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ON COMPLETENESS OF LOCALLY-INTERNAL CATEGORIES*

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Locally-internal categories over a topos **E** are regarded as categories enriched in the bicategory Span **E**. In this paper we develop some aspects of completeness. For locally-internal categories, completeness means: finite completeness over each fiber, transition functors along the maps of the base topos, and a Beck–Chevalley condition (see Bénabou, C.R. Acad. Sci. Paris 281 (1975) A897–900). We prove that this notion can be obtained by particularizing to Span **E** the general notion of completeness of enriched category theory, given in terms of indexed limits. We give also an adjoint functor theorem.

Introduction

Early in the development of category theory the necessity for the following three extensions of the notion of category became clear: (i) internal (or small) categories - whose totality of morphisms and of objects are themselves objects of a base category **E**, (ii) enriched categories - which have a naive set of objects but the totality of morphisms between each pair of objects is an object of a base category **E**, and (iii) fibrations - categories which vary over a base category **E**. (Already in Chapter 1 of [12], MacLane gives (i) and (ii) as alternatives to the naive definition of category.) Categories which are both enriched over **E** and vary over **E** are called locally internal categories, and have been studied, using a variety of formulations, by Lawvere [10], Benabou [1,2], Penon [14], Pare–Schumacher [13], Street [15]; (see also Johnstone [7]). In this and other papers we wish to show that the study of both internal categories in **E** and locally internal categories based on **E** (and the relation between these notions) may be carried out very appropriately in the context

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of categories enriched over the bicategory Span(**E**) (see also Street [16], Betti–Walters [4]). In contrast to Penon's formulation (in which the fibre over u of the locally internal category is enriched over $\mathbf{E}/u$) our description is two-sided (as implicitly indicated in Lawvere [10]).

Many of the important notions of locally internal category theory are exactly standard notions of enriched category theory. In this paper we concentrate on aspects of completeness. The universal property of Cartesian arrows, and completeness (with Beck–Chevalley condition) have natural expressions in terms of indexed limits. We give here also an adjoint functor theorem (see also Pare–Schumacher [13]). In [4] it is shown that functor categories can be defined, as usual in enriched category theory, using ends.

A feature of our development of enriched category theory over Span(**E**) is that we use only elementary properties of the base, in particular completeness with respect to internal categories. We do not make the usual assumption of external (co)completeness of the base. In subsequent papers we intend to develop in a similar elementary way the theory of categories which vary over one base and are enriched in another – enriched category theory over a base topos.

1. Preliminaries

For the general definitions of enriched category theory over a bicategory **B** see the references (for example [3,6]). To fix notation we shall denote categories enriched in **B** by $X, Y, Z, \ldots$; their objects will be denoted $x, y, z, \ldots$; x will lie over $e(x)$ in **B**; the hom in X from x to x' will be denoted $X(x,x'): e(x) \to e(x')$; the unit of x will be denoted $\iota_x: 1 \to X(x,x)$; composition will be denoted μ_{xyz}: $X(y,z)X(x,y) \to X(x,z)$. Further, **B**-functors will be denoted by $F: X \to Y$, G, $H, \ldots$; modules by $\Phi: X \nrightarrow Y$, $\Psi, \ldots$.

In this paper the main example of a base bicategory is provided by Span(**E**) where **E** is an elementary topos.

1.1. Span(E)

Objects of Span(**E**) are objects of **E**.

Arrows $\varphi: u \nrightarrow v$ are spans $(f: w \to u,\ g: w \to v)$ of maps in **E**.

2-cells $\varphi \to \psi$ (where $\psi = (f', g')$) are maps $h: w \to w'$ in **E** such that $f'h = f$ and $g'h = g$.

Composition is given by pullback and $\Delta = (1_u, 1_u)$ is the identity of u.

The assignment to each span $(f,g): u \nrightarrow v$ of the opposite span $(g,f): v \nrightarrow u$ extends in a natural way to an isomorphism $(\cdot)^\circ: \mathrm{Span}(\mathbf{E})^{\mathrm{op}} \to \mathrm{Span}(\mathbf{E})$ with the property that $(\cdot)^{\circ\circ} = 1$.

