1. a)

→ Expectation of Z:

$\mathbb{E}[Z] = \mathbb{E}[(X-Y)^2] = \mathbb{E}[X^2 + Y^2 - 2XY] =$

$\underbrace{\mathbb{E}[X^2]} + \mathbb{E}[Y^2] - 2\mathbb{E}[XY]$

$\to = \int_0^1 x^2 = \frac{1}{3}x^3 \Big|_0^1 = \frac{1}{3}$

Similarly $\mathbb{E}[Y^2] = \frac{1}{3}$

$\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y] = \left(\int_0^1 x\right)^2 = \frac{1}{4}$

Independence

Thus: $\mathbb{E}[Z] = 2 * \frac{1}{3} - \frac{2}{4} = \frac{1}{6}$

→ Variance of Z:

$Var(Z) = \mathbb{E}[Z^2] - \mathbb{E}[Z]^2 = \mathbb{E}[Z^2] - \frac{1}{36}$

To compute $\mathbb{E}[Z^2]$ we have:

$$\mathbb{E}[Z^2] = \mathbb{E}[(X-Y)^4] =$$

$$\mathbb{E}[X^4 + Y^4 - 4X^3Y - 4XY^3 + 6X^2Y^2] =$$

$$\mathbb{E}[X^4] + \mathbb{E}[Y^4] - 4\mathbb{E}[X^3Y] -$$

$$4\mathbb{E}[XY^3] + 6\mathbb{E}[X^2Y^2]$$

$$* \ \mathbb{E}[X^4] = \mathbb{E}[Y^4] = \int_0^1 x^4 = \frac{1}{5}x^5\Big|_0^1 = \frac{1}{5}$$

Independence

$$* \ 4\mathbb{E}[X^3Y] = 4\mathbb{E}[XY^3] = 4\mathbb{E}[X^3]\,\mathbb{E}[Y] =$$

$$2\,\mathbb{E}[X^3] = 2\int_0^1 x^3 = 2\cdot\frac{1}{4}x^4\Big|_0^1 = \frac{1}{2}$$

$$* \ 6\mathbb{E}[X^2Y^2] = 6\mathbb{E}[X^2]\,\mathbb{E}[Y^2] = 6\mathbb{E}[X^2]^2$$

$$= 6\left(\int_0^1 x^2\right)^2 = 6\cdot\left(\frac{1}{3}\right)^2 = \frac{2}{3}$$

Thus: $\mathbb{E}[Z^2] = 2*\frac{1}{5} - 2*\frac{1}{2} + \frac{2}{3} = \frac{1}{15}$

$$\Rightarrow \mathrm{Var}(Z) = \frac{1}{15} - \frac{1}{36} = \frac{7}{180}$$

b) $\mathbb{E}[R] = \mathbb{E}[Z_1 + \cdots + Z_d] =$

$\mathbb{E}[Z_1] + \mathbb{E}[Z_2] + \cdots + \mathbb{E}[Z_d] =$

$d\,\mathbb{E}[(X_1 - Y_1)^2] = d\,\mathbb{E}[Z] = \frac{d}{6}$

$\mathrm{Var}(R) = \mathrm{Var}(Z_1 + Z_2 + \cdots + Z_d) =$

$\mathbb{E}[(Z_1 + \cdots + Z_d)^2] - \mathbb{E}[Z_1 + \cdots + Z_d]^2 =$

$d\,\mathbb{E}[Z^2] + d(d-1)(\mathbb{E}[Z])^2 - \frac{d^2}{36} =$

$\frac{d}{15} + \frac{d(d-1)}{36} - \frac{d^2}{36} = \frac{12d - 5d}{180}$

$= \frac{7d}{180}$

3.

a) $$H(X) = \sum_x P(x) \log_2 \left(\frac{1}{P(x)}\right)$$

For each x we have: $P(x) \geq 0$ and:

$$0 \leq P(x) < 1 \Rightarrow \frac{1}{P(x)} > 1 \Rightarrow \log_2\left(\frac{1}{P(x)}\right) > 0$$

$$\Rightarrow \forall x: P(x) \log_2\left(\frac{1}{P(x)}\right) \geq 0 \Rightarrow$$

$$\sum_x P(x) \log_2\left(\frac{1}{P(x)}\right) \geq 0$$

b) First we will prove that $x \log(x)$ is a convex function. $(x > 0)$. We can easily show that by taking the second derivative of this function:

$$\frac{d^2 \, x\log(x)}{dx^2} = \frac{1}{x} > 0 \Rightarrow \text{convex}$$

Now we have:

$$\sum_x P(x) \log_2 \frac{P(x)}{q(x)} = \sum_x q(x) \frac{P(x)}{q(x)} \log_2 \frac{P(x)}{q(x)}$$

if we define $\phi(x) = x \log(x)$ then:

$$KL(P||q) = \sum_x q(x)\, \phi\left(\frac{P(x)}{q(x)}\right) \quad (*)$$

Now we can define a r.v. $Z(x) = \frac{P(x)}{q(x)}$

with the pdf of:

$$P\left(Z = \frac{P(x)}{q(x)}\right) = q(x)$$

which is legit since $\sum_x q(x) = 1$

and $q(x) > 0$.

Thus the $(*)$ expression is equal to

$E[\phi(Z)]$. From Jensen Inequality

we have: $E[\phi(Z)] \geq \phi(E[Z])$

Thus:

$$KL(P \| q) \geq \phi\left(\sum_x \cancel{q(x)} \frac{P(x)}{\cancel{q(x)}}\right) =$$

$$\phi\left(\sum_x P(x)\right) = \phi(1) = 1 \log(1) = 0$$

---

c) We have to prove that the following expression is true:

$$\underbrace{\sum P(y) \log_2 \frac{1}{P(y)}}_{(*)} + \sum_x \sum_y P(x,y) \log_2 \frac{P(x,y)}{P(x)}$$

$$=$$

$$\sum_x \sum_y P(x,y) \log_2 \frac{P(x,y)}{P(x)P(y)} \leftarrow \text{RHS}$$

We have:

$$\text{RHS} = \sum_x \sum_y P(x,y) \log_2 \frac{P(x,y)}{P(x)} -$$

$$\sum_x \sum_y P(x,y) \log_2 P(y)$$

Therefore we have to show that the following expression holds:

$$\sum_{y} P(y) \log_2 \frac{1}{P(y)} \stackrel{?}{=} -\sum_{x} \sum_{y} P(x,y) \log_2 P(y)$$

We can easily do that by replacing the integrals in the RHS and then marginalizing out y:

$$-\sum_{x} \sum_{y} P(x,y) \log_2 P(y) =$$

$$-\sum_{y} \log_2 P(y) \sum_{x} P(x,y) =$$

$$-\sum_{y} \log_2 P(y)\, P(y) = \sum_{y} P(y) \log_2 \frac{1}{P(y)}$$