Combinatoric and poset structures for the $K(\pi, 1)$ conjecture

by

Sean O'Brien

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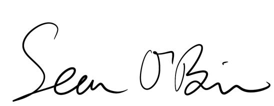


Date: August 15, 2023

Abstract This thesis concerns a very important topic.

Acknowledgement I acknowledge with thanks the inspiration of my supervisor.

Contents

Chapter 1

Introduction

In this paper we will be concerned with the $K(\pi, 1)$ conjecture for Artin groups. This states that the configuration space Y_W for any Coxeter group W is a classifying space for the Artin group G_W. A classifying space for a group G is a space X such that the fundamental group of X is G and all higher homotopy groups of X are trivial. This conjecture emerges as a generalisation of the result for Coxeter groups of type A_n and is originally attributed to Arnol'd, Pham and Thom in [vdL83]. See also [CD95] for a good overview of the history of the conjecture.

We will focus on the work of Paolini and Salvetti, *Proof of the $K(\pi, 1)$ conjecture for affine Artin groups* [PS21]. We will review some theorems therein and provide relevant background. Their main Theorem relies on proving a chain of homotopy equivalences (1), detailing these homotopy equivalences is the aim of this work. A strong theme will be the involvement of posets and related structures, hence this work's title. We will begin by providing a birds eye view of the conjecture and the main results in [PS21].

1.1 The conjecture and the objects involved

Coxeter groups emerge as generalisations of reflection groups. To each Coxeter group W, there is data encoded by a labelled graph that in turn determines an associated group presentation isomorphic to W. This data is called a Coxeter system, denoted (W, S) where S is the generating set of W. Given a Coxeter system (W, S), we can form a different group G_W, the Artin group associated to W.

For affine Coxeter groups W, the configuration space Y_W can be derived from a geometric realisation of W as a subgroup of $\mathrm{Isom}(\mathbb{E})$, the group of isometries on a Euclidean space $\mathbb{E}$. Where we consider $\mathbb{E}$ as $\mathbb{R}^n$ without the notion of origin. Specifically, W is realised as a subgroup generated by a finite set of affine reflections S. Within W, we consider the set of all reflections R (not necessarily finite). To each reflection $r_i \in R$ there is a corresponding codimension–1 space $H_i \subset \mathbb{E}$ that is the plane of reflection of r_i.

We call such spaces hyperplanes. Note there is no requirement that these hyperplanes be subspaces of $\mathbb{R}^n$.

The configuration space is realised as the complement of the complexification of all such hyperplanes H_i. It is known by work of Brieskorn [Bri71] that the fundamental group of Y_W is G_W. Thus proving the $K(\pi, 1)$ involves showing that the higher homotopy groups of Y_W are trivial. By previous work by Salvetti [Sal87, Sal94], there is a CW complex X_W called the Salvetti complex that is homotopy equivalent to Y_W. Showing homotopy equivalence to X_W thus shows homotopy equivalence to Y_W and accordingly, the Salvetti complex is the starting point of the chain of homotopy equivalences reviewed in this work.

The finishing point of this chain is the interval complex K_W. This is a space realised from certain poset structure on subsets of W. To this poset structure there is an associated group called the dual Artin group, denoted W_w. It was already known (through a now standard construction due to Garside [Gar69], extended by other authors, see [CMW02]) that K_W was a classifying space for the dual Artin group for finite W. In [PS21], the authors extend this proof to affine W. Thus, considering the result of Brieskorn, showing $Y_W \simeq K_W$ for affine W shows that (for affine W) the higher homotopy groups of Y_W are trivial and that the dual Artin group associated to W is isomorphic to the Artin group associated to W.

In the following, we will put all of this to symbols and identify the intermediate spaces used in proving $X_W \simeq K_W$.

1.2 Proof overview

Here we compile several main results from [PS21] in to two theorems. The concern of this work is Theorem 1.1 which proves that the *Salvetti complex* X_W is homotopy equivalent to the *interval complex* K_W. To form the interval complex, associated to (W, S), we must make a choice of *Coxeter element* $w \in W$. This Coxeter element is a product of all the elements of S.

For a subset $T \subset S$, the *parabolic subgroup* W_T is the subgroup of W generated only by elements of T. A parabolic Coxeter element w_T is a product of all elements of T. The space X'_W is a subspace of K_W associated to parabolic Coxeter elements associated to $T \subset S$ such that W_T is finite. Cells in X_W also correspond to such subsets, which is used in proving $X_W \simeq X'_W$.

The space K'_W is also a subspace of K_W. Preimages $\eta^{-1}(d)$ of a certain poset map $\eta\colon K_W \to \mathbb{N}$ have a linear structure as subposets of K_W. For each element $x \in \eta^{-1}(d)$, whether x is in K'_W or not is determined based on whether x comes in between two elements of X'_W in the linear structure of $\eta^{-1}(d)$.

Theorem 1.1 ([PS21])**.** *Given an affine Coxeter system* (W, S)*, the configuration space*

Y_W is homotopy equivalent to the order complex K_W.

Proof. By Theorem 2.21 the Salvetti complex X_W is homotopy equivalent to the configuration space Y_W. Therefore, we need only show $K_W \simeq X_W$. This is done through a composition of homotopy equivalences

$$X_W \overset{(a)}{\simeq} X'_W \overset{(b)}{\simeq} K'_W \overset{(c)}{\simeq} K_W \tag{1}$$

Where the results are gathered from the following sources:

(a) Theorem 3.11 [PS21, Theorem 5.5]

(b) [PS21, Theorem 8.14]

(c) Theorem 4.9 [PS21, Theorem 7.9] □

Furthermore, in the same paper another main result is shown.

Theorem 1.2 ([PS21, Theorem 6.6])**.** *Given an affine Coxeter system (W, S), corresponding affine type Artin group G_W and Coxeter element $w \in W$, the complex K_W is a classifying space for the dual Artin group W_w.*

By work of Brieskorn [Bri71] it is known that $\pi_1(Y_W) \cong G_W$. Thus considering $\pi_1(Y_W)$ and combining Theorems 1.1 and 1.2 gives

$$Y_W \simeq K(G_W, 1)$$
$$G_W \cong W_w$$

for affine G_W.

This proves the $K(\pi, 1)$ conjecture for affine Artin groups and provides a new proof than an affine Artin group is naturally isomorphic to its dual, which was already known for spherical or finite type [Bes03] and affine [MS17] cases.

Alternatives to [Bri71] proving $\pi_1(Y_W) \cong G_W$ for affine Coxeter groups and Coxeter groups of type A_n are [VD83] and [FN62b] respectively.

1.3 Coxeter groups and Artin groups

In this section we will cover the constructions and properties of Coxeter groups and Artin groups. Coxeter groups are a generalisation of *reflection groups*, which are subgroups of $GL_n(\mathbb{R})$ generated by a finite set of reflections. Although the definition of Coxeter groups is tied to an abstract group presentation, we may also think of them as groups acting on a space by compositions of reflections. For example, finite Coxeter groups can be realised as reflection groups on spheres and affine Coxeter groups can be realised as groups generated by affine reflections in $\mathbb{R}^n$ (with plane of reflection not necessarily passing through the origin). Note that the realisation of a Coxeter group as a group generated by reflections is not unique and that some Coxeter groups cannot be realised as a subgroup of $GL_n(\mathbb{R})$.

Definition 1.3. Given a finite set S, let $m\colon S\times S \to \mathbb{N}\cup\infty$ be a symmetric matrix indexed over S where $m(s,s) = 1$ for all $s \in S$ and $m(s,t)$ takes values in $\{2, 3, \dots\} \cup \{\infty\}$ for all $s \neq t$. The *Coxeter group* associated to m is the group with the following presentation.

$$W = \langle S \mid (st)^{m(s,t)} = 1 \quad \forall m(s,t) \neq \infty\rangle$$

The data of m and S along with the associated Coxeter group W is denoted (W, S) and called a *Coxeter system.*

Where $m(s,t) = \infty$, this corresponds to pairs of elements that have no explicit relations. Since $m(s,s) = 1$ for all $s \in S$, all generators have order 2. Note that the data of S and m is not uniquely determined by the isomorphism class of W, hence the need to distinguish a Coxeter system, not just a Coxeter group. The set $R \coloneqq \{wsw^{-1} \mid w \in W, s \in S\}$ is the set of *reflections* in W. Sometimes S is referred to as the set of *basic reflections*, or that a choice of S is a choice of basic reflections.

A labelled graph, called the *Coxeter diagram*, is often used to encode the data of the matrix m and its corresponding Coxeter group. In this graph, each element of S is a node and relations between pairs in S correspond to labelled edges. There are two conventions for this labelling: The *classical labelling*, where edges with $m(s,t) = 2$ are not drawn, edges with $m(s,t) = 3$ are drawn but not labelled and all other edges are drawn with the value of $m(s,t)$ as their label. And the *modern labelling*, where edges with $m(s,t) = \infty$ are not drawn, edges with $m(s,t) = 2$ are drawn but not labelled and all other edges are drawn and labelled. An example highlighting these different conventions is given in Fig. 1. We will only use the classical labelling here, but awareness of the modern labelling is useful.

Figure 1: Coxeter diagram for a certain Coxeter group with classical labelling (left) and modern labelling (right).

In the classical labelling, if the diagram has multiple connected components then W is a direct product of the groups corresponding to those components. Similarly, in the modern labelling connected components are factors in a free product. Other topological properties of these diagrams can be used, for example, in work of Huang [Hua23] which proves the $K(\pi, 1)$ conjecture for certain W with diagrams being trees or containing cycles. The property of Coxeter groups that allows us to make this graph construction is that every relation in a Coxeter group only involves two generators and is encoded by a number.

To each Coxeter system (W, S) there is an associated Artin group G_W defined as follows.

Definition 1.4. For group elements s and t, let $\Pi(s, t\,;\,n)$ be the alternating product of s and t starting with s with total length n, e.g. $\Pi(s, t\,;\,3) = sts$. Given a Coxeter system (W, S) with associated matrix m, the associated *Artin group* is

$$G_W := \langle S \mid \Pi(s, t\,;\,m(s,t)) = \Pi(t, s\,;\,m(s,t)) \;\; \forall s \neq t \text{ and } m(s,t) \neq \infty \rangle$$

Note that $m(s, s) = 1$ now carries no meaning in the presentation of G_W and that if we add the relation $s^2 = 1$ for all $s \in S$ we retrieve the original Coxeter group. The Coxeter diagram for W also encodes the data of G_W and the connected components of the diagram correspond to factors of G_W as a direct product or as a free product as with W.

Our notation for Artin groups (as with much of the notation here) is from [PS21]. Another common notation is W_Γ and A_Γ for the Coxeter and Artin group corresponding to the Coxeter diagram Γ. When classifying Artin groups, it is common to inherit properties from the corresponding Coxeter group such that "`property` (type) Artin groups" describes a family of Artin groups to which their corresponding Coxeter groups are `property`.

In particular, spherical or finite type Artin groups have associated spherical or finite Coxeter groups. Similarly, affine Artin groups have associated Coxeter groups which are affine.

1.4 Configuration space

This section contains the definition of the configuration space Y_W for a given Coxeter group W and an example where we will show, for W of type A_n, that $\pi_1(Y_W) \cong G_W$ and that $\pi_n(Y_W)$ is trivial for $n > 1$ [FN62b].

For some finite or affine Coxeter group W acting on $\mathbb{R}^n$, the set of reflections $R \in W$ acts on $\mathbb{R}^n$ by reflection through hyperplanes, one for each $r \in R$. For some $r \in R$ denote its hyperplane by $H(r) \subseteq \mathbb{R}^n$. Denote the union of all hyperplanes by $\mathcal{H} := \bigcup_{r \in R} H(r)$. Consider the tensor product $\mathbb{R}^n \otimes \mathbb{C}$. This is isomorphic to $\mathbb{C}^n$ under the natural isomorphism $x \otimes \lambda \mapsto x\lambda$. We can extend the action $W \circlearrowleft \mathbb{R}^n$ to $W \circlearrowleft (\mathbb{R}^n \otimes \mathbb{C})$ via $w \cdot (x \otimes \lambda) = (w \cdot x) \otimes \lambda$. We call this act of transporting objects related to $\mathbb{R}^n$ over to $\mathbb{R}^n \times \mathbb{C}$ (which is isomorphic to $\mathbb{C}^n$) via the tensor product *complexification*. With these tools in mind, we can then make our definition.

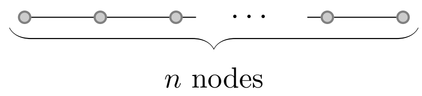


Figure 2: The classical Coxeter diagram for the Coxeter group of type A_n.

Definition 1.5. For an affine Coxeter group W and associated hyperplane system $\mathcal{H}$ as above, we define

$$Y := (\mathbb{R}^n \otimes \mathbb{C}) \setminus (\mathcal{H} \otimes \mathbb{C})$$

and define the *configuration space* Y_W to be the quotient Y/W with the action defined as above.

Note that the importance of $\mathbb{C}$ is that it is 2 dimensional. When one takes the complement of a co-dimension 1 object, you typically will not get any interesting topology. By complexifying the hyperplanes and then taking the complement within $\mathbb{R}^n \otimes \mathbb{C}$, we are effectively taking the complement of a codimension–2 object, and there is much more room for interesting topologies. The same construction can be achieved using $\mathbb{R}^{2n}$ and $\mathcal{H} \times \mathcal{H}$. A more general construction of Y_W for all Coxeter groups using the *Tits cone* can be found in [Par14].

