Coupled-Mode Theory

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Invited Paper

The principal features of coupled mode theory are reviewed, in particular as applied to passive structures, such as coupled resonators and coupled waveguides. The former are examples of coupling of modes in time, the latter are examples of coupling of modes in space. Active structures are considered briefly insofar as they obey conservation laws in terms of positive and negative energies. The complications caused by nonorthogonality of the mode energies, or powers, are brought up. These were the topic of recent publications.

The coupling of modes formalism has intuitive appeal in that the coupling equations in their simplest form can be written down by inspection. When "fine points" as, for instance, cross talk are at issue, then more formal derivations are required. The one presented here is based on a variational principle.

I. Introduction

In this paper we review the principal features of coupled mode theory. The literature on the subject is vast and it would be difficult to do justice to all aspects of coupled mode theory. When the first author, H. A. Haus, was asked to contribute this invited paper, he accepted on the grounds that aspects of coupled mode theory had intrigued him throughout his career, his first papers on the subject having been written in the 1950's. Recent work done with W. P. Huang, one of his doctorate students, prompted him to ask him to collaborate on this invited paper. Hence the present paper is an overview of coupled mode theory through the "lenses" of two "afficionados" of coupled mode theory. The review is not intended to be exhaustive due to the imposed limit on the length of the paper. It represents the view of the authors based on their own experience and knowledge.

We first give a brief historic perspective of the coupled mode theory. The development and applications of the theory in microwaves in early years and in optoelectronics and fiber optics in recent years are described. We then consider lossless coupling of two modes in time. Two coupled resonance circuits, or two coupled microwave or optical resonators, are the physical examples. Energy conservation requires that the eigenfrequencies be real. The start-up of a parametric oscillator is another example. Then we look at the formal derivation of coupled mode theory and consider the more general case when the modes are not energy-orthogonal and the energies are not necessarily positive. A more detailed account of the nonorthogonal coupled mode theory developed in the last five years for optical waveguides is given. When one resonator stores negative energy, the other positive energy, the frequencies appear in complex-conjugate pairs. A resonator containing an active medium, the latter treated in a linearized approximation, is an example.

This study is followed up by showing how the coupled mode formalism can be derived from a variational principle for the frequencies of the system. If a trial solution is introduced for the electric field in a lossless electromagnetic system that is the linear superposition of modes, the coupled mode formalism is the result. We do not consider the formal derivation for more complicated, active systems, they can be found in the literature. Indeed, the very virtue of the coupling of modes formalism lies in its intuitive appeal. Much physical insight can be obtained by inspection, without the need for a formal derivation.

Next, we consider coupling of modes in space. Here the initial conditions in time, relevant to coupling of modes in time, are replaced by boundary conditions in space. The sign of the group velocity of a wave indicates at which "end" of a structure a wave is excited. The sign of its energy indicates whether it is passive (positive energy) or active (negative energy). On the basis of such simple arguments one may distinguish the operation of a Bragg reflector, traveling wave tube, or backward wave oscillator.

The formal derivation of the coupled mode formalism in space is again restricted to the case of passive electromagnetic structures. We show how the formalism results from a variational principle. The variational principle provides a refined method for the evaluation of coupling coefficients,

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some aspects of which are easily missed by the intuitive approach. The refinements are necessary for an understanding of cross talk in optical waveguide couplers. It was in this context that the need for a more formal derivation of coupled mode theory reasserted itself.

II. Historical Perspective

The concept of coupled modes in electromagnetics may be traced back to the early 1950's. The application was initially to microwaves and developed gradually through the contributions of many people. In 1954, Pierce applied the coupled mode theory to the analysis of microwave traveling-wave tubes [1]. Later followed the work of Gould [2] on the backward-wave oscillators. The coupled mode theory was then employed to treat parametric amplifiers, oscillators, and frequency converters [3].

A parallel development happened in microwave waveguides and devices. Miller [4] first introduced the coupled mode theory to the analysis and design of microwave waveguides and passive devices. The theory was soon generalized by Louisell [5] to treat tapered waveguide structures, where the coupling coefficients depend on the length z. In the 1960's, the coupled mode theory was further developed to describe mode conversions due to various irregularities in microwave waveguides [6], [7] and periodic waveguide structures [8], [9].

The early approach to the coupled mode theory was rather heuristic. The modes of the "uncoupled systems" were identified and the coupled mode equations obeyed by the mode amplitudes were determined from power considerations.

A rigorous derivation of coupled mode theory was carried out by Schelkunoff [10]. By expanding the unknown electromagnetic fields of a coupled system in terms of the known modes of an uncoupled system, he obtained a set of generalized telegraphist's equations (a different version of the coupled mode equations) directly from Maxwell's equations. The coupling coefficients are determined unambiguously once the modes of the uncoupled system are defined. The coupled mode equations are equivalent to Maxwell's equations as long as a complete set is assumed for the mode expansion. For most applications, however, only a limited number of modes (usually two) is used in the expansion. Therefore, the coupled mode theory remains an approximate, yet insightful and often accurate mathematical description of electromagnetic oscillation and wave propagation in a coupled system. In order to put the approach on a more formal mathematical footing, one of the authors (HAH) showed in 1958 that the coupled mode theory was derivable from a variational principle set up for the propagation constant of the coupled system [11]. Because of the stationary nature of the variational principle, the errors made in an incomplete expansion do not lead to a dramatic deterioration of the accuracy for the propagation constants calculated from the coupled mode theory. If one decides to approximate the fields of the coupled system as a linear superposition of the fields of the uncoupled systems, then the best value obtainable for the propagation constant results from the coupled mode equations.

The coupled mode theory for optical waveguides was developed by Marcuse [12], [13], Snyder [14], [15], Yariv and Taylor [16], [17], and Kogelnik [18], [19] in the early 1970's. It has been successfully applied to the modeling and analysis of various guided-wave optoelectronic and fiber optical devices, such as optical directional couplers made of thin film and channel waveguides [20]–[23] and optical fibers [24]–[28], multiple waveguide lenses [29], [30], phase-locked laser arrays [31], distributed feedback lasers [31]–[37] and distributed Bragg reflectors [38]–[40], grating waveguides and couplers [41]–[45], nonparallel and tapered waveguiding structures [46]–[55], Y-branch waveguides [56]–[58], TE/TM polarization converters [59]–[63], polarization rotation in optical fibers [64]–[66], mode conversion and radiation loss in slab waveguides and fibers [67]–[71], residual coupling among scalar modes [72], [73]. It has also been used to study the wave coupling phenomena in nonlinear media such as harmonic generation in bulk [74] and guided-wave devices [75], [76], nonlinear pulse or soliton propagation [77], [78] and the modulation instability [79] in optical fibers, and nonlinear coherent couplers [80]–[86]. Many of these applications are well documented and summarized in [87]–[93].

An assumption made in the conventional coupled mode theory is that the modes of the uncoupled systems are orthogonal to each other. This may be true if the modes belong to the same reference structures. In studying the mode coupling in coupled systems, however, one often chooses the modes of the isolated systems as the basis for the mode expansion and these modes may not be orthogonal. The orthogonal coupled mode theory (OCMT) is not correct for the description of the mode-coupling process in this case. The effect of nonorthogonality between waveguide modes on cross talk in optical couplers was first recognized by Chen and Wang [94] and then studied by Haus and Whitaker [95] in a proposal to eliminate the cross talk due to this effect. Later on, several formulations of the nonorthogonal coupled mode theory (NCMT) were developed by Hardy and Streifer [96], Haus, Huang, Kawakami, and Whitaker [97], and Chuang [98]. The new nonorthogonal coupled mode theory (NCMT) is shown to yield more accurate dispersion characteristics and field patterns for the modes of the coupled waveguides. It also calls for a modification of the description of the power exchange between the waveguides.

