THE SYMMETRY OF THE CAUCHY-COMPLETION OF A CATEGORY

R. Betti and R.F.C. Walters

Cauchy-completion for enriched categories was introduced by F.W. Lawvere [3], generalizing the notion for metric spaces. In this paper, we are concerned with the case of categories based on a bicategory which is locally partially-ordered ([5],[6], [1]). A natural question that arises is whether the Cauchy-completion of a symmetric category is again symmetric. This is true for metric spaces (although the contrary is claimed in [4]), but false in general. We prove that if the base bicategory satisfies the "modular law" (as defined by P. Freyd) then symmetry is preserved by Cauchy-completion. An application is the description of sheafification in terms of Cauchy-completion.

We refer to [5],[6] for definitions not given here.

1. The modular law

Let B be a bicategory. Objects of B will be denoted u,v,w,..., and arrows $\rho,\sigma,\tau,\ldots$. Throughout this paper we will assume that B is locally a complete poset, and that suprema in each B(u,v) are preserved by intersection in B(u,v) and by composition with arrows (or both sides). We will also suppose given an involution $(\ )^{o}: B^{op} \to B$ (reversing arrows, but not order) which is the identity on objects.

Examples. The main examples we have in mind are (i) Lawvere's monoidal category R regarded as a bicategory with one object (see [3]), (ii) the bicategory, Rel, of sets and relations, and, more generally (iii) the bicategories Rel(C,J) of relations arising from a category C with a topology J, as defined in [1] or [6].

Definition. (P. Freyd) B satisfies the *modular law* if, for arrows $\rho: u \to v$, $\sigma: v \to w$ and $\tau: u \to w$, we have

$$\tau \wedge (\sigma\rho) \leq \sigma(\sigma^{o}\tau \wedge \rho).$$

Remark. The bicategories in examples (ii) and (iii) satisfy the modular law, though not example (i).

We need the following technical

Lemma. If B satisfies the modular law and $(a_i)_{i\in I}$ is a family of arrows $a_i: u \to u$ satisfying $1_u \leq \bigvee_{i\in I} a_i$ then $1_u \leq \bigvee_{i\in I} (a_i \wedge a_i^{o})$.

Proof. First we prove that if $b \leq 1_u: u \to u$ then $b = b^{o}$. To see this note that

$$b = 1_u \wedge (b1_u) \le b(b^o 1_u \wedge 1_u) \qquad \text{(modular law)}$$
$$\le 1_u(b^o \wedge 1_u) \qquad (b \le 1_u)$$
$$\le b^o .$$

But $b \le 1_u$ implies $b^o \le 1_u^o = 1_u$ and hence, as above, $b^o \le b$.

Now applying this result to $b = 1 \wedge a_i$ we get that $1_u \wedge a_i = 1_u \wedge a_i^o \le a_i^o$ and so $1_u \wedge a_i \le a_i \wedge a_i^o$.

Finally notice that $1_u \le \bigvee_{i\in I} a_i$ implies that

$$1_u = 1_u \wedge 1_u \le 1_u \wedge \left(\bigvee_{i\in I} a_i\right) = \bigvee_{i\in I}(1_u \wedge a_i) \le \bigvee_{i\in I} (a_i \wedge a_i^o) . \qquad \text{Q.E.D.}$$

2. Adjoint bimodules

Let X be a B-category. The *Cauchy-completion* PX of X (described in [1]) is defined by

(i) elements of PX over u are adjoint pairs of bimodules $\phi\colon \hat{u} \to\!\!\!\!\!+\, X$, $\psi\colon X \to\!\!\!\!\!+\, \hat{u}$, $\phi \dashv \psi$;

(ii) $d((\phi_1,\psi_1),(\phi_2,\psi_2)) = \psi_2 \cdot \phi_1$.

Now the main result of the paper depends on the following representation theorem for adjoint pairs of bimodules:

Theorem. If B satisfies the modular law and $\phi \dashv \psi\colon u \rightleftarrows X$ is an adjoint pair of bimodules, then

(i) $\phi(x) = \bigvee_{y\in X} d(y,x) \cdot (\psi^o(y) \wedge \phi(y))$,

and (ii) $\psi(x) = \bigvee_{y\in X} (\psi(y) \wedge \phi^o(y)) \cdot d(x,y)$.

Proof of (i).

Firstly, $$\phi(x) = \bigvee_y d(y,x) \cdot \phi(y) \qquad \text{(Yoneda)}$$
$$\ge \bigvee_y d(y,x)(\psi^o(y) \wedge \phi(y)).$$

Now, from the adjunction we have that $1_u \le \bigvee_y \psi(y)\phi(y)$, and hence by our lemma that $1_u \le \bigvee_y (\psi(y)\phi(y) \wedge \phi^o(y)\psi^o(y))$.