To each map $f: u \to v$ of **E** there is a corresponding arrow $(1,f): u \nrightarrow v$ in Span(**E**) and such arrows are characterized (up to isomorphism) by the fact that they have right adjoints. Further if φ has a right adjoint, then $\varphi \dashv \varphi^\circ$. We will call arrows of Span(**E**) with right adjoints *maps*, and we will identify them with arrows of E.

A useful way of thinking of Span(**E**) is to consider arrows as matrices

$$\varphi = (\varphi_{ij})_{i \in u,\, j \in v}$$

of objects of **E**. Then composition is the matrix product

$$(\psi \cdot \varphi)_{ik} = \sum_{j \in v} \psi_{jk} \times \varphi_{ij}.$$

When **E** is an elementary topos it is clear that each hom-category of Span(**E**) is finitely complete and cocomplete. Further, Span(**E**) admits right extensions and right liftings (see Kock–Wraith [9]); that is, for each pair of arrows φ and ψ as in the picture (1) (resp. (2))

$$(1) \quad \begin{tikzcd} & v \arrow[dd, "\hom^u(\varphi,\, \psi)"] \\ u \arrow[ur, "\varphi"] \arrow[dr, "\psi"'] \arrow[r, phantom, "\Downarrow"] & {} \\ & w \end{tikzcd} \qquad\qquad (2) \quad \begin{tikzcd} v \arrow[dr, "\varphi"] & \\ {} \arrow[r, phantom, "\Downarrow"] & u \\ w \arrow[uu, "\hom_u(\varphi,\, \psi)"] \arrow[ur, "\psi"'] & \end{tikzcd}$$

there exists a right extension $\hom^u(\varphi, \psi)$ (right lifting $\hom_u(\varphi, \psi)$) characterised by the universal property

$$\frac{\gamma \to \hom^u(\varphi, \psi)}{\gamma\varphi \to \psi} \qquad \left(\text{resp. } \frac{\gamma \to \hom_u(\varphi, \psi)}{\varphi\gamma \to \psi}\right).$$

First observe that because Span(**E**)$^{\mathrm{op}}$ is isomorphic to Span(**E**), the existence of right extensions implies the existence of right liftings. Next, if $\varphi = g \cdot f^\circ$, we have

$$\begin{aligned} \hom^u(g \cdot f^\circ, \psi) &\cong \hom^u(g, \psi f) \qquad \text{(extending along a composite)} \\ &\cong \prod_{g \times 1} (\psi, f). \end{aligned}$$

When $\mathbf{E} = \text{Sets}$, the formulae for right extensions and right liftings become

$$\hom^u(\varphi, \psi)_{jk} \cong \prod_i \hom(\varphi_{ij}, \psi_{ik}) \quad \text{and} \quad \hom_u(\varphi, \psi)_{kj} \cong \prod_i \hom(\varphi_{ji}, \psi_{ki}).$$

Observe that the existence of right adjoints to $\varphi \cdot (-)$ and $(-) \cdot \varphi$ ensures that composition with φ on both sides preserves colimits.

1.2. Internal categories

A category A internal to **E** is an arrow $(d_0 : A_1 \to A_0, d_1 : A_1 \to A_0)$ in Span(**E**) together with a monad structure. Thus an internal category is exactly a Span(**E**)-

category with only one object, whose underlying object is A_0 and whose hom is (d_0, d_1).

It should be noted that functors between internal categories are not directly the same as Span(**E**)-functors between the internal categories. In 2.4 we will see, however, that there is a natural way of representing functors between internal categories as Span(**E**)-functors.

Corresponding to any object u of the base there is an internal category with the hom equal to 1_u. These are just the discrete internal categories.

1.3. Modules

Recall that if X, Y are **B**-categories, then a module $\Phi : X \nrightarrow Y$ is the assignment of an arrow $\Phi(x, y) : ex \to ey$ for each pair of objects, with an action of X on the right and of Y on the left; that is, there are 2-cells

$$Y(y, y') \cdot \Phi(x, y) \to \Phi(x, y'), \qquad \Phi(x, y) \cdot X(x', x) \to \Phi(x', y)$$

satisfying the usual axioms of associativity, unity and mixed associativity.