For a concrete example, we will introduce the A_n family of Coxeter groups and show that the space Y_W for these groups is the space of configurations of $n+1$ points in $\mathbb{C}$, thus explaining the name *configuration space* for general Y_W.

The family A_n all have Coxeter diagrams of the form as in Fig. 2 and a specific A_n will have presentation.

$$A_n = \left\langle \sigma_1, \sigma_2, \ldots, \sigma_n \;\middle|\; \begin{array}{rl} \sigma_i^2 = 1 & \forall i \\ (\sigma_i \sigma_j)^2 = 1 & \forall (i+1 < j \leq n) \\ (\sigma_i \sigma_{i+1})^3 = 1 & \forall (i < n) \end{array} \right\rangle \tag{2}$$

This is well known to be a presentation for the symmetric group S_{n+1} with generators being adjacent transpositions [BB10, Proposition 1.5.4]. Accordingly, we will use the associated cycle notation for symmetric groups to talk about elements of A_n.

The action of A_n as a reflection group is realised on the space $\mathbb{R}^{n+1}$ with basis $\{e_i\}$, where $A_n \circlearrowright \mathbb{R}^{n+1}$ by permuting components with respect to that basis. The set of reflections R of A_n is all conjugations of the n adjacent generating transpositions $(l, l+1)$. So, R is the set of all transpositions (l, k). Some $(l, k) \in R$ acts on $\mathbb{R}^{n+1}$ as reflection through the plane $\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_l = x_k\}$. Thus, taking the complement of the complexification of all such planes, we have $Y = \{(\mu_1, \ldots \mu_{n+1}) \in \mathbb{C}^{n+1} \mid \forall i, j \;\; \mu_i \neq \mu_j\}$ (here Y is as in Definition 1.5). We can think of this as the space of $n+1$ distinct labelled points in $\mathbb{C}$, denoted $\mathrm{Conf}_{n+1}(\mathbb{C})$.

Historically, Artin [Art47] originally defined the braid group on n strands B_n to be $\pi_1(\mathrm{Conf}_n(\mathbb{R}^2))$. He then proved the well known presentation of the braid group.

$$B_n = \left\langle \sigma_1, \sigma_2, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{cl} \sigma_i\sigma_j = \sigma_j\sigma_i & \forall(i+1 < j \leq n) \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \forall(i < n) \end{array} \right\rangle$$

In this context, showing the validity of the presentation immediately proves $B_{n+1} \cong G_W$ and thus that $\pi_1(Y_W) \cong G_W$. See the work of Fox and Neuwirth [FN62b] for an alternative proof of the presentation.

1.5 The $K(\pi, 1)$ conjecture

Given a group G and natural number n, an *Eilenberg-MacLane space* [EM45] is a space X such that $\pi_n(X) = G$ and $\pi_i(X) = 0$ for all $1 \leq i \neq n$. We call such an X a *$K(G, n)$ space.* We will also use the terminology *classifying space for G* to mean that X is a $K(G, 1)$ space.

Conjecture 1.6 ($K(\pi, 1)$ Conjecture)**.** *For all Coxeter groups W, the space Y_W is a $K(G_W, 1)$ space.*

Admittedly, the use of π in the name of the conjecture is confusing. An equivalent formulation of the conjecture is that the universal cover of Y_W is contractible. These statements are equivalent since a cover $p\colon \tilde{X} \to X$, induces an isomorphism $p_*\colon \pi_n(\tilde{X}, \tilde{x_0}) \to \pi_n(X, x_0)$ for all $n \geq 2$ [Hat01, Proposition 4.1].

In the previous section we focused on Coxeter groups of type A_n, for which we showed $\pi_1(Y_W) \cong G_W$. We now prove the $K(\pi, 1)$ conjecture for this same family of Coxeter groups. To do so, we need to verify that the higher homotopy groups of Y_W are trivial.

Lemma 1.7. *For all $k > 1$, $\pi_k(\mathrm{Conf}_n(\mathbb{C}))$ is trivial.*

Proof. We use that $\mathrm{Conf}_n(\mathbb{C})$ is a fibre bundle over $\mathrm{Conf}_{n-1}(\mathbb{C})$ with projection p forgetting a point and fibres homeomorphic to $\mathbb{C} \setminus \{\text{n distinct points}\}$ [FN62a, Theorem 3].

The space $\mathbb{C} \setminus \{\text{n distinct points}\}$ is homotopy equivalent to $\bigvee_n S^1$, so we have the fibration $\bigvee_n S^1 \hookrightarrow \mathrm{Conf}_n(\mathbb{C}) \to \mathrm{Conf}_{n-1}(\mathbb{C})$ and the corresponding short exact sequence

$$\pi_k(\textstyle\bigvee_{n-1} S^1) \longrightarrow \pi_k(\mathrm{Conf}_n(\mathbb{C})) \xrightarrow{p} \pi_k(\mathrm{Conf}_{n-1}(\mathbb{C})) \tag{3}$$

for all k. We prove that $\pi_k(\mathrm{Conf}_n \mathbb{C}) = 0$ for all k by induction on n. We know $\pi_k(\bigvee_n S^1) = 0$ for all $k > 1$ and for all n. So the leftmost term in (3) is always trivial. Our base case is $n = 2$. The rightmost term in (3) is $\mathrm{Conf}_1(\mathbb{C}) \cong \mathbb{C}$, which has trivial higher homotopy, thus $\mathrm{Conf}_2(\mathbb{C})$ also has trivial higher homotopy. Assuming now that $\pi_k(\mathrm{Conf}_{n-1}(\mathbb{C})) = 0$ for all $k > 1$, the inductive step follows immediately from the short exact sequence. $\square$

Note that so far we have only proved that Y (from Definition 1.5) has trivial higher homotopy, not Y_W.

Theorem 1.8. *The $K(\pi, 1)$ conjecture holds for Coxeter groups of type A_n.*

Proof. The action $A_n \circlearrowleft Y$ is by permutation of points, therefore Y_W is the space of configurations of $n+1$ *unlabelled* points. Let $q\colon \tilde{Y} \to Y$ be the universal cover for Y. Let $r\colon Y \to Y_W$ be the quotient map induced by the group action. We see that r is a covering map such that $r^{-1}(z)$ is finite for all $z \in Y_W$. Thus, by [Mun00, Exercise 53.4] the composition $r \circ q$ is also a covering map so $\tilde{Y}$ is also the universal cover of Y_W and so Y_W also has trivial higher homotopy by Lemma 1.7. $\square$

A vital result due to Deligne states that this holds for all finite Coxeter groups.

Theorem 1.9 ([Del72])**.** *The $K(\pi, 1)$ conjecture holds for all finite Coxeter groups W.*

The paper of interest to us, [PS21], proves the $K(\pi, 1)$ conjecture for affine Coxeter groups.

Chapter 2

Geometric realisations of poset structures

In Section 1.4 we used the realisation of the Coxeter group W as a reflection group on a space V. We considered the planes of the defining reflections of W as affine subspaces of V and used these to define Y_W, the configuration space. The $K(\pi, 1)$ conjecture concerns with the homotopic properties of Y_W. To explore these, we will first construct a new space X_W, the *Salvetti complex*, which is homotopy equivalent to Y_W. This was originally defined [Sal87, Sal94] similarly using the realisation of W on a space. However, the Salvetti complex turns out to have another formulation based on algebraic properties of W and its defining relations. These structures arise from giving a partial order to W, and with this in mind, we start with some definitions.

2.1 Posets

A partially ordered set or *poset* $(P, \leq)$ is a set P with a relation $\leq$ on pairs in P which encodes the topology of $\mathbb{R}$. The textbook [Grä11] provides a good introduction. An important note is that there is no requirement for every pair to be related, hence the name *partial order*. We will use P as shorthand for $(P, \leq)$ where possible.

In a poset P we define the *interval* between two elements $[x, y]$ as $[x, y] \coloneqq \{u \in P \mid x \leq u \leq y\}$, which is itself a poset. For convenience, we define $[-\infty, w] \coloneqq \{u \in P \mid u \leq w\}$ and equivalently for $[w, \infty]$. A *chain* is a subset $C \subseteq P$ that is a totally ordered, i.e. every pair in $(u, v) \in C \times C$ satisfies $u \leq v$ or $v \leq u$. The *covering relations* of P, denoted $\mathcal{E}(P)$ are defined as follows.

$$\mathcal{E}(P) = \{(x, y) \in P \times P \mid x \leq y \text{ and } [x, y] = \{x, y\}\}$$

These are strictly ordered pairs $x < y$ such that there does not exist any $z \in P$ such that $x < z < y$. If $(x, y) \in \mathcal{E}(P)$, we write $x \lessdot y$. We will call a chain C *saturated* if for

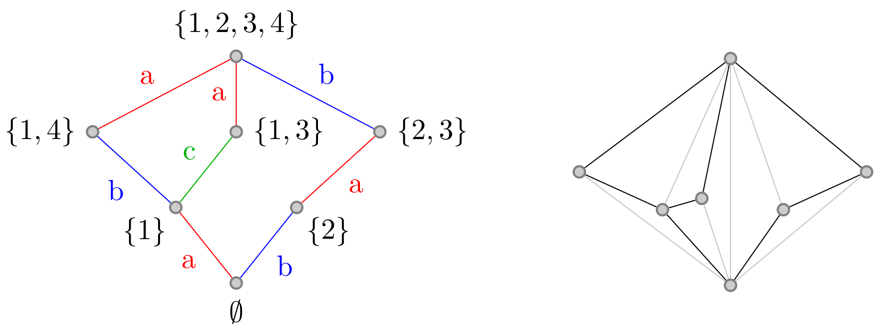


Figure 3: A simple example of a bounded and graded edge labelled poset where we have taken $\leq$ to be $\subseteq$ and $A = \{a, b, c\}$ (left). The same poset with all non-covering 1–chains drawn in faint lines to aid visualising $\Delta(P)$ introduced in Section 2.2 (right).

all $x, y \in C$ such that $x < y$, there exists $z \in C$ such that $x \lessdot z$, i.e. there are no 'gaps' in the chain.

By transitivity, the covering relations encode the whole poset structure, which can in turn be drawn in a diagram.

Definition 2.1. Given a poset P, the *Hasse Diagram* is the directed graph encoding $\mathcal{E}(P)$ in the following way: For each element $x \in P$ draw a vertex. For each pair $(x, y) \in \mathcal{E}(P)$ draw a directed edge from x to y.

As is typical, we will draw Hasse diagrams such that for each edge $x \lessdot y$, the vertex x will be at a lower position on the page than y. Thus, we will not need to draw arrows to show direction. In this work, we will only with deal with graded and bounded posets. *Bounded* means that there are minimal and maximal elements, denoted $\hat{0}$ and $\hat{1}$ such that $\hat{0} \leq x \leq \hat{1}$ for all $x \in P$, and *graded* means that every saturated chain from $\hat{0}$ to $\hat{1}$ has the same (finite) length. In the Hasse diagram for a bounded, graded poset, we will draw $\hat{0}$ at the bottom, $\hat{1}$ at the top, and put all other elements in discrete vertical levels between these based on the position in the saturated chains between $\hat{0}$ and $\hat{1}$ where each element occurs. See Fig. 8 for an example. Graded posets have a natural notion of a *rank function* $\mathrm{rk}\colon P \to \mathbb{N}$ that encodes the height above $\hat{0}$ at which an element $p \in P$ occurs in the Hasse diagram. Rank is also well–defined for posets with multiple minimal or maximal elements, so long as all saturated chains from any minimal element to any maximal element have the same length.

Definition 2.2. We define an *edge labelled poset* to be a triple $(P, \leq, l)$ where $(P, \leq)$ is a poset and the function $l : \mathcal{E}(P) \to A$ a labelling of covering relations with alphabet A.

We will use P as a shorthand for $(P, \leq, l)$ where possible. Given an edge labelled poset P, we can construct a group encoded by its labelling the and geometry of its Hasse diagram.

A *word corresponding to a saturated chain* is the word of the labels traversed in the Hasse diagram while tracing out that saturated chain.

Definition 2.3. Given some edge labelled poset $(P, \leq, l\colon \mathcal{E}(P) \to A)$, define the *poset group* $G(P)$ to be the group generated by $\mathrm{Im}(l)$ with relations equating words corresponding to saturated chains in the Hasse diagram of P which start and end at the same vertices.

In the example given in Fig. 3, the poset group is $G(P) = \langle a, b, c \mid aba = bab,\, ba = ca\rangle$.

2.2 Poset complex

For some edge labelled poset P, we can construct a cell complex $K(P)$ from P such that $\pi_1(K(P))$ is $G(P)$. To do this, we must first make some definitions. An *abstract simplicial complex* is a family of sets that is closed under taking arbitrary subsets.

Definition 2.4. Given a finite abstract simplicial complex X, the *geometric realisation* of that simplicial complex is defined as follows: For each single element set in X assign a point. For each two element set assign an open edge between the two vertices it contains. For each three element set assign an open triangle, the interior of the three edges of its three subsets of size two. In this way, continue constructing simplices of dimension n for each $n + 1$ size set in X.

