Formal Tambara theory

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1 Pseudomonoids in monoidal bicategories

A *pseudomonoid* in a monoidal bicategory is a categorification of a monoid in a monoidal category. Let $(\mathcal{K}, \circ, \otimes)$ be a monoidal bicategory. A pseudomonoid in $\mathcal{K}$ consists of an object A, 1-cells $m \colon A \otimes A \to A$ and $i \colon I \to A$ and invertible 2-cells and invertible 2-cells $\alpha \colon m \circ (m \otimes \mathrm{id}) \to m \circ (\mathrm{id} \otimes m)$, $\lambda \colon m \circ (i \otimes \mathrm{id}) \to \mathrm{id}$ and $\rho \colon m \circ (\mathrm{id} \otimes i) \to \mathrm{id}$, satisfying suitable coherence equations, see [DS97]. Pseudomonoids in **Cat** are precisely monoidal categories.

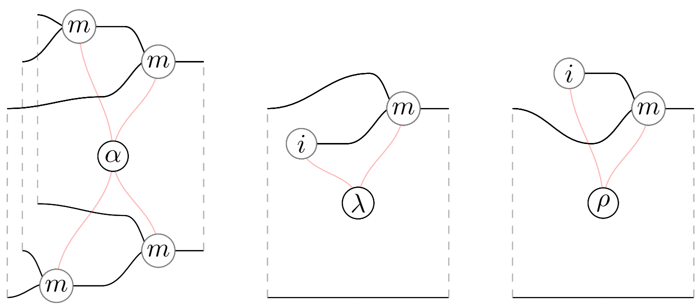

Figure 1: Associator and unitors for a pseudomonoid. We employ surface diagrams as in [Wil08] or [Bar14].

In monoidal bicategories, apart from pseudomonoids, one can also consider *map pseudomonoids*, where the multiplication and unit are left adjoints (see Figures 2 and 3). Monoidal categories are map pseudomonoids in **Prof**, but the converse only holds when the domain is Cauchy-complete (see [Bor94]).

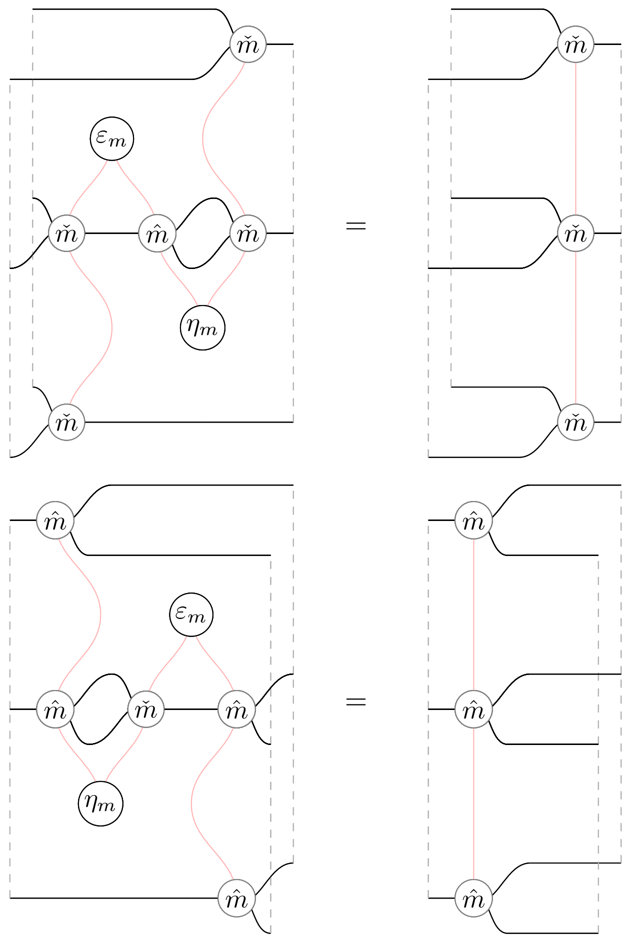

Figure 2: Adjunction for the product of a map pseudomonoid.

2 Tambara modules

The original definition of Tambara module [PS08] is in terms of enriched profunctors and (co)end calculus [Lor19]. Let us first recall it in Definition 2.1. We will then abstract this definition from the monoidal bicategory of profunctors **Prof** to an arbitrary monoidal bicategory in Definition 2.2.