When $\Phi : X \nrightarrow Y$ and $\Psi : Y \nrightarrow Z$ are modules, their composition $\Psi\Phi : X \nrightarrow Z$ is defined (if it exists) as follows: $(\Psi \cdot \Phi)(x, z)$ is the coequalizer in the category $\mathbf{B}(ex, ez)$ of the two actions

$$\sum_{y', y''} \Psi(y'', z) \cdot Y(y', y'') \cdot \Phi(x, y') \rightrightarrows \sum_{y} \Psi(y, z) \cdot \Phi(x, y).$$

A morphism $\Phi \to \Psi$ of modules $X \nrightarrow Y$ is given by a family of 2-cells $\Phi(x, y) \to \Psi(x, y)$ which is compatible with the actions.

Remark. A **B**-functor $F : X \to Y$ gives rise to two modules $Y(F, 1) : X \nrightarrow Y$ and $Y(1, F) : Y \nrightarrow X$, defined by $Y(F, 1)(x, y) = Y(Fx, y)$ and $Y(1, F)(y, x) = Y(y, Fx)$.

Remark. With base Span(**E**), **E** an elementary topos, composites of the type $X \nrightarrow A \nrightarrow Y$ always exist when A is an internal category, or even a category with a finite number of objects. There are other special composites which always exist. Given $F : X \to Y$, $\Phi : Y \nrightarrow Z$, and $G : T \to Z$ we have that $(Z(1, G) \cdot \Phi \cdot Y(F, 1))(x, t) \cong \Phi(Fx, Gt)$. Denote this last module by $\Phi(F, G)$. If $F : X \to Y$ and $G : Y \to Z$ then $Z(G, 1)Y(F, 1) \cong Z(GF, 1)$ and $Y(1, F)Z(1, G) \cong Z(1, GF)$. As usual there is a bijection between natural transformations from F to F' and 2-cells from $Y(1, F)$ to $Y(1, F')$. (For more details about the calculus of modules see Carboni, Kasangian, Walters [6].)

From the fact that composition in the base preserves local colimits it follows that, given $\Phi : X \nrightarrow Y$, $\Psi : Y \nrightarrow Z$, $\Theta : Z \nrightarrow W$, if $\Psi\Phi$ and $\Theta\Psi$ exist, then $(\Theta\Psi)\Phi$ exists if and only if $\Theta(\Psi\Phi)$ exists, and then $(\Theta\Psi)\Phi \cong \Theta(\Psi\Phi)$. This fact justifies our calculations with adjoint modules in this situation where not all composites exist.

If A is an internal category and $F: A \to X$ is a functor, then $X(F, 1)$ is left adjoint to $X(1, F)$ (the composites required to state this adjunction exist, the unit of the adjunction is the effect of F on arrows, the counit is composition in X).

1.4. *Right liftings and extensions of modules*

Proposition. *If A and C are internal categories and $\Phi : A \nrightarrow C$, $\Psi : A \nrightarrow X$ are modules, then the right extension* $\hom^A(\Phi, \Psi)$ *exists.*

Proof. An explicit formula for $\hom^A(\Phi, \Psi)(x)$ is provided by the equalizer of the following parallel pair of arrows in $\mathrm{Span}(\mathbf{E}) - (C_0, ex)$:

$$\hom^{A_0}(\Phi, \Psi x) \rightrightarrows \hom^{A_0}(\Phi, \hom^{A_0}(d_1^\circ d_0, \Psi x));$$

one of the arrows is induced by the action of A on Φ, the other by the action of A on Ψ. □

An analogous statement holds for right liftings.

1.5. *Examples*

(i) Arrows in the base $u \nrightarrow v$ are modules between discrete categories.

(ii) When A is an internal category, then modules $A \nrightarrow I$ (I is the terminal object in $\mathbf{E}$) are exactly internal presheaves.

(iii) When the base bicategory is not $\mathrm{Span}(\mathbf{E})$ but Ab, the monoidal category of abelian groups, a one-object category is just a ring and a module $\Phi : A \nrightarrow C$ is a right-A, left-C module. Composition is the tensor product of modules. When A, C, X are rings and Φ, Ψ are modules, then $\hom^A(\Phi, \Psi)$ is the right-C, left-X module of A-linear maps.

