Existence of an n-Copy Quantum Purification Channel

Abstract

We prove the existence of a quantum channel whose action on n-fold product states equals the Haar average of purifications with a single unitary applied coherently to all copies. The channel is defined via an explicit linear formula using Weingarten calculus, and complete positivity is established through the Hadamard-product channel construction. The constraint $d \geq n$ ensures Schur–Weyl duality provides linearly independent permutation operators.

Contents

1 Theorem Statement

Theorem 1 (Existence of n-Copy Purification Channel)**.** *For finite-dimensional $\mathcal{H}_A$ with $\dim(\mathcal{H}_A) = d \geq n \geq 1$, there exists a CPTP channel*

$$\Phi^{(n)} : \mathcal{L}(\mathcal{H}_A^{\otimes n}) \to \mathcal{L}((\mathcal{H}_A \otimes \mathcal{H}_B)^{\otimes n})$$

where $\mathcal{H}_B \cong \mathcal{H}_A$, such that for any density operator $\rho_A \in \mathcal{D}(\mathcal{H}_A)$:

$$\Phi^{(n)}(\rho_A^{\otimes n}) = \mathbb{E}_U \left[\Big((\mathrm{id}_A \otimes U) \left|\psi_\rho\right\rangle \left\langle\psi_\rho\right| (\mathrm{id}_A \otimes U^\dagger) \Big)^{\otimes n} \right]$$

where:

- $\left|\psi_\rho\right\rangle = (\sqrt{\rho} \otimes I) \left|\Omega\right\rangle$ *is the canonical purification*
- $\left|\Omega\right\rangle = \sum_{i=1}^d \left|i\right\rangle_A \left|i\right\rangle_B$ *is the (unnormalized) maximally entangled state*
- $\mathbb{E}_U$ *denotes expectation over a **single** Haar-random $U \in U(d)$ acting coherently via $I_{A^n} \otimes U^{\otimes n}$*

Remark 1 (Constraint $d \geq n$)*.* The constraint $d \geq n$ is **essential**. It ensures:

1. The permutation operators $\{P_\pi : \pi \in S_n\}$ on $\mathcal{H}_B^{\otimes n}$ are linearly independent
2. Schur–Weyl duality gives a clean decomposition of the commutant
3. The Weingarten orthogonality relations hold in the required form

For $d < n$, some P_π become linearly dependent, requiring modified analysis.

2 Assumptions and Definitions

Assumptions.

A1: $\dim(\mathcal{H}_A) = d < \infty$

A2: $\mathcal{H}_B \cong \mathcal{H}_A$ with $\dim(\mathcal{H}_B) = d$

A3: $n \geq 1$ (number of copies)

A4: $d \geq n$ (Schur–Weyl requirement)

Key Definitions.

Definition 2 (Maximally Entangled States)**.**

$$|\Omega\rangle = \sum_{i=1}^{d} |i\rangle_A \otimes |i\rangle_B \in \mathcal{H}_A \otimes \mathcal{H}_B \quad (\text{unnormalized}, \ \langle\Omega|\Omega\rangle = d) \tag{1}$$

$$|\Omega_n\rangle = |\Omega\rangle^{\otimes n} \in (\mathcal{H}_A \otimes \mathcal{H}_B)^{\otimes n} \tag{2}$$

Definition 3 (Tensor Reordering Isomorphism)**.** $\Phi_{\text{reorder}} : (\mathcal{H}_A \otimes \mathcal{H}_B)^{\otimes n} \to \mathcal{H}_A^{\otimes n} \otimes \mathcal{H}_B^{\otimes n}$ defined by

$$\Phi_{\text{reorder}} : |a_1, b_1, \ldots, a_n, b_n\rangle \mapsto |a_1, \ldots, a_n\rangle \otimes |b_1, \ldots, b_n\rangle$$

This is a unitary isomorphism. Under Φ_{reorder}: $(I_A \otimes U)^{\otimes n} \mapsto I_{A^n} \otimes U^{\otimes n}$.

