A Faster-Than Relation for Semi-Markov Decision Processes*

Giorgio Bacci[1], Kim G. Larsen[1], Radu Mardare[1], and Mathias R. Pedersen[1]

1 **Department of Computer Science, Aalborg University, Denmark**
`{grbacci,kgl,mardare,mrp}@cs.aau.dk`

Abstract

When modeling concurrent or cyber-physical systems, non-functional requirements such as time are important to consider. In order to improve the timing aspects of a model, it is necessary to have some notion of what it means for a process to be faster than another, which can guide the stepwise refinement of the model. To this end we define a trace-based faster-than relation for semi-Markov decision processes and show that (i) this relation preserves reachability properties, (ii) it has a succinct logical characterization, and (iii) we give an algorithm for guaranteeing that one process is faster than another. However, under our notion of faster-than, it may be the case that some process becomes slower than some other process, which it was previously faster than, when placed in some context. This is known as a parallel timing anomaly and is a well-known phenomenon in the areas of scheduling and processor architecture design. We give examples of such parallel timing anomalies under different kinds of process composition, and identify a sufficient set of conditions which are decidable for avoiding parallel timing anomalies.

1998 ACM Subject Classification F.1.1 Models of Computation

Keywords and phrases Semi-Markov decision processes, continuous-time Markov decision processes, faster-than preorder, compositionality, timing anomalies

Digital Object Identifier 10.4230/LIPIcs.CVIT.2016.23

1 Introduction

Timing aspects are increasingly important in many kinds of systems such as real-time systems and cyber-physical systems, where it can be crucial that tasks are finished within some time bound, especially in the case of safety-critical systems [19]. Improving the worst-case guarantees of such systems is therefore of interest to ensure the safety or performance of the system. However, doing so requires an understanding of what it means for one system to be faster than another, so that we can compare the improved system with the original system and show that the improved one has the same behaviour as the original system, except it operates faster. This will allow incremental speed improvements by replacing a system with a progressively faster one.

Concurrent systems are often described by multiple components running in parallel with each other. An important tool in understanding such concurrent systems is that of compositionality, which allows one to understand a complex system by breaking it into smaller components that are more easily analyzed [6]. However, it is not always the case that an analysis of the smaller components carries over to the full, composite system. In the case

* This research was supported by the Danish FTP project ASAP, the ERC Advanced Grant LASSO as well as the Sino-Danish Basic Research Center IDEA4CPS.

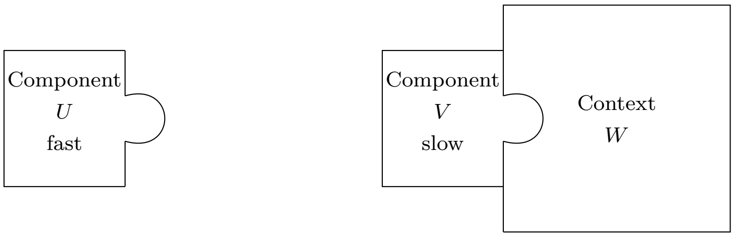

Figure 1 The context W operates in parallel with the component V. If the component U is faster than V, then if we replace V with U, we would expect the overall behaviour to also be faster.

of timing properties, an example of this is the case where locally faster behaviour leads to globally slower behaviour. This is known as a *timing anomaly* and occurs in many real-time systems such as scheduling for processors [5, 20]. When timing anomalies are caused by compositional issues, they are known as parallel timing anomalies [17].

In this paper, we therefore consider timing and compositional aspects together. The situation we are interested in can be depicted as in Figure 1, where we have a composite system consisting of a context W and a component V, and we want to understand how we can show that another component U is faster than V, and what happens when we replace V by U.

We therefore give a novel definition of what it means for one process to be faster than another in the setting of semi-Markov decision processes, in which any trace of actions that the slow process can do within some time bound, the fast process can do within a lower time bound and with a higher probability. This definition turns out to be closely connected to the study of stochastic orders of convolutions.

We study the properties of this faster-than relation, and show that it preserves time-bounded reachability properties. We also show that the faster-than relation can be characterized by a simple modal logic with a modal operator on paths that speaks about time and an operator on states that speaks about the probability of paths. In order to be able to give guarantees on timing aspects, we identify a subset of processes for which we give decidable conditions which guarantee that one process is faster than another.

Finally, we consider compositional aspects and show that under many of the commonly used ways of composition, timing anomalies can occur. We then consider different sufficient conditions for avoiding timing anomalies, and show that one of these is decidable.

Related Work. The notion of faster-than has been studied in many different contexts throughout the literature. The work most closely related to ours is that of Baier et al. [1], in which they define, among other relations, a simulation relation for continuous-time Markov chains which can be interpreted as a faster-than relation. However, they do not consider compositional aspects which is our focus. For process algebras, discrete-time faster-than relations have been defined for variations of Milner's CCS, and shown to be precongruences with respect to parallel composition [8, 21, 23, 27]. Lüttgen and Vogler [22] attempt to unify some of these process algebraic approaches and also consider the issue of parallel timing anomalies. For Petri nets, Vogler [29, 30] considers a testing preorder as a faster-than relation and shows that this is a precongruence with respect to parallel composition. Geilen et al. [10] introduces a refinement principle for timed actor interfaces under the slogan "the earlier, the better", which can also be seen as an example of a faster-than relation. In the case of timed automata, Guha et al. [13] study a faster-than prebisimulation for which parallel timing anomalies can happen, and give sufficient conditions for avoiding parallel timing anomalies.

They also show that it is decidable whether two states are comparable or not under their faster-than relation, but the decidability of their relation itself is still open. In fact, the related but different problems of language inclusion for probabilistic automata [3, 7] and trace refinement for Markov decision processes [9] are undecidable, indicating that decidability issues for these kinds of faster-than relations are hard.

Work on timing anomalies date back to at least 1969 [12], but the most influential paper in the area is probably that of Lundqvist and Stenström [20], in which they show that timing anomalies can occur in dynamically scheduled processors, contrary to what most people assumed at the time. More recent work has focused on compositional aspects [17] and defining timing anomalies formally, using transition systems as the formalism [5, 26].

2 Notation and Preliminaries

In this section, we fix some notation and recall concepts that are used throughout the rest of the paper. Let $\mathbb{N}$ denote the natural numbers and let $\mathbb{R}_{\geq 0}$ denote the non-negative real numbers. Then $(\mathbb{R}_{\geq 0}, \mathbb{B})$ is the measurable space of non-negative real numbers equipped with the standard Borel σ-algebra $\mathbb{B}$. Let $\Delta(\mathbb{R}_{\geq 0})$ denote the set of probability measures on $\mathbb{R}_{\geq 0}$ and for any set X, let $D(X)$ denote the set of discrete probability measures on X. For an element $x \in X$ of some set X, we will use δ_x to denote the Dirac measure at x defined as $\delta_x(y) = 1$ if $x = y$ and $\delta_x(y) = 0$ otherwise. We fix a non-empty, countable set L of *transition labels* or *actions* and equip them with the discrete σ-algebra Σ_L.

For a probability measure $\mu \in \Delta(\mathbb{R}_{\geq 0})$, we denote by F_μ its *cumulative distribution function (CDF)* defined as $F_\mu(t) = \mu([0,t])$, for all $t \in \mathbb{R}_{\geq 0}$. The *convolution* of two probability measures $\mu, \nu \in \Delta(\mathbb{R}_{\geq 0})$, written $\mu * \nu$, is the probability measure on $\mathbb{R}_{\geq 0}$ given by $(\mu * \nu)(B) = \int_{-\infty}^{\infty} \nu(B - x)\ \mu(\mathrm{d}x)$, for all $B \in \mathbb{B}$ [2]. Convolution is associative, i.e., $\mu * (\nu * \eta) = (\mu * \nu) * \eta$, and commutative, i.e., $\mu * \nu = \nu * \mu$.

3 Semi-Markov Decision Processes

In this section, we recall the definition of semi-Markov decision processes.

▶ **Definition 1.** A *semi-Markov decision process (SMDP)* is a tuple $M = (S, \tau, \rho)$ where

- S is a non-empty, countable set of *states*,
- $\tau : S \times L \to D(S)$ is a *transition probability function*, and
- $\rho : S \to \Delta(\mathbb{R}_{\geq 0})$ is a *residence-time probability function.*

The operational behaviour of an SMDP $M = (S, \tau, \rho)$ is as follows. The process in the state $s \in S$ reacts to an external input $a \in L$ provided by the environment by changing its state to $s' \in S$ within time $t \in \mathbb{R}_{\geq 0}$ with probability $\tau(s,a)(s') \cdot \rho(s)([0,t])$.

