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Coinverters and Categories of Fractions for Categories with Structure

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Abstract. A category of fractions is a special case of a *coinverter* in the 2-category **Cat**. We observe that, in a cartesian closed 2-category, the product of two *reflexive* coinverter diagrams is another such diagram. It follows that an equational structure on a category $\mathbf{A}$, if given by operations $\mathbf{A}^n \to \mathbf{A}$ for $n \in \mathbf{N}$ along with natural transformations and equations, passes canonically to the category $\mathbf{A}[\Sigma^{-1}]$ of fractions, provided that Σ is closed under the operations. We exhibit categories with such structures as algebras for a class of 2-monads on **Cat**, to be called *strongly finitary monads.*

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1. Coinverters, Reflexive Coinverters, and Categories of Fractions

1.1. Introduction

One of the most important constructions constructions in mathematics is the formation of quotients. For sets we factor out by an equivalence relation. For sets with structure we factor out by a congruence: that is, by an equivalence relation respecting the structure. For a category, where the objects have in themselves more tenuous individuality, it is often rather a question of universally inverting certain arrows; this process is called forming a *category of fractions* [4]. There is a more general notion, that of a *coinverter*: namely the functor universal among those whose composite with a given natural transformation is invertible. When the natural transformation is in some 2-category of categories with extra structure, we may ask how the coinverter behaves with respect to this extra structure. When the extra structure consists in the category's possessing finite limits, or being regular or abelian, such questions have been dealt with in [4] and in more detail in [1]; our present concern is rather that where the extra structure is equational, and given by the action of the kind of 2-monad that we shall call *strongly finitary*. In such cases we show that the category of fractions $\mathbf{A}[\Sigma^{-1}]$ has itself a structure of the given kind, provided that Σ contains the identities and is closed under the operations, and

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that then $\mathbf{A}[\Sigma^{-1}]$ is the "category of fractions" in the 2-category of structures. The results of this paper contain as a special case those of [3].

1.2. Coinverters and Categories of Fractions

We start by recalling the definition of a category of fractions; see for example [4].

DEFINITION 1.1. Given a collection Σ of arrows in a category $\mathbf{A}$, we say that a functor $P_\Sigma : \mathbf{A} \to \mathbf{A}[\Sigma^{-1}]$ exhibits $\mathbf{A}[\Sigma^{-1}]$ as the *category of fractions* of $\mathbf{A}$ for Σ if P_Σ induces, for each category $\mathbf{B}$, an *isomorphism* of categories between the functor category $[\mathbf{A}[\Sigma^{-1}], \mathbf{B}]$ and the full subcategory of $[\mathbf{A}, \mathbf{B}]$ determined by those functors which invert the arrows in Σ.

Categories of fractions do exist in the 2-category **Cat** of categories in some given universe; we recall from [4, pp. 6–7] that $\mathbf{A}[\Sigma^{-1}]$ may be constructed as follows. Let $\mathbf{G}$ be the graph whose objects are those of $\mathbf{A}$ and whose arrows are of two kinds: for each arrow a of $\mathbf{A}$ there is an arrow a^* of $\mathbf{G}$, and for each $f \in \Sigma$ there is an arrow f^{-1} of $\mathbf{G}$. Then $\mathbf{A}[\Sigma^{-1}]$ is the category generated by $\mathbf{G}$ subject to the relations $(ab)^* = a^*b^*$, $(1_A)^* = 1_A$, $f^*f^{-1} = 1$, and $f^{-1}f^* = 1$.

It is convenient to consider a more general notion, that of a *coinverter* [6]. This can be defined in an arbitrary 2-category.

DEFINITION 1.2. In a 2-category $\mathcal{A}$, a 1-cell $p : A \to C$ is said to be the coinverter of a 2-cell $\alpha : s \to t : S \to A$ if p induces an isomorphism of categories between $\mathcal{A}(C, B)$ and the full subcategory of $\mathcal{A}(A, B)$ given by those 1-cells q which "coinvert" α; that is, for which $q\alpha$ is invertible.

Briefly, the coinverter is the 1-cell universal among those which coinvert α; sometimes it is the object C that is loosely called the coinverter. Coinverters are particular finite weighted colimits (see [6]), and exist in any finitely cocomplete 2-category – in particular, in **Cat**. Coinverters in **Cat** include categories of fractions: given a collection Σ of arrows in $\mathbf{A}$ to be inverted, let S be the discrete category of these arrows, let s send each arrow to its domain, and t to its codomain, and let the component of α at an arrow be that arrow; then the coinverter is just P_Σ.