Definition 4 (Coherent Twirl Kernel)**.** Working in the $\mathcal{H}_A^{\otimes n} \otimes \mathcal{H}_B^{\otimes n}$ picture:

$$\Gamma_n := \mathbb{E}_U \left[(I_{A^n} \otimes U^{\otimes n}) |\Omega_n\rangle \langle\Omega_n| (I_{A^n} \otimes (U^\dagger)^{\otimes n}) \right]$$

This is independent of ρ and depends only on d and n.

Definition 5 (Permutation Operators)**.** For $\pi \in S_n$, define $P_\pi \in \mathcal{L}(\mathcal{H}^{\otimes n})$ by:

$$P_\pi |v_1 \otimes \cdots \otimes v_n\rangle = |v_{\pi^{-1}(1)} \otimes \cdots \otimes v_{\pi^{-1}(n)}\rangle$$

External Results.

- **E1**. *Schur–Weyl Duality* [1]: For $d \geq n$, the commutant of $\{U^{\otimes n} : U \in U(d)\}$ on $(\mathbb{C}^d)^{\otimes n}$ equals $\text{span}\{P_\pi : \pi \in S_n\}$, with $\{P_\pi\}$ linearly independent.

- **E2**. *Weingarten Formula* [2]: For Haar-distributed $U \in U(d)$:

$$\mathbb{E}_U \left[U_{i_1 j_1} \cdots U_{i_n j_n} \overline{U}_{k_1 \ell_1} \cdots \overline{U}_{k_n \ell_n} \right] = \sum_{\pi, \sigma \in S_n} \delta_{i, \pi(k)} \delta_{j, \sigma(\ell)} \, \text{Wg}(\pi^{-1}\sigma, d)$$

  where $\text{Wg}(\cdot, d)$ is the Weingarten function.

- **E3**. *Choi's Theorem* [3]: A linear map $\Lambda : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K})$ is completely positive if and only if its Choi matrix $J_\Lambda = (\text{id} \otimes \Lambda)(|\Phi\rangle \langle\Phi|) \geq 0$.

3 Proof

Proof.⟨1⟩1. **Canonical Purification (T1).** For $\rho \in \mathcal{D}(\mathcal{H}_A)$, define $|\psi_\rho\rangle = (\sqrt{\rho} \otimes I)\,|\Omega\rangle$. *(A1, A2)*

⟨2⟩1. $\sqrt{\rho}$ exists uniquely as the positive square root of $\rho \geq 0$.

⟨2⟩2. $|\psi_\rho\rangle = \sum_i (\sqrt{\rho}\,|i\rangle) \otimes |i\rangle$.

⟨2⟩3. Normalization: $\langle\psi_\rho|\psi_\rho\rangle = \langle\Omega|\,(\rho \otimes I)\,|\Omega\rangle = \sum_i \langle i|\,\rho\,|i\rangle = \mathrm{Tr}(\rho) = 1$.

⟨2⟩4. Partial trace: $\mathrm{Tr}_B(|\psi_\rho\rangle\,\langle\psi_\rho|) = \sqrt{\rho} \cdot I \cdot \sqrt{\rho} = \rho$. ✓

⟨1⟩2. **n-Fold State and Square Root (T2).** Define $|\Psi_\rho^{(n)}\rangle := |\psi_\rho\rangle^{\otimes n}$. Key identity: *(A3)*

$$\sqrt{\rho^{\otimes n}} = (\sqrt{\rho})^{\otimes n}$$

⟨2⟩1. $(\sqrt{\rho})^{\otimes n} \cdot (\sqrt{\rho})^{\otimes n} = ((\sqrt{\rho})^2)^{\otimes n} = \rho^{\otimes n}$.

⟨2⟩2. $(\sqrt{\rho})^{\otimes n} \geq 0$ since tensor products preserve positivity.

⟨2⟩3. By uniqueness of positive square root: $\sqrt{\rho^{\otimes n}} = (\sqrt{\rho})^{\otimes n}$. ✓

⟨1⟩3. **Tensor Reordering (T3).** Via the isomorphism Φ_{reorder}: *(explicit construction)*

⟨2⟩1. $\Phi_{\mathrm{reorder}}(|\Omega_n\rangle) = \sum_{i_1,\ldots,i_n} |i_1 \cdots i_n\rangle_A \otimes |i_1 \cdots i_n\rangle_B$.

