Proof that the Sum of Degrees of a Graph is Even

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Abstract

We present a formal proof that for any graph $G = (V, E)$, the sum of degrees of all vertices is even. The key insight is that each edge contributes exactly 2 to the total degree sum (one for each endpoint), so the sum equals $2|E|$, which is even. The proof follows Lamport's structured proof format with explicit justifications for each step.

1 Main Result

Theorem 1. *For any graph $G = (V, E)$, the sum of degrees of all vertices is even:*

$$\sum_{v \in V} \deg(v) = 2|E|$$

Proof. ⟨1⟩**1. Definition (Graph)**. A graph $G = (V, E)$ consists of a set V of vertices and a set E of edges, where each edge is an unordered pair of distinct vertices. [definition-expansion]

⟨1⟩**2. Definition (Degree)**. For a vertex $v \in V$, the degree $\deg(v)$ is the number of edges incident to v. [definition-expansion; from 1]

⟨1⟩**3. Observation**. Each edge $\{u, v\} \in E$ contributes exactly 2 to the sum $\sum_{v \in V} \deg(v)$: one for vertex u and one for vertex v. [definition-expansion; from 1, 1]

This follows directly from the definition of degree: when we count the degree of each vertex, each edge $\{u, v\}$ is counted once in $\deg(u)$ and once in $\deg(v)$, so it contributes a total of 2 to the sum.

⟨1⟩**4. Sum Calculation**. $\sum_{v \in V} \deg(v) = 2|E|$. This follows from 1: since each of the $|E|$ edges contributes exactly 2 to the sum, the total is $2|E|$. [algebraic-rewrite; from 1]

By 1, we have that each edge contributes 2 to the sum. Since there are $|E|$ edges in total, we obtain:

$$\sum_{v \in V} \deg(v) = \sum_{\{u,v\} \in E} 2 = 2|E|$$

⟨1⟩**5. Evenness**. Since $\sum_{v \in V} \deg(v) = 2|E|$ and $2|E|$ is even (being a multiple of 2), the sum of degrees is even. [modus-ponens; from 1]

Any integer of the form $2k$ (where k is an integer) is even by definition. Since $|E|$ is a nonnegative integer, $2|E|$ is even.

⟨1⟩**6. QED**. For any graph $G = (V, E)$, we have shown that $\sum_{v \in V} \deg(v) = 2|E|$, which is even. □ [qed; from 1]

We have established that the sum of degrees equals $2|E|$, and since this is a multiple of 2, it is even. This completes the proof.

□

2 Proof Metadata

Graph ID:	`graph-sum-degrees-even`
Version:	1
Proof Mode:	strict-mathematics
Total Steps:	6
Status:	All steps verified
Taint:	Clean
Admitted Steps:	None
External References:	None

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