The set of all chains in a poset P is an abstract simplicial complex. We define $\Delta(P)$ to be the geometric simplicial complex corresponding to the set of all chains in P where each n–simplex is an n–chain of P. Note that as in [MS17, Definition 1.7], we define an n–chain to have $n - 1$ elements, e.g. $(\{1\} \subseteq \{1, 2\})$ is a 1–chain.

For example, in Fig. 3, $\Delta(P)$ would be three 3–simplices all sharing an edge (a 1–simplex) corresponding to the 1–chain $(\emptyset \subseteq \{1, 2, 3, 4\})$ with two of them sharing a face corresponding to the 2–chain $(\emptyset \subseteq \{1\} \subseteq \{1, 2, 3, 4\})$. For a two–dimensional example, consider the following poset P and corresponding $\Delta(P)$. Here we forget about edge labelling in P for a moment.

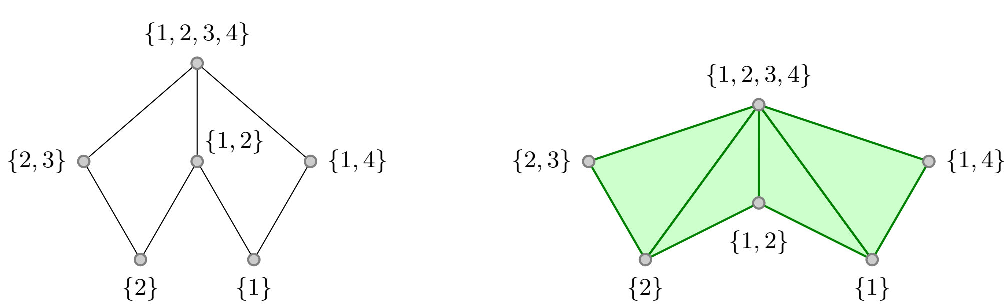


Figure 4: An example poset P (left) and corresponding $\Delta(P)$ (right).

We continue, now using an edge labelling on P (Fig. 5 (left)).

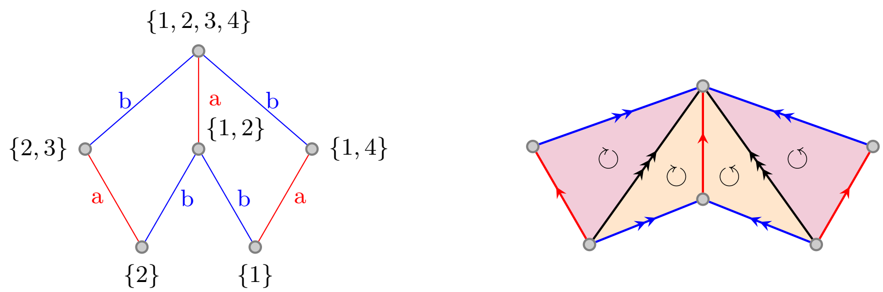


Figure 5: The poset in Fig. 4 with edge labelling (left) and the corresponding space $K(P)$ (right).

To construct $K(P)$, first we define a labelling on chains in P which extends the edge labelling in P. Let A^* denote the set of all words corresponding to the alphabet A.

Definition 2.5. Given some edge–labelled poset $(P, \leq, l\colon \mathcal{E}(P) \to A)$ and some chain $C \subseteq P$, the *extended label* $\mathcal{L}(C) \subseteq A^*$ is the language of all words corresponding to all saturated chains that contain every element of C.

For example, consider the chain $(\{2\} \subseteq \{1,2,3,4\})$ in the context of Fig. 5 (left). There are two corresponding saturated chains, $(\{2\} \subseteq \{1,2\} \subseteq \{1,2,3,4\})$ and $(\{2\} \subseteq \{2,3\} \subseteq \{1,2,3,4\})$, which respectively correspond to the words ba and ab. So $\mathcal{L}(\{2\} \subseteq \{1,2,3,4\}) = \{ba, ab\}$. Here are some illustrative examples:

- $\mathcal{L}(\{1\} \subseteq \{1,2\}) = \mathcal{L}(\{2\} \subseteq \{1,2\}) = \{b\}$.
- $\mathcal{L}(\{1\} \subseteq \{1,2,3,4\}) = \mathcal{L}(\{2\} \subseteq \{1,2,3,4\}) = \{ba, ab\}$.
- $\mathcal{L}(\{1\} \subseteq \{1,2\} \subseteq \{1,2,3,4\}) = \mathcal{L}(\{2\} \subseteq \{1,2\} \subseteq \{1,2,3,4\}) = \{ba\}$.
- $\mathcal{L}(\{1\}) = \mathcal{L}(\{2\}) = \mathcal{L}(\{1,2\}) = \cdots = \emptyset$.

This extended labelling on chains naturally extends to a labelling on simplices in $\Delta(P)$. Using this labelling and the orientation induced on a chain by $\leq$, we can define $K(P)$.

Definition 2.6 (Poset Complex [Mcc05, Definition 1.6])**.** Given some finite height edge–labelled poset P, the poset complex $K(P)$ is the quotient space $\Delta(P)/\sim$ where $\sim$ pointwise identifies simplices of the same dimension that share the same extended label, using the orientation on simplices induced by $\leq$.

In the example in Fig. 5, three red edges are identified, four blue edges are identified, two black edges are identified, two orange triangles are identified and two purple triangles are identified. The orientation of the triangles is indicated by a $\circlearrowright$ symbol.

We see that this space is homeomorphic to a torus, which has fundamental group $\mathbb{Z}^2 \cong \langle a, b \mid ab = ba \rangle$, which is also the $G(P)$ for this edge–labelled poset. This fact holds in general.

Lemma 2.7. *Given an edge labelled poset* $(P, \leq, l\colon \mathcal{E}(P) \to A)$*, the group* $\pi_1(K(P))$ *is generated by a set of loops in bijection with* $\mathrm{Im}(l)$*.*

Proof. Each vertex $p \in \Delta(P)$ has extended label $\emptyset$ and so there is only one vertex in $K(P)$, denote this point p_0. The 1–skeleton of $K(P)$ will be a wedge of circles, one for each extended label in $\{\mathcal{L}(C) \mid C \text{ is a 1–chain}\}$. The following is the setup of our proof.

1. Let Σ denote 1–chains in P.
2. Let Ω denote paths between vertices in $\Delta(P)$ and along edges in $\Delta(P)$ such that the direction along the edge corresponding to $(x \leq y)$ is from x to y.
3. Let Λ denote loops in $K(P)$ that start and end at p_0. If $\lambda \in \Lambda$, then $[\lambda] \in \pi_1(K(P), p_0)$.

Let α, be the map from 1–chains to corresponding paths in $\Delta(P)$ and $\beta\colon \Delta(P) \to K(P)$ be the quotient map. We have the following diagram.

$$\Sigma \xrightarrow{\alpha} \Omega \xrightarrow{\beta} \Lambda$$

By our remarks on the 1–skeleton of $K(P)$, we see that $\pi_1(K(P), p_0)$ is generated by loops in $\mathrm{Im}(\beta \circ \alpha)$. Let $\sigma = (x \leq y) \in \Sigma$ be a 1–chain and let λ be such that $\beta(\alpha(\sigma)) = \lambda$. Let concatenation of loops λ and λ' be denoted $\lambda\lambda'$ such that $[\lambda][\lambda'] = [\lambda\lambda']$. We will show that λ has a factorisation where one of the factors is $\beta(\alpha(x \lessdot z))$ for some $z \in P$.

There exists $z \in P$ such that $x \lessdot z \leq y$. If $z \neq y$, there is a 2–simplex in $\Delta(P)$ corresponding to the 2–chain $(x \lessdot z < y)$. One of the edges of this 2–simplex corresponds to the 1–chain $(x \lessdot z)$. The path $\alpha(x \leq y)$ is homotopic (through the 2–simplex $(x \lessdot z < y)$) to the path $\alpha(x \lessdot z)\alpha(z < y)$. Let H witness this homotopy. We have that $\beta \circ H$ is well–defined (and continuous), so the loop λ is homotopic to the loop $\beta(\alpha(x \lessdot z)\alpha(z < y)) = \beta(\alpha(x \lessdot z))\beta(\alpha(z < y))$. We repeat the process replacing σ with $(z < y)$. Eventually this must stop since our poset is of finite height. After this, we achieve a factorisation of λ, entirely in factors of the form $\beta(\alpha(r \lessdot s))$.

We see that $\pi_1(K(P), p_0)$ is generated by the loops in $\mathrm{Im}(\beta \circ \alpha|_{\mathcal{E}(P)})$. Each covering relation $(r \lessdot s)$ has extended label $\{l(r \lessdot s)\}$, therefore $\mathrm{Im}(\beta \circ \alpha|_{\mathcal{E}(P)})$ is in bijection with $\mathrm{Im}(l)$. □

Lemma 2.8. *For an edge-labelled poset* P*, there exists a surjective homomorphism* $\varphi\colon G(P) \to \pi_1(K(P), p_0)$ *where* p_0 *is as in the previous lemma.*

Proof. Let us follow from the notation in the proof of Lemma 2.7 and let $\theta\colon \mathrm{Im}(l) \to \pi_1(K(P), p_0)$ be the bijection at the end of that proof such that we have the group presentation $\pi_1(K(P), p_0) = \langle\, \mathrm{Im}(\theta \circ l)\, \rangle$.

Let $\sigma = (x_1 \lessdot \cdots \lessdot x_i)$ and $\sigma' = (x'_1 \lessdot \cdots \lessdot x'_j)$ be two saturated chains such that $x_1 = x'_1$ and $x_i = x'_j$ with corresponding words $w = w_1 \cdots w_{i-1}$, and $w' = w'_1 \cdots w'_{j-1}$ in A^* such that $l(x_k \lessdot x_{k+1}) = w_k$ and similarly for σ'. Recall that w and w' are words that are identified by the relations in the defining presentation for $G(P)$. We want to show there exists a homotopy between the loop $\theta(w_1) \cdots \theta(w_i)$ and the loop $\theta(w'_1) \cdots \theta(w'_j)$.

By doing the process in the proof of Lemma 2.7 in reverse, we get a homotopy between $\theta(w_1) \cdots \theta(w_i)$ and $\beta(\alpha(x_1 \leq x_i))$ through the two skeleton of $K(P)$. By the same argument, we get a homotopy between $\theta(w'_1) \cdots \theta(w'_j)$ and $\beta(\alpha(x'_1 \leq x'_j))$. Since $x_1 = x'_1$ and $x_i = x'_j$, we have $\theta(w_1) \cdots \theta(w_i) \sim \theta(w'_1) \cdots \theta(w'_j)$.

We have shown that $\pi_1(K(P), p_0)$ has all necessary relations to extend θ to a surjective homomorphism $\varphi \colon G(P) \to \pi_1(K(P), p_0)$. □

To finish our proof, it is necessary to extend our toolset for dealing with extended labels. Previously, extended labels labelled edges in $\Delta(P)$, now they will also label edge loops in $K(P)$. Let ℓ be an extended label corresponding to some edge $e \in \Delta(P)$. Let ω be the path along e with direction increasing with respect to $\leq$. Let ℓ also label the loop $\beta \circ \omega$ in $K(P)$. Define the opposite orientation of ω to be ω^{-1} and give this the extended label $\ell^{-1} \coloneqq \{w^{-1} \mid w \in \ell\}$ where w^{-1} is the formal inverse of the word w. We also define concatenation of extended labels ℓ and ℓ' as $\ell\ell' \coloneqq \{ww' \mid w \in \ell \text{ and } w' \in \ell'\}$ where ww' is concatenation of words proceeded by word reduction, as in free groups. Previously we defined the extended label for all vertices in $\Delta(P)$ to be $\emptyset$. Denote the trivial loop on to $p_0 \in K(P)$ as 0. Give 0 the extended label $\{\varepsilon\}$ where ε is the empty word. Note that concatenation of paths does not directly correspond to concatenation of labels.

Theorem 2.9. *Given an edge labelled poset* $(P, \leq, l \colon \mathcal{E}(P) \to A)$*, we have* $\pi_1(K(P)) \cong G(P)$*.*

Proof. Let φ be the surjective homomorphism as in Lemma 2.8. We need to show that $\ker(\varphi)$ is trivial.

Suppose there is some word $a_1 a_2 \cdots a_i \in A^*$ such that $\varphi(a_1) \cdots \varphi(a_i) \sim 0$. The loop $\varphi(a_1) \cdots \varphi(a_i)$ traverses edges in $K(P)$ corresponding to covering relations.

Consider a 2–simplex in $e^2 \in K(P)$ with edges e^1_i, $i \in \{0, 1, 2\}$ oriented as below.

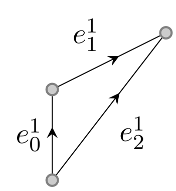


Figure 6: An oriented 2–simplex in $K(P)$. The arrows on the edges denote orientation not identification. All vertices are identified.

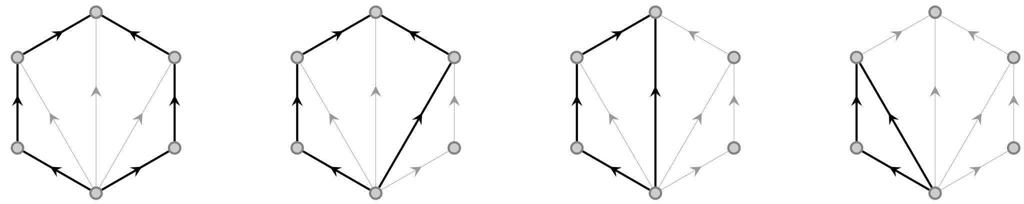


Figure 7: A contractible subcomplex $X \subseteq K(P)$ and successive application of homotopies permitted in (4) giving a homotopy from the boundary of X to a nullhomotopic loop around a single simplex. The arrows denote edge orientation as in Fig. 6. The thick line indicates the nullhomotopic loop, which traverses each diagram in a clockwise direction (not in the direction of the arrows).

