Composing optics

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Abstract

Optics constitute a compositional representation of bidirectional data accessing; they are divided into multiple families, each one encapsulating some particular pattern (accessing subfields, pattern matching, iterating, ...). Previous work has justified the composition of optics of the same family, but composition of optics of different families, which is arguably the most useful use case, has not been directly addressed.

Our main idea is that categories of optics should be regarded as a family of monoids in the bicategory of profunctors. This family is closed under two usual ways of composing monoids: coproducts and distributive laws, and we show these correspond to the two ways of composing optics that are implicitly used in programming practice.

This is work in progress.

Contents

1 Introduction

In functional programming, **optics** are a compositional representation of bidirectional data accessors, provided by libraries such as [Kme18]. Optics are divided into various *families*; each one of them encapsulating some data accessing pattern. For instance, *lenses* access subfields, *prisms* pattern match, and *traversals* iterate over containers. The usefulness of optics comes from the fact that any two of them can be composed, even if they are from different *families*. Optics function as building blocks for constructing complex data accessors, as it is exemplified in Figure 1.

```
let venues =
  [ "19 Albany Street, Pasadena, MD 21122"
  , "7 Hamilton Court Park, NY 11374"
  , "very.common@example.com"
  , "48 Grove Lane, Wallingford, CT 06492"
  ]

each :: Traversal [String] String
address :: Prism String Address
street :: Lens Address String

>>> venues ^. each . address . street %~ uppercase
[ "19 ALBANY STREET, Pasadena, MD 21122"
, "7 HAMILTON COURT PARK, NY 11374"
, "Error! not a postal address"
, "48 GROVE LANE, Wallingford, CT 06492"
]
```

Figure 1: A traversal (`each`), a prism (`address`) and a lens (`street`) are composed into a single optic that iterates over a list of strings, parses each one of them into some data structure, and modifies one of their subfields.

The existing literature [BG18, Ril18, CEG+20] successfully addresses the problem of modelling the composition of optics of the same family, both in their profunctor and existential descriptions. However, composition of optics of different families is never addressed explicitly, even when it is being used in practice. The purpose of this text is to model how optics from different families compose and to answer many questions that appear in programming practice, such as the following ones.

- Haskell performs some notion of composition of optics by joining the constraints of Tambara modules. Is this sound? How does it work and which kind of optic are we getting?
- Sometimes we can see more than one "natural" way of composing two optics. Are these two equivalent? Which should be the criterion for choosing how to compose them? How to categorically describe these two possible compositions?
- Is the composition of optics of different families still an optic of some other family? If so, how to describe this family?
- It used to be generally accepted that lenses and prisms composed into traversals. Pickering, Gibbons and Wu [PGW17] conjectured that they could be composed into a stricly more concrete optic that was later discovered and called *affine traversal*. The commonly accepted folklore is that lenses and prisms compose into affine traversals; in which sense is this right? Is this the best we can do or is there an even more concrete optic they compose into?

1.1 Contributions

- We develop diagrammatic proofs of many facts about optics using the already existing language of the bicategory *Cat* enhanced with functor boxes (Section 2).
- We describe in terms of category theory the two ways in which optics are composed in practice: direct composition (Section 3) and distributive laws (Section 4). We argue that the way in which Haskell performs this composition is the first one (Section 3.1), but the claim that lenses and prisms compose into affine traversals is only true under the second one 4.
- We spell out the notion of *distributive law* between monoidal actions (Section 4.3) and we see how it induces distributive laws between their associated optics. We motivate this construction with two examples from the literature: affine traversals (Section 4.1) and glasses (Section 4.2).
- Categories of optics are the free teleological categories, after the comprehensive work of Riley [Ril18]; but their definition is slightly inconvenient for reasoning with them. We propose a different definition of *teleological category* [Ril18, Hed17] (Section 5) and study what examples would be under a reasonable graphical calculus for them (Section 5.2).
- We extend the definition of teleological category to a *teleological* structure on profunctors, such that promonads with teleological structure correspond to teleological categories (Section 6). We give a monoidal adjunction between the category of actions $Cat/[\mathcal{C}, \mathcal{C}]$ and the category of teleological profunctors that gives an universal property of optics (Section 6.3).

2 Optics

2.1 Optic for an action

All optics them fit into a common definition that was described with slight variations by [Mil17, BG18, Ril18, Rom19, CEG+20]. We will find more convenient during this text to work with a slightly generalized definition that does not require the action to be monoidal.

Definition 1. Let $Ⓜ \colon \mathcal{M} \times \mathcal{C} \to \mathcal{C}$ be a functor and let $A, B, S, T \in \mathcal{C}$. An **optic** from (A, B) to (S, T) is an element of the set

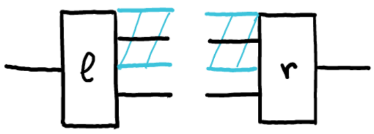
$$Optic_{Ⓜ}((S,T),(A,B)) := \int^{M \in \mathcal{M}} \mathcal{C}(S, M Ⓜ A) \times \mathcal{C}(M Ⓜ B, T).$$

In other words, optics are pairs $\langle l \mid r \rangle$, where $l \in \mathcal{C}(S, M Ⓜ A)$ and $r \in \mathcal{C}(M Ⓜ B, T)$, quotiented by the equivalence relation generated by $\langle (\alpha Ⓜ \mathrm{id}) \circ l \mid r \rangle \sim \langle l \mid r \circ (\alpha Ⓜ \mathrm{id}) \rangle$ for every $\alpha \in \mathcal{M}(M, N)$.

2.2 Diagrams for optics

In order to depict optics, we shall employ the graphical calculus for bicategories [**?**] and specifically for the bicategory of categories **Cat**, in which 0-cells are categories, 1-cells are functors and 2-cells are natural transformations. We shall also make use of monoidal functor boxes [**?**] on the monoidal category $\mathbf{Cat}(\mathcal{C}, \mathcal{C})$ for a fixed category $\mathcal{C}$.

Let us describe the specific elements that come into play when representing optics. The action $(Ⓜ) \colon \mathcal{M} \times \mathcal{C} \to \mathcal{C}$ will be seen as a functor $\mathcal{M} \to [\mathcal{C}, \mathcal{C}]$ represented by a functor box. Objects on the category $\mathcal{C}$ will be represented as functors (wires) from $\mathbf{1}$, the terminal category. The different categories we are using are usually represented by coloring the regions. However, given that ambiguity will not be a problem, we prefer to avoid coloring the regions in order to make diagrams clearer. After these considerations, an optic $\langle l \mid r \rangle \in Optic_{Ⓜ}((A, B), (S, T))$ can be depicted as the pair of functions $l \colon S \to M Ⓜ A$ and $r \colon M Ⓜ B \to T$,

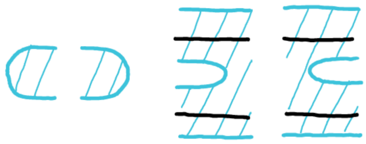


where the naturality condition quotients optics by the equivalence relation given by any natural transformation $f \colon M \to N$ travelling through the upper wire.

