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VARIATION THROUGH ENRICHMENT*

Renato BETTI
Istituto Matematico "Federigo Enriques", via Saldini 50, Milano, Italy

Aurelio CARBONI
Istituto Matematico "Federigo Enriques", via Saldini 50, Milano, Italy

Ross STREET
School of Mathematics and Physics, Macquarie University, North Ryde, N.S.W. 2113, Australia

Robert WALTERS
Department of Pure Mathematics, University of Sydney, N.S.W. 2006, Australia

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Introduction

This paper continues the authors' various works [3, 4, 12, 14] on categories enriched in bicategories. We treat the elements of the theory again, here from a more algebraic (logical) and less geometric viewpoint. For a bicategory $\mathscr{W}$ we first develop $\mathscr{W}$-matrices before passing on to $\mathscr{W}$-modules, an approach which allows a simple proof of the cocompleteness of the 2-category $\mathscr{W}$-Cat of $\mathscr{W}$-categories. When $\mathscr{W}$ has precisely one object (and so is a monoidal category) the main results are in works of Bénabou [2], Lawvere [6], and Wolff [15], although a uniform treatment even in this case has not been published.

The second part of the paper relates *variable categories* with *enriched categories*. For the purposes of this paper a variable category is taken to mean a fibration over a fixed parameter category $\mathbf{C}$. We show that the domain of variation can be organized into a bicategory $\mathscr{W}(\mathbf{C})$ such that categories varying over $\mathbf{C}$ and $\mathscr{W}(\mathbf{C})$-enriched categories appear on opposing sides of a biadjunction which tries very hard to be a biequivalence. In fact, if we impose the mild completeness condition of splitting idempotents on the fibres of the variable categories, the adjunction does restrict to a biequivalence with the "cauchy-complete" $\mathscr{W}(\mathbf{C})$-categories.

Our terminology for bicategories and 2-categories is that of [5] and [10].

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1. Matrices and graphs over a bicategory

Let Set denote the category of small sets.

A bicategory $\mathscr{W}$ is said to be *locally small-cocomplete* when each hom-category $\mathscr{W}(U, V)$ has small colimits and, for all arrows $f: U' \to U$, $g: V \to V'$ in $\mathscr{W}$, the functor $\mathscr{W}(f, g): \mathscr{W}(U, V) \to \mathscr{W}(U', V')$ preserves small colimits.

Let $\mathscr{W}$ denote a locally small-cocomplete bicategory with a small set $\mathscr{U}$ of objects.

The category Set$/\mathscr{U}$ has as objects families X of small sets X_U indexed by $U \in \mathscr{U}$; an element $x \in X_U$ is called *an element of X over U.*

The bicategory $\mathscr{W}$-Mat of *$\mathscr{W}$-matrices* is defined as follows. The objects are the objects of Set$/\mathscr{U}$. An arrow $S: X \to Y$ assigns to each pair x, y of elements of X, Y over U, V, respectively, an arrow $S(y, x): U \to V$ in $\mathscr{W}$. A 2-cell $\sigma: S \to S'$ is a family of 2-cells $\sigma_{y,x}: S(y,x) \to S'(y,x)$ in $\mathscr{W}$. Composition of 2-cells $S \to S' \to S''$ is componentwise that of $\mathscr{W}$. Composition of arrows

$$X \xrightarrow{S} Y \xrightarrow{T} Z$$

is "matrix multiplication":

$$(TS)(z, x) = \sum_{y \in Y} T(z, y) S(y, x).$$

The latter composition is compatible with 2-cells; it is associative and has identities up to coherent natural isomorphisms.

Small colimits in $\mathscr{W}$-Mat(X, Y) are constructed componentwise in the hom-categories of $\mathscr{W}$. It follows that $\mathscr{W}$-Mat is *locally small-cocomplete.*

There is a homomorphism of bicategories

$$\mathrm{Set}/\mathscr{U} \to \mathscr{W}\text{-Mat}$$

which is the identity on objects and takes an arrow $h: X \to Y$ in Set$/\mathscr{U}$ to the matrix $h_*: X \to Y$ given as follows

$$h_*(y, x) = \begin{cases} 1_U : U \to U & \text{when } y = hx, \\ 0 : U \to V & \text{otherwise,} \end{cases}$$

where x, y are elements over U, V and 0 denotes the initial object of the category $\mathscr{W}(U, V)$. Matrices of the form $h_*: X \to Y$ have right adjoints $h^*: Y \to X$ in $\mathscr{W}$-Mat: the formula for h^* is the reverse of that for h_*. (In general, not all arrows with right adjoints in $\mathscr{W}$-Mat are of the form h_*.) If h is monic then the unit $1_X \to h^* h_*$ is invertible. If h is epic the the counit $h_* h^* \to 1$ is a retraction. (The converses of the last two sentences are also true provided $\mathscr{W}$ has no objects whose identity arrows are initial.)

For each small set Y over $\mathscr{U}$ there is a category $\mathscr{P}Y$ over $\mathscr{U}$ whose objects over U are functions S which assign to each element y of Y an arrow $S(y): U \to V$ where y is over V, and whose arrows over U are families of 2-cells in $\mathscr{W}$. There is a pseudo-natural equivalence of categories:

$$\mathscr{W}\text{-Mat}(X, Y) \simeq (\mathrm{CAT}/\mathscr{V}\ )(X, \mathscr{P}Y)$$

where CAT is a suitably large 2-category of categories.

Proposition 1. *The homomorphism of bicategories*

$$\mathrm{Set}/\mathscr{V} \to \mathscr{W}\text{-Mat}$$

preserves bicolimits. The initial object 0 *of* Set/$\mathscr{V}$ *is biterminal in* $\mathscr{W}$-Mat. *For all objects* X, Y *of* Set/$\mathscr{V}$, *the coproduct diagram*

$$X \xrightarrow{i} X+Y \xleftarrow{j} Y$$

has the following properties:

(a) i^*j_*, j^*i_* *are initial in* $\mathscr{W}$-Mat(Y, X), $\mathscr{W}$-Mat(X, Y), *and the units* $1_X \to i^*i_*$, $1_Y \to j^*j_*$ *are invertible.*

(b) *The* 2-*cell* $i_*i^* + j_*j^* \to 1_{X+Y}$, *induced by the counits, is invertible.*

(c) *The diagram*

$$X \xleftarrow{i^*} X+Y \xrightarrow{j^*} Y$$

is a biproduct in $\mathscr{W}$-Mat.

Proof. The assignment $Y \mapsto \mathscr{P}Y$ provides a relative right biadjoint for Set/$\mathscr{V} \to \mathscr{W}$-Mat modulo a change of universe. This suffices for the first sentence of the Proposition. The second sentence is trivial.

The units in (a) are invertible since i and j are monic. The remainder of (a) follows from the fact that the pushout

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & X \arrow[d, "i"] \\ Y \arrow[r, "j"'] & X+Y \end{tikzcd}$$

becomes a bipushout in $\mathscr{W}$-Mat.

Since $X+Y$ is a bicoproduct, the 2-cell of (b) is invertible if and only if its composites with both i_* and j_* are invertible. But the composite with i_* is the composite isomorphism:

$$(i_*i^* + j_*j^*)i_* \cong i_*i^*i_* + j_*j^*i_* \cong i_*1 + j_*0 \cong i_*.$$

Similarly, the composite with j_* is invertible. This gives (b).

