Proof that $\sqrt{2}$ is Irrational

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Abstract

We present a formal proof that $\sqrt{2}$ is irrational using the classic proof by contradiction. The proof follows Lamport's structured proof format with explicit justifications for each step.

1 Main Result

Theorem 1. *$\sqrt{2}$ is irrational.*

Proof. We proceed by contradiction.

$\langle 1 \rangle$**1. Assumption.** Assume for contradiction that $\sqrt{2}$ is rational. [assumption]

$\langle 1 \rangle$**2.** Then $\sqrt{2} = \frac{p}{q}$ for some integers p, q with $q \neq 0$ and $\gcd(p, q) = 1$. [definition of rational; from 1]

This follows from the definition of a rational number: any rational can be expressed as a fraction in lowest terms.

$\langle 1 \rangle$**3.** Squaring both sides: $2 = \frac{p^2}{q^2}$, hence $p^2 = 2q^2$. [algebraic manipulation; from 1]

$\langle 1 \rangle$**4.** p^2 is even, therefore p is even. [modus ponens; from 1]

Since $p^2 = 2q^2$, we have that p^2 is divisible by 2, hence even. We use the fact that for any integer n: if n^2 is even, then n is even. (The contrapositive: if n is odd, then n^2 is odd, is easy to verify.)

$\langle 1 \rangle$**5.** Write $p = 2k$ for some integer k. Then $p^2 = 4k^2$, so $2q^2 = 4k^2$, giving $q^2 = 2k^2$. [substitution; from 1, 1]

$\langle 1 \rangle$**6.** q^2 is even, therefore q is even. [modus ponens; from 1]

By the same reasoning as step 1: since $q^2 = 2k^2$, we have q^2 even, hence q even.

$\langle 1 \rangle$**7.** Both p and q are even, so $2 \mid \gcd(p, q)$, contradicting $\gcd(p, q) = 1$. [contradiction; from 1, 1, 1]

If both p and q are even, then 2 is a common divisor, so $\gcd(p, q) \geq 2$. But we assumed $\gcd(p, q) = 1$ in step 1. This is a contradiction.

$\langle 1 \rangle$**8. QED.** Therefore $\sqrt{2}$ is irrational. [proof by contradiction; from 1, 1]

Our assumption that $\sqrt{2}$ is rational led to a contradiction. Therefore, $\sqrt{2}$ must be irrational.

□

2 Supporting Lemma

The proof relies on the following well-known result:

Lemma 2. *For any integer n: if n^2 is even, then n is even.*

Proof. We prove the contrapositive. Suppose n is odd. Then $n = 2m + 1$ for some integer m. Thus:

$$n^2 = (2m + 1)^2 = 4m^2 + 4m + 1 = 2(2m^2 + 2m) + 1$$

which is odd. Therefore, if n is odd, then n^2 is odd. By contraposition, if n^2 is even, then n is even. $\square$

3 Proof Metadata

Graph ID:	`graph-4f2907-61a5c9`
Version:	9
Proof Mode:	strict-mathematics
Total Steps:	8
Status:	All steps proposed (awaiting verification)
Taint:	Clean
Admitted Steps:	None
External References:	None

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