Note that each e_i^1 is not necessarily distinct in $K(P)$. Let each $\lambda_i \in \Lambda$ be the loop along e_i^1, following its orientation. The only homotopies in terms of other edge paths are the following.

$$\begin{aligned} \lambda_0 &\sim \lambda_2\lambda_1^{-1} \\ \lambda_1 &\sim \lambda_0^{-1}\lambda_2 \\ \lambda_2 &\sim \lambda_0\lambda_1 \end{aligned} \tag{4}$$

With equivalent homotopies for each λ_i^{-1}.

Let ℓ_i be the extended label corresponding to each λ_i. If the orientation in Fig. 6 is inherited from the poset order, then we have the following.

$$\begin{aligned} \ell_0\ell_1 &\subseteq \ell_2 \\ \ell_2\ell_1^{-1} \cap \ell_0 &\neq \emptyset \\ \ell_0^{-1}\ell_2 \cap \ell_1 &\neq \emptyset \end{aligned} \tag{5}$$

With equivalent relations for each ℓ_i^{-1}.

Let H witness the homotopy $\varphi(a_1)\cdots\varphi(a_i) \sim 0$. We see that H defines a contractible subcomplex X with boundary $\mathrm{Im}(\varphi(a_1)\cdots\varphi(a_i))$. Suppose the loop $\varphi(a_1)\cdots\varphi(a_i)$ passes through an edge of a particular simplex $e^2 \subseteq X$. Within e^2, the homotopy H must restrict to the subloop through e^2 according to (4). Considering the movement of the loops $H_{t\times I}$ as t increases from 0 to 1, we see that H can be mimicked by successive application of the allowed homotopies in (4) on to the simplices of X in such a way that reduces the number of edges in our path. Eventually, we achieve a homotopy from $\varphi(a_1)\cdots\varphi(a_i)$ to the path around a single simplex (which would look like $\lambda_0\lambda_1\lambda_2^{-1}$ in Fig. 6) which is nullhomotopic by (4). See Fig. 7. By examining the form of the allowed homotopies and the effect on extended label in (5) we see that the reduction of the total concatenation $a_1a_2\cdots a_i$ must be in the extended label for 0, so $a_1a_2\cdots a_i$ reduces to ε. So the word $a_1a_2\cdots a_i$ corresponds to the identity in $G(P)$ and $\ker(\varphi)$ is trivial. $\square$

Note that our proof of the above was ambivalent to the identification of n–simplices for $n > 2$. Indeed, these identifications do not affect the fundamental group, but they do ensure that higher homotopy groups are trivial. We can see if we did not identify the 2–simplices in Fig. 5, $\pi_2(K(P))$ would be non-trivial.

2.3 Interval Complex

Starting from a Coxeter group W generated by S, we wish to give W a labelled–poset structure and use the constructions from the previous section. The edge labelled Hasse diagram for W will embed in to the Cayley graph $\mathrm{Cay}(W, S)$, and it is useful to be able to swap between these two objects, as we will do. First we must define an order on a group.

Definition 2.10 (Word length in a group)**.** For a group G generated by S, the word length with respect to S is the function $l_S : G \to \mathbb{Z}$ where

$$l_S(g) \coloneqq \min\{k \mid s_1 s_2 \ldots s_k = g\,,\, s_i \in S \cup S^{-1}\}.$$

We will often omit the S in l_S where it is obvious from context.

Definition 2.11 (Order on a group)**.** For a group G generated by S, we define the partial order $x \leq y \iff l(x) + l(x^{-1}y) = l(y)$.

It can be readily checked that this does indeed define a partial order on G. This order encodes closeness to $e \in G$ along geodesics in $\mathrm{Cay}(G, S)$. We have $x \leq y$ precisely when there exists a geodesic in $\mathrm{Cay}(G, S)$ from e to y with x as an intermediate vertex, or to put it another way, when a minimal factorisation of x in to elements of S is a prefix of a minimal factorisation of y. For some $w \in W$ we define the poset $[1, w]^W$ to be the interval in W up to w with respect to this order. We give this poset an edge labelling such that the edge between w and ws is labelled s for some $s \in S$. The Hasse diagram thus embeds in to $\mathrm{Cay}(W, S)$.

Definition 2.12 (Coxeter element)**.** For some Coxeter group W generated by S, we define a *Coxeter element* $w \in W$ to be any product of all the elements of S without repetition.

These Coxeter elements are what we will use as the upper bound of our interval. We will also need to consider W as the group generated by R, the set of all reflections, rather than just the set of simple reflections S. See Fig. 8 for an example of such a poset. In principle there are many choices of Coxeter element depending on what order we multiply the elements of S. However, we will see that many structures resulting from $[1, w]^W$ are independent of that choice.

We apply the steps from Section 2.2 to $[1, w]^W$ to form a space.

Definition 2.13 (Interval Complex)**.** For a Coxeter group W generated by all reflections R with $w \in W$, we call $K_W \coloneqq K([1, w]^W)$ the *interval complex* where $K([1, w]^W)$ is as in Definition 2.6.

If W is infinite, then R is infinite and so K_W may have an infinite number of cells. We will later show that K_W deformation retracts to a finite subcomplex. Note that as in [PS21] we have dropped w from our notation K_W even though it depends on w. This is eventually justified (Theorem 1.1) since the homotopy type of K_W is independent of w.

Certain properties of the poset permit a simplified notation for the simplices within K_W. In this context, for two chains $C = (C_1 \leq C_2 \leq \cdots \leq C_m)$ and $C' = (C'_1 \leq C'_2 \leq \cdots \leq C'_n)$ we have $\mathcal{L}(C) = \mathcal{L}(C')$ exactly when $(C_1)^{-1}C_m = (C'_1)^{-1}C'_n$. Thus, we can label 1–simplices in K_W with group elements $x \in [1, w]^W$, we can label 2–simplices with factorisations of group elements in $[1, w]^W$ in to two parts (with the first part also in $[1, w]^W$) and so on. We denote an n–simplex $[x_1|x_2|\cdots|x_n]$ as in [PS21, Definition 2.8]. This notation also gives the gluing of the faces of $[x_1|x_2|\cdots|x_n]$ in the following way. A codimention 1 face of $[x_1|x_2|\cdots|x_n]$ is a subchain of $x_1 \leq x_1x_2 \leq \cdots \leq x_1x_2\ldots x_n$ consisting of $n-1$ elements. There are three ways to obtain such a subchain.

1. Remove the first element of the chain to get $x_2 \leq x_2x_3 \leq \cdots \leq x_2x_3\ldots x_n$.

2. Remove the last element of the chain to get $x_1 \leq x_1x_2 \leq \cdots \leq x_1x_2\ldots x_{n-1}$.

3. Multiply two adjacent elements x_i and x_{i+1} to get the chain $x_1 \leq \cdots \leq x_1\ldots x_{i-1} \leq x_1\ldots x_{i-1}x_ix_{i+1} \leq \cdots \leq x_1\cdots x_n$.

So the n–simplex $[x_1|x_2|\cdots|x_n]$ glues to $[x_2|x_3|\cdots|x_n]$, $[x_1|x_2|\cdots|x_{n-1}]$ and $[x_1|\cdots|x_ix_{i+1}|\cdots|x_n]$ for all $i < n$.

The particular poset group intervals $[1, w]^W$ we will consider will be *balanced.* A balanced group interval is such that $x \in [1, w]^W$ iff $l(g^{-1}x) + l(x) = l(g)$. I.e. all minimal factorisation of $x \in [1, w]^W$ also appear as a suffix in a minimal factorisation of w and all suffixes also appear as a prefix.

Where the interval is balanced, any such symbol $[x_1|x_2|\cdots|x_n]$ corresponds to an n–simplex in K_W given it satisfies the following [PS21, Definition 2.8]:

i) $x_i \neq 1$ for all i.

ii) $x_1x_2\cdots x_n \in [1, w]^W$

iii) $l(x_1x_2\cdots x_n) = l(x_1) + l(x_2) + \cdots + l(x_n)$

Hopefully the first two requirements are obvious. The third is because we require the chain $x_1 \leq x_1x_2 \leq \cdots \leq x_1x_2\ldots x_n$ to be contained in $[1, w]^W$ which translates to every subword of $x_1\cdots x_n$ also being in $[1, w]^W$. By ii and iii we have that there is some y such that $x_1\cdots x_ny = w$ and there is a minimal factorisation of w that respects the factors in $x_1\cdots x_ny$. We can take prefixes of this factorisation (and thus prefixes of $x_1\cdots x_ny$) and stay within $[1, w]^W$. We can use the balanced condition to move the suffix $x_2\cdots x_ny$ to the

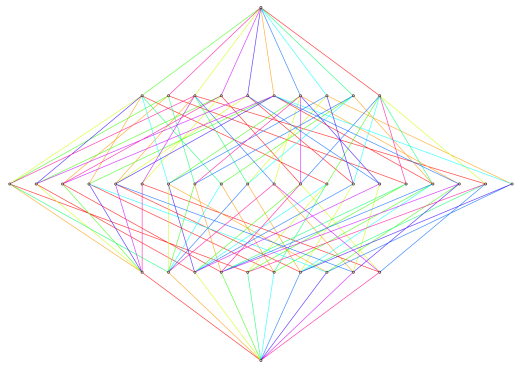


Figure 8: The interval $[1,(1,2)(2,3)(3,4)(4,5)]^{A_4}$ considering A_4 generated by all reflections, which label the edges by colour. Generated using `Sage` and `GAP` [The20, The22]

front, i.e. there exists y_2 such that $x_2 \cdots x_n y y_2 = w$ and there is a minimal factorisation of w that respects those factors. These steps can be repeated to show every subword of $x_1 \cdots x_n$ is in $[1,w]^W$.

Definition 2.14. For a Coxeter group W generated by all reflections R with Coxeter element $w \in W$. Define $[1,w]^W$ as above. The *dual Artin group* W_w is the poset group $G([1,w]^W)$ with G defined as in Definition 2.3.

From the closing remarks of Section 2.2, the fundamental group of K_W is W_w. Furthermore, for finite cases it is a classifying space.

Theorem 2.15 ([PS21, Theorems 2.9 and 2.14])**.** *For a finite Coxeter group W, the interval complex is a $K(W_w,1)$ space.*

The same theorem for affine cases is proved in [PS21], here Theorem 1.2. It is also known that for certain cases, the dual Artin group is isomorphic to the Artin group.

Theorem 2.16. *For a finite [Bes03] or affine [MS17] Coxeter group W and Coxeter element $w \in W$, the dual Artin group is isomorphic to the Artin group G_W (and thus it does not depend on the choice of w).*

In general, it is not known whether $G_W \cong W_w$, whether the isomorphism class of W_w depends on w or even if the isomorphism class of $[1,w]^W$ depends on w.

2.4 Salvetti Complex

Here we will define the *Salvetti Complex* for a Coxeter group W generated by S, which is homotopy equivalent to Y_W. First we must define a notion on subsets of S. For some subset $T \subseteq S$ define the *parabolic subgroup* of W with respect to T, W_T, to be the subgroup of W generated by T with all relations for W containing only elements of T. If Γ is the

Coxeter diagram for W, then W_T is the Coxeter group corresponding to the complete subgraph of Γ containing the vertices T. From here we follow [Pao17, Section 2.3], with notation from [PS21].

Definition 2.17. For a Coxeter group W generated by S, define Δ_W to be the family of subsets $T \subseteq S$ such that W_T is finite.

For some $T \subseteq S$, we say some $w \in W$ is T–minimal if w is the unique element of minimum length (with respect to S) in the coset wT. Uniqueness is shown in [Bou08]. Define an order on the set $W \times \Delta_W$ by the following: $(u, X) \leq (v, Y)$ iff $X \subseteq Y$, $v^{-1}u \in W_Y$ and $v^{-1}u$ is X–minimal.

Definition 2.18 (Pre-Salvetti Complex [Pao17, Definition 2.19])**.** For a Coxeter group W, define $\mathrm{Sal}(W)$ to be $\Delta(W \times \Delta_W)$ under the order $\leq$ prescribed above, where the first Δ is as in Definition 2.4.

The Salvetti complex was originally defined (and refined) in [Sal87,Sal94]. In the latter of these papers, the Salvetti complex was defined to be the quotient of a space related to the action o W on a vector space. In [Par14, Theorem 3.3] it was shown that the definition we give generates a space homeomorphic to that in the original definition. Let us quote some results that help us to interpret the definition.