Given $S: Z \to X$, $T: Z \to Y$ in $\mathscr{W}$-Mat, we obtain $i_*S + j_*T: Z \to X+Y$ with:

$$i^*(i_*S + j_*T) \cong i^*i_*S + i^*j_*T \cong S,$$

$$j^*(i_*S + j_*T) \cong j^*i_*S + j^*j_*T \cong T.$$

The 2-cell condition is also easily checked, yielding (c). □

A *$\mathcal{W}$-graph* $\mathcal{G}$ is a "square" matrix; that is, an endo-arrow in $\mathcal{W}$-Mat. The object of Set/$\mathcal{W}$ *and* the matrix from it to itself will both be denoted by $\mathcal{G}$. So, for each object U of $\mathcal{W}$, we have a small set $\mathcal{G}_U$ of objects of $\mathcal{G}$ over U; and, for objects A, B of $\mathcal{G}$ over U, V, we have an arrow $\mathcal{G}(B, A): U \to V$ in $\mathcal{W}$. *An arrow* $H: \mathcal{G} \to \mathcal{G}'$ *of $\mathcal{W}$-graphs* consists of an arrow $H: \mathcal{G} \to \mathcal{G}'$ in Set/$\mathcal{W}$ together with a 2-cell

$$H: H_* \mathcal{G} H^* \to \mathcal{G}'$$

in $\mathcal{W}$-Mat. So, for each object A of $\mathcal{G}$ over U, we have an object HA of $\mathcal{G}'$ over U, and, for objects A, B of $\mathcal{G}$ over U, V, we have a 2-cell

$$H_{BA}: \mathcal{G}(B, A) \to \mathcal{G}'(HB, HA)$$

in $\mathcal{W}$. This defines a category $\mathcal{W}$-Gph of $\mathcal{W}$-graphs.

Proposition 2. *The category* $\mathcal{W}$-Gph *has small colimits.*

Proof. Suppose $D: \mathcal{C} \to \mathcal{W}$-Gph is a functor from a small category $\mathcal{C}$. Let X denote the colimit of the composite of D with the forgetful functor $\mathcal{W}$-Gph$\to$Set/$\mathcal{W}$. There are coprojections $HC: DC \to X$ in Set/$\mathcal{W}$. There is a functor

$$\mathcal{C} \to (\mathcal{W}\text{-Mat})(X, X)$$

which takes $n: C \to C'$ to the composite

$$(HC)_*(DC)(HC)^* \xrightarrow[\cong]{} (HC')_*(Dn)_*(DC)(Dn)^*(HC')$$

$$\xrightarrow[(HC')_*(Dn)(HC')^*]{} (HC')_*(DC')(HC')^*.$$

The colimit of the last functor gives an endo-arrow of X and hence determines a $\mathcal{W}$-graph $\mathcal{G}$. The coprojections HC together with the coprojections $(HC)_*(DC)(HC)^* \to \mathcal{G}$ determine arrows $HC: DC \to \mathcal{G}$ in $\mathcal{W}$-Gph which can be checked to provide the coprojections of a colimit for D. □

2. Categories enriched over a bicategory

The following definitions occur in an equivalent, but more usual, form in [12].

A *$\mathcal{W}$-category* $\mathcal{A}$ is a $\mathcal{W}$-graph $\mathcal{A}$ together with 2-cells $\eta: 1 \to \mathcal{A}$, $\mu: \mathcal{A}\mathcal{A} \to \mathcal{A}$ in $\mathcal{W}$-Mat which satisfy the axioms for a monad in $\mathcal{W}$-Mat. Note that $\mathcal{A}_U$ becomes the set of objects for a category whose arrows $f: A \to B$ are 2-cells $1_U \to \mathcal{A}(A, B)$ in $\mathcal{W}$ and whose composition is determined by μ. It will be convenient to write $\mathcal{A}_U$ for this category and not merely for the set of objects of $\mathcal{A}$ over U.

A *$\mathcal{W}$-functor* $F: \mathcal{A} \to \mathcal{B}$ between $\mathcal{W}$-categories $\mathcal{A}, \mathcal{B}$ is an arrow of $\mathcal{W}$-graphs which respects η, μ. The arrow $F_*: \mathcal{A} \to \mathcal{B}$ and 2-cell $\tilde{F}: F_* \mathcal{A} \to \mathcal{B} F_*$ (corresponding to $F: F_*\mathscr{A}F^* \to \mathscr{B}$ under $F_* \dashv F^*$) determine a "monad opfunctor" in $\mathscr{W}$-Mat (in the terminology of [9]).

For $\mathscr{W}$-functors $F, G: \mathscr{A} \to \mathscr{B}$, a ***$\mathscr{W}$-natural transformation*** $\theta: F \to G$ is a 2-cell $\theta: F_*\mathscr{A} \to \mathscr{B}G_*$ in $\mathscr{W}$-Mat such that the following diagram commutes.

$$\begin{tikzcd}
F_*\mathscr{A}\mathscr{A} \arrow[r, "\theta\mathscr{A}"] \arrow[d, "\tilde{F}\mathscr{A}"'] & \mathscr{B}G_*\mathscr{A} \arrow[r, "\mathscr{B}\tilde{G}"] & \mathscr{B}\mathscr{B}G_* \arrow[d, "\mu G_*"] \\
\mathscr{B}F_*\mathscr{A} \arrow[r, "\mathscr{B}\theta"'] & \mathscr{B}\mathscr{B}G_* \arrow[r, "\mu G_*"'] & \mathscr{B}G_*
\end{tikzcd}$$

Notice that there is a bijection between such θ and 2-cells $\bar{\theta}: F_* \to \mathscr{B}G_*$ satisfying

$$\mu G_* \cdot \mathscr{B}\bar{\theta} \cdot \tilde{F} = \mu G_* \cdot \mathscr{B}\tilde{G} \cdot \bar{\theta}\mathscr{A};$$

the bijection is given by the equations:

$$\bar{\theta} = \theta \cdot F_*\eta, \qquad \theta = \mu G_* \cdot \mathscr{B}\bar{\theta} \cdot \tilde{F}.$$

With obvious compositions, we have defined a 2-category $\mathscr{W}$-Cat of $\mathscr{W}$-categories, $\mathscr{W}$-functors and $\mathscr{W}$-natural transformations.

A monad $m: U \to U$ in the bicategory $\mathscr{W}$ can be identified with a $\mathscr{W}$-category $\mathscr{A}$ which has precisely one object A such that A is over U and $\mathscr{A}(A, A) = m$. In particular, each object U of $\mathscr{W}$ determines *a $\mathscr{W}$-category which we also denote by U* corresponding to the identity monad on U. There is an obvious isomorphism of categories:

$$(\mathscr{W}\text{-Cat})(U, \mathscr{A}) \cong \mathscr{A}_U.$$

Proposition 3. *The forgetful functor from the category* $|\mathscr{W}\text{-Cat}|$ *of $\mathscr{W}$-categories and $\mathscr{W}$-functors to the category* $\mathscr{W}$-Gph *has a left adjoint $\mathscr{F}$ whose value at a square matrix $\mathscr{G}: X \to X$ is the geometric series $\sum_{n \in \mathbb{N}} \mathscr{G}^n : X \to X$.*

