C–520

M. A./M. Sc. (Second Semester) (Main/ATKT) EXAMINATION, May-June, 2019

MATHEMATICS

Paper Second

(Real Analysis—II)

Time : Three Hours] [*Maximum Marks : 80*

Note : Attempt all Sections as directed.

Section—A 1 each

(Objective/Multiple Choice Questions)

Note : Attempt all questions.

1. Let I = [0, 1] and let $f, \alpha : I \to \mathbf{R}$ be function such that $f(x) = x, \alpha(x) = x^2$. Then the value of $\int_0^1 f d\alpha$ is :

   (a) $\frac{1}{3}$

   (b) $\frac{2}{3}$

   (c) 2

   (d) $\frac{1}{2}$

2. Let $f \in \Re(\alpha)$ on $[a, b]$, then :

   (a) $|f| \in \Re(\alpha)$

   (b) $\left|\int_a^b f d\alpha\right| < \int_a^b |f| d\alpha$

   (c) Both (a) and (b)

   (d) None of the above

3. If $P_1, P_2 \in \Re[a, b]$. Then P^* is their common refinement if :

   (a) $P^* = P_1 \cap P_2$

   (b) $P^* = P_1 \cup P_2$

   (c) Both (a) and (b)

   (d) Neither (a) not (b)

4. If the set of numbers $\Lambda_\gamma(P)$ is unbounded for all partitions $P \in P[a, b]$, then the curve γ is said to be :

   (a) rectifiable

   (b) non-rectifiable

   (c) arc

   (d) None of the above

5. Singleton set $\{x\}$ has its measures :

   (a) 0

   (b) 1

   (c) $\frac{1}{2}$

   (d) 2

6. Cantor's set has measure :

(a) 0

(b) 1

(c) $\frac{1}{3}$

(d) $\frac{2}{3}$

7. An extended real-valued function $f : E \to \mathbf{R}$ defined on measurable set E. Let $E = \mathbf{R}$, then the set $E(f > a)$ is :

(a) open set

(b) closed set

(c) Both (a) and (b)

(d) None of the above

8. A continuous function :

(a) is never measurable

(b) may not be measurable

(c) is always measurable

(d) None of the above

9. Let φ and ψ be simple functions then :

(a) $\int (\varphi + \psi)\, dx < \int \varphi\, dx + \int \psi\, dx$

(b) $\int (\varphi + \psi)\, dx > \int \varphi\, dx + \int \psi\, dx$

(c) $\int (\varphi + \psi)\, dx = \int \varphi\, dx + \int \psi\, dx$

(d) None of the above

10. If $f(x) = 1$ almost everywhere on a set E of finite measure, then $\int_E f$ is :

(a) 0

(b) 1

(c) m (E)

(d) None of the above

11. Consider the following statements :

(I) Every simple function is also a step function.

(II) Every step function is also a simple function.

(a) (I) is true and (II) is false

(b) (I) is false and (II) is true

(c) (I) and (II) both are false

(d) (I) and (II) both are true

12. Let $(X, \mathbf{B}, \mu)$ be measurable space. A measure μ is called σ-finite if :

(a) there is a sequence $\{X_n\}$ of measurable sets in $\mathbf{B}$ such that $X = \bigcup_{n=1}^{\infty} X_n$

(b) there is a sequence $\{X_n\}$ of measurable sets in $\mathbf{B}$ such that $X = \bigcup_{n=1}^{\infty} X_n$ and $\mu(X_n) < \infty$

(c) there is a sequence $\{X_n\}$ of measurable sets in $\mathbf{B}$ such that $X = \bigcap_{n=1}^{\infty} X_n$

(d) there is a sequence $\{X_n\}$ of measurable sets in $\mathbf{B}$ such that $X = \bigcap_{n=1}^{\infty} X_n$ and $\mu(X_n) < \infty$

13. Consider the following statements :

(I) If f is an absolutely continuous function, then it has a derivative almost everywhere.

(II) If f is an absolutely continuous on $[a, b]$ and $f'=0$ almost everywhere, then f is a constant function.

(a) (I) is true and (II) is false

(b) (I) is false and (II) is true

(c) (I) and (II) both are false

(d) (I) and (II) both are true

14. Consider the following statements :

(I) A continuous function may not be of bounded variation.

(II) A function of bounded variations continuous.

(a) (I) is true and (II) is false

(b) (I) is false and (II) is true

(c) (I) and (II) both are false

(d) (I) and (II) both are true

15. Consider the following statements :

(I) Every bounded variation function is absolutely continuous.

(II) Every absolutely continuous function is of bounded variation.

(a) (I) is true and (II) is false

(b) (I) is false and (II) is true

(c) (I) and (II) both are false

(d) (I) and (II) both are true

16. Consider the following statements :

(I) If a function F (x) is an indefinite integral, then it is an absolutely continuous.

(II) If a function F (x) is an absolutely continuous, then it is need not be an indefinite integral.

(a) (I) is true and (II) is false

(b) (I) is false and (II) is true

(c) (I) and (II) both are false

(d) (I) and (II) both are true

17. Consider the following statements :

(I) If $\{f(x)\}$ converges almost everywhere to $f(x)$, then it converges in measure to $f(x)$.