Notice that Markov decision processes (MDPs) are a special case of SMDPs where for all $s \in S$, $\rho(s) = \delta_0$ (i.e. transitions happen instantaneously), and continuous-time Markov decision processes (CTMDPs) are another special case of SMDPs where, for all states $s \in S$, $F_{\rho(s)} = \mathit{Exp}(\theta_s)$ is an exponential distribution with rate $\theta_s > 0$.

The executions of an SMDP $M = (S, \tau, \rho)$ are infinite timed transition sequences of the form $\pi = (s_1, t_1, a_1)(s_2, t_2, a_2) \cdots \in (S \times \mathbb{R}_{\geq 0} \times L)^\omega$, representing the fact that M waited in state s_i for t_i time units after the action a_i was selected. For $i \in \mathbb{N}$, let $\pi[i] = s_i$, $\pi\langle i \rangle = t_i$, $\pi[\![i]\!] = a_i$, $\pi|_i = (s_1, t_1, a_1) \ldots (s_i, t_i, a_i)$, and $\pi|^i = (s_i, t_i, a_i)(s_{i+1}, t_{i+1}, a_{i+1}) \ldots$. We let $\Pi(M)$ denote the set of all executions or *timed action paths* in M, and denote by $\Pi_n(M) = \{\pi|_n \mid \pi \in \Pi(M)\}$ the set of all prefixes of length n. Hereafter, we refer to timed action paths simply as paths, unless we wish to distinguish between different kinds of paths.

Next we recall the standard construction of the measurable space of paths. A *cylinder set* of rank $n \geq 1$ is the set of all paths whose nth prefix is contained in a common subset $E \subseteq \Pi_n(M)$, and is given by $\mathfrak{C}(E) = \{\pi \in \Pi(M) \mid \pi|_n \in E\}$. It will be convenient to denote *rectangular cylinders* of the form $\mathfrak{C}(S_1 \times L_1 \times R_1 \times \cdots \times S_n \times L_n \times R_n)$, for $S_i \subseteq S$, $L_i \subseteq L$, and $R_i \subseteq \mathbb{R}_{\geq 0}$, as $\mathfrak{C}(S_1 \ldots S_n, L_1 \ldots L_n, R_1 \ldots R_n)$. We also have the special notation λ for the cylinder of rank 0, defined as $\lambda = \Pi(M)$.

We denote by $(\Pi(M), \Sigma)$ the *measurable space of timed action paths*, where Σ is the smallest σ-algebra generated by the cylinders of the form $\mathfrak{C}(S_1 \ldots S_n, L_1 \ldots L_n, R_1 \ldots R_n)$ for $S_i \in 2^S$, $L_i \in 2^L$, and $R_i \in \mathbb{B}$.

In this paper we assume that external choices are resolved by means of memoryless stochastic schedulers, however all the results we present still hold for memoryful schedulers.

▶ **Definition 2.** Given an SMDP $M = (S, \tau, \rho)$, a *scheduler* for M is a function $\sigma : S \to D(L)$ that assigns to each state a probability distribution over action labels.

We will use the notation $\tau^\sigma(s, a)(s')$ as shorthand for $\tau(s, a)(s') \cdot \sigma(s)(a)$ to denote the probability of moving from state s to s' under the stochastic choice of a given by σ. Given an SMDP M and a scheduler σ for it, the probabilistic execution of a path starting from the state s is governed by the probability $\mathbb{P}^\sigma_M(s)$ on $(\Pi(M), \Sigma)$ defined as follows.

▶ **Definition 3.** Let $M = (S, \tau, \rho)$ be an SMDP. Given a scheduler σ for M and a state $s \in S$, $\mathbb{P}^\sigma_M(s)$ is defined as the unique probability measure on $(\Pi(M), \Sigma)$ such that for all $S_i \in 2^S$, $L_i \in 2^L$, and $R_i \in \mathbb{B}$, with $1 \leq i \leq n$, we have

$$\begin{aligned}
&\mathbb{P}^\sigma_M(s)(\mathfrak{C}(S_1 \ldots S_n, L_1 \ldots L_n, R_1 \ldots R_n)) \\
&= \rho(s)(R_1) \cdot \sum_{a \in L_1} \sum_{s' \in S_1} \tau^\sigma(s, a)(s') \cdot \mathbb{P}^\sigma_M(s')(\mathfrak{C}(S_2 \ldots S_n, L_2 \ldots L_n, R_2 \ldots R_n)),
\end{aligned}$$

and $\mathbb{P}^\sigma_M(s)(\mathfrak{C}(S_1, L_1, R_1)) = \rho(s)(R_1) \cdot \sum_{a \in L_1} \sum_{s' \in S_1} \tau^\sigma(s, a)(s')$.

Intuitively, to get the probability $\mathbb{P}^\sigma_M(s)(\mathfrak{C}(S_1 \ldots S_n, L_1 \ldots L_n, R_1 \ldots R_n))$, we first take the probability that s takes a transition at a time point in R_1, given by $\rho(s)(R_1)$, after which we sum over the probabilities of all the possible transitions that can be taken by choosing a label $a \in L_1$ and a state $s' \in S_1$, and then the rest of the probability is given inductively by continuing on s'. For the rest of the paper, we will omit the subscript M in $\mathbb{P}^\sigma_M$ whenever it is clear from the context which SMDP is being referred to.

4 A Faster-Than Relation

It is our goal to have a relation which agrees with the intuitive idea of an SMDP U being "faster than" another SMDP V. For a process U to be faster that V, it must be able to execute any sequence of actions $a_1, \ldots, a_n$ in less time than V. Of course, since we are dealing with probabilistic systems, we must speak of the probability of executing a sequence of actions within some time bound.

Consider the two simple SMDPs U and V in Figure 2 with just a single transition label and initial states u_0 and v_0, respectively. Here μ, ν, η are arbitrary probability measures on $\mathbb{R}_{\geq 0}$, representing the residence-time distributions at each state. An arrow with (p, a) next to it means that when a is chosen as the action, then the SMDP takes the transition given by the arrow with probability p. The only finite sequences of actions that can be executed in these SMDPs are of the form a^n for $n > 0$.

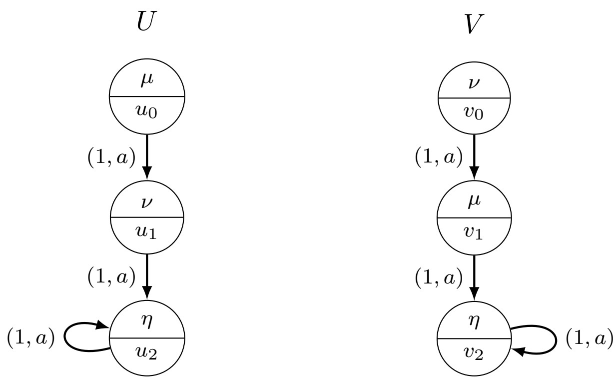

Figure 2 U is faster than V in the first states, and after that their probabilities are the same, so U is faster than V.

For U to be faster than V, it should be the case that for any time bound x and no matter which scheduler σ we choose for V, we must be able to find a scheduler σ' for U such that there is an earlier time bound $x' \leq x$ which allows U to execute any sequence a^n within time x' with higher or equal probability than that of V executing the same sequence of actions within time x. Formally, this amounts to saying that $\mathbb{P}^{\sigma'}(u_0)(\mathfrak{C}(a^n, x')) \geq \mathbb{P}^{\sigma}(v_0)(\mathfrak{C}(a^n, x))$, where $\mathfrak{C}(a_1 \dots a_n, t)$ denotes the event of executing the sequence of actions $a_1, \dots, a_n$ within time t. Hence, the type of events on which we want to focus are the following.

▶ **Definition 4.** For any finite sequence of actions $a_1, \dots, a_n$, and $x \in \mathbb{R}_{\geq 0}$, we say that

$$\mathfrak{C}(a_1 \dots a_n, x) = \{\pi \in \Pi(M) \mid \forall 1 \leq i \leq n,\ \pi[\![i]\!] = a_i \text{ and } \sum_{j=1}^{n} \pi\langle j \rangle \in [0, x]\}$$

is a *time-bounded cylinder*. The *length* of a time-bounded cylinder is the length of the sequence of actions in the time-bounded cylinder.

