Roll No. ....

D–983

M. A./M. Sc. (Fourth Semester) (Main/ATKT) EXAMINATION, May-June, 2020

MATHEMATICS

Paper First

(Functional Analysis—II)

Time : Three Hours] [*Maximum Marks : 80*

Note : Attempt all Sections as directed.

Section—A 1 each

(Objective/Multiple Choice Questions)

Note : Attempt all questions.

Choose the correct answer :

1. Let T be a closed linear map of Banach space X into Banach space Y, then :
   (a) T is closed
   (b) T is open
   (c) T is continuous
   (d) All of the above

2. Let X be an arbitrary normed linear space the mapping $f : X \to X^{**}$ is isometric isomorphics from X into X** if :
   (a) It is linear
   (b) It is bounded
   (c) It preserves distance
   (d) All of the above

3. Let X and Y be normed linear space and $D \subset X$, then linear transformation $T : D \to Y$ is closed if and only if its graph G_T is :
   (a) open
   (b) closed
   (c) bounded
   (d) None of these

4. Let $\{T_n\}$ be a sequence of continuous linear operator of Banach space X into Banach space Y such that $\lim_{n\to\infty} T_n x = Tx$ exists for every $x \in X$, then T is continuous linear operator and :
   (a) $\|T\| \le \lim_{n\to\infty} \inf \|T_n\|$
   (b) $\|T\| \le \lim_{n\to\infty} \sup \|T_n\|$
   (c) $\|T\| \ge \lim_{n\to\infty} \inf \|T_n\|$
   (d) $\|T\| \ge \lim_{n\to\infty} \sup \|T_n\|$

5. Let X be a normed space over field K and S be a linear subspace of X. Suppose that $Z \in X$ and dist $(Z, S) = d > 0$, then there exist $g \in X^*$ such that :

(a) $g(s) = \{0\}, g(z) = d, \|g\| = 1$

(b) $g(s) \neq \{0\}, g(z) = d, \|g\| = 1$

(c) $g(s) = \{0\}, g(z) = d, \|g\| \neq 1$

(d) $g(s) \neq \{0\}, g(z) = d, \|g\| \neq 1$

6. Let X be a normed space, then the set of all bounded linear functional on X constitutes a normed space with the norm defined by :

(a) $\|f\| = \sup\left\{\frac{|f(x)|}{\|x\|} : x \in X;\ x \neq 0\right\}$

(b) $\|f\| = \sup\left\{|f(x)| : x \in X; \|x\| = 1\right\}$

(c) Both (a) and (b)

(d) None of these

7. Let X and Y be normed space over the field K and $T : X \to Y$ be a boudned linear operator, then :

(a) The adjoint T* is bounded linear from Y* to X*

(b) $\|T^*\| = \|T\|$

(c) The mapping T of T* is an isometric isomorphism of B (X, Y) into B (Y*, X*)

(d) All of the above

8. Let Y be a linear subspace of normed linear space X and let f be a functional defined on Y, then f can be extended to functional F defined on the whole space X such that :

(a) $\|f\| = \|F\|$

(b) $\|f\| \neq \|F\| = 1$

(c) $\|f\| \neq \{0\}, \|F\| = 1$

(d) None of these

9. If x and y are any two vectors in an inner product space X, then :

$$|< x, y >| \leq \|x\| . \|y\|$$

The above inequality is known as :

(a) Parallelogram law

(b) Cauchy-Schwarz's inequality

(c) Polarisation identity

(d) Bessel's inequality

10. A normed space is an inner product space if and only if the norm of the normed space satisfy the .... equation.

(a) $\|x + y\| \leq \|x\| + \|y\|$

(b) $|< x, y >| \leq \|x\| . \|y\|$

(c) $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$

(d) $\langle x, y \rangle = \frac{1}{4}\left[\|x + y\|^2 - \|x - y\|^2\right]$,

where K = R

11. If A is a subset of an inner product space X, then which statement is incorrect ?

(a) $A \subseteq A^{\perp\perp}$

(b) $(A^{\perp})^{\perp} = A^{\perp\perp}$

(c) $A = A^{\perp\perp}$

(d) None of these

12. Let H be a Hilbert space and let $\{e_i\}$ be an orthonormal set in H, then :

$$\|x\|^2 = \Sigma \left| < x_i e_i > \right|^2$$

This equality is known as :

(a) Parseval's identity

(b) Apollonius identity

(c) Holder's inequality

(d) None of these

13. Let X be an inner product space. Which is incorrect statement ?

(a) If $x \perp y \Leftrightarrow y \perp x, \ \forall x, y \in X$

(b) $x \perp 0, \ \forall x \in X$

(c) 0 is the only vector in X which is orthogonal to itself.

