Notes on categories with feedback

Work in progress

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Abstract

We revisit the construction of $\mathsf{Fbk}(\mathbf{C})$, the free category with feedback over a symmetric monoidal category $\mathbf{C}$, using string diagrams. We generalize the construction in terms of monoidal actions. We first describe $\mathsf{Fbk}(\mathbf{Set})$, the free category with feedback over the category of sets, as a category of Meely automata. We show that $\mathsf{Fbk}(\mathsf{Span}(\mathbf{Set}))$, the free category with feedback over spans of sets, is equivalent to the full subcategory of $\mathsf{Span}(\mathbf{Graph})$ on graphs with a single vertex, which corresponds to automata synchronized on their boundaries. This is work in progress

1 Introduction

The motivation for *categories with feedback* is to add notions such as *feedback*, *delay* and *fixed-point* to the process interpretation of monoidal categories [KSW02]. This is achieved via a *feedback* operator that resembles the more common *trace operator* of a traced monoidal category [ASV96]. A similar *fixpoint* operation has been considered by Elgot [Elg75], Bloom and Ésik [BÉ93]; this one requiring a comonoid structure.

The only difference between traced categories and categories with feedback is that the latter ones are not required to satisfy the *yanking* equation.

Because of this, any traced category, and thus any compact closed category, is a category with feedback; while the converse is not true.

Traced categories can be embedded into compact closed categories with the Int construction, which justifies their notation as loops. The same idea can be extended to categories with feedback: there exists a chain of left adjoints constructing the free compact closed category over a symmetric monoidal category [KSW02].

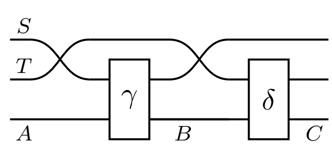
$$\mathsf{Monoidal} \xrightarrow[\dashv]{\mathsf{Circ}} \mathsf{Feedback} \xrightarrow[\dashv]{\mathsf{Yank}} \mathsf{Traced} \xrightarrow[\dashv]{\mathsf{Int}} \mathsf{CompactClosed}$$

2 Motivation

2.1 Meely automata

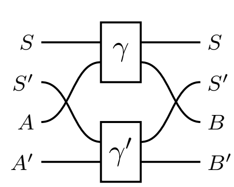
Definition 2.1. We define a **Meely automaton** $(S, \delta)\colon A \to B$ with inputs in A and outputs in B to be given by a set S of *states* together with $\delta\colon S \times A \to S \times B$, a *transition* function.

We can compose two automata $(S, \delta)\colon A \to B$ and $(T, \gamma)\colon B \to C$ sequentially into a single automaton $(S, \gamma); (T, \delta)\colon A \to C$, where the space of states is given by the product $S \times T$, and the transition function is given by the following diagram.



A transition for the resulting automaton under an input $a \in A$ is a transition for the first automaton under the input $a \in A$, and a transition for the second automaton under the input received from the output of the first automaton.

We can compose two automata $(S, \delta)\colon A \to B$ and $(S', \delta')\colon A' \to B'$ in parallel into a single automaton $(S, \delta) \otimes (S', \delta')\colon A \otimes A' \to B \otimes B'$, where the space of states is again given by the product $S \times T$ and the transition function is given by the following diagram.



A transition for the combined parallel automaton is a transition for each one of the two.

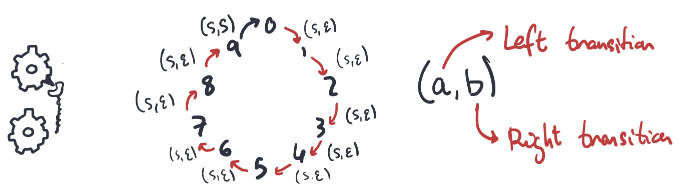
Definition 2.2. At this point, we can think of introducing a notion of equivalent that disregards isomorphic space states. We say that two automaton (S, δ) and (T, γ) are *equivalent* if there exist some isomorphism $h \colon S \to T$ such that $\delta; (h \otimes \mathrm{id}) = (h \otimes \mathrm{id}); \gamma$. This indeed forms an equivalence relation.