Proof. The monoidal category $\mathscr{W}$-Mat(X, X), whose tensor-product (that is, composition) preserves small colimits, is such that the free monoid on an object $\mathscr{G}$ is $\sum \mathscr{G}^n = \mathscr{F}\mathscr{G}$. The identity of X together with the coprojection $\mathscr{G} \to \mathscr{F}\mathscr{G}$ for $n = 1$ provide an arrow $N: \mathscr{G} \to \mathscr{F}\mathscr{G}$ of $\mathscr{W}$-graphs. Suppose $H: \mathscr{G} \to \mathscr{B}$ is an arrow of $\mathscr{W}$-graphs into a $\mathscr{W}$-category $\mathscr{B}$. Then $H^*\mathscr{B}H_*: X \to X$ is a monoid in $\mathscr{W}$-Mat(X, X), so the arrow $\mathscr{G} \to H^*\mathscr{B}H_*$ (arising from H) extends to a unique monoid arrow $\mathscr{F}\mathscr{G} \to H^*\mathscr{B}H_*$ which, together with H on objects, determines a unique $\mathscr{W}$-functor $H': \mathscr{F}\mathscr{G} \to \mathscr{B}$ with $H'N = H$. □

Lemma 4. *Suppose $F, G: \mathscr{A} \to \mathscr{B}$ are monoid arrows in* $\mathscr{W}$-Mat(X, X) *and let $H: \mathscr{B} \to \mathscr{C}$ be the coequalizer of F, G in* $\mathscr{W}$-Mat(X, X). *The $\mathscr{W}$-graph $\mathscr{C}$ possesses a unique monoid structure such that H becomes a monoid arrow if and only if*

$H\cdot\mu\cdot\mathscr{B}F=H\cdot\mu\cdot\mathscr{B}G$ and $H\cdot\mu\cdot F\mathscr{B}=H\cdot\mu\cdot G\mathscr{B}$. Furthermore, in this case, this monoid arrow is a coequalizer of F, G in $|\mathscr{W}\text{-Cat}|$.

Proof. Composition in $\mathscr{W}$-Mat preserves coequalizers, so the rows and columns of the following diagram are all coequalizers.

$$\begin{tikzcd}
\mathscr{A}\mathscr{A} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \mathscr{A}\mathscr{B} \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \mathscr{A}\mathscr{C} \arrow[d, shift left] \arrow[d, shift right] \\
\mathscr{B}\mathscr{A} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \mathscr{B}\mathscr{B} \arrow[r] \arrow[d] & \mathscr{B}\mathscr{C} \arrow[d] \\
\mathscr{C}\mathscr{A} \arrow[r, shift left] \arrow[r, shift right] & \mathscr{C}\mathscr{B} \arrow[r] & \mathscr{C}\mathscr{C}
\end{tikzcd}$$

The existence of a unique $\mu:\mathscr{C}\mathscr{C}\to\mathscr{C}$ such that the square

$$\begin{tikzcd}
\mathscr{B}\mathscr{B} \arrow[r, "HH"] \arrow[d, "\mu"'] & \mathscr{C}\mathscr{C} \arrow[d, "\mu"] \\
\mathscr{B} \arrow[r, "H"'] & \mathscr{C}
\end{tikzcd}$$

commutes is equivalent to the condition that the composite

$$\mathscr{B}\mathscr{B} \xrightarrow{\mu} \mathscr{B} \xrightarrow{H} \mathscr{C}$$

should equalize both of the pairs

$$\begin{tikzcd}
\mathscr{B}\mathscr{A} \arrow[r, shift left, "\mathscr{B}F"] \arrow[r, shift right, "\mathscr{B}G"'] & \mathscr{B}\mathscr{B},
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{A}\mathscr{B} \arrow[r, shift left, "F\mathscr{B}"] \arrow[r, shift right, "G\mathscr{B}"'] & \mathscr{B}\mathscr{B}.
\end{tikzcd}$$

As we must, define $\eta:1\to\mathscr{C}$ to be $H\eta$. From the construction in Proposition 2 we see that H is the coequalizer of F, G in the category of $\mathscr{W}$-graphs. It is easy to see that $K:\mathscr{C}\to\mathscr{D}$ is a $\mathscr{W}$-functor if and only if KH is, for any arrow K of $\mathscr{W}$-graphs into a $\mathscr{W}$-category $\mathscr{D}$. $\square$

Proposition 5. *The category $|\mathscr{W}\text{-Cat}|$ has coequalizers.*

Proof. Take two $\mathscr{W}$-functors $F, G:\mathscr{A}\to\mathscr{B}$ and form the coequalizer $\mathscr{C}$ of the underlying arrows of $\mathscr{W}$-graphs (Proposition 2). Let $\mathscr{D}$ be the coequalizer of the

$\mathscr{V}$-graph arrows $\mathscr{F}F, \mathscr{F}G : \mathscr{F}\mathscr{A} \to \mathscr{F}\mathscr{B}$. Then we have the following diagram in $\mathscr{V}$-Cat:

$$\begin{tikzcd}
\mathscr{F}\mathscr{F}\mathscr{A} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \mathscr{F}\mathscr{F}\mathscr{B} \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \mathscr{F}\mathscr{D} \arrow[d, shift left] \arrow[d, shift right] \\
\mathscr{F}\mathscr{A} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \mathscr{F}\mathscr{B} \arrow[r] \arrow[d] & \mathscr{F}\mathscr{C} \\
\mathscr{A} \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] & \mathscr{B} &
\end{tikzcd}$$

The category of monoids in $\mathscr{V}$-Mat(X, X) is monadic over $\mathscr{V}$-Mat(X, X) (since tensoring with a fixed object on either side preserves countable coproducts). So the first two columns of the above diagrams are coequalizers which are absolute (split) at the underlying level. Since $\mathscr{F}$ is a left adjoint, the first two rows are also coequalizers. Lemma 4 applies to the two arrows in the third column of the above diagram (since it applies to the first two columns) to yield the coequalizer of those two arrows in $|\mathscr{V}\text{-Cat}|$. By commutativity, an arrow from $\mathscr{B}$ into this coequalizer is induced. By the "3×3-diagram lemma" this arrow is then the coequalizer of F, G. □

Theorem 6. *The forgetful functor* $|\mathscr{V}\text{-Cat}| \to \mathscr{V}\text{-Gph}$ *is monadic.*

Proof. Consider again the diagram in proof of Proposition 5, this time with F, G a split pair at the $\mathscr{V}$-graph level. Then the top two rows are split coequalizers. By Lemma 4 the columns are coequalizers at both the $|\mathscr{V}\text{-Cat}|$ *and* $\mathscr{V}$-Gph levels. By the "3×3-diagram lemma", the coequalizer of F, G is preserved by the forgetful functor. Since the forgetful functor reflects isomorphisms and in view of Proposition 3, the result follows from Beck's Theorem [8; p. 151 Ex. 6]. □

Theorem 7. *The 2-category* $\mathscr{V}$-Cat *admits all small colimits.*

Proof. That the category $|\mathscr{V}\text{-Cat}|$ has all small colimits follows from Proposition 2, Theorem 6, Proposition 5, and Linton [7; p. 81].

