A Detailed Proofs

▶ **Lemma 37.** *If $x \geq y$, then $\mathbb{P}^\sigma(s)(\mathfrak{C}(a_1 \dots a_n, x)) \geq \mathbb{P}^\sigma(s)(\mathfrak{C}(a_1 \dots a_n, y))$.*

Proof. If $x \geq y$, then

$$\mathfrak{C}(S \dots S, \{a_1\} \dots \{a_n\}, B_x^n) \supseteq \mathfrak{C}(S \dots S, \{a_1\} \dots \{a_n\}, B_y^n),$$

and the result then follows from monotonicity of probability measures. ◀

Proof of Proposition 6. Clearly, if for all schedulers σ for V there exists a scheduler σ' for U such that $\mathbb{P}^\sigma(u_0)(C) \geq \mathbb{P}^{\sigma'}(v_0)(C)$ for all time-bounded cylinders C, then $U \leq_{\mathrm{ft}} V$ by taking $C' = C$. If $U \leq_{\mathrm{ft}} V$, then consider an arbitrary scheduler σ, and time-bounded cylinder $C = \mathfrak{C}(a_1 \dots a_n, x)$. There exists a scheduler σ' and $x' \in \mathbb{R}_{\geq 0}$ such that $x \geq x'$ and $\mathbb{P}^{\sigma'}(u_0)(a_1 \dots a_n, x') \geq \mathbb{P}^\sigma(v_0)(a_1 \dots a_n, x)$. By monotonicity, $x \geq x'$ implies that $\mathbb{P}^{\sigma'}(u_0)(a_1 \dots a_n, x) \geq \mathbb{P}^{\sigma'}(u_0)(a_1 \dots a_n, x')$, and hence $\mathbb{P}^{\sigma'}(u_0)(C) \geq \mathbb{P}^\sigma(v_0)(C)$. ◀

▶ **Proposition 38.** *For any SMDP M, scheduler σ for M, and $s \in S$, we have*

$$\begin{aligned}
&\mathbb{P}^\sigma(s)(\mathfrak{C}(S \dots S, \{a_1\} \dots \{a_n\}, R_1 \dots R_n)) \\
&= \sum_{s_n \in S} \cdots \sum_{s_1 \in S} \tau^\sigma(s, a_1)(s_1) \cdots \tau^\sigma(s_{n-1}, a_n)(s_n) \\
&\quad \cdot \rho(s) \times \rho(s_1) \times \cdots \times \rho(s_{n-1})(R_1 \times R_2 \times \cdots \times R_n).
\end{aligned}$$

Proof. The proof is by induction on the length n of the cylinder. If the cylinder has length $n = 1$ then

$$\mathbb{P}^\sigma(s)(\mathfrak{C}(S, \{a_1\}, R_1)) = \sum_{s_1 \in S} \tau^\sigma(s, a_1)(s_1) \cdot \rho(s)(R_1).$$

If the cylinder has length $n = k + 1$, then

$$\begin{aligned}
&\mathbb{P}^\sigma(s)(\mathfrak{C}(S \dots S, \{a_1\} \dots \{a_{k+1}, R_1 \dots R_{k+1}\})) \\
&= \rho(s)(R_1) \cdot \sum_{s_1 \in S} \tau^\sigma(s, a_1)(s_1) \cdot \\
&\quad \mathbb{P}^\sigma(s_1)(\mathfrak{C}(S \dots S, \{a_2\} \dots \{a_{k+1}\}, R_2 \dots R_{k+1})) \\
&= \rho(s)(R_1) \cdot \sum_{s_1 \in S} \tau^\sigma(s, a_1)(s_1) && \text{(ind. hyp.)} \\
&\quad \cdot \sum_{s_{k+1} \in S} \cdots \sum_{s_2 \in S} \tau^\sigma(s_1, a_2)(s_2) \cdots \tau^\sigma(s_k, a_{k+1})(s_{k+1}) \\
&\quad \cdot \rho(s_1) \times \cdots \times \rho(s_{k+1})(R_2 \times \cdots \times R_{k+1}) \\
&= \sum_{s_{k+1} \in S} \cdots \sum_{s_1 \in S} \tau^\sigma(s, a_1)(s_1) \cdots \tau^\sigma(s_k, a_{k+1})(s_{k+1}) \\
&\quad \cdot \rho(s) \times \rho(s_1) \times \cdots \times \rho(s_{k+1})(R_1 \times \cdots \times R_{k+1}).
\end{aligned}$$