Lemma 2.19 ([Pao17, Lemma 2.18])**.** *Consider both of these objects as geometric simplicial complexes inside* $\mathrm{Sal}(W)$*.*

$$C(v, Y) \coloneqq \{(u, X) \in W \times \Delta_W \mid (u, X) \leq (v, Y)\}$$
$$\partial C(v, Y) \coloneqq \{(u, X) \in W \times \Delta_W \mid (u, X) < (v, Y)\}$$

There is a homeomorphism $C(v, Y) \to D^n$ *that restricts to a homeomorphism* $\partial C(v, Y) \to S^{n-1}$ *where* $n = |\,Y\,|$*.*

This allows us to construct a CW complex for $\mathrm{Sal}(W)$ where each $C(w, X)$ is a $|\,X\,|$–cell for each $X \in \Delta_W$. Let us see what these cells look like. Note that the cells of the CW-complex and the simplices in $\Delta(W \times \Delta_W)$ as in Definition 2.18 comprise a completely different cell structure for $\mathrm{Sal}(W)$. We define $\langle \emptyset \rangle \coloneqq \{1\}$ to give $W_\emptyset$ meaning as the trivial subgroup inside W.

Each $C(w, \emptyset)$ is a 0–cell. We will denote these cells w as a shorthand. In general, we have that $(u, X) \leq (v, X) \implies (u, X) = (v, X)$ since we require $v^{-1}u \in X$ we have $v^{-1}uX = X$. So if $v^{-1}u$ is minimal in $v^{-1}uX$ then $v^{-1}u = 1$. In particular, there is no $(u, X) < (w, \emptyset)$, so these $w = C(w, \emptyset)$ are 0–simplices in $\Delta(W \times \Delta_W)$ as well.

Now consider each 1–cell $C(w, \{s\})$. Since $W_{\{s\}} = \{1, s\} \cong \mathbb{Z}/2$ we have $\{s\} \in \Delta_W$ for all $s \in S$. For some (u, X) to be less than $(w, \{s\})$, recall we require $w^{-1}u \in W_{\{s\}}$. So we have $u \in \{w, ws\}$. Locally, the Hasse diagram and (since we only have 1–chains here) $\Delta(W \times \Delta_W)$ both look like as in Fig. 9 (left). In the CW complex there would be only one 1–cell, labelled $C(w, \{s\})$ oriented from w to ws. Note that $C(ws, \{s\})$ also connects these two vertices, but is a different 1–cell. This doubling up will be inconsequential after we define the Salvetti complex, which will quotient away any such doubling.

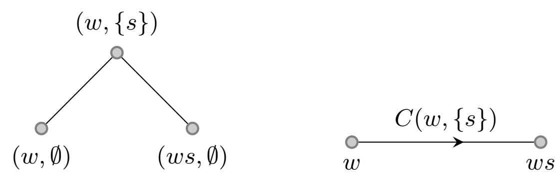


Figure 9: A local picture of $\mathrm{Sal}(W)$ as $\Delta(W \times \Delta_W)$ (which also resembles the Hasse diagram) (left). The corresponding 1–cell in the CW complex for $\mathrm{Sal}(W)$ (right).

Now consider the 2–cells in the CW complex. We have that $W_{\{s,t\}} \cong D_{2m(s,t)}$, the dihedral group of corresponding to the $m(s,t)$–gon (recall $m(s,t)$ from Definition 1.3). Thus, $\{s,t\} \in \Delta_W$ iff $m(s,t) \neq \infty$. We have $(u, \{s\})$ or $(v, \{t\})$ are less than $(w, \{s,t\})$ only when $u = wd$ for some $d \in W_{\{s,t\}}$, similarly for v. The second requirement then is that d is ($\{s\}$ or $\{t\}$)–minimal. In the general case, if $(u, X) \leq (v, Y)$ then $X \subseteq Y$ so $W_X \subseteq W_Y$. So the coset $v^{-1}uW_X \subseteq W_Y$. Due to the nature of Definition 1.3, only relations relevant to W_Y could have relevance to the word length of elements in W_Y. Thus, to determine if $v^{-1}u$ is X minimal we need only consider everything within W_Y, not the entire Coxeter group W. In particular, to tell if $(wd, \{s\}) \leq (w, \{s,t\})$ for some $d \in W_{\{s,t\}}$, we need only consider if d is $\{s\}$–minimal in the dihedral group $W_{\{s,t\}}$.

Considering for a moment s and t as letters only, a normal form comprising minimal length words for $W_{\{s,t\}}$ is

$$\{\Pi(s,t,n) \mid n \leq m(s,t)\} \cup \{\Pi(s,t,n) \mid n \leq m(s,t)\}$$

recalling the meaning of $\Pi(s,t,n)$ from Definition 1.4. Note that $\Pi(t,s,m(s,t))$ is also a minimal length word but is not included for the above to be a normal form. Thus, any $sts \cdots s$ is $\{t\}$–minimal if the total length of $sts \cdots s$ is strictly less than $m(s,t)$. Similarly, $sts \cdots t$ is $\{s\}$–minimal if the total length of $sts \cdots t$ is strictly less than $m(s,t)$, with equivalent results for $tst \cdots s$ and $tst \cdots t$ depending on the last letter in the word. A picture of the Hasse diagram for the interval $[-\infty, (w, \{s,t\})]$ corresponding to the cell $C(w, \{s,t\})$ where $m(s,t) = 3$ is shown in Fig. 10. The CW cell itself has been drawn in Fig. 11.

There is a natural action $W \circlearrowleft \mathrm{Sal}(W)$ with $w \cdot (u, T) := (wu, T)$. We can now define the following

Definition 2.20 (Salvetti Complex)**.** For a Coxeter group W define the *Salvetti Complex* X_W to be $\mathrm{Sal}(W)/W$ under the action specified above.

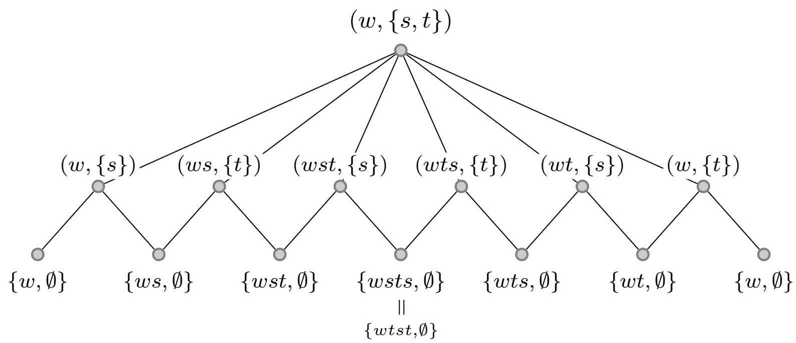


Figure 10: The local Hasse Diagram corresponding to the CW 2–cell $C(w, \{s, t\})$ where $m(s, t) = 3$. Note that w has been drawn twice for clarity in the picture. C.f. Fig. 11

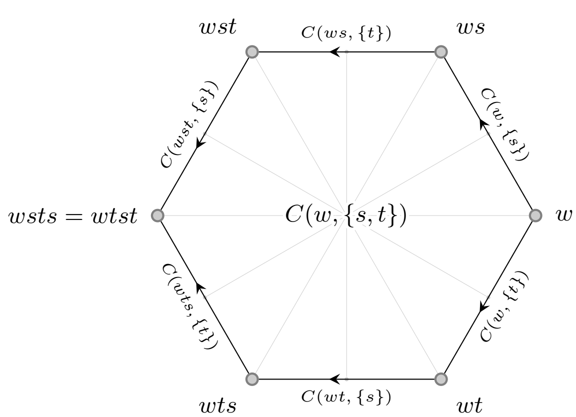


Figure 11: The 2–cell $C(w, \{s, t\})$ in the CW complex for $\mathrm{Sal}(W)$ where $m(s, t) = 3$. The faint lines are the simplices from $\Delta(W \times \Delta_W)$, which have all been incorporated in to one CW cell.

The action is cellular, thus we have a CW structure for X_W as well. We now quote the following important result.

Theorem 2.21 ([Par14, Corollary 3.4] [Sal87])**.** *For a Coxeter group W, the Salvetti complex X_W is homotopy equivalent to the configuration space Y_W.*

Now let us consider the cell structure of X_W. There is one $|\,T\,|$–cell for each $T \in \Delta_W$, in particular, there is one 0–cell corresponding to the trivial group $W_\emptyset$. Attached to this is a 1–cell for each $s \in S$ forming $\bigvee_{s\in S} S^1$ as the 1–skeleton. Then to this wedge is attached 2–cells following the procedure above for each $m(s,t) \neq \infty$. Each two cell corresponding to $\{s,t\}$ is attached to two 1–cells corresponding to $\{s\}$ and $\{t\}$. From examining this 2–skeleton it should be clear that $\pi_1(X_W) \cong G_W$. Thus combining with the previous theorem we have re-proved $\pi_1(Y_W) \cong G_W$.

Chapter 3

Implementing the CW Complexes

We will now begin to bridge the gap between some of the objects we have defined. Ultimately, we wish to show a homotopy equivalence between the space K_W and the Salvetti complex X_W, which is already known to be homotopy equivalent to Y_W. For now, we will define a subspace $K'_W \subseteq K_W$, inspired by our definition of the Salvetti complex, using subsets $T \subseteq S$ such that W_T is finite.

3.1 The subcomplex X'_W

The definition of K_W depends on the data of some Coxeter element w and thus implicitly some generating set S. For some $w = s_1 s_2 \cdots s_n$ and some $T \subseteq S$, define $w_T \in W$ to be the (non–consecutive) subword of w consisting of elements that are in T, respecting the original order in w.

Definition 3.1 (The subcomplex)**.** Define X'_W to be the finite subcomplex of K_W consisting only of simplices $[x_1|\cdots|x_n]$ such that $x_1 x_2 \cdots x_n \in [1, w_T]^W$ for some $T \in \Delta_W$. Recalling Δ_W from Definition 2.17.

Note again the absence of w from the notation of X'_W. That will be justified in this section as we will show that, up to homotopy equivalence, X'_W has no w dependence.

Lemma 3.2 ([PS21, Lemma 5.2])**.** *For a Coxeter group W and any parabolic subgroup W_T, with sets of reflections R_W and R_{W_T} respectively. For some Coxeter element $w \in W$ and corresponding $w_T \in W_T$, all minimal factorisations of w_T in R_W consist of only elements from R_{W_T}.*

An immediate consequence of this is that the intervals $[1, w_T]^W$ and $[1, w_T]^{W_T}$ agree. This allows us to decompose X'_W in a useful way. For each $T \in \Delta_W$ and corresponding W_T, the space corresponding to the whole interval $[1, w_T]^W = [1, w_T]^{W_T}$ is a subspace inside X'_W. This subspace is exactly X'_{W_T} (with respect to w_T). Thus, we can think of X'_W as some union of all X'_{W_T} for $T \in \Delta_W$.

Each X'_{W_T} is exactly the same as its interval complex K_{W_T} since all subgroups of T generate finite Coxeter groups. Thus, using known results for finite Coxeter groups, X'_{W_T} is a classifying space for the dual Artin group W_w by Theorem 2.15. Furthermore, G_w is isomorphic to the Artin group G_{W_T} by Theorem 2.16.

In a very similar way, the Salvetti complex consists of subspaces corresponding to elements of Δ_W. For each $T \in \Delta_W$, the Salvetti complex X_{W_T} is a $|\,T\,|$–cell attached to all cells corresponding to $R \subseteq T$ is the appropriate way. This is a cellular subspace of X_W, and since W_T is finite, by Theorem 1.9, $Y_{W_T} \simeq X_{W_T}$ is a $K(G_{W_T}, 1)$. The following remark summarises these observations.

Remark 3.3. The Salvetti complex X_W decomposes in to cellular subspaces X_{W_T} which are $K(G_{W_T}, 1)$ spaces. These subspaces are in bijection with cellular subspaces X'_{W_T} of X'_W, which are also $K(G_{W_T}, 1)$ spaces.

The following section will help us to exploit this similarity to show that $X_W \simeq X'_W$.

3.2 An Adjunction Homotopy Equivalence

Here we will show a fundamental link between homomorphisms in to groups G and maps in to classifying spaces for G. This result shows that in a certain way, the homotopy type of classifying spaces is unique. We will use this result to then show an important homotopy equivalence, which is an intermediate step in proving the main result of [PS21].

Lemma 3.4. *A null homotopic map $\rho\colon S^n \to X$ can be extended to a map $\sigma\colon D^{n+1} \to X$.*

Proof. Let $H\colon S^n \times I \to X$ witness the null homotopy with $H|_{S^n\times\{1\}}\colon S^n \to \{x_0\}$. We have that H factors uniquely through $(S^n \times I)/(S^n \times \{1\}) \cong D^{n+1}$. With σ being the necessary map as below.

$$\begin{tikzcd} S^n \times I \arrow[r, "H"] \arrow[d, "q"] & X \\ (S^n \times I)/(S^n \times \{1\}) \arrow[ur, dotted, "\exists!\sigma"'] & \end{tikzcd}$$

□

Theorem 3.5 ([Hat01, Proposition 1B.9]). *Let Y be a $K(G, 1)$ space and X a finite dimensional CW complex consisting of one 0–cell, the point x_0. Any homomorphism $\varphi\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is induced by a map $\tilde{\varphi}\colon X \to Y$ where $\tilde{\varphi}$ is unique up to homotopy fixing x_0.*

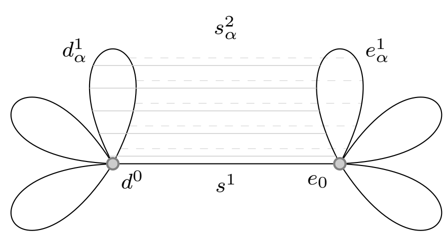
Proof. Clearly we must have $\tilde{\varphi}(x_0) = y_0$. The 1–skeleton X^1 will be a wedge of circles and there is thus a presentation of $\pi_1(X, x_0)$ with each cell e^1_α corresponding to a generator

$[e^1_\alpha] \in \pi_1(X, x_0)$. We can choose $\tilde{\varphi}(e_\alpha)$ to trace out a path corresponding to $\varphi([e^1_\alpha]) \in \pi_1(Y, y_0)$ for each $e^1_\alpha \in X^1$.

