Name: Key

10.2 COMBINATIONS and the BINOMIAL THEOREM

It is not always important to count all of the different orders that a group of objects can be arranged. A **combination** is a selection of *r* objects from a group of *n* objects where the order is not important.

Combinations of n Objects Taken r at a Time
The number of combinations of r objects taken from a group of n distinct objects is denoted by ${}_nC_r$ and is given by the formula:

$$\frac{n!}{r!(n-r)!}$$

1. A standard deck of 52 playing cards has 4 suits with 13 different cards in each suit. If the order in which the cards are dealt is not important, how many different 5-card hands are possible?

choosing 5 out of 52 cards
$n=52$, $r=5$

$${}_{52}C_5 = \frac{52!}{5!(52-5)!} = 2{,}598{,}960$$ hands

2. A teacher is choosing 3 representatives from their homeroom class. If there are 28 students in the class, how many combinations of students are possible?

$${}_{28}C_3 = \frac{28!}{3!(28-3)!} = 3276$$ combinations of 3 students

Multiple Events
- When finding the number of ways both an event A **and** an event B can occur, you need to **multiply.**
- When finding the number of ways that event A **or** event B can occur, you **add.**

3. How many 5 card hands are possible where all 5 cards are the same color?

choosing 1 out of 2 colors and 5 out of 26 in that color:

$${}_2C_1 \cdot {}_{26}C_5 = 131{,}560$$ hands

4. How many 5 card hands are possible where all 5 cards are the same suit?

choosing 1 of 4 suits and 5 out of 13 in that suit:

$${}_4C_1 \cdot {}_{13}C_5 = 5148$$ hands

5. How many 5 cards hands are there that have either all spades or all diamonds?

choosing 5 out of 13 spades or 5 out of 13 diamonds

$${}_{13}C_5 + {}_{13}C_5 = 2574$$ hands

6. How many 5 card hands have 3 clubs?

3 out of 13 clubs and 2 out of 39 other cards

$${}_{13}C_3 \cdot {}_{39}C_2 = 286 \cdot 741 = 211926$$ hands

7. You are going to toss 10 different coins. How many different ways will *at least* 4 of the coins show heads?

at least 4 show heads means 4 heads or 5 heads or 6 .....
easier way:
subtract possibilities we dont want from the $\underbrace{\text{total \# of possibilities}}_{\text{(use the counting principle)}}$

$$2^{10} - \underbrace{{}_{10}C_0}_{1} + \underbrace{{}_{10}C_1}_{10} + \underbrace{{}_{10}C_2}_{45} + \underbrace{{}_{10}C_3}_{120} = 848$$

Pascal's Triangle

Many of the relationships among combinations can be seen in the array of numbers known as Pascal's Triangle.

$_0C_n$ 1
$_1C_n$ 1 1
$_2C_n$ 1 2 1
$_3C_n$ 1 3 3 1
$_4C_n$ 1 4 6 4 1
$_5C_n$ 1 5 10 10 5 1
$_6C_n$ 1 6 15 20 15 6 1
⋮ 1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

$_0C_0$
$_1C_0$ $_1C_1$
$_2C_0$ $_2C_1$ $_2C_2$
$_3C_0$ $_3C_1$ $_3C_2$ $_3C_3$
$_4C_0$ $_4C_1$ $_4C_2$ $_4C_3$ $_4C_4$
$_5C_0$ $_5C_1$ $_5C_2$ $_5C_3$ $_5C_4$ $_5C_5$
⋮

*what's $_7C_5$? 21

8. You're getting t-shirts made but the print shop can only print 2 colors on a shirt. How many combinations of 2 colors can you make from a choice of 6?

$_6C_2 = 15$

Binomial Expansions – We'll now explore the connection between Pascal's triangle and binomial expansions.

What are the coefficients?

$(x+y)^0 = 1$ 1

$(x+y)^1 = x+y$ 1 1

$(x+y)^2 = (x+y)(x+y) = x^2+2xy+y^2$ 1 2 1

$(x+y)^3 = (x^2+2xy+y^2)(x+y) = x^3+x^2y+2x^2y+2xy^2+y^2x+y^3$ 1 3 3 1

$= x^3+3x^2y+3xy^2+y^3$

$(x+y)^4 = x^4+4x^3y+6x^2y^2+4xy^3+y^4$ 1 4 6 4 1

Binomial Theorem

For any positive integer n, the binomial expansion of $(a+b)^n$ is:

$_nC_0 a^n b^0 + {_nC_1} a^{n-1} b^1 + {_nC_2} a^{n-2} b^2 + {_nC_3} a^{n-3} b^3 + \ldots + {_nC_n} a^0 b^n$

1. Expand $(x+y)^5$ $= x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$

2. Expand $(3x-2)^4$

$1(3x)^4(-2)^0 + 4(3x)^3(-2)^1 + 6(3x)^2(-2)^2 + 4(3x)^1(-2)^3 + 1(3x)^0(-2)^4$

$= 81x^4 + 4(27x^3(-2)) + 6(9x^2)(4) + 4(3x)(-8) + (16)$

$= 81x^4 - 216x^3 + 216x^2 - 96x + 16$

3. Expand $(5\cancel{x}-2y)^3$

$1(5x)^3(-2y)^0 + 3(5x)^2(-2y)^1 + 3(5x)^1(-2y)^2 + 1(5x)^0(-2y)^3$

$125x^3 + 3(25x^2)(-2y) + 3(5x)(4y^2) + (-8y^3)$

$= 125x^3 - 150x^2y + 60xy^2 - 8y^3$