◀

Hence, for any Borel set $R^n = R_1 \times \cdots \times R_n \in \mathbb{R}^n_{\geq 0}$, it makes sense to write

$$\begin{aligned}
&\mathbb{P}^\sigma(s)(\mathfrak{C}(S\dots S,\{a_1\}\dots\{a_n\},R^n))\\
&=\sum_{s_n\in S}\cdots\sum_{s_1\in S}\tau^\sigma(s,a_1)(s_1)\cdots\tau^\sigma(s_{n-1},a_n)(s_n)\cdot\rho(s)\times\rho(s_1)\times\cdots\times\rho(s_{n-1})(R^n)\\
&=\mathbb{P}^\sigma(s)(\mathfrak{C}(S\dots S,\{a_1\}\dots\{a_n\},R_1\dots R_n)).
\end{aligned}$$

Proof of Proposition 7. By Proposition 38, we know that

$$\begin{aligned}
&\mathbb{P}^\sigma(s)(\mathfrak{C}(a_1\dots a_n,x))\\
&=\sum_{s_n\in S}\cdots\sum_{s_1\in S}\tau^\sigma(s,a_1)(s_1)\cdots\tau^\sigma(s_{n-1},a_n)(s_n)\\
&\quad\cdot\rho(s)\times\rho(s_1)\times\cdots\times\rho(s_{n-1})(B^n_x).
\end{aligned}$$

Hence, if we can show that

$$\rho(s)\times\rho(s_1)\times\cdots\times\rho(s_{n-1})(B^n_x)=(\rho(s)*\rho(s_1)*\cdots*\rho(s_{n-1}))([0,x]),$$

the proof is done.

The proof now proceeds by induction on the length n of the time-bounded cylinder $\mathfrak{C}(a_1\dots a_n,x)$. If $n=1$, then

$$\rho(s)(B^1_x)=\rho(s)([0,x]).$$

If $n=k+1$, then

$$\begin{aligned}
&(\rho(s)\times\rho(s_1)\times\cdots\times\rho(s_k))(B^{k+1}_x)\\
&=\int_0^x(\rho(s_1)\times\cdots\times\rho(s_k))(B^k_{x-t})\,\rho(s)(\mathrm{d}t) && \text{(Fubini)}\\
&=\int_0^x(\rho(s_1)*\cdots*\rho(s_k))([0,x-t])\,\rho(s)(\mathrm{d}t) && \text{(ind. hyp.)}\\
&=(\rho(s)*(\rho(s_1)*\cdots*\rho(s_k)))([0,x]) && \text{(def. of convolution)}\\
&=(\rho(s)*\rho(s_1)*\cdots*\rho(s_k))([0,x]). && \text{(associativity)}
\end{aligned}$$

◀

Proof of Lemma 13. Assume first that $\pi\in\Diamond^M_{\le x}a$. Then there exists $i\in\mathbb{N}$ such that $\pi[\![i]\!]=a$ and $\pi[\![k]\!]\neq a$ for all $k<i$, and $\sum_{j=1}^i\pi\langle j\rangle\le x$. By letting $k=i-1$, it follows that $\pi\in\mathfrak{C}(a_1\dots a_ka,x)$.

Next assume that $\pi\in\bigcup_{k\in\mathbb{N}_0}\bigcup_{(a_1,\dots,a_k)\in(L\setminus\{a\})^k}\mathfrak{C}(a_1\dots,a_ka,x)$. Then there exists $k\in\mathbb{N}_0$ and $(a_1,\dots,a_k)\in(L\setminus\{a\})^k$ such that $\pi\in\mathfrak{C}(a_1\dots a_ka,x)$. Now let $i=k+1$. Then $i\in\mathbb{N}$, $\pi[\![i]\!]=a$, $\pi[\![k]\!]\neq a$ for all $k<i$, and $\sum_{j=1}^i\pi\langle j\rangle\le x$, so $\pi\in\Diamond^M_{\le x}a$. ◀

