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FIX-POINT SEMANTICS FOR PROGRAMS IN DISTRIBUTIVE CATEGORIES

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Abstract. In an earlier paper Walters introduced a family of imperative programming languages based on concepts from distributive categories. The development in that, and in a subsequent paper by Walters and Khalil, was carried out within the category **Set** of sets and total functions. In this paper we investigate the generalization of the results of those earlier papers to more general distributive categories using fix-point constructions to replace the specific calculations used earlier.

1. Introduction

A family of imperative languages based on distributive categories was introduced by Walters in [9]. Some of the programming constructs used, in writing programs, in the family of languages were identified and analyzed in the distributive category, **Set**, of sets and functions between them, in [5]. The main programming construct used is a program called a *pseudofunction* or *functional processor.*

A pseudofunction, $e : X \longrightarrow\!\!\!\!\!\!+\!\!> Y$, with local states U, is a function of the form $e : X+U+Y \longrightarrow X+U+Y$ in **Set** with the property that $e \circ k = k$, where $k : Y \longrightarrow X + U + Y$ is the injection, and for any $x \in X$ there exists $0 < n_x \in \mathbb{N}$ such that $e^{n_x}(x) \in Y$. The pseudofunction e then defines a function $\overline{e} : X \longrightarrow Y$, $x \longmapsto e^{n_x}(x)$, called the function that e *calculates.*

For $h = 1, 2$ given pseudofunctions $e_h : X_h \longrightarrow\!\!\!\!\!\!+\!\!> Y_h$, $f : X \longrightarrow\!\!\!\!\!\!+\!\!> Y$ and $g : Y \longrightarrow\!\!\!\!\!\!+\!\!> Z$, operations on pseudofunctions in the set $\{;, \vee, \wedge\}$ were introduced in [5] that produce pseudofunctions that satisfy $\overline{f\,;g} = \overline{g} \circ \overline{f}$, $\overline{e_1 \vee e_2} = \overline{e_1} + \overline{e_2}$ and $\overline{e_1 \wedge e_2} = \overline{e_1} \times \overline{e_2}$. Given a pseudofunction $e : X + U + Y \longrightarrow\!\!\!\!\!\!+\!\!> X + U + Y$, such that the function

$\overline{e}: X+U+Y \to X+U+Y$ is a pseudofunction $\overline{e} : X \dashrightarrow Y$, an additional operation, denoted $\dagger$, referred to as the *iteration* of e, was introduced in [6] where it was shown to satisfy $\overline{e^\dagger} = \overline{\overline{e}} : X \to Y$.

The definition of a pseudofunction given in **Set** is given in terms of elements and does not readily apply to a more general setting. There are two approaches taken in characterizing pseudofunctions in a more general setting than **Set**. One approach was introduced by Khalil in [2], [3] and[4] where functional processors and operations in the set $\{\vee, \wedge\}$ were given in countable infinite extensive categories with finite products.

The other approach, will be described here. This approach avoids the dependence on elements by taking a fix point approach to semantics in distributive categories.

Section 2 provides some underlying definitions and notation. Section 3 gives the basic definitions of fix-point solution for a program and the corresponding semantics, in an arbitrary distributive category $\mathbf{D}$. In Section 4 we show that where $\mathbf{D} = \mathbf{Set}$ these definitions provide the same semantics for pseudofunctions as in [5]. However, programs in **Set** that are not pseudofunctions have infinitely many fix-point solutions each providing a different semantics. In particular, if $f : X+U+Y \to X+U+Y$ is a function with $f \circ k = k$, for the injection $k : Y \rightarrowtail X+U+Y$, then a semantics for f will be a function $\overline{f}$ such that, if $x \in X$ such that there exists $0 < n_x \in \mathbb{N}$ such that $f^{n_x}(x) \in Y$ then $\overline{f}(x) = f^{n_x}(x)$, while if no such n_x exists, then $\overline{f}(x)$ may be any element of Y. Thus pseudofunctions have a unique semantics but the other functions have multiple semantics. We also look at categories other than **Set**. We show, in particular, that in every distributive category there are programs with unique semantics but also that, for every X and Y there is a program that has every morphism in $\mathbf{D}(X,Y)$ as a semantics. In section 5 we show that the operations in the set $\{;, \vee, \wedge, \dagger\}$ described in [5] and [6] carry over to this more general setting. Finally we show that for an arbitrary distributive category $\mathbf{B} \subseteq \mathbf{D}$ the collection of all morphisms $\overline{e}$, such that there exists a program e in $\mathbf{B}$ with semantics $\overline{e}$, forms a distributive category $\mathbf{B}^\dagger$, $\mathbf{B} \subseteq \mathbf{B}^\dagger \subseteq \mathbf{D}$, such that, if $e : X+U+Y \to X+U+Y$ in $\mathbf{B}^\dagger$ then every semantics for e is also in $\mathbf{B}^\dagger$. In section 6 we introduce a precise notion of what it means for a program to have *unique semantics*. We show that for $\mathbf{D}$=**Set** this captures the notion of a pseudofunction. It will also be shown that, in arbitrary distributive categories, that the set of programs with unique semantics are closed under the operations in the set $\{;, \vee\}$. In section 7 we investigate the possibility of strengthening this by adding additional structure to $\mathbf{D}$. In particular, we describe functional processors and operations in the set $\{; \vee, \wedge\}$ in countably infinite extensive categories with finite products. We will then show that in a countably infinite Boolean extensive category $\mathbf{D}$ with finite products a program is a functional processor if and only if it has unique semantics. Then we show that all operations in the set $\{;, \vee, \wedge\}$ immediately carry over to this setting. Thus in this setting, but not in the general setting, we can show that for any distributive category $\mathbf{B} \subseteq \mathbf{D}$ the collection of all morphisms $\overline{e}$, where e is a program in $\mathbf{B}$ with unique semantics $\overline{e}$, forms a distributive subcategory of $\mathbf{B}^\dagger$ as defined above.

2. Notational Preliminaries

A category $\mathbf{D}$ is distributive if it has finite products and sums and if for all objects X, Y and Z, if $i : X \to X+Y$ and $j : Y \to X+Y$ are the indicated coproduct injections, then

$$\delta_{X,Y,Z} = [1_Z \times i,\ 1_Z \times j] : (Z \times X) + (Z \times Y) \to Z \times (X + Y)$$

is an isomorphism.

Given an object X in a category with coproducts, we write ∇_X for the codiagonal $\nabla_X : X + X \to X$, the unique morphism such that $\nabla \circ i_{X,X} = 1_X = \nabla \circ j_{X,X}$.

Let $\mathbf{D}$ be a distributive category. Given $X, Y, Z \in Obj(\mathbf{D})$, let $X + Y + Z =_{def} (X + Y) + Z$, with injections $i_{X,Y,Z} =_{def} i_{X+Y,Z} \circ i_{X,Y}$, $j_{X,Y,Z} =_{def} i_{X+Y,Z} \circ j_{X,Y}$ and, $k_{X,Y,Z} =_{def} j_{X+Y,Z}$. Given $\alpha : X \to W$, $\beta : Y \to W$, and $\gamma : Z \to W$, let

$$[\ \alpha, \beta, \gamma\ ] =_{def} \nabla_W \circ (\nabla_W + 1_W) \circ ((\alpha + \beta) + \gamma).$$

The following lemma provides the principle tools for manipulating expressions of the form $[\ \alpha, \beta, \gamma\ ]$.

