1,

$$\begin{aligned}(P/S) &\cong (P_1/S_1) \quad \text{by LEMMA 4.2} \\ (P_1/S_1) &\cong (P_2/S_2) \quad \text{by LEMMA 4.3.} \\ \text{But} \quad (P_2/S_2) &= (P'/S').\end{aligned}$$

□

LEMMA 4.4 Given a presentation P and equations S on the graph G, if we produce a presentation P' and equations S' by applying the quotient procedure to P and S, and a presentation $\overline{P}'$ and equations $\mathcal{S}$ using the same procedure on the $\overline{P}$ and S, it follows that

$$(\overline{P'}/S') \cong (\overline{P}'/\mathcal{S})$$

Proof In **Step 1**, we produce P_1,S_1 and $(\overline{P})_1$, $\mathcal{S}_1$.
Clearly $\overline{P_1} = (\overline{P})_1$ and for each $A \in G_B$,

$$\mathcal{S}_1 A = S_1 A \cup \{(\overline{P}g(u), \overline{P}g(v)) \mid Pg(u) \text{ or } Pg(v) \text{ undefined, for } \operatorname{cod} g = A\}$$

The pairs in the last set are required to be equivalent to obtain a *closed* equivalence relation since $u — v$, thus

$$(\overline{P})_1/\mathcal{S}_1 \cong (\overline{P})_1/S_1$$

and so

$$(\overline{P})_1/\mathcal{S}_1 \cong \overline{P_1}/S_1.$$

Following **Step 2** we also have

$$\mathcal{S}_2 A = S_2 A \cup \{(\overline{P}g(u), \overline{P}g(v)) \mid Pg(u) \text{ or } Pg(v) \text{ undefined, for } \operatorname{cod} g = A\}$$

and further,

$$(\overline{P})_2 A = \overline{P_2} A \cup \{g_n \dots g_1 v \mid Pg_1(v) \text{ undefined, } \operatorname{cod} g_n = A\}$$

All the terms $g_n \dots g_1 v$ in the last set will, however be made equivalent to $\overline{P}g_n \dots \overline{P}g_1(u)$ by the expression given above for S_2. Hence

$$(\overline{P})_2/S_2 \cong \overline{P_2}/S_2.$$

from which the result follows. □

THEOREM 2 Given a presentation P and equations S on the graph G, and producing a presentation P' and equations S' using the quotient procedure, it follows that

$$\overline{(P/S)} \cong \overline{(P'/S')}.$$

Proof We use the notation of the previous lemma, and

$$\begin{aligned}
\overline{(P/S)} &\cong (\overline{P}/S) && \text{by THEOREM 1} \\
&\cong (\overline{P}'/S) && \text{by LEMMA 4.3} \\
&\cong (\overline{P'}/S') && \text{by LEMMA 4.4} \\
&\cong \overline{(P'/S')} && \text{by THEOREM 1.}
\end{aligned}$$

□

We conclude the analysis of the procedure by considering the question of termination.

THEOREM 3 The procedure terminates after a finite number of steps.

Proof Each time the two steps of the procedure are carried out, one of the PA decreases in size by one. Since $(PA)_{A\in G}$ is a finite family of finite sets, the procedure must terminate after a finite number of steps. □

§5 Left Kan Extensions

The procedure for computing left Kan extensions described in [1] in the style of the traditional Todd-Coxeter procedure for Coset Enumeration (filling out tables etc.) deals with the following problem: given a graph G_A which generates a category $\mathbf{A}$ and a graph G_B along with equations which determine a category $\mathbf{B}$, and functors $F\colon \mathbf{A} \to \mathbf{B}$ and $X\colon \mathbf{A} \to \mathbf{Set}$ given on generators (i.e. on the graphs), compute the values of the left Kan extension $\mathrm{Lan}_F X$ on the generators of $\mathbf{B}$.

The left Kan extension procedure begins with a presentation on G_B given by

$$PB = \sum_{\substack{A\in \mathbf{A} \text{ s.t.} \\ FA=B}} XA \qquad \text{for all } B \in G_B$$

and

$$Pg = \emptyset \qquad \text{for all } g \text{ in } G_B$$

For this presentation,

$$\overline{P}B = \sum_{A \in \mathbf{A}} \mathcal{F}G_B(FA, B) \times XA$$

Thus it is clear that if S consists of the equations derived from relations between generators of B and ones of the form

$$(1_{FA_2}, Xf(x)) — (Ff, x) \quad \text{for each } f: A_1 \to A_2$$

then the quotient $\overline{P}/S$ is simply the coend formula for left Kan extensions, so

$$\begin{aligned} \overline{P}/S &\cong \int^A \mathbf{B}(FA, \_) \cdot XA \\ &\cong \mathrm{Lan}_F X. \end{aligned}$$

The procedure for calculating $\mathrm{Lan}_F X$ is to calculate $\overline{P}/S$ by progressively forming quotients of $\overline{P}$ by subsets of S using the procedure described in this paper. If done in a sufficiently systematically manner, this procedure will in fact terminate to give $\mathrm{Lan}_F X$ on the generators of B in the case when the sets $\mathrm{Lan}_F XB$ are finite for each $B \in \mathbf{B}$. This has a simple connection to the traditional Todd-Coxeter procedure for Coset Enumeration. If we have groups $H \leq G$ and consider them as categories **H** and **G** with one object $*$ with inclusion functor I and have the trivial one-point action $T: \mathbf{H} \to \mathbf{Set}$ then $\mathrm{Lan}_I T(*) \cong G/H$. That is to say, the left Kan extension is simply the action of G on its (left) cosets given by premultiplication.

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This paper is in final form and will not be published elsewhere.