Definition 2.1. Let $\mathbf{C}$ be a monoidal category. A **Tambara module** is a profunctor $T\colon \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{Set}$ equipped with a family of morphisms

$$t_{A,B,M}\colon\ T(A,B) \to \int_{M\in\mathbf{M}} T(M \otimes A, M \otimes B)$$

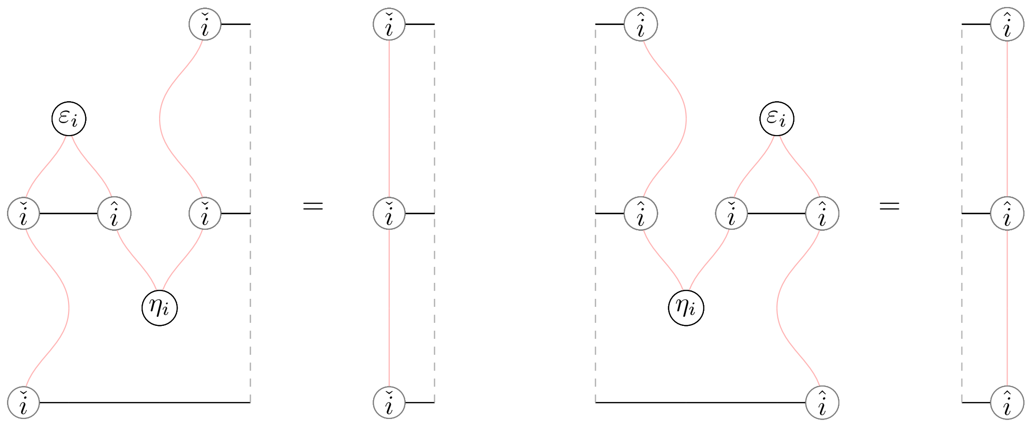

Figure 3: Adjunction for the unit of a map pseudomonoid.

natural on both A and B and dinatural on M. They are asked to satisfy a pair of conditions that ensure that they interplay nicely with the monoidal structure. A similar definition can be given for the case of actegories [PS08, Ril18, Rom19].

Definition 2.2. Let $(\mathcal{K}, \circ, \otimes)$ be an arbitrary monoidal bicategory, and let $(C, \check{m}, \check{\imath}, \alpha, \lambda, \rho)$ be a map pseudomonoid, with $\check{m} \dashv \hat{m}$ and $\check{\imath} \dashv \hat{\imath}$. A Tambara module is a 1-cell $T\colon \mathbf{C} \to \mathbf{C}$ equipped with a 2-cell defined as in Figure 4. They are subject to the equations described in Figures 5 and 6.

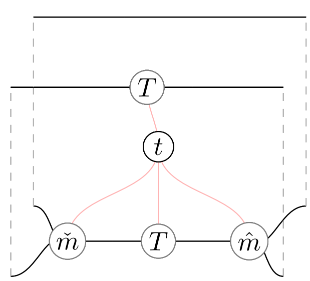

Figure 4: Tambara module.

References

[Bar14] Bruce Bartlett. Quasistrict Symmetric Monoidal 2-Categories Via Wire Diagrams. *arXiv preprint 1409.2148*, 2014.

