Part I

Special Relativity

1 Lorentz-Boost

If a 'stationary' observer in reference frame I describes an event with coordinates (t, x), then another observer I', that moves with velocity v relative to I records the same event with coordinates

$$t' = \gamma \left(t - \frac{vx}{c^2} \right) \tag{1}$$

$$x' = \gamma (x - vt) \tag{2}$$

with the Lorentz factor

$$\gamma := \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{3}$$

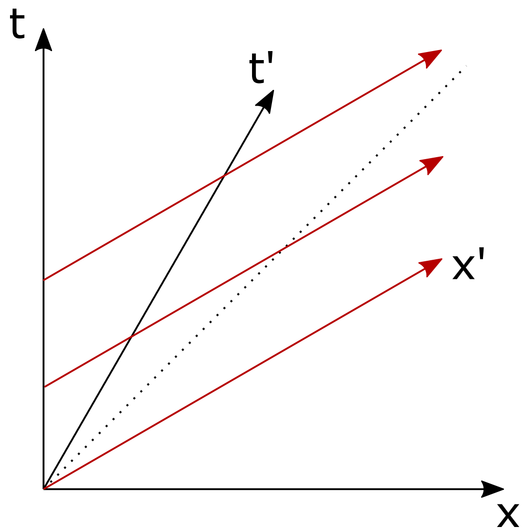
Note: $1 \leq \gamma < \infty$
This is called the Lorentz boost. For the sake of symmetry we use
$betaeta := v/c$ and write

$$ct' = \gamma (ct - \beta x) \tag{4}$$

$$x' = \gamma (x - \beta ct) . \tag{5}$$

2 Comparison of Lorentz- & Galileo-Transformation

3 Relativity of Simultaneity and Equilocality

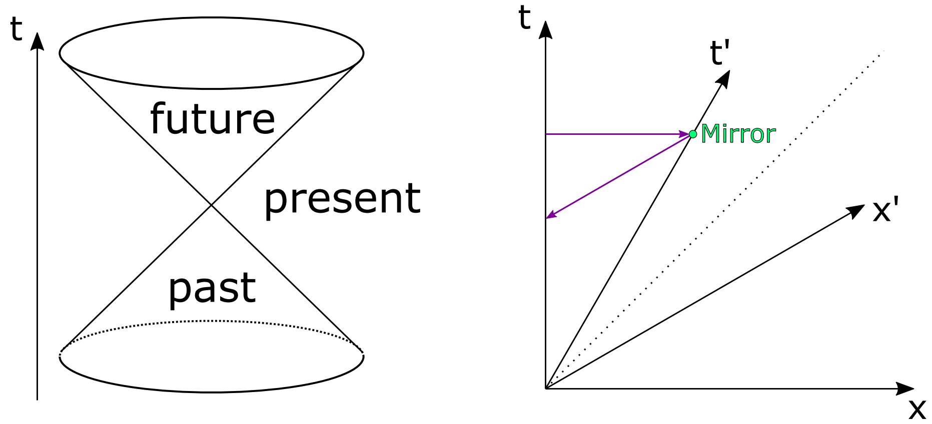


The red lines are of constant time in the moving frame.

One can always rotate the x-axis such that any event A lies on it.

One can always rotate the t-axis such that any event A lies on it.

4 Light Cone



(a) (b)

$$t^2 = x^2 + y^2 + z^2 \tag{6}$$

This induces the causal structure of special relativity.

Superluminal velocity can influence the past as can be seen in fig. 2b.

