A Faster-Than Relation for Semi-Markov Decision Processes

QAPL 2019

April 7, 2019

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Agenda

Introduction

Introduction
Background

Semi-Markov decision processes have

- real-time behaviour,
- probabilistic behaviour, and
- non-deterministic behaviour.

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Semi-Markov decision processes have

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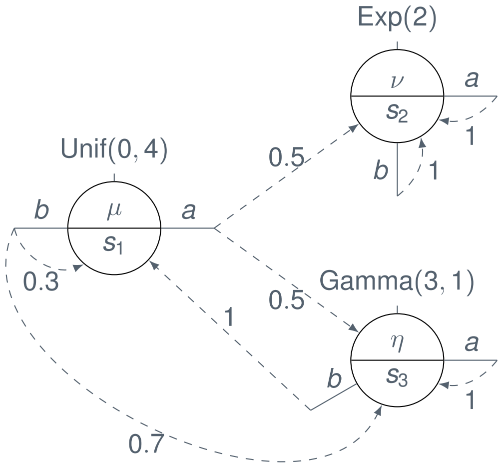
- probabilistic behaviour, and
- non-deterministic behaviour.

They have been used to model

- power plants,
- transportation infrastructure,
- revenue management systems,
- bridge maintenance,
- and more.

Introduction
Semi-Markov decision processes



Introduction

Semi-Markov decision processes

A Semi-Markov decision process is given by

- S a countable set of states,
- $\tau : S \times A \to \mathcal{D}(S)$ the transition function, and
- $\rho : S \to \mathcal{D}(\mathbb{R}_{\geq 0})$ the residence-time function

Introduction
Semi-Markov decision processes

We focus on

- the real-time behaviour of systems and
- a trace-based semantics.

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Semi-Markov decision processes

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- the real-time behaviour of systems and
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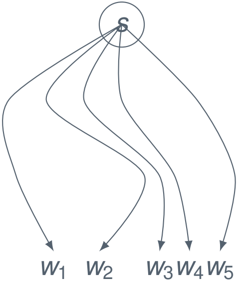
When is one process faster than another?

Faster-than relation

Faster-than relation

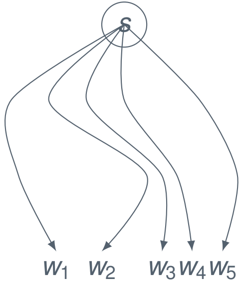
Probability



$$\mathbb{P}^{\sigma}(s)(\mathfrak{C}(a_1 \ldots a_n, t)) = \sum_i w_i \cdot F_i(t)$$

Faster-than relation

Probability



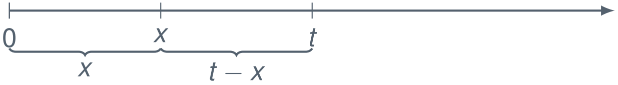
$$\mathbb{P}^\sigma(s)(\mathfrak{C}(a_1 \ldots a_n, t)) = \sum_i w_i \cdot F_i(t)$$

$$= \sum_{s_1 \in S} \cdots \sum_{s_n \in S} \tau^\sigma(s)(a_1, s_1) \cdots \tau^\sigma(s_{n-1})(a_n, s_n) \cdot (\rho(s) * \cdots * \rho(s_{n-1}))([0, t])$$

Faster-than relation
Convolution

The convolution of two measures μ and ν is given by

$$(\mu * \nu)([0, t]) = \int_0^t \nu([0, t - x]) \, \mu(\mathrm{d}x)$$



$$\mathbb{P}(X_\mu + X_\nu \in [0, t]) = (\mu * \nu)([0, t])$$

Faster-than relation

Definition

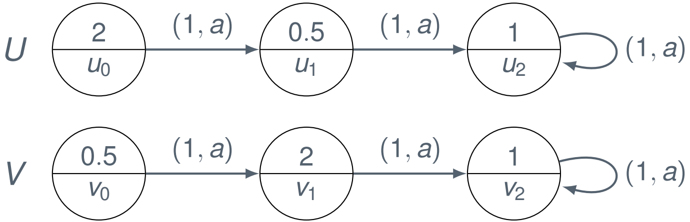
U is faster than V, written $U \preceq V$,

if and only if

for all σ, t, and $a_1 \ldots a_n$, there exists σ' such that

$$\mathbb{P}^{\sigma'}(u_0)(\mathfrak{C}(a_1 \ldots a_n, t)) \geq \mathbb{P}^{\sigma}(v_0)(\mathfrak{C}(a_1 \ldots a_n, t))$$

