F - 309

M.A./M.Sc. (First Semester)
Examination, Dec.-Jan., 2021-22
MATHEMATICS
Paper First
(Advanced Abstract Algebra-I)

Max.Marks: 80

Time : Three Hours

Min. Marks: 16

Note: Attempt all Parts as directed.

Section-A 1 each

(Objective/Multiple Choice Questions)

Note: Choose one correct answer out of four alternative answers (A) through (D).

1. A non-abelian group of order 6 is isomorphic to :

   (A) S_3

   (B) S_4

   (C) S_5

   (D) None of the above

2. If an abelian group G is simple then possible order of G is :

   (A) 4

   (B) 9

   (C) 5

   (D) 12

3. Which one of the following is incorrect?

   (A) Every cyclic group is abelian.

   (B) Subgroup of an abelian group is normal.

   (C) Subgroup of a cyclic group is normal.

   (D) Every normal subgroup is cyclic.

4. A polynomial $f(x) \in F[x]$ is reducible over the field F, then :

   (A) degree of $f(x)$ is always two

   (B) it has root in F

   (C) it has root in $F[x]$

   (D) None of the above

5. The polynomial $f(x) = x^3 + 5x^2 + 5x + 1$ defined over $\mathbb{Z}$ is :

   (A) irreducible over $\mathbb{Q}$

   (B) reducible over $\mathbb{Z}$

   (C) irreducible over $\mathbb{Z}$

   (D) reducible over $\mathbb{N}$

6. If $\omega = cos\dfrac{2\pi}{n} + isin\dfrac{2\pi}{n}$, and $u = cos\dfrac{2\pi}{n}$.Then $[Q(\omega) : Q(u)] =$

   (A) 1

   (B) 2

   (C) 3

   (D) 4

7. If E is an extension of a field F then

   (A) E is a subfield of F

   (B) F is a vector space over E

   (C) F is a subfield of E

   (D) None of the above

8. If $[E : F] = 3$, then :

   (A) there exist exactly one proper field K between E and F.

   (B) there does not exist any proper field between E and F.

   (C) there exist exactly 2 proper fields between E and F.

   (D) there exist exactly 3 proper fields between E and F.

9. If K is algebraically closed field then every polynomial $f(x)$ of positive degree over K

   (A) does not have root in K

   (B) has at least one root in K

   (C) is irreducible over K

   (D) None of the above

10. If E is field of complex numbers,then :

    (A) Algebraic closure of E is itself.

    (B) Algebraic closure of E does not exist.

    (C) Algebraic closure of E is countable.

    (D) Algebraic closure of E is field of rational numbers.

11. The prime field of a field F :

(A) may be isomorphic to $\mathbb{Q}$

(B) can not be isomorphic to $\mathbb{Q}$

(C) isomorphic to $\mathbb{R}$ always

(D) isomorphic to $\mathbb{C}$ always

12. Let F be a field with 5^{15} elements. Then how many subfields does F have?

(A) 5

(B) 4

(C) 1

(D) 10

13. If F is a field with each of its algebraic extension is separable, then :

(A) F is perfect field

(B) F is not perfect field

(C) such F does not exist

(D) F has characteristic 2 always

14. Read the following statements :

I. Every algebraic extension of a field is finite extension.

II. Every finite extension of a field is an algebraic extension.

Choose the correct option.

(A) Only I is true

(B) Only II is true

(C) Both I and II are true

(D) Both I and II are false

15. Every polynomial $f(x)$ over a field of characteristic zero

(A) is not separable

(B) is separable

(C) have multiple roots

(D) None of the above

16. Any reducible polynomial over set of integers is :

(A) reducible over $\mathbb{R}$

(B) reducible over $\mathbb{C}$

(C) reducible over $\mathbb{Q}$

(D) All of the above

17. Which of the following statement is incorrect?

(A) Any quatric over F is not solvable by radicals.

(B) The general polynomial of degree $n \geq 5$ is not solvable by radicals.

(C) A finite normal and separable extension E of a field F is a Galois extension of F.)

(D) If p is a prime number and if a subgroup G of S_p is a transitive group of permutations containing a transposition (a, b), then $G = S_p$.

18. An automorphism is :

(A) homomorphism but not one-one.

(B) homomorphism,one-one but not onto.

(C) One-one, onto but not homomorphism.

(D) homomorphism, one-one and onto.

19. Which of the following statement is correct?

(A) The fixed field of a group of automorphism of field K is not a subfield of K.

(B) $G(E/F)$ is a subgroup of the group of all automorphism of E.

(C) The fixed field of $G(E/F)$ not contains F.

(D) None of the above

20. The group $G(Q(\alpha)/Q)$, where $\alpha^5 = 1$ and $\alpha \neq 1$, is isomorphic to the cyclic group of the order

(A) 2

(B) 3

(C) 4

(D) 5

Section - B (Very Short Answer Type Questions)

2 each

1. Define maximal normal subgroup.

2. Define solvable group.

3. Define Eisenstein criterion for irreducibility of a polynomial.

4. State Kronecker theorem.

5. State Uniqueness theorem for splitting field.

6. Define multiplicity of a root.
7. Define separable polynomial.
8. Define radical extension.

Section - C (Short Answer Type Questions) 3 each

Note: Attempt all questions.

1. Show that every finite group has a composition series.
2. Let H be a normal subgroup of a group G. Show that, if both H and G/H are solvable, then G is also solvable.
3. Show that $\sqrt{2}+\sqrt[3]{5}$ is algebraic over $\mathbb{Q}$, also find $[\mathbb{Q}(\sqrt{2}+\sqrt[3]{5}):\mathbb{Q}]$.
4. Prove that an algebraically closed field can not be finite.
5. Show that the polynomial $x^7-10x^5+15x+5$ is not solvable by radicals over $\mathbb{Q}$.
6. If $f(x)\in F[x]$ is an irreducible polynomial over a finite field F, then show that all roots of $f(x)$ are distinct.
7. Let $F=\mathbb{Z}/(2)$. Show that the splitting field of $x^3+x^2+1\in F[x]$ is a finite field with eight elements.
8. Any element $a\in K$ is a root of a polynomial $p(x)$ over F of positive degree if and only if $(x-a)|p(x)$ in $K[x]$.

Section - D (Long Answer Type Questions) 5 each

Note: Attempt all questions.

1. State and prove Jordan Holder theorem for finite group.

OR

Let F be a field. Then show that, there exists an algebraically closed field K, containing F as a subfield.

2. Let F be a field, $p(x)$ an irreducible polynomial in F of degree $n\geq 1$. Prove that there exists an extension E of F, such that $[E:F]=n$, in which $p(x)$ has a root.

OR

Show that every finite separable extension of a field is necessarily a simple extension.

3. Prove that the prime field of a field F is either isomorphic to $\mathbb{Q}$ or $\mathbb{Z}/(p)$, p is prime.

OR

State and prove Artin theorem.

4. Prove that the group of automorphisms of a field F with p^n elements is cyclic of order n and generated by ϕ, where $\phi(x)=x^p$, $x\in F$.

OR

State and prove Fundamental theorem of algebra.

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