Diagrammatic Tambara theory

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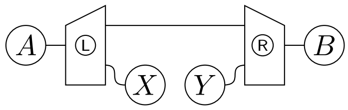
For the specialized reader, it may be of interest to consider the definition of a **mixed optic**, the generalized version of an optic. Let $\mathbf{M}$ a monoidal category acting with two monoidal actions $Ⓛ\colon \mathbf{M} \times \mathbf{C} \to \mathbf{C}$ and $Ⓡ\colon \mathbf{M} \times \mathbf{D} \to \mathbf{D}$. A mixed optic $(A, B) \to (X, Y)$ is an element of the following set.

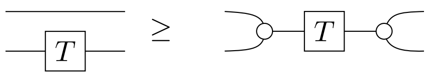


The theory of optics generalizes in the expected way to mixed optics [CEG+20]; the graphical calculus helps us explain why that should be the case, as all the derivations we need hold the same for monoidal actions and monoidal products.

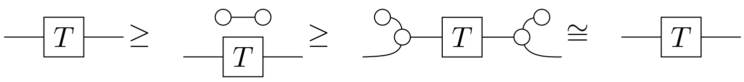
During the following section we fix a pair of actions $Ⓛ\colon \mathbf{M} \times \mathbf{C} \to \mathbf{C}$ and $Ⓡ\colon \mathbf{M} \times \mathbf{D} \to \mathbf{D}$. They are both, together with the monoidal structure of $\mathbf{M}$, to be represented by a white dot. In most cases, they are precisely the monoidal structure on $\mathbf{C}$, and the added complexity of the notation would add nothing interesting to the proof.

1 Tambara modules and the Pastro-Street monad

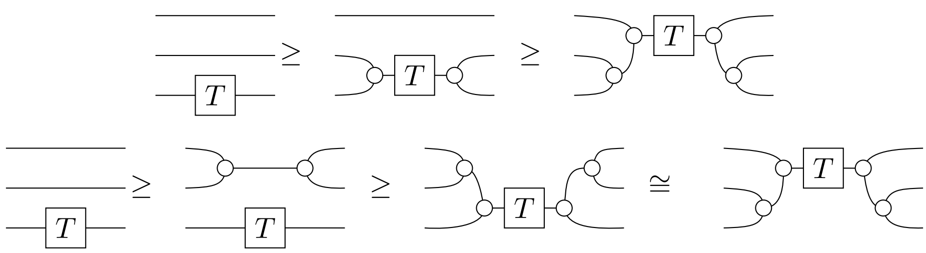
Definition 1.1 (Tambara module)**.** A **Tambara module** is a profunctor $T\colon \mathbf{C}^{op} \times \mathbf{D} \to \mathbf{Set}$ equipped with a natural transformation as follows.



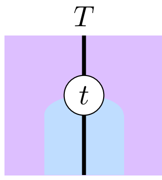
Such that the following morphism is the identity.



And such that the following two morphisms coincide

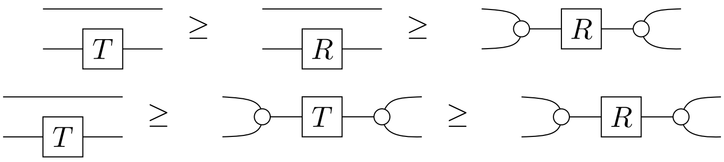


Shifting perspective, a Tambara module is a $T\colon \mathbf{C}^{op} \times \mathbf{D} \to \mathbf{Set}$ equipped with a natural transformation as follows.



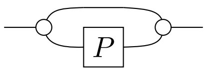
Such that the following equations hold.

Definition 1.2 (Morphism of Tambara modules)**.** A morphism of Tambara modules $T \to R$ is a natural transformation between the profunctors such that the following two morphisms coincide.

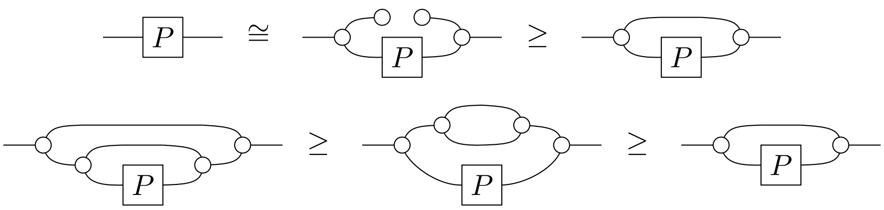


Tambara modules form a category. It can be checked that the composition of two morphisms of Tambara modules is a morphism of Tambara modules. We call た (hiragana for "ta") to the category of Tambara modules.