Faster-than relation
Example



Faster-than relation
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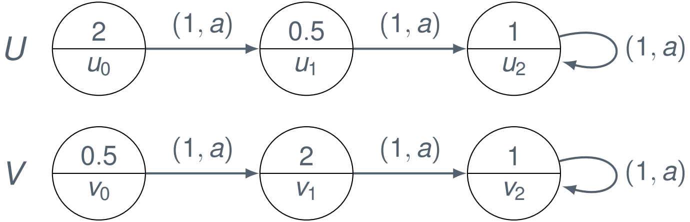

$$\mathbb{P}(u_0)(\mathfrak{C}(a,t)) \geq \mathbb{P}(v_0)(\mathfrak{C}(a,t))$$

Faster-than relation

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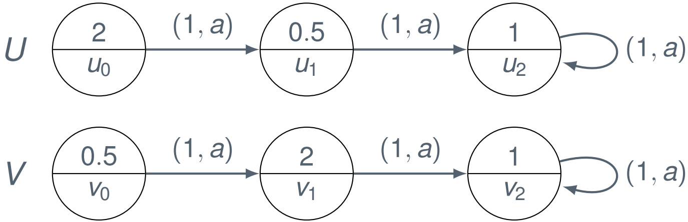

$$\mathbb{P}(u_0)(\mathfrak{C}(a, t)) \geq \mathbb{P}(v_0)(\mathfrak{C}(a, t))$$

$$\mathbb{P}(u_0)(\mathfrak{C}(aa, t)) = \mathbb{P}(v_0)(\mathfrak{C}(aa, t))$$

Faster-than relation

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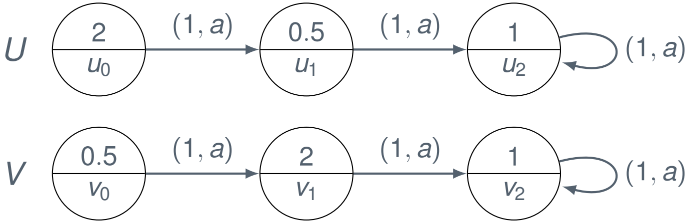


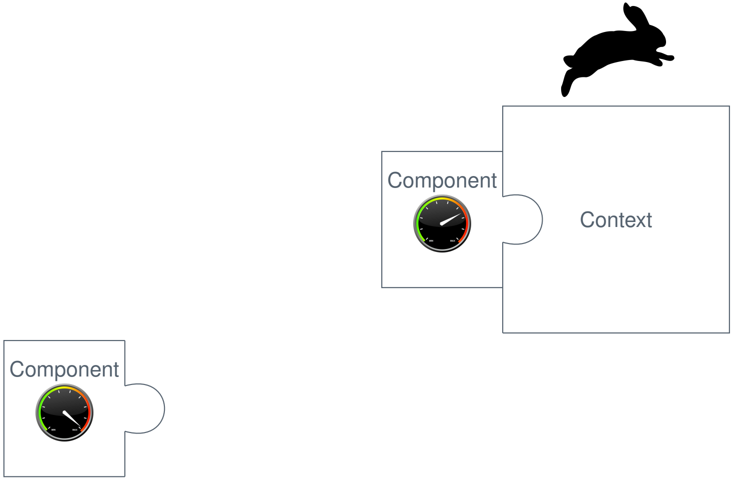
$$\mathbb{P}(u_0)(\mathfrak{C}(a,t)) \geq \mathbb{P}(v_0)(\mathfrak{C}(a,t))$$

$$\mathbb{P}(u_0)(\mathfrak{C}(aa,t)) = \mathbb{P}(v_0)(\mathfrak{C}(aa,t))$$

$$\mathbb{P}(u_0)(\mathfrak{C}(aaa^n,t)) = \mathbb{P}(v_0)(\mathfrak{C}(aaa^n,t))$$