There were some discrepancies among the different formulations at the early stage of the development. Some were superficial and soon resolved by reformulation [99]; some are more subtle [97], [99]–[101]. Snyder, Ankiewicz, and Altintasl [102] showed that the nonorthogonal formulations could lead to erroneous results for the coupling length of the TM modes of parallel slabs when the index discontinuity is large. The origin of the error is apparent in this case since the waveguide modes used as the trial solution in the coupled mode theory are subject to serious error when the index steps are large. But they also demonstrated in the same example that the conventional orthogonal coupled mode theory based on the same trial solution gives excellent prediction about the coupling length. This unexpected result triggered a series of debates in the field [103]–[110]. It was later resolved by Haus, Huang, and Snyder [111].

Despite the controversies, there have been intense research activities in the past few years in developing and applying the nonorthogonal coupled mode theory in the area of optoelectronics and fiber optics. Simplified scalar versions that may be applied to weakly guiding structures [112]–[115] and a modified vector version for the strongly guided structures [111], [115] were developed. Generalizations of the theory to multiwaveguide and/or multimode structures [116]–[121], anisotropic media [122], [123], periodic and tapered structures [124]–[128], and nonlinear couplers [129] were attempted. The nonorthogonal coupled mode theory has been employed in the analysis and design of optical guided-wave devices [130]–[138] and fiber optical couplers [139]–[146]. The experimental verification of the theory was carried out by Marcatili [147] and Syms [148]–[150].

III. Coupling of Modes in Time

We shall first address coupling of modes in time starting with a few intuitively obvious postulates. From these we can develop a formalism that describes coupling of two resonators, parametric amplification and oscillation. Then we shall show how the coupling of modes of passive electromagnetic structures can be derived from a variational principle. This variational principle provides rules for the evaluation of the coupling coefficients when intuitive arguments are not sufficiently precise to provide these coefficients.

Consider two weakly coupled lossless resonators. Denote the amplitude with the time dependence $\exp(j\omega_1 t)$ in one resonator by a_1, the amplitude in the other resonator with the time dependence $\exp(j\omega_2 t)$ by a_2. These are the positive frequency components of the electric field amplitudes. They obey the differential equations:

$$\frac{da_1}{dt} = j\omega_1 a_1 \tag{2.1}$$

$$\frac{da_2}{dt} = j\omega_2 a_2. \tag{2.2}$$

When the two resonators are coupled, their time dependence changes. When the coupling is weak, it has to be of the form:

$$\frac{da_1}{dt} = j\omega_1 a_1 + j\kappa_{12} a_2 \tag{2.3}$$

$$\frac{da_2}{dt} = j\omega_2 a_2 + j\kappa_{21} a_1. \tag{2.4}$$

At first it may not be obvious why the coupling should be proportional to the amplitude of the other resonator, rather than to the time derivative, or the time integral of the amplitude, or any complicated operator operating on the amplitude. However, if the coupling is weak, the time dependence is perturbed only weakly and the coupling term is important only when $\omega_1 \approx \omega_2$. Then, the derivative of a_2 is approximately equal to $j\omega_2 \times a_2$, its integral equal to $1/j\omega_2 \times a_2$. In the coupling term of (2.3), the coupling coefficient is small compared with both ω_1 and ω_2, and thus the replacement of the derivative or integral by its approximate values causes an error of higher order in the coupling and can be ignored. This is why the coupling of modes formalism can be set up so simply in the limit of weak coupling.

A. Coupling of Energy Orthogonal Modes of Positive Energy

We think of energy, generally, as a positive quantity. This is the case we shall address first. There are many perfectly realistic situations in which the energy must be considered negative. We shall come back to such cases later on.

We concentrate first on energy orthogonal modes for which the energy can be written

$$W = |a_1|^2 + |a_2|^2 \tag{2.5}$$

even in the presence of the coupling. Here we have normalized the electric field amplitudes a_1 and a_2 so that their squares are equal to the energies in the modes. No cross terms are assumed to exist, the modes are energy orthogonal. If the coupling is lossless, the only case we shall treat here, energy must be conserved:

$$\begin{aligned}\frac{d}{dt}(|a_1|^2 + |a_2|^2) =& j\kappa_{12} a_1^* a_2 + j\kappa_{21} a_2^* a_1 \\ &- j\kappa_{12}^* a_1 a_2^* - j\kappa_{21}^* a_2 a_1^* = 0\end{aligned} \tag{2.6}$$

where we have used (2.3) and (2.4). Because the initial conditions can be picked arbitrarily, (2.6) can only be obeyed when

$$\kappa_{12} = \kappa_{21}^* = \kappa. \tag{2.7}$$

This is the constraint on the coupling coefficients of two modes coupled in a lossfree way. When (2.7) is introduced into (2.3) and (2.4), a time dependence of the form $\exp(j\omega t)$ is assumed and the determinantal equation is solved one finds the two roots for ω:

$$\omega = \frac{\omega_1 + \omega_2}{2} \pm \sqrt{\left(\frac{\omega_1 - \omega_2}{2}\right)^2 + |\kappa|^2}. \tag{2.8}$$

The two frequencies are real, as they must be when two modes of positive energy couple and energy is to be conserved. Figure 1 shows the roots ω_s (lower curve) and ω_a (upper curve) of the symmetric and antisymmetric normal modes for the case when one of the resonance frequencies (ω_1) is changed, the other is kept fixed. This is the kind of diagram familiar also from quantum mechanical perturbation theory. When an energy level crossing occurs in a coupled system, the "eigenvalues" split, crossing is prevented. At the crossover point, for real κ, the solutions are the symmetric and antisymmetric combinations of a_1 and a_2. Farther from the crossover point the solutions

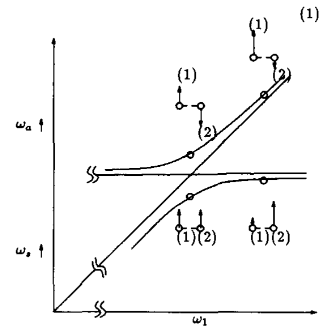


Fig. 1. The resonance frequencies of the symmetric (lower curve) and antisymmetric (upper curve) mode of a coupled resonator as a function of detuning ω_1.

acquire the character of the mode to whose frequency the eigenvalue is the closest.

It is of interest to ask for the implications of this simple coupled mode formalism. Consider first the case when the two resonators have equal frequencies. Then, if resonator 1 is excited at $t = 0, a_1 = 1$, and resonator 2 is unexcited, $a_2 = 0$, the initial conditions are matched by a superposition of equal amounts of the symmetric and antisymmetric solutions. The phases of the two solutions evolve at different rates and after the time $t = \pi/2|\kappa|$ all of the excitation will have been transferred to the other resonator. The excitation oscillates back and forth between the two resonators. When the frequencies of the two uncoupled resonators are not equal, and initially only one resonator is excited, the transfer is not complete.

