

SHEAVES ON SITES AS CAUCHY-COMPLETE CATEGORIES

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In [9] I showed that sheaves on a space H can be regarded as *symmetric Cauchy-complete B -categories* for a certain bicategory B formed from H . Here I extend this result to sheaves on a general site. More particularly, from a site C , with pretopology P , I construct a bicategory of ‘relations’, $\mathbf{Rel}(C, P)$. Then the category of sheaves on C is biequivalent to the bicategory of symmetric Cauchy-complete $\mathbf{Rel}(C, P)$ -categories.

There are several ways of looking at this result. Firstly it extends, to general sites, the idea [4, 3] (derived from Boolean-valued set theory) that sheaves on a space are sets with equality taking values in the open-set lattice of the space. Secondly, it places topos theory (and its logic) in the context of generalized logic [6], where equality exists as symmetric hom. Thirdly, it exhibits the sheaf condition as a further example of Cauchy-completeness for categories (which property was so named by Lawvere [6] since it includes the usual notion of Cauchy-completeness for metric spaces).

1. The bicategory of relations

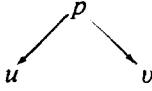
Let C be a locally small (but not necessarily small) category with pullbacks, and P a pretopology on C (see, for example, p. 12 of [5]).

If u, v are objects in C then a *crible* R from u to v is a set of spans from u to v such that

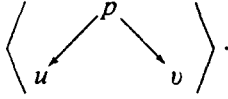
$$\begin{array}{c} \begin{array}{ccc} & p & \\ \swarrow & & \searrow \\ u & & v \end{array} \in R \quad \text{implies} \quad \begin{array}{ccc} & p' & \\ \downarrow & & \\ & p & \\ \swarrow & & \searrow \\ u & & v \end{array} \in R \end{array}$$

for any arrow $p' \rightarrow p$ with codomain p .

Of course any span



generates a *principal crible* denoted



Given a crible R from u to v , the *reverse crible* R^* from v to u is defined by

$$R^* = \{(\alpha, \beta); (\beta, \alpha) \in R\}.$$

We define a *closure operation* on the poset $\text{Cribles}(u, v)$ (order is inclusion) as follows:

$$\bar{R} = \left\{ \begin{array}{c} p \\ \swarrow \quad \searrow \\ u \quad v \end{array} ; \exists \text{ a cover } (p_i \rightarrow p)_i \text{ such that } \begin{array}{c} p_i \\ \downarrow \\ p \\ \swarrow \quad \searrow \\ u \quad v \end{array} \in R \text{ for all } i \right\}.$$

Call a closed crible a *relation*.

Examples. If C is a locale, or a Grothendieck topos (or even a lex-total category [8, 7]) and P is the canonical topology, then the relations are all principal – that is, generated as cribles by a single span. In fact, a relation R from u to v is generated by the union of those subobjects of $u \times v$ which are in R . Further a crible generated by a subobject of $u \times v$ is closed. Hence there is a bijection between relations from u to v and subobjects of $u \times v$.

Composition of relations. Given cribles R from u to v and S from v to w we define

$$S \circ R = \{(\alpha, \beta); \exists \gamma \text{ such that } (\alpha, \gamma) \in R, (\gamma, \beta) \in S\}.$$

It is straightforward to check the following properties:

- (i) $T^\circ (S \circ R) = (T^\circ S) \circ R$.
- (ii) $\langle (1_v, 1_v) \rangle \circ R = R = R \circ \langle (1_u, 1_u) \rangle$.
- (iii) If $S \subseteq S'$ and $R \subseteq R'$ then

$$S \circ R \subseteq S' \circ R \quad \text{and} \quad S \circ R \subseteq S \circ R'.$$

- (iv) $(\bigcup_i S_i) \circ R = \bigcup_i (S_i \circ R)$, $S \circ \bigcup_i R_i = \bigcup_i (S \circ R_i)$
- (v) $(S \circ R)^* = R^* \circ S^*$.
- (vi) $\overline{S \circ R} = \overline{S} \circ \overline{R}$.

Definition of $B = \text{Rel}(C, P)$. The objects of B are the objects of C . The arrows from u to v are the relations. The 2-cells are inclusions.