Proof of Theorem 14. Let σ be an arbitrary scheduler for V. By Lemma 13, we have

$$\mathbb{P}^\sigma(v_0)(\Diamond_{\le x}a)=\sum_{k\in\mathbb{N}_0}\sum_{(a_1,\dots,a_k)\in L^k}\mathbb{P}^\sigma(v_0)(\mathfrak{C}(a_1\dots a_ka,x))$$

since all the cylinders $\mathfrak{C}(a_1\dots a_ka,x)$ are disjoint. Because $U\le_{\mathrm{ft}}V$, we have that there exists a scheduler σ' for U such that

$$\mathbb{P}^{\sigma'}(u_0)(\mathfrak{C}(a_1\dots a_ka,x))\ge\mathbb{P}^\sigma(v_0)(\mathfrak{C}(a_1\dots a_ka,x))$$

for all k and cylinders $\mathfrak{C}(a_1 \dots a_k a, x)$. Hence we get

$$\sum_{k\in\mathbb{N}_0} \sum_{(a_1,\dots,a_k)\in L^k} \mathbb{P}^{\sigma}(v_0)(\mathfrak{C}(a_1 \dots a_k a, x)) \leq \sum_{k\in\mathbb{N}_0} \sum_{(a_1,\dots,a_k)\in L^k} \mathbb{P}^{\sigma'}(u_0)(\mathfrak{C}(a_1 \dots a_k a, x)) \\ = \mathbb{P}^{\sigma'}(u_0)(\Diamond_{\leq x} a)$$

◀

Proof of Theorem 18. Note first that $U \leq_{\mathrm{ft}} V$ if and only if for all σ there exists σ' such that $\mathbb{P}^{\sigma}(v_0)(C) \geq q$ implies $\mathbb{P}^{\sigma'}(u_0)(C) \geq q$ for all cylinders C and $q \in \mathbb{Q}_{\geq 0}$.

($\Longrightarrow$) Assume that for all cylinders C, all $q \in \mathbb{Q}_{\geq 0}$, and all schedulers σ, there exists σ' such that $\mathbb{P}^{\sigma}(v_0)(C) \geq q$ implies $\mathbb{P}^{\sigma'}(u_0)(C) \geq q$. Assume that $V, v_0 \models \psi$. ψ must be of the form $\psi = \mathcal{P}_{\geq q}(\varphi)$ for some path formula φ. Then $\mathbb{P}^{\sigma}(v_0)(\mathfrak{C}(\varphi)) \geq q$ for some scheduler σ. Hence, by assumption, there must exist a scheduler σ' such that $\mathbb{P}^{\sigma}(u_0)(\mathfrak{C}(\varphi)) \geq q$, which means that $U, u_0 \models \psi$.

($\Longleftarrow$) Let $\mathfrak{C}(a_1 \dots a_n, x)$ be a cylinder and $q \in \mathbb{Q}_{\geq 0}$. Let σ be an arbitrary scheduler and assume that $\mathbb{P}^{\sigma}(v_0)(\mathfrak{C}(a_1 \dots a_n, x)) \geq q$. If $x = 0$, then $\mathbb{P}^{\sigma}(v_0)(\mathfrak{C}(a_1 \dots a_n, x)) = 0$, implying that $q = 0$, and therefore it follows that $\mathbb{P}^{\sigma'}(u_0)(\mathfrak{C}(a_1 \dots a_n, x)) \geq q$ for any scheduler σ'. If $x > 0$, let $\varepsilon > 0$, and let δ be given by right-continuity. Now choose $x_1, \dots, x_n \in \mathbb{Q}_{\geq 0}$ such that $x_1 + \dots + x_n = x'$ and $x < x' < x + \delta$ and let $\varphi_\varepsilon = \langle a_1 \rangle_{x_1} \dots \langle a_n \rangle_{x_n} \top$. Then $\mathfrak{C}(\varphi_\varepsilon) = \mathfrak{C}(a_1 \dots a_n, x')$, and by monotonicity we have $\mathbb{P}^{\sigma}(v_0)(\mathfrak{C}(\varphi_\varepsilon)) \geq \mathbb{P}^{\sigma}(v_0)(\mathfrak{C}(a_1 \dots a_n, x)) \geq q$. Therefore we get $V, v_0 \models P_{\geq q}(\varphi_\varepsilon)$, implying by assumption that $U, u_0 \models P_{\geq q}(\varphi_\varepsilon)$, and hence there exists σ' such that $\mathbb{P}^{\sigma'}(u_0)(\mathfrak{C}(\varphi_\varepsilon)) \geq q$. By right-continuity, we have

$$|\mathbb{P}^{\sigma'}(u_0)(\mathfrak{C}(\varphi_\varepsilon)) - \mathbb{P}^{\sigma'}(u_0)(\mathfrak{C}(a_1 \dots a_n, x))| < \varepsilon. \tag{1}$$