A monad $\mathscr{A} : X \to X$ in $\mathscr{V}$-Mat leads to a monad

$$\begin{pmatrix} \mathscr{A} & \mathscr{A} \\ 0 & \mathscr{A} \end{pmatrix} : X + X \to X + X$$

in $\mathscr{V}$-Mat which is easily verified to have the property required of $\mathbf{2} \otimes \mathscr{A}$ in $\mathscr{V}$-Cat;

$$(\mathscr{V}\text{-Cat})(\mathbf{2} \otimes \mathscr{A}, \mathscr{B}) \cong [\mathbf{2}, \mathscr{V}\text{-Cat}(\mathscr{A}, \mathscr{B})].$$

It remains to prove that small colimits in $|\mathscr{V}\text{-Cat}|$ are preserved by the categoryvalued representables $\mathcal{V}$-Cat$(-,\mathcal{A})$ and hence are colimits in $\mathcal{V}$-Cat. This will follow if we can prove that the functor

$$\mathbf{2}\otimes - : |\mathcal{V}\text{-Cat}| \to |\mathcal{V}\text{-Cat}|$$

preserves small colimits. That it preserves small coproducts is trivial. That it preserves coequalizers of the type in Lemma 4 follows from the straightforward observation that the functor $\mathcal{V}$-Gph$\to \mathcal{V}$-Gph which takes

$$\mathcal{G} : X \to X \quad \text{to} \quad \begin{pmatrix} \mathcal{G} & \mathcal{G} \\ 0 & \mathcal{G} \end{pmatrix} : X+X \to X+X$$

preserves coequalizers (see Proposition 2). Using the construction of Proposition 5 and these facts, we deduce that $\mathbf{2}\otimes -$ preserves all coequalizers. □

3. Modules

Suppose $\mathcal{A}$, $\mathcal{B}$ are $\mathcal{V}$-categories; that is, monads $\mathcal{A} : X \to X$, $\mathcal{B} : Y \to Y$ in $\mathcal{V}$-Mat. Composition with $\mathcal{A}$, $\mathcal{B}$ on the right, left (respectively) determines a monad $\mathcal{V}$-Mat$(\mathcal{A}, \mathcal{B})$ on the category $\mathcal{V}$-Mat(X, Y). The category of Eilenberg–Moore algebras for this monad is denoted by:

$$\mathcal{V}\text{-Mod}(\mathcal{A}, \mathcal{B}).$$

An object Φ of $\mathcal{V}$-Mod$(\mathcal{A}, \mathcal{B})$ is called a *$\mathcal{V}$-module* from $\mathcal{A}$ to $\mathcal{B}$; it consists of a matrix $\Phi : X \to Y$ together with compatible actions $\varrho : \Phi\mathcal{A} \to \Phi$, $\lambda : \mathcal{B}\Phi \to \Phi$.

For $\mathcal{V}$-modules $\Phi : \mathcal{A} \to \mathcal{B}$, $\Psi : \mathcal{B} \to \mathcal{C}$, there is a *composite $\mathcal{V}$-module* $\Psi\Phi : \mathcal{A} \to \mathcal{C}$ defined in the familiar "tensor-product-like" manner; that is, it is made up of the coequalizer in $\mathcal{V}$-Mat(X, Z) of the pair

$$\Psi\lambda, \varrho\Phi : \Psi\mathcal{B}\Phi \to \Psi\Phi,$$

the ϱ induced by the ϱ of Φ, and the λ induced by the λ of Ψ.

This defines a bicategory $\mathcal{V}$-Mod whose objects are $\mathcal{V}$-categories and whose arrows are $\mathcal{V}$-modules.

The category $\mathcal{V}$-Mod$(\mathcal{A}, \mathcal{B})$ has small colimits since $\mathcal{V}$-Mat(X, Y) has small colimits and $\mathcal{V}$-Mat$(\mathcal{A}, \mathcal{B})$ preserves them. Composition with a $\mathcal{V}$-module preserves the small colimits since coequalizers commute with colimits. So $\mathcal{V}$-Mod *is locally small-cocomplete.*

Each $\mathcal{V}$-functor $F : \mathcal{A} \to \mathcal{B}$ determines a $\mathcal{V}$-module $F_* : \mathcal{A} \to \mathcal{B}$ whose underlying matrix is the composite

$$X \xrightarrow[F_*]{} Y \xrightarrow[\mathcal{B}]{} Y,$$

and whose actions ϱ, λ are the composites

$$\mathcal{B}F_*\mathcal{A} \xrightarrow[\mathcal{B}\bar{F}]{} \mathcal{B}\mathcal{B}F_* \xrightarrow[\mu F_*]{} \mathcal{B}F_*, \qquad \mathcal{B}\mathcal{B}F_* \xrightarrow[\mu F_*]{} \mathcal{B}F_*.$$

Modules of the form $F_*: \mathscr{A} \to \mathscr{B}$ have right adjoints $F^*: \mathscr{B} \to \mathscr{A}$. The $\mathscr{W}$-functor F is *fully faithful* if and only if the unit $1_{\mathscr{A}} \to F^*F_*$ is invertible. If the $\mathscr{W}$-functor F is *bijective on objects*, then the counit gives a coequalizer diagram:

$$F_*F^*F_*F^* \rightrightarrows F_*F^* \to 1_{\mathscr{B}}$$

in $\mathscr{W}$-Mod$(\mathscr{B}, \mathscr{B})$; for this is now the Eilenberg–Moore category $\mathscr{W}$-Mod$(\mathscr{A}, \mathscr{B})^{F^*F_*}$. For $\mathscr{W}$-functors $F, G: \mathscr{A} \to \mathscr{B}$, there are natural bijections between 2-cells $F_* \to G_*$ in $\mathscr{W}$-Mod, 2-cells $G^* \to F^*$ in $\mathscr{W}$-Mod, and $\mathscr{W}$-natural transformations $F \to G$.

[We have extended the "hyperdoctrine" Set$/\mathscr{U} \to \mathscr{W}$-Mat of Section 1 to a "hyperdoctrine" $\mathscr{W}$-Cat$\to \mathscr{W}$-Mod.]

As remarked just before Proposition 3, objects A, B of $\mathscr{A}$ over U, V can be regarded as $\mathscr{W}$-functors $A: U \to \mathscr{A}$, $B: V \to \mathscr{A}$. Observe further that $\mathscr{A}(A, B) \cong A^*B_*$. Given a cospan:

$$\mathscr{B} \xrightarrow{G} \mathscr{C} \xleftarrow{F} \mathscr{A}$$

in $\mathscr{W}$-Cat, it is therefore consistent to denote the $\mathscr{W}$-module $G^*F_*: \mathscr{A} \to \mathscr{B}$ by $\mathscr{C}(G, F)$. We shall now see that every $\mathscr{W}$-module has this form.

The mapping cone $\mathrm{Cn}(\Phi)$ of a $\mathscr{W}$-module $\Phi: \mathscr{A} \to \mathscr{B}$ is the $\mathscr{W}$-category defined as follows. Suppose $\mathscr{A}, \mathscr{B}$ are monads on X, Y in $\mathscr{W}$-Mat. Then $\mathrm{Cn}(\Phi)$ is the monad on $Y+X$ made up of the matrix

$$\begin{pmatrix} \mathscr{B} & \Phi \\ 0 & \mathscr{A} \end{pmatrix} : Y+X \to Y+X,$$

with unit

$$\begin{pmatrix} \eta & 0 \\ 0 & \eta \end{pmatrix},$$

and multiplication

$$\begin{pmatrix} \mu & (\varrho, \lambda) \\ 0 & \mu \end{pmatrix}.$$

In an obvious way we obtain a cospan

$$\mathscr{B} \xrightarrow{J} \mathrm{Cn}(\Phi) \xleftarrow{I} \mathscr{A}$$

in $\mathscr{W}$-Cat, and $\mathrm{Cn}(\Phi)(J, I) = J^*I_* \cong \Phi$.