5 Time Dilation

While simultaneity and locality are observer-dependent, the measure

$$c^2t^2 - x^2 = c^2t'^2 - x'^2 \tag{7}$$

is invariant, which inspires us to define a **line element**

$$ds^2 := -cdt^2 + dx^2. \tag{8}$$

The **proper time**

$$d\tau^2 := -\frac{ds^2}{c^2} \tag{9}$$

is then an invariant measure for the length of a world line.
Note: Light has proper time $ds^2 = 0$. A 'normal' movement $x(t)$ has proper time

$$ds^2 = -dt^2 + dx^2 = dt^2 \left(-1 + \left(\frac{dx}{dt}\right)^2\right) \tag{10}$$

$$d\tau = dt\sqrt{1 - v^2} = \frac{1}{\gamma} dt < dt \tag{11}$$

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$$ds^2 = -dt^2 + dx^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta$$

Matrix:

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Contravariant:

$$u_\alpha = \eta_{\alpha\beta} u^\beta$$

scalar product:

$$u \cdot w = u_\alpha w^\alpha = \eta_{\alpha\beta} u^\beta w^\alpha$$

derivative:

$$\phi_{,\alpha} = \frac{\partial \phi(x)}{\partial x^\alpha}$$

6 Special Relativistic Mechanics

system I with $\vec{x}(t), \vec{v}(t)$ It is more convenient to use proper time τ over t because it is invariant:

$$d\tau = -dt^2 + dx^2$$

So $x^\alpha(\tau)$ is used with the four-velocity

$$x^\alpha(\tau) := \frac{dx^\alpha}{d\tau}$$

The components are: $u^0 = \frac{dt}{d\tau} = \frac{1}{\sqrt{1-v^2(t)}} = \gamma(t)$
$u^i = \frac{dx^i}{dt}\frac{dt}{d\tau} = v^i \gamma(t)$
Note:

$$u^2 = \eta_{\alpha\beta} \frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = -1$$

because

$$d\tau^2 = -\eta_{\alpha\beta} dx^\alpha dx^\beta$$

The four-acceleration is

$$a^\alpha = \frac{d^2 x^\alpha}{d\tau^2}$$

Differentiate $u^2 = -1$ by τ yields $u \cdot a = 0$.

Dynamics?

$$m\frac{du}{d\tau} = f \leftrightarrow ma = f$$

where m is the rest mass. example: a simple accelerated world line: $t(\theta) = \frac{1}{a}sinh(\theta)$ $x(\theta) = \frac{1}{a}cosh(\theta)$ $x^2 - t^2 = \frac{1}{a^2}$ hyperbola: $x(t) = \sqrt{t - \frac{1}{a^2}}$ proper time $d\tau^2 = dt^2 - dx^2 = \frac{d\theta^2}{a^2} \rightarrow \tau = \frac{\theta}{a}$

$$t(\tau) = \frac{1}{a}sinh(a\tau), \qquad x(\tau) = \frac{1}{a}cosh(a\tau)$$

$$v = dx/dt = \frac{dx}{d\tau}\frac{d\tau}{dt} = sinh(a\tau)/cosh(a\tau) = tanh(a\tau)$$

$$a = \frac{du}{d\tau} = \gamma\frac{du}{dt}$$

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7 Covariant Maxwell Equations

The known Maxwell equations of electrodynamics

$$\nabla\vec{B} = 0, \qquad \nabla \times \vec{E} + \frac{\partial B}{\partial t} = 0$$

$$\nabla\vec{E} = 4\pi\rho, \qquad \nabla \times \vec{B} - \frac{\partial E}{\partial t} = 4\pi\vec{j}$$

$$E = -\nabla\phi - \frac{\partial\vec{A}}{\partial t}, \qquad \vec{B} = \nabla \times \vec{A}$$

are in this formulation above not Lorentz invariant. By defining

$$A^\mu := \begin{pmatrix}\phi \\ \vec{A}\end{pmatrix} = \underline{A}$$

$$j^\mu := (\rho, \vec{j}) = \underline{j}$$

the homogeneous Maxwell-equations can be rewritten as

$$\partial_{[\rho}F_{\mu,\nu]} = \partial_\rho F_{\mu\nu} + \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} = 0$$

with field strength tensor $F_\mu\nu := \partial_\mu A_\nu - \partial_\nu A_\mu = -F_{\nu\mu}$.