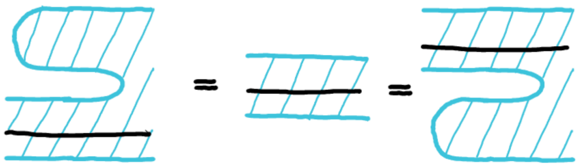
Remark 2. There exist some other graphical calculi for optics on the literature [Hed17, Boi19]. Our proposal is not to be understood as a graphical calculus for categories of optics but a way of understanding what optics are in the already existing language of a bicategory.

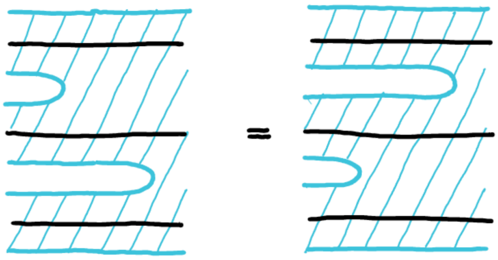
We have defined optics in full generality, allowing $\textcircled{\tiny M}$ to be an arbitrary functor from an arbitrary category $\mathcal{M}$. However, the most interesting case, and the one commonly studied when one talks about optics, is the one where $\mathcal{M}$ is a monoidal category and the functor is a *monoidal action*. In that case, we can endow optics over an action with category structure.

Definition 3. An action $\textcircled{\tiny M}\colon \mathcal{M} \times \mathcal{C} \to \mathcal{C}$ from a monoidal category $\mathcal{M}$ is a (strong) monoidal action when the associated functor $\mathcal{M} \to [\mathcal{C}, \mathcal{C}]$ is strong monoidal. In other words, it comes equipped with natural isomorphisms $\varepsilon_A\colon A \to I \textcircled{\tiny M} A$ and $\mu_{M,N,A}\colon M \textcircled{\tiny M} N \textcircled{\tiny M} A \to M \otimes N \textcircled{\tiny M} A$.



These isomorphisms must satisfy the usual unitality and associativity requirements, which can be translated into the graphical calculus as saying that the following equalities between diagrams hold.



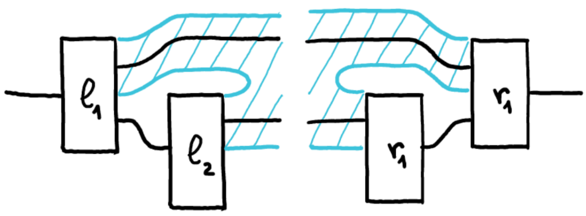


Proposition 4. *Let* $(\textcircled{\tiny M})\colon \mathcal{M} \times \mathcal{C} \to \mathcal{C}$ *be a monoidal action.* $Optic_{\textcircled{\tiny M}}$ *can be given category structure.*

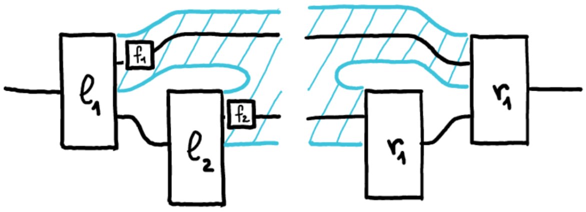
Proof. We start by proving that $Optic_{\textcircled{\tiny M}}$ defines a category. Let

$$\langle l_1 \mid r_1 \rangle \in Optic_{\textcircled{\tiny M}}((A,B),(S,T)), \quad \text{and} \quad \langle l_2 \mid r_2 \rangle \in Optic_{\textcircled{\tiny M}}((X,Y),(A,B)),$$

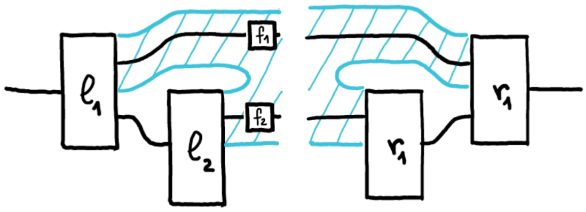
we define their composition in $Optic((X,Y),(S,T))$ to be the following.



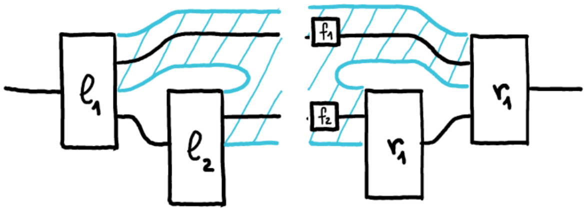
This is well defined, as it preserves the equivalence relation.



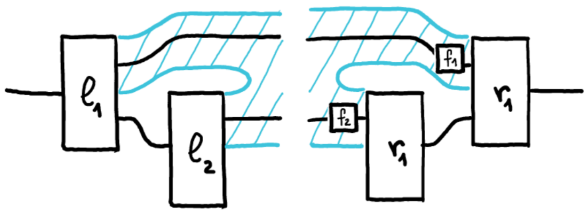
= (Naturality)



= (Equivalence relation)

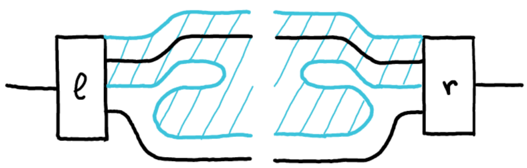


= (Naturality)

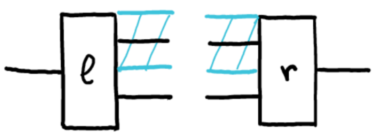


The identity, on the other hand, is defined as follows.

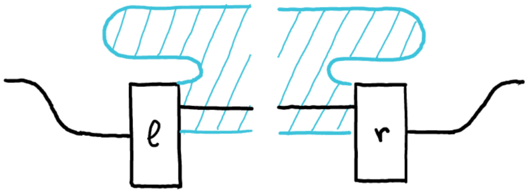
Composing with the identity leaves the optic unchanged on both sides.



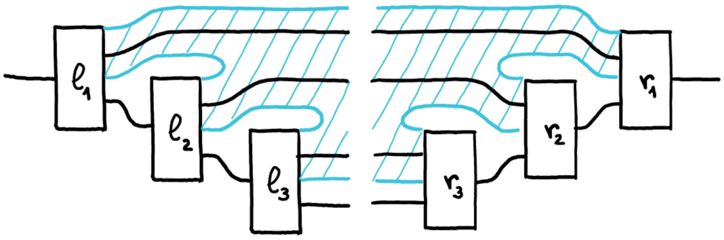
= (Topology)



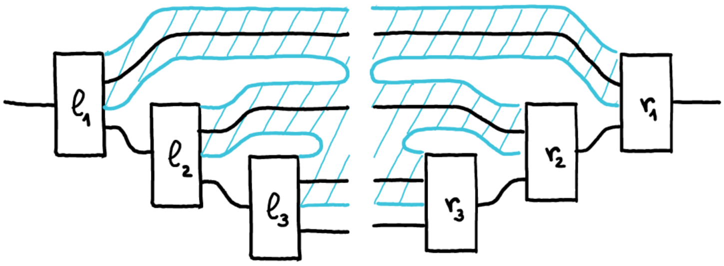
$=$ (Topology)



We also prove associativity of composition.