1.3. Reflexive Coinverters

The particular type of coinverter in which we are interested is called a *reflexive coinverter.*

DEFINITION 1.3. A 2-cell $\alpha : s \to t : S \to A$ in a 2-category $\mathcal{A}$ is said to be *reflexive* if there is a 1-cell $i : A \to S$ such that $si = ti = 1_A$ and $\alpha i = id$.

The coinverter of a 2-cell is, by an abuse of language, said to be *reflexive* if the 2-cell is reflexive.

Although reflexivity is a strong condition on the 2-cell α, nevertheless (if coproducts exist) every coinverter may be expressed, canonically, as the coinverter of a reflexive 2-cell.

LEMMA 1.4. *In a 2-category with binary coproducts, $p : A \to B$ is the coinverter of $\alpha : s \to t : S \to A$ if and only if it is the coinverter of the reflexive 2-cell $(id, \alpha) : (1_A, s) \to (1_A, t) : A + S \to A$.*

Proof. Clearly any $f : A \to C$ inverts α just when it inverts $(1_A, \alpha)$, and the injection $i_A : A \to A + S$ is a reflexion for (id, α). □

REMARK 1.5. In the case of categories of fractions, this lemma corresponds to the statement that $\mathbf{A}[\Sigma^{-1}] \cong \mathbf{A}[\Sigma_1^{-1}]$, where Σ_1 is the union of Σ and the collection of identities of $\mathbf{A}$.

2. The 3×3 Lemma

There is a well-known lemma [5, p. 16] which says that given a 3×3 diagram commuting in the obvious way, in which the rows and columns are coequalizer diagrams, with the pairs coequalized in the first row and first column being reflexive, the diagonal is a coequalizer diagram. Coinverters are a 2-dimensional generalization of coequalizers, and in this section we generalize the lemma to deal with reflexive coinverters in an arbitrary 2-category.

Consider in a 2-category a diagram

$$\begin{tikzcd}
A_{11} \arrow[r, bend left, "S_1"] \arrow[r, bend right, "T_1"'] \arrow[r, phantom, "\Downarrow \alpha_1"] \arrow[d, bend right, "M_1"'] \arrow[d, bend left, "N_1"] \arrow[d, phantom, "\overset{\beta_1}{\Rightarrow}"] & A_{12} \arrow[r, "P_1"] \arrow[d, bend right, "M_2"'] \arrow[d, bend left, "N_2"] \arrow[d, phantom, "\overset{\beta_2}{\Rightarrow}"] & A_{13} \arrow[d, bend right, "M_3"'] \arrow[d, bend left, "N_3"] \arrow[d, phantom, "\overset{\beta_3}{\Rightarrow}"] \\
A_{21} \arrow[r, bend left, "S_2"] \arrow[r, bend right, "T_2"'] \arrow[r, phantom, "\Downarrow \alpha_2"] \arrow[d, "Q_1"'] & A_{22} \arrow[r, "P_2"] \arrow[d, "Q_2"'] & A_{23} \arrow[d, "Q_3"'] \\
A_{31} \arrow[r, bend left, "S_3"] \arrow[r, bend right, "T_3"'] \arrow[r, phantom, "\Downarrow \alpha_3"] & A_{32} \arrow[r, "P_3"] & A_{33},
\end{tikzcd}$$

built as follows. We suppose that

$$M_2\alpha_1 = \alpha_2 M_1,\ N_2\alpha_1 = \alpha_2 N_1,\ S_2\beta_1 = \beta_2 S_1,\ T_2\beta_1 = \beta_2 T_1\ ,$$

in which are of course implicit the equalities of the various domains and codomains of these 2-cells, such as $M_2S_1 = S_2M_1$; and from which follows the equality $\beta_2\alpha_1 = \alpha_2\beta_1$. We suppose further that P_1, P_2, Q_1, Q_2 are the coinverters of $\alpha_1, \alpha_2, \beta_1, \beta_2$ respectively; whereupon M_3, N_3, and β_3 are the maps and the 2-cell induced in the evident way, as are S_3, T_3, and α_3. Finally let P_3 be the coinverter of α_3, and Q_3 the induced map; it is easy to see that Q_3 is then the coinverter of β_3.

