Combinatoric and Poset Structures for the $K(\pi, 1)$ Conjecture

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1 Introduction

In this paper we will be concerned with the $K(\pi, 1)$ conjecture for Artin groups. This states that the configuration space Y_W for any Coxeter group W is a $K(G_W, 1)$ space, where G_W is the Artin group associated to W. This conjecture emerges as a generalisation of the result for Coxeter groups of type $\tilde{A}_n$ and is originally attributed to Arnol'd, Pham and Thom in [Lek83]. See also [CD95] for a good overview of the history of the conjecture.

The specific focus here will be on [PS21] in which the authors prove the $K(\pi, 1)$ conjecture for Artin groups of affine type. Here, we will review some theorems in that paper and provide background such that someone not familiar with the field will be able to follow along. Much of this will involve proving a chain of homotopy equivalences. A strong theme will be the involvement of posets in these proofs and related structures in these proofs, hence the title of this paper. We will begin by providing a birds eye view of the main results in [PS21], after which we will give an introduction to the objects involved.

1.1 Overview of Main Results

Here we compile the main results from [PS21] in to a few theorems. Many of the objects here are not yet defined. Those mentioned in this section and defined in this paper are; W: 1.2.1, G_W: 1.2.2, Y_W: 1.3.1, K_W: 2.3.4 X_W: 2.4.4, and X'_W: 3.1.1.

Theorem 1.1.1 ([PS21]). Given an affine Coxeter group W, the configuration space Y_W is homotopy equivalent to the order complex K_W.

Proof. By Theorem 2.4.5 the Salvetti complex X_W is homotopy equivalent to the configuration space Y_W. Therefore, we need only show $K_W \simeq X_W$. This is done through a composition of homotopy equivalences

$$X_W \overset{\text{(a)}}{\simeq} X'_W \overset{\text{(b)}}{\simeq} K'_W \overset{\text{(c)}}{\simeq} K_W \tag{1.1}$$

Where the results are gathered from the following sources:

(a): Theorem 3.2.8 [PS21, Theorem 5.5] (b): [PS21, Theorem 8.14] (c): [PS21, Theorem 7.9] □

Furthermore, in the same paper another main result is shown.

Theorem 1.1.2 ([PS21, Theorem 6.6])**.** Given an affine Coxeter group W, corresponding affine Artin group G_W and Coxeter element $w \in W$, the complex K_W is a classifying space for the dual Artin group W_w. I.e.

$$K_W \simeq K(W_w, 1)$$

It was already known [Bri71] that $\pi_1(Y_W) = G_W$. Thus considering $\pi_1(Y_W)$ and combining Theorems 1.1.1 and 1.1.2 gives

$$Y_W \simeq K(G_W, 1)$$
$$G_W \cong W_w$$

for affine G_W.

This proves the $K(\pi, 1)$ conjecture for affine Artin groups and provides a new proof than an affine Artin group is naturally isomorphic to its dual, which was already known for finite [Bes03] and affine [MS17] cases.

The proof of $\pi_1(Y_W) \cong G_W$ for all W in [Bri71] is in German and only German or Russian translations are available. This result is fundamental and non-trivial. Alternative proofs for Coxeter groups of type A_n [FN62] or affine type [Viê83] are available in English.

1.2 Coxeter Groups and Artin Groups

In this section we will cover the constructions and linked properties of the two groups of interest to this paper. Coxeter groups are a generalisation of *reflection groups*, which are subgroups of $O(\mathbb{R}, n)$ generated by a finite set of reflections. Although the definition of Coxeter groups is tied to an abstract group presentation, we must also think of them as groups with a natural reflection action on some space. For example, finite Coxeter groups can be realised as reflection groups on spheres and affine Coxeter groups can be realised as groups generated by affine (with plane of reflection not necessarily passing through the origin) reflections in $\mathbb{R}^n$. Note that the realisation of a Coxeter group as a group generated by reflections is not unique and that some Coxeter groups cannot be realised as a subgroup of $O(\mathbb{R}, n)$.

Definition 1.2.1 (Coxeter Group)**.** For a finite set S, a Coxeter group W generated by S is a group with presentation of the form

$$W = \langle S \mid (st)^{m(s,t)} = 1 \quad \forall m(s,t) \neq \infty \rangle$$

where $m\colon S \times S \to \mathbb{N}$ is a symmetric matrix indexed over S where $m(s,s) = 1$ for all $s \in S$ and $m(s,t)$ takes values in $\{2, 3, \ldots\} \cup \{\infty\}$ for all $s \neq t$.

The infinities correspond to pairs of elements that have no explicit relations. The ones along the diagonal of m ensure that all generators have order 2. The set $R \coloneqq \{wsw^{-1} \mid w \in W, s \in S\}$ is the set of *reflections* in W. Sometimes S is referred to as the set of *basic reflections*, or that a choice of S is a choice of basic reflections.

A graph, called the *Coxeter diagram*, is often used to encode the data of the matrix m and its corresponding Coxeter group. In this graph, each element of S is a node and relations between pairs in S correspond to labelled edges. There are two conventions for this labelling: The *classical labelling*, where edges with $m(s,t) = 2$ are not drawn, edges with $m(s,t) = 3$ are drawn but not labelled and all other edges are drawn with the value of $m(s,t)$ as their label. And the *modern labelling*, edges with $m(s,t) = \infty$ are not drawn, edges with $m(s,t) = 2$ are drawn but not labelled and all other edges are drawn and labelled. An example highlighting these differences is given in Fig. 1.1. We will only use the classical labelling here, but awareness of the modern labelling is useful.

In the classical labelling, if the diagram has multiple connected components then W is a direct product of the groups corresponding to those components. Similarly, in the modern labelling connected components are factors in a free product. Other topological properties of these diagrams can be used, for example in [Hua23] which proves the $K(\pi, 1)$ conjecture for certain W with diagrams being trees or containing cycles. The property of Coxeter groups that allows us to make this graph construction is that every relation in a Coxeter group only involves two generators, and that each relation is encoded by a number and a pair of generators.

Figure 1.1: Coxeter diagram for a certain Coxeter group with classical labelling (left) and modern labelling (right).

To each Coxeter group W there is an associated Artin group G_W defined as follows

Definition 1.2.2 (Artin Group)**.** For a given Coxeter group W generated by S with associated matrix m, the associated Artin group is

$$G_W \coloneqq \langle S \mid \Pi(s,t,m(s,t)) = \Pi(t,s,m(s,t)) \ \ \forall s \neq t \text{ and } m(s,t) \neq \infty \rangle$$

where $\Pi(s,t,n)$ is defined to be an alternating product of s and t starting with s with total length n. E.g. $\Pi(s,t,3) = sts$.

Note that the ones along the diagonal of m now carry no meaning in the presentation and that if we add the relation $s^2 = 1$ for all $s \in S$ we retrieve the original Coxeter group. The Coxeter diagram for W also encodes the data of G_W and the topology of the diagram holds similar meaning for G_W. Our notation for Artin groups, G_W (shared in much of the literature), seems to imply the data for the Artin group is inherited from its Coxeter group. While each presentation determines the other, in principle there is no precedence, but practically we often start by defining a Coxeter group. Often "`property` type Artin groups" describes a family of Artin groups to which their corresponding Coxeter groups are `property`.

In particular, *spherical* or *finite* type Artin groups have associated spherical or finite type Coxeter groups. Similarly, affine type Artin groups have affine associated Coxeter groups.

1.3 Configuration Space

Here we will give the definition of the configuration space Y_W for a given Coxeter group W and go through an example where we will show that $\pi_1(Y_W) \cong G_W$ for W of type A_n [FN62].

For some finite or affine Coxeter group W acting on $\mathbb{R}^n$, the set of reflections $R \in W$ acts on $\mathbb{R}^n$ by reflection through hyperplanes, one for each $r \in R$. For some $r \in R$ denote its hyperplane by $H(r) \subseteq \mathbb{R}^n$. Denote the union of all hyperplanes by $\mathcal{H} \coloneqq \bigcup_{r \in R} H(r)$. We associate $\mathbb{R}^n \otimes \mathbb{C}$ with $\mathbb{C}^n$ under the natural isomorphism $x \otimes \lambda \mapsto x\lambda$. This also extends the action $W \circlearrowleft \mathbb{R}^n$ to $W \circlearrowleft \mathbb{C}^n$ via $w \cdot (x \otimes \lambda) = (w \cdot x) \otimes \lambda$. We call this act of transporting objects related to $\mathbb{R}^n$ over to $\mathbb{C}^n$ via the tensor product *complexification.* With these tools in mid, we can then make our definition.

Definition 1.3.1 (Configuration space)**.** For some Coxeter group W and associated hyperplane system $\mathcal{H}$ as above, we define

$$Y \coloneqq \mathbb{C}^n \setminus (\mathcal{H} \otimes \mathbb{C})$$

and define the configuration space Y_W to be the quotient Y/W with the action defined as above.

It is important to note that the importance of $\mathbb{C}$ here is that it is 2 dimensional. When one takes the complement of a co-dimension 1 object, you typically will

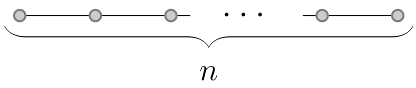


Figure 1.2: The clasical Coxeter diagram for the Coxeter group of type A_n.

not get any interesting topology. By complexifying the hyperplanes and then taking the complement within $\mathbb{C}^n$, we are effectively taking the complement of a co-dimension 2 object, and there is much more room for interesting topologies. The same construction can be achieved using $\mathbb{R}^{2n}$ and $\mathcal{H} \times \mathcal{H}$ but here we choose $\mathbb{C}$. A more general construction for all Coxeter groups can be found in [Par14].

