Initialized feedback categories

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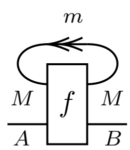
Abstract

Definition of initialized feedback categories and construction of the free initialized feedback category over some symmetric monoidal category. This is work in progress complementing [Rom20].

1 Categories with initialized feedback

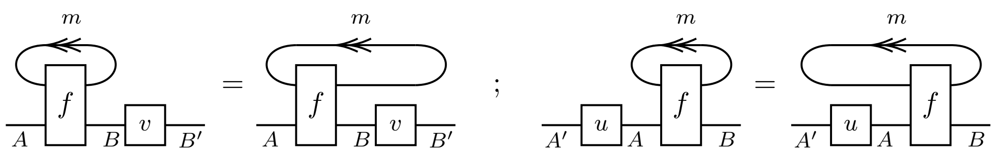
1.1 Initialized feedback

Definition 1.1. A **category with initialized feedback** is a symmetric monoidal category $\mathcal{C}$ endowed with an operator $\mathsf{fbk}^{(m,M)}_{A,B} \colon \mathcal{C}(M \otimes A, M \otimes B) \to \mathcal{C}(A, B)$ for every $A, B, M \in \mathcal{C}$ and every $m \in \mathcal{C}(I, M)$, which we call *feedback initialized to m*.

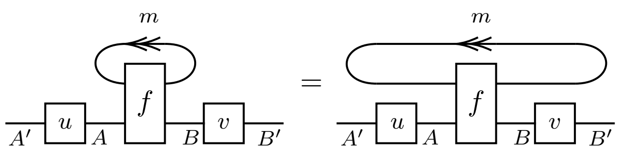
$\mathsf{fbk}^{m,M}_{A,B}(f) \;\coloneqq$ 

The feedback operator is subject to the following axioms.

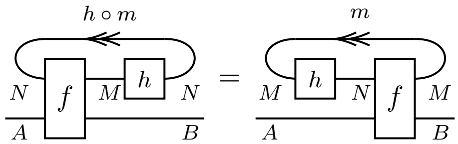
- *Left and right tightening*, meaning that $v \circ \mathsf{fbk}^{m,M}_{A,B}(f) = \mathsf{fbk}^{m,M}_{A,B'}((\mathrm{id} \otimes v) \circ f)$ for any $v \in \mathcal{C}(B, B')$; and that $\mathsf{fbk}^{m,M}_{A,B}(f) \circ u = \mathsf{fbk}^{m,M}_{A',B}(f \circ (\mathrm{id} \otimes u))$ for any $u \in \mathcal{C}(A', A)$. This is to say that the feedback operator is natural in $A, B \in \mathcal{C}$.



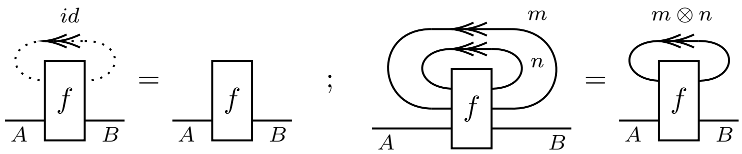
Alternatively, the two tightening rules can be combined into a single *tightening* rule, saying that $\mathsf{fbk}^{m,M}_{A',B'}((\mathrm{id} \otimes v) \circ f \circ (\mathrm{id} \otimes u)) = v \circ \mathsf{fbk}^{m,M}_{A,B}(f) \circ u$.



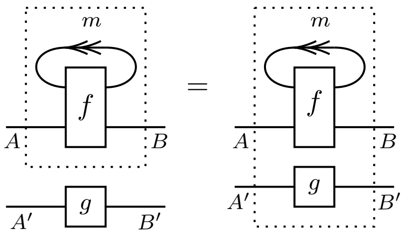
- *Sliding*, meaning that $\mathsf{fbk}^{h\circ m,N}_{A,B}((h\otimes \mathrm{id})\circ f) = \mathsf{fbk}^{m,M}_{A,B}(f\circ(h\otimes \mathrm{id}))$. This is to say that the feedback operator is dinatural in $(m,M)\in \mathcal{C}/I$.



