MATH 3353-001/2, Fall 2016
Lab # 5: **DUE 11/9/16**

To receive credit for this lab, you must turn in your diary that demonstrates you running these commands. Please name it as follows: <Your_last_name>_lab5_diary.txt (for example, `Barreiro_lab5_diary.txt`). There will be a place on Canvas for you to submit your file. I recommend that you save your commands in a `.m` file (for example, `myscript.m`): this will allow you to work out any mistakes before you run it and save your diary. Then run:

```
>> diary Barreiro_lab5_diary.txt
>> myscript
>> diary off
```

and you're done! **Don't forget to erase your diary before you do your submission run**

Introduction: In this lab, we will use linear algebra to model a seasonally appropriate topic....the zombie apocalypse!!! We will model the spread of the zombie virus through the population using a *Markov chain* (such models are discussed in §4.9 of your text). To begin with, we divide the population into 4 parts: susceptible (S), infected (I), zombie (Z) and dead/destroyed (D). We will interpret these numbers as fractions of the total population, so that $S + I + Z + D = 1$.

Next, we consider how people can shift from one population to another: we'll assume that the number of individuals that undergo each process per unit of time can be described as a linear function of the number of eligible individuals: for example, the number of susceptible individuals that become infected is rS, for some constant r. The possible transitions that can occur are:

- Susceptible individuals can be infected (at rate r).
- Infected individuals can become zombies (at rate z).
- Zombies can be killed (at rate k).

The matrix of rates we just described looks like:

$$A = \begin{bmatrix} -r & 0 & 0 & 0 \\ r & -z & 0 & 0 \\ 0 & z & -k & 0 \\ 0 & 0 & k & 0 \end{bmatrix} \tag{1}$$

For now, we assume that the time scale is so short that — for practical purposes — there are no births. Therefore the population (when deceased persons are included) will remain constant: each column of A must sum to zero.

We define the state of the population at time 0 as a vector in $\mathbb{R}^4$:

$$\mathbf{x}_0 = \begin{bmatrix} S \\ I \\ Z \\ D \end{bmatrix} \tag{2}$$

and the population at subsequent time steps is defined using the stochastic matrix $P \equiv I + A$:

$$\mathbf{x}_{n+1} = P\mathbf{x}_n, \qquad \text{for } n = 0, 1, 2, \ldots \tag{3}$$

1. Begin with the following parameters:

$$r = 0.05, z = 0.3, k = 0.1$$

Use the equation (3) to track the evolution of the population for 28 time units. Use the initial condition $(0.999, 0.001, 0, 0)$. Print the following to your diary:

   (a) The stochastic matrix P.

   (b) The population vector $\mathbf{x}_n$ at $n = 7, 14$, and 28.

   (c) **Optional**: Plot your results: for example, if `X` is stored as an 4 x 29 matrix, use:
   `>> figure; plot(X');`

   (the single quote `'` takes the transpose; since `plot` will treat each column of a multi-column matrix as a separate data source, this allows us to use a single plot command for all 4 populations).

2. [**Patient Zero**] Does it just take one person to start the zombie apocalypse? Test this by choosing the initial condition to reflect the total population of a geographical region. For example:

   (a) [**A large metropolitan area**] The Dallas-Forth Worth metroplex contains about $N = 6.7 \times 10^6$ million people.

   (b) [**A small town in rural Texas...**] In contrast, the nearby small town of Edom contains just $N = 373$ people.

   For each setting, model the "patient zero" situation by defining your initial population vector, $\mathbf{x}_0$, with $I = 1/N$, $S = 1 - 1/N$. Print the population vector, $\mathbf{x}_n$, at $n = 7, 14$, and 28.

   Optional: Plot to visualize the evolution.

3. [**Zombie physics**] What if we incorporate the rules from our favorite zombie movies?

   (a) [**28 Days Later**] Once infected, people become zombies very quickly...increase z. (must have $z \leq 1$)

   (b) [**The Walking Dead**] Zombies can be made docile by removing their jaws and arms...decrease r (must have $r \geq 0$).

   For each rule, print your altered parameter (z or r) and report the population statistics at $t = 7, 14$, and 28.

   Optional: Plot to visualize the evolution.

4. [**Predicting the distant future**] What happens to the population in the long run? You can probably guess an equilibrium vector $\mathbf{q}$. See the discussion on pg. 258–260 of your text.

   (a) Check your answer by printing $\mathbf{q}$ and $P\mathbf{q}$ to your diary.

   (b) Check your answer by examining $\text{Nul}\,(P - I)$ (you can use `rref` or `null`).

   You can show that P is not a regular stochastic matrix. Nevertheless, in this case the equilibrium vector, $\mathbf{q}$, is still unique.

5. [**A slightly more hopeful scenario**] Suppose we now introduce a non-zero birth rate β. So that $\mathbf{x}$ is still a population vector, we will model this as a transition from the "dead" population as follows[1]:

$$A_\beta = \begin{bmatrix} -r & 0 & 0 & \beta \\ r & -z & 0 & 0 \\ 0 & z & -k & 0 \\ 0 & 0 & k & -\beta \end{bmatrix} \tag{4}$$

As parameters use:

$$r = 0.05, z = 0.3, k = 0.1, \beta = 0.01$$

(a) First, use Matlab commands to show that $P_\beta \equiv I + A_\beta$ is a *regular stochastic matrix* (see definition on pg. 260).

(b) Find the unique equilibrium vector $\mathbf{q}_\beta$ (see Theorem 18 on pg. 261).

(c) Pick **one** modeling problem above to repeat with A_β (2(a), 2(b), 3(a) or 3(b)). Does using A_β vs. A make any difference?

6. [**and on that note...**] Happy Halloween!

[1] This is not very good modeling — we could do better by allowing nonlinear interactions — but it lets us stay within the framework of §4.9