(d) $A \cap A^{\perp}$ is neither $\{0\}$ nor ϕ

i. e. $A \cap A^{\perp} \neq \{0\}$ or ϕ

14. A subspace M of a Hilbert space H is closed in H if :

(a) $M = M^{\perp\perp}$

(b) $H = M \oplus M^{\perp}$

(c) Both (a) and (b)

(d) None of these

15. A Banach space X is said to be reflexive if it is isometrically isomorphic to :

(a) X*

(b) X**

(c) X***

(d) All of the above

16. Let H_1 and H_2 be Hilbert space and T and S are element of $B(H_1, H_2)$ and $\alpha \in K$, then which statement is incorrect ?

(a) $(T + S)^* = T^* + S^*$

(b) $(TS)^* = S^* T^*$

(c) $T^{**} = T$

(d) $(\alpha T)^* = \alpha T^*$

17. An operator T is called unitary if :

(a) $T = T^*$

(b) $T^*T = TT^*$

(c) $T^*T = TT^* = I$

(d) $\langle T_x, x \rangle \geq 0, \ \forall x \in H$

18. If T is a positive operator on a Hilbert space H, then :
    (a) I + T is non-singular
    (b) I – T is non-singular
    (c) I + T is singular
    (d) None of these
19. Let $P \in B(H)$ be a projection operator, then :
    (a) R (P) and N (P) are closed subspace of H
    (b) I – P is a projection
    (c) R (P) = N (I – P)
    (d) All of the above
20. Which statement is incorrect ?
    (a) Every positive operator is self-adjoint.
    (b) Every self-adjoint operator is normal.
    (c) Every normal operator is unitary.
    (d) Every unitary operator is normal.

Section—B 2 each

(Very Short Answer Type Questions)

Note : Attempt all questions. Answer in 2 to 3 sentences.

1. Give an example of a closed operator which is not bounded.
2. Define dual space and normed linear space.
3. Define weak sequential compactness.
4. Define inner product space.

5. If A and B are subset of inner product space X such that $A \subset B$ then show that $A^{\perp} \supset B^{\perp}$.
6. Write the statement of Riesz representation theorem.
7. Prove that every positive operator is self-adjoint.
8. Define normal operator on Hilbert space.

Section—C 3 each

(Short Answer Type Questions)

Note : Attempt all questions. Answer in 75 words.

1. Let X be a Banach space over the field K. If :

$$\{T_n\} \in B(X, Y)$$

be a sequence such that :

$$\lim_{n \to \infty} T_n x = Tx$$

where $x \in X$ exists, then prove that :

$$T \in B(X, Y).$$

2. Show that a Banach space is reflexive if and only if its dual space is reflexive.

3. Show that in an inner product space X $x \perp y$ if and only if $\|x + \alpha y\| \geq \|x\| \forall$ scalar $\alpha \; \forall \, x, y \in X$.

4. Show that the linear space l^P, where $1 \leq P < \infty$ and $P \neq 2$ where norm is defined by :

$$\|x\|_P = \left(\sum |\xi_i|^P\right)^{\frac{1}{P}}$$

$$x = \langle \xi_i \rangle \in l^P$$

is not an inner product space and hence it is not a Hilbert space.

5. Let H_1 and H_2 be Hilbert space and $T \in B(H_1, H_2)$, then prove that :

$$\|T^* T\| = \|T\|^2 = \|TT^*\|$$

6. Show that the mapping :

$$\psi : H \to H^*$$

defined by $\psi(y) = f_y$, where $f_y(x) = \langle x, y \rangle$ is one-one onto but not linear and an isometry.

7. Show that the product of two bounded self-adjoint operators S and T on a Hilbert space H is self-adjoint if and only if the operators commute.

8. Let H be complete Hilbert space and $T \in B(H)$, then the following statements are equivalent :

   (a) T is normal

   (b) T* is normal

   (c) $\|T^* x\| = \|Tx\| \; \forall \, x \in H$

Section—D 5 each

(Long Answer Type Questions)

Note : Attempt all questions. Answer in 150 words.

1. State and prove closed graph theorem.

Or

State and prove uniform boundedness principle.

2. Let M is a closed linear subspace of a normed linear space N and x_0 is a vector not in M, then there exist a functional f_0 in N* such that :

$$f_0(M) \neq 0$$

and $$f_0(x_0) \neq 0.$$

Or

State and prove closed rang theorem for Banach space.

3. Prove that every Hilbert space is reflexive.

Or

Show that a closed convex subset C of Hilbert space H contains a unique vector of smallest norm.

4. Let y be a fixed vector in Hilbert space H and let f_y be scalar defined by :

$$f_y(x) = \langle x, y \rangle \forall x \in \mathrm{H}$$

then show that :

$$f_y \in \mathrm{B}(\mathrm{H}, \mathrm{K})$$

i.e. $$f_y \in \mathrm{H}^*.$$

Further show that :

$$\|y\| = \|f_y\|$$

Or

A bounded linear operator T on a complex Hilbert space H is unitary if and only if T is isometry and surjective.