The inhomogeneous Maxwell equations then become $\partial_\nu F^{\nu\mu} = 4\pi j^\mu$

$$\partial_\mu\partial_\nu F^{\mu\nu} = 4\pi\partial_{mu}j^\mu = 0 \rightarrow \dot{\rho} + \nabla\vec{j} = 0$$

Lorentz force law:

$$\frac{dp^\mu}{d\tau} = f^\mu = qF^{\mu\nu}u_\nu$$

small r:

$$\frac{d\vec{p}}{dt} = \vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

Energy momentum tensor:

$$T^{\mu\nu}(x) = \begin{pmatrix} \text{energy density} & \text{energy flux} \\ \text{momentum density} & \text{stress tensor} \end{pmatrix}$$

stress tensor T^{ij} i'th component of the force per unit are exerted across a surface with normal in direction j

Energy flux = momentum density: $T^{\mu\nu} = T^{\nu\mu}$ is symmetric

conservation of energy and momentum

$$betaoxed\frac{\partial T^{\mu\nu}}{\partial x^\nu} = 0$$

Electrodynamics:

$$4\pi T^{\mu\nu}_{EM} = T^{\mu\nu}F^{\nu\lambda} - \frac{1}{4}\eta^{\mu\nu}F^{\lambda\sigma}F_{\lambda\sigma}$$

Minkowski:

$$F^{\mu\lambda} = \eta^{\mu\sigma}\eta\lambda\nu F_{\sigma\nu}F^{\nu}_{\lambda} = \eta^{\nu\sigma}F_{\sigma\lambda}$$

8 Gravity as Geometry

Twin problem in SR: Twin (1) stays on earth, twin (2) travels to space and returns to earth. Who is older?

We need to compute the proper time along the timelike world line

$$\tau_{AB} = \int_A^B d\tau = \int_A^B \sqrt{dt^2 - dx^2} = \int_{t_a}^{t_\beta} dt\sqrt{1 - v^2}$$

In earth-frame coordinate time t, clock 1 shows $\tau^1_{AB} = \int_{t_A}^{t_B} dt = t_B - t_A$ while clock 2 shows $\tau^2_{AB} = \int_{t_A}^{t_B} dt\sqrt{1^2_v} < t_B - t_A$ so twin 2 is younger. This is not a symmetric situation because (2) is not in an inertial system! His rest frame is the (t', x')-system, so $d\tau^2 \neq dt'^2 - dx'^2$.

Consider a small acceleration g in x-direction

$$(t, x, y, z) \mapsto (t', x', y, z) = (t, x - \frac{1}{2}gt^2, y, z)$$

The world-length is invariant:

$$\begin{aligned} ds^2 = -dt^2 + dx^2 + ... &= -dt'^2 + dx'^2 + ... \\ &= (-1 + g^2 t'^2)dt'^2 + 2gt'dx'dt' + dx'^2 + dy'^2 + dz'^2 \\ &\neq -dt'^2 + dx'^2 + ... \end{aligned}$$

This is why we generalize the definition of the length element to

$$ds^2 = g_{\alpha\beta}(x')dx'^{\alpha}dx'^{\beta} = \eta_{\alpha betaeta}dx^{\alpha}dx^{\beta} \tag{12}$$

with the metric tensor $g_{\alpha betaeta}(x')$. For this acceleration transformation we get

$$g'_{00} = -1 + g^2t'^2 \qquad g'_{01} = gt' \qquad g'_{ii} = 1 \qquad g'_{other} = 0$$

So the clock at rest in (2) ($dx' = 0$) does not show $\int dt'$ but $d\tau = dt'\sqrt{1 - g^2t'^2}$ so $\int_A^B d\tau = \int_{t'_A}^{t'_B} dt'\sqrt{1 - g^2t'^2} + \int dt'$

general coordinates:

$$ds^2 = g'_{\alpha\beta}(x')dx'^{\alpha}dx'betaeta = \eta_{\rho\delta}dx^{\rho}dx^{\delta} = \eta_{\gamma\delta}\frac{\partial x^{\rho}}{\partial x'^{\alpha}}\frac{\partial x^{\delta}}{\partial x'betaeta}dx'^{\alpha}dx'betaeta$$

8.1 Equivalence Principle

There is a local equivalence of accelerated system and gravitational field. In this notion we can use the metric $ds^2 = g'_{a\beta}(x')dx'^{\alpha}dx'betaeta$ to describe the gravitational field. There are 10 independent components per spacetime point. In general there is no global inertial system in the presence of gravity!