Note that time-bounded cylinders $\mathfrak{C}(a_1 \dots a_n, x)$ are measurable in $(\Pi(M), \Sigma)$, since they are given as the measurable pre-image $(\pi|_n \circ f)^{-1}(S^n \times \{(a_1, \dots, a_n)\} \times B_x^n)$, where B_x^n is the Borel set $\{(r_1, \dots, r_n) \in \mathbb{R}^n_{\geq 0} \mid \sum_{j=1}^{n} r_j \in [0, x]\}$ and f is the canonical measurable isomorphism from $\Pi_n(M) = (S \times L \times \mathbb{R}_{\geq 0})^n$ to $S^n \times L^n \times \mathbb{R}^n_{\geq 0}$.

We will use the notation (M, s_0) to indicate that $M = (S, \tau, \rho)$ is an SMDP with initial state $s_0 \in S$ and call it *pointed SMDP*. For the rest of the paper, we fix three SMDPs $M = (S, \tau, \rho)$, $U = (S_U, \tau_U, \rho_U)$, and $V = (S_V, \tau_V, \rho_V)$, with initial states $s_0 \in S$, $u_0 \in S_U$, $v_0 \in S_V$, respectively. Now we are ready to define what it means for an SMDP to be "faster than" another one.

▶ **Definition 5** (Faster-than). We say that U is *faster than* V, written $U \leq_{\text{ft}} V$, if for all schedulers σ for V, time bounds x, and sequences of actions $a_1 \dots a_n$, there exists a scheduler σ' for U and time bound $x' \leq x$, such that $\mathbb{P}^{\sigma'}(u_0)(\mathfrak{C}(a_1 \dots a_n, x')) \geq \mathbb{P}^{\sigma}(v_0)(\mathfrak{C}(a_1 \dots a_n, x))$.

The following proposition gives a different characterisation of the faster-than relation that is often easier to work with.

▶ **Proposition 6.** *$U \leq_{\text{ft}} V$ iff for all schedulers σ for V there exists a scheduler σ' for U such that $\mathbb{P}^{\sigma'}(u_0)(C) \geq \mathbb{P}^{\sigma}(v_0)(C)$, for all time-bounded cylinders C.*

Before showing an example of an SMDP being faster than another one, we provide an easier analytic way for computing the probability over time-bounded cylinders in terms of convolutions of the residence time distributions.

▶ **Proposition 7.** *For any SMDP $M = (M, \tau, \rho)$, scheduler σ for M, and $s \in S$ we have*

$$\begin{aligned}&\mathbb{P}^\sigma(s)(\mathfrak{C}(a_1 \dots a_n, x))\\ &= \sum_{s_1 \in S} \cdots \sum_{s_n \in S} \tau^\sigma(s, a_1)(s_1) \cdots \tau^\sigma(s_{n-1}, a_n)(s_n) \cdot (\rho(s) * \rho(s_1) * \cdots * \rho(s_{n-1}))([0, x])\,.\end{aligned}$$

Proposition 7 intuitively says that the absorption-time of any path of length n through the SMDP is distributed as the n-fold convolution of its residence-time probabilities. Therefore, the probability of doing transitions with labels $a_1, \dots, a_n$ within time x is the sum of the probabilities of taking a path of length n with labels $a_1, \dots, a_n$ through the SMDP, weighted by the probability of reaching the end of each of these paths within time x. This is similar in spirit to a result on phase-type distributions, see e.g. [25, Proposition 2.11].

▶ **Example 8.** Consider the pointed SMDPs (U, u_0) and (V, v_0) that are depicted in Figure 2. Assuming that $F_\mu \geq F_\nu$, we now show that $U \leq_{\text{ft}} V$. To compare U and V, first notice that we only need to consider time-bounded cylinders of the form $\mathfrak{C}(a^n, x)$, for $n \geq 1$. Since the set of actions is $L = \{a\}$, the only possible valid scheduler σ for both U and V is the one assigning the Dirac measure δ_a to all states. We consider two cases.

(Case $n = 1$**)** In this case we get $\mathbb{P}^\sigma(u_0)(\mathfrak{C}(a, x)) = F_\mu(x)$ and $\mathbb{P}^\sigma(v_0)(\mathfrak{C}(a, x)) = F_\nu(x)$. Since we assumed $F_\mu \geq F_\nu$, we have $\mathbb{P}^\sigma(u_0)(\mathfrak{C}(a, x)) \geq \mathbb{P}^\sigma(v_0)(\mathfrak{C}(a, x))$.

(Case $n > 1$**)** By Proposition 7 we have that $\mathbb{P}^\sigma(u_0)(\mathfrak{C}(a^n, x)) = (\mu * \nu * \eta^{*(n-2)})([0, x])$ and $\mathbb{P}^\sigma(v_0)(\mathfrak{C}(a^n, x)) = (\nu * \mu * \eta^{*(n-2)})([0, x])$, where η^{*n} is the n-fold convolution of η, defined inductively by $\eta^{*0} = \delta_0$ and $\eta^{*(n+1)} = \eta * \eta^{*n}$. Since convolution is commutative and associative, and δ_0 is the identity for convolution, we obtain that $\mathbb{P}^\sigma(u_0)(\mathfrak{C}(a^n, x)) = \mathbb{P}^\sigma(v_0)(\mathfrak{C}(a^n, x))$.

We therefore conclude that $U \leq_{\text{ft}} V$.

The standard notions used to compare processes are bisimulation [24] and simulation [1]. We next recall their definitions for SMDPs.

▶ **Definition 9.** For an SMDP M, a relation $R \subseteq S \times S$ is a *bisimulation relation* (resp. *simulation relation*) on M if for all $(s_1, s_2) \in R$ we have

- $F_{\rho(s_1)}(x) =$ (resp. $\leq$)$F_{\rho(s_2)}(x)$ and
- for all $a \in L$ there exists a weight function $\Delta_a : S \times S \to [0, 1]$ such that
  - $\Delta_a(s, s') > 0$ if and only if (resp. implies) $(s, s') \in R$,
  - $\tau(s_1, a)(s) = \sum_{s' \in S} \Delta_a(s, s')$ for all $s \in S$, and
  - $\tau(s_2, a)(s') = \sum_{s \in S} \Delta_a(s, s')$ for all $s' \in S$.

If there is a bisimulation relation (resp. simulation relation) R such that $(s_1, s_2) \in R$, then we say that s_1 is *bisimilar* (resp. *simulates*) s_2 and write $s_1 \sim$ (resp. $\lesssim$)s_2.

We lift bisimulation and simulation relations to SMDPs by considering the disjoint union of the two and comparing their initial states. We denote by $\sim$ the largest bisimulation relation and by $\lesssim$ the largest simulation relation. Furthermore, we say that U and V are *equally fast* and write $U \equiv_{\text{ft}} V$ if $U \leq_{\text{ft}} V$ and $V \leq_{\text{ft}} U$.

▶ **Example 10.** Consider the two SMDPs U and V in Figure 3 with the same probability measure μ in all states. It is easy to see that U is bisimilar to V, and hence U also simulates

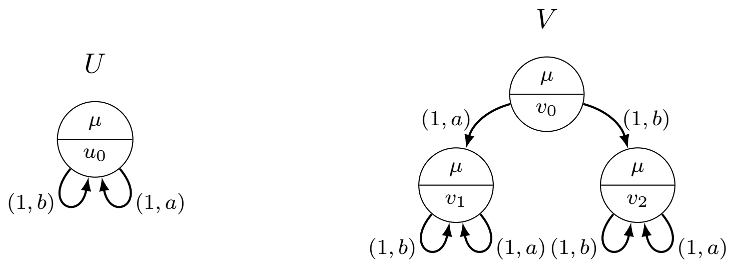

Figure 3 Example showing that the faster-than relation and and the simulation relation are incomparable.