$=$ (Topology)



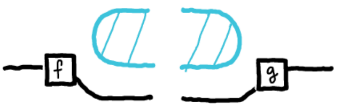
$\square$

2.3 Identity-on-objects embedding

An important, and possibly overlooked detail on the theory of optics, is the existence of an identity-on-objects functor that embeds the category $\mathcal{C} \times \mathcal{C}^{op}$ into $Optic_{Ⓜ}$. Monoids in the bicategory of profunctors, which we will call *promonads*, can be characterized to be equivalent to identity-on-objects functors. This is to say that the following result makes $Optic_{Ⓜ}$ a promonad.

Theorem 5. *There exists an identity-on-objects functor* $i \colon \mathcal{C} \times \mathcal{C}^{op} \to Optic_{Ⓜ}$.

Proof. The embedding of a morphism of $\mathcal{C} \times \mathcal{C}^{op}$ given by a pair of functions (f, g) is determined by the following diagram.



It is routine to check that this defines in fact a functor. $\square$

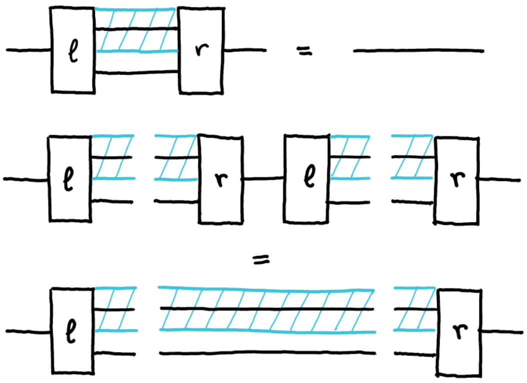
2.4 Lawful optics

Optics, and in particular lenses, were originally considered to be particularly well-behaved if they were to satisfy some extra axioms. In practice, these axioms are used to ensure that optics behave as the final user expects them to (the *lens* library [Kme18] examplifies this convention). An important contribution in the work of Riley [Ril18] is to characterize the laws of optics as the axioms of a comonoid homomorphism. For completeness, we will depict them following the graphical calculus we just introduced.

Definition 6. Consider a type-invariant three-leg variant of optics where elements are the elements of a coend given as follows [Ril18]. Elements of this type can be written as triples quotiented by the equivalence relation of the coend, and depicted as triples of diagrams.

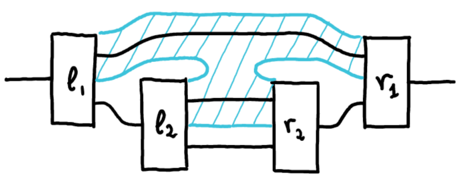
$$\mathit{Optic}^2_{Ⓜ}(S,A) := \int^{M_1,M_2\in\mathcal{M}} \mathcal{C}(S, M_1 Ⓜ A) \times \mathcal{C}(M_1 Ⓜ A, M_2 Ⓜ A) \times \mathcal{C}(M_2 Ⓜ A, S).$$

Definition 7. An optic $\langle l \mid r\rangle$ is **lawful** when $r \circ l = \mathrm{id}$ and $\langle l \mid r \circ l \mid r\rangle = \langle l \mid \mathrm{id} \mid r\rangle$. That is to say that the following diagrammatic equations hold.



Proposition 8. *Assume* (Ⓜ) *is a monoidal action. Lawful optics form a subcategory of the category of optics.*

Proof. We prove that the composition of two lawful optics is again a lawful optic.



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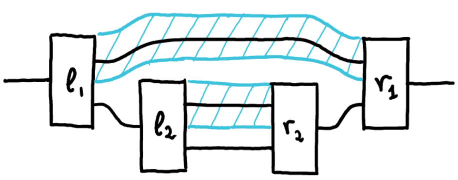

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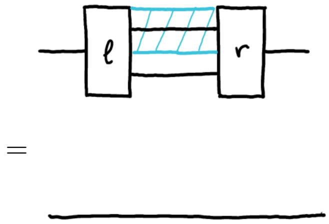


□

3 Direct composition of optics

In the same way that we compose optics from the same family, we could think of composing optics from different families. Consider Example 1, where a lens, a prism and a traversal compose. The full representation of what is happening there can be drawn in the same graphical calculus we were using before. We call this technique the "direct composition" of optics. That is, the direct composition of optics is the composition in *Cat* of the natural transformations they determine. We will see that this direct composition of optics is again a optic for the coproduct of the actions.

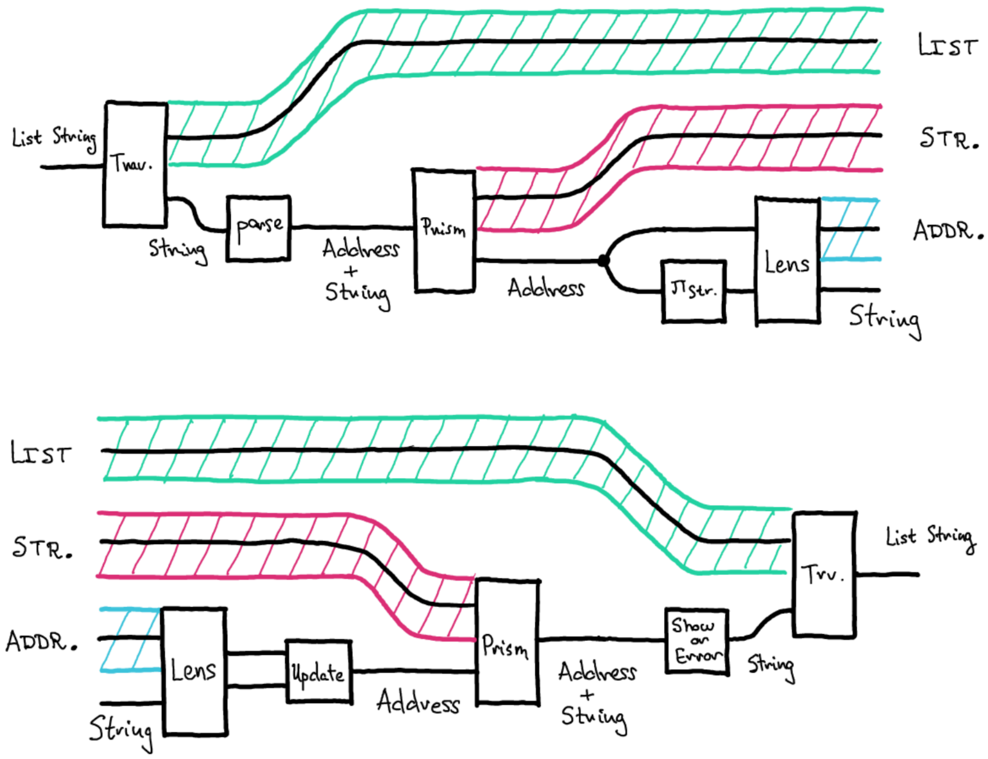


Figure 2: An example of direct composition in monoidal notation.

It should be noted how we have separated each optic in two parts. One is given by the *plain* functions that the optic applies to its inputs or outputs; that is simply a morphism in $\mathcal{C}^{op} \times \mathcal{C}$. The second one is given by a tautological optic of the form $(A, B) \to (M ⓜ A, M ⓜ B)$. It is the case that every optic admits this decomposition.