8.2 Gravitational Redshift

A Rocket with constant acceleration g without gravity. All calculations in the frame of the rocket at rest at $t = 0$ A photon arrives at B at $t = \frac{1}{c}(h + \frac{1}{2}gt^2)$ then B has w.r.t. the IS a velocity $v = gt = \frac{gh}{c}$ so B recedes from the arriving photon which experiences a Doppler redshift: $z := \frac{\Delta\lambda}{\lambda_A} = \frac{\lambda_B - \lambda_A}{\lambda_A} \approx \frac{v}{c} \approx \frac{gh}{c^2}$ The equivalence principle suggests that there is also a redshift in a homogeneous gravitational field with acceleration g: $z = \frac{gh}{c^2} = \frac{\phi}{c^2}$

Alternative derivation: A body of mass m travels from B to A with $v(t = 0) = 0$ It has the kinetic energy mgh in A. Then it is transformed into a photon which flies back to B. If no interaction between photon and the gravitational field, it would be transformed back to mass and kinetic energy and energy would be gained! Demand energy conservation: $E_A = E_B + mgh = mc^2 + mgh + mc^2(1 + \frac{gh}{c^2})$ $\frac{\lambda_B}{\lambda_A} = \frac{\hbar\omega_A}{\hbar\omega_B} = \frac{E_A}{E_B} = 1 + \frac{gh}{c^2}$ So the photon loses energy when climbing upwards.

Lets consider n waves, sent from A to B with two equal clocks. The clock in B runs faster. EP clocks are slowed down in gravitational field. More precisely: worldline is shorter (than for inertial motion) if at rest in the gravitational field.

8.3 Motion in the gravitational field

Freely falling systems <-> local inertial motion -> free particles move uniformly on straight lines SR: $\tau_{AB} = \int_A^B \sqrt{dt^2 - dx^2}$ is maximized for inertial motions. can be obtained from variational principle: $\delta \int_A^B d\tau = 0$ also holds in GR when no global inertial system is available. $d\tau^2 = -g_{\mu\nu}(x) dx^\mu dx^\nu$ The curve is described by $x^\mu(\lambda), \dot{x}^\mu = \frac{dx^\mu}{d\lambda}$

$$0 = \delta \int_A^B d\tau = \delta \int_{\lambda_1}^{\lambda_2} d\lambda \underbrace{\sqrt{-g_{\mu\nu}(x) dx^\mu dx^\nu}}_{L}$$

-> Euler Lagrange equation:

$$\frac{\partial L}{\partial x^\alpha} - \frac{d}{d\lambda}\frac{\partial L}{\partial \hat{x}^\alpha} = 0$$

$$\frac{\partial L}{\partial x^\alpha} = -g_{\mu\nu,\alpha}\hat{x}^\mu \hat{x}^\nu \frac{1}{2L} = \frac{-L}{2} g_{\mu\nu,\alpha}\dot{x}^\mu \dot{x}^\nu$$

$$\frac{\partial L}{\partial \hat{x}^\alpha} = -2 g_{\alpha\beta}\hat{x}^\beta \frac{1}{2L} = -g_{\alpha\beta}\dot{x}^\beta$$

$$\frac{d}{d\lambda}\frac{\partial L}{\partial \hat{x}^\alpha} = L\frac{d}{d\tau}\frac{\partial L}{\partial \hat{x}^\alpha} = -L(g_{\alpha\beta}\ddot{x}^\beta + g_{\alpha\beta,\gamma}\dot{x}^\gamma \dot{x}^\beta)$$

combine:

$$\ddot{x}^\beta + \underbrace{\frac{1}{2} g^{\alpha\beta}\left[-g_{\mu\nu,\alpha} + g_{\nu\alpha,\mu} + g_{\alpha\mu,\nu}\right]\dot{x}^\mu \dot{x}^\nu}_{\Gamma^\beta{}_{\mu\nu}} = 0$$