Compositionality

Compositionality
Overview



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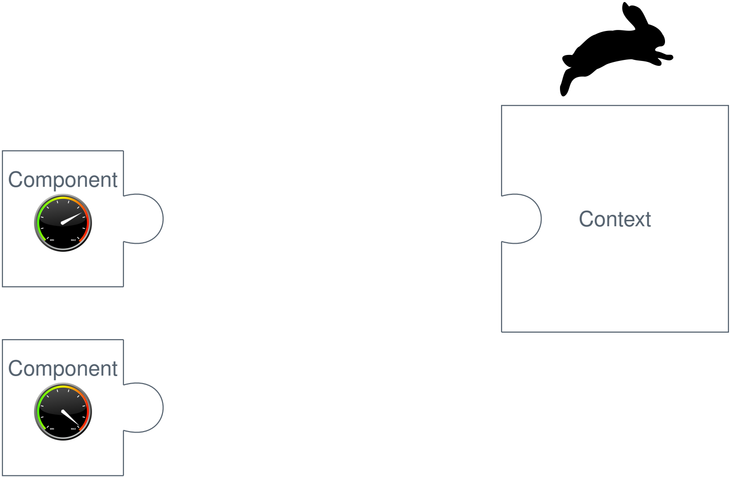

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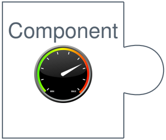

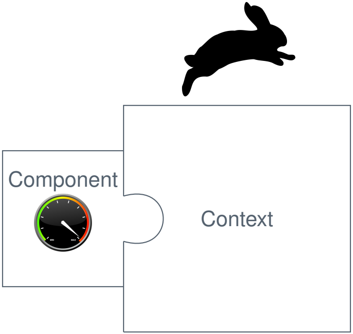

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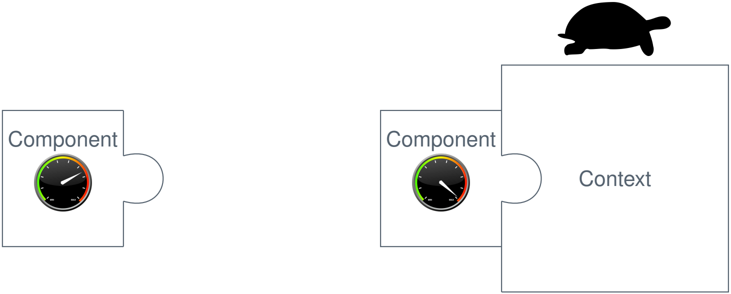

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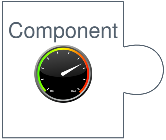

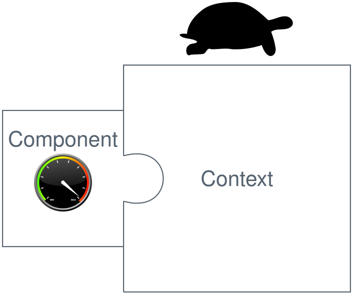

(Parallel) timing anomaly

Compositionality

Composing time behaviour

Definition

A function $\star : \mathcal{D}(\mathbb{R}_{\geq 0}) \times \mathcal{D}(\mathbb{R}_{\geq 0}) \to \mathcal{D}(\mathbb{R}_{\geq 0})$ is a *residence-time composition function* if $\star(\mu, \nu) = \star(\nu, \mu)$ for all $\mu, \nu \in \mathcal{D}(\mathbb{R}_{\geq 0})$.

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- Maximum composition: $F_{\star(\mu,\nu)}(t) = \max(F_\mu(t), F_\nu(t))$.
- Product rate composition: $F_{\star(\mu,\nu)}(t) = Exp[\theta \cdot \theta'](t)$.
- Minimum rate composition: $F_{\star(\mu,\nu)}(t) = Exp[\min(\theta, \theta')](t)$.
- Maximum rate composition: $F_{\star(\mu,\nu)}(t) = Exp[\max(\theta, \theta')](t)$.

