Diss

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1 Introduction

1.1 Coxeter Groups and Artin Groups

Here we will introduce the definitions for the groups of interest to this paper.

> **Definition 1.1.1** (Coxeter Group)**.** For a finite set of generators S, a Coxeter group W generated by S is a group with presentation of the form
>
> $$W = \langle S \mid (st)^{m(s,t)} = 1 \quad \forall m(s,t) \neq \infty \rangle$$
>
> where $m \colon S \times S \to \mathbb{N}$ is a symmetric matrix indexed over S where $m(s,s) = 1$ for all $s \in S$ and $m(s,t)$ takes values in $\{2, 3, \ldots\} \cup \{\infty\}$ for all $s \neq t$.

The infinities correspond to pairs of elements that have no explicit relations. The 1s along the diagonal of m ensure that all generators have order 2. The set $R \coloneqq \{wsw^{-1} \mid w \in W, s \in S\}$ is the set of *reflections* in W. Sometimes S is referred to as the set of *basic reflections*.

A graph, called the *Coxeter diagram*, is often used to encode the data of the matrix m and its corresponding Coxeter group. In this graph each element of S is a node and relations between pairs in S correspond to labelled edges. There are two conventions for this labelling: The *classical labelling*, where edges with $m(s,t) = 2$ are not drawn, edges with $m(s,t) = 3$ are drawn but not labelled and all other edges are drawn with the value of $m(s,t)$ as their label. And the *modern labelling*, edges with $m(s,t) = \infty$ are not drawn, edges with $m(s,t) = 2$ are drawn but not labelled and all other edges are drawn and labelled. An example highlighting these differences is given in Fig. 1.1. In the classical labelling, if the diagram has multiple connected components then W is a direct product of the groups corresponding to those components. Similarly, in the modern labelling connected components are factors in a free product. Other topological properties of these diagrams can be used, for example in [Hua23] which proves the $K(\pi, 1)$ conjecture for certain W with diagrams being trees or containing cycles.

Figure 1.1: Coxeter diagram for a certain Coxeter group with classical labelling (left) and modern labelling (right).

To each Coxeter group W there is an associated Artin group G_W defined as follows

Definition 1.1.2 (Artin Group)**.** For a given Coxeter group W generated by S with associated matrix m, the associated Artin group is

$$G_W \coloneqq \langle S \mid \Pi(s, t, m(s,t)) = \Pi(t, s, m(s,t)) \;\; \forall m(s,t) \neq \infty \rangle$$

where $\Pi(s, t, n)$ is defined to be an alternating product of s and t starting with s with total length n. E.g. $\Pi(s, t, 3) = sts$.

Note that the 1s along the diagonal of m now carry no meaning in the presentation and that if we add the relation $s^2 = 1$ for all $s \in S$ we retrieve the original Coxeter group. The Coxeter diagram for W also encodes the data of G_W and the topology of the diagram holds similar meaning. Our notation for Artin groups, G_W (shared in much of the literature), seems to imply the data for the Artin group is inherited from its Coxeter group. In principle there is no precedence, but practically we often start by defining a Coxeter group. We will see that often "`property` Artin groups" describes a family of Artin groups to which their corresponding Coxeter groups are `property`.

1.2 Configuration Space

Here we will give the definition of the configuration space Y_W for a given Coxeter group W and go through an example where we will show that $\pi_1(Y_W) \cong G_W$ for W of type A_n following from [FN62].

For some Coxeter group W acting on $\mathbb{R}^n$, the set of reflections $R \in W$ acts on $\mathbb{R}^n$ by reflection through hyperplanes. For some $r \in R$ denote its hyperplane by $H(r) \subseteq \mathbb{R}^n$. Denote the union of all hyperplanes by $\mathcal{H} \coloneqq \bigcup_{r \in R} H(r)$. We associate $\mathbb{R}^n \otimes \mathbb{C}$ with $\mathbb{C}^n$ under the natural isomorphism $x \otimes \lambda \mapsto x\lambda$. This also extends the action $W \circlearrowleft \mathbb{R}^n$ to $W \circlearrowleft \mathbb{C}^n$ via $w \cdot (x \otimes \lambda) = (w \cdot x) \otimes \lambda$. We call this act of transporting objects related to $\mathbb{R}^n$ over to $\mathbb{C}^n$ via the tensor product *complexification*. With these tools in mid, we can then make our definition.