Christoffel symbol of first kind:

$$\Gamma_{\gamma\alpha\beta} = \Gamma_{\gamma\alpha\beta}\mu\beta\alpha = \frac{1}{2}\left(g_{\mu\alpha,\beta} + g_{\beta\mu,\alpha} - g_{\alpha\beta,\mu}\right)$$

Light? Moves along null world lines $ds^2 = 0$ then the geodesic equation is:

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu{}_{\nu\rho}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda} = 0$$

this is the null geodesic.

8.4 Newtonian Limit (Heuristic Discussion)

Geometry in the solar System is approximately Euclidean and time independent. Ansatz: $g_{\alpha betaeta}(x) = \eta_{\alpha\beta} + 2\Psi_{\alpha\beta}(x)$ with $|\Psi_{\alpha betaeta}| << 1$ assume also that the considered velocities are much smaller than the speed of light. SR: $\underline{v} = \gamma \begin{pmatrix} 1 \\ \vec{v} \end{pmatrix}$ take $\dot{x}^\mu = \frac{dx^\mu}{d\tau} \approx \begin{pmatrix} 1 \\ \vec{0} \end{pmatrix} \Rightarrow d\tau = dt$

$$\ddot{x}^\mu + \underbrace{{\Gamma^\mu}_{\nu\lambda} \dot{x}^\nu \dot{x}^\lambda}_{{\Gamma^\mu}_{00}} = 0$$

For $\mu = 0$: $\frac{dx^0}{d\tau} \approx 1 \Rightarrow \frac{d^2x^0}{d\tau^2} \approx 0$ so ${\Gamma^0}_{00} \approx 0$ For $\mu = i$: ${\Gamma^i}_{00} = g^{i\mu}\Gamma_{\mu 00} \approx \eta^{i\mu}\Gamma_{\mu 00} = \Gamma_{i00} = -\frac{1}{2}g_{00,i} \approx -\Psi_{00,i}$ so $\frac{d^2x^i}{dt^2} \approx \frac{\partial \Psi_{00}}{\partial x^i} = -\frac{\partial \Phi}{\partial x^i}$

$$\boxed{\Psi_{00} \approx -\Phi + const}$$

For isolated systems, Ψ_{00} and Φ should vanish at large distances. $g_{00} = \eta_{00} + 2\Psi_{00} \approx -1 + 2\Phi$ ->$ds^2 = (-1 + 2\Phi)dt^2 + ...$ Consider Φ of a mass M: $\Phi(r) = -\frac{GM}{r}$ re-insert c: $ds^2 = -(1 - \frac{2GM}{rc^2})c^2dt^2 + ...$ Correction to flat spacetime already in the newtonian limit!

Schwarzschild-radius

$$R_s := \frac{2GM}{c^2}$$

$$ds^2 = -(1 - \frac{R_s}{r})c^2dt^2 + ...$$

Sun: $R = 7 \cdot 10^5 km, \quad R_s = 3\,km \Rightarrow \dfrac{R_s}{R} < 10^{-5}$

Earth: $R = 6378\,km, \quad R_s = 0.9\,cm \Rightarrow \dfrac{R_s}{R} < 10^{-9}$

The true Newtonian limit is $ds^2 = (1 - \frac{2GM}{rc^2})c^2dt^2 + (1 + \frac{2GM}{rc^2}dx^2)$

Newtons second law: $\ddot{\vec{x}} = -\nabla\phi$ corresponds to a motion, generated my a force. Einstein: This is free motion and $\ddot{\vec{x}} + \nabla\phi = 0$ is the limit of the general geodesic equation.