B. A Simple Example

Figure 2 shows a simple example that permits both a rigorous analysis and yields easily to the coupled mode formalism. In this way one may compare the exact answer with the approximate solution. The two resonators are metallic rectangular waveguides partially filled with a dielectric so that the empty waveguides are below cutoff at the resonance frequency. The uncoupled waveguides are each terminated in infinite air-filled waveguides. The question is as to how one may find the coupling coefficient for this structure, and thus the value of the beat frequency of the two eigenmodes. Denote the spatial distribution of the dielectric constant that forms cavity (1) by

$$\epsilon_o + \delta\epsilon_1 \tag{2.9}$$

and the one that forms resonator (2) by

$$\epsilon_o + \delta\epsilon_2. \tag{2.10}$$

The actual distribution is

$$\epsilon = \epsilon_o + \delta\epsilon_1 + \delta\epsilon_2. \tag{2.11}$$

Denote the electric field patterns of modes (1) and (2) by $\mathbf{e}_1$ and $\mathbf{e}_2$, respectively. The energy in mode (1) is

$$W_{11} = |a_1|^2 \frac{1}{2} \int \epsilon |\mathbf{e}_1|^2 dv \equiv |a_1|^2 \tag{2.12}$$

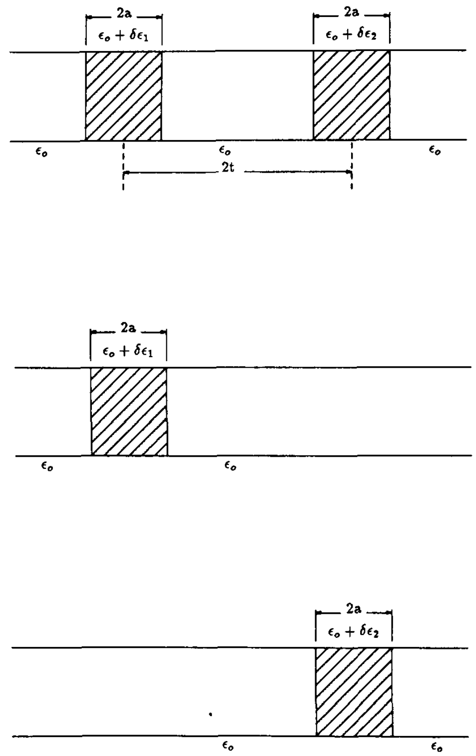


Fig. 2. Coupled metallic rectangular waveguide resonators that permit exact analysis.

by setting

$$\frac{1}{2} \int \epsilon |\mathbf{e}_1|^2 dv = 1.$$

We use energy arguments to derive the coupling. Consider the time rate of change of the energy in mode (1):

$$\frac{d}{dt}|a_1|^2 = j\kappa_{12} a_1^* a_2 - j\kappa_{12}^* a_1 a_2^*. \tag{2.13}$$

This energy change must be equal to the power fed into mode (1) by mode (2). Mode (2) finds the perturbation of dielectric constant $\delta\epsilon_1$ within resonator (1) and drives a polarization current density through that perturbation that is equal to

$$j\omega \boldsymbol{P}_{12} = j\omega \delta\epsilon_1 a_2 \mathbf{e}_2. \tag{2.14}$$

The power fed into mode (1) by this current density is equal to

$$\begin{aligned} &\frac{1}{4} \int j\omega \boldsymbol{P}_{12} \cdot a_1^* \mathbf{e}_1^* dv + c.c. \\ &\quad = j\kappa_{12} a_1^* a_2 - j\kappa_{12}^* a_1 a_2^* \\ &\quad = \frac{1}{4} \int j\omega \delta\epsilon_1 \mathbf{e}_2 \cdot \mathbf{e}_1^* dv a_1^* a_2 + c.c. \end{aligned} \tag{2.15}$$

and must be responsible for the rate of growth in time of the energy in resonator (1) due to the presence of mode (2).

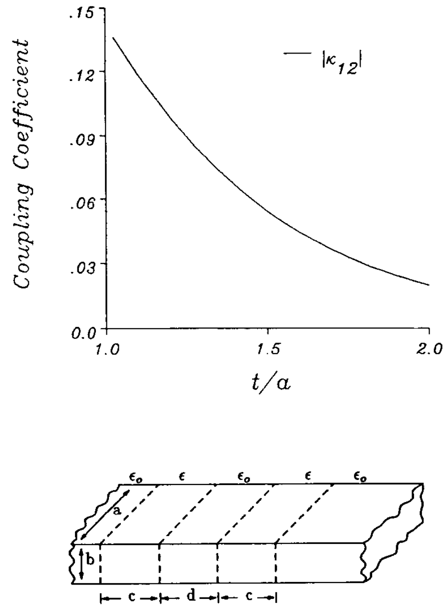


Fig. 3. The coupling coefficients κ_{12} (solid line) and κ_{11} (dashed line) as functions of the resonator separation. The parameters of the waveguides are $a/b = 1, c/b = 1$ and $\epsilon/\epsilon_o = 2.25$.

Since the excitation amplitudes are arbitrary, the equality must hold:

$$\kappa_{12} = \frac{\frac{1}{4}\omega \int \delta\epsilon_1 \boldsymbol{e}_1^* \cdot \boldsymbol{e}_2 dv}{\frac{1}{2}\int \epsilon |\boldsymbol{e}_1|^2 dv} \tag{2.16}$$

where we have written the result independent of the normalization. Figure 3 shows a plot of the coupling coefficient as a function of normalized resonator separation.

C. Nonorthogonal Modes of Positive and Negative Energy

All modes need not have positive energy. An example of negative energy is a moving electron beam or plasma. If the equations of this system are linearized, and wave solutions are found, the energy of these waves can be negative. A negative energy simply means that the energy of the system is lower when the wave is excited than when it is not: A moving electron beam stores positive kinetic energy and electromagnetic energy. The excitation of the wave decreases this energy.

For the present discussion we need not assume that there are only two coupled waves. Suppose that there are n modes of interest, their n amplitudes are arranged in a column vector of nth order $\boldsymbol{a}$. In the absence of coupling, the modes of different resonance frequencies must be energy orthogonal. If they were not, if there were cross terms of energy, they would vary with time, and the energy would not be independent of time. If the modes are of the same frequency, they can always be orthogonalized. The energy W of the system can then be written

$$W = \boldsymbol{a}^\dagger \boldsymbol{W} \boldsymbol{a} \tag{2.17}$$

where $\boldsymbol{W}$ is a square matrix of nth rank and may be positive definite, if all wave energies are positive, or indefinite, if there are wave energies of either sign. The dagger indicates a Hermitian transpose. Then the equations of motion of the modes can be written

$$\boldsymbol{W}\frac{d\boldsymbol{a}}{dt} = j\boldsymbol{H}\boldsymbol{a} \tag{2.18}$$

where $\boldsymbol{H}$ is the coupling matrix that incorporates the (small) frequency differences and the coupling. If energy is to be conserved we have

$$0 = \frac{d}{dt}\boldsymbol{a}^\dagger \boldsymbol{W}\boldsymbol{a} = j[\boldsymbol{a}^\dagger \boldsymbol{H}\boldsymbol{a} - \boldsymbol{a}^\dagger \boldsymbol{H}^\dagger \boldsymbol{a}]. \tag{2.19}$$