4. Right liftings and limits

Suppose now that $\mathscr{W}$ satisfies the following further conditions:

C1. Each hom-category $\mathscr{W}(U, V)$ has small limits.

C2. Each pair of arrows $F: U \to W$, $g: V \to W$ admits a right lifting $g \pitchfork f: U \to V$ of f through g:

$$\frac{h \to g \pitchfork f}{gh \to f}$$

Theorem 8. *The bicategories* $\mathscr{W}$-Mat *and* $\mathscr{W}$-Mod *both satisfy conditions* C1 *and* C2.

Proof. Limits in $\mathscr{W}$-Mat(X, Y) can be constructed componentwise so that C1 for $\mathscr{W}$-Mat is easy. For matrices $S: X \to Z$, $T: Y \to Z$, the formula for $T \pitchfork S: X \to Y$ is:

$$(T \pitchfork S)(y, x) = \prod_z T(z, y) \pitchfork S(z, x);$$

with this, C2 is easily checked.

Since $\mathscr{W}$-Mod$(\mathscr{A}, \mathscr{B})$ is monadic over $\mathscr{W}$-Mat(X, Y), limits are carried over; so C1 for $\mathscr{W}$-Mod follows. For modules $\Phi: \mathscr{A} \to \mathscr{C}$, $\Psi: \mathscr{B} \to \mathscr{C}$, the module $\Psi \pitchfork \Phi: \mathscr{A} \to \mathscr{B}$ is made up of the equalizer in $\mathscr{W}$-Mat of the two arrows:

$$\begin{tikzcd} \Psi \pitchfork \Phi \arrow[rr, "\lambda \pitchfork 1"] \arrow[dr, "\varrho"'] & & \mathscr{C}\Psi \pitchfork \Phi \\ & \mathscr{C}\Psi \pitchfork \mathscr{C}\Phi \arrow[ur, "1 \pitchfork \lambda"'] & \end{tikzcd}$$

the ϱ induced by the ϱ of Φ, and the λ induced by the ϱ of Ψ. Condition C2 for $\mathscr{W}$-Mod is easily checked. □

For each $\mathscr{W}$-category $\mathscr{B}$ based on the category Set of small sets, there is a $\mathscr{W}$-cateogry $\mathscr{P}\mathscr{B}$ based on SET, defined as follows:

$$(\mathscr{P}\mathscr{B})_U = \mathscr{W}\text{-Mod}(U, \mathscr{B}), \qquad (\mathscr{P}\mathscr{B})(\Psi, \Phi) = \Psi \pitchfork \Phi.$$

There is a pseudo-natural equivalence:

$$\mathscr{W}\text{-Mod}(\mathscr{A}, \mathscr{B}) \simeq \mathscr{W}\text{-CAT}(\mathscr{A}, \mathscr{P}\mathscr{B}).$$

Precisely the same arguments used in proving Proposition 1 now yield:

Proposition 9. *The homomorphism* $\mathscr{W}$-Cat $\to$ $\mathscr{W}$-Mod *satisfies all the properties listed for the homomorphism* Set$/\mathscr{W} \to \mathscr{W}$-Mat *in Proposition* 1. □

Theorem 10. *The 2-category* $\mathscr{W}$-Cat *has all small limits.*

Proof. Suppose $J: \mathscr{I} \to$ Cat, $D: \mathscr{I} \to \mathscr{W}$-Cat are functors from a small category $\mathscr{I}$. Write D_U for the composite of D with $\mathscr{W}$-Cat$(U, -)$: $\mathscr{W}$-Cat$\to$Cat. Define a $\mathscr{W}$-

category $\mathscr{L}$ as follows. An object of $\mathscr{L}$ over U is a natural transformation $A: J \to D_U$. For objects A, B of $\mathscr{L}$ over U, V, take $\mathscr{L}(B, A)$ to be the limit in $\mathscr{W}(U, V)$ of the diagram:

$$\begin{tikzcd}[column sep=small]
(DC)(B_C j, A_C j) \arrow[dr, "(DC')(1, A_{C'}\xi) \cdot D_U n"'] & & (DC')(B_{C'} j', A_{C'} j) \arrow[dl, "(DC')(B_{C'}\xi, 1)"] \\
& (DC')(B_{C'}(Jn)j, A_{C'} j') &
\end{tikzcd}$$

as n, ξ run over arrows $n: C \to C'$, $\xi: (Jn)j \to j'$ in $\mathscr{C}$, JC', respectively. One may verify the isomorphism

$$(\mathscr{W}\text{-Cat})(\mathscr{X}, \mathscr{L}) \cong [\mathscr{C}, \mathrm{Cat}](J, \mathscr{W}\text{-Cat}(\mathscr{X}, D)). \qquad \square$$

5. Fibrations as enriched categories

Let $\mathbf{C}$ denote a small category whose set of objects is $\mathscr{U}$. Rather than the 2-category of fibrations over $\mathbf{C}$, we prefer to deal with the equivalent 2-category

$$\mathscr{H}(\mathbf{C}) = \mathrm{Hom}(\mathbf{C}^{\mathrm{op}}, \mathrm{Cat})$$

of homomorphisms from $\mathbf{C}^{\mathrm{op}}$ to Cat and strong (=pseudo-natural) transformations between them. We identify the category $\hat{\mathbf{C}} = [\mathbf{C}^{\mathrm{op}}, \mathrm{Set}]$ of presheaves on $\mathbf{C}$ with a full sub-2-category of $\mathscr{H}(\mathbf{C})$ consisting of discrete objects. We also regard $\mathbf{C}$ as a full sub-2-category of $\mathscr{H}(\mathbf{C})$ consisting of representable objects.

Recall the construction of the bicategory Spn $\mathscr{A}$ from a category $\mathscr{A}$ with pullbacks (Bénabou [1, p. 22]). Our convention is to draw a span S from U to V as

$$V \longleftarrow S \longrightarrow U,$$

and to identify an arrow $f: U \to V$ in $\mathscr{A}$ with the span

$$V \xleftarrow{f} U \xrightarrow{1} U.$$

It is a straightforward calculation to verify the following assertion (the case $\mathscr{A} = \mathrm{Set}$ suffices):

An arrow S in Spn $\mathscr{A}$ *has a right adjoint if and only if it is isomorphic to an arrow in $\mathscr{A}$.*

Let $\mathscr{W}(\mathbf{C})$ denote the full subbicategory of Spn $\hat{\mathbf{C}}$ determined by the objects which are actually in $\mathbf{C}$. Arrows in $\mathscr{W}(\mathbf{C})$ are spans in $\hat{\mathbf{C}}$ between objects of $\mathbf{C}$.

An arrow in $\mathscr{W}(\mathbf{C})$ has a right adjoint if and only if it is isomorphic to an arrow in $\mathbf{C}$. (This follows from the above assertion about adjunctions in Spn $\mathscr{A}$ and the Yoneda Lemma.)

The properties required of $\mathscr{W}$ in Section 1 and properties C1, C2 of Section 4 are satisfied by $\mathscr{W}(\mathbf{C})$ since $\hat{\mathbf{C}}$ is a Grothendieck topos.

Our purpose now is to study the relationship between $\mathscr{M}(\mathbf{C})$ and $\mathscr{W}(\mathbf{C})$-Cat. This study begins with the 2-functor

$$L:\mathscr{M}(\mathbf{C})\to\mathscr{W}(\mathbf{C})\text{-Cat}$$

defined below.