(II) If $\{f(x)\}$ converges in measure to $f(x)$, then it converges almost everywhere to $f(x)$.

(a) (I) is true and (II) is false

(b) (I) is false and (II) is true

(c) (I) and (II) both are false

(d) (I) and (II) both are true

18. If $f(x) \in L^p$ and $g(x) \in L^q$, where $\frac{1}{p}+\frac{1}{q}=1, p>1$, then :

(a) $f(x)\,g(x) \in L^p$

(b) $f(x)\,g(x) \in L^q$

(c) $f(x)\,g(x) \in L$

(d) None of the above

19. If $f(x) \in L^p$ and $g(x) \in L^p$, then :

(a) $f(x) - g(x) \in L^p$

(b) $f(x) + g(x) \in L^p$

(c) Both (a) and (b)

(d) None of the above

20. If $f(x) \in L^q$ and $g(x) \in L^q$, then :

(a) $f(x)g(x) \in L^{\frac{q}{2}}$

(b) $f(x)g(x) \in L^q$

(c) $f(x)g(x) \in L$

(d) None of the above

Section—B

$1\frac{1}{2}$ each

(Very Short Answer Type Questions)

Note : Attempt all questions.

1. Define Riemann-Stieltjes integral of a real bounded function define on $[a, b]$.
2. Write the statement of mean value theorem for Riemann-Stieltjes integrals.
3. Define Borel measurable.
4. Define regularity of a measure.
5. Define complete measure.
6. Write the statement of Caratheodory extension theorem.
7. State Vitali's Covering theorem.
8. State Lebesgue differentiation theorem.
9. State Hölder's inequality for L^p-space.
10. Define L^p-space and give example.

Section—C

$2\frac{1}{2}$ each

(Short Answer Type Questions)

Note : Attempt all questions.

1. Let f be monotonic on $[a, b]$ and let α be continuous and monotonically increasing on $[a, b]$. Then prove that $f \in \mathbf{R}(\alpha)$.
2. Let f be a constant function on $[a, b]$ define by $f(x) = k$ and α a monotonically increasing function on $[a, b]$. Then prove that $\int_a^b f d\alpha$ exists and :
$$\int_a^b f d\alpha = k[\alpha(b) - \alpha(a)].$$
3. Prove that the intersection of a finite number of measurable sets is measurable.
4. Let f and g be measurable functions on a common domain E. Then prove that the set $E(f > g) = \{x \in E \mid f(x) > g(x)\}$ is measurable.
5. Let $(X, \mathbf{B}, \mu)$ be a measure space. If $E_i \in \mathbf{B}$, for all $i \in \mathbf{N}$, then prove that :
$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i).$$
6. Let $\{f_n\}$ be a sequence of non-negative measurable function and let $f = \sum_{n=1}^{\infty} f_n$. Then prove that :
$$\int f = \sum_{n=1}^{\infty} \int f_n$$

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7. Find the four derivatives for the function $f: \mathrm{R} \to \mathrm{R}$ defined by :

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x > 0 \\ 0 & \text{if } x = 0. \\ x \sin\left(\frac{1}{x}\right) & \text{if } x < 0 \end{cases}$$

8. Prove that every absolutely continuous function is of bounded variation.

9. Let $f, g \in L^p[a, b]$, then prove that :

$$f + g \in L^p[a, b]$$

10. If $f \in L^2[0, 1]$, then show that :

$$\left|\int_0^1 f(x)\, dx\right| \le \left[\int_0^1 |f(x)|^2\, dx\right]^{\frac{1}{2}}$$

4 each

Section—D

(Long Answer Type Questions)

Note : Attempt all questions.

1. Let $f \in \mathbf{R}$ on $[a, b]$. For $a \le x \le b$, put :

$$F(x) = \int_a^x f(t)\, dt.$$

Or

Then prove that F is continuous on $[a, b]$; furthermore, if f is continuous at a point x_0 of $[a, b]$, then prove that F is differentiable at x_0 and :

$$F'(x_0) = f(x_0).$$

$$\Lambda_\gamma(a, b) = \Lambda_\gamma(a, c) + \Lambda_\gamma(c, b).$$

2. Prove that a continuous function defined on a measurable set is measurable but converse is not true.

Or

Prove that the interval (α, ∞) is measurable.

3. Let (X, S, μ) be a measure space and let μ^* be the outer measure generated by μ. There for $E \subset X$, show the following statements are equivalent :

(i) E is μ^*-measurable.

(ii) $\mu(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$ holds for all $A \in S$ with $\mu(A) < \infty$.

Or

State and prove Lebesgue-Monotone Convergence theorem.

4. State and prove Jordan-Decomposition theorem.

Or

If f is of bounded variation on $[a, b]$, then prove that $T_a^b = P_a^b + N_a^b$ and $f(b) - f(a) = P_a^b - N_a^b$.

5. State and prove Riesz theorem.

Or

State and prove Riesz-Fischer theorem.

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