This puts a constraint on the matrix $\boldsymbol{H}$. Because the initial conditions are arbitrary, the matrix $\boldsymbol{a}$ is arbitrary and one must have

$$\boldsymbol{H} = \boldsymbol{H}^\dagger. \tag{2.20}$$

The energy nonorthogonality may appear naturally as a consequence of the coupling, as we shall see later on. Hence it is of importance to understand its consequences. We shall look at these consequences in greater detail when we address coupling of modes in space, in the context of which power nonorthogonality [96]–[98] was first recognized as an important effect. However, the formalism can always be cast into a form that is energy orthogonal. Let us briefly look at the orthogonalization. Because the energy W is real, the matrix $\boldsymbol{W}$ is Hermitian. A Hermitian matrix can always be diagonalized by a unitary transformation. Denote the matrix that does it by $\boldsymbol{U}$. Then,

$$\boldsymbol{U}\boldsymbol{W}\boldsymbol{U}^\dagger = \boldsymbol{P} \tag{2.21}$$

where $\boldsymbol{P}$ is a diagonal matrix. It may be indefinite, possessing both positive and negative diagonal elements. With no loss of generality one may assume that the elements are all either $+1$ or -1, because the original amplitudes a_i of the modes can be properly normalized for that purpose. Then, of course,

$$\boldsymbol{P}^\dagger \boldsymbol{P} = \boldsymbol{P}\boldsymbol{P}^\dagger = \boldsymbol{I} \tag{2.22}$$

where $\boldsymbol{I}$ is the identity matrix. Now define the new amplitude matrix $\boldsymbol{b}$:

$$\boldsymbol{b} = \boldsymbol{U}\boldsymbol{a}. \tag{2.23}$$

Multiplying both sides of (2.18) by $\boldsymbol{U}$ we obtain an equation for the amplitude matrix $\boldsymbol{b}$:

$$\frac{d}{dt}\boldsymbol{b} = j\boldsymbol{M}\boldsymbol{b} \tag{2.24}$$

where

$$\boldsymbol{M} \equiv \boldsymbol{P}^\dagger \boldsymbol{U}\boldsymbol{H}\boldsymbol{U}^\dagger. \tag{2.25}$$

The new coupling matrix has to obey the constraint imposed by energy conservation. Indeed let us study the matrix $\boldsymbol{P}\boldsymbol{M}$:

$$\boldsymbol{P}\boldsymbol{M} = \boldsymbol{U}\boldsymbol{H}\boldsymbol{U}^\dagger \tag{2.26}$$

and thus

$$PM = M^{\dagger}P. \tag{2.27}$$

We have thus derived a coupling of modes formalism that is energy orthogonal and allows for both positive and negative energy modes. In the case when $\boldsymbol{P}$ is positive-definite, or the identity matrix, we see that $\boldsymbol{M}$ is Hermitian. A Hermitian matrix can have only real eigenvalues, i.e, all frequencies of the coupled oscillators must be real. When $\boldsymbol{P}$ is indefinite, $\boldsymbol{M}$ is not Hermitian, the eigenvalues can be complex. One can show that the complex eigenvalues appear in complex-conjugate pairs; for every solution growing in time one finds a solution that decays in time.

Now that we have developed the general formalism for coupling of modes that do not all have positive energy, it is of interest to look at a special case of two modes, one of which has positive energy, the other negative energy. As mentioned earlier, this could be the case of an electromagnetic mode of a cavity through which moves a plasma with a wave that carries negative energy. Clearly the coupling of modes in time implies that the two respective modes change uniformly over all of space. This means that the plasma may have to be reentrant. A more realistic model is a parametric oscillator, in which a signal mode at frequency ω_s, and an idler mode at frequency ω_i are coupled by a pump of frequency ω_p with

$$\omega_p = \omega_s + \omega_i. \tag{2.28}$$

This case is not strictly one of energy conservation, but rather the case of a system obeying the Manley–Rowe relations [151]. The Manley–Rowe relations were derived originally from classical considerations, and can be deduced from the Hamiltonian of the (lossless nonlinear) system [152]. However, they may be stated in terms of photon number "conservation" [153]. In a parametric oscillator, the pump is stimulated to emit into the idler and into the signal mode. There is a simultaneous generation of signal photons and idler photons. Thus if $|b_1|^2$ stands for the number of signal photons, $|b_2|^2$ for the number of idler photons, then the relation holds:

$$\frac{d}{dt}|b_1|^2 = \frac{d}{dt}|b_2|^2. \tag{2.29}$$

But this looks like an energy conservation relation with one mode having negative energy. Now that we have stated the type of system we are analyzing, let us look at the solution of the determinantal equation. The equations are of the form of (2.3) and (2.4), except that the a_i's have to be replaced by b_i's, and that, according to (2.25):

$$\kappa_{12} = -\kappa_{21}^* \equiv \kappa. \tag{2.30}$$

The solution of the determinantal equation is now

$$\omega = \frac{\omega_1 + \omega_2}{2} \pm \sqrt{\left(\frac{\omega_1 - \omega_2}{2}\right)^2 - |\kappa|^2}. \tag{2.31}$$

When the two modes have nearly the same frequency, $|\omega_1 - \omega_2| < 2|\kappa|$, the eigenvalues are complex, there is a growing mode and a decaying mode. One should point out that "synchronism," i.e., $\omega_1 = \omega_2$, in the case of the parametric oscillator, occurs when (2.28) is strictly obeyed.

The growth and decay found in the present case is the signature of an active unstable system. Positive energy (or photon number) in one mode grows or decays in consonance with the growth or decay of the (negative) energy (or photon number) in the other mode.

IV. The Variational Principle for Positive Energy Modes

Thus far we developed the intuitive approach to coupled mode theory. Clearly, if the theory is indeed physically correct, then it ought to be possible to derive it from the fundamental equations of the system. This is indeed possible. Several approaches are possible. In the case of a purely electromagnetic system, Schelkunoff derived the coupling of modes formalism from an expansion of the field in terms of a complete set of modes [10]. Another approach is to derive the coupled mode formalism from a variational principle. This has been done for the case of an electron beam in a microwave structure [11], and can be done rather easily for the case of a purely electromagnetic structure with a positive definite energy matrix $\boldsymbol{W}$. We shall concentrate on this case here because it is simple and gives insight into the procedure.

A variational principle provides an approximate value for an eigenvalue of a differential equation that is more accurate than the trial solution used in its evaluation. This is an important aspect of the use of a variational principle. Indeed, when one assumes that the excitation of a coupled system is given by a linear superposition of the modes of an uncoupled system one uses field patterns that have been obtained in the absence of coupling, field patterns that ignore the coupling. Thus, how does one expect to obtain a reliable prediction of the time evolution of the system, if the fields employed in the evaluation ignore the coupling? The variational principle lays this criticism to rest.

The equation for the electric field in a medium of dielectric tensor $\bar{\bar{\epsilon}}$, a function of position, is

$$\nabla \times (\nabla \times \boldsymbol{E}) = \omega^2 \mu_o \bar{\bar{\epsilon}} \cdot \boldsymbol{E}. \tag{3.1}$$

Suppose further that the system is enclosed in an enclosure consisting of perfect electric and/or magnetic conductors. Then, on the enclosure the boundary conditions are satisfied:

$$\boldsymbol{n} \times (\nabla \times \boldsymbol{E}) = 0 \tag{3.2a}$$

on the magnetic conductor, and

$$\boldsymbol{n} \times \boldsymbol{E} = 0 \tag{3.2b}$$

on the electric conductor. Finally, consider the constraint on the $\bar{\bar{\epsilon}}$ tensor imposed by the losslessness of the medium. It is well known that the ϵ tensor written in Cartesian coordinates forms a Hermitian matrix obeying the relationship:

$$\bar{\bar{\epsilon}}^{\dagger} = \bar{\bar{\epsilon}}. \tag{3.3}$$

Here, the dagger indicates, as usual, the complex-conjugate transpose of the tensor. Let us now prove the variational character of (2.18). By dot-multiplying (3.1) by $\boldsymbol{E}^*$, integrating over the volume enclosed by the enclosure, and by integration by parts, using the boundary conditions (3.2) one obtains

$$\omega^2 = \frac{\int(\nabla \times \boldsymbol{E}^*)\cdot(\nabla \times \boldsymbol{E})dv}{\mu_o \int \boldsymbol{E}^* \cdot \bar{\bar{\epsilon}} \cdot \boldsymbol{E}dv}. \tag{3.4}$$