Deterministic automata, quotiented under equivalence, form a symmetric monoidal category. This symmetric monoidal category is equivalent to $\mathsf{Fbk}(\mathbf{Set})$, and it is the free category with feedback over the category of sets.

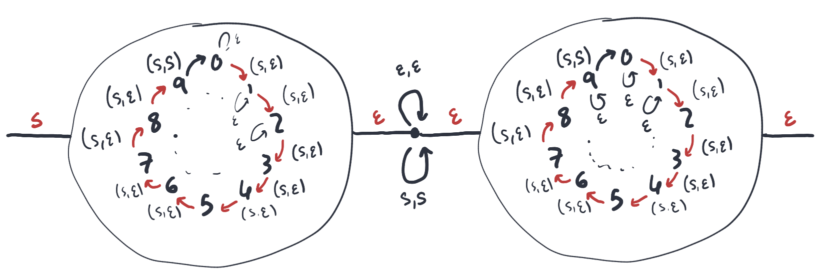
2.2 Span(Graph)

The modelling of automata as input/output machines, however, is of limited use for purposes such as synchronization. Oppossed to the usual process interpretation of monoidal categories, where processes have inputs and outputs, the approach that can be found in spans of graphs addresses transition systems that synchronize on their boundaries [KSW97].

Blaise Pascal invented the *Pascaline,* a calculator machine made up of gears that allowed for carrying. Every time one gear reached ten, there would be a small incision that would transfer the movement to the next gear.



The approach using spans of graphs lets the gears syncronize over the boundaries. The composite system can be now described from the parts.



The central graph (which usually has a single node) contains two transitions that describe the syncronization between the two parts. Every transition produces some behaviour on two different ports.

Formally, systems are spans of graphs, where the boundaries are graphs with a single node. They compose by the usual pullback composition of spans. In other words, a process is then a set of edges E, labelled under the inputs A and the outputs B, with source and target on a set of states S. This is precisely a span $S \times A \to S \times B$.

(Graph)
$$\begin{tikzcd} E \arrow[r] \arrow[d, bend right] \arrow[d, bend left] & A \times B \\ S & \end{tikzcd}$$

(Span)
$$\begin{tikzcd} & E \arrow[dl] \arrow[dr] & \\ A \times S & & B \times S \end{tikzcd}$$

Figure 1: Equivalence between graphs labelled on A and B with vertices in S and spans $A \times S \nrightarrow B \times S$.

We can propose a conceptual explanation for how this category happens to capture composition and synchronization. The full subcategory of **Span**(**Graph**) on graphs with a single vertex, and quotiented by the equivalence given by isomorphisms of spans, is equivalent to $\mathsf{Fbk}(\mathbf{Span})$, the free category with feedback over spans of sets. (TODO, write and link to the theorem)

3 Categories with feedback

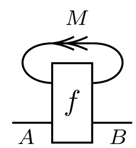
Following the work of Katis, Sabadini and Walters [KSW02], we will define a category with feedback as a structure over a symmetric monoidal category. Along with the axioms, we propose a graphical notation for the feedback operation similar to that of traces.

Definition 3.1. A **category with feedback** [KSW02] is a symmetric monoidal category $\mathbf{C}$ endowed with some given operator

$$\mathsf{fbk}^M_{A,B} \colon \mathbf{C}(M \otimes A, M \otimes B) \to \mathbf{C}(A, B),$$

which we call *feedback*. Through this definition, and when using string diagrams, we will denote the feedback operation as a loop. Justification for this choice will follow from the axioms that we will impose on the feedback operator.

$$\mathsf{fbk}^{M}_{A,B}(f) \;\coloneqq$$



A feedback operator must satisfy the following set of axioms. We will be using both string diagrams and formulaic descriptions; for consistency, composition is written in diagrammatic order.

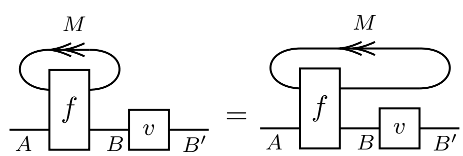
- *Left and right tightening.* Let $f\colon M \otimes A \to \otimes B$. For any $v\colon B \to B'$,

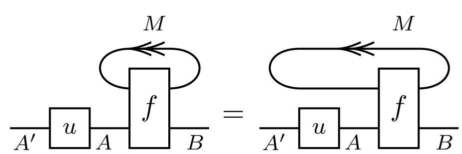
  $$(\mathsf{fbk}^{M}_{A,B}(f); v) = \mathsf{fbk}^{M}_{A,B'}(f; (\mathrm{id} \otimes v)),$$

  and for any $u\colon A' \to A$,

  $$u; \mathsf{fbk}^{M}_{A,B}(f) = \mathsf{fbk}^{M}_{A',B}((\mathrm{id} \otimes u); f).$$

  The following is the diagrammatic depiction of these two rules.