Rotation system: ${\Gamma^\mu}_{\nu\lambda}$ contains centrifugal and Coriolis forces.

Inertial system: ${\Gamma^\mu}_{00} = \dfrac{\partial\phi}{\partial x^i}$

Unification of inertial and gravity by ${\Gamma^\mu}_{\nu\lambda}$

8.5 Curved spaces

Euclids 5th postulate: parallel postulate ->non euclidean geometry. Straight lines on curved spaces: geodesics (minimal length) Example1: sphere S^2: space of a constant positive curvature then a geodesic is a part of along tudinal circle. This is a violation of the 5th postulate since every great circle through P intersects a given great circle ("no parallels") Ex. ") Pseudo-sphere H^2: space of constant negative curvature can be embedded into a 3-dim Minkowski spacetime, but no into a 3-dim Euclidean space.

$(x^0)^2 - (x^1)^2 - (x^2)^2 = r^2$ (hyperboloid) Through any point on the pseudosphere, there are infinitely many geodesicsthat do not intersect a given geodesic Gauss: Disquisitiones pure intrinsic description of two dimensional surfaces (no embedding needed) Surface of disk: Area A, circumference C: Euclidean Geometry: $C/R = 2\pi$, S^2: $C/R < 2\pi$, H^2: $C/R > 2\pi$ Later this was generalized to higher dimensions by Riemann.

9 Differential Geometry - Classic Formulation

Definition: An n-dimensional manifold M^n in a (second countable, Hausdorff topological) space in which every point P has a neighbourhood that can be mapped by a homeomorphic map σ to an open set of the n-dimensional Euclidean space $\mathbb{R}^n$.
The σ is a coordinate system or chart in the neighbourhood of P: $\sigma(P) = \left(x^1(P), ..., x^n(P)\right)$.
For an overlap of the neighbourhoods of two points P, Q with charts σ, σ' we demand $x'^a = f^a(x^\beta), x^\beta = \left(f^{-1}\right)^\beta (x'^a)$ being differentiable to all orders. Then M^n is called a differentiable manifold and a specific collection of charts that covers M^n is called an atlas.

Definition: An m-dimensional submanifold $M^m \in M^n$ is defined by $x^a = f^a(u^1, ..., u^m)$, $rank(\frac{\partial f^n}{\partial u^k}) = m$
Example are curves ($m = 1$) and hypersurfaces ($m = n-1$). The coordinate differentials transform as $dx'^a = \frac{\partial x'^a}{\partial x^\beta} dx^\beta = p^a_\beta(x) dx^\beta$ where we identify p^a_β as a generalization of the L^a_β in SR.
Note: $p^a_\beta p^\beta_c = \frac{\partial x'^a}{\partial x^\beta}\frac{\partial x^\beta}{\partial x'^c} = \delta^a_c$. $p^a_\beta := \frac{\partial x^a}{\partial x'^\beta}$ is the inverse, p^β_a is the inverse transposed of p^a_β.

Let $\Phi : M^n \to N^n$ be a map between manifolds. If Φ is bijective and has a C^∞ inverse it is called a diffeomorphism.

10 Vector- and Tensorfields

SR defined w.r.t. Lorentz transformation L^a_β, Here we have general coordinate transformations $p^a_\beta(x)$ There are different objects:

1. Scalar field: $\phi(x) = \phi'(x')$

2. Contravariant vector field: transforms like coordinate differentials: $v'^a(x') := \frac{\partial x'^a}{\partial x^\beta} v^\beta(x) = p^a_\beta(x) v^\beta(x)$
3. Covariant vector field: Transforms like the gradient of a scalar field: $w_a(x) :=) \frac{\partial \phi(x)}{\partial x^a}$
4. General tensor field of type (m,n): $T'^{a\beta...}{}_{cd...} = p^a_i p^\beta_c p^k_c p^l_d T^{ij...}{}_{kl...}$