Let $\psi_\beta \colon S^1 \to X^1$ be an attaching map for a 2–cell $e^2_\beta \subseteq X$. Let $i \colon X^1 \hookrightarrow X$ be the inclusion. We have that i_* is the surjetcion from the free group generated by each e^1_α to $\pi_1(X, x_0)$. The attaching of the 2–cell e^2_β provides a null homotopy for the path traced by ψ_β. In the presentation of $\pi_1(X, x_0)$ as above, each relation corresponds to the path of a ψ_β. Thus, $i_*([\psi_\beta]) = 0$ and so $\tilde{\varphi}_*([\psi_\beta]) = \varphi \circ i_*([\psi_\beta]) = 0$. Thus, $\tilde{\varphi} \circ \psi_\beta$ is null homotopic and so can be extended over all of the closure of e^2_β by Lemma 3.4. This is an extension of $\tilde{\varphi}$ and repeating this allows us to extend $\tilde{\varphi}$ over all of X^2.

To extend $\tilde{\varphi}$ over e^3_γ we use that S^2 is simply connected (as for any S^n with $n \geq 2$) and so for the attaching map $\psi_\gamma \colon S^2 \to X^2$ we have that $\tilde{\varphi} \circ \psi_\gamma$ lifts to the universal cover of Y, which is contractible since Y is a $K(G, 1)$, so $\tilde{\varphi} \circ \psi_\gamma$ is null homotopic. This same argument applies for any e^n_δ for $n \geq 3$. We can thus extend $\tilde{\varphi}$ over the 3–cells and proceeding inductively, over all of X.

Now we turn to the uniqueness of $\tilde{\varphi}$ up to homotopy. Let φ be some homomorphism and $\tilde{\varphi}_0$ and $\tilde{\varphi}_1$ be any such maps constructed as above. Clearly $\tilde{\varphi}_0(x_0) = \tilde{\varphi}_1(x_0)$ and $\tilde{\varphi}_0|_{X^1} \sim \tilde{\varphi}_1|_{X^1}$ by the restrictions of our construction. Let H witness this homotopy. Give $X \times I$ the following CW structure: Let $X \times \{0\}$ and $X \times \{1\}$ both have the same cell structure as X with cells notated d^n_α and e^n_α respectively. Connect d^0 to e^0 with a 1–cell s^1, called *the spine*. Connect a 2–cell s^2_α along d^1_α, then s^1 then e^1_α then back along s^1 with opposite orientations on d^1_α and e^1_α such that $d^0 \cup e^0 \cup s^1 \cup d^1_\alpha \cup e^1_\alpha \cup s^2_\alpha \cong S^1 \times I$. The spine now consists of $s_1 \cup s^2_\alpha$. Repeat this for each 1–cell in X and then repeat for each 2–cell and so on, attaching an s^n_β along d^{n-1}_β, e^{n-1}_β and s^{n-1}_β, inductively building up the spine. A picture of this CW complex completed for one s^2_α is below.



We can now extend the domain of H from $X^1 \times I$ to all of $X \times I$ using this cell structure. Note that now we have two 0–cells, but this does not cause any issues. Let H have domain $X^1 \times I \subseteq X \times I$. Now extend H such that $H|_{X \times \{0\}}$ agrees with $\tilde{\varphi}_0$ and $H|_{X \times \{1\}}$ agrees with $\tilde{\varphi}_1$. This is possible because H is a homotopy between restrictions of these maps. Note that now H is defined on the whole 2–skeleton of $X \times I$. We can extend H to all the higher dimensional cells by the exact same argument as before, using the contractability of the universal cover of Y. Thus, we have a continuous function $H \colon X \times I \to Y$ witnessing the homotopy $\tilde{\varphi}_0 \sim \tilde{\varphi}_1$. □

Corrolary 3.6. *Let X and Y both be $K(G,1)$ spaces. Any isomorphism $\varphi\colon \pi_1(X,x_0) \to \pi_1(Y,y_0)$ induces a homotopy equivalence witnessing $X \simeq Y$.*

Proof. We have maps $\tilde{\varphi}\colon X \to Y$ and $\widetilde{(\varphi^{-1})}\colon Y \to X$ with $(\tilde{\varphi} \circ \widetilde{(\varphi^{-1})})_* = \mathrm{Id}_{\pi_1(Y,y_0)}$. Thus, since the homotopy class of such maps is determined by the induced action on their fundamental groups $\tilde{\varphi} \circ \widetilde{(\varphi^{-1})} \sim \mathrm{Id}_Y$. Similarly, $\widetilde{(\varphi^{-1})} \circ \tilde{\varphi} \sim \mathrm{Id}_X$. □

Lemma 3.7. *Let $T \in \Delta_W \setminus \emptyset$. Let $\varphi\colon \bigcup_{Q \subsetneq T} X_{W_Q} \to \bigcup_{Q \subsetneq T} X'_{W_Q}$ be a homotopy equivalence. We can extend φ to a homotopy equivalence $\psi\colon X_{W_T} \to X'_{W_T}$ such that the following diagram commutes.*

$$\begin{tikzcd} \bigcup_{Q \subsetneq T} X_{W_Q} \arrow[r, "\varphi"] \arrow[d, hook] & \bigcup_{Q \subsetneq T} X'_{W_Q} \arrow[d, hook] \\ X_{W_T} \arrow[r, "\psi"] & X'_{W_T} \end{tikzcd}$$

Proof. We prove this by cases. By Remark 3.3, X_{W_T} and X'_{W_T} are classifying spaces.

i) If $|T| = 1$ then Q is uniquely $\emptyset$. Let ψ be any map witnessing $X_{W_T} \simeq X'_{W_T}$ that fixes the point corresponding to X_{W_Q}.

ii) If $|T| = 2$ then we can extend φ to ψ such that $\psi_*\colon \pi_1(X_{W_T}, X_\emptyset) \to \pi_1(X'_{W_T}, X'_\emptyset)$ is an isomorphism using the same argument as in the proof of Theorem 3.5. This map is a homotopy equivalence by Corollary 3.6.

iii) If $|T| \geq 3$ then we can extend φ to some map ψ using the same methods as in Theorem 3.5. In this case, $\bigcup_{Q \subsetneq T} X_{W_Q}$ contains the 2–skeleton of X_T and similarly for $\bigcup_{Q \subsetneq T} X'_{W_Q}$ and X'_T. So $\pi_1(\bigcup_{Q \subsetneq T} X_{W_Q}, X_{W_\emptyset}) = \pi_1(X_T, X_{W_\emptyset})$ and similarly for X'_T. By assumption φ is a homotopy equivalence and so φ_* is an isomorphism. Therefore, ψ_* is an isomorphism [Hat01, Corollary 4.12] and thus ψ a homotopy equivalence by Corollary 3.6.

□

Definition 3.8 (Adjunction Space)**.** For two spaces X and U, with a continuous map $f\colon A \to U$ for some subspace $A \subseteq X$. The *adjunction space* $X \sqcup_f U$ is the space formed by gluing X and U via the map f.

$$X \sqcup_f U := (X \sqcup Y)/(a \sim f(a))$$

An adjunction space is associated to the commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"] & U \arrow[d, "\bar{i}"] \\ X \arrow[r, "\bar{f}"] & X \sqcup_f U \end{tikzcd} \tag{6}$$

where i is inclusion of A in to X and $\bar{i}$ is inclusion of U in to $X \sqcup_f U$. Suppose we also have the adjunction space $Y \sqcup_g V$ with $g\colon B \to V$ and $B \subseteq Y$. Suppose further that we have maps $\varphi_1\colon X \to Y$, $\varphi_2\colon A \to B$ and $\varphi_3\colon U \to V$ such that the following diagram commutes.

$$
\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "\varphi_2"'] \arrow[dd, "i"'] & & U \arrow[dr, "\varphi_3"] \arrow[dd, "\bar{i}"' near start] & \\
& B \arrow[rr, "g" near start] \arrow[dd, "j" near start] & & V \arrow[dd, "\bar{j}"] \\
X \arrow[rr, "\bar{f}"' near end] \arrow[dr, "\varphi_1"'] & & X \sqcup_f U \arrow[dr, "\varphi"] & \\
& Y \arrow[rr, "\bar{g}"'] & & Y \sqcup_g V
\end{tikzcd}
\tag{7}
$$

If all φ_1, φ_2 and φ_3 are homotopy equivalences then the following lemma tells us that φ is also a homotopy equivalence.

Lemma 3.9 ([Bro06, Theorem 7.5.7])**.** *Consider a commutative diagram as in* (7) *where the front and back faces define an adjunction space as in* (6)*. If i and j are closed cofibrations and φ_1, φ_2 and φ_3 are homotopy equivalences, then the φ as determined by the diagram is also a homotopy equivalence.*

The restriction of i and j being closed cofibrations is quite mild. In the cases important to us, i and j will be cellular inclusions in to finite CW complexes, and thus closed cofibrations. See [Bro06] for more details on pushout squares and adjunction spaces.

To use Lemma 3.9 we must be able to construct X_W and X'_W as a sequence of adjunction spaces. Consider X_W in the following example.

Example 3.10. Let $\Delta_W = \{\emptyset, \{s\}, \{t\}, \{u\}, \{s,t\}, \{s,u\}, \{t,u\}\}$ and let $\Delta_W^n := \{T \in \Delta_W \mid |T| = n\}$. Clearly we have that $X_W = \bigcup_{T \in \Delta_W^2} X_{W_T}$ with the appropriate gluing. Suppose we had the 1–skeleton $X_W^1 \subsetneq X_W$, and some ordering on $\Delta_W^2 = (\{s,t\}, \{s,u\}, \{t,u\})$. To construct X_W, we would first glue $X_{\{s,t\}}$ to X_W^1 as an adjunction space in the following way.

$$
\begin{tikzcd}
X_{\{s\}} \cup X_{\{t\}} \arrow[r, "f"] \arrow[d, "i_1"] & X_{\{s,t\}} \arrow[d, "\overline{i_1}"] \\
X_W^1 \arrow[r, "\bar{f}"] & X_W^1 \sqcup_f X_{\{s,t\}}
\end{tikzcd}
$$

Where f is inclusion of those 1–cells in to $X_{\{s,t\}}$, which in this case makes $X_W^1 \sqcup_f X_{\{s,t\}} \cong X_{\{s,t\}}$. Note that $X_{\{s\}} \cup X_{\{t\}}$ is really shorthand for another adjunction space, which we assume to have been already constructed. We can then add $X_{\{t,u\}}$ to the preceding adjunction space in the following way.

$$\begin{tikzcd} X_{\{t\}} \cup X_{\{u\}} \arrow[r, "g"] \arrow[d, "i_2"] & X_{\{t,u\}} \arrow[d, "\overline{i_2}"] \\ X_W^1 \sqcup_f X_{\{s,t\}} \arrow[r, "\bar{g}"] & (X_W^1 \sqcup_f X_{\{s,t\}}) \sqcup_g X_{\{t,u\}} \end{tikzcd} \tag{8}$$

After which we would continue with $\{u, v\}$ in the same manner. In the final space, $X_{\{s,t\}}$ is glued to $X_{\{t,u\}}$ along $X_{\{t\}}$. In general, for $T_1, T_2 \in \Delta_W^n$, we have that X_{T_1} and X_{T_2} are glued along $X_{T_1 \cap T_2} \subseteq X_W^{n-1}$ where $T_1 \cap T_2 \in \Delta_W^{n-1}$. We can always construct the n–skeleton from the $(n-1)$–skeleton in exactly this way.

The exact same construction works for X'_W. This construction may seem too abstracted, in that much of the structure is hidden away in the maps f and g. However, as it turns out, we can use this adjunction structure without considering the details of these maps.

Theorem 3.11 ([PS21, Theorem 5.5])**.** *For a Coxeter group W, the space X'_W as in Definition 3.1 is homotopy equivalent to the Salvetti complex X_W.*

Proof. We achieve this inductively. To tidy our notation, in this proof we drop W so that X, X', X_T and X'_T correspond to X_W, X'_W, X_{W_T} and X'_{W_T} respectively. Let Δ_W^n be as in Example 3.10.

Suppose we have the $(n-1)$–skeletons X^{n-1} and $(X')^{n-1}$ and a homotopy equivalence $\varphi\colon X^{n-1} \to (X')^{n-1}$. We wish to show that we can extend φ to a homotopy equivalence for the respective n–skeletons. We do so by constructing the n–skeletons as adjunction spaces of the $(n-1)$–skeletons as in Example 3.10 and use Lemma 3.9.