For a concrete example, we will introduce the A_n family of Coxeter groups and show that the space Y_W for these groups is the space of configurations of $n+1$ points in $\mathbb{C}$, thus explaining the name *configuration space* for general Y_W.

The family A_n all have Coxeter diagrams of the form as in Fig. 1.2 and a specific A_n will have presentation.

$$A_n = \left\langle \sigma_1, \sigma_2, \ldots, \sigma_n \;\middle|\; \begin{array}{rl} \sigma_i^2 = 1 & \forall i \\ (\sigma_i\sigma_j)^2 = 1 & \forall (i+1 < j \leqslant n) \\ (\sigma_i\sigma_{i+1})^3 = 1 & \forall (i < n) \end{array} \right\rangle \tag{1.2}$$

This is well known to be a presentation for the symmetric group S_{n+1} with generators being adjacent transpositions [BB05, Proposition 1.5.4]. Accordingly, we will use the associated cycle notation for symmetric groups to talk about elements of A_n.

The action of A_n as a reflection group is realised on the space $\mathbb{R}^{n+1}$ with basis $\{e_i\}$, where $A_n \circlearrowleft \mathbb{R}^{n+1}$ by permuting components with respect to that basis. The set of reflections R of A_n is all conjugations of the n adjacent generating transpositions $(l, l+1)$. So, R is the set of all transpositions (l, n). Some $(l, n) \in R$ acts on $\mathbb{R}^{n+1}$ as reflection through the plane $\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_l = x_n\}$. Thus, taking the complement of the complexification of all such planes, we have $Y = \{(\mu_1, \ldots \mu_{n+1}) \in \mathbb{C}^{n+1} \mid \forall i, j \;\; \mu_i \neq \mu_j\}$ (here Y is as in Definition 1.3.1). We can think of this as the space of $n+1$ distinct labelled points in $\mathbb{C}$. The action $A_n \circlearrowleft \mathbb{C}^{n+1}$ also permutes components, so we can think of the configuration space Y_W as the set of $n+1$ distinct *unlabelled* points in $\mathbb{C}$, denoted $\mathrm{Conf}_{n+1}(\mathbb{C})$.

Historically, Emile Artin [Art47] originally defined the braid group on n strands B_n to be $\pi_1(\mathrm{Conf}_n(\mathbb{R}^2))$. He then showed the validity of the well known presentation of the braid group.

$$B_n = \left\langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{rll} \sigma_i\sigma_j &= \sigma_j\sigma_i & \forall (i+1 < j \leqslant n) \\ \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} & \forall (i < n) \end{array} \right\rangle$$

In this context, showing the validity of that presentation immediately proves $B_{n+1} \cong G_W$ and thus that $\pi_1(Y_W) \cong G_W$. This proof by Artin is often considered dubious and other proofs are available. One good example is [FN62]. Importantly, this is also true in the general case.

Theorem 1.3.2 ([Bri71])**.** For any Coxeter group W, we have $\pi_1(Y_W) \cong G_W$.

The paper cited is in German and only German or Russian translations are available. Alternative proofs for Coxeter groups of affine type [Viê83] are available in English.

1.4 The $K(\pi, 1)$ Conjecture

For a group G, an *Eilenberg-MacLane space* [EM45] for G is a space X such that $\pi_n(X) = G$ for some n and $\pi_i(X) = 0$ for all $i \neq n$. We will use the terminology "X is a $K(G, n)$ space". We will also use the terminology "X is a *classifying space* for G" to mean that X is a $K(G, 1)$ space.

Conjecture 1.4.1 ($K(\pi, 1)$ Conjecture)**.** For all Coxeter groups W, the space Y_W is a $K(G_W, 1)$ space.

Admittedly, the use of π in the name of the conjecture is confusing. An equivalent formulation of the conjecture is that the universal cover of Y_W is contractible. This would give us that the higher homotopy groups of Y_W are trivial by [Hat01, Porposition 4.1]. A–priori this is a stronger statement than the conjecture as we have given it. But since weak homotopy equivalences are homotopy equivalences for CW complexes [Hat01, Theorem 4.5], and Y_W has a CW structure (Section 2.4), these two statements of the conjecture are equivalent.

In the previous section we focused on Coxeter groups of type $\tilde{A}_n$ and saw that for these groups $\pi_1(Y_W) \cong G_W$. We wish to focus again on the $\tilde{A}_n$ family and prove the $K(\pi, 1)$ conjecture in this case. To do so, we need only verify that the higher homotopy groups of Y_W are trivial. This can be done by observing that $\mathrm{Conf}_n(\mathbb{C})$ is a fibre bundle over $\mathrm{Conf}_{n-1}(\mathbb{C})$ with projection p forgetting a point and fibres homeomorphic to $\mathbb{C}\backslash\{\text{n distinct points}\}$, as spelled out in [Sin10].

The space $\mathbb{C}\backslash\{\text{n distinct points}\}$ is homotopy equivalent to $\bigvee_n S^1$, so we can use the fibration

$$\bigvee_{n-1} S^1 \hookrightarrow \mathrm{Conf}_n(\mathbb{C}) \xrightarrow{p} \mathrm{Conf}_{n-1}(\mathbb{C})$$

to build a tower of fibrations

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
\bigvee_3 S^1 \arrow[r, hook] & \mathrm{Conf}_4(\mathbb{C}) \arrow[d] \\
\bigvee_2 S^1 \arrow[r, hook] & \mathrm{Conf}_3(\mathbb{C}) \arrow[d] \\
S^1 \arrow[r, hook] & \mathrm{Conf}_2(\mathbb{C})
\end{tikzcd}$$

where there is a short exact sequence in homotopy groups starting at each $\pi_k(\bigvee_n S^1)$ going right and down to $\pi_k(\mathrm{Conf}_n(\mathbb{C}))$ for any k. We note that $\mathrm{Conf}_2(\mathbb{C}) \simeq S^1$ and so has trivial homotopy above π_1. Similarly, $\bigvee_2 S^1$ has trivial higher homotopy. So $\pi_k(\mathrm{Conf}_3(\mathbb{C})) \cong 0$ for $k > 1$, and we can continue up the tower inductively to show $\pi_k(\mathrm{Conf}_n(\mathbb{C})) \cong 0$ for $k > 1$ for all n. So indeed $\mathrm{Conf}_{n+1}(\mathbb{C}) = Y_W$ is a $K(G_W, 1)$ for $W = A_n$.

Theorem 1.4.2 ([Del72])**.** The $K(\pi, 1)$ conjecture holds for all finite Coxeter groups W.

The paper of interest to us, [PS21], proves the $K(\pi, 1)$ conjecture for affine type Coxeter groups.

2 Geometric Realisations of Poset Structures

In Section 1.3 we used the realisation of the Coxeter group W as a reflection group on a space V. We considered the planes of the defining reflections of W as affine subspaces of V and used these to define Y_W, the configuration space. The $K(\pi, 1)$ conjecture concerns with the homotopic properties of Y_W. To explore these, we will first construct a new space X_W, the *Salvetti complex*, which is homotopy equivalent to Y_W. This was originally defined [Sal87]; [Sal94] similarly using the realisation of W on a space. However, the Salvetti complex turns out to have a more useful formulation based more on algebraic properties of W and its defining relations. These structures arise from giving a partial order to W, and with this in mid, we will start with some definitions.

2.1 Posets

A partially ordered set or *poset* $(P, \leqslant)$ is a set P with a relation $\leqslant$ on pairs in P which encodes the topology of $\mathbb{R}$. The textbook [Grä11] provides a good introduction. An important note is that there is no requirement for every pair to be related, hence partial. We will use P as shorthand for $(P, \leqslant)$ where possible.

In a poset P we define the *interval* between two elements $[x, y]$ as $[x, y] \coloneqq \{u \in P \mid x \leqslant u \leqslant y\}$, which is itself a poset. For convenience, we define $[-\infty, w] \coloneqq \{u \in P \mid u \leqslant w\}$ and equivalently for $[w, \infty]$. A *chain* is a subset $C \subseteq P$ that is a totally ordered, i.e. every pair in $(u, v) \in C \times C$ satisfies $u \leqslant v$ or $v \leqslant u$. The *covering relations* of P, denoted $\mathcal{E}(P)$ are defined,

$$\mathcal{E}(P) = \{(x, y) \in P \times P \mid x \leqslant y \text{ and } [x, y] = \{x, y\}\}$$

i.e. ordered pairs such that there is nothing in between them in the poset. If $(x, y) \in \mathcal{E}(P)$, we write $x \lessdot y$. We will call a chain C *saturated* if for all $x, y \in C$ such that $x < y$, there exists $z \in C$ such that $x \lessdot z$, i.e. there are no gaps in the chain.

By transitivity, the covering relations encode the whole poset structure, which can in turn be drawn in a diagram which we will now define.

Definition 2.1.1 (Hasse Diagram)**.** Given a poset P, the *Hasse Diagram* is the directed graph encoding $\mathcal{E}(P)$ in the following way: For each element $x \in P$ draw a vertex. For each pair $(x, y) \in \mathcal{E}(P)$ draw an edge connecting x to y.

As is typical, we will draw the lesser (with respect to $\leqslant$) elements towards the bottom of the plane and visa versa. Thus, we will not need to draw arrows to show direction. In our case, the drawing of these Hasse diagrams is made easier, as we will concern ourselves only with graded and bounded posets. *Bounded* meaning that there are minimal and maximal elements, denoted $\hat{0}$ and $\hat{1}$ such that $\hat{0} \leqslant x \leqslant \hat{1}$ for all $x \in P$, and *graded* meaning that every saturated chain from $\hat{0}$ to $\hat{1}$ has the same (finite) length. In the Hasse diagram for a bounded, graded poset, we will draw $\hat{0}$ at the bottom, $\hat{1}$ at the top, and put all other elements in discrete vertical levels between these based on the position in the saturated chains between $\hat{0}$ and $\hat{1}$ each element occurs. See Fig. 2.4 for an example. Graded posets have a natural notion of a *rank function* $\mathrm{rk}\colon P \to \mathbb{N}$ that encodes the height above $\hat{0}$ that an element $p \in P$ occurs in the Hasse diagram.