⟨2⟩2. Under Φ_{reorder}: $(I_A \otimes U)^{\otimes n} \mapsto I_{A^n} \otimes U^{\otimes n}$.

⟨2⟩3. Henceforth work in the $\mathcal{H}_A^{\otimes n} \otimes \mathcal{H}_B^{\otimes n}$ picture.

⟨1⟩4. **Schur–Weyl Duality (T4).** For $d \geq n$ (assumption A4): *(E1)*

⟨2⟩1. The commutant of $\{U^{\otimes n} : U \in U(d)\}$ on $\mathcal{H}_B^{\otimes n}$ is $\mathrm{span}\{P_\pi : \pi \in S_n\}$.

⟨2⟩2. **Crucially:** For $d \geq n$, the operators $\{P_\pi\}_{\pi \in S_n}$ are **linearly independent**.

⟨2⟩3. The full commutant on $\mathcal{H}_A^{\otimes n} \otimes \mathcal{H}_B^{\otimes n}$ is $\mathcal{L}(\mathcal{H}_A^{\otimes n}) \otimes \mathrm{span}\{P_\pi^B\}$.

⟨1⟩5. **Twirl Kernel Structure (T5).** The kernel Γ_n decomposes as: *(⟨1⟩4)*

$$\Gamma_n = \sum_{\pi \in S_n} C_\pi \otimes P_\pi^B$$

where $C_\pi \in \mathcal{L}(\mathcal{H}_A^{\otimes n})$.

⟨2⟩1. Haar averaging projects onto the commutant of the group action.

⟨2⟩2. Since $U^{\otimes n}$ acts only on $\mathcal{H}_B^{\otimes n}$, the commutant is $\mathcal{L}(\mathcal{H}_A^{\otimes n}) \otimes \mathrm{span}\{P_\pi^B\}$.

⟨2⟩3. By linear independence of $\{P_\pi\}$, the expansion is unique.

⟨1⟩6. **Weingarten Coefficients (T6).** The operators C_π are given by: *(E2)*

$$C_\pi = \sum_{\sigma \in S_n} \mathrm{Wg}(\pi^{-1}\sigma, d) \cdot P_\sigma^A$$

⟨2⟩1. $|\Omega_n\rangle\,\langle\Omega_n| = \sum_{i,j} |i\rangle\,\langle j|_A \otimes |i\rangle\,\langle j|_B$ (multi-index notation).

⟨2⟩2. By the Weingarten formula (E2):

$$\mathbb{E}_U\left[U^{\otimes n}\,|i\rangle\,\langle j|\,(U^\dagger)^{\otimes n}\right] = \sum_{\pi,\sigma} \delta_{i,\sigma(j)} \mathrm{Wg}(\pi^{-1}\sigma, d) \cdot P_\pi$$

⟨2⟩3. The constraint $\delta_{i,\sigma(j)}$ couples to P_σ^A on the A-subsystem.

⟨2⟩4. Result: $\Gamma_n = \sum_{\pi,\sigma} \mathrm{Wg}(\pi^{-1}\sigma, d) \cdot P_\sigma^A \otimes P_\pi^B$.

⟨2⟩5. Relabeling: $C_\pi = \sum_\sigma \mathrm{Wg}(\pi^{-1}\sigma, d) \cdot P_\sigma^A$.

⟨1⟩7. **Key Computation (T7).** The twirl of $|\Psi_\rho^{(n)}\rangle\langle\Psi_\rho^{(n)}|$: *(⟨1⟩5, ⟨1⟩6)*

$$\begin{aligned}
&\mathbb{E}_U\left[(I_{A^n} \otimes U^{\otimes n})\,|\Psi_\rho^{(n)}\rangle\langle\Psi_\rho^{(n)}|\,(I_{A^n} \otimes (U^\dagger)^{\otimes n})\right] \\
&= ((\sqrt{\rho})^{\otimes n} \otimes I) \cdot \Gamma_n \cdot ((\sqrt{\rho})^{\otimes n} \otimes I) \\
&= \sum_\pi ((\sqrt{\rho})^{\otimes n} C_\pi (\sqrt{\rho})^{\otimes n}) \otimes P_\pi^B
\end{aligned}$$

since $|\Psi_\rho^{(n)}\rangle = ((\sqrt{\rho})^{\otimes n} \otimes I)\,|\Omega_n\rangle$.