- *Vanishing*, meaning first that $\mathsf{fbk}^{\mathrm{id},I}_{A,B}(\lambda_B\circ f\circ \lambda_A^{-1}) = f$, which is to say that feedback on the unit, initialized to the identity, does nothing. *Vanishing* also means that $\mathsf{fbk}^{m,M}_{A,B}(\mathsf{fbk}^{n,N}_{M\otimes A,M\otimes B}(f)) = \mathsf{fbk}^{m\otimes n,M\otimes N}_{A,B}(f)$, which is to say that feedback on a monoidal pair is the same as two consecutive feedbacks.



- *Strength*, meaning that $\mathsf{fbk}^{m,M}_{A,B}(f)\otimes g = \mathsf{fbk}^{m,M}_{A\otimes A',B\otimes B'}(f\otimes g)$.



Remark 1.2. The category $\mathcal{C}/I$ plays the role of the category over which feedback can be taken, but it can be easily substituted with any monoidal category $\mathcal{D}$ endowed with a strong monoidal functor $\mathcal{D}\to\mathcal{C}$ (see Appendix 4). The case where $\mathcal{D} = \mathsf{Core}(\mathcal{C})$ yields the categories with feedback of Katis, Sabadini and Walters [KSW02]. Arguably, the case where $\mathcal{D}=\mathcal{C}$ yields a more general notion of feedback.

Definition 1.3. An *initialized feedback functor* between two initialized feedback categories $\mathcal{C}$ and $\mathcal{D}$ is a strong monoidal functor $F\colon \mathcal{C}\to\mathcal{D}$ such that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{C}(M\otimes A, M\otimes B) \arrow[r, "\mathsf{fbk}^{m,M}"] \arrow[d, "F"] & \mathcal{C}(A,B) \arrow[dd, "F"] \\
\mathcal{D}(F(M\otimes A), F(M\otimes B)) \arrow[d, "\cong"] & \\
\mathcal{D}(FM\otimes FA, FM\otimes FB) \arrow[r, "\mathsf{fbk}^{Fm,FM}"] & \mathcal{D}(FA,FB)
\end{tikzcd}$$

Note that a strong monoidal functor $F\colon\mathcal{C}\to\mathcal{D}$ induces a strong monoidal functor $F\colon \mathcal{C}/I\to\mathcal{D}/J$. Categories with initialized feedback form a category with initialized feedback functors between them.

1.2 Kyb construction

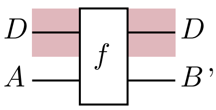
If we consider the morphisms of a symmetric monoidal category $\mathcal{C}$ to be *processes*, the following construction freely adds to the processes in $\mathcal{C}$ the ability to use some of their outputs as inputs. The outputs that can be used by a process to self-regulate are described by a different category $\mathcal{D}$ that maps monoidally into $\mathcal{C}$. As an intuition, these are processes of $\mathcal{C}$ freely allowing for feedback over a category $\mathcal{D}$. We will call the resulting construction, Kyb, a category of *cybernetic processes* over $\mathcal{C}$.

Definition 1.4. Let $(\mathcal{C}, \otimes, I)$ and $(\mathcal{D}, \boxtimes, J)$ be symmetric monoidal categories and let $U\colon \mathcal{D} \to \mathcal{C}$ be a strong monoidal functor. The Kyb category over $\mathcal{C}$ with feedback on $\mathcal{D}$ has the same objects as $\mathcal{C}$ but morphisms given by

$$\mathsf{Kyb}_{\mathcal{C}}^{\mathcal{D}}(A, B) := \int^{D \in \mathcal{D}} \mathcal{C}(UD \otimes A, UD \otimes B).$$

In other words, elements $[D, f] \in \mathsf{Kyb}(A, B)$ are equivalence classes of pairs given by an object $D \in \mathcal{D}$ and a morphism $f \in \mathcal{D}(UM \otimes A, UM \otimes B)$ quotiented by the equivalence relation generated by $[D', (d \otimes \mathrm{id}) \circ f] \sim [D, f \circ (d \otimes \mathrm{id})]$ for every $d \in \mathcal{M}(D, D')$.

