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Reviews

Normed Space $(\vec{x}, ||\cdot||)$ $(\vec{x} \in X)$ is induced from inner product $(\vec{x}, <\cdot,\cdot>)$ $(<\cdot,\cdot>: X \times X \to \mathbb{R})$

$||\vec{x}|| = \sqrt{<\vec{x},\vec{x}>}$

Cauchy-Schwarz Inequality
$|<\vec{x},\vec{y}>| \leq ||\vec{x}|| \cdot ||\vec{y}||$

Proof $||\vec{x}+\vec{y}|| \leq ||\vec{x}|| + ||\vec{y}||$:

$||\vec{x}+\vec{y}||^2 = <\vec{x}+\vec{y}, \vec{x}+\vec{y}>$

$||\vec{x}+\vec{y}||^2 = <\vec{x},\vec{x}> + <\vec{x},\vec{y}> + <\vec{y},\vec{x}> + <\vec{y},\vec{y}>$

$||\vec{x}+\vec{y}||^2 = ||\vec{x}||^2 + ||\vec{y}||^2 + 2<\vec{x},\vec{y}>$

$||\vec{x}+\vec{y}||^2 \leq ||\vec{x}||^2 + ||\vec{y}||^2 + 2|<\vec{x},\vec{y}>| \quad (a \leq |a|)$

Using $|<\vec{x},\vec{y}>| \leq ||\vec{x}|| \cdot ||\vec{y}||$

$||\vec{x}+\vec{y}||^2 \leq ||\vec{x}||^2 + ||\vec{y}||^2 + 2||\vec{x}|| \cdot ||\vec{y}||$

$||\vec{x}+\vec{y}||^2 \leq (||\vec{x}|| + ||\vec{y}||)^2$

$||\vec{x}+\vec{y}|| \leq ||\vec{x}|| + ||\vec{y}||$

Matrix Multiplication

Not Communtative: $AB \neq BA$

Associative: $(AB)C = A(BC)$

$\vec{x} \in \mathbb{R}^d$
$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix}$$
$\vec{x}^T = [x_1, x_2, ..., x_d]$

$\vec{x}^T \cdot \vec{x} = x_1^2, x_2^2, ..., x_d^2$

$$\vec{x} \cdot \vec{x}^T = \begin{bmatrix} x_1^2 & x_1x_2 & \dots & x_1x_d \\ x_2x_1 & x_2^2 & \dots & x_2x_d \\ \vdots & \vdots & \ddots & \vdots \\ x_dx_1 & x_dx_2 & \dots & x_d^2 \end{bmatrix}$$

Vector Space

$$\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$\vec{a}, \vec{b}$,and $\vec{c}$ are basis of $\vec{v}$

$dim(\vec{v})$= size of basis

Basis property
- Every vector is a linear combination of basis.

Linear Independent
- Span(B) $= < e_1, e_2, ..., e_n > = \vec{v}$
- e_i is linear independent if
- There exists a constant $c_1, c_2, ..., c_n$ not all zero such that $\sum_{i=1}^{n} c_i e_i \neq 0$

Basically write a vector as multiple of others

Orthogonal Basis

Example: $\vec{e_1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \vec{e_2} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \vec{e_3} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$||\vec{e_1}|| = ||\vec{e_2}|| = ||\vec{e_3}|| = 1$

$< \vec{e_1}, \vec{e_2} > = < \vec{e_1}, \vec{e_3} > = < \vec{e_2}, \vec{e_3} > = 0 => \vec{e_1} \perp \vec{e_2} \perp \vec{e_3}$

Grand-Schmidt

$\vec{v}$ basis $\vec{e_i}, \vec{e_j}, \vec{e_k}$ $||\vec{e_1}||^2 = 1$

$< \vec{e_1}, \vec{e_2} > = 0 = \delta_{ij}$
$\delta_{ij} = 1$ if $i = j$
$\delta_{ij} = 0$ if $i \neq j$

$\vec{v} = a_1 \vec{v_1} + a_2 \vec{v_2} + ... + a_n \vec{v_n} = \sum_{k=1}^{n} a_k v_k$