10.1 Tensor Algebra

Properties of a tensor at a field point:

1. (m,n)-Tensors form a vector space over the real numbers: $\alpha A^a_{bc} + \beta B^a_{bc} = C^a_{bc}$
2. Product:$A^{a...}{}_{bc...} B^{i...}{}_{jk...} = D^{ai...}{}_{bcjk...}$
3. Contraction: $A^{a\beta...}{}_{ac...} = B^{b...}{}_{c...}$
4. Quotient Theorem: Let D be some object, if for any tensor B of type (r,s) $A^{ab...}{}_{cd...} := D^{ab...ij...}{}_{cd...kl...} B^{kl...}{}_{ij...}$ is a tensor of type (m,n), then D is a tensor of type $(n+s, m+r)$.

We demand now $ds^2 = g_{\mu,\nu}(x) dx^\mu dx^\nu$ so there should be a diagonal ($g_{ab} = g_{ba}$) tensor field of type $(0,2)$ with $det(g_{ab}(x)) \neq 0$.

11 Riemannian Normal Coordinates

EP-> $\delta \int d\tau = 0 \rightarrow \frac{dx^\alpha}{d\tau^2} + \Gamma^\alpha{}_{\beta\gamma} \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau} = 0$ Euclidean space: straight lines are convenient coordinate axes.
Here we use geodesics as coordinate lines. This is called Riemannian coordinate system (RCS). A geodesic $x^i(s)$ brings a tangent vector at x_0 which we call ξ^i.

$$g_{\alpha\beta} \xi^\alpha \xi^\beta = 1$$

Expand $x^i(s)$ in a neighbourhood of x_0 wrt s:

$$x^i(s) = x^i_0 + \xi^i s - \frac{1}{2} \Gamma^i{}_{jk}|_0 \, \xi^j \xi^k s^2 + ...$$

Define $y^i := \xi^i \cdot s$

$$x^i(s) = x^i_0 + y^i s - \frac{1}{2} \Gamma^i{}_{jk}|_0 \, y^j y^k + ...$$

defines a coordinate transformation $x \mapsto y(x)$ if n geodesics are considered. Find inverse transformations recursively, neglecting $y^i y^j$ we have $y^i = x^i - x^i_0$

$$y^i(s) = x^i - x^i_0 s + \frac{1}{2} \Gamma^i{}_{jk}|_0 \, (x^j - x^j_0)(x^k - x^k_0) + ...$$

The $\{y^i\}$ span a RCS because a geodesic through x_0 is given by $y^i(s) = \xi^i s$. $\overline{g}_{ij}(y)$ resp. $\overline{\Gamma}^i{}_{kl}(y)$ metric Christoffel symbol in the y-system.

$$\frac{d^2y^i}{ds^2} + \overline{\Gamma}^i{}_{kl}(y)\frac{dy^k}{ds}\frac{dy^l}{ds} = 0 \tag{13}$$

$$0 + \overline{\Gamma}^i{}_{kl}(\xi^m s)\xi^k\xi^l = 0 \tag{14}$$

$$(s=0) \quad \Rightarrow \quad \overline{\Gamma}^i{}_{kl}(0) = 0 \tag{15}$$

$$\Gamma_{ikl} + \Gamma_{lki} = g_{il,k} \Rightarrow \overline{g}_{il,k} = 0 \tag{16}$$

11.1 RNCS (Riemann normal coordinate system)

$\{x^i\} \to \{x'^i\}$ defines a new RCS: $y'^i = \xi'^i s$

$$y'^i = \left.\frac{\partial x'^i}{\partial x^k}\right|_0 \underbrace{\left.\frac{dx^k}{ds}\right|_0}_{\xi^k} s = p^i_k\big|_0 \xi^k s = p^i_k\big|_0 y^k$$

Linear transformation with constant coefficients. Choose this such that $\overline{g}_{\mu\nu}(0)$ is transformed to $\eta_{\mu\nu}$

Inhomogeneity of the gravitational field:

$$\overline{g}_{ik}(y) = \overline{g}_{ik}(0) + c_{iklm}y^l y^m + ...$$

This realizes a local inertial system:
Geodesics: $\frac{d^2y^\alpha}{ds^2} = 0 \to$ straight lines (freely falling frame).