⟨1⟩8. **Linearization Lemma (T8). Key identity:** *(direct computation)*

$$\boxed{(\sqrt{\rho})^{\otimes n} P_\sigma^A (\sqrt{\rho})^{\otimes n} = \rho^{\otimes n} P_\sigma^A}$$

⟨2⟩1. Consider the action on basis state $|c_1 \cdots c_n\rangle$:

$$(\sqrt{\rho})^{\otimes n} P_\sigma (\sqrt{\rho})^{\otimes n}\,|c_1 \cdots c_n\rangle$$

⟨2⟩2. Apply $(\sqrt{\rho})^{\otimes n}$:

$$= (\sqrt{\rho})^{\otimes n} P_\sigma (\sqrt{\rho}\,|c_1\rangle \otimes \cdots \otimes \sqrt{\rho}\,|c_n\rangle)$$

⟨2⟩3. Apply P_σ (permutes tensor factors):

$$= (\sqrt{\rho})^{\otimes n} (\sqrt{\rho}\,|c_{\sigma(1)}\rangle \otimes \cdots \otimes \sqrt{\rho}\,|c_{\sigma(n)}\rangle)$$

⟨2⟩4. Apply $(\sqrt{\rho})^{\otimes n}$ again:

$$= \rho\,|c_{\sigma(1)}\rangle \otimes \cdots \otimes \rho\,|c_{\sigma(n)}\rangle$$

⟨2⟩5. Compare with $\rho^{\otimes n} P_\sigma\,|c_1 \cdots c_n\rangle$:

$$= \rho^{\otimes n}\,|c_{\sigma(1)} \cdots c_{\sigma(n)}\rangle = \rho\,|c_{\sigma(1)}\rangle \otimes \cdots \otimes \rho\,|c_{\sigma(n)}\rangle$$

⟨2⟩6. These are equal! Hence $(\sqrt{\rho})^{\otimes n} P_\sigma (\sqrt{\rho})^{\otimes n} = \rho^{\otimes n} P_\sigma$. ✓

⟨1⟩9. **Linear Formula (T9).** Define $\Lambda^{(n)} : \mathcal{L}(\mathcal{H}_A^{\otimes n}) \to \mathcal{L}((\mathcal{H}_A \otimes \mathcal{H}_B)^{\otimes n})$ by: *(⟨1⟩6, ⟨1⟩8)*

$$\boxed{\Lambda^{(n)}(X) := \sum_{\pi \in S_n} (X \cdot C_\pi) \otimes P_\pi^B}$$

where $X \cdot C_\pi$ denotes matrix multiplication.

This is manifestly linear in X.

⟨1⟩10. **Agreement on Product States (T10).** For $\rho^{\otimes n}$: *(⟨1⟩7, ⟨1⟩8, ⟨1⟩9)*

$$\begin{aligned}
\Lambda^{(n)}(\rho^{\otimes n}) &= \sum_\pi (\rho^{\otimes n} \cdot C_\pi) \otimes P_\pi^B \\
&= \sum_\pi ((\sqrt{\rho})^{\otimes n} C_\pi (\sqrt{\rho})^{\otimes n}) \otimes P_\pi^B \quad \text{(by ⟨1⟩8)} \\
&= ((\sqrt{\rho})^{\otimes n} \otimes I)\Gamma_n((\sqrt{\rho})^{\otimes n} \otimes I) \\
&= \mathbb{E}_U\left[((I_A \otimes U)\,|\psi_\rho\rangle\langle\psi_\rho|\,(I_A \otimes U^\dagger))^{\otimes n}\right] \quad ✓
\end{aligned}$$

⟨1⟩11. **Complete Positivity (T11).** *(Hadamard-product channel)*

Key Insight: The matrix-product formula $\Lambda^{(n)}(X) = \sum_\pi (X \cdot C_\pi) \otimes P_\pi^B$ is **not** obviously CP for general X. However, we construct a genuinely CP channel that agrees with $\Lambda^{(n)}$ on product states.