V. However, we can show that $U \not\leq_{\mathrm{ft}} V$ in the following way. Construct the scheduler σ for V by letting

$$\sigma(v_0)(a) = 0.5, \ \sigma(v_0)(b) = 0.5, \ \sigma(v_1)(a) = 1, \ \text{ and } \ \sigma(v_2)(b) = 1.$$

Now, for any scheduler σ' for U, we must have either $\sigma'(u_0)(a) < 1$ or $\sigma'(u_0)(b) < 1$. If $\sigma'(u_0)(a) < 1$, then $\sigma'(u_0)(a) > (\sigma'(u_0)(a))^2 > \cdots > (\sigma'(u_0)(a))^n$. Now, $\mathbb{P}^{\sigma}(v_0)(\mathfrak{C}(a^n, x)) = 0.5 \cdot \mu^{*n}(x)$ and $\mathbb{P}^{\sigma'}(u_0)(\mathfrak{C}(a^n, x)) = (\sigma'(u_0)(a))^n \cdot \mu^{*n}(x)$ for $n > 1$. Take some n such that $(\sigma'(u_0)(a))^n < 0.5$. In that case we get $\mathbb{P}^{\sigma'}(u_0)(\mathfrak{C}(a^n, x)) < \mathbb{P}^{\sigma}(v_0)(\mathfrak{C}(a^n, x))$. The same procedure can be used in case $\sigma'(u_0)(b) < 1$. Hence we conclude that $U \not\leq_{\mathrm{ft}} V$, and therefore also that $U \not\equiv_{\mathrm{ft}} V$.

Example 10 also works for schedulers with memory, although the argument has to be modified a bit. In that case, in each step either the probability of a trace consisting only of a's or the probability of a trace consisting only of b's must decrease in U, so after some number of steps, the probability of one of these two must decrease below 0.5, and then the rest of the argument is the same.

▶ **Example 11.** Consider the SMDPs U and V in Figure 2 and let $F_\mu = \mathit{Exp}(\theta_1)$ and $F_\nu = \mathit{Exp}(\theta_2)$ be exponential distributions with rates $\theta_1 > \theta_2 > 0$. Then, as shown in Example 8, it holds that $U \leq_{\mathrm{ft}} V$. However, we have both $U \not\lesssim V$ and $U \not\sim V$.

From Examples 10 and 11, we get the following theorem.

▶ **Theorem 12.** *$\lesssim$ and $\leq_{\mathrm{ft}}$ are incomparable, $\sim$ and $\leq_{\mathrm{ft}}$ are incomparable, and $\sim \not\subseteq \equiv_{\mathrm{ft}}$.*

4.1 Reachability

If a process U is faster than another process V, then we would expect it to be the case that if V can do something within a given time bound, then U should also be able to do the same thing within the given time bound. This is the problem of time-bounded reachability. In the case of probabilistic systems, this must of course be cast in terms of probabilities saying that the probability of U doing something within the time bound should be greater than the probability of V doing the same thing within the time bound. Let

$$\Diamond^M_{\leq x} a = \{\pi \in \Pi(M) \mid \exists i \in \mathbb{N}.\pi[\![i]\!] = a \text{ and } \pi[k] \neq a \text{ for all } k < i, \text{ and } \sum_{j=1}^{i} \pi\langle j \rangle \leq x\}$$

be the set of all paths in M where the first occurence of the action a happens within x time units. $\Diamond^M_{\leq x} a$ is thus the set of all paths that reach an a-transition within x time units. First we show how $\Diamond^M_{\leq x} a$ can be expressed in terms of time-bounded cylinders.

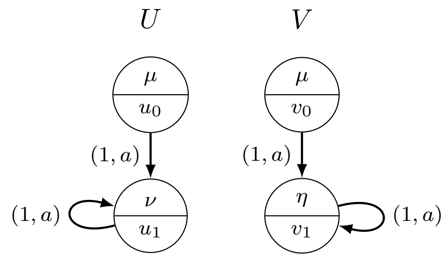

Figure 4 Reachability properties in V are preserved in U, but U is not faster than V.

▶ **Lemma 13.** *For any $x \in \mathbb{R}_{\geq 0}$ and $a \in L$ it holds that*

$$\Diamond^M_{\leq x} a = \bigcup_{k\in\mathbb{N}_0} \bigcup_{(a_1,\ldots,a_k)\in(L\setminus\{a\})^k} \mathfrak{C}(a_1 \ldots a_k a, x)$$

where $\bigcup_{(a_1,\ldots,a_k)\in(L\setminus\{a\})^k} \mathfrak{C}(a_1 \ldots a_k a, x) = \mathfrak{C}(a, x)$ *if* $k = 0$.

In particular, if $L = \{a\}$, then $\Diamond^M_{\leq x} a = \mathfrak{C}(a, x)$. We can now use this result to show that the faster-than relation preserves reachability properties.

▶ **Theorem 14.** *Let $x \in \mathbb{R}_{\geq 0}$ and $a \in L$. Then $U \leq_{\text{ft}} V$ implies that for all schedulers σ for V there exists a scheduler σ' for U such that $\mathbb{P}^{\sigma'}(u_0)(\Diamond^U_{\leq x} a) \geq \mathbb{P}^{\sigma}(v_0)(\Diamond^V_{\leq x} a)$.*

The following example shows that the converse of Theorem 14 does not hold.

▶ **Example 15.** Consider the SMDPs U and V given in Figure 4 with residence-time distributions given by $F_\mu = Exp(1)$, $F_\nu = Exp(\theta_1)$, and $F_\eta = Exp(\theta_2)$ where $0 < \theta_1 < \theta_2$. For any x we have

$$\mathbb{P}(u_0)(\Diamond^U_{\leq x} a) = \mathbb{P}(u_0)(\mathfrak{C}(a, x)) = Exp(1) = \mathbb{P}(v_0)(\mathfrak{C}(a, x)) = \mathbb{P}(v_0)(\Diamond^V_{\leq x} a),$$

so reachability properties are preserved. However, if we consider the cylinder $\mathfrak{C}(aa, x)$, then we see that the product of the first two rates of U is strictly less than the product of the first two rates of V, i.e. $1 \cdot \theta_1 < 1 \cdot \theta_2$, which implies by [4, Theorem 1] that there exists an x such that $\mathbb{P}(u_0)(\mathfrak{C}(aa, x)) < \mathbb{P}(v_0)(\mathfrak{C}(aa, x))$, and hence $U \not\leq_{\text{ft}} V$.

4.2 Logical Characterisation

Next we wish to give a logical characterisation of the faster-than relation. The logic we use to do this has the following succinct syntax where $a \in L$ and $x, q \in \mathbb{Q}_{\geq 0}$.

(Path formulas) $\varphi ::= \top \mid \langle a \rangle_x \varphi$ (State formulas) $\psi ::= \mathcal{P}_{\geq q}(\varphi)$

As usual for non-symmetric relations, we can not allow negation in the logic. In order to give the semantics for the logic, we introduce some definitions for time-bounded cylinders.

▶ **Definition 16.** Let $\mathfrak{C}(a_1 \ldots a_n, x)$ and $\mathfrak{C}(b_1 \ldots b_m, y)$ be two time-bounded cylinders. Then *concatenation of cylinders* is given by

$$\mathfrak{C}(a_1 \ldots a_n, x) :: \mathfrak{C}(b_1 \ldots b_m, y) = \mathfrak{C}(a_1 \ldots a_n b_1 \ldots b_m, x + y)$$

and $\mathfrak{C}(a_1 \ldots a_n, x) :: \lambda = \mathfrak{C}(a_1 \ldots a_n, x)$.

▶ **Definition 17.** Given a path formula φ, we define the *characteristic cylinder of* φ as

$$\mathfrak{C}(\varphi) = \begin{cases} \lambda & \text{if } \varphi = \top \text{ and} \\ \mathfrak{C}(a, x) :: \mathfrak{C}(\varphi') & \text{if } \varphi = \langle a \rangle_x \varphi'. \end{cases}$$

Now we can define the semantics of our logic.

$$\begin{array}{lll} \pi \models \top & & \text{always,} \\ \pi \models \langle a \rangle_x \varphi & \text{iff} & \pi[\![1]\!] = a, \pi\langle 1 \rangle = x, \text{ and } \pi|^2 \models \varphi, \text{ and} \\ M, s \models \mathcal{P}_{\geq q}(\varphi) & \text{iff} & \mathbb{P}^\sigma(s)(\mathfrak{C}(\varphi)) \geq q \text{ for some scheduler } \sigma. \end{array}$$

Intuitively, $\pi \models \langle a \rangle_x \varphi$ says that the first transition label of π is a, the first time delay of π is x, and then the rest of π satisfies φ. A path formula φ therefore describes a set of paths. This idea is used in Definition 17 to describe a time-bounded cylinder which contains these paths. $M, s \models \mathcal{P}_{\geq q}(\varphi)$ then says that the probability of this cylinder starting from s is greater than or equal to q, under some scheduler. The following theorem shows that although the logic is quite simple, it characterizes exactly those states that are in a faster-than relation with each other.