Graphically, we can use a functor box for $U\colon \mathcal{D} \to \mathcal{C}$. We are considering morphisms of the form



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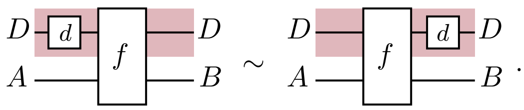
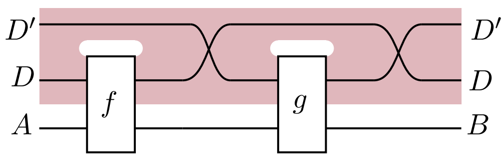
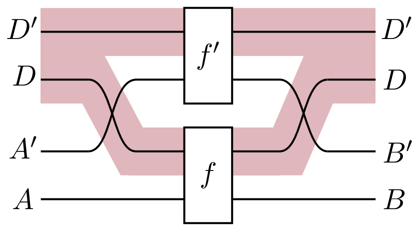


We can compose two of these elements $[D, f] \in \mathsf{Kyb}(A, B)$ and $[D', g] \in \mathsf{Kyb}(B, C)$ into $[D' \otimes D, (\sigma \otimes \mathrm{id}_C) \circ (\mathrm{id}_D \otimes g) \circ (\sigma \otimes \mathrm{id}_B) \circ (\mathrm{id}_{D'} \otimes f)]$.

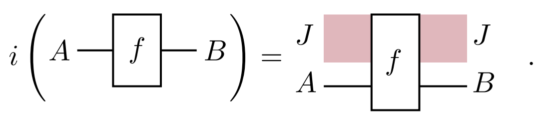


The category is also symmetric monoidal, with the tensor product $(f' \otimes f)$ defined as follows.



Remark 1.5. In particular, $\mathsf{Kyb}_{\mathcal{C}}^{\mathsf{Core}(\mathcal{C})}$ is precisely the Circ construction of [KSW02].

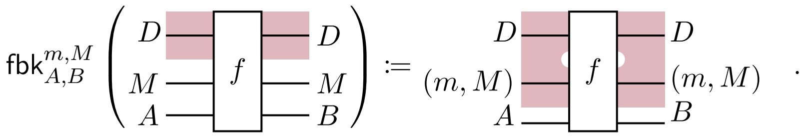
Proposition 1.6. *There exists a functor $i\colon \mathcal{C} \to \mathsf{Kyb}_{\mathcal{C}}^{\mathcal{C}\backslash I}$ that is an identity-on-objects and acts on morphisms as*



1.3 Freeness of the Kyb construction

Lemma 1.7. *The category $\mathsf{Kyb}_{\mathcal{C}}^{\mathcal{C}\backslash I}$ is a category with initialized feedback.*

Proof. Let us define the feedback operator as



which makes it natural on A and B, but also dinatural on $(m, M) \in \mathcal{C}\backslash I$. Vanishing and strength can be shown to also hold. $\square$

Proposition 1.8. *The category $\mathsf{Kyb}_{\mathcal{C}}^{\mathcal{C}\backslash I}$ is the free category with initialized feedback.*

Proof. Assume a functor $F: \mathcal{C} \to \mathcal{D}$, where $\mathcal{D}$ is a category with initialized feedback. We will show that it factors uniquely through $\mathcal{C} \to \mathsf{Kyb}_{\mathcal{C}}^{\mathcal{C}\backslash I}$ via an initialized-feedback functor.

$$\begin{tikzcd}
\mathsf{Kyb}_{\mathcal{C}}^{\mathcal{C}\backslash I} \arrow[r, dashed, "\exists!\tilde{F}"] & \mathcal{D} \\
\mathcal{C} \arrow[u, "i"] \arrow[ur, "F"'] &
\end{tikzcd}$$

The image of the functor is already determined on objects. Now, every morphism $[D, f] \in \mathsf{Kyb}_{\mathcal{C}}^{\mathcal{C}\backslash I}(A, B)$ can be written equivalently as $[(D, d), f] = \mathsf{fbk}^{D,d}_{A,B}(i(f))$. This means that its image is determined as

$$\begin{aligned}
\tilde{F}[(D, d), f] &= \tilde{F}\left(\mathsf{fbk}^{D,d}_{A,B}(i(f))\right) \\
&= \mathsf{fbk}^{D,d}_{A,B}(\tilde{F}(i(f))) \\
&= \mathsf{fbk}^{D,d}_{A,B} F f.
\end{aligned}$$

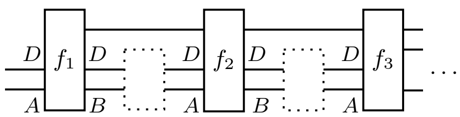
The fact that this functor is well-defined follows from the axioms of an initialized-feedback category. $\square$

2 Semantics on ∞-combs

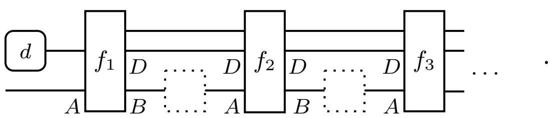
We outline now the intended semantics of the free category with feedback $(\mathcal{C}, \otimes, I)$ in the symmetric monoidal category of ∞-combs. We first consider $\mathsf{invComb}_\infty$ as the full category on constant families of objects.