Lemma 2.1 *Given $\alpha : X \to W$, $\beta : Y \to W$, $\gamma : Z \to W$, and $\rho : X + Y + Z \to W$ together with $\alpha' : X' \to X$, $\beta' : Y' \to Y$, $\gamma' : Z' \to Z$, and $\tau : W \to V$ we have*

1. $[\ \alpha, \beta, \gamma\ ] \circ i_{X,Y,Z} = \alpha$ 2. $[\ \alpha, \beta, \gamma\ ] \circ j_{X,Y,Z} = \beta$
3. $[\ \alpha, \beta, \gamma\ ] \circ k_{X,Y,Z} = \gamma$ 4. $[\ \rho \circ i_{X,Y,Z},\ \rho \circ j_{X,Y,Z},\ \rho \circ k_{X,Y,Z}\ ] = \rho$
5. $(\ \alpha' + \beta' + \gamma'\ ) \circ i_{X',Y',Z'} = i_{X,Y,Z} \circ \alpha'$ 6. $(\ \alpha' + \beta' + \gamma'\ ) \circ j_{X',Y',Z'} = j_{X,Y,Z} \circ \beta$
7. $(\ \alpha' + \beta' + \gamma'\ ) \circ k_{X',Y',Z'} = i_{X,Y,Z} \circ \gamma$ 8. $\tau \circ [\ \alpha, \beta, \gamma\ ] = [\ \tau \circ \alpha,\ \tau \circ \beta,\ \tau \circ \gamma\ ]$
9. $[\alpha, \beta, \gamma] \circ (\alpha' + \beta' + \gamma') = [\alpha \circ \alpha',\ \beta \circ \beta',\ \gamma \circ \gamma']$

In what follows we will generally omit the subscripts on the coproduct injections i, j and k.

3. Abstract Pseudofunctions

For the remainder of the paper let $\mathbf{D}$ be a distributive category.

Definition 3.2 A *program* in $\mathbf{D}$ is a morphism $e : X + U + Y \to X + U + Y$ such that $e \circ k = k$ and $Y \neq 0$ (the initial object of $\mathbf{D}$).

A program $e : X + U + Y \to X + U + Y$, has a *fix-point solution* if there exists a morphism $W : X + U + Y \to Y$ in $\mathbf{D}$ such that

3.2.2 $W \circ e = W$,

3.2.3 $W \circ k = 1_Y$.

When such W exists we say that it is a *fix-point solution* for e, and that $\overline{e} = W \circ i$ is the *fix-point semantics of e corresponding to W*. We shall generally omit the adjective "fix-point". □

4. On the Nature of Fix-point Solutions

Our fix-point definition of semantics for programs captures pseudofunctions in the sense that when $\mathbf{D}$=$\mathbf{Set}$ then every pseudofunction has the same semantics under either definition. However, there are programs in $\mathbf{Set}$ that are not pseudofunctions, and these programs will, in general, have many fix-point solutions with differing semantics. The proof that every program in $\mathbf{Set}$ has a solution follows from the following definition and result for arbitrary distributive categories.

Definition 4.3 Let $\mathbf{D}$ be a distributive category with terminal object 1, and let X and Y be objects in $\mathbf{D}$. By an *element of* X we mean a morphism $x : 1 \to X$. Given an element $e : 1 \to X + Y$ we say that e *is in* X (resp. e is in Y) if there is an element x in X such that $i \circ x = e$ (resp. there is an element y in Y such that $j \circ y = e$). □

Proposition 4.4 *Let $e : X + U + Y \to X + U + Y$ be a program with solution W and semantics $\overline{e}$, then for every element of X, if there exists an natural number n such that $e^n \circ i \circ x$ is in Y (i.e. there exists an element y of Y such that $e^n \circ i \circ x = k \circ y$) then $\overline{e} \circ x = y$.*

Proof We have

$$\overline{e} \circ x = W \circ i \circ x = W \circ e^n \circ i \circ x = W \circ k \circ y = 1_Y \circ y = y.$$

□

Proposition 4.5 *Every program $e : X + U + Y \to X + U + Y$ in* **Set** *has a solution (and thus at least one semantics).*

Proof Pick $y_0 \in Y$ (recall we assumed $Y \neq 0(= \emptyset)$, then for every $z \in X + U$, if there exists a natural number n_z such that $e^{n_z}(z) \in Y$, take $W(z) = e^{n_z}(z)$, while, if no such n_z exists, take $W(z) = y_0$. Finally, take $W \circ k = 1_Y$. □

Proposition 4.6 *A program $e : X + U + Y \to X + U + Y$ in an arbitrary distributive category* $\mathbf{D}$ *may have many distinct solutions or none.*

Proof Let $e = 1_{X+U+Y} = [i, j, k]$, then every morphism $W : X + U + Y \to Y$ such that $W \circ k = 1_Y$ is a solution for e. Thus, in particular, if $U = 0$, the initial object of $\mathbf{D}$, then for every $f : X \to Y$, $[f, !, 1_Y]$ is a solution for e with $\overline{e} = f$, where ! denotes the unique morphism $0 \to Y$.

Let S be a finite set, then the power set $\mathcal{P}(S)$, ordered by inclusion is a distributive category with union as + and intersection as $\times$. Then a program $e : X + U + Y \to X + U + Y$ has a solution $W : X + U + Y \to Y$ iff $X \cup U \subseteq Y$. □

In the next section we shall see examples of programs with unique solutions.

5. Closure Results

In this section we generalize the operations in the set $\{;, \vee, \wedge\}$ on pseudofunctions, in **Set**, to arbitrary distributive categories. To all intents and purposes, the proofs utilize the same constructions as those in [5], however the notation and the requisite greater level of detail may obscure this fact.

Proposition 5.7 *Let* $\mathbf{D}$ *be a distributive category and let $f : X \to Y$ be a morphism in* $\mathbf{D}$. *Then the program $e = [k \circ f,\ k] : X + 0 + Y \to X + 0 + Y$ has a unique solution, $W = [f,\ 1_Y]$, and a unique semantics $\overline{e} = f$.*

Proof We have

$$W \circ e = [f,\ 1_Y] \circ [k \circ f,\ k] = [f,\ 1_Y] \circ k \circ [f,\ 1_Y] = 1_Y \circ [f,\ 1_Y] = W.$$

Now let $W' = [W' \circ i,\ W' \circ k] = [W' \circ i, k]$ be any solution for e then

$$W' = W' \circ e = W' \circ [k \circ f,\ k] = W' \circ k \circ [f,\ 1_Y] = 1_Y \circ [f,\ 1_Y] = W.$$

Which establishes the uniqueness of W. □

Theorem 5.8 *For $h = 1, 2$, let $\overline{e_h} : X_h \to X_{h+1}$ be the semantics of a program $e_h : X_h + U_h + X_{h+1} \to X_h + U_h + X_{h+1}$ with solution $W_h : X_h + U_h + X_{h+1} \to X_h + U_h + X_{h+1}$.*

For $h = 1, 2$ let the injections for the indicated coproducts be,

$$\begin{aligned} i_h &: X_h && \to X_h + U_h + X_{h+1}, \\ j_h &: U_h && \to X_h + U_h + X_{h+1}, \\ k_h &: X_{h+1} && \to X_h + U_h + X_{h+1} \end{aligned}$$

and let

$$\begin{aligned} i_3 &: U_1 && \to U_1 + X_2 + U_2 \\ j_3 &: X_2 && \to U_1 + X_2 + U_2 \\ k_3 &: U_2 && \to U_1 + X_2 + U_2, \end{aligned}$$

$$\begin{aligned} \hat{\imath}_1 &: X_1 && \to X_1 + (U_1 + X_2 + U_2) + X_3, \\ \hat{\jmath}_1 &: (U_1 + X_2 + U_2) && \to X_1 + (U_1 + X_2 + U_2) + X_3, \\ \hat{k}_1 &: X_3 && \to X_1 + (U_1 + X_2 + U_2) + X_3, \end{aligned}$$

$$\begin{aligned} \hat{\imath}_2 &: (X_1 + U_1 + X_2) && \to (X_1 + U_1 + X_2) + U_2 + X_3, \\ \hat{\jmath}_2 &: U_2 && \to (X_1 + U_1 + X_2) + U_2 + X_3, \\ \hat{k}_2 &: X_3 && \to (X_1 + U_1 + X_2) + U_2 + X_3 \end{aligned}$$