2. Locally internal categories

2.1. *Substitution*

Definition. Let X be a $\mathrm{Span}(\mathbf{E})$-category, x an object over v and $f: u \to v$ a map. A substitution of f in x is an object x_f over u, together with an isomorphism of modules $X(x, 1) \cdot f \cong X(x_f, 1)$.

We say that X has substitutions when for each x and each map f a substitution x_f exists.

Remark. There is an equivalent dual condition for x to be the substitution of f in x; namely, the existence of an isomorphism

$$X(1, x_f) \cong f^\circ \cdot X(1, x).$$

The equivalence of the two conditions is an immediate consequence of the fact that $X(x, 1) \dashv X(1, x)$, $f \dashv f^\circ$, and $X(x_f, 1) \dashv X(1, x_f)$.

Proposition. *Functors between* Span(**E**)*-categories preserve any substitutions which exist.*

Proof. Let $f: u \to v$ be a map, $x: v \to X$ an object of X and $G: X \to Y$ a functor. Then

$$Y(1, Gx_f) \cong X(1, x_f) \cdot Y(1, G) \cong f^\circ \cdot X(1, x) \cdot Y(1, G)$$
$$\cong f^\circ \cdot Y(1, Gx) \cong Y(1, (Gx)_f). \qquad \square$$

2.2. Locally internal categories

Definition. A category locally internal to a topos **E** is a Span(**E**)-category with substitutions.

Let us briefly compare our definition with those of Benabou [1], Penon [14], Pare–Schumacher [13].

Given a Span(**E**)-category X with substitutions we can construct a fibration $p: \mathbf{F} \to \mathbf{E}$ as follows. The objects of **F** are the objects of X. Suppose x lies over u and y lies over v. Then the arrows in **F** from x to y are pairs (f, α) where f is a map from u to v, and α is a 2-cell $1_u \to f^\circ \cdot X(x, y)$. The composite $(g, \beta) \cdot (f, \alpha)$ in **F** is

$$1 \xrightarrow{\alpha} f^\circ \cdot X(x, y) \xrightarrow{1 \cdot \beta \cdot 1} f^\circ \cdot g^\circ \cdot X(y, z)X(x, y) \xrightarrow{1 \cdot \mu} (gf)^\circ \cdot X(x, z).$$

Further $p(x) = e(x)$ and $p(f, \alpha) = f$. If x_f is a substitution of f in x, then $(f, \iota_{x_f}): x_f \to x$ is a Cartesian arrow.

Let us verify this last fact. Suppose $(g, \beta): z \to x$ is an arrow in **F**, and that g factorizes as $g = fh$ in **E**. Note that β is a 2-cell from 1_{ez} to $g^\circ \cdot X(z, x)$. Then (g, β) has a unique factorization in the form $(f, \iota_{x_f}) \cdot (h, \beta')$; in fact, β' is the 2-cell

$$1_{ez} \xrightarrow{\beta} g^\circ \cdot X(z, x) = h^\circ f^\circ X(z, x) \cong h^\circ \cdot X(z, x_f).$$

Denote by $\mathbf{F}_u$ (or X_u, when thinking in terms of Span(**E**)-categories) the fibre of p over 1_u. The fibration $p: \mathbf{F} \to \mathbf{E}$ is locally small; that is, for each x, y, the functor $[x, y]: \mathrm{Span}(\mathbf{E})^{\mathrm{op}}(u, v) \to \mathrm{SETS}$ which takes (f, g) to $\mathbf{F}_u(f^*x, g^*y)$ (u is the common domain of f and g) is represented, the representing object being $X(x, y)$.

Conversely, given a locally small fibration $p: \mathbf{F} \to \mathbf{E}$ we can define a Span(**E**)-category with substitutions as follows. Take the objects of X to be the objects of **F**. Define $e(x)$ to be $p(x)$. Take $X(x, y)$ to be a representing object for the functor $[x, y]$. Given x over v and a map $f: u \to v$ in **E**, f^*x is a substitution of f in x.

Notice that passing from fibrations to Span(**E**)-categories there is a choice of representing span for $[x, y]$ for each x, y in **F**. Apart from this choice the two processes described above are inverse. Further, Cartesian functors between fibrations correspond (in view of the proposition of 2.1) to Span(**E**)-functors.