Definition 2.1.2 (Edge Labelled Poset)**.** We define an edge labelled poset to be a triple $(P, \leqslant, l)$ where $(P, \leqslant)$ is a poset and the function $l : \mathcal{E}(P) \to A$ is the data of our labels with A being the alphabet of our labels.

We will use P as a shorthand for $(P, \leqslant, l)$ where possible. Given an edge labelled poset P, we can construct a group encoded by its labelling and geometry.

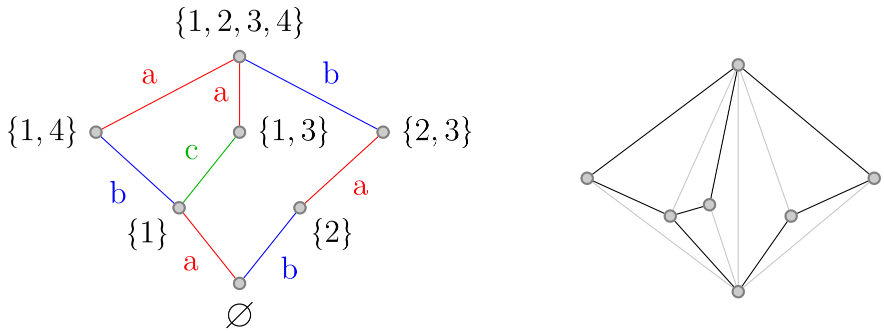


Figure 2.1: A simple example of a bounded and graded edge labelled poset where we have taken $\leqslant$ to be $\subseteq$ (left). The same poset with all chains drawn in light lines to aid visualising $\Delta(P)$ (right).

Definition 2.1.3 (Poset group)**.** Given some edge labelled poset $(P, \leqslant, l\colon \mathcal{E}(P) \to A)$, let the poset group $G(P)$ be the group generated by $\mathrm{Im}(l)$ with relations equating words corresponding to saturated chains going up the Hasse diagram of P which start and end at the same vertices.

A *word corresponding to a saturated chain* is the word of the labels traversed in the Hasse diagram while tracing out that saturated chain. In the example given in Fig. 2.1, the poset group is $G(P) = \langle a, b, c \mid aba = bab,\ ba = ca\rangle$.

2.2 Poset Complex

For some edge labelled poset P, we can construct a cell complex $K(P)$ from P such that $\pi_1(K(P))$ is $G(P)$. To do this, we must begin by initially defining a *geometric simplicial complex* $\Delta(P)$. An *abstract simplicial complex* is a family of sets that is closed under taking arbitrary subsets. Following from this, we can make our definition.

Definition 2.2.1 (Geometric Simplicial Complex)**.** Given a finite abstract simplicial complex X, the *geometric realisation* of that simplicial complex is defined as follows: For each single element set in X assign a point. For each two element set assign an open edge between the two vertices it contains. For each three element set assign an open triangle, the interior of the three edges of its three subsets of size two. In this way, continue constructing simplices of dimension n for each $n+1$ size set in X.

The set of all chains in a poset P is an abstract simplicial complex. We define $\Delta(P)$ to be the geometric simplicial complex corresponding to the set of all chains in P where each n–simplex is an n–chain of P. Note that as in [MS17, Definition 1.7], we define an n–chain to have $n-1$ elements. E.g. $(\{1\} \subseteq \{1,2\})$ is a 1–chain.

For example, in Fig. 2.1, $\Delta(P)$ would be three solid tetrahedrons all sharing an edge (a 1–simplex) corresponding to the 1–chain $(\varnothing \subseteq \{1, 2, 3, 4\})$ with two of them sharing a face corresponding to the 2–chain $(\varnothing \subseteq \{1\} \subseteq \{1, 2, 3, 4\})$. For a more two–dimensional example, consider the following poset P and corresponding $\Delta(P)$. Here we forget about edge labelling in P for a moment.

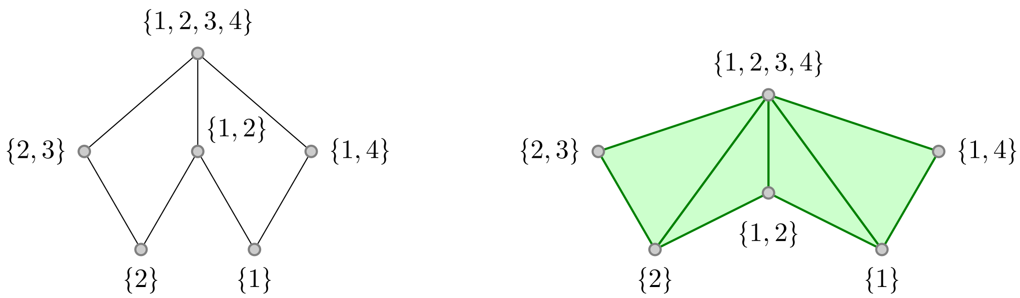


Figure 2.2: An example poset P (left) with corresponding $\Delta(P)$ (right).

We continue, now using an edge labelling on P, to generate a quotient space $K(P)$ of $\Delta(P)$. Let us put some arbitrary edge labelling on P to progress with this, shown in Fig. 2.3 (left).

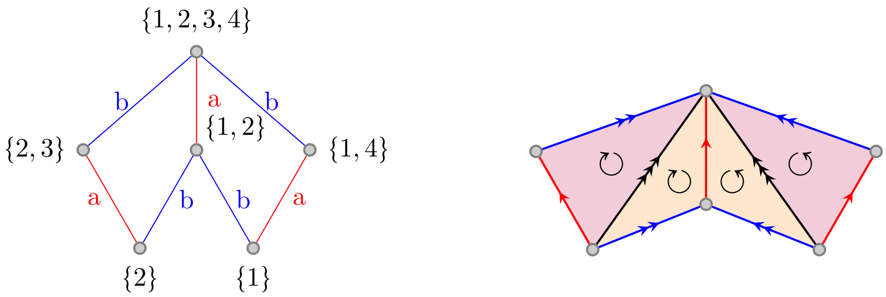


Figure 2.3: The poset in Fig. 2.2 with edge labelling (left) and the corresponding space $K(P)$ (right).

To construct $K(P)$, first we define a labelling on chains in P which extends from the edge labelling in P.

> **Definition 2.2.2** (Extended Labelling)**.** Given some edge–labelled poset $(P, \leqslant, l\colon \mathcal{E}(P) \to A)$ and some chain $C \subseteq P$, the *extended label* $\mathcal{L}(C) \subseteq A^*$ is the language of all words corresponding to all saturated chains that contain every element of C.

Here A^* denotes the set of all words in the alphabet A. For an example on extended labels, consider the chain $(\{2\} \subseteq \{1, 2, 3, 4\})$ in the context of Fig. 2.3. There are two corresponding saturated chains, $(\{2\} \subseteq \{1, 2\} \subseteq \{1, 2, 3, 4\})$ and $(\{2\} \subseteq \{2, 3\} \subseteq \{1, 2, 3, 4\})$, which respectively correspond to the words ba and ab. So $\mathcal{L}(\{2\} \subseteq \{1, 2, 3\}) = \{ba, ab\}$. Here are some illustrative examples:

- $\mathcal{L}(\{1\} \subseteq \{1,2\}) = \mathcal{L}(\{2\} \subseteq \{1,2\}) = \{b\}$.
- $\mathcal{L}(\{1\} \subseteq \{1,2,3,4\}) = \mathcal{L}(\{2\} \subseteq \{1,2,3,4\}) = \{ba, ab\}$.
- $\mathcal{L}(\{1\} \subseteq \{1,2\} \subseteq \{1,2,3,4\}) = \mathcal{L}(\{2\} \subseteq \{1,2\} \subseteq \{1,2,3,4\}) = \{ba\}$.
- $\mathcal{L}(\{1\}) = \mathcal{L}(\{2\}) = \mathcal{L}(\{1,2\}) = \cdots = \varnothing$.

This extended labelling on chains naturally extends to a labelling on simplices in $\Delta(P)$. Using this labelling and the orientation induced on a chain by $\leqslant$, we can define $K(P)$.

> **Definition 2.2.3** (Poset Complex [Mcc, Definition 1.6])**.** For an edge–labelled poset P the poset complex $K(P)$ is the quotient space $\Delta(P)/\sim$ where $\sim$ identifies pointwise simplices of the same dimension that share the same extended label, using the orientation on simplices induced by $\leqslant$.

In the example in Fig. 2.3, three red edges are identified, four blue edges are identified, two black edges are identified, two orange triangles are identified and two purple triangles are identified. The orientation of the identification on triangles is denoted by a $\circlearrowleft$ symbol.

We see that this space is homeomorphic to a torus, which has fundamental group $\mathbb{Z}^2 \cong \langle a, b \mid ab = ba \rangle$, which is also the $G(P)$ for this edge–labelled poset.