Now assume towards a contradiction that $\mathbb{P}^{\sigma'}(u_0)(\mathfrak{C}(a_1 \dots a_n, x)) < q$. Then there exists a $q' \in \mathbb{Q}_{\geq 0}$ such that $\mathbb{P}^{\sigma'}(u_0)(\mathfrak{C}(a_1 \dots a_n, x)) < q' < q$. Pick an $\varepsilon' > 0$ such that $\mathbb{P}^{\sigma'}(u_0)(\mathfrak{C}(a_1 \dots a_n, x)) < q' < q' + \varepsilon' < q \leq \mathbb{P}^{\sigma'}(u_0)(\mathfrak{C}(\varphi_{\varepsilon'}))$. Then $|\mathbb{P}^{\sigma'}(u_0)(\mathfrak{C}(\varphi_{\varepsilon'})) - \mathbb{P}^{\sigma'}(u_0)(\mathfrak{C}(a_1 \dots a_n, x))| > q' + \varepsilon' - q' > 0$, contradicting (1). Hence we conclude that $\mathbb{P}^{\sigma'}(u_0)(\mathfrak{C}(a_1 \dots a_n, x)) \geq q$. ◀

Proof of Lemma 22. Since there are $|S_U| \cdot |S_V|$ ways of choosing a pair $(u_i, v_j) \in S_U \times S_V$ of states from U and V, if we pair the states of π_U and π_V such that we get the pairs $(\pi_U[1], \pi_V[1])$, $(\pi_U[2], \pi_V[2])$, $\dots$, $(\pi_U[n], \pi_V[n])$, there must be two of these pairs that are the same because $n > |S_U| \cdot |S_V|$. Hence we get states $\pi_U[i] = \pi_U[j]$ and $\pi_V[i] = \pi_V[j]$ with $i < j \leq n$. It also follows that i and j can be chosen so that $i < j \leq |S_U| \cdot |S_V|$, because otherwise we would have $j - i > |S_U| \cdot |S_V|$ different pairs $(\pi_U[i], \pi_V[i]), (\pi_U[i+1], \pi_V[i+1]), \dots, (\pi_U[j], \pi_V[j])$, contradicting the fact that there are only $|S_U| \cdot |S_V|$ such different pairs. ◀

Proof of Theorem 23. Because we are considering a singleton set L as transition labels, there is only one possible scheduler, and hence we will omit discussions of schedulers.

Let $C = \mathfrak{C}(a_1 \dots a_k, x)$ be a cylinder of length $k > |S_U| \cdot |S_V|$. Because U and V have deterministic Markov kernels, there will exist some paths π_U in U and π_V in V of length k such that

$$\mathbb{P}(u_0)(C) = (\rho_U(\pi_U[1]) * \dots * \rho_U(\pi_U[k]))([0, x])$$

and

$$\mathbb{P}(v_0)(C) = (\rho_V(\pi_V[1]) * \dots * \rho_V(\pi_V[k]))([0, x]).$$

By Lemma 22, there must be some index $i < j \leq |S_U| \cdot |S_V|$ such that $\pi_U[i] = \pi_U[j]$ and $\pi_V[i] = \pi_V[j]$. For convenience, let $u_i = \pi_U[i]$, $v_i = \pi_V[i]$, and $\rho_M^{m,n} = (\rho(\pi_M[m]) * \cdots * \rho(\pi_M[n]))$. This means that we can split π_U into several parts, each of length less than or equal to $|S_U| \cdot |S_V|$, and all starting in either $\pi_U[1]$ or u_i:

$$(\rho_U(\pi_U[1]) * \cdots * \rho_U(\pi_U[k]))([0,x]) = (\rho_U^{1,i-1} * \rho_U^{i,j-1} * \cdots * \rho_U^{i,j-1} * \rho_U^{i,k})([0,x])$$

and

$$(\rho_V(\pi_V[1]) * \cdots * \rho_V(\pi_V[k]))([0,x]) = (\rho_V^{1,i-1} * \rho_V^{i,j-1} * \cdots * \rho_V^{i,j-1} * \rho_V^{i,k})([0,x]).$$

There must exist cylinders C_1, C_2, and C_3 such that

$$\mathbb{P}(u_0)(C_1) = \rho_U^{1,i-1}([0,x]) \text{ and } \mathbb{P}(v_0)(C_1) = \rho_V^{1,i-1}([0,x]),$$

$$\mathbb{P}(u_i)(C_2) = \rho_U^{i,j-1}([0,x]) \text{ and } \mathbb{P}(v_i)(C_2) = \rho_V^{i,j-1}([0,x]),$$

and

$$\mathbb{P}(u_i)(C_3) = \rho_U^{i,k}([0,x]) \text{ and } \mathbb{P}(v_i)(C_3) = \rho_V^{i,k}([0,x]).$$

