MODELS OF CONTINUOUS AND DISCRETE DYNAMICAL SYSTEMS

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The authors have introduced in [KSW97a], KSW97b] two complementary algebras for systems with boundaries. The first is loops in the suspension of a monoidal category $(C,\otimes)$, denoted $\Omega(\Sigma(C,\otimes))$, an algebra of deterministic systems in which the natural feedback operation has delay. The second is **Span(Graph)**, an algebra of specifications of systems (nondeterministic transition systems), which is a discrete cartesian bicategory, in which the natural feedback is instantaneous. This algebra provides a compositional calculus of transition systems with interfaces, which generalizes the operation of product of automata of [A94], [Z87], and Kurshan. Further, to an expression in each algebra there is a graphic representation, which corresponds to the distributed nature of the system.
We show here that in addition to the discrete examples described earlier also continuous linear systems (including RLC circuits) are examples of the algebra. The main difference between this approach and the traditional one [Rugh96] is that our theory is completely compositional, and includes in a common framework the continuous and the discrete. This approach provides a clear algebraic setting in which to develop the theory of hybrid systems.

CONTINUOUS LINEAR SYSTEM THEORY

The mathematical structure of a graph can, in some cases, model continuous systems: the domain and codomain functions of a graph can be thought of as giving the position p and velocity v of an continuous motion rather than source and target of a discrete motion.
In this note we consider input-output systems governed by a system of linear differential (and integral) equations [Rugh96]. The matrix of coefficients of a system is described as an arrow in $\Omega(\Sigma(Vect,\oplus))$ where $Vect$ is finite dimensional real vector spaces, and $\oplus$ is the direct sum. We describe two functors D and I to **Span(Graph**$(Vect,\oplus)$**)** such that behaviours of the image spans are solutions respectively of differential and integral equations with those coefficients. **Span(Graph)** is an environment in which these different systems may be composed. Within the theory of continuous linear systems is of course the theory of linear RLC circuits; in fact, we shall see that there are a class of arrows in $\Omega(\Sigma(Vect,\oplus))$ whose image under D are (models of) capacitors and whose image under I are inductors. The geometry associated to an expression in our algebra is the classical circuit picture.

Given a graph $(p,v): M \to S^2$ in $Vect$ we define a behaviour of this graph to be a differentiable function $f:\mathbf{R}\to M$ ($\mathbf{R}$ the real numbers) such that $(p\cdot f)' = v\cdot f$. By a behaviour of a span of graphs we mean a behaviour of the head of the span. It is clear that the notion of behaviour is compositional, in the sense that if $\alpha : X \to Y$ and $\beta : Y \to Z$ are spans then to give a behaviour of $\beta\cdot\alpha$ is to give behaviours of α and β that agree on the common boundary Y.

We now define the functors D and I. Recall that an arrow in $\Omega(\Sigma(Vect,\oplus)$ from X to Y is a linear transformation of the form $X\oplus U \to U\oplus Y$, that is, a matrix of real numbers. Noting that $U\oplus Y$ is in fact a product, we will write such arrows as $(\alpha,\beta): X\oplus U \to U\oplus Y$. Let $D:\Omega(\Sigma(Vect,\oplus)) \to \mathbf{Span}(\mathbf{Graph}(Vect))$ be the homomorphism which maps an arrow $(\alpha,\beta): X\oplus U \to U\oplus Y$ to the span from X to Y with head $(\pi_2,\alpha): X\oplus U \to U^2$, left leg $\pi_1 : X\oplus U \to X$ and right leg $\beta : X\oplus U \to Y$. Clearly, to give a behaviour of $D(\alpha,\beta)$ is to give a pair of differentiable functions $(f,g):\mathbf{R}\to X\oplus U$ such that $g' = \alpha(f,g)$, that is, a solution of a system of differential equation. Let $I:\Omega(\Sigma(Vect,\oplus)) \to \mathbf{Span}(\mathbf{Graph}(Vect))$ be defined by: $I(\alpha,\beta)$ is the dual graph of $D(\alpha,\beta)$. Then I of a span can be regard as an integral equation.