and

$$\begin{aligned} \hat{\imath}_3 &: X_1 && \to X_1 + U_1 + (X_2 + U_2 + X_3), \\ \hat{\jmath}_3 &: U_1 && \to X_1 + U_1 + (X_2 + U_2 + X_3) \\ \hat{k}_3 &: (X_2 + U_2 + X_3) && \to X_1 + U_1 + (X_2 + U_2 + X_3) \end{aligned}$$

be coproducts as indicated. Then $\overline{e}_2 \circ \overline{e}_1 : X_1 \to X_3$ is the semantics of the program

$$e_1; e_2 =_{def} [\hat{\imath}_1,\ \hat{\jmath}_1 \circ i_3,\ [\hat{\jmath}_1 \circ j_3,\ \hat{\jmath}_1 \circ k_3,\ \hat{k}_1]] \circ (1_{X_1} + 1_{U_1} + e_2) \circ$$
$$[[\hat{\imath}_3,\ \hat{\jmath}_3,\ \hat{k}_3 \circ i_2],\ \hat{k}_3 \circ j_2,\ \hat{k}_3 \circ k_2] \circ (e_1 + 1_{U_2} + 1_{X_3}) \circ [\hat{\imath}_2 \circ i_1,\ [\hat{\imath}_2 \circ j_1,\ \hat{\imath}_2 \circ k_1,\ \hat{\jmath}_2],\ \hat{k}_2]$$
$$: X_1 + (U_1 + X_2 + U_2) + X_3 \to X_1 + (U_1 + X_2 + U_2) + X_3$$

and this program has a solution

$$W =_{def} W_2 \circ (W_1 + 1_{U_2} + 1_{X_3}) \circ [\hat{\imath}_2 \circ i_1,\ [\hat{\imath}_2 \circ j_1,\ \hat{\imath}_2 \circ k_1,\ \hat{\jmath}_2],\ \hat{k}_2]$$
$$W : X_1 + (U_1 + X_2 + U_2) + X_3 \to X_3$$

Proof We break the proof up into a series of lemmas.

Lemma 5.9 *Let $\kappa = [\hat{\jmath}_1 \circ j_3,\ \hat{\jmath}_1 \circ k_3,\ \hat{k}_1]$, and $\gamma = [\hat{\imath}_1,\ \hat{\jmath} \circ i_3,\ \kappa \circ e_2 \circ i_2]$, then*

$$e_1; e_2 = [\ [\hat{\imath}_1,\ \hat{\jmath}_1 \circ i_3,\ \kappa \circ e_2 \circ i_2] \circ e_1 \circ i_1,\ [\ [\hat{\imath}_1,\ \hat{\jmath}_1 \circ i_3,\ \kappa \circ e_2 \circ i_2] \circ e_1 \circ j_1,\ \kappa \circ e_2 \circ i_2,\ \kappa \circ e_2 \circ j_2],\ \hat{k}_1]$$
$$= [\gamma \circ e_1 \circ i_1,\ [\gamma \circ e_1 \circ j_1,\ \kappa \circ e_2 \circ i_2,\ \kappa \circ e_2],\ \hat{k}_1].$$

Proof

$$e_1; e_2 =_{def} [\hat{\imath}_1,\ \hat{\jmath}_1 \circ i_3,\ [\hat{\jmath}_1 \circ j_3,\ \hat{\jmath}_1 \circ k_3,\ \hat{k}_1]] \circ (1_{X_1} + 1_{U_1} + e_2) \circ$$
$$[[\hat{\imath}_3,\ \hat{\jmath}_3,\ \hat{k}_3 \circ i_2],\ \hat{k}_3 \circ j_2,\ \hat{k}_3 \circ k_2] \circ (e_1 + 1_{U_2} + 1_{X_3}) \circ [\hat{\imath}_2 \circ i_1,\ [\hat{\imath}_2 \circ j_1,\ \hat{\imath}_2 \circ k_1,\ \hat{\jmath}_2],\ \hat{k}_2]$$

Now

$$\begin{aligned}
&[\hat{\imath}_1,\ \hat{\jmath}_1 \circ i_3,\ [\hat{\jmath}_1 \circ j_3,\ \hat{\jmath}_1 \circ k_3,\ \hat{k}_1]] \circ (1_{X_1} + 1_{U_1} + e_2)\\
&= \ [\hat{\imath}_1,\ \hat{\jmath}_1 \circ i_3,\ \kappa] \circ (1_{X_1} + 1_{U_1} + e_2)\\
&= \ [\hat{\imath}_1,\ \hat{\jmath}_1 \circ i_3,\ \kappa \circ e_2]
\end{aligned}$$

and

$$\begin{aligned}
&[\hat{\imath}_1,\ \hat{\jmath}_1 \circ i_3,\ \kappa \circ e_2] \circ [\ [\hat{\imath}_3,\ \hat{\jmath}_3,\ \hat{k}_3 \circ i_2],\ \hat{k}_3 \circ j_2,\ \hat{k}_3 \circ k_2]\\
&= \ [\ [\hat{\imath}_1,\ \hat{\jmath}_1 \circ i_3,\ \kappa \circ e_2 \circ i_2],\ \kappa \circ e_2 \circ j_2,\ \kappa \circ e_2 \circ k_2]
\end{aligned}$$

and

$$\begin{aligned}
&[\ [\hat{\imath}_1,\ \hat{\jmath}_1 \circ i_3,\ \kappa \circ e_2 \circ i_2],\ \kappa \circ e_2 \circ j_2,\ \kappa \circ e_2 \circ k_2] \circ (e_1 + 1_{U_2} + 1_{X_3})\\
&= \ [\ [\hat{\imath}_1,\ \hat{\jmath}_1 \circ i_3,\ \kappa \circ e_2 \circ i_2] \circ e_1,\ \kappa \circ e_2 \circ j_2,\ \kappa \circ e_2 \circ k_2]
\end{aligned}$$

Now, taking $\gamma = [\hat{\imath}_1,\ \hat{\jmath} \circ i_3,\ \kappa \circ e_2 \circ i_2]$, we get

$$\begin{aligned}
&[\ [\hat{\imath}_1,\ \hat{\jmath}_1 \circ i_3,\ \kappa \circ e_2 \circ i_2] \circ e_1,\ \kappa \circ e_2 \circ j_2,\ \kappa \circ e_2 \circ k_2]\\
&\qquad \circ [\hat{\imath}_2 \circ i_1,\ [\hat{\imath}_2 \circ j_1,\ \hat{\imath}_2 \circ k_1,\ \hat{\jmath}_2],\ \hat{k}_2]\\
&= \ [\ \gamma \circ e_1,\ \kappa \circ e_2 \circ j_2,\ \kappa \circ e_2 \circ k_2] \circ [\hat{\imath}_2 \circ i_1,\ [\hat{\imath}_2 \circ j_1,\ \hat{\imath}_2 \circ k_1,\ \hat{\jmath}_2],\ \hat{k}_2]\\
&= \ [\gamma \circ e_1 \circ i_1,\ [\gamma \circ e_1 \circ j_1,\ \gamma \circ e_1 \circ k_1,\ \kappa \circ e_2 \circ j_2],\ \kappa \circ e_2 \circ k_2]
\end{aligned}$$

but $\gamma \circ e_1 \circ k_1 = \gamma \circ k_1 = \kappa \circ e_2 \circ i_2$, and $\kappa \circ e_2 \circ k_2 = \kappa \circ k_2 = \hat{k}_1$ so the above equals

$$\begin{aligned}
&[\gamma \circ e_1 \circ i_1,\ [\gamma \circ e_1 \circ j_1,\ \kappa \circ e_2 \circ i_2,\ \kappa \circ e_2 \circ j_2],\ \hat{k}_1]\\
&= \ [\ [\hat{\imath}_1,\ \hat{\jmath}_1 \circ i_3,\ \kappa \circ e_2 \circ i_2] \circ e_1 \circ i_1,\\
&\qquad [\ [\hat{\imath}_1,\ \hat{\jmath}_1 \circ i_3,\ \kappa \circ e_2 \circ i_2] \circ e_1 \circ j_1,\ \kappa \circ e_2 \circ i_2,\ \kappa \circ e_2 \circ j_2],\ \hat{k}_1]
\end{aligned}$$

as desired. □

Lemma 5.10 $W = [W_2 \circ i_2 \circ W_1 \circ i_1,\ [W_2 \circ i_2 \circ W_1 \circ j_1,\ W_2 \circ i_2,\ W_2 \circ j_2],\ 1_{X_3}]$.