Notation. From now on we will use the notation f^*x instead of x_f for the substitution of f in x.

A leading example of locally internal category is obtained as follows. Consider a category Sets of small sets and suppose that **C** is an ordinary category which is locally small with respect to Sets. The Span(Sets)-category we will describe will be denoted Fam(**C**). The objects of Fam(**C**) over u are families $x=(x_i)_{i\in u}$ of objects of **C**. If x over u and y over v are two objects of Fam(**C**), then Fam(**C**)(x, y) is the span from u to v whose fibre over i, j is $\mathbf{C}(x_i, y_j)$. Compositions and identities are straightforward. Fam(**C**) has substitutions: if $f: u\to v$ is a map and y is an object over v, then $(f^*y)_i=y_{fi}$.

2.3. *Internal presheaves*

If A is an internal category, then the Span(**E**)-category PA of internal presheaves on A is defined as follows: objects over u are modules $A \mathrel{+\!\!\!\rightarrow} u$; if $\Phi: A\mathrel{+\!\!\!\rightarrow} u$ and $\Psi: A\mathrel{+\!\!\!\rightarrow} v$ are two modules, then $PA(\Phi, \Psi)=\hom^A(\Phi, \Psi): u\mathrel{+\!\!\!\rightarrow} v$.

If X is a Span(**E**)-category and A is an internal category, then there is a bijection between Span(**E**)-functors $F: X\mathrel{+\!\!\!\rightarrow} PA$ and modules $\Phi: A\mathrel{+\!\!\!\rightarrow} X$ (natural in X) given by $\Phi(x, a)=(Fx)(a)$.

Proposition. *PA is a locally internal category.*

Proof. Given objects $\Phi: A\mathrel{+\!\!\!\rightarrow} v$ and $\Psi: A\mathrel{+\!\!\!\rightarrow} w$ of PA, and a map $f: u\to v$, then since $f\dashv f^\circ$ it follows that $\hom^A(\Phi, \Psi)\cdot f\cong \hom^A(f^\circ\cdot\Phi, \Psi)$. Hence $f^\circ\cdot\Phi$ is a substitution of f in Φ. $\square$

Example (PI). Let I be the terminal object of **E**. Then PI may be thought of **E** itself regarded as a locally internal category. Its objects over u are arrows $I\mathrel{+\!\!\!\rightarrow} u$; that is, maps in **E** with codomain u. If $f: w\mathrel{+\!\!\!\rightarrow} u$ and $g: w'\mathrel{+\!\!\!\rightarrow} v$ are two objects, then $PI(f, g)\cong \prod_{f\times 1}(1\times g)$. (When **E** = Sets this formula can be written $PI(f, g)_{ij}=\hom(f^{-1}(i), g^{-1}(j))$.) Substitution is pullback.

2.4. *Adjoining substitutions*

We will now describe how to adjoin substitutions freely to a Span(**E**)-category. The construction is as follows (see also Street [16]). Given a category X, the objects of LX over v are pairs (x, h) where $h: v\to u$ is a map of **E** and x is an object of X over u. The hom is given by $LX((x, h), (y, k))=k^\circ\cdot X(x, y)\cdot h$.

Substitutions in LX are given by $f^*(x, h)=(x, hf)$.

There is a functor $\Delta: X\to LX$ given by $x\to(x, 1)$.

Proposition. *LX is the free category with substitutions generated by X.*

Proof. Suppose $F: X \to Y$ is any functor and Y has substitutions. Then we can define $G: LX \to Y$ by $G(x, h) = h^*(Fx)$ and check that $G \cdot \Delta = F$. So Span(**E**)-cat(X, Y)) $\cong$ Span(**E**)-cat(LX, Y). $\square$

Example (Internal categories). An internal functor between internal categories A and C is exactly a Span(**E**)-functor from A to LC. In fact we have

$$\mathrm{IntCat}(A, C) \cong \mathrm{Span}(\mathbf{E})\text{-}\mathrm{Cat}(A, LC) \simeq \mathrm{Span}(\mathbf{E})\text{-}\mathrm{Cat}(LA, LC).$$

Example (Internal full subcategory). If X is a locally internal category and x is an object of X, we can consider the 'internal full subcategory' determined by x by taking a one-object category with the same underlying object as x, and $X(x, x)$ as hom.