It is true in general that $\pi_1(K(p)) \cong G(P)$. We can determine $\pi_1(K(P))$ from its 2–skeleton [Hat01, Corollary 4.12]. The 1–skeleton will be a wedge of circles, one for each $a \in \mathrm{Im}(l)$ and some corresponding to unsaturated 1–chains in P (the black edges in Fig. 2.3). Only labelled edges will contribute generators to $\pi_1(K(P))$ since a labelled path can always be deformed to an unlabelled 1–chain through the simplex in $K(P)$. If two n–chains start and end at the same points, they will share an edge in an n–simplex corresponding to an unlabelled 1–chain. So one of the paths can be deformed to the path corresponding to the edge of the unlabelled 1–chain, and then through that shared edge to the other path, making the paths homotopic. E.g. in Fig. 2.3 we can deform $(\{2\} \subseteq \{1,2\} \subseteq \{1,2,3,4\})$ through $(\{2\} \subseteq \{1,2,3,4\})$ to $(\{2\} \subseteq \{2,3\} \subseteq \{1,2,3,4\})$. Identification of n–simplices for $n > 1$ does not affect the fundamental group, but does ensure that that higher homotopy groups are trivial. We can see if we did not identify the 2–simplices in Fig. 2.3, $\pi_2(K(P))$ would be non-trivial.

2.3 Interval Complex

Starting from a Coxeter group W generated by S, we wish to give W a labelled–poset structure and use the constructions from the previous section. The edge labelled Hasse diagram for W will embed in to the Cayley graph $\mathrm{Cay}(W, S)$, and it is useful to be able to swap between these two objects, as we will do. First we must define an order on a group.

Definition 2.3.1 (Word length in a group)**.** For a group G generated by S, the word length with respect to S is the function $l_S : G \to \mathbb{Z}$ where $l_S(g) = \min\{k \mid s_1 s_2 \dots s_k = g\,,\, s_i \in S\}$.

We will often omit the S in l_S where it is obvious from context.

Definition 2.3.2 (Order on a group)**.** For a group G generated by S, we define the order $x \leqslant y \iff l(x) + l(x^{-1}y) = l(y)$.

It can be readily checked that this does indeed define an order on G. This order encodes closeness to $e \in G$ along geodesics in $\mathrm{Cay}(G, S)$. We have $x \leqslant y$ precisely when there exists a geodesic in $\mathrm{Cay}(G, S)$ from e to y with x as an intermediate vertex, or to put it another way, when a minimal factorisation of x in to elements of S is a prefix of a minimal factorisation of y. For some $w \in W$ we define the poset $[1, w]^W$ to be the interval in W up to w with respect to this order. We give this poset an edge labelling such that the edge between w and ws is labelled s for some $s \in S$. The Hasse diagram thus embeds in to $\mathrm{Cay}(W, S)$.

Definition 2.3.3 (Coxeter element)**.** For some Coxeter group W generated by S, we define a *Coxeter element* $w \in W$ to be any product of all the elements of S without repetition.

These Coxeter elements are what we will use as the upper bound of our interval. We will also need to consider W as the group generated by R, the set of all reflections, rather than just the set of simple reflections S. See Fig. 2.4 for an example of such a poset. In principle there are many choices of Coxeter element depending on what order we multiply the elements of S. However, we will see that many structures resulting from $[1, w]^W$ are independent of that choice.

We apply the steps from Section 2.2 to $[1, w]^W$ to form a space.

Definition 2.3.4 (Interval Complex)**.** For a Coxeter group W generated by all reflections R with $w \in W$, we call $K_W \coloneqq K([1, w]^W)$ the *interval complex* where $K([1, w]^W)$ is as in Definition 2.2.3.

If W is infinite, then R is infinite and so K_W may have an infinite number of cells. We will later show that K_W deformation retracts to a finite subcomplex. Note that as in [PS21] we have dropped w from our notation K_W even though it depends on w. This is eventually justified (Theorem 1.1.1) since the homotopy type of K_W is independent of w.

Certain properties of the poset permit a simplified notation for the simplices within K_W. In this context, for two chains $C = (C_1 \leqslant C_2 \leqslant \cdots \leqslant C_m)$

and $C' = (C'_1 \leqslant C'_2 \leqslant \cdots \leqslant C'_n)$ we have $\mathcal{L}(C) = \mathcal{L}(C')$ exactly when $(C_1)^{-1}C_m = (C'_1)^{-1}C'_n$. Thus, we can label 1–simplices in K_W with group elements $x \in [1,w]^W$, we can label 2–simplices with factorisations of group elements in $[1,w]^W$ in to two parts (with the first part also in $[1,w]^W$) and so on. We denote an n–simplex $[x_1|x_2|\cdots|x_n]$ as in [PS21, Definition 2.8]. This notation also gives the gluing of the faces of $[x_1|x_2|\cdots|x_n]$ in the following way. A codimention 1 face of $[x_1|x_2|\cdots|x_n]$ is a subchain of $x_1 \leqslant x_1x_2 \leqslant \cdots \leqslant x_1x_2\ldots x_n$ consisting of $n-1$ elements. There are three ways to obtain such a subchain.

1. Remove the first element of the chain to get $x_2 \leqslant x_2x_3 \leqslant \cdots \leqslant x_2x_3\ldots x_n$.
2. Remove the last element of the chain to get $x_1 \leqslant x_1x_2 \leqslant \cdots \leqslant x_1x_2\ldots x_{n-1}$.
3. Multiply two adjacent elements x_i and x_{i+1} to get the chain $x_1 \leqslant \cdots \leqslant x_1\ldots x_{i-1} \leqslant x_1\ldots x_{i-1}x_ix_{i+1} \leqslant \cdots \leqslant x_1\cdots x_n$.

So the n–simplex $[x_1|x_2|\cdots|x_n]$ glues to $[x_2|x_3|\cdots|x_n]$, $[x_1|x_2|\cdots|x_{n-1}]$ and $[x_1|\cdots|x_ix_{i+1}|\cdots|x_n]$ for all $i < n$.

The particular poset group intervals $[1,w]^W$ we will consider will be *balanced*. A balanced group interval is such that $x \in [1,w]^W$ iff $l(g^{-1}x) + l(x) = l(g)$. I.e. all minimal factorisation of $x \in [1,w]^W$ also appear as a suffix in a minimal factorisation of w and all suffixes also appear as a prefix.

Where the interval is balanced, any such symbol $[x_1|x_2|\cdots|x_n]$ corresponds to an n–simplex in K_{W_w} given it satisfies the following [PS21, Definition 2.8]:

i) $x_i \neq 1$ for all i.

ii) $x_1x_2\cdots x_n \in [1,w]^W$

iii) $l(x_1x_2\cdots x_n) = l(x_1) + l(x_2) + \cdots + l(x_n)$

Hopefully the first two requirements are obvious. The third is because we require the chain $x_1 \leqslant x_1x_2 \leqslant \cdots \leqslant x_1x_2\ldots x_n$ to be contained in $[1,w]^W$ which translates to every subword of $x_1\cdots x_n$ also being in $[1,w]^W$. By ii and iii we have that there is some y such that $x_1\cdots x_ny = w$ and there is a minimal factorisation of w that respects the factors in $x_1\cdots x_ny$. We can take prefixes of this factorisation (and thus prefixes of $x_1\cdots x_ny$) and stay within $[1,w]^W$. We can use the balanced condition to move the suffix $x_2\cdots x_ny$ to the front. I.e. there exists y_2 such that $x_2\cdots x_nyy_2 = w$ and there is a minimal factorisation of w that respects those factors. We can then repeat these steps to show every subword of $x_1\cdots x_n$ is in $[1,w]^W$.

Definition 2.3.5 (Dual Artin group)**.** For a Coxeter group W generated by all reflections R with Coxeter element $w \in W$. Define $[1,w]^W$ as above. The *dual Artin group* W_w is the poset group $G([1,w]^W)$ with G defined as in Definition 2.1.3.

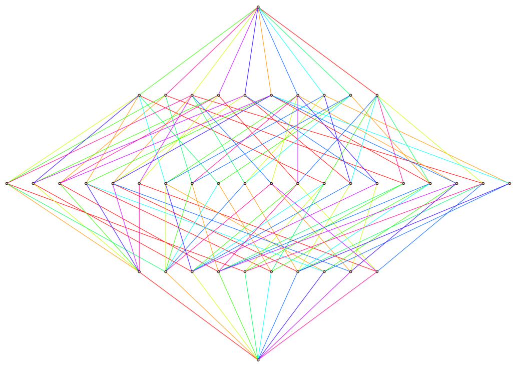


Figure 2.4: The interval $[1,(1,2)(2,3)(3,4)(4,5)]^{A_4}$ considering A_4 generated by all reflections, which label the edges by colour. Generated using `Sage` and `GAP` [Sag20]; [GAP22]

From the closing remarks of Section 2.2, the fundamental group of K_W is W_w. Furthermore, for finite cases it is a classifying space.

Theorem 2.3.6 ([PS21, Theorems 2.9 and 2.14])**.** For a finite Coxeter group W, the interval complex is a $K(W_w, 1)$ space.

The same theorem for affine cases is proved in [PS21], here Theorem 1.1.2. It is also known that for certain cases, the dual Artin group is isomorphic to the Artin group.

Theorem 2.3.7. For a finite [Bes03] or affine [MS17] Coxeter group W and Coxeter element $w \in W$, the dual Artin group is isomorphic to the Artin group G_W (and thus it does not depend on the choice of w).

In general, it is not known whether $G_W \cong W_w$, whether the isomorphism class of W_w depends on w or even if the isomorphism class of $[1, w]^W$ depends on w.

2.4 Salvetti Complex

Here we will define the *Salvetti Complex* for a Coxeter group W generated by S, which is homotopy equivalent to Y_W. First we must define a notion on subsets of S. For some subset $T \subseteq S$ define the *parabolic subgroup* of W with respect to T, W_T, to be the subgroup of W generated by T with all relations for W containing only elements of T. If Γ is the Coxeter diagram for W, then W_T is the Coxeter group corresponding to the complete subgraph of Γ containing the vertices T. From here we follow [Pao17, Section 2.3], with notation from [PS21].