3.1 Composition by Tambara modules

In functional programming, and more specifically in Haskell, the approach to composition of optics is to compose them in terms of Tambara modules. For our purposes, Tambara modules [PS08] are the presheaves of a category of optics, but they are also algebra structures for a certain monad.

Proposition 9. *A Tambara module for an action* Ⓜ *is a coalgebra for the following comonad.*

$$\Theta_{Ⓜ} P(A,B) := \int_{M \in \mathcal{M}} P(M Ⓜ A, M Ⓜ B).$$

Tambara modules for a fixed action form a category $\mathit{Tamb}_{Ⓜ}$ *with morphisms given by the coalgebra morphisms.*

Tambara modules are the copresheaves of the category $\mathit{Optic}_{Ⓜ}$. This means that every optic of a certain family can be written as a single polymorphic function that works for every Tambara module for its action. This is the content of the **profunctor representation theorem**, and this single function is called the *profunctor representation* of the optic.

$$\mathit{Optic}_{Ⓜ}(A,B,S,T) \cong \int_{P \in \mathit{Tamb}Ⓜ} \mathit{Set}(P(A,B), P(S,T))$$

In practice, optics are written in this form and composition of optics of different families works in the following way. Let Ⓜ and Ⓝ be two different actions, and assume we have two optics of types

$$\int_{P \in \mathit{Tamb}Ⓜ} \mathit{Set}(P(X,Y), P(A,B)), \quad \text{and} \quad \int_{P \in \mathit{Tamb}Ⓝ} \mathit{Set}(P(A,B), P(S,T)).$$

We can compose both into an optic of the following type

$$\int_{P \in \mathit{BiTamb}(Ⓜ,Ⓝ)} \mathit{Set}(P(A,B), P(S,T)),$$

where $\mathit{BiTamb}(Ⓜ,Ⓝ)$ is the category of profunctors that have two Tambara structures, for Ⓜ and Ⓝ, respectively. We will show that $\mathit{BiTamb}(Ⓜ,Ⓝ) \cong \mathit{Tamb}(Ⓜ+Ⓝ)$. In other words, this way of composing optics gets us an optic for the coproduct of the actions. Moreover, we will show in Corollary 20 that the coproduct promonad of two promonads of optics is the promonad of optics for the action of the coproduct.

The direct composition of optics can always be interpreted to be an optic for the coproduct of the actions. In other words, the profunctor composition of optics coincides with the direct composition.

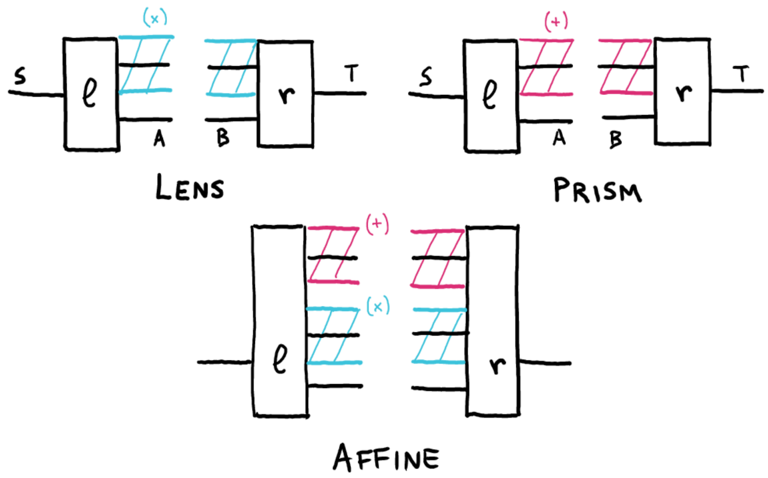
4 Distributive laws between families of optics

4.1 Lenses and prisms

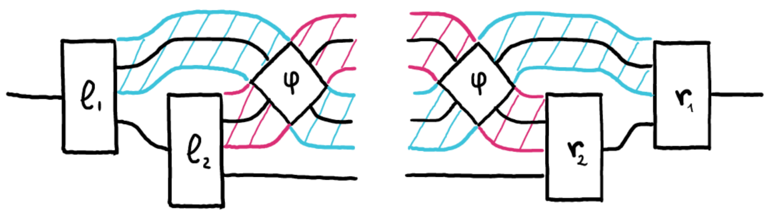
There exist two clear examples of distributive law of optics on the literature. The first one is the one between *lenses* and *prisms*. In [PGW17], the authors compose lenses and prisms into traversals and ask whether a more refined composition is possible. The question was answered positively by showing that lenses and prisms can be composed into *affine traversals*, an optic for an action $(+\times)\colon \mathcal{C}\times\mathcal{C}\to[\mathcal{C},\mathcal{C}]$ defined as

$$(X,Y)(+\times)A \coloneqq X+Y\times A.$$

There exists a way of distributing products over coproducts, $\phi_{X,Y,A}\colon Y\times(X+A)\cong Y\times X+Y\times A$, and that can be translated into a way of distributing lenses over prisms.



How to compose lenses after prisms into affine traversals can be seen directly, and only the second case, composing prisms after lenses, requires the use of the ϕ natural transformation.



4.2 Grates and lenses

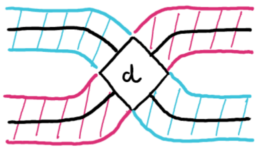
The second one is between *lenses* and *grates*. It was used in [**?**] to describe an optic called *glass*. Glasses are optics for an action $(\times\to)\colon \mathcal{C}\times\mathcal{C}\to[\mathcal{C},\mathcal{C}]$ defined as

$$(X,Y)(\times\to)A \coloneqq X\times(Y\to A).$$

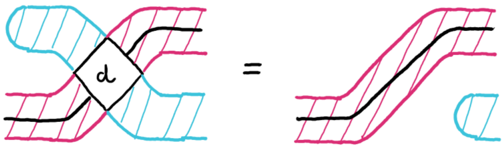
The natural transformation determining a way of distributing exponential over coproducts is given by $\phi_{X,Y,A}\colon Y\to X\times A \coloneqq (Y\to X)\times(Y\to A)$.

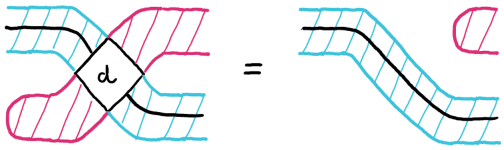
4.3 Distributive laws of monoidal actions

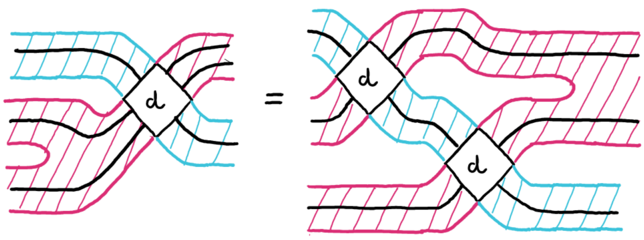
Definition 10. A **distributive law** between two monoidal actions $(Ⓜ)\colon \mathcal{M}\times\mathcal{C}\to\mathcal{C}$ and $(Ⓝ)\colon \mathcal{N}\times\mathcal{C}\to\mathcal{C}$ is given by a strong monoidal functor $F\colon \mathcal{M}\times\mathcal{N}\to\mathcal{N}\times\mathcal{M}$ and a monoidal natural isomorphism $d_{M,N,A} \colon M \textcircled{M} N \textcircled{N} A \to F(M,N) \textcircled{N} \textcircled{M} A$, satisfying some equations we can depict graphically.