Proof

$$\begin{aligned}
&W\\
&= \ W_2 \circ (W_1 + 1_{U_2} + 1_{X_3}) \circ [\hat{\imath}_2 \circ i_1,\ [\hat{\imath}_2 \circ j_1,\ \hat{\imath}_2 \circ k_1,\ \hat{\jmath}_2],\ \hat{k}_2]\\
&= \ W_2 \circ [i_2 \circ W_1 \circ i_1,\ [i_2 \circ W_1 \circ j_1,\ i_2 \circ W_1 \circ k_1,\ j_2],\ k_2]\\
&= \ [W_2 \circ i_2 \circ W_1 \circ i_1,\ [W_2 \circ i_2 \circ W_1 \circ j_1,\ W_2 \circ i_2 \circ W_1 \circ k_1,\ W_2 \circ j_2],\ W_2 \circ k_2]\\
&= \ [W_2 \circ i_2 \circ W_1 \circ i_1,\ [W_2 \circ i_2 \circ W_1 \circ j_1,\ W_2 \circ i_2,\ W_2 \circ j_2],\ 1_{X_3}] \qquad \text{by 3.2.3}
\end{aligned}$$

□

Lemma 5.11 $\overline{e_1; e_2} = \overline{e}_2 \circ \overline{e}_1$.

Proof

$$\begin{aligned}
&\overline{e_1; e_2}\\
&= \ W \circ \hat{\imath}_1\\
&= \ [W_2 \circ i_2 \circ W_1 \circ i_1,\ [W_2 \circ i_2 \circ W_1 \circ j_1,\ W_2 \circ i_2,\ W_2 \circ j_2],\ 1_{X_3}] \circ \hat{\imath}_1 \qquad \text{by 5.10}\\
&= \ W_1 \circ i_2 \circ W_1 \circ i_1\\
&= \ \overline{e}_2 \circ \overline{e}_1. \qquad \text{by def. } e_h
\end{aligned}$$

□

Lemma 5.12 $W \circ \kappa = W_2$

Proof

$$\begin{array}{rll} & W \circ \kappa & \\ = & W \circ [\hat{\jmath}_1 \circ j_3,\ \hat{\jmath}_1 \circ k_3,\ \hat{k}_1] & \\ = & [W_2 \circ i_2,\ W_2 \circ j_2,\ 1_{X_3}] & \text{by } 3.2.3 \\ = & [W_2 \circ i_2,\ W_2 \circ j_2,\ W_2 \circ k_2] & \\ = & W_2 & \end{array}$$

□

Lemma 5.13 $W \circ \gamma \circ e_1 = W_2 \circ i_2 \circ W_1$.

Proof

$$\begin{array}{rll} & W \circ \gamma \circ e_1 & \\ = & W \circ [\hat{\imath}_1,\ \hat{\jmath}_1 \circ i_3,\ \kappa \circ e_2 \circ i_2] \circ e_1 & \\ = & [W_2 \circ i_2 \circ W_1 \circ i_1,\ W_2 \circ i_2 \circ W_1 \circ j_1,\ W_2 \circ i_2] \circ e_1 & \text{by 5.10 and 5.12 and 3.2.2} \\ = & W_2 \circ i_2 \circ [W_1 \circ i_1, W_1 \circ j_1,\ 1_{X_2}] \circ e_1 & \\ = & W_2 \circ i_2 \circ [W_1 \circ i_1, W_1 \circ j_1,\ W_1 \circ k_1] \circ e_1 & \text{by } 3.2.3 \\ = & W_2 \circ i_2 \circ W_1 \circ e_1 & \\ = & W_2 \circ i_2 \circ W_1 & \text{by } 3.2.2 \end{array}$$

□

Lemma 5.14 $W \circ (e_1; e_2) = W$

Proof

$$\begin{array}{rl} & W \circ (e_1; e_2) \\ = & W \circ [\gamma \circ e_1 \circ i_1, [\gamma \circ e_1 \circ j_1,\ \kappa \circ e_2 \circ i_2,\ \kappa \circ e_2 \circ j_2],\ \hat{k}_1] \\ = & [W_2 \circ i_2 \circ W_1 \circ i_1,\ [W_2 \circ i_2 \circ W_1 \circ j_1,\ W_2 \circ e_2 \circ i_2,\ W_2 \circ e_2 \circ j_2], 1_{X_3}] \\ & \qquad \text{by 5.13,5.12 and 5.10} \\ = & W \end{array}$$

□

which completes the proof of the main parts of the proposition. □

Lemma 5.15 *For $h = 1, 2$ let X_h, U_h and, Y_h be objects of* D *and let*

$$\begin{array}{l} i_h : X_h \to X_h + U_h + Y_h \\ j_h : U_h \to X_h + U_h + Y_h \\ k_h : Y_h \to X_h + U_h + Y_h \end{array}$$

be the indicated coproduct injections. Then $a = [i_1{+}i_2,\ j_1{+}j_2, k_1{+}k_2]$ is an isomorphism $a : (X_1 + X_2) + (U_1 + U_2) + (Y_1 + Y_2) \to (X_1 + U_1 + Y_1) + (X_2 + U_2 + Y_2)$.

Proposition 5.16 *For $h = 1, 2$ let $X_h, U_h,\ Y_h,\ i_h,\ j_h,$ and k_h, and a be as above. Furthermore, for $h = 1, 2$ let $e_h : X_h + U_h + Y_h \to X_h + U_h + Y_h$ and define*

$$(e_1 \vee e_2) = a^{-1} \circ (e_1 + e_2) \circ a$$
$$(e_1 \vee e_2) : (X_1 + X_2) + (U_1 + U_2) + (Y_1 + Y_2) \to (X_1 + X_2) + (U_1 + U_2) + (Y_1 + Y_2).$$

If, for $h = 1, 2$, e_h has a solution W_h with corresponding semantics $\overline{e}_h = W_h \circ i_h$, then $(e_1 \vee e_2)$ has a solution $W = (W_1 + W_2) \circ a$ with corresponding semantics $\overline{(e_1 \vee e_2)} = \overline{e}_1 + \overline{e}_2$.

Proof Let

$$\begin{aligned}
i &: X_1 + X_2 \to (X_1 + X_2) + (U_1 + U_2) + (Y_1 + Y_2) \\
j &: U_1 + U_2 \to (X_1 + X_2) + (U_1 + U_2) + (Y_1 + Y_2) \\
k &: Y_1 + Y_2 \to (X_1 + X_2) + (U_1 + U_2) + (Y_1 + Y_2)
\end{aligned}$$

be the indicated coproduct injections. We must show: 1) $(e_1 \vee e_2) \circ k = k$, 2) $W \circ (e_1 \vee e_2) = W$, 3) $W \circ k = 1_{Y_1+Y_2}$, and $W \circ i = \overline{e}_1 + \overline{e}_2$.