▶ **Theorem 18.** *Then $U \leq_{\mathrm{ft}} V$ if and only if $V, v_0 \models \psi$ implies that $U, u_0 \models \psi$ for all ψ.*

4.3 Toward Decidability

We saw in Example 8 that the simple cases of SMDPs in Figure 2 gave rise to convolution of distributions. In order to decide the faster-than relation, one therefore has to consider how to compare convolutions of distributions as well as how to give guarantees on all paths by just looking at a finite number of these, and the quantification over schedulers complicates this even more. In this section we will therefore impose some restrictions on the SMDPs we consider.

▶ **Definition 19.** Let $M = (S, \tau, \rho)$ be an SMDP. If S is a finite set, we will say that M is *finite*. Furthermore, if for all $s \in S$ there is some $a \in L$ and $s' \in S$ such that $\tau(s, a)(s') = 1$, then we will say that M has a *deterministic Markov kernel*.

For most distributions, it is currently unknown whether stochastic orderings between these are decidable, although significant work has been done on the subject [4, 18, 16]. In this section we will therefore instead formulate sufficient conditions for being faster-than, and show that these sufficient conditions are decidable, which can then be used to give guarantees on timing behaviour. To do this, we first introduce some terminology and notation that we will also use later in the paper.

▶ **Definition 20.** A *state path in* M is a sequence of states $s_1, s_2, \ldots$ where for all $i \in \mathbb{N}$ there exists a label $a \in L$ such that $\tau(s_i, a)(s_{i+1}) > 0$. For a state path $\pi = s_1, s_2, \ldots$, we let $\pi[i] = s_i$, $\pi|^i = s_i, s_{i+1}, \ldots$, $\pi|_i = s_1, s_2, \ldots, s_i$, and we let $\Pi[M]$ denote the set of all state paths in M. For a state $s \in S$, we let $\Pi[s] = \{\pi \in \Pi[M] \mid \pi[1] = s\}$ be the set of state paths in M that start in the state s, and we let $\Pi_n[s] = \{\pi|_n \mid \pi \in \Pi[s]\}$.

By using the notion of a state path, we can write Proposition 7 as

$$\begin{aligned} &\mathbb{P}^\sigma(s)(\mathfrak{C}(a_1 \ldots a_n, x)) \\ &= \sum_{\pi \in \Pi_{n+1}[s]} \tau^\sigma(\pi[1], a_1)(\pi[2]) \cdots \tau^\sigma(\pi[n], a_n)(\pi[n+1]) \cdot (\rho(\pi[1]) * \cdots * \rho(\pi[n]))([0, x]). \end{aligned}$$

▶ **Definition 21.** A state s is called *recurrent* if there exists a state path π such that $\pi[1] = s$ and $\pi[k] = s$ for some k.

A crucial part of showing decidability is showing that for finite SMDPs, paths will eventually start repeating. This will allow us to only consider a finite number of paths to decide the faster-than relation.

▶ **Lemma 22.** *Let U and V be two finite, pointed SMDPs. For any state paths π_U and π_V of length $l > |S_U| \cdot |S_V|$, there will be $i < j \leq |S_U| \cdot |S_V|$ such that $\pi_U[i] = \pi_U[j]$, $\pi_V[i] = \pi_V[j]$.*

Now we can use this to give sufficient conditions for $U \leq_{\text{ft}} V$ in the case where there is no non-determinism and U and V are finite and have a deterministic Markov kernel.

▶ **Theorem 23.** *Let L be a singleton and let U and V be finite pointed SMDPs with a deterministic Markov kernel. Then we have $U \leq_{\text{ft}} V$ if $\mathbb{P}(u)(C) \geq \mathbb{P}(v)(C)$ for all recurrent states $u \in S_U$ and $v \in S_V$ as well as $u = u_0$ and $v = v_0$, and cylinders of length $l \leq |S_U| \cdot |S_V|$.*

As mentioned in the beginning of the section, the probabilities that occur in Theorem 23 give rise to convolutions of distributions. Hence, in order for us to be able to decide these conditions, we must have some decidable way of guaranteeing that one convolution is pointwise greater than another.

▶ **Theorem 24.** *Let L be a singleton and let U and V be finite pointed SMDPs with a deterministic Markov kernel. If for all state paths $\pi_U \in \Pi_k[U]$ and $\pi_V \in \Pi_k[V]$ where $k \leq |S_U| \cdot |S_V|$ there exist decidable sufficient conditions for when*

$$(\rho_U(\pi_U[1]) * \cdots * \rho_U(\pi_U[k]))([0, x]) \geq (\rho_V(\pi_V[1]) * \cdots * \rho_V(\pi_V[k]))([0, x]) \text{ for all } x,$$

then there exists decidable sufficient conditions for when $U \leq_{\text{ft}} V$.

Such decidable sufficient conditions exist for convolutions of Dirac distributions, uniform distributions [18], exponential distributions [4], and gamma distributions [16]. Only in the case of Dirac distributions are the conditions also necessary.

5 Compositionality

Next we introduce the notion of composition of SMDPs. As argued in [28], the style of synchronous CSP composition is the most natural one to consider for reactive probabilistic systems, so this is the one we will adopt. However, we leave the composition of the residence-times general, so that we can compare different kinds of composition.

▶ **Definition 25.** A function $\star : \Delta(\mathbb{R}_{\geq 0}) \times \Delta(\mathbb{R}_{\geq 0}) \to \Delta(\mathbb{R}_{\geq 0})$ is called a *residence-time composition function* if it is commutative, i.e. $\star(\mu, \mu') = \star(\mu', \mu)$ for all $\mu, \mu' \in \Delta(\mathbb{R}_{\geq 0})$.

One example of such a composition function is when $\star$ is a coupling, which is a joint probability measure such that its marginals are μ and μ'. A simple special case of this is the product measure $\star(\mu, \mu') = \mu \times \mu'$, which is defined by $(\mu \times \mu')(x) = \mu(x) \cdot \mu(x)$.

In order to model the situation in which we want the composite system only to take a transition when both components can take a transition, it is natural to take the minimum of the two probabilities, which corresponds to waiting for the slowest of the two. In that case, we let $\star(\mu, \mu') = \nu$ be such that $F_\nu(x) = \min\{F_\mu(x), F_{\mu'}(x)\}$, and we call this *minimum composition.* Likewise, if we only require one of the components to be able to take a transition, then it is natural to take the maximum of the two probabilities by letting $\star(\mu, \mu') = \nu$ be such that $F_\nu(x) = \max\{F_\mu(x), F_{\mu'}(x)\}$, which we call *maximum composition*. A special case of minimum composition is the composition on rates used in PEPA [15], and a special case of maximum composition is the composition on rates used in TIPP [11].

If one has more knowledge about the processes that are being composed, one can of course create more specific composition functions. As an example, if we know that the components only have exponential distributions, then we can make composition functions that work directly on the rates of the distributions. If μ has an exponential CDF F_μ^θ and μ' has an exponential CDF $F_{\mu'}^{\theta'}$ with rates $\theta, \theta' > 0$, then one could for example let $\star(\mu, \mu')$ be such that it has an exponential CDF $F_{\star(\mu,\mu')}^{\theta''}$ with rate $\theta'' = \theta \cdot \theta'$. This corresponds to the composition on rates that is used in SPA [14], and we will call it *product composition*. Note that product composition is not given by the product measure.

▶ **Definition 26.** Let $\star$ be a residence-time composition function. The $\star$*-composition* of U and V, denoted by $U \parallel_\star V = (S, \tau, \rho)$, is then given by

- $S = U \times V$,
- $\tau((u, v), a)((u', v')) = \tau_U(u, a)(u') \cdot \tau_V(v, a)(v')$ for all $a \in L$ and $(u', v') \in S$, and
- $\rho((u, v)) = \star(\rho_U(u), \rho_V(v))$

for all $(u, v) \in S$.

When considering the composite SMDP $U \parallel_\star V$ of two SMDPs U and V, we will also write $u \parallel_\star v$ to denote the composite state (u, v) of $U \parallel_\star V$ where $u \in S_U$ and $v \in S_V$.