This is Penon's notion [14]. The original notion, due to Benabou, is concerned with $X = PI$. Given $f: v \to u$ in **E**, consider it as an object of I. $\mathrm{Full}_{\mathbf{E}}(f)$ is the internal category determined by the object f.

3. Completeness

3.1. Definitions and simple properties

The correct notion of limit for enriched categories is that of indexed limit (Street [16], Betti–Walters [4]) generalizing the analogous notion for categories based on a monoidal category (Borceux–Kelly [5], Kelly [8]).

Definition. Given functor $G: A \to X$ and module $\Phi: C \dashrightarrow\!\!\!\!\!\!+\ \ X$ and module $\Phi: C \,{+\!\!\!\to}\, A$

$$\begin{tikzcd} & X & \\ C \arrow[ur, "{\{\Phi, G\}}"] \arrow[rr, "\Phi"', "|" marking] & & A \arrow[ul, "G"'] \end{tikzcd}$$

the limit of G indexed by Φ is a functor $\{\Phi, G\}: C \to X$ which represents the right lifting of $X(1, G)$ through Φ; that is,

$$X(1, \{\Phi, G\}) \cong \hom_A(\Phi, X(1, G)).$$

Remark. Observe that with base Span(**E**), **E** an elementary topos, the lifting in the definition does exist when A has a finite number of objects. Notice further that in calculating limits we may without loss of generality assume that the domain category C of the module is discrete.

Definition. A Span(**E**)-category X is said to be *complete* if it admits all limits indexed by modules whose codomain category A has a finite number of objects.

Lemma (Iterated limits). *If X is a complete* Span(**E**)*-category, $G : A \to X$ a functor, and $\Phi : B \to\!\!\!\!\!\!+\; A$, $\Psi : C \to\!\!\!\!\!\!+\; B$ are modules between categories with a finite number of objects, then*

$$\{\Phi\Psi, G\} \cong \{\Psi, \{\Phi, G\}\}.$$

Proof. The result is immediate from the fact that

$$\hom_A(\Phi\Psi, X(1, G) \cong \hom_B(\Psi, \hom_A(\Phi, X(1, G)). \quad \square$$

Theorem. *If A is a* Span(**E**)*-category with a finite number of objects, then PA is complete.*

Proof. Given functor $F : C \to PA$ and module $\Psi : u \to\!\!\!\!\!\!+\; C$ where C has a finite number of objects, consider the module Φ associated to F (as in 2.2). Then $\{\Psi, F\} \cong \hom_C(\Psi, \Phi)$. $\square$

Remark. As usual there is a formula for pointwise right Kan extensions of functors in terms of indexed limits. Suppose $G : A \to C$ is a functor between categories with a finite number of objects, and $F : A \to X$ is a functor with a complete codomain. Then $\mathrm{Ran}_G(F) \cong \{C(G, 1), F\}$.

Finally let us look at some examples of indexed limit.

Example (Substitution). The substitution of a map $f : u \to v$ in an object x over v is an example of an indexed limit. The defining property of x_f is that $X(1, x_f) \cong f^\circ \cdot X(1, x)$. But since $f \dashv f^\circ \cdot (f^\circ, X(1, x)) \cong \hom_u(f, X(1, x))$. Hence a complete Span(**E**)-category is always a locally internal category.

Example (Cauchy sequences). Another example of indexed limit, where the base is not Span(**E**), is the (usual) limit of a Cauchy sequence. Recall (Lawvere [11]) that a metric space is a category enriched over $\mathbb{R}^+$ (non-negative real numbers, preordered by $\geq$ and monoidal with $+$). Let N be the null sequence $(1/n)$ of real numbers, considered as an $\mathbb{R}^+$-category. Then a functor $x : N \to X$ is a Cauchy sequence in X dominated by this null sequence (and each Cauchy sequence is equivalent to such a sequence). Consider moreover the module $\Phi : I \to N$ (I is the trivial R-category with one-object) whose components are $\Phi(1/n) = 1/n$. Then $\lim_{n\to\infty}(x_n) = \{\Phi, x\}$. To test Cauchy-completeness of a metric space it is thus sufficient to check the existence of limits indexed by this particular module Φ.