Definition 2.4.1. For a Coxeter group W generated by S, define Δ_W to be the family of subsets $T \subseteq S$ such that W_T is finite.

For some $T \subseteq S$, we say some $w \in W$ is T–minimal if w is the unique element of minimum length (with respect to S) in the coset wT. Uniqueness is shown in [Bou08]. Define an order on the set $W \times \Delta_W$ by the following: $(u, X) \leqslant (v, Y)$ iff $X \subseteq Y$, $v^{-1}u \in W_Y$ and $v^{-1}u$ is X–minimal.

> **Definition 2.4.2** (Pre-Salvetti Complex [Pao17, Definition 2.19])**.** For a Coxeter group W, define $\mathrm{Sal}(W)$ to be $\Delta(W \times \Delta_W)$ under the order $\leqslant$ prescribed above, where the first Δ is as in Definition 2.2.1.

The Salvetti complex was originally defined (and refined) in [Sal87]; [Sal94]. In the latter of these papers, the Salvetti complex was defined to be the quotient of a space related to the action o W on a vector space. In [Par14, Theorem 3.3] it was shown that the definition we give generates a space homeomorphic to that in the original definition. Let us quote some results that help us to interpret the definition.

> **Lemma 2.4.3** ([Pao17, Lemma 2.18])**.** Consider both of these objects as geometric simplicial complexes inside $\mathrm{Sal}(W)$.
>
> $$\begin{aligned} C(v, Y) &:= \{(u, X) \in W \times \Delta_W \mid (u, X) \leqslant (v, Y)\} \\ \partial C(v, Y) &:= \{(u, X) \in W \times \Delta_W \mid (u, X) < (v, Y)\} \end{aligned}$$
>
> There is a homeomorphism $C(v, Y) \to D^n$ that restricts to a homeomorphism $\partial C(v, Y) \to S^{n-1}$ where $n = |\, Y \,|$.

This allows us to construct a CW complex for $\mathrm{Sal}(W)$ where each $C(w, X)$ is a $|\, X \,|$–cell for each $X \in \Delta_W$. Let us see what these cells look like. Note that the cells of the CW-complex and the simplices in $\Delta(W \times \Delta_W)$ as in Definition 2.4.2 comprise a completely different cell structure for $\mathrm{Sal}(W)$. We define $\langle \varnothing \rangle := \{1\}$ to give $W_\varnothing$ meaning as the trivial subgroup inside W.

Each $C(w, \varnothing)$ is a 0–cell. We will denote these cells w as a shorthand. In general, we have that $(u, X) \leqslant (v, X) \implies (u, X) = (v, X)$ since we require $v^{-1}u \in X$ we have $v^{-1}uX = X$. So if $v^{-1}u$ is minimal in $v^{-1}uX$ then $v^{-1}u = 1$. In particular, there is no $(u, X) < (w, \varnothing)$, so these $w = C(w, \varnothing)$ are 0–simplices in $\Delta(W \times \Delta_W)$ as well.

Now consider each 1–cell $C(w, \{s\})$. Since $W_{\{s\}} = \{1, s\} \cong \mathbb{Z}/2$ we have $\{s\} \in \Delta_W$ for all $s \in S$. For some (u, X) to be less than $(w, \{s\})$, recall we require $w^{-1}u \in W_{\{s\}}$. So we have $u \in \{w, ws\}$. Locally, the Hasse diagram and (since we only have 1–chains here) $\Delta(W \times \Delta_W)$ both look like as in Fig. 2.5 (left). In the CW complex there would be only one 1–cell, labelled $C(w, \{s\})$ oriented from w to ws. Note that $C(ws, \{s\})$ also connects these two vertices, but is a different 1–cell. This doubling up will be inconsequential after we define the Salvetti complex, which will quotient away any such doubling.

Now consider the 2–cells in the CW complex. We have that $W_{\{s,t\}} \cong D_{2m(s,t)}$, the dihedral group of corresponding to the $m(s,t)$–gon (recall $m(s,t)$ from

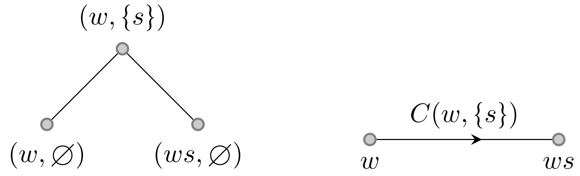


Figure 2.5: A local picture of $\mathrm{Sal}(W)$ as $\Delta(W \times \Delta_W)$ (which also resembles the Hasse diagram) (left). The corresponding 1–cell in the CW complex for $\mathrm{Sal}(W)$ (right).

Definition 1.2.1). Thus, $\{s, t\} \in \Delta_W$ iff $m(s, t) \neq \infty$. We have $(u, \{s\})$ or $(v, \{t\})$ are less than $(w, \{s, t\})$ only when $u = wd$ for some $d \in W_{\{s,t\}}$, similarly for v. The second requirement then is that d is ($\{s\}$ or $\{t\}$)–minimal. In the general case, if $(u, X) \leqslant (v, Y)$ then $X \subseteq Y$ so $W_X \subseteq W_Y$. So the coset $v^{-1}uW_X \subseteq W_Y$. Due to the nature of Definition 1.2.1, only relations relevant to W_Y could have relevance to the word length of elements in W_Y. Thus, to determine if $v^{-1}u$ is X minimal we need only consider everything within W_Y, not the entire Coxeter group W. In particular, to tell if $(wd, \{s\}) \leqslant (w, \{s, t\})$ for some $d \in W_{\{s,t\}}$, we need only consider if d is $\{s\}$–minimal in the dihedral group $W_{\{s,t\}}$.

Considering for a moment s and t as letters only, a normal form comprising minimal length words for $W_{\{s,t\}}$ is

$$\{\Pi(s, t, n) \mid n \leqslant m(s, t)\} \cup \{\Pi(s, t, n) \mid n \leqslant m(s, t)\}$$

recalling the meaning of $\Pi(s, t, n)$ from Definition 1.2.2. Note that $\Pi(t, s, m(s, t))$ is also a minimal length word but is not included for the above to be a normal form. Thus, any $sts \cdots s$ is $\{t\}$–minimal if the total length of $sts \cdots s$ is strictly less than $m(s, t)$. Similarly, $sts \cdots t$ is $\{s\}$–minimal if the total length of $sts \cdots t$ is strictly less than $m(s, t)$, with equivalent results for $tst \cdots s$ and $tst \cdots t$ depending on the last letter in the word. A picture of the Hasse diagram for the interval $[-\infty, (w, \{s, t\})]$ corresponding to the cell $C(w, \{s, t\})$ where $m(s, t) = 3$ is shown in Fig. 2.6. The CW cell itself has been drawn in Fig. 2.7.

There is a natural action $W \circlearrowleft \mathrm{Sal}(W)$ with $w \cdot (u, T) \coloneqq (wu, T)$. We can now define the following

Definition 2.4.4 (Salvetti Complex)**.** For a Coxeter group W define the *Salvetti Complex* X_W to be $\mathrm{Sal}(W)/W$ under the action specified above.

The action is cellular, thus we have a CW structure for X_W as well. We now quote the following important result.

Theorem 2.4.5 ([Par14, Corollary 3.4][Sal87])**.** For a Coxeter group W, the Salvetti complex X_W is homotopy equivalent to the configuration space Y_W.

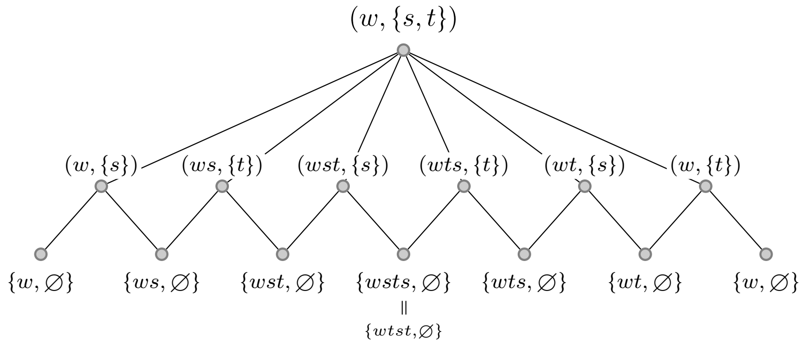


Figure 2.6: The local Hasse Diagram corresponding to the CW 2–cell $C(w, \{s,t\})$ where $m(s,t) = 3$. Note that w has been drawn twice for clarity in the picture. C.f. Fig. 2.7

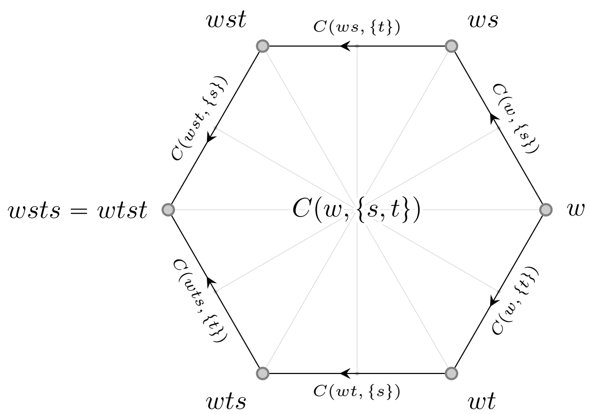


Figure 2.7: The 2–cell $C(w, \{s,t\})$ in the CW complex for $\mathrm{Sal}(W)$ where $m(s,t) = 3$. The faint lines are the simplices from $\Delta(W \times \Delta_W)$, which have all been incorporated in to one CW cell.