1)

$$\begin{aligned}
&(e_1 \vee e_2) \circ k \\
&= a^{-1} \circ (e_1 + e_2) \circ a \circ k \\
&= a^{-1} \circ (e_1 + e_2) \circ (k_1 + k_2) \\
&= a^{-1} \circ (e_1 \circ k_1 + e_2 \circ k_2) \\
&= a^{-1} \circ (k_1 + k_2) \\
&= k.
\end{aligned}$$

2)

$$\begin{aligned}
&W \circ (e_1 \vee e_2) \\
&= (W_1 + W_2) \circ a \circ a^{-1} \circ (e_1 + e_2) \circ a \\
&= (W_1 + W_2) \circ (e_1 + e_2) \circ a \\
&= (W_1 \circ e_1 + W_2 \circ e_2) \circ a \\
&= (W_1 + W_2) \circ a \\
&= W.
\end{aligned}$$

3)

$$\begin{aligned}
&W \circ k \\
&= (W_1 + W_2) \circ a \circ k \\
&= (W_1 + W_2) \circ (k_1 + k_2) \\
&= (W_1 \circ k_1 + W_2 \circ k_2) \\
&= 1_{Y_1} + 1_{Y_2} \\
&= 1_{Y_1+Y_2}.
\end{aligned}$$

4)

$$\begin{aligned}
&W \circ i \\
&= (W_1 + W_2) \circ (i_1 + i_2) \\
&= (W_1 \circ i_1 + W_2 \circ i_2) \\
&= \overline{e}_1 + \overline{e}_2.
\end{aligned}$$

□

In the following let b be the isomorphism from

$$\begin{gathered}
T = \\
(X_1 \times X_2) + ((X_1 \times (U_2 + Y_2)) + (U_1 \times (X_2 + U_2 + Y_2)) \\
+(Y_1 \times (X_2 + U_2))) + (Y_1 \times Y_2).
\end{gathered}$$

to

$$(X_1 + U_1 + Y_1) \times (X_2 + U_2 + Y_2),$$

which exists by virtue of **D** being a distributive category.

Proposition 5.17 *For $h = 1, 2$ let X_h, U_h and, Y_h be objects of* $\mathbf{D}$ *and let $e_h : X_h + U_h + Y_h \to X_h + U_h + Y_h$ and define*

$$(e_1 \wedge e_2) = b^{-1} \circ (e_1 \times e_2) \circ b$$
$$(e_1 \wedge e_2) : T \to T.$$

If, for $h = 1, 2$, e_h has a solution W_h with corresponding semantics $\overline{e}_h = W_h \circ i_h$, then $(e_1 \wedge e_2)$ has a solution $W = (W_1 \times W_2) \circ b$ with corresponding semantics $\overline{(e_1 \wedge e_2)} = \overline{e}_1 \times \overline{e}_2$.

Proof We have

$$\begin{aligned} & W \circ (e_1 \wedge e_2) \\ = \ & (W_1 \times W_2) \circ b \circ b^{-1} \circ (e_1 \times e_2) \circ b \\ = \ & (W_1 \times W_2) \circ (e_1 \times e_2) \circ b \\ = \ & (W_1 \circ e_1 \times W_2 \circ e_2) \circ b \\ = \ & (W_1 \times W_2) \circ b \\ = \ & W. \end{aligned}$$

Now, for $h = 1, 2$, let

$$\begin{aligned} i &: (X_1 \times X_2) &&\to T, \\ k &: (Y_1 \times Y_2) &&\to T, \\ i_h &: X_h &&\to (X_h + U_h + Y_h), \\ j_h &: U_h &&\to (X_h + U_h + Y_h), \\ k_h &: Y_h &&\to (X_h + U_h + Y_h), \end{aligned}$$

be the indicated coproduct injections. Then we have

$$\begin{aligned} & W \circ k \\ = \ & (W_1 \times W_2) \circ b \circ k \\ = \ & (W_1 \times W_2) \circ (k_1 \times k_2) \\ = \ & (W_1 \circ k_1 \times W_2 \circ k_2) \\ = \ & (1_{Y_1} \times 1_{Y_2}) \\ = \ & 1_{Y_1 \times Y_2}. \end{aligned}$$

and

$$\begin{aligned} & W \circ i \\ = \ & (W_1 \times W_2) \circ b \circ i \\ = \ & (W_1 \times W_2) \circ (i_1 \times i_2) \\ = \ & (W_1 \circ i_1 \times W_2 \circ i_2) \\ = \ & (\overline{e}_1 \times \overline{e}_2). \end{aligned}$$

□

Proposition 5.18 *If $e : (X+U+Y)+U'+(X+U+Y) \to (X+U+Y)+U'+(X+U+Y)$ has a solution W with semantics $\overline{e} : X + U + Y \to X + U + Y$, and $\overline{e}$ has a solution W' with semantics $\overline{\overline{e}}$, then there exists $e^\dagger : X + (U + Y + U' + X + U) + Y \to X + (U + Y + U' + X + U) + Y$ with $\overline{e^\dagger} = \overline{\overline{e}}$.*

More specifically, let $\gamma : X+(U+Y+U'+X+U)+Y \to (X+U+Y)+U'+(X+U+Y)$ *be the evident isomorphism, and let*

$$\begin{array}{ll} i : X & \to X+U+Y, \\ j : U & \to X+U+Y, \\ k : Y & \to X+U+Y, \\ i' : X & \to X+(U+Y+U'+X+U)+Y \\ j' : (U+Y+U'+X+U) & \to X+(U+Y+U'+X+U)+Y \\ k' : Y & \to X+(U+Y+U'+X+U)+Y, \\ \hat{\imath} : (X+U+Y) & \to (X+U+Y)+U'+(X+U+Y) \\ \hat{\jmath} : U' & \to (X+U+Y)+U'+(X+U+Y) \\ \hat{k} : (X+U+Y) & \to (X+U+Y)+U'+(X+U+Y), \end{array}$$

be the indicated coproduct injections. Then we claim that

$$e^{\dagger} = \gamma^{-1} \circ [\hat{\imath},\ \hat{\jmath},\ [\hat{\imath} \circ i,\ \hat{\imath} \circ j,\ \hat{k} \circ k]] \circ e \circ \gamma$$

is the desired program and that it has solution $W^{\dagger} = W' \circ W \circ \gamma$.

Proof Consider that,

$$\begin{array}{ll} & W^{\dagger} \circ e^{\dagger} \\ = & W' \circ W \circ \gamma \circ \gamma^{-1} \circ [\hat{\imath},\ \hat{\jmath},\ [\hat{\imath} \circ i,\ \hat{\imath} \circ j,\ \hat{k} \circ k]] \circ e \circ \gamma \\ = & W' \circ W \circ [\hat{\imath},\ \hat{\jmath},\ [\hat{\imath} \circ i,\ \hat{\imath} \circ j,\ \hat{k} \circ k]] \circ e \circ \gamma \\ = & W' \circ [W \circ \hat{\imath},\ W \circ \hat{\jmath},\ [W \circ \hat{\imath} \circ i,\ W \circ \hat{\imath} \circ j,\ W \circ \hat{k} \circ k]] \circ e \circ \gamma \\ = & W' \circ [W \circ \hat{\imath},\ W \circ \hat{\jmath},\ [\overline{e} \circ i,\ \overline{e} \circ j,\ \overline{e} \circ k]] \circ e \circ \gamma \\ = & W' \circ [W \circ \hat{\imath},\ W \circ \hat{\jmath},\ \overline{e}] \circ e \circ \gamma \\ = & [W' \circ W \circ \hat{\imath},\ W' \circ W \circ \hat{\jmath},\ W' \circ \overline{e}] \circ e \circ \gamma \\ = & [W' \circ W \circ \hat{\imath},\ W' \circ W \circ \hat{\jmath},\ W'] \circ e \circ \gamma \\ = & W' \circ [W \circ \hat{\imath},\ W \circ \hat{\jmath},\ W \circ \hat{k}] \circ e \circ \gamma \\ = & W' \circ W \circ e \circ \gamma \\ = & W' \circ W \circ \gamma \\ = & W^{\dagger}. \end{array}$$

$$W^{\dagger} \circ k' = W' \circ W \circ \gamma \circ k' = W' \circ W \circ \hat{k} \circ k = W' \circ k = 1_Y$$

$$W^{\dagger} \circ i' = W' \circ W \circ \gamma \circ i' = W' \circ W \circ \hat{\imath} \circ i = W' \circ \overline{e} \circ i = W' \circ i = \overline{\overline{e}}.$$

□

Corollary 5.19 *Let* **D** *be a distributive category, and let* **B** *be a distributive category of* **D**. *Then the collection of all morphisms* $\overline{e}$ *in* **D** *such that* $\overline{e}$ *is a semantics for some program* e *in* **B** *forms a distributive category* $\mathbf{B}^{\dagger}$ *such that* $\mathbf{B} \subseteq \mathbf{B}^{\dagger} \subseteq \mathbf{D}$. *Furthermore, if* $e : X+U+Y \to X+U+Y$ *is a program in* $\mathbf{B}^{\dagger}$, *then every semantics* $\overline{e} : X \to Y$ *for* e *is also in* $\mathbf{B}^{\dagger}$.