Definition 1.2.1 (Configuration space)**.** For some Coxeter group W and associated hyperplane system $\mathcal{H}$ as above, we define

$$Y \coloneqq \mathbb{C}^n \setminus (\mathcal{H} \otimes \mathbb{C})$$

and define the configuration space Y_W to be the quotient Y/W.

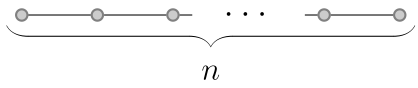


Figure 1.2: The clasical Coxeter diagram for the Coxeter group of type A_n.

It is important to note that the importance of $\mathbb{C}$ here is that it is 2 dimensional. When one takes the complement of a co-dimension 1 object, you typically will not get any interesting topology. By complexifying the hyperplanes and then taking the complement within $\mathbb{C}^n$, we are effectively taking the complement of a co-dimension 2 object, and there is much more room for interesting topologies. The same construction can be achieved using $\mathbb{R}^{2n}$ and $\mathcal{H} \times \mathcal{H}$ but here we choose $\mathbb{C}$ because once spelled out explicitly, the construction is more intuitive.

For a concrete example, we will introduce the A_n family of Coxeter groups and show that the space Y_W for these groups is the space of configurations of $n+1$ points in $\mathbb{C}$, thus explaining the name *configuration space* for general Y_W.

The family A_n all have Coxeter diagrams of the form as in Fig. 1.2 and a specific A_n will have presentation.

$$A_n = \left\langle \sigma_1, \sigma_2, \ldots, \sigma_n \;\middle|\; \begin{array}{rl} \sigma_i^2 = 1 & \forall i \\ (\sigma_i \sigma_j)^2 = 1 & \forall (i+1 < j \leqslant n) \\ (\sigma_i \sigma_{i+1})^3 = 1 & \forall (i < n) \end{array} \right\rangle \tag{1.1}$$

This is well known to be a presentation for the symmetric group S_{n+1} with generators being adjacent transpositions [BB05, Proposition 1.5.4]. Accordingly, we will use the associated cycle notation for symmetric groups to talk about elements of A_n.

The action of A_n as a reflection group is realised on the space $\mathbb{R}^{n+1}$ with basis $\{e_i\}$, where $A_n \circlearrowright \mathbb{R}^{n+1}$ by permuting components with respect to that basis. The set of reflections R of A_n is all conjugations of the n adjacent generating transpositions $(l, l+1)$. So, R is the set of all transpositions (l, n). Some $(l, n) \in R$ acts on $\mathbb{R}^{n+1}$ as reflection through the plane $\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_l = x_n\}$. Thus, taking the complement of the complexification of all such planes, we have $Y = \{(\mu_1, \ldots \mu_{n+1}) \in \mathbb{C}^{n+1} \mid \forall i, j \;\; \mu_i \neq \mu_j\}$ (here Y is as in Definition 1.2.1). We can think of this as the space of $n+1$ distinct labelled points in $\mathbb{C}$. The action $A_n \circlearrowright \mathbb{C}^{n+1}$ also permutes components, so we can think of the configuration space Y_W as the set of $n+1$ distinct *unlabelled* points in $\mathbb{C}$, denoted $\mathrm{Conf}_{n+1}(\mathbb{C})$.