3.2. A formula for limits

This section is concerned with analysing the notion of completeness and verifying that it agrees with the definition given by other authors.

Suppose $f: u \to v$ is a map. Notice that the assignment to y (over v) of f^*y (over u) is a functor $f^*(-)$ from X_v to X_u.

Definition. If the functor $f^*: X_v \to X_u$ has a right adjoint we denote its adjoint by Π_f.

Definition. A locally internal category X for which Π_f exists for each map f satisfies the *Beck–Chevalley condition* if, given a pullback square in E

$$\begin{tikzcd} \cdot \arrow[r, "k"] \arrow[d, "h"'] & w \arrow[d, "g"] \\ u \arrow[r, "f"'] & v \end{tikzcd}$$

the canonical transformation $f^* \cdot \Pi_g \to \Pi_h \cdot k^*$ is an isomorphism.

Theorem. *A locally internal category X has limits indexed by modules between discrete categories (= discrete indexed limits) if and only if*

(i) *Π_f exists for each map f,*

(ii) *X satisfies the Beck–Chevalley condition.*

Proof. First suppose that X admits the stated limits. Then if $f: u \to v$ is a map, and x lies over u, $\Pi_f x \cong \{f^\circ, x\}$. Just note the bijections:

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{y \to \{f^\circ, x\} \quad (\text{in } X_v)}{1_v \to X(y, \{f^\circ, x\})}}{1_v \to \hom_u(f^\circ, X(y, x))}}{f^\circ \to X(y, x)}}{1 \to X(y, x) \cdot f \quad (\text{since } f \dashv f^\circ)}}{1 \to X(f^*y, x)}}{f^*y \to x \quad (\text{in } X_u).}$$

Further, to verify the Beck–Chevalley condition, consider the pullback square given above. Using the lemma on iterated limits, note that

$$f^*(\Pi_g y) \cong \{f, \{g^\circ, y\}\} \cong \{g^\circ f, y\} \cong \{kh^\circ, y\} \cong \{h^\circ, \{k, y\}\} \cong \Pi_h(k^*y).$$

Now suppose that X satisfies conditions (i) and (ii) above. We will show that $\{f^\circ, x\} \cong \Pi_f x$. Consider z in X, any arrow $qp^\circ : e(z) \to v$ and suppose $f^\circ q \cong sr^\circ$, where p, q, r, s are maps. Then the result follows by noting the following bijections:

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{qp^\circ \to X(z, \Pi_f x)}{1 \to q^\circ \cdot X(z, \Pi_f x)p} \quad (\text{since } p \dashv p^\circ, q \dashv q^\circ)}{1 \to X(p^*z, q^*(\Pi_f x))}}{1 \to X(p^*z, \Pi_r s^*x)} \quad (\text{Beck–Chevalley})}{1 \to X(r^*p^*z, s^*x)}}{1 \to s^\circ \cdot X(z, x)pr}}{f^\circ qp^\circ \cong sr^\circ p^\circ \to X(z, x)}}{qp^\circ \to \hom_u(f^\circ, X(z, x)).} \qquad \square$$

Theorem. *A locally internal category X is complete iff*
(i) *Π_f exists for each map f,*
(ii) *X satisfies the Beck–Chevalley condition,*
(iii) *each category X_u has finite limits, which are preserved by functors of the form f^*.*

Proof. Suppose first that X is complete. To verify that (iii) holds consider a finite diagram F, with domain $\mathbf{D}$, in X_v. We can form the free Span(**E**)(v, v)-category $\mathbf{D} \cdot 1_v$ on $\mathbf{D}$ (since this base is finitely cocomplete). Let $\tilde{F} : \mathbf{D} \cdot 1_v \to X$ be the Span(**E**)-functor corresponding to F. Then as usual $\lim F \cong \{\Delta \cdot 1_v, \tilde{F}\}$ where $\Delta \cdot 1_v$ is the obvious module from the discrete category on v to $\mathbf{D} \cdot 1_v$ all of whose components are 1_v.