Now let us consider the cell structure of X_W. There is one $|\,T\,|$–cell for each $T \in \Delta_W$, in particular, there is one 0–cell corresponding to the trivial group $W_\varnothing$. Attached to this is a 1–cell for each $s \in S$ forming $\bigvee_{s\in S} S^1$ as the 1–skeleton. Then to this wedge is attached 2–cells following the procedure above for each $m(s,t) \neq \infty$. Each two cell corresponding to $\{s,t\}$ is attached to two 1–cells corresponding to $\{s\}$ and $\{t\}$. From examining this 2–skeleton it should be clear that $\pi_1(X_W) \cong G_W$. Thus combining with the previous theorem we have re-proved Theorem 1.3.2.

3 Implementing the CW Complexes

We will now begin to bridge the gap between some of the objects we have defined. Ultimately, we wish to show a homotopy equivalence between the space K_W and the Salvetti complex X_W, which is already known to be homotopy equivalent to Y_W. For now, we will define a subspace $K'_W \subseteq K_W$, inspired by our definition of the Salvetti complex, using subsets $T \subseteq S$ such that W_T is finite.

3.1 The subcomplex X'_W

The definition of K_W depends on the data of some Coxeter element w and thus implicitly some generating set S. For some $w = s_1 s_2 \cdots s_n$ and some $T \subseteq S$, define $w_T \in W$ to be the (non–consecutive) subword of w consisting of elements that are in T, respecting the original order in w.

Definition 3.1.1 (The subcomplex)**.** Define X'_W to be the finite subcomplex of K_W consisting only of simplices $[x_1|\cdots|x_n]$ such that $x_1x_2\cdots x_n \in [1,w_T]^W$ for some $T \in \Delta_W$. Recalling Δ_W from Definition 2.4.1.

Note again the absence of w from the notation of X'_W. That will be justified in this section as we will show that, up to homotopy equivalence, X'_W has no w dependence.

Lemma 3.1.2 ([PS21, Lemma 5.2])**.** For a Coxeter group W and any parabolic subgroup W_T, with sets of reflections R_W and R_{W_T} respectively. For some Coxeter element $w \in W$ and corresponding $w_T \in W_T$, all minimal factorisations of w_T in R_W consist of only elements from R_{W_T}.

An immediate consequence of this is that the intervals $[1,w_T]^W$ and $[1,w_T]^{W_T}$ agree. This allows us to decompose X'_W in a useful way. For each $T \in \Delta_W$ and corresponding W_T, the space corresponding to the whole interval $[1,w_T]^W = [1,w_T]^{W_T}$ is a subspace inside X'_W. This subspace is exactly X'_{W_T} (with respect to w_T). Thus, we can think of X'_W as some union of all X'_{W_T} for $T \in \Delta_W$.

Each X'_{W_T} is exactly the same as its interval complex K_{W_T} since all subgroups of T generate finite Coxeter groups. Thus, using known results for finite Coxeter groups, X'_{W_T} is a classifying space for the dual Artin group W_w by Theorem 2.3.6. Furthermore, G_w is isomorphic to the Artin group G_{W_T} by Theorem 2.3.7.

In a very similar way, the Salvetti complex consists of subspaces corresponding to elements of Δ_W. For each $T \in \Delta_W$, the Salvetti complex X_{W_T} is a $|T|$–cell attached to all cells corresponding to $R \subseteq T$ is the appropriate way. This is a cellular subspace of X_W, and since W_T is finite, by Theorem 1.4.2, $Y_{W_T} \simeq X_{W_T}$ is a $K(G_{W_T}, 1)$. The following remark summarises these observations.

Remark 3.1.3. The Salvetti complex X_W decomposes in to cellular subspaces X_{W_T} which are $K(G_{W_T}, 1)$ spaces. These subspaces are in bijection with cellular subspaces X'_{W_T} of X'_W, which are also $K(G_{W_T}, 1)$ spaces.

The following section will help us to exploit this similarity to show that $X_W \simeq X'_W$.

3.2 An Adjunction Homotopy Equivalence

Here we will show a fundamental link between homomorphisms in to groups G and maps in to classifying spaces for G. This result shows that in a certain way, the homotopy type of classifying spaces is unique. We will use this result to then show an important homotopy equivalence, which is an intermediate step in proving the main result of [PS21].

Lemma 3.2.1. A null homotopic map $\rho\colon S^n \to X$ can be extended to a map $\sigma\colon D^{n+1} \to X$.

Proof. Let $H\colon S^n \times I \to X$ witness the null homotopy with $H|_{S^n \times \{1\}}\colon S^n \to \{x_0\}$. We have that H factors uniquely through $(S^n \times I)/(S^n \times \{1\}) \cong D^{n+1}$. With σ being the necessary map as below.

$$\begin{tikzcd} S^n \times I \arrow[r, "H"] \arrow[d, "q"] & X \\ (S^n \times I)/(S^n \times \{1\}) \arrow[ur, dotted, "\exists! \sigma"'] & \end{tikzcd}$$

□

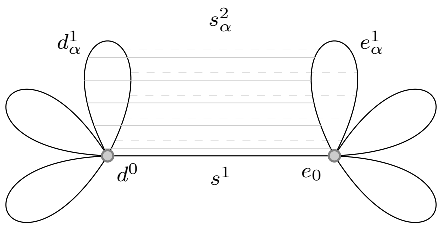
Theorem 3.2.2 ([Hat01, Proposition 1B.9])**.** Let Y be a $K(G, 1)$ space and X a finite dimensional CW complex consisting of one 0–cell, the point x_0. Any homomorphism $\varphi\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is induced by a map $\tilde{\varphi}\colon X \to Y$ where $\tilde{\varphi}$ is unique up to homotopy fixing x_0.

Proof. Clearly we must have $\tilde{\varphi}(x_0) = y_0$. The 1–skeleton X^1 will be a wedge of circles and there is thus a presentation of $\pi_1(X, x_0)$ with each cell e^1_α corresponding to a generator $[e^1_\alpha] \in \pi_1(X, x_0)$. We can choose $\tilde{\varphi}(e_\alpha)$ to trace out a path corresponding to $\varphi([e^1_\alpha]) \in \pi_1(Y, y_0)$ for each $e^1_\alpha \in X^1$.

Let $\psi_\beta \colon S^1 \to X^1$ be an attaching map for a 2–cell $e^2_\beta \subseteq X$. Let $i \colon X^1 \hookrightarrow X$ be the inclusion. We have that i_* is the surjetcion from the free group generated by each e^1_α to $\pi_1(X, x_0)$. The attaching of the 2–cell e^2_β provides a null homotopy for the path traced by ψ_β. In the presentation of $\pi_1(X, x_0)$ as above, each relation corresponds to the path of a ψ_β. Thus, $i_*([\psi_\beta]) = 0$ and so $\tilde{\varphi}_*([\psi_\beta]) = \varphi \circ i_*([\psi_\beta]) = 0$. Thus, $\tilde{\varphi} \circ \psi_\beta$ is null homotopic and so can be extended over all of the closure of e^2_β by Lemma 3.2.1. This is an extension of $\tilde{\varphi}$ and repeating this allows us to extend $\tilde{\varphi}$ over all of X^2.

To extend $\tilde{\varphi}$ over e^3_γ we use that S^2 is simply connected (as for any S^n with $n \geqslant 2$) and so for the attaching map $\psi_\gamma \colon S^2 \to X^2$ we have that $\tilde{\varphi} \circ \psi_\gamma$ lifts to the universal cover of Y, which is contractible since Y is a $K(G, 1)$, so $\tilde{\varphi} \circ \psi_\gamma$ is null homotopic. This same argument applies for any e^n_δ for $n \geqslant 3$. We can thus extend $\tilde{\varphi}$ over the 3–cells and proceeding inductively, over all of X.

Now we turn to the uniqueness of $\tilde{\varphi}$ up to homotopy. Let φ be some homomorphism and $\tilde{\varphi}_0$ and $\tilde{\varphi}_1$ be any such maps constructed as above. Clearly $\tilde{\varphi}_0(x_0) = \tilde{\varphi}_1(x_0)$ and $\tilde{\varphi}_0|_{X^1} \sim \tilde{\varphi}_1|_{X^1}$ by the restrictions of our construction. Let H witness this homotopy. Give $X \times I$ the following CW structure: Let $X \times \{0\}$ and $X \times \{1\}$ both have the same cell structure as X with cells notated d^n_α and e^n_α respectively. Connect d^0 to e^0 with a 1–cell s^1, called *the spine*. Connect a 2–cell s^2_α along d^1_α, then s^1 then e^1_α then back along s^1 with opposite orientations on d^1_α and e^1_α such that $d^0 \cup e^0 \cup s^1 \cup d^1_\alpha \cup e^1_\alpha \cup s^2_\alpha \cong S^1 \times I$. The spine now consists of $s_1 \cup s^2_\alpha$. Repeat this for each 1–cell in X and then repeat for each 2–cell and so on, attaching an s^n_β along d^{n-1}_β, e^{n-1}_β and s^{n-1}_β, inductively building up the spine. A picture of this CW complex completed for one s^2_α is below.



We can now extend the domain of H from $X^1 \times I$ to all of $X \times I$ using this cell structure. Note that now we have two 0–cells, but this does not cause any issues. Let H have domain $X^1 \times I \subseteq X \times I$. Now extend H such that $H|_{X \times \{0\}}$ agrees with $\tilde{\varphi}_0$ and $H|_{X \times \{1\}}$ agrees with $\tilde{\varphi}_1$. This is possible because H is a homotopy between restrictions of these maps. Note that now H is defined on the whole 2–skeleton of $X \times I$. We can extend H to all the higher dimensional cells by the exact same argument as before, using the contractability of the universal cover of Y. Thus, we have a continuous function $H\colon X \times I \to Y$ witnessing the homotopy $\tilde{\varphi}_0 \sim \tilde{\varphi}_1$. $\square$

Corrolary 3.2.3. Let X and Y both be $K(G,1)$ spaces. Any isomorphism $\varphi\colon \pi_1(X,x_0) \to \pi_1(Y,y_0)$ induces a homotopy equivalence witnessing $X \simeq Y$.