Historically, Emile Artin [Art47] originally defined the braid group on n strands B_n to be $\pi_1(\mathrm{Conf}_n(\mathbb{R}^2))$. He then showed the validity of the well known presentation of the braid group.

$$B_n = \left\langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \forall(i+1 < j \leqslant n) \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \forall(i < n) \end{array} \right\rangle$$

In this context, showing the validity of that presentation immediately proves $B_{n+1} \cong G_W$ and thus that $\pi_1(Y_W) \cong G_W$. This proof by Artin is often considered dubious and other proofs are available. One good example is [FN62]. Importantly, this is also true in the general case.

Theorem 1.2.2 ([Bri71])**.** For any Coxeter group W, we have $\pi_1(Y_W) \cong G_W$.

The paper cited is in German and only German or Russian translations are available. Alternative proofs for Coxeter groups of affine type [Viê83] are available in English.

1.3 The $K(\pi, 1)$ Conjecture

For a group G, an Eilenberg-MacLane space [EM45] for G is a space X such that $\pi_n(X) = G$ for some n and $\pi_i(X) = 0$ for all $i \neq n$. We will use the terminology "X is a $K(G, n)$ space". The $K(\pi, 1)$ conjecture for Artin groups states that for all Coxeter groups W, the space Y_W is a $K(G_W, 1)$ space. Admittedly the use of π in the name of the conjecture is confusing.

We have already seen in Section 1.2 that indeed $\pi_1(Y_W) \cong G_W$, thus to prove the $K(\pi, 1)$ conjecture for type A_n Coxeter groups we need to verify that the higher homotopy groups of Y_W are trivial. This can be done by observing that $\mathrm{Conf}_n(\mathbb{C})$ is a fibre bundle over $\mathrm{Conf}_{n-1}(\mathbb{C})$ with projection p forgetting a point and fibres homeomorphic to $\mathbb{C}\backslash\{\text{n distinct points}\}$, as spelled out in [Sin10].

The space $\mathbb{C}\backslash\{\text{n distinct points}\}$ is homotopy equivalent to $\bigvee_n S^1$, so we can use the fibration

$$\bigvee_{n-1} S^1 \hookrightarrow \mathrm{Conf}_n(\mathbb{C}) \xrightarrow{p} \mathrm{Conf}_{n-1}(\mathbb{C})$$

to build a tower of fibrations

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
\bigvee_3 S^1 \arrow[r, hook] & \mathrm{Conf}_4(\mathbb{C}) \arrow[d] \\
\bigvee_2 S^1 \arrow[r, hook] & \mathrm{Conf}_3(\mathbb{C}) \arrow[d] \\
S^1 \arrow[r, hook] & \mathrm{Conf}_2(\mathbb{C})
\end{tikzcd}$$

where there is a short exact sequence in homotopy groups starting at each $\pi_k(\bigvee_n S^1)$ going right and down to $\pi_k(\mathrm{Conf}_n(\mathbb{C}))$ for any k. We note that $\mathrm{Conf}_2(\mathbb{C}) \simeq S^1$ and so has trivial homotopy above π_1. Similarly, $\bigvee_2 S^1$ has trivial higher homotopy. So $\pi_k(\mathrm{Conf}_3(\mathbb{C})) \cong 0$ for $k > 1$, and we can continue up the tower inductively to show $\pi_k(\mathrm{Conf}_n(\mathbb{C})) \cong 0$ for $k > 1$ for all n. So indeed $\mathrm{Conf}_{n+1}(\mathbb{C}) = Y_W$ is a $K(G_W, 1)$ for $W = A_n$.

Theorem 1.3.1 ([Del72])**.** *The $K(\pi, 1)$ conjecture holds for all finite Coxeter groups W.*

The paper of interest to us, [PS21], proves the $K(\pi, 1)$ for affine type Coxeter groups.

2 Geometric Realisations of Poset Structures

In Section 1.2 we used the realisation of the Coxeter group W as a reflection group on a space V. We considered the planes of the defining reflections of W as affine subspaces of V and used these to define Y_W, the configuration space. The $K(\pi, 1)$ conjecture concerns with the homotopic properties of Y_W. To explore these, we will first construct a new space X_W, the *Salvetti complex*, which is homotopy equivalent to Y_W. This was originally defined [Sal87]; [Sal94] similarly using the realisation of W on a space. However, the Salvetti complex turns out to have a more useful formulation based more on algebraic properties of W, where we only turn these algebraic structures in to spaces at the very end. These structures arise from giving a partial order to W, and with this in mid, we will make some definitions.