LEMMA 2.1. *Suppose that, in the circumstances above, the 2-cells α_1 and β_1 are reflexive. Then Q_3P_2 $(= P_3Q_2)$ is the coinverter of $\beta_2\alpha_1$ $(= \alpha_2\beta_1)$.*

Proof. Let $I : A_{12} \to A_{11}$ and $J : A_{21} \to A_{11}$ be the reflexions. Suppose that $F : A_{22} \to T$ coinverts $\beta_2\alpha_1$. Then also F coinverts $\beta_2\alpha_1 I = \beta_2$, and so F factorizes uniquely as $F = F_1Q_2$. Now F_1 coinverts $Q_2\beta_2\alpha_1 = Q_2\alpha_2\beta_1$, and so also coinverts $Q_2\alpha_2\beta_1 J = Q_2\alpha_2 = \alpha_3 Q_1$. But Q_1 is a coinverter, and so by the two-dimensional part of the universal property of coinverters, F_1 coinverts α_3. Thus F_1 factorizes uniquely as $F'P_3$. We now have a unique factorization $F = F'P_3Q_2$. Thus P_3Q_2 is the coinverter, since the two-dimensional part of the universal property is immediate. □

The major application of the 3×3 lemma is the following result.

COROLLARY 2.2. *If p and q are the coinverters of reflexive 2-cells $\rho : s \to t : A \to B$ and $\sigma : u \to v : D \to E$ in a cartesian closed 2-category, then $p \times q$ is the coinverter of $\rho \times \sigma$.*

Proof. Apply the above lemma starting with $\alpha_1 = \rho \times id_D$, $\alpha_2 = \rho \times id_E$, $\beta_1 = id_A \times \sigma$, and $\beta_2 = id_B \times \sigma$. □

REMARK 2.3. The same thing clearly applied in a monoidal 2-category, with $p \otimes q$ replacing $p \times q$, provided that each $A \otimes -$ and each $- \otimes B$ preserves colimits.

REMARK 2.4. Given coinverters as in Corollary 2.2, which are not necessarily reflexive, and converting them as before into reflexive ones, we see that $p \times q$ is the coinverter of $(id_A, \rho) \times (id_D, \sigma)$.

3. Categories with Structure

In this section we work, for concreteness, with categories of fractions, and consider structures given by a collection of operations of the form $\mathbf{A}^n \to \mathbf{A}$, along with natural transformations and equations. In the next section, where the formal result is given, we return to the more general coinverters, and use the precise description of such structures given in the preceding article [7] – namely, algebras for a strongly-finitary 2-monad on **Cat**.

3.1. Categories with n-ary operations and equations

In the case of categories of fractions, we are working in **Cat**, which is cartesian closed. To say, in the situation of Remark 2.4, that $p \times q$ is the coinverter of $(id_A, \rho) \times (id_D, \sigma)$ is, loosely, to say that $\mathbf{A}[\Sigma^{-1}] \times \mathbf{B}[\Omega^{-1}] \cong (\mathbf{A} \times \mathbf{B})\,[\Sigma_1^{-1} \times \Omega_1^{-1}]$, where Σ_1 is Σ together with the identities of $\mathbf{A}$, and likewise Ω_1. In other words, if Σ and Ω contain the identities, then $\mathbf{A}[\Sigma^{-1}] \times \mathbf{B}[\Omega^{-1}] \cong (\mathbf{A} \times \mathbf{B})\,[\Sigma^{-1} \times \Omega^{-1}]$. By induction we have $(\mathbf{A}[\Sigma^{-1}])^n \cong \mathbf{A}^n[(\Sigma^n)^{-1}]$ if Σ contains the identities. Given now a functor $a : \mathbf{A}^n \to \mathbf{A}$ that maps Σ^n into Σ, there is a unique map $a_\Sigma : (\mathbf{A}[\Sigma^{-1}])^n \to \mathbf{A}[\Sigma^{-1}]$ making commutative the diagram

$$\begin{tikzcd}
\mathbf{A}^n \arrow[r, "a"] \arrow[d, "p^n"'] & \mathbf{A} \arrow[d, "p"] \\
\mathbf{A}[\Sigma^{-1}]^n \arrow[r, "a_\Sigma"'] & \mathbf{A}[\Sigma^{-1}].
\end{tikzcd}$$

Thus if we are given a family of "basic operations" on $\mathbf{A}$ of the form $\mathbf{A}^n \to \mathbf{A}$, under which Σ is closed, then there is a canonical way of giving a family of such basic operations on $\mathbf{A}[\Sigma^{-1}]$. Similarly, if we have natural transformations between operations derived from the basic ones on $\mathbf{A}$, there is a unique way of extending them to give natural transformations between the corresponding derived operations on $\mathbf{A}[\Sigma^{-1}]$. Any equations between such natural transformations in $\mathbf{A}$ will be preserved when we move to $\mathbf{A}[\Sigma^{-1}]$. This intuitively obvious construction will be made precise later, when we state it using the theory of monads. First, however, we give a concrete example.