Suppose we are gluing on the cells corresponding to some $T \in \Delta_W^n$. So any $Q \subsetneq T$ will correspond to a cell in X^{n-1}. Let Y and Y' be some intermediate steps in the adjunction gluing, such as the bottom–left term of (8). Suppose we have a homotopy equivalence $\varphi_1\colon Y \to Y'$ such that $\varphi_1|_{X^{n-1}} = \varphi$ so the following commutes.

$$\begin{tikzcd} \bigcup_{Q \subsetneq T} X_Q \arrow[r, "\varphi"] \arrow[d, hook] & \bigcup_{Q \subsetneq T} X'_Q \arrow[d, hook] \\ Y \arrow[r, "\varphi_1"] & Y' \end{tikzcd} \tag{9}$$

We wish to extend φ to a homotopy equivalence ψ such that the following commutes.

$$\begin{tikzcd} \bigcup_{Q \subsetneq T} X_Q \arrow[r, "\varphi"] \arrow[d, hook] & \bigcup_{Q \subsetneq T} X'_Q \arrow[d, hook] \\ X_T \arrow[r, "\psi"] & X'_T \end{tikzcd}$$

This is possible by Lemma 3.7. Now we have the following commutative diagram.

$$
\begin{tikzcd}
\bigcup_{Q \subsetneq T} X_Q \arrow[rr, "f"] \arrow[dr, "\varphi"'] \arrow[dd, hook] & & X_T \arrow[dr, "\psi"] \arrow[dd, hook] & \\
& \bigcup_{Q \subsetneq T} X'_Q \arrow[rr, "g"] \arrow[dd, hook] & & X'_T \arrow[dd, hook] \\
Y \arrow[rr, "\bar{f}"'] \arrow[dr, "\varphi_1"'] & & Y \sqcup_f X_T \arrow[dr, "\sigma"] & \\
& Y' \arrow[rr, "\bar{g}"'] & & Y' \sqcup_g X'_T
\end{tikzcd}
\tag{10}
$$

Where the induced map σ is a homotopy equivalence by Lemma 3.9. At the next inductive step, $Y \sqcup_f X_T$ and $Y' \sqcup_g X'_T$ replace Y and Y' respectively. Accordingly, σ replaces φ_1. Suppose we are next going to glue the cells corresponding to $\tilde{T} \in \Delta_W$. To proceed inductively, there are two possible outcomes:

1. We are still constructing $X^n \simeq (X')^n$ and $\tilde{T} \in \Delta_W^n$.

2. We completely constructed $X^n \simeq (X')^n$ in the previous step and $\tilde{T} \in \Delta_W^{n+1}$.

In Case 1, we have that any $Q \subsetneq \tilde{T}$ corresponds to cells in X^{n-1}. By the inductive hypothesis, we can restrict $\varphi_1 \colon Y \to Y'$ to X^{n-1}, and thus we can do the same for σ and so the restriction $\sigma|_{X^{n-1}}$ is well–defined, we can get (9) with the appropriate replacements and proceed inductively.

In Case 2, $Y \sqcup_f X_T$ and $Y' \sqcup_g X'_T$ are X^n and $(X')^n$ respectively. Some $Q \subsetneq \tilde{T}$ will correspond to cells in X^n, but σ is exactly the restriction $\sigma|_{X^n}$ so, the restriction is well–defined. We get (9) with the appropriate replacements and proceed inductively.

The base case is $X_\emptyset \simeq X'_\emptyset \simeq \{\bullet\}$. $\square$

Chapter 4

Discrete Morse Theory

In this section we will prove the next homotopy equivalence along the chain in (1). This will again involve the use of posets and their combinatorics. Morse theory for smooth manifolds gives a way to infer topological properties of manifolds from analytical properties of certain smooth functions on that manifold. Discrete Morse theory is a CW (non–smooth) analogue. Certain functions on the (discrete) set of cells of a CW complex can tell us topological facts about the CW complex. Here we will only give a brief introduction to the main results of this theory that are relevant to us.

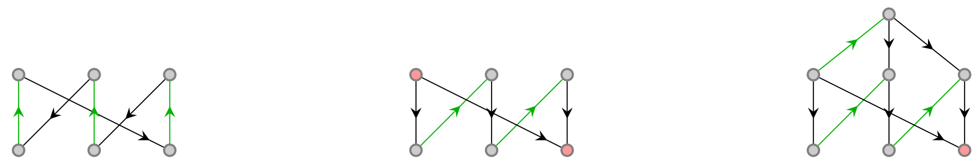
4.1 The Face Poset and Acyclic Matchings

In previous sections we gave constructions that formed spaces from posets, we now give a construction in the opposite direction. Given a CW complex X denote the set of open cells as X^*.

Definition 4.1 (The Face Poset)**.** Given a CW complex X, the *face poset* $\mathcal{F}(X)$ is an ordering on X^* where $\tau \leq \sigma$ when $\bar{\tau} \subseteq \bar{\sigma}$.

For a finite dimensional and connected CW complex, $\mathcal{F}(X)$ is a bounded and graded poset with rank function $\mathrm{rk}(\sigma) = \dim(\sigma)$. Let P denote $\mathcal{F}(X)$. Q Here the use of rank is quite different compared to its introduction in Section 2.1. I think this sleight of hand is OK but can be changed if necessary Consider some subset of the covering relations $\mathcal{M} \subseteq \mathcal{E}(P)$. We consider this as a set of edges in the Hasse diagram for P, which is denoted H. From $\mathcal{M}$ we define an ordering on the graph H such that $p \lessdot q$ is oriented from p to q if $(p \lessdot q) \in \mathcal{M}$, and otherwise in the opposite direction. We denote this oriented graph $H_{\mathcal{M}}$. We call $\mathcal{M}$ a *matching* if for all $p \in P$, at most one $m \in \mathcal{M}$ contains p. A matching is *acyclic* if $H_{\mathcal{M}}$ contains no directed cycles. Furthermore, a matching is *proper* if for all $p \in P$, the set of all nodes in $H_{\mathcal{M}}$ reachable by a directed path from p is finite. Fig. 12 gives some (non)examples of matchings.

We observe that the requirement of being a matching means that any path through $H_{\mathcal{M}}$ will never consecutively go through two edges in $\mathcal{M}$. A cycle in $H_{\mathcal{M}}$ must clearly start and end at the same rank. Since edges in $\mathcal{M}$ increase rank and edges in $\mathcal{E}(P) \setminus \mathcal{M}$ decrease rank, a cycle must therefore be (cyclically) alternating between edges in $\mathcal{M}$ and edges in $\mathcal{E}(P) \setminus \mathcal{M}$. Therefore, if a cycle is to start at $p \in P$, it must completely occur in $\{q \in P \mid \operatorname{rk}(q) - \operatorname{rk}(p) \in \{0, 1\}\}$ or completely in $\{q \in P \mid \operatorname{rk}(q) - \operatorname{rk}(p) \in \{0, -1\}\}$. I.e. the horizontal bands above or below p in H. Since it is alternating, a cycle must also comprise an even number of edges.



(a) A cyclic matching.

(b) An acyclic matching with invalid subcomplex.

(c) An acyclic matching with valid subcomplex.

Figure 12: Directed Hasse diagrams corresponding to face posets and choices of $\mathcal{M}$. Green edges are those in $\mathcal{M}$ and red nodes are critical cells.

We call σ a *face* of τ if $\sigma \lessdot \tau$. Consider Φ, the characteristic map of some n–cell τ with $\Phi \colon D^n \to X$. We call σ a *regular face* of τ if it is a face and the following hold.

1) $\Phi|_{\Phi^{-1}(\sigma)} \colon \Phi^{-1}(\sigma) \to \sigma$ is a homeomorphism.

2) $\overline{\Phi^{-1}(\sigma)}$ is homeomorphic to D^{n-1} as a subset of D^n.

For a matching $\mathcal{M}$, any $p \in P$ that is disjoint from all of $\mathcal{M}$ is called *critical*. In this context, a *critical cell*. We can now state the version of discrete Morse theory we will use.

Theorem 4.2 ([PS21, Theorem 2.4])**.** *Consider a CW complex X, a subcomplex $Y \subseteq X$, and a proper, acyclic matching $\mathcal{M}$ on X. If $\mathcal{F}(Y) \subseteq \mathcal{F}(X)$ is the set of critical cells in X with respect to $\mathcal{M}$ and every if σ is a regular face of τ for every $(\sigma \lessdot \tau) \in \mathcal{M}$, then X deformation retracts on to Y.*

You may notice that this seems to have no link to discrete functions $X^* \to \mathbb{N}$, as promised in the prologue of this section. Indeed, this statement is a reformulation of Discrete Morse Theory due [Cha00] and [Bat02]. The original formulation of Discrete Morse Theory is due to [For98]. The exact wording of Theorem 4.2 is important, we explore this in the following example.

Example 4.3. The Hasse diagrams in Fig. 12 correspond to the obvious CW complex for a triangle. Figs. 12a and 12b are for a hollow 1–dimensional triangle and Fig. 12c is for a filled 2–dimensional triangle. We know that a hollow triangle cannot deformation retract on to any of its subcomplexes, thus the required construction for Theorem 4.2 should fail for Figs. 12a and 12b. We see that Fig. 12a is a cyclic matching, but we achieve an acyclic (vacuously proper) matching in Fig. 12b. Importantly, the space corresponding to the union of the critical cells, which are highlighted in the figure, is not a valid subcomplex, thus Theorem 4.2 does not apply. For a subset of cells $Y^* \subseteq X^*$ to correspond to a valid subcomplex $Y \subseteq X$, we require

$$\bigcup_{y \in \mathcal{F}(Y)} [-\infty, y] = \mathcal{F}(Y) \tag{11}$$

where $[-\infty, y]$ is taken within the poset $\mathcal{F}(X)$. For Fig. 12b, the left–hand side of (11) would include the bottom left cell in the Hasse diagram, which we see is not critical. In Fig. 12c, we have a valid subcomplex and the critical cell corresponds to a vertex in the 2–dimensional triangle, which is of course a valid deformation retract of the whole complex.

We will also require the following standard tool for forming acyclic matchings. For this we introduce the notion of a *poset map*, which a map between posets $P \to Q$ that respects the poset structure. Given such a map φ, we call preimages of single elements *fibres*.

Theorem 4.4 (Patchwork Theorem [Koz08, Theorem 11.10])**.** *Given a poset map $\varphi\colon P \to Q$, assume we have acyclic matchings on all fibres $\varphi^{-1}(q)$, where the matching need only be acyclic within the fibre itself. The union of all of these matchings is an acyclic matching on P.*

4.2 A New Subcomplex, K'_W

We now introduce the particular poset map we will use. Recall that w is a choice of Coxeter element. Given the standard ordering of $\mathbb{N}$, the map $\eta\colon \mathcal{F}(K_W) \to \mathbb{N}$ with

$$\eta([x_1|x_2|\cdots|x_d]) \coloneqq \begin{cases} d & \text{if } x_1x_2\dots x_d = w \\ d+1 & \text{otherwise} \end{cases}$$

is a poset map. This can be readily checked. We call a connected component of $\eta^{-1}(d)$ a *d–fibre component*. We now investigate the form of these d–fibre components. Again, denote $\mathcal{F}(K_W)$ by P. Consider such a d–fibre component C such that $x \coloneqq [x_1|x_2|\cdots|x_d] \in C$. Clearly $x_1x_2\dots x_d = w$ and so x is not the face of anything in P. The faces of x are $[x_2|\cdots|x_d]$, $[x_1|\cdots|x_{d-1}]$ and $[x_1|\cdots|x_ix_{i+1}|\cdots|x_d]$ for all $i < d$. Only the first two faces will be in C since $\eta([x_1|\cdots|x_ix_{i+1}|\cdots|x_d]) = d - 1$. Therefore, a d–fibre component will look like Fig. 13.

To each fibre component there is an associated sequence $(x_i)_{i\in\mathbb{Z}}$ such that every product of d consecutive elements is equal to w and each corresponding cell from the sequence is in K_W. Denote the set of all such sequences S_d. For each fibre component, the choice of $s \in S_d$ is unique up translation of indices.

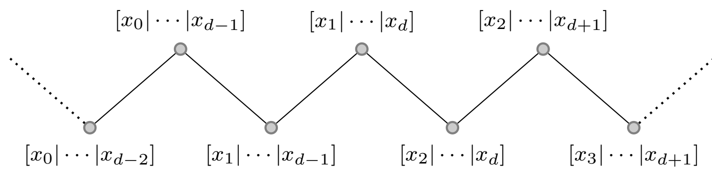


Figure 13: The general form of a connected d–fibre component.

Remark 4.5. Consider some cell $[x_1|\cdots|x_{d-1}]$, i.e. a cell in the bottom row of Fig. 13. To form the whole fibre component, you may choose to move up and to the right along the Hasse diagram from to $[x_1|\cdots|x_d]$. This is always possible and completely deterministic, since $x_1x_2\ldots x_{d-1}$ is a prefix of a minimal factorisation of w and the product $x_1x_2\ldots x_d$ must equal w. Similarly, a movement up and leftward is always possible and deterministic using the balanced property of K_W. Thus, a fibre component does not have initial of final nodes and its Hasse diagram will either be homeomorphic to $\mathbb{R}$ or S^1. A finite fibre component will have a repeating sequence. If the sequence repeats every n, then the Hasse diagram of the fibre component will contain $2n$ nodes. An infinite fibre component will be non–repeating, though an individual group element x_i may be repeated in the sequence.