2.1 Posets

A partially ordered set or *poset* $(P, \leqslant)$ is a set P with a relation $\leqslant$ on pairs in P which encodes the topology of $\mathbb{R}$. The textbook [Grä11] provides a good introduction. An important note is that there is no requirement for every pair to be related, hence partial. We will use P as shorthand for $(P, \leqslant)$ where possible.

In a poset P we define the *interval* between two elements $[x, y]$ as $[x, y] \coloneqq \{u \in P \mid x \leqslant u \leqslant y\}$, which is itself a poset. A *chain* is a subset $C \subseteq P$ that is a totally ordered, i.e. every pair in $(u, v) \in C \times C$ satisfies either $u \leqslant v$ or $v \leqslant u$. The *covering relations* of P, denoted $\mathcal{E}(P)$ are defined,

$$\mathcal{E}(P) = \{(x, y) \in P \times P \mid x \leqslant y \text{ and } [x, y] = \{x, y\}\}$$

i.e. ordered pairs such that there is nothing in between them in the order. If $(x, y) \in \mathcal{E}(P)$, we write $x \lessdot y$. We will call a chain C *saturated* if for all $x, y \in C$ such that $x < y$, there exists $z \in C$ such that $x \lessdot z$, i.e. there are no gaps in the chain.

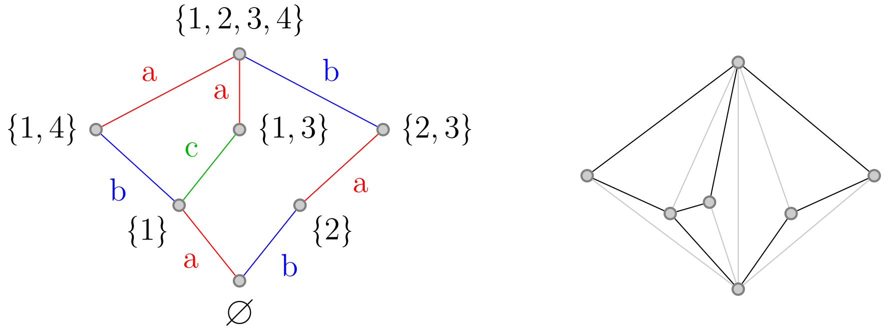


Figure 2.1: A simple example of an edge labelled poset where we have taken $\leqslant$ to be $\subseteq$ (left). The same poset with all chains drawn in light lines to aid visualising $\Delta(P)$ (right).

By transitivity, the covering relations encode the whole poset structure, which can in turn be drawn in a diagram which we will now define.

Definition 2.1.1 (Hasse Diagram)**.** Given a poset P, the *Hasse Diagram* is the directed graph encoding $\mathcal{E}(P)$ in the following way: For each element $x \in P$ draw a vertex. For each pair $(x, y) \in \mathcal{E}(P)$ draw an edge connecting x to y.

As is typical, we will draw the lesser (with respect to $\leqslant$) elements towards the bottom of the plane and visa versa. Thus, we will not need to draw arrows to show direction. The drawing of these Hasse diagrams is made easier as we will concern ourselves only with graded, bounded posets. *Bounded* meaning that there are minimal and maximal elements, denoted $\hat{0}$ and $\hat{1}$ such that $\hat{0} \leqslant x \leqslant \hat{1}$ for all $x \in P$, and *graded* meaning that every saturated chain from $\hat{0}$ to $\hat{1}$ has the same (finite) length. In the Hasse diagram for a bounded, graded poset, we will draw $\hat{0}$ at the bottom, $\hat{1}$ at the top, and put all other elements in discrete vertical levels between these based on the position in the saturated chains between $\hat{0}$ and $\hat{1}$ each element occurs. See Fig. 2.4 for an example.