Composition of relations R from u to v , S from v to w is given by

$$S \cdot R = \overline{S \circ R}.$$

The identity of u is $\langle \overline{(1_u, 1_u)} \rangle$.

Using the properties listed above, it is easy to prove that B is a bicategory, locally a cocomplete poset (with $\bigvee R_i = \bigcup R_i$), and that composition (on either side) preserves suprema. Further $(\)^*: B \rightarrow B$ is an involution which is the identity on objects.

2. B -categories and modules

I recall here briefly some of the theory of categories based on a bicategory B [1, 2, 9]. I assume that B satisfies the conditions of the last paragraph.

A B -category X is a set X with functions $e: X \rightarrow \text{obj } B$ and $d: X \times X \rightarrow \text{morph } B$ satisfying:

- (i) $d(x_1, x_2): e(x_1) \rightarrow e(x_2)$,
- (ii) $1_{e(x)} \leq d(x, x)$,
- (iii) $d(x_2, x_3) \cdot d(x_1, x_2) \leq d(x_1, x_3)$.

We say that X is *small* if $e^{-1}(u)$ is small for all $u \in B$.

A B -functor, $f: X \rightarrow Y$, is a function $f: X \rightarrow Y$ satisfying

- (i) $e(x) = e(f(x))$,
- (ii) $d(x_1, x_2) \leq d(fx_1, fx_2)$.

A B -transformation exists from f to $g: X \rightarrow Y$ (and we write $f \leq g$) if $1_{e(x)} \leq d(fx, gx)$ all $x \in X$.

So B -categories form a bicategory $B\text{-Cat}$, and it makes sense to talk about B -equivalent B -categories. A B -category X is called *skeletal* if, whenever $e(x) = e(x') = u$ and $1_u \leq d(x, x')$, $1_u \leq d(x', x)$, then $x = x'$. As usual we have that every B -category is B -equivalent to a skeletal B -category.

A B -category X is *symmetric* if

$$d(x_1, x_2) = (d(x_2, x_1))^* \quad \text{all } x_1, x_2 \text{ in } X.$$

To each object u of B there is a special B -category denoted \hat{u} defined by $\hat{u} = \{*\}$, $e(*) = u$, $d(*, *) = 1_u$.

Now to describe Cauchy-completeness I need to define the notion 'adjoint pair of modules' (see [6]) between B -categories. For brevity I will look only at the special case when one of the B -categories is of the form \hat{u} . An *adjoint pair of modules from \hat{u} to X* is a pair of functions, $\phi, \psi: X \rightarrow \text{morph } B$, satisfying

- AM(i) $\phi(x): u \rightarrow e(x)$, $\psi(x): e(x) \rightarrow u$.
- AM(ii) $d(x, x') \cdot \phi(x) \leq \phi(x')$, $\psi(x) \cdot d(x', x) \leq \psi(x')$.
- AM(iii) $1_u \leq \bigvee_x \psi(x) \cdot \phi(x)$.
- AM(iv) $\phi(x') \cdot \psi(x) \leq d(x, x')$.

A B -category X is *Cauchy-complete* if for each $u \in B$ and each adjoint pair of bi-modules ϕ, ψ from \hat{u} to X there is an element $x \in X$ such that $e(x) = u$ and

$$\phi(y) = d(x, y), \quad \psi(y) = d(y, x) \quad (\text{all } y \in X).$$

3. Sheaves

We define a functor

$$L : \mathbf{Shv}(C, P) \rightarrow B\text{-}\mathbf{Cat} \quad (B = \mathbf{Rel}(C, P))$$

as follows: If F is a sheaf then

$$L(F) = \coprod_{u \in C} F(u).$$

Further, if $s \in F(u)$, $t \in F(v)$ then

$$e(s) = u, \quad d(s, t) = \left\{ \begin{array}{c} p \\ \alpha \swarrow \quad \searrow \beta \\ u \qquad \qquad v \end{array} ; F\alpha(s) = F\beta(t) \right\}.$$

It is straightforward to show that $d(s, t)$ is a relation, and in fact that $L(F)$ is a B -category. The extension of L to morphisms is immediate and it is easy to see that L is a fully-faithful functor. Notice that $d(t, s) = (d(s, t))^*$; and that if s and t have $e(s) = e(t) = u$, $1_u \leq d(s, t)$ then $s = t$. Hence L lands on symmetric skeletal B -categories.