Each object T of $\mathscr{M}(\mathbf{C})$ determines a $\mathscr{W}(\mathbf{C})$-category LT defined as follows. An object of LT over U is an object of TU which we can also view as an arrow $U\to T$ in $\mathscr{M}(\mathbf{C})$ (using the bicategorical Yoneda lemma). For objects x, y of TU, TV, the arrow $(LT)(x,y): V\to U$ in $\mathscr{W}(\mathbf{C})$ is the span from V to U obtained as the comma object of $x:U\to T$, $y: V\to T$ in $\mathscr{M}(\mathbf{C})$:

$$\begin{tikzcd}
(LT)(x,y) \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\overset{\lambda}{\Rightarrow}"] & V \arrow[d, "y"] \\
U \arrow[r, "x"'] & T
\end{tikzcd}$$

Since U, V have values in Set, so does $(LT)(x,y)$. More explicitly,

$$(LT)(x,y)S=\{(u,\theta,v)\mid u:S\to U, v:S\to V \text{ in } \mathbf{C} \text{ and}$$
$$\theta:(Tu)x\to(Tv)y \text{ in } TS\}.$$

Composition for LT is given by:

$$((LT)(x,y)\circ(LT)(y,z))S\to(LT)(x,z)S,$$
$$((u,\theta,v),(v,\phi,w))\mapsto(u,\phi\theta,w).$$

For each arrow $\sigma: T\to T'$ in $\mathscr{M}(\mathbf{C})$, there is a $\mathscr{W}(\mathbf{C})$-functor $L\sigma:LT\to LT'$. The object x of LT over U is taken to $(L\sigma)x=\sigma_U x$, and the function

$$(L\sigma)_{xy}S:(LT)(x,y)S\to(LT')(\sigma_U x,\sigma_V y)S$$

takes (u,θ,v) to (u,θ',v), where θ' is the composite

$$(T'u)(\sigma_U x)\cong\sigma_S(Tu)x\xrightarrow[\sigma_S(\theta)]{}\sigma_S(Tv)y\cong(T'v)(\sigma_V y).$$

Theorem 11. *The 2-functor* $L:\mathscr{M}(\mathbf{C})\to\mathscr{W}(\mathbf{C})$-Cat *has a right adjoint with fully faithful unit.*

Proof. Since $\mathbf{C}$ is a small full dense sub-2-category of $\mathscr{M}(\mathbf{C})$ and $\mathscr{W}(\mathbf{C})$-Cat is small cocomplete (Theorem 7), a right adjoint Γ for L must have the form:

$$\Gamma\mathscr{A}=\mathscr{W}(\mathbf{C})\text{-Cat}(L-,\mathscr{A}):\mathbf{C}^{\mathrm{op}}\to\mathrm{Cat}.$$

The unit $\eta:1\to\Gamma L$ has component at T given by the composite:

$$T-\simeq\mathscr{M}(\mathbf{C})(-,T)\xrightarrow{L}\mathscr{W}(\mathbf{C})\text{-Cat}(L-,LT)=\Gamma(LT)-.$$

There is a $\mathscr{W}$-functor $i_U : U \to LU$ for each object U of $\mathbf{C}$ which takes the one object of U to 1_U as an object of LU over U. (The objects of LU over V are arrows $V \to U$ in $\mathbf{C}$.)

To see that $\eta_T : T \to \Gamma LT$ is fully faithful, take $x, y : U \to T$ in $\mathscr{H}(\mathbf{C})$ and $\theta : Lx \Rightarrow Ly$ in $\mathscr{W}(\mathbf{C})$-Cat. This gives $\theta i_U : x = (Lx)i_U \to (Ly)i_U = y$ in $(LT)_U$, which means a map of spans $1_U \to (LT)(x, y)$ from U to U:

$$\begin{tikzcd}
U \arrow[drr, bend left, "1"] \arrow[dr, "\theta"] \arrow[ddr, "1"'] & & \\
& (LT)(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & U \arrow[d, "y"] \\
& U \arrow[r, "x"'] & T
\end{tikzcd}$$

Thus we obtain a unique 2-cell

$$\begin{tikzcd}
U \arrow[r, bend left, "x"{name=A}] \arrow[r, bend right, "y"'{name=B}] \arrow[from=A, to=B, phantom, "\Downarrow\phi"] & T
\end{tikzcd}$$

in $\mathscr{H}(\mathbf{C})$ with $L\phi = \theta$. This completes the proof that η_T is fully faithful.

The $\mathscr{W}(\mathbf{C})$-functor $\varepsilon_{\mathscr{A}} : L\Gamma\mathscr{A} \to \mathscr{A}$ takes an object $A : LU \to \mathscr{A}$ over U to the object of $\mathscr{A}$ over U corresponding to $Ai_U : U \to \mathscr{A}$. Given $A : LU \to \mathscr{A}$, $B : LV \to \mathscr{B}$ in $\mathscr{W}(\mathbf{C})$-Cat, we must describe an arrow of spans

$$(L\Gamma\mathscr{A})(A, B) \to \mathscr{A}(Ai_U, Bi_V)$$

from V to U in $\hat{\mathbf{C}}$. Elements of $(L\Gamma\mathscr{A})(A, B)S$ are triples (u, θ, v) where u, v make S into a span in $\mathbf{C}$ from V to U and $\theta : A \cdot Lu \Rightarrow B \cdot Lv$ is a $\mathscr{W}(\mathbf{C})$-natural transformation. Composing with $i_S : S \to LS$, we obtain a 2-cell $(A \cdot i_U)_* u \Rightarrow (B \cdot i_V)_* v$ in $\mathscr{W}(\mathbf{C})$-Mod. This gives a 2-cell $uv^* \Rightarrow (Ai_U)^*(Bi_V)_*$ in $\mathscr{W}(\mathbf{C})$-Mod between arrows from V to U. But $\mathscr{W}(\mathbf{C})\text{-Mod}(V, U) = \mathscr{W}(\mathbf{C})(V, U)$. So we have an element of $\mathscr{A}(Ai_U, Bi_V)S \cong (Ai_U)^*(Bi_V)_* S$.

The adjunction identities can be routinely checked. $\square$

Theorem 12. *The 2-functor* $L : \mathscr{H}(\mathbf{C}) \to \mathscr{W}(\mathbf{C})$-Cat *preserves small limits.*

Proof. Since the construction of L involves comma objects which are themselves limits in $\mathscr{H}(\mathbf{C})$, the verification is routine. $\square$

6. Cofibrations and cauchy completeness

For any small bicategory $\mathscr{H}$, *fibrations in* $\mathrm{Hom}(\mathscr{H}^{\mathrm{op}}, \mathrm{Cat})$ were extensively studied in [11]. A bicategory $\mathrm{DFib}(\mathrm{Hom}(\mathscr{H}^{\mathrm{op}}, \mathrm{Cat}))$ was constructed having the same objects as $\mathrm{Hom}(\mathscr{K}^{\mathrm{op}}, \mathrm{Cat})$ and having the bidiscrete fibrations as arrows. To each homomorphism $T: \mathscr{K}^{\mathrm{op}} \to \mathrm{Cat}$ was associated its *cooperative homomorphism* $\# T: \mathscr{K}^{\mathrm{co}} \to \mathrm{Cat}$ which provided the following *representation of bidiscrete fibrations:*

$$\mathrm{DFib}(\mathrm{Hom}(\mathscr{K}^{\mathrm{op}}, \mathrm{Cat}))(S, T) \simeq \mathrm{Hom}(\mathscr{K}^{\mathrm{op}}, \mathrm{CAT})(S, [(\# T)^{\mathrm{op}}, \mathrm{Set}]).$$

A fibration in $\mathscr{K}$ is a span in $\mathscr{K}$ which is taken to a fibration by $\mathscr{K} \to \mathrm{Hom}(\mathscr{K}^{\mathrm{op}}, \mathrm{Cat})$. This agrees with the definition in [10] where the fibration property is expressed in terms of finite bilimits in $\mathscr{K}$. A finitely bicomplete and finitely bicocomplete bicategory $\mathscr{K}$ was called *fibrational* when bipullback along a leg of a fibration preserved the bicolimit involved in the definition of fibrational composition. Under these conditions one obtained a bicategory $\mathrm{DFib}(\mathscr{K})$ with the same objects as $\mathscr{K}$ and with bidiscrete fibrations in $\mathscr{K}$ as arrows.