Proof By Proposition 5.7 we have that if $f : X \to Y$ in **B** then the program $e = [k \circ f,\ k]$ has semantics $\overline{e} = f$. Clearly $e \in \mathbf{B}$ and thus $\mathbf{B} \subseteq \mathbf{B}^{\dagger}$.

That $\mathbf{B}^{\dagger}$ is category follows from the above together with Proposition 5.8 since these results show that $e_1; e_2 \in \mathbf{B}$ if $e_1, e_2 \in \mathbf{B}$.

The distributivity of $\mathbf{B}^{\dagger}$ follows from Propositions 5.16 and 5.17 once we note that the isomorphisms a and b used, respectively, in the definitions of $\vee$ and $\wedge$ are in **B**, and thus, for any $e_1, e_2 \in \mathbf{B}^{\dagger}$, that $e_1 \times e_2$ and $e_1 + e_2$ are in $\mathbf{B}^{\dagger}$.

Finally, by Proposition 5.18, if $\overline{e} : X+U+Y \to X+U+Y$ is a program in $\mathbf{B}^{\dagger}$, then every semantics $\overline{\overline{e}} : X \to Y$ for $\overline{e}$ is also in $\mathbf{B}^{\dagger}$ since every semantics for $\overline{e}$ is a semantics for $e^{\dagger} \in \mathbf{B}$.

□

6. Semantic Uniqueness

Definition 6.20 Let $e : X + U + Y \to X + U + Y$ be a program in a distributive category $\mathbf{D}$. We say that e *has unique semantics* if, for all Z and $V, W : X+U+Y \to Z$, if $V \circ e = V$, $W \circ e = W$ and $V \circ k = W \circ k$ then $V \circ i = W \circ i$. Note that this does not preclude the possibility that e has no solutions and thus no semantics. □

Proposition 6.21 *In* **Set** *a program* $e : X+U+Y \to X+U+Y$ *has unique semantics if, and only if, for every* $x \in X$ *there exists a natural number* n_x *such that* $e^{n_x}(x) \in Y$.

Proof Assume that for each $x \in X$ there exists n such that $e^n(x) \in Y$ and let n_x be the least such. Now let $V : X + U + Y \to Z$ such that $V \circ e = V$. Then for each $x \in X$ we have

$$(V \circ i)(x) = (V \circ e^{n_x} \circ i)(x) = (V \circ k)(e^{n_x} \circ i)(x).$$

¿From which it follows that if $V, W : X + U + Y \to Z$, if $V \circ e = V$, $W \circ e = W$ and $V \circ k = W \circ k$ then $V \circ i = W \circ i$.

Assume now that there exists $x \in X$ such that for all natural numbers n, $e^n(x) \notin Y$ then, by the same means as in the proof of Proposition 4.5 we can, for any set Z with $card(Z) \geq 2$, construct functions $V, W : X + U + Y \to Z$ with $V \circ e = V$, $W \circ e = W$ and $V(x) \neq W(x)$. □

Proposition 6.22 *Let* $e : X + U + Y \to X + U + Y$ *be a program in* $\mathbf{D}$ *with unique semantics then,*

1. *If* e *has any solutions then its semantics* $\overline{e} : X \to Y$ *of* e *is the same for all solutions.*

2. *If* $V : X + U + Y \to Z$ *such that* $V \circ e = V$, *then* $V \circ i = (V \circ k) \circ \overline{e}$.

Proof If the semantics of e is not uniquely defined then there must exist solutions $W_1, W_2 : X + U + Y \to Y$ for e with different semantics, i.e., with $W_1 \circ i \neq W_2 \circ i$. But this would contradict the assumption that e has unique semantics.

Given $V : X+U+Y \to Z$ such that $V \circ e = V$, let W be a solution for e and define $V' = (V \circ k) \circ W$. Clearly $V' : X+U+Y \to Z$, $V' \circ k = (V \circ k) \circ W \circ k = (V \circ k) \circ 1_Y = V \circ k$ and $V' \circ e = (V \circ k) \circ W \circ e = (V \circ k) \circ W = V'$. So, since e has unique semantics, $V \circ i = V' \circ i$, but then, $V \circ i = V' \circ i = (V \circ k) \circ W \circ i = (V \circ k) \circ \overline{e}$. □

Corollary 6.23 *Let* $e : X + U + Y \to X + U + Y$ *and let* $W : X + U + Y \to Y$ *be a solution for* e *with semantics* $\overline{e} = W \circ i$, *then* e *has unique semantics if, and only if, for every* Z *and* $V : X + U + Y \to Z$, $V \circ e = V$ *implies* $V \circ i = (V \circ k) \circ \overline{e}$.

Proposition 6.24 *If* $e_1 : X_1 + U_1 + X_2 \to X_1 + U_1 + X_2$ *and* $e_2 : X_2 + U_2 + X_3 \to X_2+U_2+X_3$ *have unique semantics then so does* $e = e_1; e_2 : X_1+(U_1+X_2+U_2)+X_3 \to X_1 + (U_1 + X_2 + U_2) + X_3$.

Proof We will use the same notation for coproduct injections as in Theorem 5.8. Let $V : X_1 + (U_1 + X_2 + U_2) + X_3 \to Z$ such that $V \circ (e_1; e_2) = V$. By Corollary 6.23 it suffices to show that $V \circ \hat{\imath}_1 = (V \circ \hat{k}_1) \circ \overline{e}$.

For $h = 1, 2$, let $W_h : X_h + U_h + Y_h \to Y_h$ be a solution for e_h and let $\overline{e}_h : X_h \to X_{h+1}$ be the corresponding semantics. From Lemma 5.9 we have

$e_1; e_2 = [\,[\hat{\imath}_1,\ \hat{\jmath}_1 \circ i_3,\ \kappa \circ e_2 \circ i_2] \circ e_1 \circ i_i,\ [\,[\hat{\imath}_1,\ \hat{\jmath}_1 \circ i_3,\ \kappa \circ e_2 \circ i_2] \circ e_1 \circ j_1,\ \kappa \circ e_2 \circ i_2,\ \kappa \circ e_2 \circ j_2],\ \hat{k}_1]$ where $\kappa = [\hat{\jmath}_1 \circ j_3,\ \hat{\jmath}_1 \circ k_3,\ \hat{k}_1]$. ¿From which it follows by direct computation that

$$V \circ \hat{\imath}_1 = V \circ (e_1; e_2) \circ \hat{\imath}_1 = V \circ [\hat{\imath}_1,\ \hat{\jmath}_i \circ i_3,\ \kappa \circ e_2 \circ i_2] \circ e_1 \circ i_1, \tag{1}$$

and

$$V \circ \hat{\jmath}_1 \circ i_3 = V \circ (e_1; e_2) \circ \hat{\jmath}_1 \circ i_3 = V \circ [\hat{\imath}_1,\ \hat{\jmath}_i \circ i_3,\ \kappa \circ e_2 \circ i_2] \circ e_1 \circ j_1,$$

and, since $e_1 \circ k_1 = k_1$, that

$$V \circ \kappa \circ e_2 \circ i_2 = V \circ [\hat{\imath}_1,\ \hat{\jmath}_i \circ i_3,\ \kappa \circ e_2 \circ i_2] \circ e_1 \circ k_1.$$