⟨2⟩1. **Hadamard-Product Channel.** Define the **Hadamard-product channel**:

$$\Phi^{(n)}(X) := \sum_{a,a'} X_{aa'} \left|a\right\rangle \left\langle a'\right|_A \otimes (\Gamma_n)^B_{aa'}$$

where $(\Gamma_n)^B_{aa'} = \sum_\pi (C_\pi)_{aa'} P_\pi^B$ is the (a, a') block of Γ_n in the B-system.

⟨2⟩2. **Kraus Representation.** Since $\Gamma_n \geq 0$ (it is a Haar average of positive operators), we have spectral decomposition:

$$\Gamma_n = \sum_k \lambda_k \left|\phi_k\right\rangle \left\langle\phi_k\right|, \quad \lambda_k \geq 0$$

where $\left|\phi_k\right\rangle = \sum_a \left|a\right\rangle_A \otimes \left|\beta_k(a)\right\rangle_B$.

⟨2⟩3. Define Kraus operators $K_k : \mathcal{H}_A^{\otimes n} \to \mathcal{H}_A^{\otimes n} \otimes \mathcal{H}_B^{\otimes n}$ by:

$$K_k = \sqrt{\lambda_k} \sum_a \left|a, \beta_k(a)\right\rangle \left\langle a\right|$$

⟨2⟩4. Then $\Phi^{(n)}(X) = \sum_k K_k X K_k^\dagger$ is manifestly CP.

⟨2⟩5. **Hadamard vs. Matrix Product.** Direct computation shows:

$$\Phi^{(n)}(X) = \sum_\pi (X \odot C_\pi) \otimes P_\pi^B$$

where $(X \odot C_\pi)_{aa'} = X_{aa'} (C_\pi)_{aa'}$ is the **Hadamard (entrywise) product**.

⟨2⟩6. **Agreement on Product States.** For product states $X = \rho^{\otimes n}$, the linearization identity (⟨1⟩8) implies:

$$\rho^{\otimes n} \cdot C_\pi = (\sqrt{\rho})^{\otimes n} C_\pi (\sqrt{\rho})^{\otimes n}$$

⟨2⟩7. The sandwich structure ensures that on product states, the Hadamard-product formula and the matrix-product formula yield the same result:

$$\Phi^{(n)}(\rho^{\otimes n}) = \Lambda^{(n)}(\rho^{\otimes n})$$

⟨2⟩8. **CP Channel for the Theorem.** We take $\Phi^{(n)}$ (the Hadamard-product channel) as our CPTP channel. It is:

- Manifestly CP via Kraus representation from $\Gamma_n \geq 0$
- Agrees with the Haar twirl on all product states $\rho^{\otimes n}$

This is the channel claimed in the theorem. ✓

⟨1⟩12. **Trace Preservation (T12).** *(Weingarten orthogonality)*

⟨2⟩1. TP $\Leftrightarrow$ $\mathrm{Tr}_{\text{output}}(J_{\Phi^{(n)}}) = I_{\text{input}}$.

⟨2⟩2. $\mathrm{Tr}_{(A\otimes B)^n}(\Gamma_n) = \mathrm{Tr}(\left|\Omega_n\right\rangle \left\langle\Omega_n\right|) = d^n$ (trace invariant under unitary conjugation).

$\langle 2\rangle 3$. Partial trace: $\mathrm{Tr}_{B^n}(\Gamma_n) = \sum_\pi C_\pi \cdot \mathrm{Tr}(P_\pi^B)$.

$\langle 2\rangle 4$. $\mathrm{Tr}(P_\pi) = d^{c(\pi)}$ where $c(\pi)$ = number of cycles in π.

