Timed Comparisons of Semi-Markov Processes

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Agenda

Introduction

Introduction
Background

Semi-Markov processes have

- real-time behaviour and
- probabilistic behaviour.

They have been used to model

- power plants,
- transportation infrastructure,
- revenue management systems,
- bridge maintenance,
- and more.

Introduction
Semi-Markov Processes

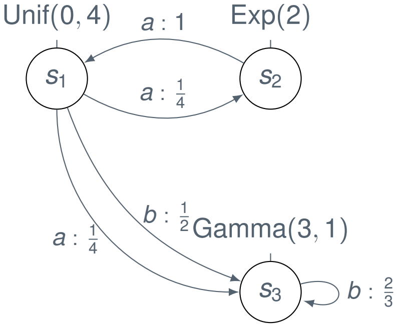

Figure: A semi-Markov process with different residence-time distributions.

Introduction

Semi-Markov Processes

A Semi-Markov process is given by

- S a finite set of states,
- $\mathtt{Out}$ a finite set of output labels,
- $\Delta : S \to \mathcal{D}(S \times \mathtt{Out})$ the transition function, and
- $\rho : S \to \mathcal{D}(\mathbb{R}_{\geq 0})$ the residence-time function.

Introduction
Semi-Markov Processes

We focus on

- the real-time behaviour of systems and
- a trace-based semantics.

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Semi-Markov Processes

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- the real-time behaviour of systems and
- a trace-based semantics.

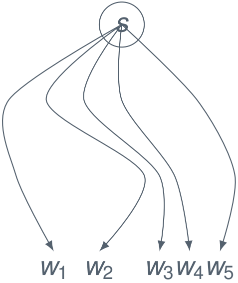
When is one process faster than another?

When everything that can be done by the slow process within some time bound can be done by the faster process with a higher probability.

Faster-Than Relation

Faster-Than Relation

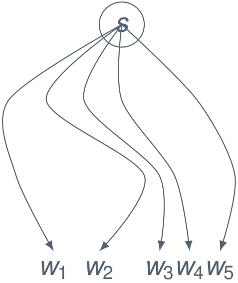
Probability



$$\mathbb{P}(s, a_1 \ldots a_n, t) = \sum_i w_i \cdot F_i(t)$$

Faster-Than Relation

Probability



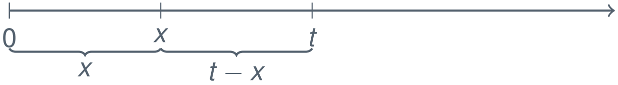
$$\mathbb{P}(s, a_1 \ldots a_n, t) = \sum_i w_i \cdot F_i(t)$$

$$= \sum_{s_1 \in S} \cdots \sum_{s_n \in S} \Delta(s)(a_1, s_1) \cdots \Delta(s_{n-1})(a_n, s_n) \cdot (\rho(s) * \cdots * \rho(s_{n-1}))([0, t])$$

Faster-Than Relation
Convolution

The convolution of two measures μ and ν is given by

$$(\mu * \nu)([0, t]) = \int_0^t \nu([0, t - x])\, \mu(\mathrm{d}x)$$



$$\mathbb{P}(X_\mu + X_\nu \in [0, t]) = (\mu * \nu)([0, t])$$

Faster-Than Relation
Definition

s is faster than s', written $s \preceq s'$ if and only if

$$\text{for all } w \text{ and all } t, \quad \mathbb{P}(s, w, t) \geq \mathbb{P}(s', w, t)$$

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Everything that can be done by s' within some time bound can be done by s with a higher probability.

Faster-Than Relation

Example

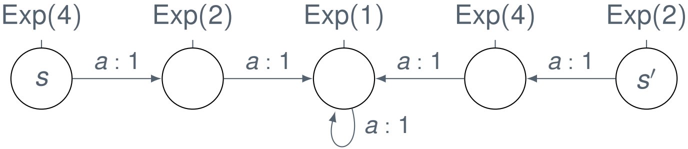

Figure: s is faster than s'.

Faster-Than Relation

Example

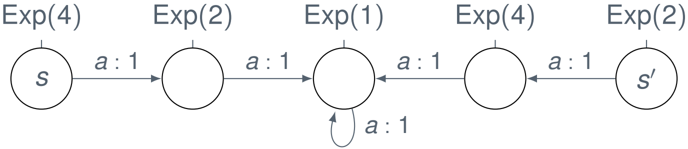

Figure: s is faster than s'.

$$\mathbb{P}(s, a, t) \geq \mathbb{P}(s', a, t)$$

Faster-Than Relation
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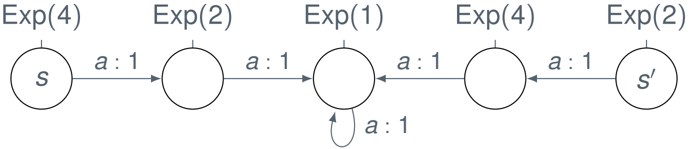

Figure: s is faster than s'.

$$\mathbb{P}(s, a, t) \geq \mathbb{P}(s', a, t)$$
$$\mathbb{P}(s, aa, t) = \mathbb{P}(s', aa, t)$$

Faster-Than Relation
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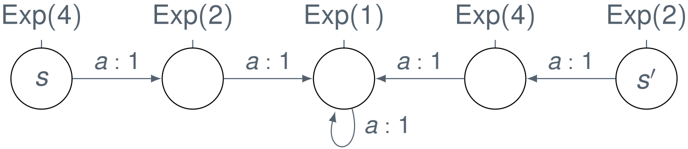


Figure: s is faster than s'.

$$\mathbb{P}(s, a, t) \geq \mathbb{P}(s', a, t)$$

$$\mathbb{P}(s, aa, t) = \mathbb{P}(s', aa, t)$$

$$\mathbb{P}(s, aaa^n, t) = \mathbb{P}(s', aaa^n, t)$$

Faster-Than Relation
Decision Problem

Faster-than problem:

Given s and s', determine whether $s \preceq s'$.