¿From this it follows that

$$V \circ [\hat{\imath}_1,\ \hat{\jmath}_i \circ i_3,\ \kappa \circ e_2 \circ i_2] \circ e_1 = V \circ [\hat{\imath}_1,\ \hat{\jmath}_i \circ i_3,\ \kappa \circ e_2 \circ i_2] \tag{2}$$

and so, since e_1 has unique semantics, that

$$V \circ [\hat{\imath}_1,\ \hat{\jmath}_i \circ i_3,\ \kappa \circ e_2 \circ i_2] \circ i_1 = V \circ \kappa \circ e_2 \circ i_2 \circ \overline{e}_1 \tag{3}$$

Now consider that

$$\begin{aligned}
& V \circ \kappa \\
= \quad & V \circ [\hat{\jmath}_1 \circ j_3,\ \hat{\jmath}_3, \circ k_3,\ \hat{k}_1] \\
= \quad & [V \circ \hat{\jmath}_1 \circ j_3,\ V \circ \hat{\jmath}_3, \circ k_3,\ V \circ \hat{k}_1] \\
= \quad & [V \circ (e_1; e_2) \circ \hat{\jmath}_1 \circ j_3,\ V \circ (e_1; e_2) \circ \hat{\jmath}_3, \circ k_3,\ V \circ (e_1; e_2) \circ \hat{k}_1] \\
= \quad & [V \circ \kappa \circ e_2 \circ i_2,\ V \circ \kappa \circ e_2 \circ j_2,\ V \circ \kappa \circ e_2 \circ k_2] \\
= \quad & V \circ \kappa \circ e_2
\end{aligned}$$

and $V \circ \kappa \circ e_2 \circ k_2 = V \circ \kappa \circ k_2 = V \circ \hat{k}_1$. So, since e_2 has unique semantics we have

$$V \circ \kappa \circ e_2 \circ i_2 = (V \circ \kappa \circ k_2) \circ \overline{e}_2 = (V \circ \hat{k}_1) \circ \overline{e}_2. \tag{4}$$

But then, from above, we have

$$\begin{aligned}
& V \circ \hat{\imath}_1 \\
= \quad & V \circ [\hat{\imath}_1,\ \hat{\jmath}_i \circ i_3,\ \kappa \circ e_2 \circ i_2] \circ e_1 \circ i_1 && \text{by (1)} \\
= \quad & V \circ [\hat{\imath}_1,\ \hat{\jmath}_i \circ i_3,\ \kappa \circ e_2 \circ i_2] \circ i_1 && \text{by (2)} \\
= \quad & V \circ \kappa \circ e_2 \circ i_2 \circ \overline{e}_1 && \text{by (3)} \\
= \quad & (V \circ \hat{k}_1) \circ \overline{e}_2 \circ \overline{e}_1 && \text{by (4)} \\
= \quad & (V \circ \hat{k}_1) \circ \overline{(e_1; e_2)},
\end{aligned}$$

as desired. □

Proposition 6.25 *If for $h = 1, 2$, $e_h : X_h + U_h + Y_h \to X_h + U_h + Y_h$ has unique semantics then so does $(e_1 \vee e_2)$.*

Proof For $h = 1, 2$ let i_h, j_h, k_h, i, j, k and a be as in the proof of Proposition 5.16. Furthermore, let

$$\hat{\imath} : (X_1 + U_1 + Y_1) \to (X_1 + U_1 + Y_1) + (X_2 + U_2 + Y_2)$$

and

$$\hat{\jmath} : (X_2 + U_2 + Y_2) \to (X_1 + U_1 + Y_1) + (X_2 + U_2 + Y_2)$$

be the indicated coproduct injections. What we must show is that for any Z and $V : (X_1 + X_2) + (U_1 + U_2) + (Y_1 + Y_2) \to Z$ such that $V \circ (e_1 \vee e_2) = V$ that $V \circ i = (V \circ k) \circ \overline{(e_1 \vee e_2)}$.

But,

$$\begin{aligned} & V \\ = \;& V \circ (e_1 \vee e_2) \\ = \;& V \circ a^{-1} \circ (e_1 + e_2) \circ a \\ = \;& [V \circ a^{-1} \circ \hat{\imath} \circ e_1,\ V \circ a^{-1} \circ \hat{\jmath} \circ e_2] \circ a \end{aligned}$$

So, $[V \circ a^{-1} \circ \hat{\imath} \circ e_1,\ V \circ a^{-1} \circ \hat{\jmath} \circ e_2] = V \circ a^{-1}$. But this implies

$$V \circ a^{-1} \circ \hat{\imath} \circ e_1 = V \circ a^{-1} \circ \hat{\imath}$$

and

$$V \circ a^{-1} \circ \hat{\jmath} \circ e_2 = V \circ a^{-1} \circ \hat{\jmath}$$

which, from the assumption that e_1 and e_2 have unique semantics, yields

$$V \circ a^{-1} \circ \hat{\imath} \circ i_1 = V \circ a^{-1} \circ \hat{\imath} \circ k_1 \circ \overline{e}_1$$

and

$$V \circ a^{-1} \circ \hat{\jmath} \circ i_2 = V \circ a^{-1} \circ \hat{\imath} \circ k_2 \circ \overline{e}_2$$

from which it follows that

$$\begin{aligned} & V \circ i \\ = \;& V \circ a^{-1} \circ (i_1 + i_2) \\ = \;& [V \circ a^{-1} \circ \hat{\imath} \circ i_1,\ V \circ a^{-1} \circ \hat{\jmath} \circ i_2] \\ = \;& [V \circ a^{-1} \circ \hat{\imath} \circ k_1 \circ \overline{e}_1,\ V \circ a^{-1} \circ \hat{\jmath} \circ k_2 \circ \overline{e}_2] \\ = \;& V \circ a^{-1} \circ (k_1 + k_2) \circ (\overline{e}_1 + \overline{e}_2) \\ = \;& (V \circ k) \circ \overline{(e_1 \vee e_2)} \end{aligned}$$

as desired. □

We have not been able to show, in this general setting, that a similar result holds for $\wedge$, i.e., that if for $h = 1, 2$, $e_h : X_h + U_h + Y_h \to X_h + U_h + Y_h$ has unique semantics then so does $(e_1 \wedge e_2)$. However the following section sketches a proof in a setting with additional structure.

As shown by the proof of Proposition 4.6, there are programs in every non-trivial distributive category with more than one semantics. Coupling that result with Proposition 5.7 we see that e having unique semantics $\overline{e}$ does not imply that $\overline{e}$ has unique semantics. However a reasonable, and still open, question is whether or not if $\overline{e}$ has unique semantics implies that $e^{\dagger}$ (see Proposition 5.18) has unique semantics.

7. Functional Processors and Semantic Uniqueness

In this section we give a brief description of the approach taken in characterising pseudofunctions, described in [2], [3] and [4], and give a proof of its relation to programs with unique semantics.

We assume $\mathbf{D}$ to be a *countably extensive category with finite products.* Details of the definition and properties of extensive categories can be found in [1]. We now list some of the essential features of extensive categories that are needed to characterise pseudofunctions.

1. An extensive category with finite products is distributive; similarly a countably infinite extensive category with products is countably distributive.

2. If $(y_h : Y_h \longrightarrow Y,\ h = 1,2)$ is a coproduct and $f : Z \longrightarrow Y$ then we can form an object $f^{-1}(Y_h)$, by taking the pullback of f along the injection y_h. In **Set** the object $f^{-1}(Y_h)$, is the set $\{z \in Z : f(z) \in Y_h\}$.

3. If Y is also the coproduct $(y'_h : Y'_h \longrightarrow Y,\ h = 1,2)$ then we can form the object $Y_h \cap Y'_h$ by taking the pullback of y_h along y'_h.

4. In this notation countable infinite extensivity implies that if $f : Z \longrightarrow \Sigma_{n=1}^{\infty} A_n$ and there are injections $x : X \longrightarrow Z,\ y_i : Y_i \longrightarrow Z$ then $f^{-1}(\Sigma_{n=1}^{\infty} A_n) \cong \Sigma_{n=1}^{\infty} f^{-1}(A_n)$ and $X \cap \Sigma_i Y_i \cong \Sigma_i (X \cap Y_i)$.