By a change of universe, the construction of $\mathrm{DFib}(\mathscr{K})$ can be made even when $\mathscr{K}$ is not small and agrees with that of the first paragraph of this section when applied to $\mathrm{Hom}(\mathscr{K}^{\mathrm{op}}, \mathrm{Cat})$.

Fibrations in $\mathscr{K}^{\mathrm{op}}$ are called *cofibrations in* $\mathscr{K}$, and bidiscrete fibrations in $\mathscr{K}^{\mathrm{op}}$ will be called *modules in* $\mathscr{K}$. When $\mathscr{K}^{\mathrm{op}}$ is fibrational, we obtain a bicategory $\mathrm{DFib}(\mathscr{K}^{\mathrm{op}})$; set

$$\mathrm{Mod}(\mathscr{K}) = \mathrm{DFib}(\mathscr{K}^{\mathrm{op}})^{\mathrm{co}}.$$

If both $\mathscr{K}$ and $\mathscr{K}^{\mathrm{op}}$ and fibrational, there is a homomorphism

$$\mathrm{Mod}(\mathscr{K}) \to \mathrm{DFib}(\mathscr{K})$$

which is the identity on objects and which takes each module to the bicomma object of its underlying cospan. The dual construction gives a left biadjoint for this homomorphism.

Theorem 13. *For any small category* $\mathbf{C}$, *the bicategories* $\mathscr{K}(\mathbf{C})$, $\mathscr{K}(\mathbf{C})^{\mathrm{op}}$ *are both fibrational and the homomorphism of the last paragraph provides a biequivalence:*

$$\mathrm{Mod}(\mathscr{K}(\mathbf{C})) \sim \mathrm{DFib}(\mathscr{K}(\mathbf{C})).$$

Proof. It was proved in [10] that Cat and $\mathrm{Cat}^{\mathrm{op}}$ are both fibrational. Every module in Cat is the cocomma object of its comma object. This gives the result for "constant categories" ($\mathbf{C} = 1$). The "variable" case is then straightforward after [11; 3.8]. □

Theorem 14. *Suppose* $\mathscr{W}$ *is a locally small-cocomplete bicategory with a small set of objects that satisfies* C1, C2 *of Section* 4. *Then* $(\mathscr{W}\text{-}\mathrm{Cat})^{\mathrm{op}}$ *is a fibrational bicategory and there is a biequivalence*

$$\mathscr{W}\text{-}\mathrm{Mod} \sim \mathrm{Mod}(\mathscr{W}\text{-}\mathrm{Cat})$$

which is the identity on objects and takes each $\mathscr{W}$*-module to its mapping cone.*

Proof. The case where $\mathscr{W}$ has one object was dealt with in [10; §6]. The generalization here provides no difficulties. □

A module from A to B in a bicategory $\mathscr{H}$ is called *cauchy* when it has a right adjoint in Mod($\mathscr{H}$). A module from A to B in $\mathscr{H}$ is called *convergent* when there exists an arrow $f: A \to B$ in $\mathscr{H}$ such that the module is equivalent to the bicocomma object of the span

$$B \xleftarrow{f} A \xrightarrow{1_A} A.$$

Every convergent module is cauchy. Call an object B of $\mathscr{H}$ *cauchy-complete* when every cauchy module into B is convergent. Write $\mathscr{H}_{\mathrm{cc}}$ for the full subbicategory of $\mathscr{H}$ consisting of the cauchy-complete objects.

Corollary 15. *The 2-functor* $L: \mathscr{H}(\mathbf{C}) \to \mathscr{W}(\mathbf{C})$-Cat *induces a homomorphism of bicategories*

$$L: \mathrm{Mod}\,\mathscr{H}(\mathbf{C}) \to \mathscr{W}(\mathbf{C})\text{-Mod}.$$

Proof. Since Mod($\mathscr{H}$) is constructed from $\mathscr{H}$ using finite bilimits and finite bicolimits, the result follows from Theorems 11, 12, 14. □

Proposition 16. (a) *An object T of $\mathscr{H}(\mathbf{C})$ is cauchy-complete if and only if, for all objects W of $\mathbf{C}$, idempotents split in the category TW.*

(b) *An object $\mathscr{A}$ of $\mathscr{W}$-Cat is cauchy-complete if and only if, for all objects W of $\mathscr{W}$, each cauchy $\mathscr{W}$-module $W \to \mathscr{A}$ is convergent.*

Proof. Part (b) follows from the fact that the objects W of $\mathscr{W}$-Cat can be used to detect convergence of modules; as a special case, an object of Cat is cauchy-complete if and only if each module from 1 into it is convergent. It can be calculated from this (as is well known) that cauchy-complete categories are those in which idempotents split.

To prove part (a), take $T \in \mathscr{H}(\mathbf{C})$. Suppose idempotents split in each TW. For each object W of $\mathbf{C}$, the evaluation homomorphism $\mathrm{ev}_W: \mathscr{H}(\mathbf{C}) \to \mathrm{Cat}$ preserves finite limits and colimits, and so an arrow $E: X \to T$ with a right adjoint E^* in Mod($\mathscr{H}(\mathbf{C})$) gives an arrow $E_W: XW \to TW$ with a right adjoint in Mod(Cat). Since TW is a cauchy-complete category, there exists a functor $f_W: XW \to TW$ such that E_W, E^*_W are isomorphic to the discrete fibrations associated with the comma categories $TW/f_W, f_W/TW$, respectively. Since E, E^* are homomorphisms, it follows that the functors f_W are the components of a strong transformation $f: X \to T$. Clearly E converges to f. So T is cauchy-complete.

Conversely, suppose T is cauchy-complete in $\mathscr{H}(\mathbf{C})$. An idempotent in TW amounts to an idempotent in $\mathscr{H}(\mathbf{C})(W, T)$. This gives an idempotent between convergent modules whose splitting gives a cauchy module $W \to T$. Since T is cauchy-complete, this splitting converges giving a splitting of the idempotent in TW. $\square$

7. The main biequivalence

For each object T of $\mathscr{W}(\mathbf{C})$, there is a homomorphism of bicategories $\mathscr{P}T:\mathbf{C}^{\mathrm{op}}\to\mathrm{CAT}$ whose value at W is given by

$$(\mathscr{P}T)W=\mathrm{Mod}\,\mathscr{W}(\mathbf{C})(W,T)\simeq[(W\# T)^{\mathrm{op}},\mathrm{Set}].$$

This determines a homomorphism

$$\mathscr{P}:\mathscr{W}(\mathbf{C})^{\mathrm{coop}}\to\mathrm{Hom}(\mathbf{C}^{\mathrm{op}},\mathrm{CAT})$$

which is part of a Yoneda structure [11; §6]. Recall also the definition of $\mathscr{P}$ for enriched categories given earlier (Section 4). For each $T\in\mathscr{W}(\mathbf{C})$, there is a comparison $\mathscr{W}(\mathbf{C})$-functor

$$L\mathscr{P}T\to\mathscr{P}LT$$

determined using the fact that both $\mathscr{P}$'s represent modules and using Corollary 15.