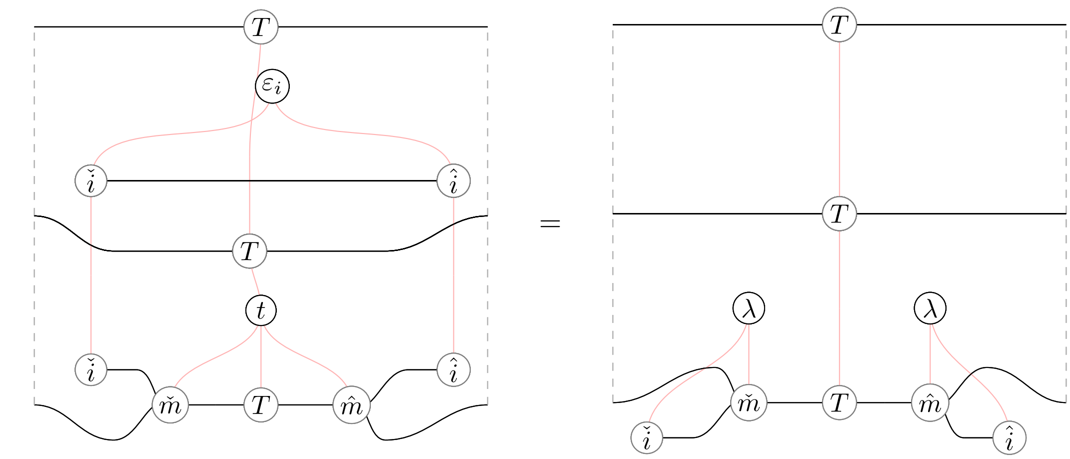

Figure 5: First axiom for a Tambara module.

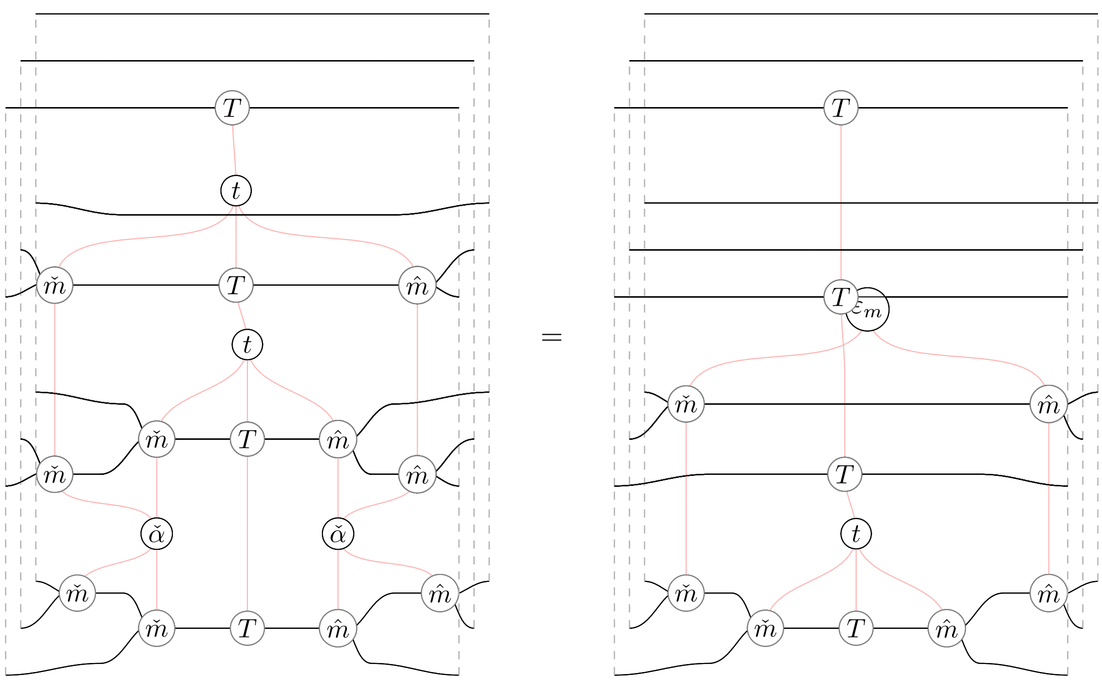

Figure 6: Second axiom for a Tambara module.

[Bor94] Francis Borceux. *Handbook of categorical algebra: volume 1, Basic category theory*, volume 1. Cambridge University Press, 1994.

[DS97] Brian Day and Ross Street. Monoidal bicategories and Hopf algebroids. *Advances in Mathematics*, 129(1):99–157, 1997.

[Lor19] Fosco Loregian. Coend calculus. *arXiv preprint arXiv:1501.02503*, 2019.

[PS08] Craig Pastro and Ross Street. Doubles for monoidal categories. *Theory and applications of categories*, 21(4):61–75, 2008.

[Ril18] Mitchell Riley. Categories of Optics. *arXiv preprint arXiv:1809.00738*, 2018.

[Rom19] Mario Román. Profunctor optics and traversals. *Master's thesis, University of Oxford*, 2019.

[Wil08] Simon Willerton. A Diagrammatic Approach to Hopf Monads. *arXiv preprint arXiv:0807.0658*, 2008.