5.1 Parallel Timing Anomalies

If we have two components U and V, and we know that U is faster than V, then if U is in parallel with some context W, we would expect this composition to become faster when we replace the component U with the component V. However, sometimes this fails to happen, and we will call such an occurrence a *parallel timing anomaly*.

In this section, we will show that parallel timing anomalies can occur for some of the kinds of composition that we gave as examples in Section 5 by giving different contexts W for the SMDPs U and V from Figure 2, for which it was shown in Example 8 that $U \leq_{\text{ft}} V$. Our examples make no use of non-determinism or transition probabilities, thus showing that the parallel timing anomalies have to do inherently with the timing behaviour of the SMDPs. For ease of presentation, we let the set of labels L consist only of the label a in this section.

Consider the two SMDPs U and V depicted in Figure 2. For the examples in this section, let $F_\mu = Exp(2)$, $F_\nu = Exp(0.5)$, and let η be arbitrary.

▶ **Example 27** (Product composition). Let $\star$ be product composition and let the context (W, w_0) be given by Figure 5, where $F_{\mu'} = Exp(10)$, $F_{\nu'} = Exp(0.1)$ and $\eta = \eta'$. In $U \parallel_\star W$, the rates in the first two states will then be 20 and 0.05, and in $V \parallel_\star W$ they will be 5 and 0.5. Consider the time-bounded cylinder $\mathfrak{C}(aa, 2)$. Then we see that $\mathbb{P}(u_0 \parallel_\star w_0)(\mathfrak{C}(aa, 2)) \approx 0.09$ and $\mathbb{P}(v_0 \parallel_\star w_0)(\mathfrak{C}(aa, 2)) \approx 0.30$, showing that $U \parallel_\star W_1 \not\leq_{\text{ft}} V \parallel_\star W_1$. Hence we have a parallel timing anomaly. What happens is that in the process V the probability of taking a transition before time 2 with rate 5 is already very close to 1, so the process U does not gain much by having a rate of 20, whereas in the next step, V gains a lot of probability by having a rate of 0.5 compared to the rate 0.05 of U.

▶ **Example 28** (Minimum composition). Let $\star$ be minimum composition and let the context (W, w_0) be given by Figure 5, where $F_{\mu'} = Exp(1)$, $F_{\nu'} = Exp(2)$, and $\eta = \eta'$. Then $\mathbb{P}(u_0 \parallel_\star w_0)(\mathfrak{C}(aa, 2)) \approx 0.40$ and $\mathbb{P}(v_0 \parallel_\star w_0)(\mathfrak{C}(aa, 2)) \approx 0.51$, so $U \parallel_\star W_2 \not\leq_{\text{ft}} V \parallel_\star W_2$.

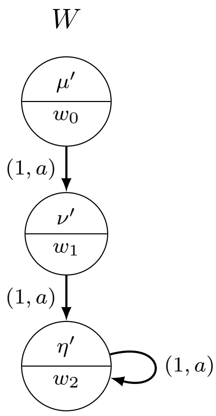

Figure 5 For different instantiations of μ', ν', and η', the context W leads to parallel timing anomalies for product, minimum, and maximum rate composition, respectively.

▶ **Example 29** (Maximum composition). Let $\star$ be product composition and let the context (W, w_0) be given by Figure 5, where $F_{\mu'} = \mathit{Exp}(2)$, $F_{\nu'} = \mathit{Exp}(1)$, and $\eta = \eta'$. Then $\mathbb{P}(u_0 \parallel_\star w_0)(\mathfrak{C}(aa, 2)) \approx 0.75$ and $\mathbb{P}(v_0 \parallel_\star w_0)(\mathfrak{C}(aa, 2)) \approx 0.91$, so $U \parallel_\star W_3 \not\lesssim_{\mathrm{ft}} V \parallel_\star W_3$.

5.2 Avoiding Parallel Timing Anomalies

We have seen in the previous section that parallel timing anomalies can occur. We now wish to understand what kind of contexts do not lead to timing anomalies. More precisely, given a residence-time composition function $\star$ and pointed SMDPs (U, u_0) and (V, v_0), we want to find conditions on (W, w_0) such that $U \leq_{\mathrm{ft}} V$ will imply $U \parallel_\star W \leq_{\mathrm{ft}} V \parallel_\star W$. For this section, we assume that the set L of transition labels is a finite set. Also, we fix a residence-time composition function $\star$ and two additional SMDPs $(W, w_0) = (S_W, \tau_W, \rho_W)$ and $(W', w_0') = (S_{W'}, \tau_{W'}, \rho_{W'})$ which should be thought of as contexts.

We first give conditions that over-approximate the faster-than relation between the composite systems by requiring that when U and W are put in parallel, then the composite system is point-wise faster than U along all paths. Likewise, we require that when V and W are put in parallel, the composite system is point-wise slower than V along all paths. If we already know that U is faster than V, then this will imply by transitivity that $U \parallel_\star W$ is faster than $V \parallel_\star W$. We have already seen in Example 8 that a process U need not be point-wise faster than V along all paths in order for U to be faster than V. However, by imposing this condition, we do not need to compare convolutions of distributions, but can compare the distributions directly, which makes decidability much easier.

▶ **Definition 30.** Let $n \in \mathbb{N}$. We say that $\star$ is *n-monotonic in U, V, W, and W'*, written $(U, W) \lessapprox_\star^n (V, W')$, if W' has a deterministic Markov kernel and for all state paths $\pi_U \in \Pi_n[u_0]$, $\pi_V \in \Pi_n[v_0]$, $\pi_W \in \Pi_n[w_0]$, and $\pi_{W'} \in \Pi_n[w_0']$ it holds that

- $F_{\rho(\pi_U[i] \parallel_\star \pi_W[i])}(x) \geq F_{\rho_U(\pi_U[i])}(x)$ and $F_{\rho_V(\pi_V[i])}(x) \geq F_{\rho(\pi_V[i] \parallel_\star \pi_{W'}[i])}(x)$ for all $x \in \mathbb{R}_{\geq 0}$ and $1 \leq i \leq n$,
- for all schedulers σ_U for U there exists a scheduler $\sigma_{U,W}$ for $U \parallel_\star W$ such that we have $\tau^{\sigma_{U,W}}(\pi_U[i] \parallel_\star \pi_W[i], a)(\pi_U[i+1] \parallel_\star \pi_W[i+1]) \geq \tau_U^{\sigma_U}(\pi_U[i], a)(\pi_U[i+1])$, and
- for all schedulers $\sigma_{V,W'}$ for $V \parallel_\star W'$ there exists a scheduler σ_V for V such that we have $\tau_V^{\sigma_V}(\pi_V[i], a)(\pi_V[i+1]) \geq \tau^{\sigma_{V,W'}}(\pi_V[i] \parallel_\star \pi_{W'}[i], a)(\pi_V[i+1] \parallel_\star \pi_{W'}[i+1])$

for all $a \in L$ and $1 \leq i < n$. Furthermore, we will say that $\star$ is *monotonic in U, V, W, and W'* and write $(U, W) \lessapprox_\star (V, W')$, if it is n-monotonic in U, V, W, and W' for all $n \in \mathbb{N}$.

Clearly, if $(U, W) \lessapprox_\star^n (V, W')$, then $(U, W) \lessapprox_\star^m (V, W')$ for all $m \leq n$. The next result shows that if $(U, W) \lessapprox_\star (V, W')$, then we are guaranteed to avoid parallel timing anomalies.

▶ **Theorem 31.** *If $(U, W) \lessapprox_\star (V, W')$ as well as $U \leq_{\mathrm{ft}} V$ and $W \leq_{\mathrm{ft}} W'$, then $U \parallel_\star W \leq_{\mathrm{ft}} V \parallel_\star W'$.*

The special case where $W = W'$ shows that this condition is sufficient to avoid parallel timing anomalies. We do not know if it is decidable whether $(U, W) \lessapprox_\star (V, W')$. However, we have a stronger condition which we can show is decidable in the case of finite SMDPs.