$||\vec{v_1}||^2 = 1$
$< \vec{v_2}, \vec{v_k} > = < \vec{v_2}, \sum_{k=1}^{n} a_n \vec{v_n} > = \sum_{k=i}^{n} a < \vec{v_i}, \vec{v_j} > = \sum_{k=i}^{n} a \delta_{ij} = a_j$

Matrices

$M = M^T$: M is symmetric matrix;

Diagonal matrix

$$\begin{bmatrix} d_{1,1} & 0 & \cdots & 0 \\ 0 & d_{2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{n,n} \end{bmatrix}$$

Identity matrix

$$\mathbb{I}_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Inverse of matrix A is when:
$A^{-1} A = \mathbb{I}_n$
To calculate if matrices are invertible or singular, we calculate determinant
$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$det(A) = ad - bc$

$det(A) \neq 0$ if matrix A is invertible

Invertible meant columns of A are linear independent

Rank of a matrix = number of linearly independent columns in a matrix Full-Rank = rank(A)

In the following, this system of equations will be described.

$a_{11}x_1 + a_{12}x_2 = b_1$
$a_{21}x_1 + a_{22}x_2 = b_2$
The system can be written as $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ or $Ax = b$
We can solve for x by $x = A^{-1}b$
Therefore, if matrix A is not invertible, the system can't be solve, ie two lines never cross.

Quadratic Form
$x \in \mathbb{R}^d$
$\vec{x}^T \vec{x} = \vec{x}^T \mathbb{I} \vec{x}$

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$
$\vec{x}^T M \vec{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
$= x_1(m_{11}x_1 + m_{12}x_2) + x_2(m_{21}x_1 + m_{22}x_2) = (m_{11}x_1)^2 + m_{12}x_2x_1 + m_{21}x_1x_2 + (m_{22}x_2)^2$

Hence,
$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + 2x_1x_2 + x_2^2 = (x_1 + x_2)^2 \geq 0$
$M \in \mathbb{R}^{d \times d}$ is a positive semi-definite matrix
If $x \in \mathbb{R}^d$ then $\vec{x}^T M \vec{x} \geq 0$ or M has only $\lambda \geq 0$

Eigenvalues and Eigenvectors

$A\vec{x} = \lambda \vec{x}$ such that $\vec{x} \neq 0$
λ: Eigenvalue
$\vec{x}$: Eigenvector

Spectral decomposition

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$
A has 2 eigenvalues: λ_1, λ_2
$A\vec{v_1} = \lambda_1 \vec{v_1}$
$A\vec{v_2} = \lambda_2 \vec{v_2}$
$V = \begin{bmatrix} 1 & 1 \\ \vec{v_1} & \vec{v_2} \\ 1 & 1 \end{bmatrix}$
$AV = \begin{bmatrix} 1 & 1 \\ A\vec{v_1} & A\vec{v_2} \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \lambda\vec{v_1} & \lambda\vec{v_2} \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \vec{v_1} & \vec{v_2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$
$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$
$AV = V\Lambda$
$A = V\Lambda V^{-1}$ is eigen decomposition
If A is symmetrical
$A^T = (V\Lambda V^{-1})^T = V^T \Lambda V^{-1^T}$

Traces
Sum of the diagonal of the matrix
$A = \begin{bmatrix} a_{11} & a_{21} \\ a_{21} & a_{22} \end{bmatrix}$
$tr(A) = a_{11} + a_{22}$
$tr(ABC) = tr(CAB) = tr(BCA)$
$tr(A) = tr(V\Lambda V^{-1}) = tr(\mathbb{I}\Lambda) = \sum_{i=1}^{n} \lambda_i$

$A = V\Lambda V^{-1}$
$det(A) = det(V\Lambda V^{-1})$
$det(A) = det(V)det(\Lambda)det(V^{-1})$
$det(A) = det(VV^{-1})det(\Lambda)$
$det(A) = det(\mathbb{I})det(\Lambda)$
$det(A) = 1\prod_{i=1}^{\alpha}\lambda_i$