12 Tensor Analysis

Consider a constant coordinate transformation as we know it from SR:

$$A'^\mu(x') = L^\mu_\nu A^\nu(x) \tag{17}$$

$$\frac{\partial A'^\mu(x')}{\partial x'^\alpha} = L^\mu_\nu \frac{\partial A^\nu(x)}{\partial x^\beta}\frac{\partial x^\beta}{\partial x'^\alpha} = L^\mu_\nu L^\beta_\alpha \frac{\partial A^\nu}{\partial x^\beta} \tag{18}$$

Now turn to a general coordinate transformation $L^\mu_\nu \to p^\mu_\nu(x)$:

$$A'^\mu(x') = p^\mu_\nu(x) A^\nu(x) \tag{19}$$

$$\frac{\partial A'^\mu(x')}{\partial x'^\alpha} = \frac{\partial}{\partial x'^\alpha} p^\mu_\nu(x) A^\nu(x) = p^\beta_\alpha p^\mu_\nu A^\nu_{,\beta} + p^\beta_\alpha p^\mu_{\nu,\beta} A^\nu \tag{20}$$

The second tern means that partial derivatives do not transform as tensor fields. Our goal is now to construct a new derivative ('covariant derivative') that transforms as a tensor:

1. Scalar field: $\boxed{\phi_{;\alpha} = \phi_{,\alpha}}$
2. Contravariant vector field:

$$\boxed{A^{\mu}{}_{;\alpha} := A^{\mu}{}_{,\alpha} + \Gamma^{\mu}{}_{\alpha\lambda} A^{\lambda}}$$

3. Covariant vector field:

$$\phi_{;\alpha} = (A^{\beta} B_{\beta})_{;\alpha} = A^{\beta}_{;\alpha} B_{\beta} + A^{\beta} B_{\beta;\alpha} = \phi_{,\alpha} = A^{\beta}_{,\alpha} B_{\beta} + A^{\beta} B_{\beta,\alpha} \tag{21}$$

$$A^{\beta} B_{\beta;\alpha} = A^{\beta}_{,\alpha} B_{\beta} + A^{\beta} B_{\beta,\alpha} - B_{\beta}(A^{\beta}_{,\alpha} + \Gamma^{\beta}_{\alpha\nu} A^{\nu}) = A^{\beta} B_{\beta,\alpha} - \Gamma^{\beta}{}_{\alpha\nu} B_{\beta} A^{\nu} \tag{22}$$

$$\boxed{B_{\beta;a} = B_{\beta,\alpha} - \Gamma^{\rho}{}_{\beta\alpha} B_{\rho}} \tag{23}$$

4. General tensor field:

$$T^{\alpha\beta\ldots}{}_{\mu\nu\ldots;\lambda} = T^{\alpha\beta\ldots}{}_{\mu\nu\ldots,\lambda} + \Gamma^{\alpha}{}_{\lambda\rho} T^{\rho\beta\ldots}{}_{\mu\nu\ldots} + \ldots - \Gamma^{\rho}{}_{\lambda\mu} T^{\alpha\beta\ldots}{}_{\rho\nu\ldots} - \ldots$$

One can show that $g_{\mu\nu;\rho} = 0$, $g^{\mu\nu}{}_{;\rho} = 0$ advantage indices with covariant derivation can be raised and lowered as ordinary indices eg $A^{\alpha}{}_{;\beta} = (g^{\alpha,\nu} A_{\nu})_{;\beta} = g^{\alpha\nu} A_{\nu;\beta}$, etc.