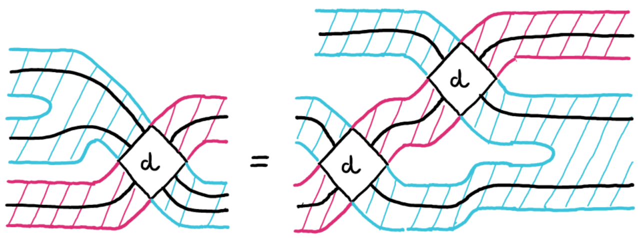


The following equations implicitly use the fact that F is a strong monoidal functor to keep the number of wires invariant on each side of the natural transformation.







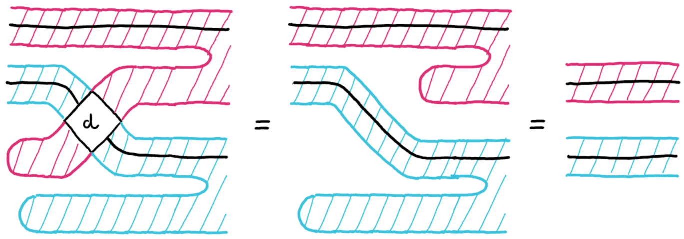


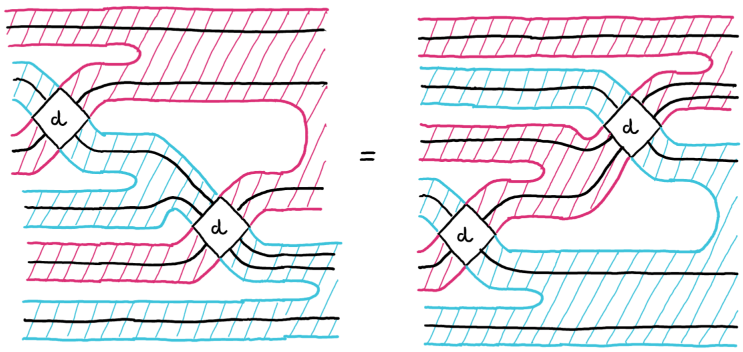
It should be pointed that this matches the definition of a distributive law between monoids on the category **MonCat**$/[\mathcal{C},\mathcal{C}]$ of strict monoidal actions.

4.4 Family defined by a distributive law

Proposition 11. *Given a distributive law between monoidal actions* $(\textcircled{M}) \colon \mathcal{M} \times \mathcal{C} \to \mathcal{C}$ *and* $(\textcircled{N}) \colon \mathcal{N} \times \mathcal{C} \to \mathcal{C}$*, the functor* $(\textcircled{N}\textcircled{M}) \colon \mathcal{N} \times \mathcal{M} \times \mathcal{C} \to \mathcal{C}$ *is again a monoidal action.*

Proof. The unit of the monoidal action is given by a pair of units. The product of the monoidal action is given by the distributive law followed by the product. We can check this applying the axioms for a distributive law we defined previously.





□

4.5 Inclusions into the distributive family

It can be shown that, given a distributive law between monoidal actions $(\textcircled{M}) \colon \mathcal{M} \times \mathcal{C} \to \mathcal{C}$ and $(\textcircled{N}) \colon \mathcal{N} \times \mathcal{C} \to \mathcal{C}$, we have morphisms of monoidal actions $(\textcircled{M}) \to (\textcircled{N}\textcircled{M})$ and $(\textcircled{N}) \to (\textcircled{N}\textcircled{M})$. These induce in turn functors between the categories of optics $\mathit{Optic}_{\textcircled{M}} \to \mathit{Optic}_{\textcircled{N}\textcircled{M}}$ and $\mathit{Optic}_{\textcircled{N}} \to \mathit{Optic}_{\textcircled{N}\textcircled{M}}$. We can compose monoidal optics of different families into a monoidal optic if we push them both to the monoidal optic that their distributive law has induced.

5 Teleological categories

5.1 Teleological categories

We ask what is the universal property of a category of optics for a monoidal action. As profunctors, optics are freely adding a family of morphisms $(M \textcircled{M} A, M \textcircled{M} B) \to (A, B)$ to the category.

Definition 12. Let $(\textcircled{M}) \colon \mathcal{M} \times \mathcal{C} \to \mathcal{C}$ a monoidal action. A **teleological category** is a category $\mathcal{T}$ equipped with an identity-on-objects functor $\mathcal{C}^{op} \times \mathcal{C} \to \mathcal{T}$ and a family of morphisms

$$t_{M,A,B} \colon \begin{pmatrix} M \textcircled{M} A \\ M \textcircled{M} B \end{pmatrix} \to \begin{pmatrix} A \\ B \end{pmatrix},$$

which we call its teleological morphisms. We require the teleological morphisms to be natural on A and B, and to be dinatural on M. Additionally, we require them to make the following diagrams commute.

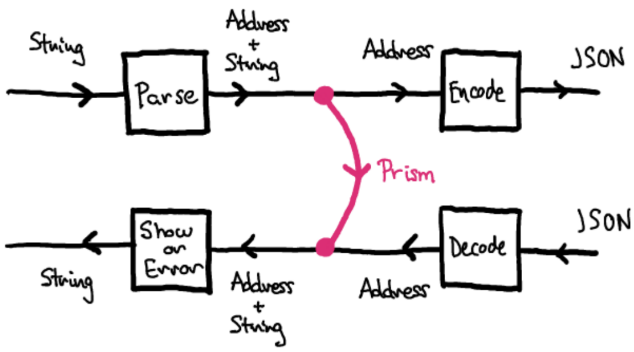
$$
\begin{tikzcd}
\begin{pmatrix} N \textcircled{\scriptsize M} M \textcircled{\scriptsize M} A \\ N \textcircled{\scriptsize M} M \textcircled{\scriptsize M} B \end{pmatrix} \arrow[r, "t_N"] \arrow[d] & \begin{pmatrix} M \textcircled{\scriptsize M} A \\ M \textcircled{\scriptsize M} B \end{pmatrix} \arrow[d, "t_M"] \\
\begin{pmatrix} N \otimes M \textcircled{\scriptsize M} A \\ N \otimes M \textcircled{\scriptsize M} B \end{pmatrix} \arrow[r, "t_{N \otimes M}"] & \begin{pmatrix} A \\ B \end{pmatrix}
\end{tikzcd}
\qquad
\begin{tikzcd}
\begin{pmatrix} I \textcircled{\scriptsize M} A \\ I \textcircled{\scriptsize M} B \end{pmatrix} \arrow[d, bend left, "t_I"] \\
\begin{pmatrix} A \\ B \end{pmatrix} \arrow[u, bend left, "{(\lambda,\lambda^{-1})}"]
\end{tikzcd}
$$