▶ **Definition 32.** We say that $\star$ is *strongly* n-monotonic in U, V, W, and W' and write $(U, W) \leqq_\star^n (V, W')$ if W' has a deterministic Markov kernel and for all state paths $\pi_U \in \Pi_n[u_0]$, $\pi_V \in \Pi_n[v_0]$, $\pi_W \in \Pi_n[w_0]$, and $\pi_{W'} \in \Pi_n[w'_0]$, the first condition of Definition 30 is satisfied and

- for all schedulers σ_U for U and all schedulers $\sigma_{U,W}$ for $U \parallel_\star W$, it is the case that $\tau^{\sigma_{U,W}}(\pi_U[i] \parallel_\star \pi_W[i], a)(\pi_U[i+1] \parallel_\star \pi_W[i+1]) \geq \tau_U^{\sigma_U}(\pi_U[i], a)(\pi_U[i+1])$, and
- for all schedulers $\sigma_{V,W'}$ for $V \parallel_\star W'$ and all schedulers σ_V for V, it is the case that $\tau_V^{\sigma_V}(\pi_V[i], a)(\pi_V[i+1]) \geq \tau^{\sigma_{V,W'}}(\pi_V[i] \parallel_\star \pi_{W'}[i], a)(\pi_V[i+1] \parallel_\star \pi_{W'}[i+1])$

for all $a \in L$ and $1 \leq i < n$. If $(U, W) \leqq_\star^n (V, W')$ for all $n \in \mathbb{N}$, then we say that $\star$ is *strongly* monotonic in U, V, W, and W' and write $(U, W) \leqq_\star (V, W')$.

The conditions of Definition 32 are simply the second and third conditions from Definition 30 with the existential quantifier strengthened to a universal quantifier. It is obvious that $(U, W) \leqq_\star (V, W')$ implies $(U, W) \lessapprox_\star (V, W')$, and hence we get the following corollary.

▶ **Corollary 33.** *If $(U, W) \leqq_\star (V, W')$ as well as $U \leq_{\mathrm{ft}} V$ and $W \leq_{\mathrm{ft}} W'$, then $U \parallel_\star W \leq_{\mathrm{ft}} V \parallel_\star W'$.*

▶ **Example 34.** Let U and V be given by Figure 2 with $F_\mu \geq F_\nu$ as in Example 8. Let $\star$ be minimum rate composition and consider the context W from Figure 5, where $\mu' = \mu$, $\nu' = \nu$, and $\eta' = \eta$. There is only one possible scheduler σ, which is the Dirac measure at a, and hence it is clear that the second and third conditions are satisfied. We also find that

$$F_{\rho(u_0 \parallel_\star w_0)}(x) = F_{\rho_U(u_0)}(x), \quad F_{\rho_V(v_0)}(x) = F_{\rho(v_0 \parallel_\star w_0)}(x), \quad F_{\rho(u_1 \parallel_\star w_1)}(x) = F_{\rho_U(u_1)}(x),$$

$$F_{\rho_V(v_1)}(x) = F_{\rho(v_1 \parallel_\star w_1)}(x), \quad F_{\rho(u_2 \parallel_\star w_2)}(x) = F_{\rho_U(u_2)}(x), \quad F_{\rho_V(v_2)}(x) = F_{\rho(v_2 \parallel_\star w_2)}(x),$$

and hence the first condition is also satisfied, so $(U, W) \leqq_\star (V, W)$.

We now wish to show that it is decidable whether $(U, W) \leqq_\star (V, W')$ for finite SMDPs, thereby giving a decidable condition for avoiding timing anomalies. To do this, we first show that in order to establish strong monotonicity, it is enough to consider paths up to length $m = \max\{|S_U| \cdot |S_W|, |S_V| \cdot |S_{W'}|\} + \max\{|S_U|, |S_V|, |S_W|, |S_{W'}|\}$, due to the fact that they start repeating, as we showed in Lemma 22.

▶ **Lemma 35.** *Let (U, u_0), (V, v_0), (W, w_0), and (W', w'_0) be finite, pointed SMDPs. If $(U, W) \leqq_\star^m (V, W')$, then $(U, W) \leqq_\star (V, W')$.*

We can now use the first-order theory of the reals to show that strong monotonicity is a decidable property.

▶ **Theorem 36.** *Let (U, u_0), (V, v_0), (W, w_0), and (W', w'_0) be finite pointed SMDPs. If for all state paths $\pi_U \in \Pi_m[u_0]$, $\pi_V \in \Pi_m[v_0]$, $\pi_W \in \Pi_m[w_0]$, and $\pi_{W'} \in \Pi_m[w'_0]$ we have that $\{x \in \mathbb{R}_{\geq 0} \mid F_{\rho(\pi_U[i] \|_\star \pi_W[i])}(x) \geq F_{\rho_U(\pi_U[i])}(x)\}$ and $\{x \in \mathbb{R}_{\geq 0} \mid F_{\rho_V(\pi_V[i])}(x) \geq F_{\rho(\pi_V[i] \|_\star \pi_{W'}[i])}(x)\}$ are semialgebraic sets for all $1 \leq i \leq m$, then it is decidable whether $(U, W) \leqq_\star (V, W')$.*

For uniform and exponential distributions with minimum or maximum composition, the corresponding sets are all semialgebraic, and the same is true for exponential distributions with product composition. Theorem 36 can therefore be used for these types of composition.

6 Conclusion and Future Work

In this paper, we have investigated the notion of a process being faster than another process in the context of semi-Markov decision processes. We have given a trace-based definition of a faster-than relation, and shown that this definition is closely connected to convolutions of distributions. We have also shown that the faster-than relation enjoys some nice properties such as preserving reachability properties, having a logical characterisation by a very simple modal logic, as well as giving decidable conditions that guarantee, in some special cases, that one process is faster than another. By considering composition as being parametric in how the residence times of states are combined, we have given examples showing that our faster-than relation gives rise to parallel timing anomalies for many of the popular ways of composing rates. We have therefore given sufficient conditions for how such parallel timing anomalies can be avoided, and we have shown that these conditions are decidable.

The decidability of the faster-than relation itself is still an open question, although we conjecture that it is undecidable. However, even decidability of some simple special cases would be of enormous interest, as this could allow one to decide stochastic orders between convolutions of distributions, which is currently not possible for most kinds of distributions.

The conditions we have given for avoiding timing anomalies do not look at the context in isolation, but depends on the processes that are being swapped. It would be preferable to have conditions on a context that would guarantee the absence of parallel timing anomalies no matter what processes are being swapped. Furthermore, the algorithm for avoiding parallel timing anomalies makes use of the first-order theory of the reals and thus has double-exponential complexity. It would therefore be of interest to reduce this complexity, either by improving the algorithm or by using a different set of conditions.

Acknowledgements. The authors would like to thank Reza Pulungan, Bo Friis Nielsen, and Robert Furber for insightful discussions and pointers to the literature.

References

1. Christel Baier, Joost-Pieter Katoen, Holger Hermanns, and Verena Wolf. Comparative branching-time semantics for markov chains. *Inf. Comput.*, 200(2):149–214, 2005. `doi:10.1016/j.ic.2005.03.001`.
2. Patrick Billingsley. *Probability And Measure*. Wiley-Interscience, 3rd edition, 1995.
3. Vincent D. Blondel and Vincent Canterini. Undecidable problems for probabilistic automata of fixed dimension. *Theory Comput. Syst.*, 36(3):231–245, 2003. `doi:10.1007/s00224-003-1061-2`.
4. Jean-Louis Bon and Eugen Păltănea. Ordering properties of convolutions of exponential random variables. *Lifetime Data Analysis*, 5(2):185–192, 1999. `doi:10.1023/A:1009605613222`.

5 Franck Cassez, René Rydhof Hansen, and Mads Chr. Olesen. What is a timing anomaly? In Tullio Vardanega, editor, *12th International Workshop on Worst-Case Execution Time Analysis, WCET 2012, July 10, 2012, Pisa, Italy*, volume 23 of *OASICS*, pages 1–12. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2012. doi:10.4230/OASIcs.WCET.2012.1.

6 Edmund M. Clarke, David E. Long, and Kenneth L. McMillan. Compositional model checking. In *Proceedings of the Fourth Annual Symposium on Logic in Computer Science (LICS '89), Pacific Grove, California, USA, June 5-8, 1989*, pages 353–362. IEEE Computer Society, 1989. doi:10.1109/LICS.1989.39190.

7 Anne Condon and Richard J. Lipton. On the complexity of space bounded interactive proofs (extended abstract). In *30th Annual Symposium on Foundations of Computer Science, Research Triangle Park, North Carolina, USA, 30 October - 1 November 1989*, pages 462–467. IEEE Computer Society, 1989. doi:10.1109/SFCS.1989.63519.