Equation (3.4) is a variational expression for the frequency ω. Indeed suppose that we substitute into it a field $\boldsymbol{E}+\delta\boldsymbol{E}$, where $\delta\boldsymbol{E}$ is an error, a deviation from the exact solution. Then we can show that ω^2 will be unaffected to first order in $\delta\boldsymbol{E}$. Taking a perturbation of (3.4) we find

$$\begin{aligned}\delta\omega^2\mu_o \int \boldsymbol{E}^* \cdot \bar{\bar{\epsilon}} \cdot \boldsymbol{E}dv &+ \omega^2\mu_o \int \delta\boldsymbol{E}^* \cdot \bar{\bar{\epsilon}} \cdot \boldsymbol{E}dv \\ &+ \omega^2\mu_o \int \boldsymbol{E}^* \cdot \bar{\bar{\epsilon}} \cdot \delta\boldsymbol{E}dv \\ &= \int(\nabla \times \delta\boldsymbol{E}^*)\cdot(\nabla \times \boldsymbol{E})dv \\ &+ \int(\nabla \times \boldsymbol{E}^*)\cdot(\nabla \times \delta\boldsymbol{E})dv \end{aligned} \tag{3.5}$$

Because of (3.3), the order of the vectors on the left-hand side of (3.5) may be reversed. Further, integration by parts with the aid of the boundary conditions (3.2) gives

$$\begin{aligned}\delta\omega^2 \int \boldsymbol{E}^* \cdot \bar{\bar{\epsilon}} \cdot \boldsymbol{E}dv &+ \int dv\delta\boldsymbol{E}^* \cdot [\omega^2\mu_o\bar{\bar{\epsilon}} \cdot \boldsymbol{E} - \nabla \times (\nabla \times \boldsymbol{E})] \\ &+ \int dv\delta\boldsymbol{E} \cdot [\omega^2\mu_o\bar{\bar{\epsilon}}^* \cdot \boldsymbol{E}^* - \nabla \times (\nabla \times \boldsymbol{E})^*] = 0. \end{aligned} \tag{3.6}$$

The integrands in the last two expressions of (3.6) vanish and thus $\delta\omega^2$ must be zero to first order in $\delta\boldsymbol{E}$. This completes the proof of the variational character of (3.4).

We shall now apply the formalism to a multimode resonator. The first part of the analysis will be general, with no specific situation in mind. Then we shall specialize the analysis to the case of two dielectric cavities that are coupled by their fringing fields.

The field patterns we use for the trial solution consist of the fields in some spatially varying dielectric medium, in general a different medium for every mode. We denote the modes by $\boldsymbol{e}_i$ and the dielectric constant distribution by $\bar{\bar{\epsilon}}_i$. The dielectric tensors $\bar{\bar{\epsilon}}_i$ are assumed to be those of lossless media. The modes are assumed to obey the wave equation

$$\nabla \times (\nabla \times \boldsymbol{e}_i) = \omega_i^2\mu_o\bar{\bar{\epsilon}}_i \cdot \boldsymbol{e}_i \tag{3.7}$$

where ω_i is the resonance frequency of the mode. The modes obey the proper boundary conditions over the surface enclosing the volume of the system. We assume as the trial field the linear superposition of these two field patterns:

$$\boldsymbol{E} = \sum_i a_i\boldsymbol{e}_i. \tag{3.8}$$

When this trial solution is introduced into (3.4) one obtains

$$\omega^2 = \frac{\boldsymbol{a}^\dagger\boldsymbol{K}\boldsymbol{a}}{\boldsymbol{a}^\dagger\boldsymbol{W}\boldsymbol{a}} \tag{3.9}$$

where

$$W_{ij} = \int \boldsymbol{e}_i^* \cdot \bar{\bar{\epsilon}} \cdot \boldsymbol{e}_j dv \tag{3.10}$$

and

$$\begin{aligned}\mu_o K_{ij} &= \int(\nabla \times \boldsymbol{e}_i^*)\cdot(\nabla \times \boldsymbol{e}_j)dv \\ &= \omega_j^2\mu_o \int \boldsymbol{e}_i^* \cdot \bar{\bar{\epsilon}}_j \cdot \boldsymbol{e}_j dv. \end{aligned} \tag{3.11}$$

Clearly, both $\boldsymbol{W}$ and $\boldsymbol{K}$ are Hermitian matrices. Because both $\boldsymbol{W}$ and $\boldsymbol{K}$ are also positive definite the value of ω^2 is real and positive, as it ought to be. An stationary value is found for the eigenvalue ω^2 by differentiating the right hand side of (3.9) with respect to the amplitudes and phases of the a_i's. This can be done formally by differentiating (3.9) with respect to the a_i^*'s, keeping the a_j's constant. The Appendix shows that this is indeed legitimate. The result is

$$\omega^2\boldsymbol{W}\boldsymbol{a} = \boldsymbol{K}\boldsymbol{a} \tag{3.12}$$

We shall now show that the above expression can be written as a coupling of modes equation. Indeed, assume that perturbation theory is valid, that all frequencies cluster around a typical value ω_o. Then one may write

$$\omega^2 - \omega_o^2 \simeq 2\omega_o\delta\omega \tag{3.13}$$

where $\delta\omega$ is the deviation from this frequency. However, $j\delta\omega$ can be interpreted as a time derivative, and (3.12) can be written with the help of (3.13):

$$\boldsymbol{W}\frac{d\boldsymbol{a}}{dt} = \frac{j}{2\omega_o}(\boldsymbol{K} - \omega_o^2\boldsymbol{W})\boldsymbol{a}. \tag{3.14}$$

This is clearly in the form of a coupled mode equation with a Hermitian energy matrix $\boldsymbol{W}$ and a Hermitian coupling matrix of the form $\boldsymbol{H} \equiv (\boldsymbol{K} - \omega_o^2\boldsymbol{W})/2\omega_o$. The formalism has not only led to the coupled mode equations, but also provided a recipe for the evaluation of the coupling coefficients.

B. Application to Coupled Dielectric Cavities

We now turn to the coupled mode analysis of the coupled cavities analyzed previously by the simple orthogonal mode coupling theory. Here we handle the case more carefully, allowing for the nonorthogonality introduced by the coupling, which changes the equations somewhat. We compare this improved coupled mode analysis with the previous one, and also with the exact analysis that is not difficult in this simple case.

We introduce two dielectric distributions associated with the two individual waveguides in the absence of the other (see Fig. 2).

$$\bar{\bar{\epsilon}}_1 = (\epsilon_o + \delta\epsilon_1)\boldsymbol{I} \tag{3.15a}$$

$$\bar{\bar{\epsilon}}_2 = (\epsilon_o + \delta\epsilon_2)\boldsymbol{I}. \tag{3.15b}$$

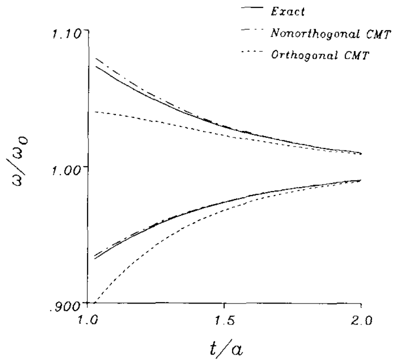


Fig. 4. The resonance frequencies of the symmetric (lower curves) and antisymmetric (upper curves) modes of the coupled resonators in Fig. 2. Solid: Exact solution; Dash–dot: Nonorthogonal coupled mode; Dashed: Orthogonal coupled mode. The parameters are the same as in Fig. 3.