Proposition 13. *Let* $(\textcircled{\scriptsize M}) \colon \mathcal{M} \times \mathcal{C} \to \mathcal{C}$ *be a monoidal action.* $\mathit{Optic}_{\textcircled{\scriptsize M}}$ *is the free teleological category for this action.*

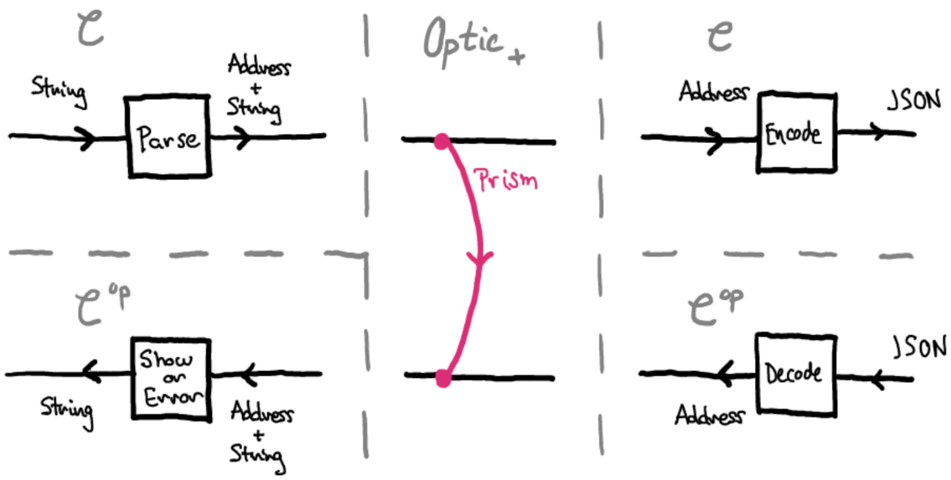
Proof. This is a consequence of the adjunction described in Proposition 16. Explicitly, it is the lifting to the category of moonids of the monoidal adjunction in Corollary 18. $\square$

5.2 Graphical notation for teleological categories

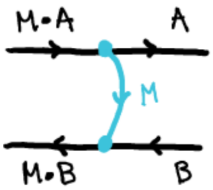
We shall represent morphisms in a teleological category as we would do in $\mathcal{C} \times \mathcal{C}^{op}$, that is, with a pair of diagrams, one in $\mathcal{C}$ and the other in $\mathcal{C}^{op}$. The only difference is the addition of these teleological morphisms that join the two. For instance, consider the following prism.



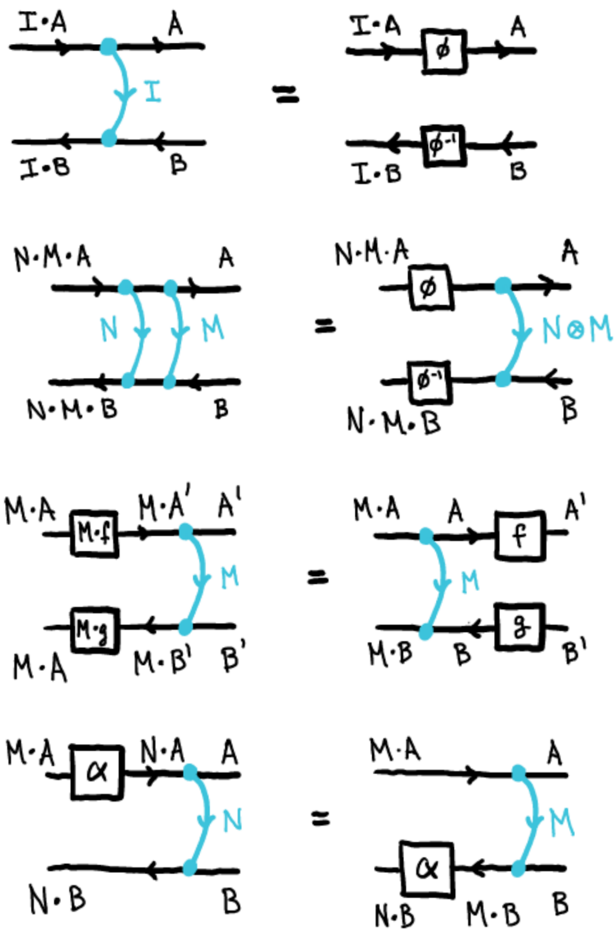
This diagram is to be read split as the composition of two morphisms in $\mathcal{C}^{op} \times \mathcal{C}$ with a formal tautological morphism that joins the two.



In general, teleological morphisms $(M \mathbin{\textcircled{\scriptsize M}} A, M \mathbin{\textcircled{\scriptsize M}} B) \to (A, B)$ will be depicted as follows.

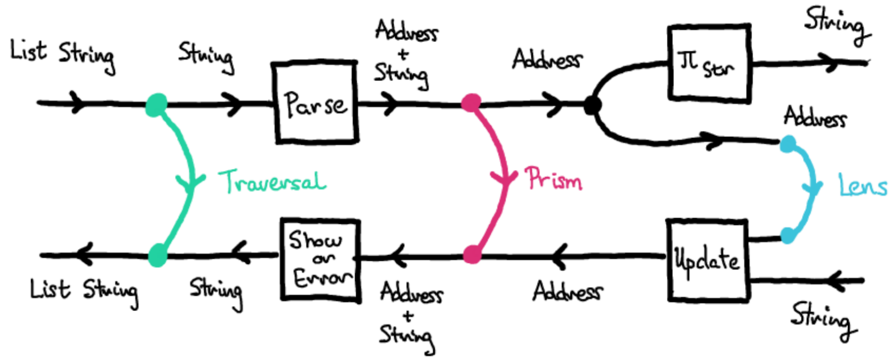


They are subject to the following equations, which represent, respectively: compatibility with the unit, compatibility with multiplication, naturality and dinaturality.



The composition of optics can be interpreted then in any category that provides an interpretation for all the teleological morphisms involved, such as the coproduct promonad.

In this notation, the previous Example 1 looks as follows.



6 Teleological profunctors

It is a straightforward check that monoids on the bicategory of profunctors, also known as *promonads*, correspond precisely to identity-on-objects functors. In order to study what is the universal property of the categories of optics, we will start regarding them as promonads. If we work on a category of profunctors with extra algebraic structure, monoids of this category will precisely be categories of optics.

Definition 14. Let $\mathcal{C}$ be an arbitrary category. A **teleological profunctor** (P, p) for an action $\circledM\colon \mathcal{M} \to [\mathcal{C}, \mathcal{C}]$ is given by

- a profunctor $P\colon \mathcal{C} \times \mathcal{C}^{op} \nrightarrow \mathcal{C} \times \mathcal{C}^{op}$;
- a family of *teleological elements* $p_{M,A,B} \in P((M \circledM A, M \circledM B), (A, B))$ dinatural on both $A, B \in \mathcal{C}$ and $M \in \mathcal{M}$.

A morphism between teleological profunctors $(P, p) \to (Q, q)$ for two possibly different actions $\circledM$ and $\circledN$ is a morphism of actions $(F, \alpha)\colon \circledM \to \circledN$ together with a natural transformation

$$\eta_{S,T,A,B}\colon P((S, T), (A, B)) \to Q((S, T), (A, B))$$

such that

$$\eta_{M\circledM A, M\circledM B, A, B}(P(\alpha^{-1}, \alpha)(p_{M,A,B})) = q_{FM,A,B}.$$

We call *TeleProf* to the category of teleological profunctors.