Compositionality
Composing SMDPs

Definition

Let $\star$ be a residence-time composition function. Then the $\star$-*composition* of U and V, denoted by $U \parallel_\star V = (S, \tau, \rho)$, is given by

- $S = U \times V$,
- $\tau((u, v), a)((u', v')) = \tau_U(u, a)(u') \cdot \tau_V(v, a)(v')$ for all $a \in L$ and $(u', v') \in S$, and
- $\rho((u, v)) = \star(\rho_U(u), \rho_V(v))$.

Timing anomalies

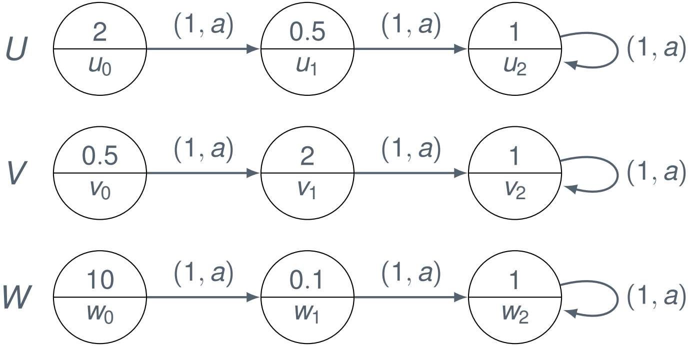
Timing anomalies
Definition

Timing anomaly:

$$U \preceq V \quad \text{but} \quad U \parallel_\star W \not\preceq V \parallel_\star W$$

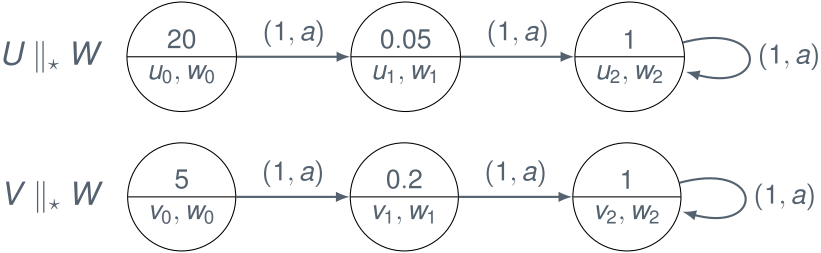
Timing anomalies

Product composition

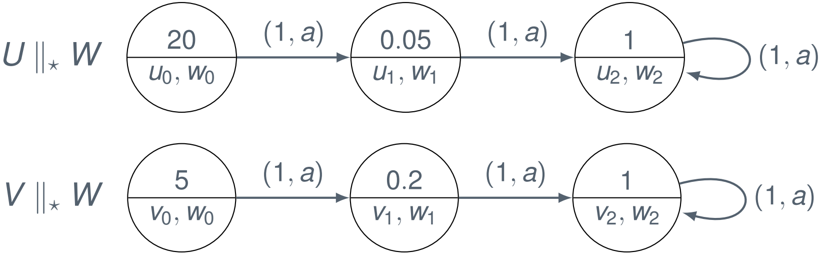


Timing anomalies

Product composition



Timing anomalies
Product composition



$$\mathbb{P}(u_0 \parallel_\star w_0)(\mathfrak{C}(aa, 2)) \approx 0.09 \not\geq \mathbb{P}(v_0 \parallel_\star w_0)(\mathfrak{C}(aa, 2)) \approx 0.30$$

Avoiding timing anomalies

Avoiding timing anomalies
The idea

$U \parallel_\star W$ $\qquad$ $V \parallel_\star W$

Avoiding timing anomalies

The idea

$$U \parallel_\star W \qquad U \preceq V \qquad V \parallel_\star W$$

Avoiding timing anomalies
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$$U \parallel_{\star} W \quad \preceq \quad U \quad \preceq \quad V \quad \preceq \quad V \parallel_{\star} W$$

Avoiding timing anomalies

The idea

$$U \parallel_\star W \quad \preceq \quad U \quad \preceq \quad V \quad \preceq \quad V \parallel_\star W$$

Problem: $\preceq$ is undecidable[1]

[1] M. R. Pedersen, N. Fijalkow, G. Bacci, K. G. Larsen, R. Mardare: Timed Comparisons of Semi-Markov Processes, LATA 2018

Avoiding timing anomalies

The idea

$$U \parallel_\star W \quad \preceq \quad U \quad \preceq \quad V \quad \preceq \quad V \parallel_\star W$$

Problem: $\preceq$ is undecidable[1]
Solution: Overapproximate $\preceq$

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Avoiding timing anomalies

Strong monotonicity

Definition

$\star$ is *strongly n-monotonic* in U, V, W, and W', written $(U, W) \leqq^n_\star (V, W')$ if W' has a deterministic Markov kernel and the following holds pointwise along all paths of length up to n:

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Strong monotonicity

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- The CDF of $U \parallel_{\star} W$ is pointwise greater than that of U.