Definition 2.1.2 (Edge Labelled Poset)**.** We define an edge labelled poset to be a triple $(P, \leqslant, l)$ where $(P, \leqslant)$ is a poset and the function $l : \mathcal{E}(P) \to A$ is the data of our labels with A being the alphabet of our labels.

We will use P as a shorthand for $(P, \leqslant, l)$ where possible. Given an edge labelled poset P, we can construct a group encoded by its labelling and geometry.

Definition 2.1.3 (Poset group)**.** Given some edge labelled poset $(P, \leqslant, l \colon \mathcal{E}(P) \to A)$, let the poset group $G(P)$ be the group generated by $\mathrm{Im}(l)$ with relations equating words corresponding to saturated chains going up the Hasse diagram of P which start and end at the same vertices.

A *word corresponding to a saturated chain* is the word of the labels traversed in the Hasse diagram while tracing out that saturated chain. In the example given in Fig. 2.1, the poset group is $G(P) = \langle a, b, c \mid aba = bab,\, ba = ca \rangle$.

2.2 Poset Complex

For some edge labelled poset P, we can construct a cell complex $K(P)$ from P such that $\pi_1(K(P))$ is $G(P)$. We do this by initially defining a simplicial complex $\Delta(P)$. An *abstract simplicial complex* is a family of sets that is closed under taking arbitrary subsets. From this, we can define

> **Definition 2.2.1** (Geometric Simplicial Complex)**.** Given an abstract simplicial complex X, the *geometric realisation* of that simplicial complex is defined as follows: For each single element set in X assign a point, for each three element set assign an edge attaching the two vertices it contains, for each two element set assign a triangle, comprising the three edges of its three subsets containing two elements. In this way continue constructing simplices of dimension n for each $n+1$ size set in X.

The set of all chains in a poset P is an abstract simplicial complex. We define $\Delta(P)$ to be the geometric simplicial complex corresponding to the set of all chains in P where each n–simplex is an n–chain of P. Note that as in [MS17, Definition 1.7], we define an n–chain to have $n-1$ elements. E.g. $(\{1\} \subseteq \{1,2\})$ is a 1–chain.

For example, in Fig. 2.1, $\Delta(P)$ would be three solid tetrahedrons all sharing an edge (a 1–simplex) corresponding to the 1–chain $(\varnothing \subseteq \{1,2,3,4\})$ with and two of them sharing a face corresponding to the 2–chain $(\varnothing \subseteq \{1\} \subseteq \{1,2,3,4\})$. For a more two–dimensional example consider the following poset P and corresponding $\Delta(P)$. Here we forget about edge labelling in P for a moment.

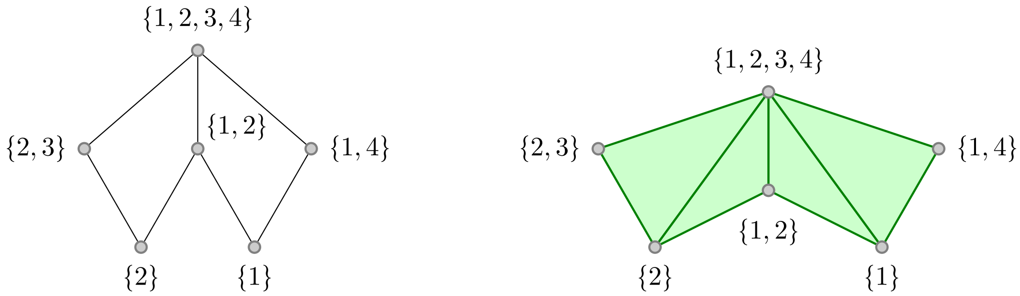


Figure 2.2: An example poset P (left) with corresponding $\Delta(P)$ (right).