The resonance frequencies are equal and can be set equal to ω_o. The coupled mode equation becomes

$$\boldsymbol{W}\frac{d}{dt}\boldsymbol{a} = j\boldsymbol{H}\boldsymbol{a} \tag{3.16}$$

with the coupling matrix

$$H_{ij} = -\frac{\omega_o}{2}\int \boldsymbol{e}_i^* \cdot \delta\bar{\bar{\epsilon}}_i \cdot \boldsymbol{e}_j dv. \tag{3.17}$$

The coupling coefficients were obtained previously from physical arguments. What was not obtained *ab initio,* when energy orthogonality was taken for granted, were the self terms on the right-hand side. They are corrections that affect the value of the eigenvalues as shown in Fig. 4.

V. Coupling of Modes in Space

Consider two waves in a linear system, with the implicit time dependence $\exp(j\omega t)$ that, uncoupled, have the spatial dependences $\exp(-j\beta_1 z)$ and $\exp(-j\beta_2 z)$, respectively. The two waves are coupled in space, either by a uniform structure, or by some periodic structure, like a grating, or a helix (in a backward oscillator) or some other periodic structure; or by a periodic space–time phenomenon, like an optical pump in a parametric amplifier. In the latter case, the common frequency ω specified above is, in fact, the signal frequency ω_s of mode (1) on one hand, and the frequency $\omega_p - \omega_i$ for mode (2), on the other hand, which is equal to ω_s at synchronism. In a uniform structure, the propagation constants β_1 and β_2 must be of same sign and approximately equal, if the waves are to affect each other. If the coupling structure is periodic with the period Λ, then coupling among space harmonics (Bloch waves or Brillouin components) becomes possible. Suppose, for example, that the coupling coefficient is of the form: $2\kappa_{12}\cos(2\pi z/\Lambda)$. The coupled mode equations in space, then read

$$\frac{d}{dz}a_1 = -j\beta_1 a_1 - j2\kappa_{12}\cos(\frac{2\pi z}{\Lambda})a_2 \tag{4.1}$$

and

$$\frac{d}{dz}a_2 = -j\beta_2 a_2 - j2\kappa_{21}\cos(\frac{2\pi z}{\Lambda})a_1. \tag{4.2}$$

All the arguments as to the form of the coupling made in connection with the coupled mode formalims in time apply to the present case. The modes may carry positive or negative power. Negative power may occur because positive energy is transported in the $-z$ direction (the group velocity is negative) or negative energy is transported forward (positive group velocity). In this connection one may ask as to how is it possible to have two waves with phase velocities of the same sign possess group velocities of opposite sign. This property is common in periodic structures that possess many "Bloch waves," some of which have phase and group velocities of opposite sign. It is coupling with some of these "backward" Bloch waves that gives rise to the physical situations to be discussed below.

Define the power matrix $\boldsymbol{P} = \mathrm{diag}(1, \pm 1)$ where the signs correspond to the sign of the power flow of the two waves, so that with proper normalization, the power can be written

$$\text{Power } = \boldsymbol{a}^\dagger \boldsymbol{P}\boldsymbol{a}. \tag{4.3}$$

Suppose that the space harmonic with the propagation constant $2\pi/\Lambda + \beta_2$ is close to synchronism with β_1. Then using the ansatz

$$a_1 = A_1 \exp\{-j(\frac{\beta_1+\beta_2}{2} + \frac{\pi}{\Lambda})z\} \tag{4.4}$$

$$a_2 = A_2 \exp\{-j(\frac{\beta_1+\beta_2}{2} - \frac{\pi}{\Lambda})z\} \tag{4.5}$$

we can write

$$\frac{dA_1}{dz} = -j(\frac{\beta_1-\beta_2}{2} - \frac{\pi}{\Lambda})A_1 - j\kappa_{12}A_2 \tag{4.6}$$

$$\frac{dA_2}{dz} = -j(\frac{\beta_2-\beta_1}{2} + \frac{\pi}{\Lambda})A_2 - j\kappa_{21}A_1. \tag{4.7}$$

Just as in the case of coupling of modes in time, the coupling matrix **M**, which in the present case is of the form:

$$\boldsymbol{M} = -\begin{bmatrix} \frac{\beta_1-\beta_2}{2} - \frac{\pi}{\Lambda} & \kappa_{12} \\ \kappa_{21} & \frac{\beta_2-\beta_1}{2} + \frac{\pi}{\Lambda} \end{bmatrix} \tag{4.8}$$

has to obey the power conservation law (compare (2.27)):

$$\boldsymbol{PM} = \boldsymbol{M}^\dagger \boldsymbol{P}. \tag{4.9}$$

The power matrix may be positive-definite or indefinite. The physical reasons for this are more varied than in the case of coupling of modes in time. Thus we may have the simple situation of two waves with colinear group velocities and positive energies. Such waves carry power in the same direction and have a positive definite $\boldsymbol{P}$ matrix. The two waves may have oppositely directed group velocities, but energies of same sign. Then $\boldsymbol{P}$ is indefinite. On the other hand, the energies may be of opposite signs, the group velocities codirectional, and then $\boldsymbol{P}$ is also indefinite.

We shall concentrate here on the case of two synchronous waves when the two modes are phase matched. For the

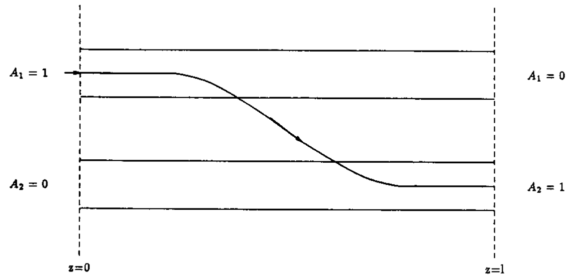


Fig. 5. Schematic of a dielectric waveguide coupler.

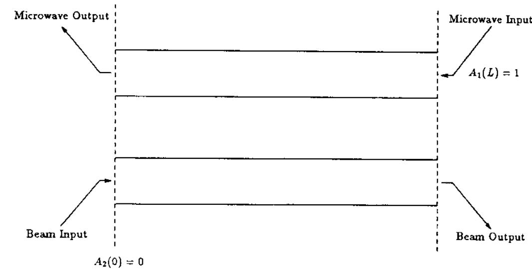


Fig. 6. Schematic of a backward wave microwave oscillator.

uniform structures where the period $\Lambda \to \infty$, the phase-matching requires that $\beta_1 = \beta_2$; for the periodic structures, it leads to $\beta_1 - \beta_2 = 2\pi/\Lambda$. Thus, coupling can occur, even when the propagation constants are widely different, but the periodicity of the structure makes up for the difference. In this way, waves with opposite phase velocities can be made to couple to each other.