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- The CDF of $U \parallel_\star W$ is pointwise greater than that of U.
- The CDF of V is pointwise greater than that of $V \parallel_\star W'$.
- For all σ_U and $\sigma_{U,W}$, the transition probability of $U \parallel_\star W$ under $\sigma_{U,W}$ is greater than that of U under σ_U.

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- For all $\sigma_{V,W'}$ and σ_V , the transition probability of V under σ_V is greater than that of $V \parallel_\star W'$ under $\sigma_{V,W'}$.

Avoiding timing anomalies
Strong monotonicity

Definition

$\star$ is *strongly n-monotonic* in U, V, W, and W', written $(U, W) \leqq_\star^n (V, W')$ if W' has a deterministic Markov kernel and the following holds pointwise along all paths of length up to n:

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$(U, W) \leqq_\star (V, W')$ if $(U, W) \leqq_\star^n (V, W')$ for all $n \in \mathbb{N}$.

Avoiding timing anomalies
Sufficient conditions

Theorem
If $(U, W) \leqq_\star (V, W')$ *as well as* $U \preceq V$ *and* $W \preceq W'$*, then* $U \parallel_\star W \preceq V \parallel_\star W'$*.*

Avoiding timing anomalies

Sufficient conditions

Theorem

If $(U, W) \leqq_\star (V, W')$ as well as $U \preceq V$ and $W \preceq W'$, then $U \parallel_\star W \preceq V \parallel_\star W'$.

Proof.

$$U \parallel_\star W \quad \preceq \quad U \quad \preceq \quad V \quad \preceq \quad V \parallel_\star W$$

$\square$

Avoiding timing anomalies

Decidability

Lemma

Let U, V, W, and W' be finite and let

$$m = \max\{|S_U| \cdot |S_W|, |S_V| \cdot |S_{W'}|\} + \max\{|S_U|, |S_V|, |S_W|, |S_{W'}|\} + 1.$$

If $(U, W) \leqq_\star^m (V, W')$, then $(U, W) \leqq_\star (V, W')$.

Avoiding timing anomalies
Decidability

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If $(U, W) \leqq_\star^m (V, W')$, then $(U, W) \leqq_\star (V, W')$.

Theorem

Let U, V, W, and W' be finite. If for all paths $u_1 \dots u_m$, $v_1 \dots v_m$, $w_1 \dots w_m$, and $w'_1 \dots w'_m$ we have that

$$\{t \mid F_{\rho(u_i \|_\star w_i)}(t) \geq F_{\rho_U(u_i)}(t)\} \quad \textit{and} \quad \{t \mid F_{\rho_V(v_i)}(t) \geq F_{\rho(v_i \|_\star w'_i)}(t)\}$$

are semialgebraic sets for all $1 \leq i \leq m$, then it is decidable whether $(U, W) \leqq_\star (V, W')$.

Avoiding timing anomalies
Limitations

Lemma

If $(U, W) \leqq_{\star} (V, W')$*, then* $|A| = 1$*, i.e. there is only one action.*

But remember: Timing anomalies can happen with only one action, and even with only one action, decidability of $\preceq$ is a difficult open problem.

Conclusion

Conclusion
Summary

Schedulers are horrible.

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Schedulers are horrible.

- The faster-than relation compares the accumulated time required for a trace.
- When composing systems, timing anomalies can occur such that a faster process makes the overall system slower.
- Timing anomalies occur for many common ways of composing systems found in the literature.
- We have identified a decidable set of conditions that are sufficient to guarantee the absence of timing anomalies.

Conclusion

Open Problems

- Clarify the power of different types of schedulers.
- Conditions that only look at the context, not the processes being swapped.
- Boundaries of decidability for conditions to avoid timing anomalies.