For the next result it is helpful to use the explicit description of $\# T:\mathbf{C}\to\mathrm{Cat}$ for $T\in\mathscr{W}(\mathbf{C})$. The value of $\# T$ at $W\in\mathbf{C}$ is the category $W\# T$ whose objects are pairs (f,x) where $f:U\to W$ is an arrow in $\mathbf{C}$ and $x\in TU$, and whose arrows $(h,\xi):(f,x)\to(f',x')$ consist of $h:U\to U'$ in $\mathbf{C}$, $\xi:x\to(Th)x'$ in TU with $f=f'h$.

Proposition 17. *The 2-functor*

$$L:\mathrm{Hom}(\mathbf{C}^{\mathrm{op}},\mathrm{CAT})\to\mathscr{W}(\mathbf{C})\text{-CAT}$$

is a logical morphism of Yoneda structures; in other words, the comparison arrow is an equivalence

$$L\mathscr{P}T\simeq\mathscr{P}LT.$$

It follows that L takes cauchy-complete objects of $\mathscr{W}(\mathbf{C})$ into cauchy-complete $\mathscr{W}(\mathbf{C})$-categories.

Proof. The comparison arrow $(L\mathscr{P}T)_W\to(\mathscr{P}LT)_W$ takes a bidiscrete fibration E from W to T to the $\mathscr{W}(\mathbf{C})$-module Φ from W to LT given by $\Phi(x)=x^*E$:

$$\begin{tikzcd}
& \Phi(x) \arrow[dl] \arrow[dr] & & \\
U \arrow[dr, "x"'] \arrow[rr, phantom, "\mathrm{p.b.}"] & & E \arrow[dl] \arrow[dr] & \\
& T & & W
\end{tikzcd}$$

On the other hand, a $\mathscr{W}(\mathbf{C})$-module Φ from W to LT determines a functor $E:(W\# T)^{\mathrm{op}}\to\mathrm{Set}$ whose value at (f,x) is given by

$$E(f,x)=\mathscr{W}(\mathbf{C})(W,U)(f^*,\Phi(x)).$$

Clearly the bidiscrete fibration corresponding to this E (under the representation theorem) is taken to an isomorph of Φ under the comparison arrow. The remaining details are easily checked. □

Proposition 18. *The right adjoint Γ of L preserves cauchy completeness.*

Proof. Let $\mathscr{A}$ be a cauchy-complete $\mathscr{W}(\mathbf{C})$-category. Then

$$(\Gamma\mathscr{A})U=\mathscr{W}(\mathbf{C})\text{-}\mathrm{Cat}(LU,\mathscr{A})=\mathscr{W}(\mathbf{C})\text{-}\mathrm{Cat}(U,\mathscr{A})$$

(since LU is the cauchy-completion of U), which is equivalent to the full subcategory of $\mathscr{W}(\mathbf{C})\text{-}\mathrm{Mod}(U,\mathscr{A})$ consisting of the cauchy modules. Now $\mathscr{W}(\mathbf{C})\text{-}\mathrm{Mod}(U,\mathscr{A})$ is small cocomplete, so certainly idempotents split therein. Suppose $\Phi:U\to\mathscr{A}$ is cauchy and $\varrho:\Phi\to\Phi$ is an idempotent. Then we have a corresponding idempotent $\varrho^*:\Phi^*\to\Phi^*$ on the right adjoint of Φ. A splitting for ϱ^* gives a right adjoint for a splitting of ϱ. □

Theorem 19. *The 2-functor $L:\mathscr{H}(\mathbf{C})\to\mathscr{W}(\mathbf{C})\text{-}\mathrm{Cat}$ restricts to a biequivalence*

$$\mathscr{H}(\mathbf{C})_{\mathrm{cc}}\simeq(\mathscr{W}(\mathbf{C})\text{-}\mathrm{Cat})_{\mathrm{cc}}.$$

Proof. The unit $\eta:1\to\Gamma L$ is fully faithful (Theorem 11). If T is cauchy-complete in $\mathscr{H}(\mathbf{C})$ then LT is cauchy complete. Since LU is the cauchy-completion of U, we have:

$$\mathscr{W}(\mathbf{C})\text{-}\mathrm{Cat}(LU,LT)\simeq\mathscr{W}(\mathbf{C})\text{-}\mathrm{Cat}(U,LT)$$

which has the same objects as TU. It follows that η_T is surjective on objects up to isomorphism.

Suppose $\mathscr{A}$ is a cauchy-complete $\mathscr{W}(\mathbf{C})$-category. Objects of $L\Gamma\mathscr{A}$ are $\mathscr{W}(\mathbf{C})$-functors $U\to\mathscr{A}$, which amounts to objects of $\mathscr{A}$. Take two objects A,B of $\mathscr{A}$ over U,V, respectively. To give a 2-cell

$$\begin{tikzcd}
S \arrow[r, "v"] \arrow[d, "u"'] \arrow[dr, phantom, "\overset{\theta}{\Rightarrow}"] & V \arrow[d, "B"] \\
U \arrow[r, "A"'] & \Gamma\mathscr{A}
\end{tikzcd}$$

in $\mathscr{S}(\mathbf{C})$ is precisely to give a 2-cell

$$\begin{tikzcd} S \arrow[r, "v"] \arrow[d, "u"'] \arrow[dr, phantom, "\Rightarrow"] & V \arrow[d, "B_*"] \\ U \arrow[r, "A_*"'] & \mathscr{A} \end{tikzcd}$$

in $\mathscr{S}(\mathbf{C})$-Mod. But a 2-cell $A_*u \Rightarrow B_*v$ amounts to a 2-cell $uv^* \Rightarrow A^*B_*$ in $\mathscr{S}(\mathbf{C})$-Mod. This is the same as a map of spans $S \to \mathscr{A}(A,B)$. So $(L\Gamma\mathscr{A})(A,B) \cong \mathscr{A}(A,B)$. Thus $\varepsilon_{\mathscr{A}}$ is an equivalence. □

Let Rel($\mathbf{C}$) denote the bicategory whose objects are the objects of $\mathbf{C}$, whose arrows are relations in $\hat{\mathbf{C}}$ between the representables, whose 2-cells are inclusions, and whose composition is the usual composition of relations. There is a homomorphism of bicategories

$$\mathscr{S}(\mathbf{C}) \to \mathrm{Rel}(\mathbf{C})$$

which is the identity on objects and is given on hom-categories by the reflection of spans into relations.

Corollary 20. *The* 2*-functor* L *induces a biequivalence of* 2*-categories*

$$[\mathbf{C}^{\mathrm{op}}, \mathrm{Poset}] \sim (\mathrm{Rel}(\mathbf{C})\text{-}\mathrm{Cat})_{\mathrm{cc}}$$

where Poset *denotes the* 2*-category of small ordered sets.* □

The result of Walters [14] characterizing presheaves on $\mathbf{C}$ as *symmetric* cauchy-complete Rel($\mathbf{C}$)-categories is obtained on restriction of the biequivalence of Corollary 20.

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