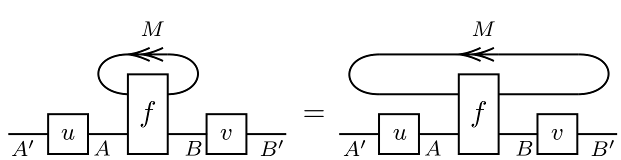
  

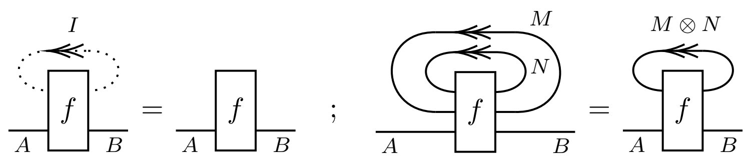
  This is to say that the feedback operator is natural in $A, B \in \mathbf{C}$. Alternatively, the two tightening rules can be combined into a single *tightening* rule, saying that

  $$u; \mathsf{fbk}^{M}_{A,B}(f); v = \mathsf{fbk}^{M}_{A',B'}((\mathrm{id} \otimes u); f; (\mathrm{id} \otimes v)).$$

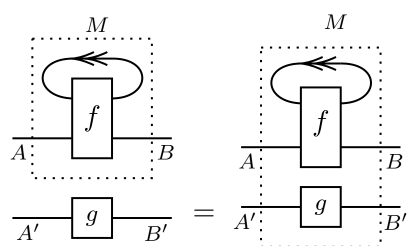
  The following is the diagrammatic depiction of this combined rule.

- *Vanishing.* Let $f\colon A \to B$. It holds that $\mathsf{fbk}^I_{A,B}(\lambda_A; f; \lambda_B^{-1}) = f$, which is to say that feedback on the unit does nothing. *Vanishing* also means that, for every $f\colon M \otimes (N \otimes A) \to \otimes(N \otimes B)$, it holds that $\mathsf{fbk}^M_{A,B}(\mathsf{fbk}^N_{M\otimes A, M\otimes B}(f)) = \mathsf{fbk}^{M\otimes N}_{A,B}(\alpha_{M,N,A}; f; \alpha^{-1}_{M,N,B})$, which is to say that feedback on a monoidal pair is the same as two consecutive feedbacks.



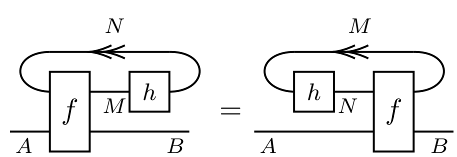
- *Strength.* For every $f\colon M \otimes A \to M \otimes B$ and $g\colon A' \to B'$, $\mathsf{fbk}^M_{A,B}(f) \otimes g = \mathsf{fbk}^M_{A\otimes A', B\otimes B'}(f \otimes g)$.



- *Sliding.* For any $f\colon N \otimes A \to M \otimes B$ and any isomorphism $h\colon M \to N$,

$$\mathsf{fbk}^N_{A,B}(f; (h \otimes \mathrm{id})) = \mathsf{fbk}^M_{A,B}((h \otimes \mathrm{id}); f).$$

  This is to say that the feedback operator is dinatural over the core subgroupoid of the category $\mathsf{Core}(\mathbf{C})$. The following is the diagrammatic depiction of this rule.



  The sliding axiom, its variants and consequences, is detailed in Section 5.

4 Free category with feedback

The construction of the free category with feedback by Katis, Sabadini and Walters [KSW02] can be regarded as a normal form theorem. This theorem states that every morphism in a category with feedback can be rewritten as a single application of feedback on a morphism on the category.