3.2. Categories with sums

The possession by a category $\mathbf{A}$ of (chosen) binary sums, although in some sense a *property* of $\mathbf{A}$, may be seen as a structure of the kind envisaged; as we have only to ensure that the diagonal $\Delta : \mathbf{A} \to \mathbf{A}$ has a left adjoint, we have only to posit a functor $+ : \mathbf{A} \times \mathbf{A} \to \mathbf{A}$, along with natural transformations $\epsilon : +\Delta \to 1$ and $\eta : 1 \to \Delta+$, subject to the triangular equations for an adjunction. Of course, the A-component of ϵ is the codiagonal $A + A \to A$, while the (A, B)-component of η is $(i : A \to A + B,\ j : B \to A + B)$ given by the coproduct injections.

If now we have a collection Σ of arrows in $\mathbf{A}$ such that Σ contains the identities and contains $f+g$ whenever it contains f and g, there is a unique induced functor $+_\Sigma : \mathbf{A}[\Sigma^{-1}] \times \mathbf{A}[\Sigma^{-1}] \to \mathbf{A}[\Sigma^{-1}]$. Similarly, by the 2-dimensional part of the universal property, the injections and the codiagonal pass to the quotient; and by the 2-dimensional aspect of the uniqueness property, they will satisfy the new triangular equations.

Thus we have a canonical structure of category with sums on $\mathbf{A}[\Sigma^{-1}]$. Moreover, the coinverter $p : \mathbf{A} \to \mathbf{A}[\Sigma^{-1}]$ strictly preserves the chosen sums, by the definition

$$\begin{tikzcd}
\mathbf{A} \times \mathbf{A} \arrow[r, "+"] \arrow[d, "p \times p"'] & \mathbf{A} \arrow[d, "p"] \\
\mathbf{A}[\Sigma^{-1}] \times \mathbf{A}[\Sigma^{-1}] \arrow[r, "+_\Sigma"'] & \mathbf{A}[\Sigma^{-1}]
\end{tikzcd}$$

of $+_\Sigma$.

Of course there is nothing special about sums: the construction works just as well for products, or sums and products, or tensor products, or any collection of n-ary operations along with natural transformations and equations. In the next section the extent of the applicability of the result will be made more precise.

4. Algebras for a Strongly Finitary 2-Monad

To make the above precise in its appropriate generality, we observe that the kind of structure we have in mind on a category $\mathbf{A}$ is just an action $a : T\mathbf{A} \to \mathbf{A}$ of a *strongly-finitary* 2-monad T on **Cat**, in the sense of the preceding article [7]. We recall that these are those 2-monads T whose endofunctor part is the left Kan extension of its restriction to the full subcategory $\mathbf{S}$ of **Cat** determined by the finite sets $n \in \mathbf{N}$ seen as discrete categories; which is to say that T satisfies

$$T = \int^{n \in \mathbf{S}} (-)^n \times Tn \,. \tag{4.1}$$

THEOREM 4.1. *Strongly-finitary endo-2-functors of* **Cat** *preserve reflexive coinverters.*

Proof. By the obvious extension of Corollary 2.2, the product functor $\Pi : \mathbf{Cat}^n \to \mathbf{Cat}$ preserves reflexive coinverters for $n \geq 2$; while it trivially does so for $n = 0, 1$. Since the diagonal $\Delta : \mathbf{Cat} \to \mathbf{Cat}^n$, being a left adjoint, preserves all colimits, the functor $(-)^n : \mathbf{Cat} \to \mathbf{Cat}$ preserves these coinverters. Since $- \times Tn$ preserves all colimits in **Cat**, and since colimits commute with colimits, it follows from (4.1) that a strongly-finitary T preserves such coinverters. □