AN ALGEBRAIC FOUNDATION FOR RLC CIRCUIT THEORY

We are now in a position to define the basic components needed to build RLC circuits. Let $X = \mathbf{R}^2$. We will be thinking of the first variable i of X as current and the second variable v as voltage. Resistors, capacitors and inductors will be defined to be certain spans $X \to X$. A behaviour of such a span induces a behaviour $(i(t), v(t), j(t), w(t))$ in $(X \oplus X)^{\mathbf{R}}$ on the boundaries; it is these behaviours that we consider below.

RESISTORS For any R in **R**, let $\rho_R : X \to X$ be the transformation $(i,v) \mapsto (i, v - Ri)$. The resistor of resistance R is the span $D(\rho_R) : X \to X$. Notice that this equals $I(\rho_R)$, and we denote this span also by ρ_R. A behaviour (i, v, j, w) of the resistor ρ_R satisfies, for all t, the condition $i = j$ and $w = v - iR$. This latter equation can be written in the more familiar form $V = v - w = Ri$.

CAPACITORS For any $C \in \mathbf{R}$, let $\chi_C : X \oplus \mathbf{R} \to \mathbf{R} \oplus X$ be the transformation $(i, v, q) \mapsto (i, i, v - q/C)$. The capacitor of capacitance C is the span $D(\chi_C) : X \to X$, which we denote by χ_C. The induced boundary behaviours (i, v, j, w) satisfy $i = j$ and $V' = (v - w)' = q'/C = i/C$.

INDUCTORS For any $L \in \mathbf{R}$, let $\iota_L : X \oplus \mathbf{R} \to \mathbf{R} \oplus X$ be the transformation $(i, v, q) \mapsto (i, i, v - Lp)$. Notice that $\iota_L = \chi_{1/L}$. The inductor of inductance L is the span $I(\iota_L) : X \to X$ which we denote by ι_L. The induced boundary behaviours (i, v, j, w) satisfy $i = j$ and $V = (v - w) = Lp = Li'$.

KIRCHHOFF WIRES The Kirchhoff diagonal wire is the span $W_{1,2} : X \to X^2$ which has as centre the graph $! : \mathbf{R}^3 \to 1^2$, as left leg the transformation $(i_1, i_2, v) \mapsto (i_1 + i_2, v)$ and as right leg the transformation $(i_1, i_2, v) \mapsto (i_1, v, i_2, v)$. It is important to note that this is not the diagonal of the discrete Cartesian structure of **Span(Graph**$(Vect, \oplus)$**)**. Other Kirchhoff wires can be constructed from $W_{1,2}$ using the compact closed structure of **Span(Graph**$(Vect, \oplus)$**)**.

RLC CIRCUITS The bicategory **RLC** of RLC circuits is the smallest self-dual compact closed sub-bicategory of **Span** containing the above basic components. Rather than just an arrow in **RLC**, by an RLC circuit we sometimes mean an expression built out of resistors, inductors, capacitors and the Kirchhoff diagonal wire using only the self-dual compact closed structure of **Span** (that is, composition, tensor product, the identity $1_X : X \to X$, the symmetry $s_X : X \oplus X \to X \oplus X$, the unit $\eta_X : 1 \to X \oplus X$, and the counit $\varepsilon_X : X \oplus X \to 1$.) Of course, the diagrams of these expressions correspond to the classical RLC circuit diagrams.

This theory permits a more subtle analysis than given by classical linear circuit theory by virtue of the fact that the internal state of a component is explicitly described. For example, the composite of two capacitors is not a capacitor as its internal state is $\mathbf{R}^2$. However, it has the same input-output behaviours as a capacitor (if started in appropriate initial states); in fact, the diagonal function $\Delta : \mathbf{R} \to \mathbf{R} \oplus \mathbf{R}$ provides a 2-cell from the capacitor $\chi_{(C_2 + C_1)/C_1C_2}$ to the composite $\chi_{C_2} \cdot \chi_{C_1}$.

The reader may like to contrast this approach to electrical circuit theory with the non-compositional one proposed in [Smale72].

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