We will define a *symmetric category of circuits* over a symmetric monoidal category, whose morphisms are implicitly the morphisms of the original category after a single application of feedback. Proving this normal form theorem will amount to show that circuits actually form a symmetric monoidal category and, moreover, that this is the free category with feedback.

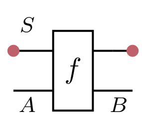
4.1 Category of circuits

Definition 4.1. Let $\mathbf{C}$ be a symmetric monoidal category. We define a *circuit* to be an element of the following set expressed as a coend.

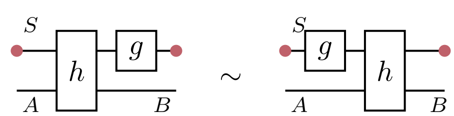
$$\mathsf{Fbk}(A, B) \coloneqq \int^{M \in \mathsf{Core}(\mathbf{C})} \mathbf{C}(M \otimes A, M \otimes B)$$

In other words, morphisms of the circuit construction are pairs of an object $S \in \mathbf{C}$ and a morphism $f \in \mathbf{C}(S \otimes A, S \otimes B)$, quotiented by the equivalence relation generated by pairs of the following form for any isomorphism $g \colon S \to T$,

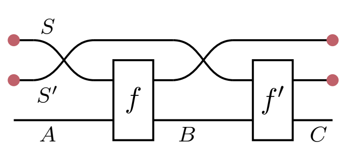
$$(S \mid (g \otimes \mathrm{id}); h) \sim (S' \mid h; (g \otimes \mathrm{id})), \text{ for each } g \in \mathsf{Core}(\mathbf{C})(S, T).$$



quotiented by



Sequential composition of two circuits $(S \mid f) \colon A \to B$ and $(S' \mid f') \colon B \to C$ is defined by the following formula.

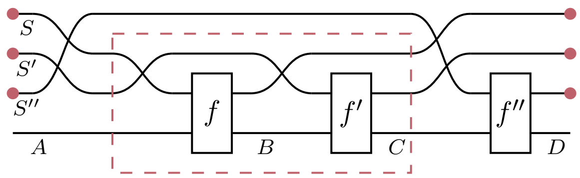


$$(S \mid f); (S' \mid f') = (S \otimes S' \mid (\sigma_{S,S'} \otimes \mathrm{id}_A); (\mathrm{id}_{S'} \otimes f); (\sigma_{S',S} \otimes \mathrm{id}_B); (\mathrm{id}_{S'} \otimes f'))$$

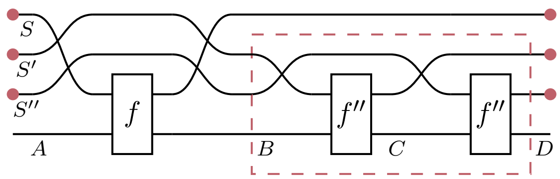
We define an *identity* circuit as $(I \mid id_{I \otimes A})$ for each $A \in \mathbf{C}$.

Proposition 4.2. *Circuits over a symmetric monoidal category* $\mathbf{C}$ *form a category* $\mathbf{Circ}(\mathbf{C})$ *with the same objects as* $\mathbf{C}$*.*

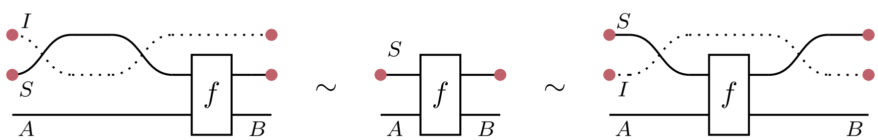
Proof. We check that sequential composition is associative. Let $(S \mid f)\colon A \to B$, $(S' \mid f')\colon B \to C$ and $(S'' \mid f'')\colon C \to D$ be three circuits.



$=$ (Topology)



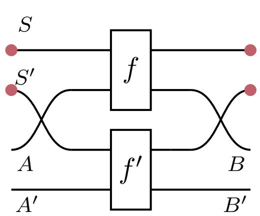
We check the identity circuit is neutral under composition. Let $f\colon S \otimes A \to S \otimes B$,



$\square$

4.2 Symmetric monoidal category of circuits

The *parallel composition* of two circuits $(S \mid f)$ and $(S' \mid f')$ is defined as



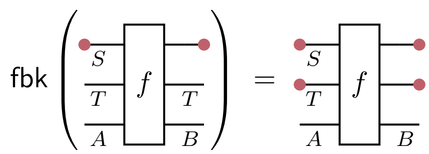
$(S \mid f) \otimes (S' \mid f') = (S \otimes S' \mid (\mathrm{id}_S \otimes \sigma_{S',A} \otimes \mathrm{id}_{A'}); (f \otimes f'); (\mathrm{id}_S \otimes \sigma_{B,S'} \otimes \mathrm{id}_{B'}))$.