Proof. We have maps $\tilde{\varphi}\colon X \to Y$ and $\widetilde{(\varphi^{-1})}\colon Y \to X$ with $(\tilde{\varphi} \circ \widetilde{(\varphi^{-1})})_* = \mathrm{Id}_{\pi_1(Y,y_0)}$. Thus, since the homotopy class of such maps is determined by the induced action on their fundamental groups $\tilde{\varphi} \circ \widetilde{(\varphi^{-1})} \sim \mathrm{Id}_Y$. Similarly, $\widetilde{(\varphi^{-1})} \circ \tilde{\varphi} \sim \mathrm{Id}_X$. $\square$

Lemma 3.2.4. Let $T \in \Delta_W \backslash \varnothing$. Let $\varphi\colon \bigcup_{Q \subsetneq T} X_{W_Q} \to \bigcup_{Q \subsetneq T} X'_{W_Q}$ be a homotopy equivalence. We can extend φ to a homotopy equivalence $\psi\colon X_{W_T} \to X'_{W_T}$ such that the following diagram commutes.

$$\begin{tikzcd} \bigcup_{Q \subsetneq T} X_{W_Q} \arrow[r, "\varphi"] \arrow[d, hook] & \bigcup_{Q \subsetneq T} X'_{W_Q} \arrow[d, hook] \\ X_{W_T} \arrow[r, "\psi"] & X'_{W_T} \end{tikzcd}$$

Proof. We prove this by cases. By Remark 3.1.3, X_{W_T} and X'_{W_T} are classifying spaces.

i) If $|T| = 1$ then Q is uniquely $\varnothing$. Let ψ be any map witnessing $X_{W_T} \simeq X'_{W_T}$ that fixes the point corresponding to X_{W_Q}.

ii) If $|T| = 2$ then we can extend φ to ψ such that $\psi_*\colon \pi_1(X_{W_T}, X_\varnothing) \to \pi_1(X'_{W_T}, X'_\varnothing)$ is an isomorphism using the same argument as in the proof of Theorem 3.2.2. This map is a homotopy equivalence by Corollary 3.2.3.

iii) If $|T| \geqslant 3$ then we can extend φ to some map ψ using the same methods as in Theorem 3.2.2. In this case, $\bigcup_{Q \subsetneq T} X_{W_Q}$ contains the 2–skeleton of X_T and similarly for $\bigcup_{Q \subsetneq T} X'_{W_Q}$ and X'_T. So $\pi_1(\bigcup_{Q \subsetneq T} X_{W_Q}, X_{W_\varnothing}) = \pi_1(X_T, X_{W_\varnothing})$ and similarly for X'_T. By assumption φ is a homotopy equivalence and so φ_* is an isomorphism. Therefore, ψ_* is an isomorphism [Hat01, Corollary 4.12] and thus ψ a homotopy equivalence by Corollary 3.2.3.

$\square$

Definition 3.2.5 (Adjunction Space)**.** For two spaces X and U, with a continuous map $f\colon A \to U$ for some subspace $A \subseteq X$. The *adjunction* *space* $X \sqcup_f U$ is the space formed by gluing X and U via the map f.

$$X \sqcup_f U \coloneqq (X \sqcup Y)/(a \sim f(a))$$

An adjunction space is associated to the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"] & U \arrow[d, "\bar{i}"] \\
X \arrow[r, "\bar{f}"] & X \sqcup_f U
\end{tikzcd} \tag{3.1}$$

where i is inclusion of A in to X and $\bar{i}$ is inclusion of U in to $X \sqcup_f U$. Suppose we also have the adjunction space $Y \sqcup_g V$ with $g\colon B \to V$ and $B \subseteq Y$. Suppose further that we have maps $\varphi_1\colon X \to Y$, $\varphi_2\colon A \to B$ and $\varphi_3\colon U \to V$ such that the following diagram commutes.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "\varphi_2"] \arrow[dd, "i"] & & U \arrow[dr, "\varphi_3"] \arrow[dd, "\bar{i}"] & \\
& B \arrow[rr, "g"] \arrow[dd, "j"] & & V \arrow[dd, "\bar{j}"] \\
X \arrow[rr, "\bar{f}"] \arrow[dr, "\varphi_1"] & & X \sqcup_f U \arrow[dr, "\varphi"] & \\
& Y \arrow[rr, "\bar{g}"] & & Y \sqcup_g V
\end{tikzcd} \tag{3.2}$$

If all φ_1, φ_2 and φ_3 are homotopy equivalences then the following lemma tells us that φ is also a homotopy equivalence.

Lemma 3.2.6 ([Bro06, Theorem 7.5.7])**.** Consider a commutative diagram as in (3.2) where the front and back faces define an adjunction space as in (3.1). If i and j are closed cofibrations and φ_1, φ_2 and φ_3 are homotopy equivalences, then the φ as determined by the diagram is also a homotopy equivalence.

The restriction of i and j being closed cofibrations is quite mild. In the cases important to us, i and j will be cellular inclusions in to finite CW complexes, and thus closed cofibrations. See [Bro06] for more details on pushout squares and adjunction spaces.

To use Lemma 3.2.6 we must be able to construct X_W and X'_W as a sequence of adjunction spaces. Consider X_W in the following example.

Example 3.2.7. Let $\Delta_W = \{\varnothing, \{s\}, \{t\}, \{u\}, \{s,t\}, \{s,u\}, \{t,u\}\}$ and let $\Delta_W^n \coloneqq \{T \in \Delta_W \mid |T| = n\}$. Clearly we have that $X_W = \bigcup_{T \in \Delta_W^2} X_{W_T}$

with the appropriate gluing. Suppose we had the 1–skeleton $X^1_W \subsetneq X_W$, and some ordering on $\Delta^2_W = (\{s,t\},\{s,u\},\{t,u\})$. To construct X_W, we would first glue $X_{\{s,t\}}$ to X^1_W as an adjunction space in the following way.

$$\begin{tikzcd}
X_{\{s\}} \cup X_{\{t\}} \arrow[r, "f"] \arrow[d, "i_1"] & X_{\{s,t\}} \arrow[d, "\overline{i_1}"] \\
X^1_W \arrow[r, "\bar{f}"] & X^1_W \sqcup_f X_{\{s,t\}}
\end{tikzcd}$$

Where f is inclusion of those 1–cells in to $X_{\{s,t\}}$, which in this case makes $X^1_W \sqcup_f X_{\{s,t\}} \cong X_{\{s,t\}}$. Note that $X_{\{s\}} \cup X_{\{t\}}$ is really shorthand for another adjunction space, which we assume to have been already constructed. We can then add $X_{\{t,u\}}$ to the preceding adjunction space in the following way.

$$\begin{tikzcd}
X_{\{t\}} \cup X_{\{u\}} \arrow[r, "g"] \arrow[d, "i_2"] & X_{\{t,u\}} \arrow[d, "\overline{i_2}"] \\
X^1_W \sqcup_f X_{\{s,t\}} \arrow[r, "\bar{g}"] & (X^1_W \sqcup_f X_{\{s,t\}}) \sqcup_g X_{\{t,u\}}
\end{tikzcd} \tag{3.3}$$

After which we would continue with $\{u,v\}$ in the same manner. In the final space, $X_{\{s,t\}}$ is glued to $X_{\{t,u\}}$ along $X_{\{t\}}$. In general, for $T_1, T_2 \in \Delta^n_W$, we have that X_{T_1} and X_{T_2} are glued along $X_{T_1 \cap T_2} \subseteq X^{n-1}_W$ where $T_1 \cap T_2 \in \Delta^{n-1}_W$. We can always construct the n–skeleton from the $(n-1)$–skeleton in exactly this way.

The exact same construction works for X'_W. This construction may seem too abstracted, in that much of the structure is hidden away in the maps f and g. However, as it turns out, we can use this adjunction structure without considering the details of these maps.

Theorem 3.2.8 ([PS21, Theorem 5.5])**.** For a Coxeter group W, the space X'_W as in Definition 3.1.1 is homotopy equivalent to the Salvetti complex X_W.

Proof. We achieve this inductively. To tidy our notation, in this proof we drop W so that X, X', X_T and X'_T correspond to X_W, X'_W, X_{W_T} and X'_{W_T} respectively. Let Δ^n_W be as in Example 3.2.7.

Suppose we have the $(n-1)$–skeletons X^{n-1} and $(X')^{n-1}$ and a homotopy equivalence $\varphi\colon X^{n-1} \to (X')^{n-1}$. We wish to show that we can extend φ to a homotopy equivalence for the respective n–skeletons. We do so by constructing the n–skeletons as adjunction spaces of the $(n-1)$–skeletons as in Example 3.2.7 and use Lemma 3.2.6.