6.1 Adjunction

We will show that the obvious forgetful functor $U\colon \mathit{TeleProf} \to \mathit{Act}$ has a left adjoint $\mathit{Optic}\colon \mathit{Act} \to \mathit{TeleProf}$ given by the optic formula that we described at the introduction.

Proposition 15. *Let $\circledM\colon \mathcal{M} \to [\mathcal{C}, \mathcal{C}]$ be an action, the Optic profunctor,*

$$\mathit{Optic}_{\circledM}((A, B), (S, T)) = \int^{M \in \mathcal{M}} \mathcal{C}(S, M \circledM A) \times \mathcal{C}(M \circledM B, T).$$

is a teleological profunctor for $(\circledM)$, with a family of tautological elements given by

$$t_{\circledM} \coloneqq \langle \lambda \mid \lambda^{-1} \rangle \in \mathit{Optic}_{\circledM}((A, B), (M \circledM A, M \circledM B)).$$

Proposition 16. *There exists an adjunction $\mathit{Optic} \dashv U$. That is to say that the profunctor $\mathit{Optic}_{\circledM}$ is the free teleological profunctor over the action $\circledM$.*

Proof. Assume some teleological profunctor (P, p) for the action $\textcircled{\scriptsize N}\colon \mathcal{N} \to [\mathcal{C}, \mathcal{C}]$. Let $\textcircled{\scriptsize M}\colon \mathcal{M} \to [\mathcal{C}, \mathcal{C}]$ be an action with a morphism of actions $\textcircled{\scriptsize M} \to \textcircled{\scriptsize N}$. We will show that it uniquely lifts to a morphism of teleological profunctors $\mathbf{Optic}_{\textcircled{\scriptsize M}} \to P$. The set of natural transformations $\mathbf{Optic}_{\textcircled{\scriptsize M}} \to P$ can be rewritten as follows by virtue of the Yoneda lemma.

$$\begin{aligned} & \left(\int^{D\in\mathcal{D}} \mathcal{C}(S, D \textcircled{\scriptsize M} A) \times \mathcal{C}(D \textcircled{\scriptsize M} B, T)\right) \to P((S,T),(A,B)) \\ \cong \quad & \text{(Yoneda)} \\ & \int_{D\in\mathcal{D}} P((A,B),(D \textcircled{\scriptsize M} A, D \textcircled{\scriptsize N} B)). \end{aligned}$$

A natural transformation of this form is then determined by the family $p_{M,A,B}$. This witnesses the natural isomorphism that determines the adjunction

$$\mathit{Act}(\textcircled{\scriptsize M}, UP) \cong \mathit{TeleProf}(\mathit{Optic}_{\textcircled{\scriptsize M}}, P). \qquad \square$$

6.2 Monoidal structure

Proposition 17. *The category TeleProf is monoidal.*

Proof. Let (hom, id, id) be the teleological profunctor given by the hom-functor $(\mathcal{C}^{op} \times \mathcal{C}) \nrightarrow (\mathcal{C}^{op} \times \mathcal{C})$; the identity action $\mathrm{id}\colon 1 \to [\mathcal{C}, \mathcal{C}]$; and the family of elements given by the identity morphisms. We define the unit of the monoidal structure to be this profunctor.

Let $(P, \textcircled{\scriptsize p}, p)$ and $(Q, \textcircled{\scriptsize q}, q)$ be two teleological profunctors. We define their monoidal product as

$$(Q, \textcircled{\scriptsize q}, q) \otimes (P, \textcircled{\scriptsize p}, p) := (\textcircled{\scriptsize q}\textcircled{\scriptsize p}, Q \diamond P, p \bullet q).$$

Here $Q \diamond P$ denotes profunctor composition and $p \bullet q$ is a family of elements

$$(p \bullet q)_{A,B,ED} \in \int^{(U,V)} P((A,B),(U,V)) \times Q((U,V),(E \textcircled{\scriptsize q} D \textcircled{\scriptsize p} A, E \textcircled{\scriptsize q} D \textcircled{\scriptsize p} B))$$

given by $t_{A,B,D} \in P((A,B),(D \textcircled{\scriptsize p} A, D \textcircled{\scriptsize p} B))$ and $k_{DA,DB,E} \in P((D \textcircled{\scriptsize p} A, D \textcircled{\scriptsize p} B),(E \textcircled{\scriptsize q} D \textcircled{\scriptsize p} A, E \textcircled{\scriptsize q} D \textcircled{\scriptsize p} B))$ when interpreted as representatives for an equivalence class under the coend.

It can be checked that this unit and multiplication define a monoidal structure. $\square$

6.3 Monoidal adjunction

Let us take a moment to consider the consequences of this adjunction. Because $\mathcal{U}$ is strong monoidal, its left adjoint **Optic** must also be oplax monoidal. In this particular case, we will see that the oplax monoidal maps induced this way are actually isomorphisms, and *Optic* will be strong monoidal. In particular, this means that the optic construction produces promonads when applied to strict monoidal actions; Kleisli objects for these promonads are precisely categories of optics. Moreover, we will now see that the two ways of composing actions translate into the two ways of composing promonads.

Corollary 18. *The adjunction is a monoidal adjunction.*

Proof. Because U is monoidal, *Optic* is oplax monoidal. We will show that the induced oplax maps are actually isomorphisms. With the Yoneda lemma, we can construct an isomorphism, $Optic_{ⓂⓃ} \cong Optic_{Ⓜ} \diamond Optic_{Ⓝ}$; it can be checked that this isomorphism sends the teleological morphisms of $Optic_{ⓂⓃ}$ to the pairs of teleological morphisms in $Optic_{Ⓜ} \diamond Optic_{Ⓝ}$. By definition of the adjunction, this isomorphism is the induced oplax map.

Via Kelly's doctrinal adjunction [Kel74], the adjunction lifts to a monoidal adjunction. □

Corollary 19. *There is an adjunction between actegories (the monoids of the category of actions) and teleological promonads (the monoids of the category of teleological profunctors).*

$$\mathsf{Mon}\,Act(Ⓓ, UP) \cong \mathsf{Mon}\,TeleProf(Optic_{Ⓓ}, P).$$

Corollary 20. *The functor Optic takes coproducts of monoidal actions into coproduct promonads, $Optic_{Ⓜ+Ⓝ} \cong Optic_{Ⓜ} \oplus Optic_{Ⓝ}$.*

Proof. Left adjoints preserve colimits. □

Corollary 21. *A distributive law between two monoidal actions $ⓂⓃ \to ⓃⓂ$ induces a distributive law between their associated promonads $Optic_{Ⓜ} \diamond Optic_{Ⓝ} \to Optic_{Ⓝ} \diamond Optic_{Ⓜ}$.*

Proof. A strong monoidal functor preserves monoids and their distributive laws. □

6.4 Fibration to actions

Proposition 22. *The forgetful functor $U\colon \to Act$ is a monoidal fibration, in the sense of Shulman [**?**].*

Proof. Let $(P, Ⓝ, p)$ be a teleological profunctor and let $Ⓜ \to Ⓝ$ be a morphism of actions. We can construct some $(P, Ⓜ, q)$ where the tautological components $t \in P((A,B),(D Ⓝ A, D Ⓝ B))$ are precisely the $k \in P((A,B),(FD Ⓜ A, FD Ⓜ B))$ after isomorphism.