Proposition 4.3. *The category* $\mathbf{Circ}(\mathbf{C})$ *is symmetric monoidal.*

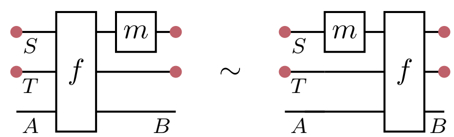
4.3 Feedback category of circuits

The operation of feedback is the one that moves a wire into the feedback.

$$\mathsf{fbk}_{N,A,B}(M \mid f) \coloneqq (M \otimes N \mid f).$$

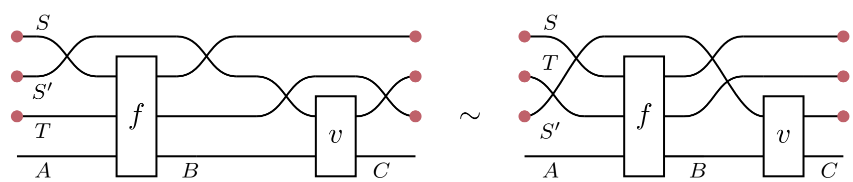


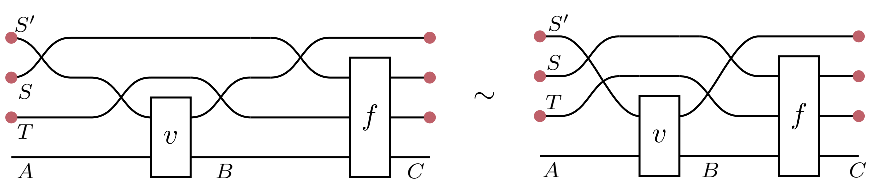
We can check that it is indeed well-defined.



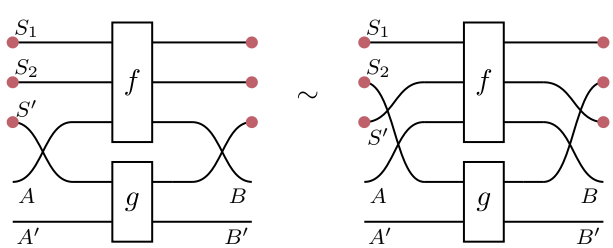
Proposition 4.4. *The category* $\mathsf{Fbk}(\mathbf{C})$ *endowed with the previous feedback operator is a category with feedback.*

Proof. Left tightening and right tightening.





Strength.



Vanishing is trivial to show under the definition of both monoidal categories. Sliding follows from the same quotienting relation. □

4.4 The circuit category is the free category with feedback

Theorem 4.5. *The category* $\mathsf{Fbk}(\mathbf{C})$ *is the free category with feedback over a symmetric monoidal category* $\mathbf{C}$*.*

Proof. Given any other category with feedback, we need to send feedback to feedback and then everything is determined because every morphism can be written as a single feedback. □

5 Generalizing categories with feedback

Slight changes on the *sliding* axiom provide variants of a category with feedback. As it stands, this sliding axiom defines a category where feedback is taken only over isomorphisms. This is the definition by Katis, Sabadini and Walters [KSW02]. Instead, we could decide to lift this restriction and consider the sliding axiom for arbitrary morphisms. These categories could also be called *categories with feedback*. The only change on the construction of the free category with feedback is that the coend is taken now over the base category instead of its core.

A strict generalization can be written where the coend is taken over an arbitrary category and two monoidal actions define the feedback.

Acknowledgements

This note has been written following conversations with Nicoletta Sabadini, Alessandro Gianola and Elena Di Lavore. Many ideas come from them or from the work of Katis, Sabadini and Walters. I may have introduced errors or misrepresented these ideas; this is still work in progress.

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