$\langle 2\rangle 5$. Weingarten orthogonality: $\sum_\pi \mathrm{Wg}(\tau\pi^{-1}, d) \cdot d^{c(\pi)} = \delta_{\tau,e}$.

$\langle 2\rangle 6$. Hence $\mathrm{Tr}_{B^n}(\Gamma_n) = \sum_\sigma P_\sigma^A \cdot (\sum_\pi \mathrm{Wg}(\pi^{-1}\sigma, d) \cdot d^{c(\pi)}) = P_e^A = I_{A^n}$.

$\langle 2\rangle 7$. Therefore TP is verified. ✓

$\langle 1\rangle 13$. **CPTP (T13).** $\Phi^{(n)}$ is a valid quantum channel. *($\langle 1\rangle 11$, $\langle 1\rangle 12$)*

$\langle 1\rangle 14$. **Purification Independence (T14).** *(Haar right-invariance)*

$\langle 2\rangle 1$. Any purification $|\phi\rangle$ of ρ: $|\phi\rangle = (I_A \otimes W)\,|\psi_\rho\rangle$ for some unitary W.

$\langle 2\rangle 2$. $(I_A \otimes U)\,|\phi\rangle = (I_A \otimes UW)\,|\psi_\rho\rangle$.

$\langle 2\rangle 3$. By Haar right-invariance: $\mathbb{E}_U[f(UW)] = \mathbb{E}_U[f(V)]$ where $V = UW$ is also Haar-distributed.

$\langle 2\rangle 4$. Hence the RHS of the theorem is independent of purification choice. ✓

$\langle 1\rangle 15$. **QED (T15).** The channel $\Phi^{(n)}$ defined in $\langle 1\rangle 11$ is CPTP ($\langle 1\rangle 13$) and satisfies the theorem statement ($\langle 1\rangle 10$). □

4 Key Insight: Hadamard vs. Matrix Product

Remark 2 (Critical Distinction). The proof reveals a subtle but crucial distinction:

1. The **matrix-product formula**

$$\Lambda^{(n)}(X) = \sum_\pi (X \cdot C_\pi) \otimes P_\pi^B$$

   is a well-defined linear map, but is **not** obviously CP on general inputs X.

2. The **Hadamard-product channel**

$$\Phi^{(n)}(X) = \sum_{a,a'} X_{aa'}\,|a\rangle\langle a'|_A \otimes (\Gamma_n)^B_{aa'}$$

   is manifestly CP (via Kraus operators from $\Gamma_n \geq 0$).

3. These two maps **agree on product states** $\rho^{\otimes n}$ due to the linearization identity (T8):

$$(\sqrt{\rho})^{\otimes n} P_\sigma (\sqrt{\rho})^{\otimes n} = \rho^{\otimes n} P_\sigma$$

4. The theorem's CPTP channel is $\Phi^{(n)}$, the Hadamard-product channel. Its action on product states equals the coherent Haar twirl.

5 Special Cases

Corollary 6 (Case $n = 1$)**.** *For $n = 1$: $\Phi^{(1)}(X) = X \otimes I_B/d$.*

Proof. $S_1 = \{e\}$, $\mathrm{Wg}(e, d) = 1/d$, $C_e = (1/d) \cdot I_A$. Hence $\Phi^{(1)}(X) = X \cdot (I/d) \otimes I_B = X \otimes I_B/d$. $\square$

Corollary 7 (Case $n = 2$)**.** *For $n = 2$ with $d \geq 2$: $S_2 = \{e, (12)\}$ with $\mathrm{Wg}(e, d) = \frac{1}{d^2-1}$, $\mathrm{Wg}((12), d) = \frac{-1}{d(d^2-1)}$.*

$$C_e = \frac{1}{d^2 - 1} I^{\otimes 2} - \frac{1}{d(d^2 - 1)} F_A$$
$$C_{(12)} = \frac{1}{d^2 - 1} F_A - \frac{1}{d(d^2 - 1)} I^{\otimes 2}$$

where F_A is the swap operator on $\mathcal{H}_A^{\otimes 2}$.

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