Recall from [2] that T-algebras form two 2-categories $\mathbf{T\text{-}Alg}_s$ and $\mathbf{T\text{-}Alg}$, where the 1-cells in the first are strict maps of T-algebras, while those in the second preserve the structure only to within coherent isomorphism. More precisely, a map f in $\mathbf{T\text{-}Alg}$ from $(\mathbf{S}, s)$ to $(\mathbf{A}, a)$ is a functor $f : \mathbf{S} \to \mathbf{A}$ along with an invertible 2-cell $\bar{f}$ as in

$$
\begin{tikzcd}
TS \arrow[r, "Tf"] \arrow[d, "s"'] \arrow[dr, phantom, "\Downarrow \bar{f}"] & TA \arrow[d, "a"] \\
S \arrow[r, "f"'] & A
\end{tikzcd}
$$

satisfying two coherence axioms; the maps in $\mathbf{T\text{-}Alg}_s$ are those in $\mathbf{T\text{-}Alg}$ for which $\bar{f}$ is the identity. Our main result is:

THEOREM 4.2. *The forgetful functor* $U : \mathbf{T\text{-}Alg} \to \mathbf{Cat}$ *creates the coinverter of reflexive 2-cells; and these coinverters are strict maps of algebras, so that the forgetful* $U_s : \mathbf{T\text{-}Alg}_s \to \mathbf{Cat}$ *also creates such coinverters.*

Proof. Let $\alpha : (f, \bar{f}) \to (g, \bar{g}) : \mathbf{S} \to \mathbf{A}$ be a reflexive 2-cell in $\mathbf{T\text{-}Alg}$; recall from [2] that α then satisfies

$$
\begin{tikzcd}
T\mathbf{S} \arrow[r, "f"] \arrow[d, "s"'] \arrow[dr, phantom, "\Downarrow \bar{f}"] & T\mathbf{A} \arrow[d, "a"] \\
\mathbf{S} \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow \alpha"] & \mathbf{A}
\end{tikzcd}
=
\begin{tikzcd}
T\mathbf{S} \arrow[r, bend left, "Tf"] \arrow[r, bend right, "Tg"'] \arrow[r, phantom, "\Downarrow T_\alpha"] \arrow[d, "s"'] \arrow[dr, phantom, "\Downarrow \bar{g}"] & T\mathbf{A} \arrow[d, "a"] \\
\mathbf{S} \arrow[r, "g"'] & \mathbf{A}
\end{tikzcd}
\tag{4.2}
$$

Let $p : \mathbf{A} \to \mathbf{C}$ be the coinverter in $\mathbf{Cat}$ of α, which is certainly a reflexive 2-cell in $\mathbf{Cat}$. Then, since $p\alpha$ is invertible and $\bar{f}$ is an isomorphism, the composite of p with the 2-cell on the left of (4.2) is invertible; so the composite of p with the 2-cell on the right of (4.2) is so. But $\bar{g}$ is an isomorphism, so $pa.T\alpha$ is invertible. But by Theorem 4.1, the coinverter of $T\alpha$ is Tp. Thus there is a unique $c : T\mathbf{C} \to \mathbf{C}$ with $c.Tp = pa$. That this is an action of T on $\mathbf{C}$ follows easily: the necessary diagrams $T^2\mathbf{C} \to \mathbf{C}$ or $\mathbf{C} \to \mathbf{C}$ commute when preceded by T^2p or p, since a is a T-action; but T^2p and p are coinverters, hence epimorphic. Clearly p is a strict map of T-algebras. If now $(h, \bar{h}) : (A, a) \to (D, d)$ is a map of algebras with $h\alpha$ invertible, we have $h = kp$ for a unique $k : \mathbf{C} \to \mathbf{D}$; and there is a unique $\bar{k} : d.Tk \to kc$ with $\bar{k}.Tp = \bar{h}$, by the two-dimensional part of the universal property of the coinverter Tp. That $(k, \bar{k})$ is a map of T-algebras follows at once, by uniqueness again. □

We return to the special case of categories of fractions. We apply the above to $\mathbf{C} = \mathbf{A}[\Sigma^{-1}]$ where $\mathbf{A}$ is a T-algebra and Σ a set of morphisms in $\mathbf{A}$, such that Σ contains the identities and is closed under the basic operations $\mathbf{A}^n \to \mathbf{A}$ involved, in the sense of [7], in a presentation of T. We conclude that $\mathbf{A}[\Sigma^{-1}]$ has a canonical T-algebra structure, that $p : \mathbf{A} \to \mathbf{A}[\Sigma^{-1}]$ is a strict map of T-algebras, and that $\mathbf{A}[\Sigma^{-1}]$ is the "category of fractions" in each of the 2-categories **T-Alg** and **T-Alg**$_s$.

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