We continue, now using an edge labelling on P, to generate a quotient space $K(P)$ of $\Delta(P)$. Let us put some arbitrary edge labelling on P to progress with this, shown in Fig. 2.3 (left).

To construct $K(P)$, first we define a labelling on chains in P which extends from the edge labelling in P.

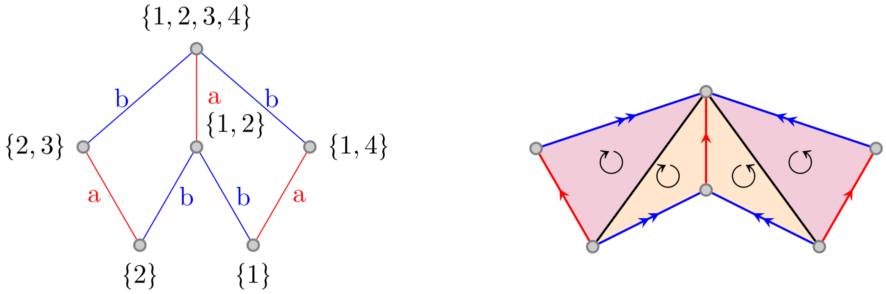


Figure 2.3: The poset in Fig. 2.2 with edge labelling (left) and the corresponding space $K(P)$ (right).

Definition 2.2.2 (Extended Labelling)**.** Given some edge–labelled poset $(P, \leqslant, l\colon \mathcal{E}(P) \to A)$ and some chain $C \subseteq P$, the *extended label* $\mathcal{L}(\rho) \subseteq A^*$ is the language of all words corresponding to all saturated chains that contain every element of C.

Here a *saturated chain* is a chain such that every relation is a covering relation. For an example on extended labels, consider the chain $(\{2\} \subseteq \{1,2,3\})$ in the context of Fig. 2.3. There are two corresponding saturated chains, $(\{2\} \subseteq \{1,2\} \subseteq \{1,2,3\})$ and $(\{2\} \subseteq \{2,3\} \subseteq \{1,2,3\})$, which respectively correspond to the words ba and ab. So $\mathcal{L}(\{2\} \subseteq \{1,2,3\}) = \{ba, ab\}$. Here are some illustrative examples:

- $\mathcal{L}(\{1\} \subseteq \{1,2\}) = \mathcal{L}(\{2\} \subseteq \{1,2\}) = \{b\}$.
- $\mathcal{L}(\{1\} \subseteq \{1,2,3,4\}) = \mathcal{L}(\{2\} \subseteq \{1,2,3,4\}) = \{ba, ab\}$.
- $\mathcal{L}(\{1\} \subseteq \{1,2\} \subseteq \{1,2,3,4\}) = \mathcal{L}(\{2\} \subseteq \{1,2\} \subseteq \{1,2,3,4\}) = \{ba\}$.
- $\mathcal{L}(\{1\}) = \mathcal{L}(\{2\}) = \mathcal{L}(\{1,2\}) = \cdots = \varnothing$.

This extended labelling on chains naturally extends to a labelling on simplices in $\Delta(P)$. Using this labelling and the orientation induced on a chain by $\leqslant$, we can define $K(P)$.

Definition 2.2.3 (Poset Complex [Mcc, Definition 1.6])**.** For an edge–labelled poset P the poset complex $K(P)$ is the quotient space $\Delta(P)/\sim$ where $\sim$ identifies pointwise simplices of the same dimension that share the same extended label, using the orientation on simplices induced by $\leqslant$.

In the example in Fig. 2.3, three red edges are indentified, four blue edges are identified, two orange triangles are identified and two purple triangles are identified. Note that the two black edges are in this case identified, but only because they belong to the two pairs of identified triangles. In the second example for $\mathcal{L}$ above, we see they have different labels.

We see that this space is homeomorphic to a torus, which has $\pi(1) \cong \mathbb{Z}^2 \cong$

$\langle a, b \mid ab = ba \rangle$, which is also the $G(P)$ for this edge–labelled poset.