Proposition 1. $L(F)$ is Cauchy-complete.

Proof. Consider an adjoint pair of modules ϕ, ψ from \hat{u} to $L(F)$. Condition AM(iii) says that

$$(1_u, 1_u) \in \overline{\bigcup_{s \in L(F)} \psi(s) \circ \phi(s)}.$$

Hence there is a cover

$$(u_i \xrightarrow{\alpha_i} u)_{i \in I}$$

such that for each $i \in I$ there is an $s_i \in L(F)$ with

$$(\alpha_i, \alpha_i) \in \psi(s_i) \circ \phi(s_i).$$

This means that for these s_i there are arrows λ_i such that $(\alpha_i, \lambda_i) \in \phi(s_i)$ and $(\lambda_i, \alpha_i) \in \psi(s_i)$.

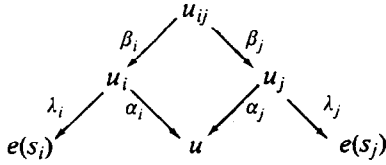
Make a choice of this data: the cover, the s_i and the λ_i . Define sections σ_i over u_i by

$$\sigma_i = F(\lambda_i)(s_i).$$

I claim that these σ_i are a compatible family of sections. First note that σ_i does not depend on the choice of λ_i . Suppose μ_i is another arrow with $(\alpha_i, \mu_i) \in \phi(s_i)$, $(\mu_i, \alpha_i) \in \psi(s_i)$. Then

$$(\lambda_i, \mu_i) \in \phi(s_i) \circ \psi(s_i) \leq d(s_i, s_i) \quad (\text{by AM(iv)}).$$

Hence $F\lambda_i(s_i) = F\mu_i(s_i)$. Now to prove the compatibility of the σ_i consider the diagram



where the diamond commutes. We need to show that $F\beta_i(\sigma_i) = F\beta_j(\sigma_j)$. But clearly

$$(\lambda_i\beta_i, \lambda_j\beta_j) \in \phi(s_j) \circ \psi(s_i) \leq d(s_i, s_j) \quad (\text{by AM(iv)}).$$

Hence

$$F\beta_i(\sigma_i) = F(\lambda_i\beta_i)(s_i) = F(\lambda_j\beta_j)(s_j) = F\beta_j(\sigma_j).$$

Since F is a sheaf we get the existence of a section σ over u such that $F\alpha_i(\sigma) = \sigma_i$.

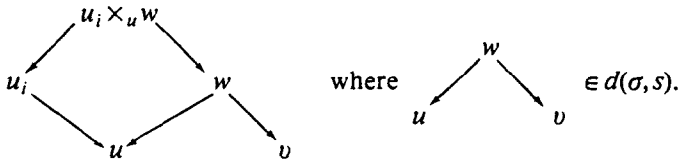
I will now prove that

$$\phi(s) = d(\sigma, s) \quad \text{and} \quad \psi(s) = d(s, \sigma).$$

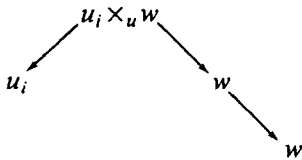
The first step is to show that

$$d(\sigma, s) = \overline{\bigcup_{i \in I} d(\sigma_i, s) \circ \langle \alpha_i, 1_{u_i} \rangle}.$$

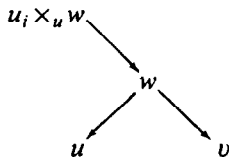
The inclusion of the right-hand-side in the left is easy. To see the opposite inclusion consider the diagram



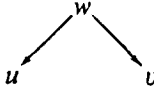
Clearly,



is in $d(\sigma_i, s)$ and hence



belongs to $d(\sigma_i, s) \circ \langle (\alpha_i, 1_u) \rangle$. But $(u_i \times_u w \rightarrow w)_i$ is a cover and so



is in the right-hand-side.