Since u_i and v_i are recurrent states, we know that

$$\mathbb{P}(u_0)(C_1) \geq \mathbb{P}(v_0)(C_1),$$

$$\mathbb{P}(u_i)(C_2) \geq \mathbb{P}(v_i)(C_2),$$

and

$$\mathbb{P}(u_i)(C_3) \geq \mathbb{P}(v_i)(C_3).$$

Hence it follows that $\mathbb{P}(u_0)(C) \geq \mathbb{P}(v_0)(C)$, and therefore $U \leq_{\text{ft}} V$. ◀

Proof of Theorem 24. By Theorem 23, it suffices to decide whether $\mathbb{P}(u)(C) \geq \mathbb{P}(v)(C)$ for the initial states, all recurrent states, and all cylinders of length $k \leq |S_U| \cdot |S_V|$. Since L is a singleton and U and V have deterministic Markov kernels, there exist paths π_U in U and π_V in V such that

$$\mathbb{P}(u)(C) = (\rho_U(\pi_U[1]) * \cdots * \rho_U(\pi_U[n]))([0,x])$$

and

$$\mathbb{P}(v)(C) = (\rho_V(\pi_V[1]) * \cdots * \rho_V(\pi_V[n]))([0,x]).$$

Hence the inequality is decidable by assumption, and since are only finitely many of these, this is decidable. ◀

Proof of Theorem 31. Let $\mathfrak{C}(a_1 \ldots a_n, x)$ be an arbitrary time-bounded cylinder, and let $\sigma_{V,W'}$ be an arbitrary scheduler for $V \parallel_\star W'$. Because $(U,W) \lesssim_\star (V,W')$, there exists a scheduler σ_V for V and a path π such that

$$\begin{aligned}
&\mathbb{P}^{\sigma_{V,W'}}(v_0 \parallel_\star w_0')(\mathfrak{C}(a_1 \dots a_n, x)) \\
&= \tau^{\sigma_{V,W'}}(\pi[1], a_1)(\pi[2]) \cdots \tau^{\sigma_{V,W'}}(\pi[n], a_n)(\pi[n+1]) && \text{(det. Markov kernel)} \\
&\quad \cdot (\rho(\pi[1]) * \cdots * \rho(\pi[n]))([0,x]) \\
&\leq \tau_V^{\sigma_V}(\pi_V[1], a_1)(\pi_V[2]) \cdots \tau_V^{\sigma_V}(\pi_V[n], a_n)(\pi[n+1]) && \text{(def. 30)} \\
&\quad \cdot (\rho_V(\pi_V[1]) * \cdots * \rho_V(\pi_V[n]))([0,x]) \\
&\leq \sum_{\pi \in \Pi_{n+1}[v_0]} \tau_V^{\sigma_V}(\pi[1], a_1)(\pi[2]) \cdots \tau_V^{\sigma_V}(\pi[n], a_n)(\pi[n+1]) \\
&\quad \cdot (\rho_V(\pi[1]) * \cdots * \rho_V(\pi[n]))([0,x]) \\
&= \mathbb{P}^{\sigma_V}(v_0)(\mathfrak{C}(a_1 \dots a_n, x)),
\end{aligned}$$

Since $U \leq_{\text{ft}} V$, there must exist some scheduler σ_U for U such that

$$\mathbb{P}^{\sigma_V}(v_0)(\mathfrak{C}(a_1 \dots a_n, x))) \leq \mathbb{P}^{\sigma_U}(u_0)(\mathfrak{C}(a_1 \dots a_n, x)).$$

Again, since $(U, W) \lessapprox_\star (V, W')$, there exists a scheduler $\sigma_{U,W}$ for $U \parallel_\star W$ such that