Suppose we are gluing on the cells corresponding to some $T \in \Delta_W^n$. So any $Q \subsetneq T$ will correspond to a cell in X^{n-1}. Let Y and Y' be some intermediate steps in the adjunction gluing, such as the bottom–left term of (3.3). Suppose we have a homotopy equivalence $\varphi_1 \colon Y \to Y'$ such that $\varphi_1|_{X^{n-1}} = \varphi$ so the following commutes.

$$\begin{tikzcd}
\bigcup_{Q \subsetneq T} X_Q \arrow[r, "\varphi"] \arrow[d, hook] & \bigcup_{Q \subsetneq T} X'_Q \arrow[d, hook] \\
Y \arrow[r, "\varphi_1"] & Y'
\end{tikzcd} \tag{3.4}$$

We wish to extend φ to a homotopy equivalence ψ such that the following commutes.

$$\begin{tikzcd}
\bigcup_{Q \subsetneq T} X_Q \arrow[r, "\varphi"] \arrow[d, hook] & \bigcup_{Q \subsetneq T} X'_Q \arrow[d, hook] \\
X_T \arrow[r, "\psi"] & X'_T
\end{tikzcd}$$

This is possible by Lemma 3.2.4. Now we have the following commutative diagram.

$$\begin{tikzcd}
\bigcup_{Q \subsetneq T} X_Q \arrow[rr, "f"] \arrow[dr, "\varphi"] \arrow[dd, hook] & & X_T \arrow[dr, "\psi"] \arrow[dd, hook] & \\
& \bigcup_{Q \subsetneq T} X'_Q \arrow[rr, "g"] \arrow[dd, hook] & & X'_T \arrow[dd, hook] \\
Y \arrow[rr, "\bar{f}"] \arrow[dr, "\varphi_1"] & & Y \sqcup_f X_T \arrow[dr, "\sigma"] & \\
& Y' \arrow[rr, "\bar{g}"] & & Y' \sqcup_g X'_T
\end{tikzcd} \tag{3.5}$$

Where the induced map σ is a homotopy equivalence by Lemma 3.2.6. At the next inductive step, $Y \sqcup_f X_T$ and $Y' \sqcup_g X'_T$ replace Y and Y' respectively. Accordingly, σ replaces φ_1. Suppose we are next going to glue the cells corresponding to $\tilde{T} \in \Delta_W$. To proceed inductively, there are two possible outcomes:

1. We are still constructing $X^n \simeq (X')^n$ and $\tilde{T} \in \Delta_W^n$.
2. We completely constructed $X^n \simeq (X')^n$ in the previous step and $\tilde{T} \in \Delta_W^{n+1}$.

In Case 1, we have that any $Q \subsetneq \tilde{T}$ corresponds to cells in X^{n-1}. By the inductive hypothesis, we can restrict $\varphi_1 \colon Y \to Y'$ to X^{n-1}, and thus we can do the same for σ and so the restriction $\sigma|_{X^{n-1}}$ is well–defined, we can get (3.4) with the appropriate replacements and proceed inductively.

In Case 2, $Y \sqcup_f X_T$ and $Y' \sqcup_g X'_T$ are X^n and $(X')^n$ respectively. Some $Q \subsetneq \tilde{T}$ will correspond to cells in X^n, but σ is exactly the restriction $\sigma|_{X^n}$ so, the restriction is well–defined. We get (3.4) with the appropriate replacements and proceed inductively.

The base case is $X_\varnothing \simeq X'_\varnothing \simeq \{\bullet\}$. □

4 Discrete Morse Theory

In this section we will prove the next homotopy equivalence along the chain in (1.1). This will again involve the use of posets and their combinatorics. Morse theory for smooth manifolds gives a way to infer topological properties of manifolds from analytical properties of certain smooth functions on that manifold. Discrete Morse theory is a CW (non–smooth) analogue. Certain functions on the (discrete) set of cells of a CW complex can tell us topological facts about the CW complex. Here we will only give a brief introduction to the main results of this theory that are relevant to us.

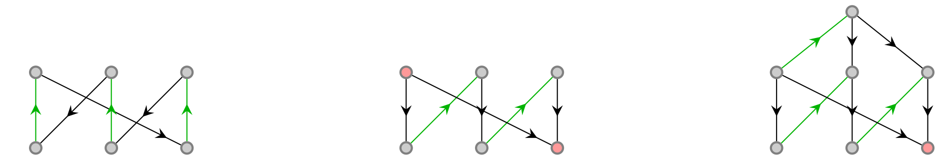
4.1 The Face Poset and Acyclic Matchings

In previous sections we gave constructions that formed spaces from posets, we now give a construction in the opposite direction. Given a CW complex X denote the set of open cells as X^*.

> **Definition 4.1.1** (The Face Poset)**.** Given a CW complex X, the *face poset* $\mathcal{F}(X)$ is an ordering on X^* where $\tau \leqslant \sigma$ when $\bar{\tau} \subseteq \bar{\sigma}$.

For a finite dimensional and connected CW complex, $\mathcal{F}(X)$ is a bounded and graded poset with rank function $\mathrm{rk}(\sigma) = \dim(\sigma)$. Let P denote $\mathcal{F}(X)$. Q Here the use of rank is quite different compared to its introduction in Section 2.1. I think this sleight of hand is OK but can be changed if necessary Consider some subset of the covering relations $\mathcal{M} \subseteq \mathcal{E}(P)$. We consider this as a set of edges in the Hasse diagram for P, which is denoted H. From $\mathcal{M}$ we define an ordering on the graph H such that $p \lessdot q$ is oriented from p to q if $(p \lessdot q) \in \mathcal{M}$, and otherwise in the opposite direction. We denote this oriented graph $H_{\mathcal{M}}$. We call $\mathcal{M}$ a *matching* if for all $p \in P$, at most one $m \in \mathcal{M}$ contains p. A matching is *acyclic* if $H_{\mathcal{M}}$ contains no directed cycles. Furthermore, a matching is *proper* if for all $p \in P$, the set of all vertices in $H_{\mathcal{M}}$ reachable by a directed path from p is finite. Fig. 4.1 gives some (non)examples of matchings.

We observe that the requirement of being a matching means that any path through $H_{\mathcal{M}}$ will never consecutively go through two edges in $\mathcal{M}$. A cycle in $H_{\mathcal{M}}$ must clearly start and end at the same rank. Since edges in $\mathcal{M}$ increase rank and edges in $\mathcal{E}(P)\backslash\mathcal{M}$ decrease rank, a cycle must therefore be (cyclically) alternating between edges in $\mathcal{M}$ and edges in $\mathcal{E}(P)\backslash\mathcal{M}$. Therefore, if a cycle is to start at $p \in P$, it must completely occur in $\{q \in P \mid \operatorname{rk}(q) - \operatorname{rk}(p) \in \{0, 1\}\}$ or completely in $\{q \in P \mid \operatorname{rk}(q) - \operatorname{rk}(p) \in \{0, -1\}\}$. I.e. the horizontal bands above or below p in H. Since it is alternating, a cycle must also comprise an even number of edges.



(a) A cyclic matching. (b) An acyclic matching with invalid subcomplex. (c) An acyclic matching with valid subcomplex.

Figure 4.1: Directed Hasse diagrams corresponding to face posets and choices of $\mathcal{M}$. Green edges are those in $\mathcal{M}$ and red nodes are critical cells.

We call σ a *face* of τ if $\sigma \lessdot \tau$. Consider Φ, the characteristic map of some n–cell τ with $\Phi\colon D^n \to X$. We call σ a *regular face* of τ if it is a face and the following hold.

1) $\Phi|_{\Phi^{-1}(\sigma)}\colon \Phi^{-1}(\sigma) \to \sigma$ is a homeomorphism.

2) $\overline{\Phi^{-1}(\sigma)}$ is homeomorphic to D^{n-1} as a subset of D^n. Q Surely 2 always follows from 1. Since $\sigma \cong \Phi^{-1}(\sigma)$ and σ is an open $(n-1)$–disk. Maybe this is to do with pathological CW complexes.

For a matching $\mathcal{M}$, any $p \in P$ that is disjoint from all of $\mathcal{M}$ is called *critical*. In this context, a *critical cell*. We can now state the version of discrete Morse theory we will use.

Theorem 4.1.2 ([PS21, Theorem 2.4])**.** Consider a CW complex X, a subcomplex $Y \subseteq X$, and a proper, acyclic matching $\mathcal{M}$ on X. If $\mathcal{F}(Y) \subseteq \mathcal{F}(X)$ is the set of critical cells in X with respect to $\mathcal{M}$ and every if σ is a regular face of τ for every $(\sigma \lessdot \tau) \in \mathcal{M}$, then X deformation retracts on to Y.

You may notice that this seems to have no link to discrete functions $X^* \to \mathbb{N}$, as promised in the prologue of this section. Indeed, this statement is a reformulation of Discrete Morse Theory due [Cha00] and [Bat02]. The original formulation of Discrete Morse Theory is due to [For98]. The exact wording of Theorem 4.1.2 is important, we explore this in the following example.

Example 4.1.3. The Hasse diagrams in Fig. 4.1 correspond to the obvious CW complex for a triangle. Figs. 4.1a and 4.1b are for a hollow

1–dimensional triangle and Fig. 4.1c is for a filled 2–dimensional triangle. We know that a hollow triangle cannot deformation retract on to any of its subcomplexes, thus the required construction for Theorem 4.1.2 should fail for Figs. 4.1a and 4.1b. We see that Fig. 4.1a is a cyclic matching, but we achieve an acyclic (vacuously proper) matching in Fig. 4.1b. Importantly, the space corresponding to the union of the critical cells, which are highlighted in the figure, is not a valid subcomplex, thus Theorem 4.1.2 does not apply. For a subset of cells $Y^* \subseteq X^*$ to correspond to a valid subcomplex $Y \subseteq X$, we require

$$\bigcup_{y \in \mathcal{F}(Y)} [-\infty, y] = \mathcal{F}(Y) \tag{4.1}$$

where $[-\infty, y]$ is taken within the poset $\mathcal{F}(X)$. For Fig. 4.1b, the left–hand side of (4.1) would include the bottom left cell in the Hasse diagram, which we see is not critical. In Fig. 4.1c, we have a valid subcomplex and the critical cell corresponds to a vertex in the 2–dimensional triangle, which is of course a valid deformation retract of the whole complex.

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