8 Flavio Corradini, Roberto Gorrieri, and Marco Roccetti. Performance preorder: Ordering processes with respect to speed. In Jirí Wiedermann and Petr Hájek, editors, *Mathematical Foundations of Computer Science 1995, 20th International Symposium, MFCS'95, Prague, Czech Republic, August 28 - September 1, 1995, Proceedings*, volume 969 of *Lecture Notes in Computer Science*, pages 444–453. Springer, 1995. doi:10.1007/3-540-60246-1_150.

9 Nathanaël Fijalkow, Stefan Kiefer, and Mahsa Shirmohammadi. Trace refinement in labelled markov decision processes. In Bart Jacobs and Christof Löding, editors, *Foundations of Software Science and Computation Structures - 19th International Conference, FOSSACS 2016, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2016, Eindhoven, The Netherlands, April 2-8, 2016, Proceedings*, volume 9634 of *Lecture Notes in Computer Science*, pages 303–318. Springer, 2016. doi:10.1007/978-3-662-49630-5_18.

10 Marc Geilen, Stavros Tripakis, and Maarten Wiggers. The earlier the better: a theory of timed actor interfaces. In Marco Caccamo, Emilio Frazzoli, and Radu Grosu, editors, *Proceedings of the 14th ACM International Conference on Hybrid Systems: Computation and Control, HSCC 2011, Chicago, IL, USA, April 12-14, 2011*, pages 23–32. ACM, 2011. doi:10.1145/1967701.1967707.

11 Norbert Götz, Ulrich Herzog, and Michael Rettelbach. Multiprocessor and distributed system design: The integration of functional specification and performance analysis using stochastic process algebras. In Lorenzo Donatiello and Randolph D. Nelson, editors, *Performance Evaluation of Computer and Communication Systems, Joint Tutorial Papers of Performance '93 and Sigmetrics '93, Santa Clara, CA, USA, May 10-14, 1993*, volume 729 of *Lecture Notes in Computer Science*, pages 121–146. Springer, 1993. doi:10.1007/BFb0013851.

12 Ronald L. Graham. Bounds on multiprocessing timing anomalies. *SIAM Journal of Applied Mathematics*, 17(2):416–429, 1969.

13 Shibashis Guha, Chinmay Narayan, and S. Arun-Kumar. On decidability of prebisimulation for timed automata. In P. Madhusudan and Sanjit A. Seshia, editors, *Computer Aided Verification - 24th International Conference, CAV 2012, Berkeley, CA, USA, July 7-13, 2012 Proceedings*, volume 7358 of *Lecture Notes in Computer Science*, pages 444–461. Springer, 2012. doi:10.1007/978-3-642-31424-7_33.

14 Holger Hermanns, Ulrich Herzog, and Vassilis Mertsiotakis. Stochastic process algebras - between LOTOS and markov chains. *Computer Networks*, 30(9-10):901–924, 1998. doi:10.1016/S0169-7552(97)00133-5.

15 Jane Hillston. *A Compositional Approach to Performance Modelling*. Distinguished Dissertations in Computer Science. Cambridge University Press, 2005.

16 Baha-Eldin Khaledi and Subhash Kochar. A review on convolutions of gamma random variables. In Haijun Li and Xiaohu Li, editors, *Stochastic Orders in Reliability and Risk:*

In Honor of Professor Moshe Shaked, pages 199–217. Springer New York, New York, NY, 2013. doi:10.1007/978-1-4614-6892-9_10.

17 Raimund Kirner, Albrecht Kadlec, and Peter P. Puschner. Precise worst-case execution time analysis for processors with timing anomalies. In *21st Euromicro Conference on Real-Time Systems, ECRTS 2009, Dublin, Ireland, July 1-3, 2009*, pages 119–128. IEEE Computer Society, 2009. doi:10.1109/ECRTS.2009.8.

18 Ramesh M. Korwar. On stochastic orders for sums of independent random variables. *Journal of Multivariate Analysis*, 80(2):344 – 357, 2002. doi:http://dx.doi.org/10.1006/jmva.2000.1982.

19 Edward A. Lee. Cyber physical systems: Design challenges. In *11th IEEE International Symposium on Object-Oriented Real-Time Distributed Computing (ISORC 2008), 5-7 May 2008, Orlando, Florida, USA*, pages 363–369. IEEE Computer Society, 2008. doi:10.1109/ISORC.2008.25.

20 Thomas Lundqvist and Per Stenström. Timing anomalies in dynamically scheduled microprocessors. In *Proceedings of the 20th IEEE Real-Time Systems Symposium, Phoenix, AZ, USA, December 1-3, 1999*, pages 12–21. IEEE Computer Society, 1999. doi:10.1109/REAL.1999.818824.

21 Gerald Lüttgen and Walter Vogler. A faster-than relation for asynchronous processes. In Kim Guldstrand Larsen and Mogens Nielsen, editors, *CONCUR 2001 - Concurrency Theory, 12th International Conference, Aalborg, Denmark, August 20-25, 2001, Proceedings*, volume 2154 of *Lecture Notes in Computer Science*, pages 262–276. Springer, 2001. doi:10.1007/3-540-44685-0_18.

22 Gerald Lüttgen and Walter Vogler. Bisimulation on speed: A unified approach. *Theor. Comput. Sci.*, 360(1-3):209–227, 2006. doi:10.1016/j.tcs.2006.03.004.

23 Faron Moller and Chris M. N. Tofts. Relating processes with respect to speed. In Jos C. M. Baeten and Jan Friso Groote, editors, *CONCUR '91, 2nd International Conference on Concurrency Theory, Amsterdam, The Netherlands, August 26-29, 1991, Proceedings*, volume 527 of *Lecture Notes in Computer Science*, pages 424–438. Springer, 1991. doi:10.1007/3-540-54430-5_104.

24 Martin R. Neuhäußer and Joost-Pieter Katoen. Bisimulation and logical preservation for continuous-time markov decision processes. In Luís Caires and Vasco Thudichum Vasconcelos, editors, *CONCUR 2007 - Concurrency Theory, 18th International Conference, CONCUR 2007, Lisbon, Portugal, September 3-8, 2007, Proceedings*, volume 4703 of *Lecture Notes in Computer Science*, pages 412–427. Springer, 2007. doi:10.1007/978-3-540-74407-8_28.

25 Muhammad Reza Pulungan. *Reduction of Acyclic Phase-Type Representations.* PhD thesis, Faculty of Natural Sciences and Technology, Saarland University, 2009.

26 Jan Reineke, Björn Wachter, Stephan Thesing, Reinhard Wilhelm, Ilia Polian, Jochen Eisinger, and Bernd Becker. A definition and classification of timing anomalies. In Frank Mueller, editor, *6th Intl. Workshop on Worst-Case Execution Time (WCET) Analysis, July 4, 2006, Dresden, Germany*, volume 4 of *OASICS*. Internationales Begegnungs- und Forschungszentrum fuer Informatik (IBFI), Schloss Dagstuhl, Germany, 2006.

27 Ichiro Satoh and Mario Tokoro. A formalism for remotely interacting processes. In Takayasu Ito and Akinori Yonezawa, editors, *Theory and Practice of Parallel Programming, International Workshop TPPP'94, Sendai, Japan, November 7-9, 1994, Proceedings*, volume 907 of *Lecture Notes in Computer Science*, pages 216–228. Springer, 1994. doi:10.1007/BFb0026571.

28 Ana Sokolova and Erik P. de Vink. Probabilistic automata: System types, parallel composition and comparison. In Christel Baier, Boudewijn R. Haverkort, Holger Hermanns, Joost-Pieter Katoen, and Markus Siegle, editors, *Validation of Stochastic Systems - A*

Guide to Current Research, volume 2925 of *Lecture Notes in Computer Science*, pages 1–43. Springer, 2004. doi:10.1007/978-3-540-24611-4_1.

29 Walter Vogler. Faster asynchronous systems. In Insup Lee and Scott A. Smolka, editors, *CONCUR '95: Concurrency Theory, 6th International Conference, Philadelphia, PA, USA, August 21-24, 1995, Proceedings*, volume 962 of *Lecture Notes in Computer Science*, pages 299–312. Springer, 1995. doi:10.1007/3-540-60218-6_22.

30 Walter Vogler. Timed testing of concurrent systems. *Inf. Comput.*, 121(2):149–171, 1995. doi:10.1006/inco.1995.1130.