Coupled Oscillations

Begin by sketching the coupled spring system.

Solving the equations

You are going to solve for the motion of this system as follows.

1. Write an equation for the acceleration of mass 1 (x_1'') in terms of the spring constant, k, the coupling constant, k_c, and the positions of the two masses, x_1 and x_2.
2. Repeat this exercise for the accelerations of x_2.
3. Rearrange the equations above to be of the forms $x_1'' + Ax_1 + Bx_2 = 0$ and $x_2'' + Cx_1 + Dx_2$. These are second order differential equations.
4. By adding and then subtracting the two equations from step 3 get two equations in terms of only $(x_1 + x_2)$ and $(x_1 - x_2)$.
5. By using the substitutions $q_1 = x_1 + x_2$ and $q_2 = x_1 - x_2$ write the equations in the form:
$$q_1'' + \omega_1^2 q_1 = 0$$
$$q_2'' + \omega_2^2 q_2 = 0$$
. How would you describe this motion in words?
6. Using your knowledge of differential equations you should be agree that the solutions to these two equations are:
$$q_1 = C_1 \cos \omega_1 t + C_2 \sin \omega_1 t$$
$$q_2 = C_3 \cos \omega_2 t + C_4 \sin \omega_2 t$$
7. Write your definitions of q_1 and q_2 to get x_1 and x_2 in terms of q_1 and q_2. Substitute in the equations above to get x_1 and x_2 in terms of sines and cosines.
8. By considering the following boundary conditions you should be able to simplify your equations:
$$x_1(0) = A \qquad x_1'(0) = 0$$
$$x_2(0) = B \qquad x_2'(0) = 0$$
9. You should now have equations, each with two terms on the right. These are the solutions to the equations - well done!

Exploring the solution

We are going to look at particular boundary conditions as follows.

Symmetric

For the solution both x_1 and x_2 when the initial conditions are $x_1(0) = x_2(0) = A$. Describe the motion by thinking about how the relative positions of x_1 and x_2 change.

Asymmetric

For this solution the initial condition is that $B = -A$. Again describe the motion.

General for one displaced mass

For this solution, the initial condition is that $x_1(0) = A$ and $x_2(0) = 0$. Substitute these values into your solutions.

Now, use the following relations to re-write your solutions as products rather than sums:

$$\cos\alpha\cos\beta = \frac{1}{2}\left(\cos\alpha - \beta + \cos\alpha + \beta\right)$$

$$\sin\alpha\sin\beta = \frac{1}{2}\left(\cos\alpha - \beta - \cos\alpha + \beta\right)$$

Can you describe this motion?