For any other $(Q, ⓞ, q)$ with a morphism of actions $ⓞ \to Ⓜ \to Ⓝ$, we can factor any morphism of teleological profunctors through $(P, Ⓜ, p)$ in a unique way; this follows from the definition.

Note: *This proof is still not complete.*

This yields a weak monoidal pseudofunctor $Cat/[\mathcal{C},\mathcal{C}] \to Cat$. An action $Ⓜ\colon \mathcal{M} \to [\mathcal{C},\mathcal{C}]$ is sent by this functor to a category of endoprofunctors $P\colon \mathcal{C}^{op} \times \mathcal{C} \nrightarrow \mathcal{C}^{op} \times \mathcal{C}$ with a family of elements $t_{A,B,D} \in P((A,B),(M Ⓜ A, M Ⓜ B))$. □

7 Contributions and related work

Our main contribution is the translation of the two ways of composing optics in terms of the well-studied ways of composing monoids. This enables us to answer questions like the following.

- *What kind of optic is the composition of a lens and a prism in Haskell?* It is an optic for the coproduct action of the product and the coproduct. In other words, for the action

$$(X_1, Y_1, \ldots, X_n, Y_n) \bullet A \coloneqq X_1 + Y_1 \times (\ldots (X_n + Y_n \times A)).$$

  This optic does not seem to have any nice concrete characterization that makes it practical.

- *What are affine traversals?* Affine traversals admit two inclusions from both lenses and prisms that can be justified in terms of distributive laws. Thus, they enable a possible composition of lenses and prisms that is strictly less general than the one described before. Their advantage is that they have a concrete characterization.
- *Is the composition of optics of different families an optic?* Always, for the two notions of composition we have discussed. This is to say that optics are a family of promonads closed under coproducts and closed under suitable distributive laws. One could still define a distributive law between the promonads that does not arise in this way, but it would not preserve the tautological morphisms in the sense described previously.

Incidentally, we have started using a diagrammatic presentation that provides some advantages such as the following ones.

- Optics of different families can be represented in a unified fashion, unlike in [Hed17, Boi19].
- We can still reason about the laws of optics diagrammatically. This is achieved in [Boi19] in an elegant way, but the technique there does not seem to allow for composition of different optics.
- Teleological categories [Hed17] and their graphical calculus can be recovered as a particular case of this presentation.

7.1 String diagrams for optics

Boisseau [Boi19] has recently presented a clever idea to give categories of optics a graphical language. The central idea is to use the monoidal structure of their presheaf completion given by Tambara modules. It is not clear there, nor to us, what is the exact relation between this graphical calculus and the ones previously presented by Hedges [Hed17] and Riley [Ril18] that are being generalized here.

The description presented by Boisseau [Boi19] clearly depicts the laws of optics as comonoid homomorphisms. However, it is not easy to use this graphical calculus for optic composition. The main obstacle is that, in this graphical calculus, an action $\mathcal{M} \times \mathcal{C} \to \mathcal{C}$ needs to be lifted to a compatible $\mathcal{M} \times \mathcal{M} \to \mathcal{M}$ in order to be composed with itself; making it unsuitable for composition of optics of different kinds. The closest we can get is to draw the optics we have been drawing in *Cat* under their embedding into the bicategory of profunctors *Prof* (Figure **??**); the resulting graphical calculus has looks more similar to that of Boisseau, but it is still different. How to compose optics of different families in this formalization remains future work.

7.2 Teleological categories

Definition 23. [Ril18, copied from Definition 2.1.4.] A teleological category is a symmetric monoidal category $(\mathcal{T}, \boxtimes, I)$, equipped with:

- A symmetric monoidal subcategory $\mathcal{T}_d$ of *dualisable morphisms* containing all the objects of $\mathcal{T}$, with an involutive symmetric monoidal functor $(-)^* \colon \mathcal{T}_d \to \mathcal{T}_d^{op}$, where – not finding a standard symbol for such a thing – we mean $\mathcal{T}_d^{op}$ to be the category with both the direction of the arrows and the order of the tensor flipped: $(A \boxtimes B)^* \cong B^* \boxtimes A^*$. Note that there is therefore also a canonical isomorphism $\varphi : I = I^*$.
- A symmetric monoidal extranatural family of morphisms $\varepsilon_X \colon X \boxtimes X^* \to I$, called *counits*, natural with respect to the dualisable morphisms.

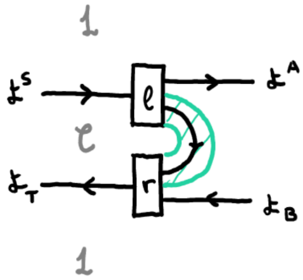


Figure 3: An optic embedded into *Prof*. Note how the coend condition becomes invisible to the diagrammatic calculus. Here we use $\text{よ}_\square$ and $\text{よ}^\square$ to denote the covariant and contravariant Yoneda embedding.

Lemma 24. *Teleological categories are precisely teleological promonads whose underlying action is a monoidal product.*

Proof. Let $\mathcal{T}$ be a teleological category. We start by noting that $\mathcal{T}_d \to \mathcal{T}$ is an identity on objects functor, and so is $\mathcal{T}_d^{op} \to \mathcal{T}_d \to \mathcal{T}$.

Consider the promonad $\Phi\colon \mathcal{T}_d^{op} \times \mathcal{T}_d \nrightarrow \mathcal{T}_d^{op} \times \mathcal{T}_d$ defined by $\Phi(A, B, S, T) := \mathcal{T}(A^* \boxtimes B, C^* \boxtimes D)$. We define the teleological morphisms $\mathcal{T}(A^* \boxtimes B, (C \boxtimes A)^* \boxtimes (C \boxtimes B))$ to be the whiskered units.

Conversely, let Φ be an teleological promonad over $\mathcal{C}^{op} \times \mathcal{C}$ for the action $(\boxtimes)$. We will show that its Kleisli category Φ can be given teleological category structure. The wide subcategory of dualisable morphisms is precisely $\mathcal{C}^{op} \times \mathcal{C}$, endowed with the obvious involutive functor $(\mathcal{C}^{op} \times \mathcal{C})^{op} \to \mathcal{C}^{op} \times \mathcal{C}$. There is a family $\varepsilon_A \in \Phi((A, A), (I, I))$ coming from the tautological morphisms. $\square$

7.3 Categories for an actegory

The notion of teleological promonad may look artificial, and one can ask if there are simpler notions from which it arises. There exists the notion of groupoid for a group action, that can be extended to the action of category for a monoid action.

Definition 25. Let $\circledM\colon M \times A \to A$ be the action of a monoid on a set. We define the category of the action $\mathsf{Act}(\circledM)$ as letting A as the underlying set of objects and letting $\mathsf{Act}(\circledM)(a, b) = \{m \in M \mid m \circledM a = a\}$ be the set of morphisms.

8 Acknowledgements

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