The previous remark already shows that each sequence $s \in S_d$ is uniquely determined by any length $(d-1)$ subsequence. This is made clearer by the following observation. Let φ denote conjugation by w, i.e. $\varphi(x) = w^{-1}xw$. Then we have $\varphi(x_i) = x_{i+d}$ by the following factorisation.

$$\varphi(x_i) = (x_{i+d-1}^{-1}x_{i+d-2}^{-1}\ldots x_i^{-1})x_i(x_{i+1}x_{i+2}\ldots x_{i+d})$$

We will now define a subcomplex $K'_W \subseteq K_W$ based on these fibre components, after which we will prove the necessary properties of K'_W. Let $F \subseteq 2^{\mathcal{F}(K)}$ be the set of all connected fibre components such that $\bigcup_{f\in F} f = \eta^{-1}(\mathbb{N})$. Recall X'_W from Definition 3.1.

Definition 4.6 (The subcomplex K'_W)**.** For each infinite $f \in F$ let f' be the elements of f in between and including the first and last elements of $f \cap X'_W$, where *first*, *last* and *in between* derive from the linear form of the fibre components as in (13). Define K'_W to be the union of all finite f and f' such that f is infinite.

Note that since for finite f, the Hasse diagram of f is homeomorphic to S^1, so for finite f there is no notion of *in between*. Recall that X'_W is finite, so $f \cap X'_W$ is finite and *first* and *last* are well–defined. The first property of K'_W we must prove is that K'_W is indeed a valid subcomplex, satisfying (11).

Lemma 4.7. *As defined above, K'_W is a valid subcomplex of K_W.*

Proof. We first concentrate on the infinite $f \in \eta^{-1}(d)$. We need to show that some infinite $f \in F$, if $\tau \in f'$ and $\sigma \lessdot \tau$, then there exists some $g \in F$ such that $\sigma \in g'$. Let $s = (x_n)_{n\in\mathbb{Z}}$

be the sequence of group elements corresponding to f and let $\tau = [x_0|\cdots|x_{d-1}]$. Let τ be between α and ω both in $f \cap X'_W$. See Fig. 14. By Lemma 3.2 all of $[-\infty, \alpha]$ and $[-\infty, \omega]$ are also in X'_W, so we may assume that both α and ω are in the *bottom row* of the Hasse diagram of f' and accordingly consist of $d-1$ group elements. Choose $a, z \in \mathbb{Z}$ such that $\alpha = [x_a|\cdots|x_{a+d-2}]$ and $\omega = [x_z|\cdots|x_{z+d-2}]$.

- First we will prove the case where τ is in the top of the Hasse diagram for f' such that $x_0x_1\ldots x_{d-1} = w$. The two faces $[x_1|\cdots|x_{d-1}]$ and $[x_0|\cdots|x_{d-2}]$ are both already in f', so we need only check the faces

$$\sigma^i = [x_0|\cdots|x_ix_{i+1}|\cdots|x_{d-1}] \lessdot \tau.$$

Clearly each $\sigma^i \in \eta^{-1}(d-1)$. From the sequence s, we define the following.

$$s^i := (y^i_n)_{n\in\mathbb{Z}} = (\ldots, x_0, x_1, \ldots, x_ix_{i+1}, \ldots x_d, \ldots, x_{d+i}x_{d+i+1}, \ldots, x_{2d}, \ldots)$$

Where we multiply each adjacent pair x_j, x_{j+1} by removing the comma wherever $j \equiv i \mod d$. We see that every product of $(d-1)$ consecutive terms in s^i is w. Each s^i is the sequence corresponding to some connected component of $\eta^{-1}(d-1)$. Each face $\sigma^i \lessdot \tau$ is associated to the connected component associated to s^i. Denote this component g^i. We need to show that there exists $\alpha', \omega' \in g^i$ such that $\alpha' \leq \alpha$ and $\omega' \leq \omega$ and thus that $\alpha', \omega' \in X'_W$. We may choose α' and ω' to be any $[y^i_k|\cdots|y^i_{k+d-3}]$, i.e. a cell on the bottom row of g^i consisting of $(d-2)$ group elements. Let us concentrate on α'. There are three possibilities, remember that $\alpha = [x_a|\cdots|x_{a+d-2}]$ and $\omega = [x_z|\cdots|x_{z+d-2}]$.

1. $a - i \equiv 1 \mod d$ in which case the relevant part of s^i looks like

   $$\ldots, x_{a-2}, x_{a-1}x_a, x_{a+1}, \ldots, x_{a+d-2}, x_{a+d-1}x_{a+d}, x_{a+d+1}, \ldots.$$

   We choose $\alpha' = [x_{a+1}|\cdots|x_{a+d-2}]$.

2. $a - i \equiv 2 \mod d$ in which case the relevant part of s^i looks like

   $$\ldots, x_{a-3}, x_{a-2}x_{a-1}, x_a, \ldots, x_{a+d-3}, x_{a+d-2}x_{a+d-1}, x_{a+d}, \ldots.$$

   We choose $\alpha' = [x_a|\cdots|x_{a+d-3}]$.

3. $a - i \equiv k \mod d$ with $3 \leq k \leq d$ in which case the relevant part of s^i looks like

   $$\ldots, x_a, x_{a+1}, \ldots, x_{a+i}x_{a+i+1}, \ldots, x_{a+d-1}, x_{a+d}, \ldots.$$

   We choose $\alpha' = [x_a|\cdots|x_{a+i}x_{a+i+1}|\cdots|x_{a+d-2}]$.

In all cases α' is a cell in the bottom row of g' and a face of α. The same argument works for ω as well. We see that σ is indeed in a fibre component between two cells in X'_W.

• For the case where τ is in the bottom row of f' we may set $\tau = [x_0|\cdots|x_{d-2}]$. We can use the same methods as before. The extra two faces $[x_1|\cdots|x_{d-2}]$ and $[x_0|\cdots|x_{d-3}]$ to consider do not pose any extra difficulty. We can choose an appropriate s^i and proceed as before.

We now focus on the case where $\tau \in f$ and f is finite. By Remark 4.5 the sequence s associated to f will be repeating. We can again case–split whether τ is in the top or bottom row of the Hasse diagram for f. In either case, a face $\sigma \lessdot \tau$ will be in a component $g \in F$ associated to an appropriately chosen s^i. This s^i will also be repeating, thus the associated g will be finite. $\square$

For the definition of K' to be well–defined, we require that the construction of each f' be well–defined. This requires that $f \cap X'_W \neq \emptyset$ for each infinite f. This happens to be a detail for which the proof relies on W being of affine type.

Lemma 4.8 ([PS21, Lemma 7.6]). *Given a Coxeter group W of affine type and $f \in F$ is a d–fibre component, then there exists a simplex $\tau = [x_1|x_2|\cdots|x_{d-1}] \in f$ such that τ is also in X'_W.*

This covers all the details of Definition 4.6 and so K'_W is a valid subcomplex of K_W. We now get to reap the benefits of working in this strange setting of face posets.

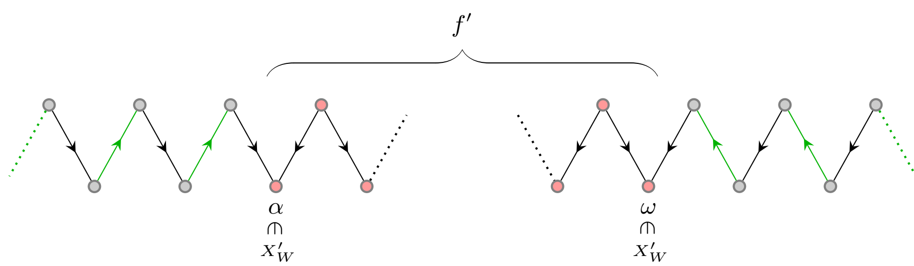


Figure 14: The unique acyclic matching $\mathcal{M}_f$ on f such that the critical cells correspond to exactly f'. Here α and ω are the relevant first and last elements in $f \cap X'_W$ as in the proof of Lemma 4.7. Critical cells are red and edges in $\mathcal{M}_f$ are green.

Theorem 4.9 ([PS21, Theorem 7.9, Lemma 7.11]). *The subcomplex K'_W is a deformation retract of K_W.*

Proof. We need to show that for each $f \in F$, we have an acyclic matching $\mathcal{M}_f$ such that the critical vertices corresponding to $\mathcal{M}_f$ are exactly those contained in K'_W. For finite f, this is trivial. We choose $\mathcal{M}_f = \emptyset$. For infinite f, choosing such an $\mathcal{M}_f$ is very simple. As shown in Fig. 14. There is a unique choice of $\mathcal{M}_f$. The union of these $\mathcal{M}_f$ is an acyclic matching on $\mathcal{F}(K_W)$ by Theorem 4.4. This acyclic matching gives a deformation retract on to the union of critical cells, which is exactly K'_W, by Theorem 4.2. $\square$

References

[Art47] E. Artin. Theory of Braids. *Annals of Mathematics*, 48(1):101–126, 1947.

[Bat02] Ekkehard Batzies. Discrete Morse theory for cellular resolutions. *Journal für die reine und angewandte Mathematik*, 2002(543):147–168, January 2002.

[BB10] Anders Bjorner and Francesco Brenti. *Combinatorics of Coxeter Groups.* Springer Berlin Heidelberg, October 2010.

[Bes03] David Bessis. The dual braid monoid. *Annales Scientifiques de l'École Normale Supérieure*, 36(5):647–683, September 2003.

[Bou08] Nicolas Bourbaki. *Elements of Mathematics. Lie Groups and Lie Algebras Chapters 4 - 6.* Springer, Berlin, Heidelberg, 1. softcover print. of the 1. english ed. of 2002 edition, 2008.

[Bri71] E. Brieskorn. Die Fundamentalgruppe des Raumes der regulären Orbits einer endlichen komplexen Spiegelungsgruppe. *Inventiones mathematicae*, 12(1):57–61, March 1971.

[Bro06] Ronald Brown. *Topology and Groupoids.* Booksurge, 2006.

[CD95] Ruth Charney and Michael W. Davis. The \$K(\pi, 1)\$-Problem for Hyperplane Complements Associated to Infinite Reflection Groups. *Journal of the American Mathematical Society*, 8(3):597–627, 1995.

[Cha00] Manoj K. Chari. On discrete Morse functions and combinatorial decompositions. *Discrete Mathematics*, 217(1):101–113, April 2000.

[CMW02] Ruth Charney, John Meier, and Kim Whittlesey. Bestvina's normal form complex and the homology of Garside groups, March 2002.

[Del72] Pierre Deligne. Les immeubles des groupes de tresses générales. *Inventiones Mathematicae*, 17:273–302, 1972.

[EM45] Samuel Eilenberg and Saunders MacLane. Relations Between Homology and Homotopy Groups of Spaces. *Annals of Mathematics*, 46(3):480–509, 1945.

[FN62a] Edward Fadell and Lee Neuwirth. Configuration Spaces. *Mathematica Scandinavica*, 10:111–118, 1962.

[FN62b] R. Fox and L. Neuwirth. The Braid Groups. *Mathematica Scandinavica*, 10:119–126, 1962.

[For98] Robin Forman. Morse Theory for Cell Complexes. *Advances in Mathematics*, 134(1):90–145, March 1998.

[Gar69] Frank Arnold Garside. The braid group and other groups. *The Quarterly Journal of Mathematics*, 20(1):235–254, January 1969.

[Grä11] George Grätzer. *Lattice Theory: Foundation*. Springer, Basel, 2011.

[Hat01] Allen Hatcher. *Algebraic Topology*. Cambridge University Press, 2001.

[Hua23] Jingyin Huang. Labeled four cycles and the \$K(\pi,1)\$-conjecture for Artin groups, May 2023.

[Koz08] Dmitry Kozlov. *Combinatorial Algebraic Topology*. Number 21 in Algorithms and Computation in Mathematics. Springer, 2008.

[Mcc05] Jon Mccammond. AN INTRODUCTION TO GARSIDE STRUCTURES. *Department of Mathematics, University of California, Santa Barbara*, 2005.

[MS17] Jon McCammond and Robert Sulway. Artin groups of Euclidean type. *Inventiones mathematicae*, 210(1):231–282, October 2017.

[Mun00] James R. Munkres. *Topology*. Prentice Hall, Inc, Upper Saddle River, NJ, 2nd ed edition, 2000.

[Pao17] Giovanni Paolini. On the classifying space of Artin monoids. *Communications in Algebra*, 45(11):4740–4757, November 2017.

[Par14] Luis Paris. \$K(\pi ,1)\$ conjecture for Artin groups. *Annales de la Faculté des sciences de Toulouse : Mathématiques*, 23(2):361–415, 2014.

[PS21] Giovanni Paolini and Mario Salvetti. Proof of the \$K(\pi,1)\$ conjecture for affine Artin groups. *Inventiones mathematicae*, 224(2):487–572, May 2021.

[Sal87] M. Salvetti. Topology of the complement of real hyperplanes in $\mathbb{C}$N. *Inventiones mathematicae*, 88(3):603–618, October 1987.

[Sal94] Mario Salvetti. The Homotopy Type of Artin Groups. *Mathematical Research Letters*, 1(5):565–577, September 1994.

[The20] The SageMath Developers. SageMath, October 2020.

[The22] The GAP Group. GAP – Groups, Algorithms, and Programming., 2022.

[VD83] Nguyêñ Viêt Dũng. The fundamental groups of the spaces of regular orbits of the affine weyl groups. *Topology*, 22(4):425–435, 1983.

[vdL83] H. van der Lek. *The Homotopy Type of Complex Hyperplane Complements*. PhD thesis, Katholieke Universiteit te Nijmegen, 1983.