Proposition 2.1. *The category $\mathsf{invComb}_\infty$ is an initialized feedback category. As a consequence, there exists a functor $\mathsf{Kyb}_{\mathcal{C}}^{\mathcal{C}\backslash I} \to \mathsf{invComb}_\infty$.*

Proof. We define the feedback operator using delay and an initializing morphism. Explicitly, we take some generic morphism



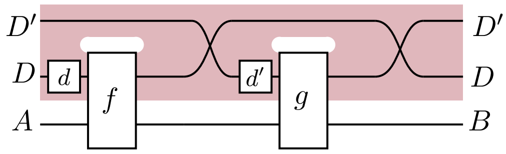
to the morphism



We can check that this assignment satisfies the axioms of an initialized feedback. $\square$

3 Appendix: diagrammatic proofs

Proposition 3.1. *Sequential composition is well-defined.*



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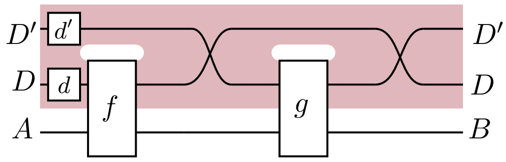

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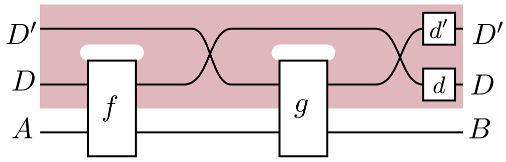

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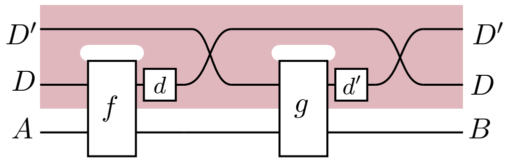
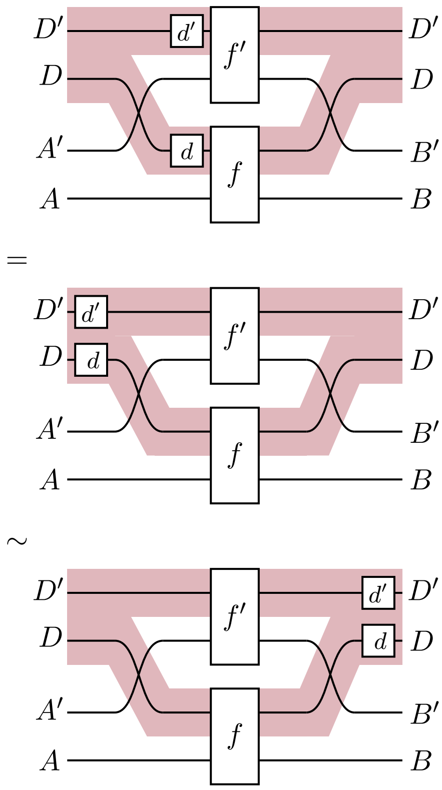
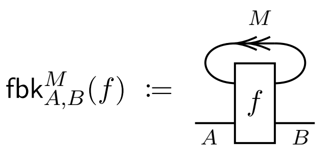


This tensor product is well-defined.



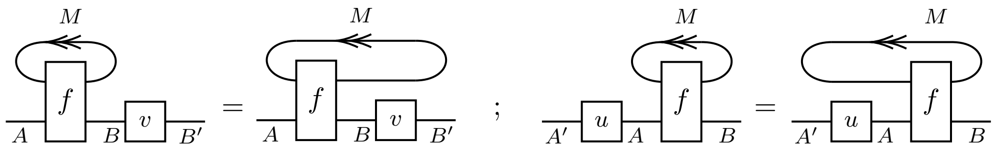
4 Appendix: general feedback categories

Definition 4.1. A **category with feedback** is a symmetric monoidal category $\mathcal{C}$ endowed with a operator $\mathsf{fbk}^M_{A,B} \colon \mathcal{C}(M \otimes A, M \otimes B) \to \mathcal{C}(A,B)$, which we call *feedback*.