$$\begin{aligned}
&\mathbb{P}^{\sigma_U}(u_0)(\mathfrak{C}(a_1 \dots a_n, x)) \\
&= \sum_{\pi \in \Pi_{n+1}[u_0]} \tau_U^{\sigma_U}(\pi[1], a_1)(\pi[2]) \cdots \tau_U^{\sigma_U}(\pi[n], a_n)(\pi[n+1]) \\
&\quad \cdot (\rho_U(\pi[1]) * \cdots * \rho_U(\pi[n]))([0,x]) \\
&\leq \sum_{\pi_W \in \Pi_{n+1}[w_0]} \sum_{\pi \in \Pi_{n+1}[u_0]} \tau_U^{\sigma_U}(\pi[1], a_1)(\pi[2]) \cdots \tau_U^{\sigma_U}(\pi[n], a_n)(\pi[n+1]) \\
&\quad \cdot (\rho_U(\pi[1]) * \cdots * \rho_U(\pi[n]))([0,x]) \\
&\leq \sum_{\pi \in \Pi_{n+1}[u_0 \parallel_\star w_0]} \tau^{\sigma_{U,W}}(\pi[1], a_1)([2]) \cdots \tau^{\sigma_{U,W}}([n], a_n)([n+1]) && \text{(def. 30)} \\
&\quad \cdot (\rho(\pi[1]) * \cdots * \rho(\pi[n]))([0,x]) \\
&= \mathbb{P}^{\sigma_{U,W}}(u_0 \parallel_\star w_0)(\mathfrak{C}(a_1 \dots a_n, x)).
\end{aligned}$$

◀

Proof of Lemma 35. Assume that $(U, W) \leqq_\star^m (V, W')$. Then $(U, W) \leqq_\star^k (V, W')$ for all $k \leq m$. Hence it remains to show that $(U, W) \leqq_\star^k (V, W')$ for all $k > m$.

Let $k > m$ and consider two state paths $\pi_U = u_1 u_2 \dots u_k$ and $\pi_W = w_1 w_2 \dots w_k$ of U and W, respectively, both of length k. By Lemma 22 there must exist $i < j \leq |S_U| \cdot |S_W|$ such that $u_i = u_j$ and $w_i = w_j$. Since there exists a state path from u_1 to u_i, it must be possible to reach this state in less than $|S_U|$ steps, and likewise for W. Hence there must exist $l \leq \max\{|S_U|, |S_W|\}$, and state paths $u_1 u_2' \dots u_l' u_i u_{i+1} \dots u_j$ and $w_1 w_2' \dots w_l' w_i w_{i+1} \dots w_j$ of length $l + (j - i) \leq \max\{|S_U|, |S_W|\} + |S_U| \cdot |S_W| \leq m$. Hence we know that the conditions of Definition 32 are satisfied for $u_i \dots u_j$ and $w_i \dots w_j$. By removing the states $u_{i+1} \dots u_j$ and $w_{i+1} \dots w_j$ from π_U and π_W we end up with two new state paths π_U' and π_W' of length $k' = k - (j - i)$. We can keep doing this as long as $k' > m$, so at some point we must end up with state paths π_U^* and π_W^* of length $k^* \leq m$, for which the conditions of Definition 32 are satisfied by assumption, and hence they are satisfied for all of π_U and π_W. The same argument can be applied to two state paths π_V and $\pi_{W'}$ of V and W', so we conclude that $(U, W) \leqq_\star (V, W')$. ◀

Proof of Theorem 36. Note first of all that since L and W' are finite, it is decidable whether W' has a deterministic Markov kernel by looking at all the states. By Lemma 35, it suffices to check whether $(U, W) \leqq_\star^m (V, W')$ where $m = \max\{|S_U| \cdot |S_W|, |S_V| \cdot |S_{W'}|\} + \max\{|S_U|, |S_V|, |S_W|, |S_{W'}|\}$. This can be done by exploiting the decidability of the first-order theory of the reals in the following way. Since L is finite and U, V, W, and W' are all finite, there are finitely many state paths $\pi_U \in \Pi_m[u_0]$, $\pi_V \in \Pi_m[v_0]$, $\pi_W \in \Pi_m[w_0]$, and $\pi_{W'} \in \Pi_m[w'_0]$. Because of this, and since the sets $\{x \in \mathbb{R}_{\geq 0} \mid F_{\rho(\pi_U[i] \|_\star \pi_W[i])}(x) \geq F_{\rho_U(\pi_U[i])}(x)\}$ and $\{x \in \mathbb{R}_{\geq 0} \mid F_{\rho_V(\pi_V[i])}(x) \geq F_{\rho(\pi_V[i] \|_\star \pi_{W'}[i])}(x)\}$, which we need to check for the first condition, were assumed to be semialgebraic, it is possible to express the conditions of Definition 32 in the first-order theory of the reals, using finitely many quantifiers and inequalities. Since the first-order theory of the reals is decidable, the truth value of the resulting formula is decidable. ◀