Hardness Results

Hardness Results

Undecidability

Theorem

The faster-than problem is undecidable.

Proof.

Reduction from the universality problem for probabilistic automata. □

The hardness results hold even for Markov processes, i.e. processes with discrete time.

Hardness Results

Recovering Decidability

We discuss three approaches to recovering decidability:

- restricting observations (output labels),
- approximation, and
- structural restrictions.

Hardness Results
Restricting Observations

Positivity problem for linear recurrence sequences: unsolved problem for at least 30 years.

Theorem

The faster-than problem is Positivity-hard for processes with a single output label.

Hardness Results
Approximation

Can the faster-than problem be approximated?

Yes

Hardness Results

Approximation

Can the faster-than problem be approximated?

Yes... and no

Hardness Results
Approximation

Can the faster-than problem be approximated?

Yes... and no

- Additive approximation: yes
- Multiplicative approximation: no

Hardness Results

Multiplicative approximation

Theorem

Let $0 < \varepsilon < \frac{1}{3}$. There is no algorithm which, given s and s'

- *returns YES if $\mathbb{P}(s, w) \geq \mathbb{P}(s', w)$ for all w*
- *returns NO if $\mathbb{P}(s, w) \leq \mathbb{P}(s', w) \cdot (1 - \varepsilon)$ for some w.*

Additive Approximation

Additive Approximation

Assumptions

Assumptions:

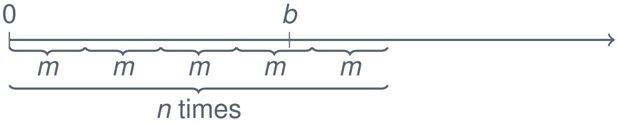
- Time-bounded: We only look at behaviours up to a given time bound.
- Slow residence-time functions: all transitions take *some* time to fire.

Time-bounded additive approximation:
Given s, s', $b \in \mathbb{R}_{\geq 0}$, and $\varepsilon > 0$, determine whether $\mathbb{P}(s, w, t) \geq \mathbb{P}(s', w, t) - \varepsilon$ for all w and $t \leq b$.

Additive Approximation

Slow Residence Time

Consider a process which uses at least m time to take a transition and only has a as output label.



- $\mathbb{P}(s, a^n, b) \to 0$ as $n \to \infty$.
- Hence we can find N such that $\mathbb{P}(s, a^n, b) \leq \varepsilon$ for all $n \geq N$.
- We only need to consider words of length $\leq N$.

Additive Approximation

Decidability

Theorem

For SMPs with slow residence-time distributions, the time-bounded additive approximation problem is decidable.

Unambiguous Processes

Unambiguous Processes
Structural Restrictions

A SMP is *unambiguous* if every output label leads to a unique successor state.

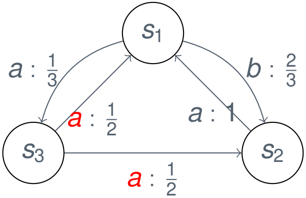


Figure: Ambiguous

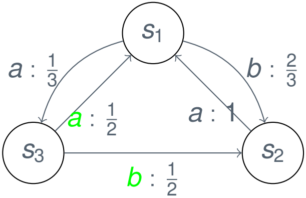


Figure: Unambiguous

Unambiguous Processes
Looping States

Let $L(s, s')$ denote the set of pairs of states reachable from (s, s') that eventually return to themselves, as well as the witnessing word for this return.

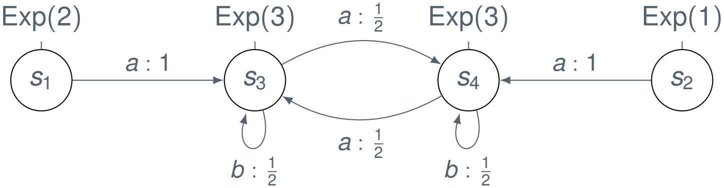


Figure: An unambiguous semi-Markov process.

$$L(s_1, s_2) = \{(s_3, s_4, ab^n ab^n), (s_4, s_3, ab^n ab^n)\}$$

Unambiguous Processes
Algorithm

Lemma
$s \preceq s'$ if and only if

- $\mathbb{P}(s, w, t) \geq \mathbb{P}(s', w, t)$ *for all* t *and* w *of length* $\leq S^2$ *and*
- $\mathbb{P}(s_1, v, t) \geq \mathbb{P}(s_2, v, t)$ *for all* t *and* $(s_1, s_2, v) \in L(s, s')$.

Unambiguous Processes
Decidability

Theorem

For unambiguous SMPs, the faster-than problem is decidable.

Conclusion

Conclusion
Summary

- We have introduced a faster-than relation for SMPs.
- The faster-than relation is undecidable.
  - Positivity-hard for one label.
  - Multiplicative approximation impossible.
- However, we can do time-bounded additive approximation.
- The relation becomes decidable for unambiguous processes.

Conclusion

Summary

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- However, we can do time-bounded additive approximation.
- The relation becomes decidable for unambiguous processes.

Open problems

- The symmetric equally-fast relation.
- Reactive models instead of generative.
- Logical aspects.
- Compositional aspects, including timing anomalies.