5. An extensive category is *boolean* if it has a terminal object and the first injection $T : I \longrightarrow I + I$ is a subobject classifier. In a boolean category, for any injection $y_1 : Y_1 \longrightarrow Y$ there is an injection $y_2 : Y_2 \longrightarrow Y$ that exhibits Y as the coproduct of Y_1 and Y_2.

In all that follows $e : X + U + Y \longrightarrow X + U + Y$, is an arrow satisfying $e \circ k = k$. Notice that, in **Set**, $(e^n \circ i)^{-1}(Y)$ is the set of elements in X that are mapped to Y in n iterations. The object $A_n = (e^n i)^{-1}(Y) \cap (e^{n-1} i)^{-1}(X + U)$ is the set

$$A_n = \{x \in X : e^n(x) \in Y\} \cap \{x \in X : e^{n-1}(x) \in X + U\},$$

that is, the set of elements in X which reach Y after *exactly* n iterations.

The following notation will be used in the remainder of this section. Let $X_n = (e^n \circ i)^{-1}(Y)$ and $X'_n = (e^n \circ i)^{-1}(X + U)$ then from the above discussion we can form the following pair of pullback squares

$$\begin{tikzcd}
A_n \arrow[r, "a_n"] \arrow[d, "a'_n"] \arrow[dr, phantom, "\lrcorner", very near start] & X_n \arrow[r, "e^n"] \arrow[d, "i_{X_n}"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "k"] \\
X'_{n-1} \arrow[r, "i_{X'_{n-1}}"] & X \arrow[r, "e^n \circ i"] & X + U + Y
\end{tikzcd}$$

where i_{X_n} and i'_{X_n} are suitable injections.

Definition 7.26

1. The arrow e is a *functional processor* or a *pseudofunction* if the family

$$(i_{X_n} \circ a_n : A_n \longrightarrow X)$$

exhibits X as a sum, the arrow e is then denoted $e : X \longrightarrow\!\!\!\!\!\!+\!\!\!\!\!\!\!\rightarrow Y$, and U is called the *local states* of e.

2. If e is a functional processor then, by the property of coproducts, the family of arrows $(e_n \circ a_n : A_n \longrightarrow Y)$ defines a unique arrow $\overline{e} : X \longrightarrow Y$.

□

It is clear that this characterises pseudofunctions in **Set** and $\overline{e}$ is the function that e calculates.

Theorem 7.27 *For $h = 1, 2$ let $e_h : X_h \twoheadrightarrow X_h$, $f : X \twoheadrightarrow Y$, and $g : Y \twoheadrightarrow Z$ then operations on functional processors in the set $\{;, \vee, \wedge\}$ produce functional processors that satisfy $\overline{f\,;g} = \overline{g} \circ \overline{f}$, $\overline{e_1 \vee e_2} = \overline{e_1} + \overline{e_2}$ and $\overline{e_1 \wedge e_2} = \overline{e_1} \times \overline{e_2}$.*

Proof Detailed proofs for the operations in the set $\{\vee, \wedge\}$ can be found in [4]. Here we give an outline of the proof that $f\,; g$ is a functional processor. Since f and g are functional processors then for all $0 < n \in \mathbb{N}$ there exists A_n and B_n such that $X \cong \Sigma A_n$ and $Y \cong \Sigma B_n$. Let $f_{A_n} = f_n \circ a_n$ where f_n is derived analogously to e_n in the above diagram. Let U and V be the local states of f and g and let $x : X \longrightarrow X+U+Y+V+Z$ be the injection then define

$$C_k = ((f\,;g)^k \circ x)^{-1}(Z) \,\cap\, ((f\,;g)^{k-1} \circ x)^{-1}(X+U+Y+V).$$

That is, in **Set**, C_k is the set of elements in X that take exactly k iterations to reach Z. Then $C_{i+j-1} \cap A_i \cong f_{A_i}^{-1}(B_j)$ and $C_k \cap A_i \cong O$ if $k < i$ (proof omitted). Therefore,

$$\begin{aligned}
\Sigma_k C_k &\cong \Sigma_k(C_k \cap X) \cong \Sigma_k(C_k \cap \Sigma_i A_i) \\
&\cong \Sigma_k \Sigma_i (C_k \cap A_i) \cong \Sigma_i \Sigma_k (C_k \cap A_i) \\
&\cong \Sigma_i \Sigma_j (C_{i+j-1} \cap A_i) \cong \Sigma_i \Sigma_j f_{A_i}^{-1}(B_j) \\
&\cong \Sigma_i f_{A_i}^{-1}(\Sigma_j B_j) \cong \Sigma_i f_{A_i}^{-1}(Y) \cong \Sigma_i A_i \\
&\cong X.
\end{aligned}$$

A detailed proof that it satisfies $\overline{f\,;g} = \overline{g} \circ \overline{f}$, can be found in [3] □

We now relate functional processors to programs with unique semantics.

Proposition 7.28 *Suppose $e : X+U+Y \longrightarrow X+U+Y$ is a functional processor, in an extensive category with products, then e has unique semantics.*

Proof Let $V, W : X+U+Y \longrightarrow Z$ be such that $V \circ e = V$, $W \circ e = W$ and $V \circ k = W \circ k$ then we show that $V \circ i = W \circ i$. To show that $V \circ i = W \circ i$ it is enough to show that $V \circ i \circ i_{A_n} = V \circ i \circ i_{A_n}$, where $i_{A_n} = i_{X_n} \circ a_n$ then by the above diagram

$$\begin{aligned}
V \circ i \circ i_{A_n} &= V \circ e^n \circ i \circ i_{A_n} \\
&= V \circ k \circ e_n \circ a_n \\
&= W \circ k \circ e_n \circ a_n \\
&= W \circ e^n \circ i \circ i_{A_n} \\
&= W \circ i \circ i_{A_n}
\end{aligned}$$

□

Proposition 7.29 *In a countable infinite extensive boolean category with products, **D**, if e has unique semantics, then e is a functional processor.*

Proof We show that if e is not a functional processor then e does not have a unique semantics. If e is not a functional processor then $L = \Sigma A_n \not\cong X$ and there exists an injection $i_L : L \longrightarrow X$. But as **D** is boolean, there exists an injection $i_{L'} : L' \longrightarrow X$ that defines X as a coproduct of L and $L' \not\cong O$. Now let $Z = I+I$, where I is a terminal object, and let $V : X+U+Y \longrightarrow I+I$ be given as $V = k_0 \circ !$ where $k_0 : I \longrightarrow I+I$ is the first injection; and let $W = [k_1 \circ !_{L'},\ k_0 \circ !_{L+U+Y}]$, where $k_1 : I \longrightarrow I+I$ is the second injection, and $!_{L'}$ and $!_{L+U+Y}$ are the obvious unique arrows into I. Clearly, $V \circ e = V$, $W \circ e = W$ and $V \circ k = W \circ k$ but $V \circ i \neq W \circ i$, hence e does not have a unique semantics and the result follows. □

Theorem 7.30 *In a countable infinite extensive boolean category with products, the operations in the set* $\{;, \vee, \wedge\}$ *produce programs with suitable unique semantics from programs with unique semantics.*

Proof The proof follows immediately from the above. $\square$

8 Concluding Remarks

We have shown how the results on programs in distributive categories developed in [9], [5] and [6] for the category **Set**, can be carried over to more general distributive categories. Furthermore we have introduced a new notion, that of programs with unique semantics. We have shown that programs with unique semantics play a role in general distributive categories analogous to that played by pseudofunctions in the category of sets. Several open problems remain, some of these are detailed at the ends of Sections 5 and 6. A problem not mentioned earlier, but clearly significant from the computer science viewpoint, is the lack of a meaningful example of the application of these ideas in a distributive category that is not a subcategory of **Set**.

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