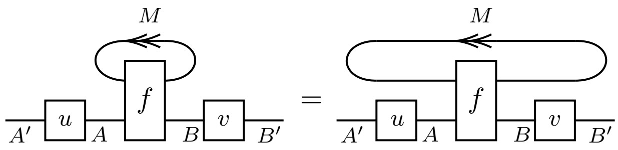


The feedback operator is subject to the following axioms.

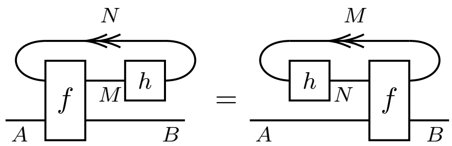
- *Left and right tightening*, meaning that $v \circ \mathsf{fbk}^M_{A,B}(f) = \mathsf{fbk}^M_{A,B'}((\mathrm{id} \otimes v) \circ f)$ for any $v \in \mathcal{C}(B,B')$; and that $\mathsf{fbk}^M_{A,B}(f) \circ u = \mathsf{fbk}^M_{A',B}(f \circ (\mathrm{id} \otimes u))$ for any $u \in \mathcal{C}(A',A)$. This is to say that the feedback operator is natural in $A, B \in \mathcal{C}$.



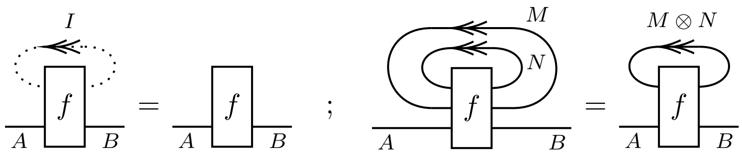
Alternatively, the two tightening rules can be combined into a single *tightening* rule, saying that $\mathsf{fbk}^M_{A',B'}((\mathrm{id} \otimes v) \circ f \circ (\mathrm{id} \otimes u)) = v \circ \mathsf{fbk}^M_{A,B}(f) \circ u$.



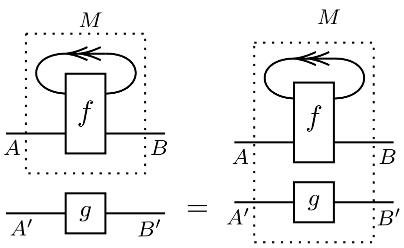
- *Sliding*, meaning that $\mathsf{fbk}^N_{A,B}((h \otimes \mathrm{id}) \circ f) = \mathsf{fbk}^M_{A,B}(f \circ (h \otimes \mathrm{id}))$. This is to say that the feedback operator is dinatural in $M \in \mathcal{C}$.



- *Vanishing*, meaning first that $\mathsf{fbk}^I_{A,B}(\lambda_B \circ f \circ \lambda_A^{-1}) = f$, which is to say that feedback on the unit does nothing. *Vanishing* also means that $\mathsf{fbk}^M_{A,B}(\mathsf{fbk}^N_{M\otimes A, M\otimes B}(f)) = \mathsf{fbk}^{M\otimes N}_{A,B}(f)$, which is to say that feedback on a monoidal pair is the same as two consecutive feedbacks.



- *Strength*, meaning that $\mathsf{fbk}^M_{A,B}(f) \otimes g = \mathsf{fbk}^M_{A\otimes A', B\otimes B'}(f \otimes g)$.



Remark 4.2. Any traced category, and thus any compact closed category, is a category with both strong and weak feedback. The converse is not true, categories with feedback do not necessarily satisfy the yanking equation.

We can consider many variants of this definition by just changing the category over which we take feedback. Let $F\colon \mathcal{D} \to \mathcal{C}$ be a strong monoidal functor, a *feedback* over $F\colon \mathcal{D} \to \mathcal{C}$ is an operator $\mathsf{fbk}^M_{A,B}\colon \mathcal{C}(M \otimes A, M \otimes B) \to \mathcal{C}(A, B)$

Remark 4.3. We say it is a category with *weak* feedback if the feedback operator is only dinatural with respect to isomorphisms. This is the definition originally proposed by Katis, Sabadini and Walters [KSW02].

References

[KSW02] Piergiulio Katis, Nicoletta Sabadini, and Robert F. C. Walters. Feedback, trace and fixed-point semantics. *ITA*, 36(2):181–194, 2002.

[Rom20] Mario Román. Comb diagrams for discrete-time feedback. *arXiv preprint arXiv:2003.06214*, 2020.