Hence $$\phi(x) = \phi(x) \cdot 1_u$$
$$\le \phi(x) \cdot \bigvee_y (\psi(y)\phi(y) \wedge \phi^o(y)\psi^o(y))$$
$$\le \phi(x) \cdot \bigvee_y [\psi(y)(\psi^o(y)\phi^o(y)\psi^o(y) \wedge \phi(y))] . \qquad \text{(modularity)}$$

But $$\psi^{o}(y)\phi^{o}(y)\psi^{o}(y) = (\psi(y)\phi(y)\psi(y))^{o}$$

$$\leq (\psi(y)d(y,y))^{o} \qquad \text{(adjunction)}$$

$$\leq \psi^{o}(y).$$

Hence $$\phi(x) \leq \bigvee_{y} \phi(x)\psi(y)(\psi^{o}(y) \wedge \phi(y))$$

$$\leq \bigvee_{y} d(y,x)(\psi^{o}(y) \wedge \phi(y)). \qquad \text{(adjunction)}$$

The proof of (ii) is similar. Q.E.D.

Remark. Although the bicategory R of example (i) does not satisfy the modular law, a very similar (though simpler) calculation to that above shows that adjoint bimodules can be represented in the same way in that case. We do not have a natural proof that includes both calculations.

3. Symmetry

For X to be *symmetric* we require that $d(y,x) = d(x,y)^{o}$ for all $x,y \in X$.

Theorem. If B satisfies the modular law and X is symmetric, then so is PX, the Cauchy-completion of X.

Proof. It is sufficient to prove that if $\phi \dashv \psi: \hat{u} \rightleftarrows X$ is an adjoint pair of bimodules then $\psi = \phi^{o}$, because then

$$d((\phi_1,\psi_1),(\phi_2,\psi_2)) = \psi_2 \cdot \phi_1 = \phi_2^{o} \cdot \phi_1$$

$$= (\phi_1^{o} \cdot \phi_2)^{o} = (\psi_1 \cdot \phi_2)^{o}$$

$$= d((\phi_2,\psi_2),(\phi_1,\psi_1))^{o}.$$

Now from the theorem in §2, assuming the symmetry of X, we have

$$\phi^{o}(x) = [\bigvee_{y} d(y,x)\cdot(\psi^{o}(y) \wedge \phi(y))]^{o}$$

$$= \bigvee_{y} (\psi(y) \wedge \phi^{o}(y))\cdot d(y,x)^{o}$$

$$= \bigvee_{y} (\phi^{o}(y) \wedge \psi(y))\cdot d(x,y) \qquad \text{(symmetry of X)}$$

$$= \psi(x).$$

Q.E.D.

That symmetry is not always preserved is shown by the following very simple counterexample suggested by S. Kasangian.

Counterexample. Let $G = \{1,a,b\}$ be the group with three elements. Let B be the bicategory with one object whose arrows are subsets of G. Let $(\ )^o$ be the identity. Let I be the one element B-category. Then adjoint pairs of bimodules $I \rightleftarrows I$ correspond to elements of G, and under this correspondence $d(g,h) = h^{-1}g$. Hence in PI, $d(1,a) = a^{-1} \neq a = d(a,1)$, so PI is not symmetric.

4. Sheafification

In [1] and [5], sheaves on a site (C,J) are shown to be symmetric Cauchy-complete $\mathrm{Rel}(C,J)$-categories. In particular, presheaves on C are sheaves for a topology J_0 on C and there is an obvious functor

$$\mathrm{Rel}(C,J_0) \to \mathrm{Rel}(C,J)$$

which induces a functor

$$\mathrm{Rel}(C,J_0)\text{-cat} \to \mathrm{Rel}(C,J)\text{-cat} \qquad \text{(change of base)}$$

This preserves symmetry.

Now from the description of sheafification in [1], using the fact that symmetry is preserved by Cauchy-completion, we get that sheafification is the composite

$$\mathrm{Preshv}(C) \subseteq \text{sym } \mathrm{Rel}(C,J_0)\text{-cat} \to \text{sym } \mathrm{Rel}(C,J)\text{-cat} \xrightarrow[\text{Cauchy-completion}]{} \mathrm{Shv}(C,J).$$

So we have a description of sheafification in terms of standard constructions of enriched category theory.

REFERENCES

1. R. Betti and A. Carboni, Cauchy-completion and the associated sheaf, to appear in *Cahiers top. et géom. diff.*

2. Denis Higgs, A category approach to boolean valued set theory, *Lecture Notes, University of Waterloo*, 1973.

3. F.W. Lawvere, Metric spaces, generalized logic, and closed categories, *Rendiconti del Seminario Matematico e Fisico di Milano* 43 (1973) 135-166.

4. H. Lindner, Morita equivalences of enriched categories, *Cahiers top. et géom. diff.* 15 (1974), 377-397.

5. R.F.C. Walters, Sheaves and Cauchy-complete categories, *Cahiers top. et géom. diff.* 22 (1981), 283-286.

6. R.F.C. Walters, Sheaves on a site as Cauchy-complete categories, to appear in *Journal of Pure and Applied Algebra*.