Next, notice that $\langle (\alpha_i, 1_{u_i}) \rangle \leq \phi(\sigma_i)$ (by AM(ii)) and so

$$d(\sigma, s) \leq \overline{\bigcup_i d(\sigma_i, s) \cdot \phi(\sigma_i)} \leq \phi(s) \quad (\text{by AM(ii)}).$$

Similarly

$$d(s, \sigma) \leq \psi(s).$$

Finally, notice that

$$\begin{aligned} \phi(s) &= \phi(s) \circ \langle (1_u, 1_u) \rangle \\ &\leq \phi(s) \circ \overline{\bigcup_i (d(\sigma_i, \sigma) \circ \langle (\alpha_i, 1_{u_i}) \rangle)} \\ &\leq \overline{\bigcup_i \phi(s) \circ d(\sigma_i, \sigma) \circ \langle (\alpha_i, 1_{u_i}) \rangle} \\ &\leq \overline{\bigcup_i \phi(s) \circ \psi(\sigma_i) \circ \langle (\alpha_i, 1_{u_i}) \rangle} \quad (\text{just proved}) \\ &\leq \overline{\bigcup_i d(\sigma_i, s) \circ \langle (\alpha_i, 1_{u_i}) \rangle} \quad (\text{AM(iv)}) \\ &= d(\sigma, s). \quad \square \end{aligned}$$

Proposition 2. *If X is a Cauchy-complete, skeletal, symmetric B -category then X is $L(F)$ for some sheaf F .*

Proof. It is clear how to define F on objects:

$$Fu = e^{-1}(u).$$

Given an arrow $\gamma: u \rightarrow v$ in C , and an element $x \in X$ over v we want to define $F\gamma(x)$ over u . To construct this element we consider the adjoint pair of bimodules from \hat{u} to X

$$\phi(y) = \{(\alpha, \beta); (\gamma\alpha, \beta) \in d(x, y)\}, \quad \psi(y) = \phi^*(y).$$

Then $F\gamma(x)$ is the element guaranteed by the Cauchy-completeness of X . Since X is skeletal $F\gamma(x)$ is uniquely determined by

$$d(F\gamma(x), y) = \phi(y) \quad \text{all } y \in X.$$

Functoriality of F follows easily.

Notice that if $\delta: u \rightarrow w$ and x' lies over w then $F\gamma(x) = F\delta(x')$ iff $(\gamma, \delta) \in d(x, x')$; so $L(F) = X$.

Now let's check that F is a sheaf. Suppose $(x_i)_{i \in I}$ is a compatible family of sections $(x_i$ lying over $u_i)$, where

$$(u_i \xrightarrow{\alpha_i} u)_{i \in I}$$

is a cover. To produce a section over u consider the adjoint pair of modules from \hat{u} to X :

$$\phi(y) = \bigcup d(x_i, y) \circ \langle \alpha_i, 1_{u_i} \rangle, \quad \psi(y) = \phi^*(y).$$

It is not hard to check that the resulting section, x say, is unique with the property that $F\alpha_i(x) = x_i$ ($i \in I$). \square

As a consequence of these two propositions we have:

Theorem. *$\mathbf{Shv}(C, P)$ is biequivalent to the bicategory of symmetric Cauchy-complete $\mathbf{Rel}(C, P)$ -categories.*

Proof. The functor L defined earlier is in fact a homomorphism of bicategories. The fully-faithfulness of L , together with the scarcity of 2-cells involving skeletal B -categories, imply that L is always an equivalence of hom-categories. Since every B -category is B equivalent to a skeletal B -category, Propositions 1 and 2 imply that the B -categories of the form $L(F)$ are exactly those which are B -equivalent to symmetric Cauchy-complete B -categories. \square

Corollary. *If E is a Grothendieck topos (or even a lex-total category) then E is biequivalent to the bicategory of small symmetric Cauchy-complete $\mathbf{Rel} E$ -categories.*

Proof. Complete the identification of $\mathbf{Rel}(E, P)$ (P the canonical topology), begun in Section 1, as the usual bicategory of relations $\mathbf{Rel} E$. Then $E \approx \mathbf{Shv}(E, P)$ (see [7] for the lex-total case). \square

Acknowledgement

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