Consider now a map $f : u \to v$. There is a module $\mathbf{D} \cdot f : \mathbf{D} \cdot 1_u \to \mathbf{D} \cdot 1_v$ whose components are copowers of f, which satisfies $(\mathbf{D} \cdot f)(\Delta \cdot 1_u) \cong (\Delta \cdot 1_v)(f)$, and such that the Span(**E**)-functor $\mathbf{D} \cdot 1_u \to X$ corresponding to f^*F is $\{\mathbf{D} \cdot f, \tilde{F}\}$. Then

$$\lim f^*F \cong \{\mathbf{D} \cdot 1_u, \{\mathbf{D} \cdot f, \tilde{F}\}\} \cong \{(\mathbf{D} \cdot f)(\mathbf{D} \cdot 1_u), \tilde{F}\}$$
$$\cong \{(\Delta \cdot 1_v)f, \tilde{F}\} \cong \{f, \{\Delta \cdot 1_v, \tilde{F}\}\} \cong f^* \lim F.$$

Conversely, assume that X satisfies (i), (ii), (iii). Consider a functor $F : A \to X$ and a module $\Psi : u \nrightarrow A$. If A has a finite number of objects there is a formula for $\{\Psi, F\}$ in terms of finite limits in the fibres, and discrete indexed limits: $\{\Psi, F\}$ is the equalizer in X_u of the following two arrows:

$$\prod_{a \in A} \{\Psi a, Fa\} \rightrightarrows \prod_{a,\, b \in A} \{X(a, b) \cdot \Psi a, Fb\};$$

one arrow arises from the action of the module and the other from the effect of the functor. □

Example. When $\mathbf{E} = \text{Sets}$ and $\mathbf{C}$ is an ordinary locally-small small-complete category, the fibres of Fam($\mathbf{C}$) are clearly finitely complete and these limits are preserved by substitution. If x is a family of objects of $\mathbf{C}$ over v and $\Psi : u \nrightarrow v$ is a span, then

$$\{\Psi, x\}_i = \prod_{j \in u} x_j^{\Psi_{ij}}.$$

Hence Fam($\mathbf{C}$) is complete as a Span(Sets)-category.

3.3. An adjoint functor theorem

Pare–Schumacher [13] have given versions of the Freyd adjoint functor theorem adapted to locally internal categories. We give here another version, the conditions of which are very simply expressed in terms of Span($\mathbf{E}$)-categories.

Theorem. *Suppose X, Y are locally internal categories, and that Y is complete. Suppose that $G : Y \to X$ is a* Span($\mathbf{E}$)*-functor which preserves indexed limits. Then G has a left adjoint if and only if the following 'solution set condition' is satisfied: for any $x : u \to X$ there exists a $y : v \to Y$ such that the 2-cell*

$$\mu \cdot (\alpha \cdot 1) : Y(y, 1) \cdot X(x, Gy) \to X(Gy, G) \cdot X(x, Gy) \to X(x, G)$$

(α is the effect of G, μ is composition) has a section.

Proof. Consider the internal full subcategory J_y (see 2.4) corresponding to the object y. Let $i_y : J_y \to Y$ be the inclusion functor. If the indexed limit $\{X(x, G), 1\}$: $u \to Y$ exists (for each x in X)), then G has a left adjoint F given by the formula $Fx = \{X(x, G), 1\}$.

That this large limit exists follows from the fact that $\{X(x, G), 1\} \cong \{X(x, Gy), i_y\}$. To see this first note that $\{Y(i_y, 1), 1\} \cong i_y$; then, using the lemma on iterated limits,

$$\{Y(i_y, 1)X(x, Gi_y), 1\} \cong \{X(x, Gi_y), i_y\}.$$

It only remains to prove that $Y(i_y, 1)X(x, Gi_y) \cong X(x, G)$, and for this it suffices to show that $\mu(\varphi \cdot 1)(y')$ is the coequalizer of the obvious pair of arrows

$$Y(y, y')\,Y(y, y)X(x, Gy) \rightrightarrows Y(y, y')X(x, Gy)$$

in the presence of the given section of $\mu(\alpha \cdot 1)$. The argument for this is an adaptation of that of Section X.2 of MacLane [12] to Span(**E**). The method is to demonstrate that the coequalizer is absolute in the sense of Pare. □

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