Under phase-matching, the coupled mode equations assume the simpler form:

$$\frac{dA_1}{dz} = -j\kappa A_2 \tag{4.9}$$

$$\frac{dA_2}{dz} = \mp j\kappa^* A_1 \tag{4.10}$$

where $\kappa = \kappa_{12}$. The two signs correspond to equal or opposite directions of power flow. The solutions are

$$A_1 = A_+ e^{-j|\kappa|z} + A_- e^{j|\kappa|z}. \tag{4.11a}$$

$$A_2 = \frac{|\kappa|}{\kappa}[A_+ e^{-j|\kappa|z} - A_- e^{j|\kappa|z}] \tag{4.11b}$$

for $\boldsymbol{P}$ positive (or negative) definite

$$A_1 = A_+ e^{-|\kappa|z} + A_- e^{|\kappa|z}. \tag{4.12a}$$

$$A_2 = -j\frac{|\kappa|}{\kappa}(A_+ e^{-|\kappa|z} - A_- e^{|\kappa|z}) \tag{4.12b}$$

for $\boldsymbol{P}$ indefinite.

Consider first the periodic solutions (4.11). If the system is passive, the energies of the waves are positive, the two group velocities must be codirectional. Suppose the system is excited from the left in mode (1). After a distance $\pi/2|\kappa|$ the excitation is all in mode (2). This is analogous to coupling of modes in time discussed earlier (see Fig. 5). Examples of structures employing this behavior are waveguide couplers used in microwaves and optics.

However, active structures with opposite group velocities have the same solution. Such is a backward wave amplifier. Suppose wave (1) is the circuit wave with negative group velocity. It is excited at the far end of the structure of length l (see Figs. 6 and 7). The active wave has positive group velocity and enters at $z = 0$, presumably unexcited. The solutions are

$$A_2 = A_o \frac{|\kappa|}{\kappa} \sin|\kappa|z \tag{4.13}$$

$$A_1 = A_o \cos|\kappa|z \tag{4.14}$$

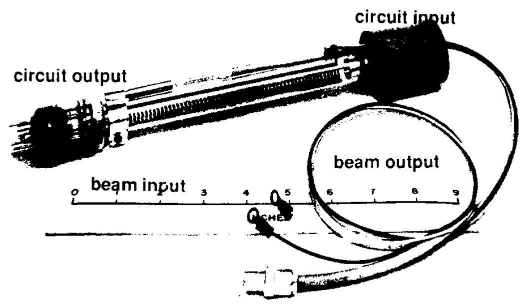


Fig. 7. A photograph of physical backward wave oscillator.

and the gain of the system is

$$\left|\frac{A_1(l)}{A_1(0)}\right|^2 = \frac{1}{\cos^2 \kappa l}. \tag{4.15}$$

The gain goes to infinity when $|\kappa| l = \pi/2$. The "backward wave" amplifier becomes an oscillator. Another physical example is a parameter amplifier with phase-matched signal and idler waves.

Finally, look at the case of exponential solutions. The passive case, where two waves of positive energy and opposite group velocities couple, the forward wave A_1 excited from the left, the backward wave A_2 unexcited from the right, leads to the solution:

$$A_2 = A_o \sinh|\kappa|(z - l) \tag{4.16}$$

$$A_1 = -j\frac{\kappa}{|\kappa|} A_o \cosh|\kappa|(z - l). \tag{4.17}$$

This solution applies to a grating coupler that couples two counterpropagating waves (see Fig. 8). The coupler has the reflectivity:

$$\left|\frac{A_2}{A_1}\right|^2 = \tanh^2|\kappa| l. \tag{4.18}$$

Conversely, if one wave has negative energy and both have positive group velocities, then both waves have to be excited from the left. The negative energy wave A_2 is the beam wave in a TWT, the positive energy wave A_1 is the circuit wave. The gain of the system is

$$\frac{|A_1(l)|^2}{|A_1(0)|^2} = \cosh^2|\kappa| l. \tag{4.19}$$

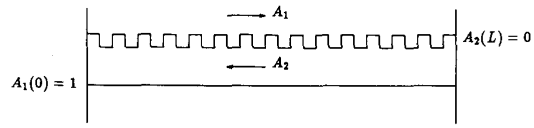


Fig. 8. Schematic of a distributed Brag reflector.

Thus far we looked at the range of different physical device characteristics covered by the simple coupling of modes formalism for coupling in space. Next we shall look at the derivation of the formalism from a variational principle. We shall confine the study to the passive case and consider some issues that have arisen in connection with this formalism.

VI. Vector Nonorthogonal Coupled-Mode Theory

Consider a typical optical waveguide structure with an index distribution that is a function of the transverse coordinates only. The Maxwell equations for the structure can be written

$$\nabla_t \times \boldsymbol{E} + j\omega\mu_o \boldsymbol{H} = \lambda \hat{\boldsymbol{z}} \times \boldsymbol{E} \quad (5.1a)$$
$$\nabla_t \times \boldsymbol{H} - j\omega\bar{\bar{\epsilon}} \cdot \boldsymbol{E} = \lambda \hat{\boldsymbol{z}} \times \boldsymbol{H} \quad (5.1b)$$

where $\bar{\bar{\epsilon}}$ is the dielectric tensor of the medium and

$$\nabla_t = \hat{\boldsymbol{x}}\frac{\partial}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y}.$$

Multiply (5.1a) by $\boldsymbol{H}^*$, (5.1b) by $\boldsymbol{E}^*$, subtract, and integrate over the entire cross section. Solving for λ one obtains

$$\lambda = \int [\boldsymbol{H}^* \cdot (\nabla_t \times \boldsymbol{E} + j\omega\mu_o \boldsymbol{H}) - \boldsymbol{E}^* \cdot (\nabla_t \times \boldsymbol{H} - j\omega\bar{\bar{\epsilon}} \cdot \boldsymbol{E})] da \cdot \left\{ \int [\boldsymbol{E} \times \boldsymbol{H}^* + \boldsymbol{E}^* \times \boldsymbol{H}] \cdot \hat{\boldsymbol{z}} da \right\}^{-1}. \quad (5.2)$$

For a lossless medium,

$$\bar{\bar{\epsilon}}^{\dagger} = \bar{\bar{\epsilon}}. \quad (5.3)$$

Expression (5.2) can be shown to be stationary if both $\hat{\boldsymbol{n}} \times \boldsymbol{E}$ or $\hat{\boldsymbol{n}} \times \boldsymbol{H}$ are continuous where $\hat{\boldsymbol{n}}$ is the normal unit vector to the boundaries across the index discontinuities, $\hat{\boldsymbol{n}} \times \boldsymbol{E}$ or $\hat{\boldsymbol{n}} \times \hat{\boldsymbol{H}}$ vanish on an external boundary or at infinity.

A. Nonorthogonal Coupled Mode Formulations

To derive the coupled mode theory from the variational principle, we use the vector waveguide modes as trial solutions. The vector waveguide modes obey the following equations:

$$\nabla_t \times \boldsymbol{e}_i + j\omega\mu_o \boldsymbol{h}_i = j\beta_i \hat{\boldsymbol{z}} \times \boldsymbol{e}_i \quad (5.4)$$
$$\nabla_t \times \boldsymbol{h}_i - j\omega\epsilon_i \boldsymbol{e}_i = j\beta_i \hat{\boldsymbol{z}} \times \boldsymbol{h}_i \quad (5.5)$$

where ϵ_i is the dielectric constant distribution that includes the dielectric constant of the ith guide, which is assumed to be lossless and isotropic. β_i is the propagation constant associated with the vector mode in the ith guide. For simplicity, only one guided-mode is assigned to each guide. More than one guided mode for each guide and even the radiation modes may be included in the summation without much modification of the derivation.