Definition 1.3 (Pastro-Street monad)**.** Consider the functorial assigment that maps a profunctor P to the following profunctor.

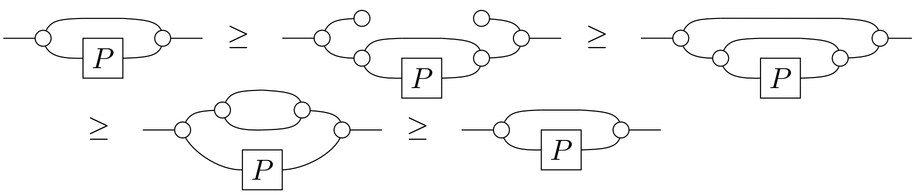


It is a monad Φ with the unit and multiplication given by the following morphisms.

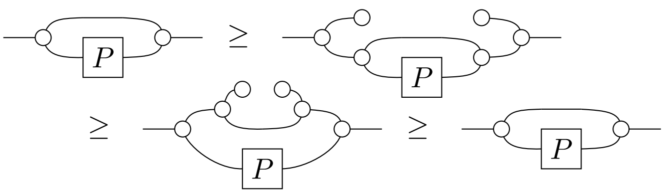


Proof. Let us prove left unitality, right unitality is analogous. The idea is to see that the following three derivations are homotopic.

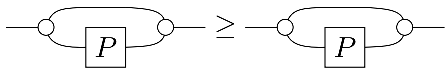
- Slice one.



- Slice two.



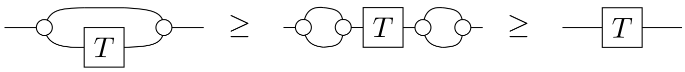
- Slice three.



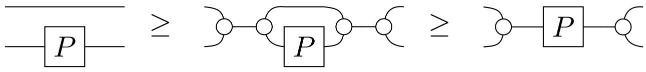
□

Theorem 1.4. *Tambara modules are the algebras of the Pastro-Street monad.*

Proof. Given a Tambara module, we construct an algebra as follows.



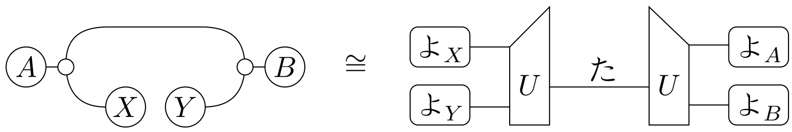
Given an algebra, we construct a Tambara module as follows.



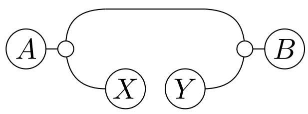
It remains to check that the two morphisms are inverse to each other, this follows from the axioms of monoidal categories. □

2 Profunctor representation

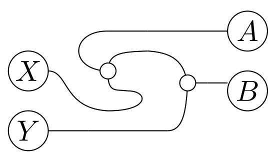
Theorem 2.1 (Profunctor Representation Theorem)**.** *There exists an isomorphism between elements of the following shape that moreover preserves composition.*



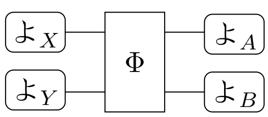
Proof. This proofs closely follows [Rom19, Theorem 5.3.1].



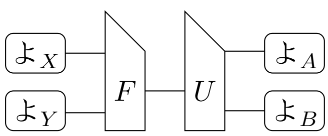
$\cong$



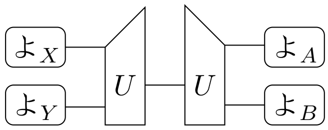
$\cong$ (Yoneda)



$\cong$ (Adjunction giving rise to Φ)



$\cong$



$\square$

References

[CEG+20] Bryce Clarke, Derek Elkins, Jeremy Gibbons, Fosco Loregian, Bartosz Milewski, Emily Pillmore, and Mario Román. Profunctor optics, a categorical update. *arXiv preprint arXiv:1501.02503*, 2020.

[Rom19] Mario Román. Profunctor optics and traversals. *Master's thesis, University of Oxford*, 2019.