In forming the linear superposition of the modes, we expand the total fields (both transverse and longitudinal) such that

$$\boldsymbol{E} = \sum a_i(z)\boldsymbol{e}_i(x,y) \quad (5.6a)$$
$$\boldsymbol{H} = \sum a_i(z)\boldsymbol{h}_i(x,y) \quad (5.6b)$$

where $\boldsymbol{e}_i$ and $\boldsymbol{h}_i$ are the total electric and magnetic fields of the ith individual guide. Substitute the trial solution (5.6) into the variational expressions (5.2).

$$\lambda = j\beta = j\frac{\sum_{i,j} a_i^* H_{ij} a_j}{\sum_{i,j} a_i^* P_{ij} a_j} \quad (5.7)$$

where

$$H_{ij} = P_{ij}\beta_j + \kappa_{ij} \quad (5.8)$$

$$P_{ij} = \frac{1}{4}\int [\boldsymbol{e}_i^* \times \boldsymbol{h}_j + \boldsymbol{e}_j \times \boldsymbol{h}_i^*] \cdot \hat{z} da \quad (5.9)$$

$$\kappa_{ij} = \frac{1}{4}\omega \int \boldsymbol{e}_i^* \cdot [\bar{\bar{\epsilon}} - \epsilon_j \boldsymbol{I}] \cdot \boldsymbol{e}_j da. \quad (5.10)$$

Next we use the stationary property of the expression for λ and differentiate with respect to a_i^* or a_j [141].

$$\lambda \sum_j P_{ij} a_j = j \sum_j H_{ij} a_j. \quad (5.11)$$

Since

$$\frac{d}{dz} = -j\beta = -\lambda.$$

The coupled mode equations results by replacing $-\lambda$ by d/dz:

$$\sum_j P_{ij}\frac{da_j}{dz} = -j\sum_j H_{ij} a_j. \quad (5.12)$$

Since the waveguides are lossless, power conservation imposes a constraint on P_{ij}:

$$\frac{d}{dz}\sum a_i^* P_{ij} a_j = 0. \quad (5.13)$$

Making use of (5.12), it follows from (5.13) that

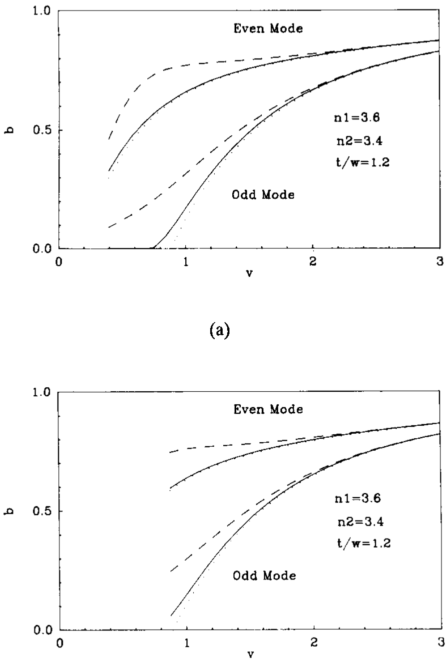
$$H_{ij} = H_{ji}^* \quad (5.14)$$

which may be further reduced to

$$P_{ij}(\beta_j - \beta_i) = \kappa_{ij} - \kappa_{ji}^*. \quad (5.15)$$

We note that the κ_{ij} matrix is Hermitian only when synchronism prevails, $\beta_i = \beta_j$.

Figure (9a) and (9b) shows the dispersion characteristics of TE and TM modes for parallel slabs. The index difference $(n_1 - n_2)/n_1$ is assumed to be small (5.6%), but the



(b)

Fig. 9. The normalized propagation constants of the symmetric (upper curves) and antisymmetric (lower curves) mode of two coupled slab waveguides. (a) TE modes. (b) TM modes. Dashed: Orthogonal coupling of modes; Dotted: Nonorthogonal coupling of modes; Solid: Exact analysis. The parameters used $t/w = 1.2$, $n_1 = 3.6$, and $n_2 = 3.4$.

two guides are very close to each other ($t/a = 1.2$). The nonorthogonal coupled mode theory yields more accurate results than the conventional theory which neglects the effect of nonorthogonality.

VII. Conclusions

We have reviewed the extensive literature on coupling of modes. Then we studied briefly the powerful insight that coupling of modes permits into the processes of interacting systems, first in the formalism of coupling of modes in time. The case of energy orthogonal modes showed that coupling of two modes with positive energies leads to a splitting of the resonance frequencies. The case of modes with energies of opposite sign is the unstable case, provided, of course, the two systems have natural frequencies that are not too different. Even though the case of the parametric oscillator is not, strictly, a case of energies of opposite sign, the Manley Rowe relations provide a conservation law that leads to a system description congruent with that of a system of coupled modes with opposite signs of energy.

Energy orthogonality provides a simple picture of coupled mode theory. If the energies are not orthogonal, the description changes. The nonorthogonality is, usually, the consequence of the coupling itself. For this reason it does not cause a qualitatively different behavior. Quantitatively, differences appear which we explored with the aid of an example of two coupled cavities that could be analyzed by both methods as well as exactly. The refinement of energy nonorthogonality calls for more systematic methods for the determination of the coupling matrices. We have shown that coupling of modes in time can be based on a variational principle that leads to the coupled mode equations when the trial solution is made up of a linear superposition of mode patterns.

Next, we considered coupling of modes in space. Here the physical phenomena covered by coupling of two modes are more numerous, because the modes can have energies of equal or opposite sign, and power flows of equal and opposite signs. The waveguide coupler, traveling wave tube, backward wave oscillator and grating reflector were examples.

Just as in the case of the coupling of modes in time formalism, the coupling of modes in space was also derived from a variational principle that was illustrated in the case of a passive lossfree electromagnetic structure. Here again the subtleties of coupling of nonorthogonal modes were mastered via a variational principle.

VIII. Appendix

A. The Extremization

Differentiation of an expression like $E \equiv a_i^* M_{ij} a_j$ must be carried out with respect to the pair of variables $|a_i|$ and $\arg a_i = \phi_i$ of the complex variable a_i. Thus for example,

$$\frac{\partial}{\partial |a_i|} E = e^{-j(\phi_i - \phi_j)} M_{ij} |a_j| \qquad i \neq j$$

and

$$\frac{\partial}{\partial |a_i|} E = 2M_{ii}|a_i| \qquad \text{for } i = j$$

and

$$\frac{\partial}{\partial \phi_i} E = -j|a_i|M_{ij}|a_j|e^{-j(\phi_i - \phi_j)} \qquad i \neq j$$

and

$$\frac{\partial}{\partial \phi_i} E = 0 \qquad i = j.$$

If we apply the above to (3.9) we find

$$e^{-j(\phi_i - \phi_j)}\{\omega^2 W_{ij}|a_j| = \kappa_{ij}|a_j|\} \qquad i \neq j$$

and

$$-je^{-j(\phi_i - \phi_i)}\{\omega^2 W_{ij}|a_j| = \kappa_{ij}|a_j|\} \qquad i \neq j.$$

These two equations are equivalent to

$$\omega^2 W_{ij} a_j = \kappa_{ij} a_j.$$

But, this equation is obtained equivalently by differentiating

$$\omega^2 a_i^* W_{ij} a_j = a_i^* \kappa_{ij} a$$

with respect to a_i^*. The argument carries through for $a_i = a_j$ as well.

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