This is true in general. We can determine $\pi_1(K(P))$ from its 2–skeleton [Hat01, Corollary 4.12]. The 1–skeleton will be a wedge of circles, one for each unlabelled 1–chain in P and one for each $a \in \mathrm{Im}(l)$. Only labelled edges will contribute generators to $\pi_1(K(P))$ since a labelled path can always be deformed to an unlabelled 1–chain through the simplex in $K(P)$. If two n–chains start and end at the same points, they will share an edge in an n–simplex corresponding to an unlabelled 1–chain. So one of the paths can be deformed to the path corresponding to the edge of the unlabelled 1–chain, and then through that shared edge to the other path, making the paths homotopic. E.g. in Fig. 2.3 we can deform $(\{2\} \subseteq \{1,2\} \subseteq \{1,2,3\})$ through $(\{2\} \subseteq \{1,2,3\})$ to $(\{2\} \subseteq \{2,3\} \subseteq \{1,2,3\})$. Identification of n–simplices for $n > 1$ does not affect the fundamental group, but does ensure that that higher homotopy groups are trivial. We can see if we did not identify the 2–simplices in Fig. 2.3, $\pi_2(K(P))$ would be non-trivial.

2.3 Interval Complex

Starting from a Coxeter group W generated by S, we wish to give W a labelled–poset structure and use the constructions from the previous section. The edge labelled Hasse diagram for W will embed in to the Cayley graph $\mathrm{Cay}(W, S)$, and it is useful to be able to swap between these two objects, as we will do. First we must define an order on our group W.

> **Definition 2.3.1** (Word length in a group)**.** For a group G generated by S, the word length with respect to S is the function $l_S : G \to \mathbb{Z}$ where $l_S(g) = \min\{k \mid s_1 s_2 \ldots s_k = g\,,\ s_i \in S\}$.

We will often omit the S in l_S where it is obvious from context.

> **Definition 2.3.2** (Order on a group)**.** For a group G generated by S, we define the order $x \leqslant y \iff l(x) + l(x^{-1}y) = l(y)$.

It can be readily checked that this does indeed define an order on G. This order encodes closeness to $e \in G$ along geodesics in $\mathrm{Cay}(G, S)$. We have $x \leqslant y$ precisely when there exists a geodesic in $\mathrm{Cay}(G, S)$ from e to y with x as an intermediate vertex, or to put it another way, when a minimal factorisation of x in to elements of S is a prefix of a minimal factorisation of y. For some $w \in W$ we define the poset $[1, w]^W$ to be the interval in W up to w with respect to this order. We give this poset an edge labelling such that the edge between w and ws is labelled s for any $s \in S$. The Hasse diagram thus embeds in to $\mathrm{Cay}(W, S)$.

> **Definition 2.3.3** (Coxeter element)**.** For some Coxeter group W generated by S, we define a *Coxeter element* $w \in W$ to be any product of all the elements of S without repetition.

These Coxeter elements are what we will use as the upper bound of our interval.

We will also need to consider W as the group generated by R, the set of all reflections, rather than just the set of simple reflections S. See Fig. 2.4 for an example of such a poset. In principle there are many choices of Coxeter element depending on what order we multiply the elements of S. However, we will see that many structures resulting from $[1,w]^W$ are independent of that choice.

We apply the steps from Section 2.2 to $[1,w]^W$ to form a space.

> **Definition 2.3.4** (Interval Complex)**.** For a Coxeter group W generated by all reflections R with $w \in W$, we call $K_{W_w} \coloneqq K([1,w]^W)$ the *interval complex* where $K([1,w]^W)$ is as in Definition 2.2.3.

Note that in [PS21] K_{W_w} is denoted K_W. We will later show that this notation is justified, in that up to homotopy equivalence, there is no dependence, of K_{W_w} on w. For now, we will stick with the notation as we have defined it. Certain properties of the poset permit a simplified notation for the simplices within K_{W_w}. In this context, for two chains $C = (C_1 \leqslant C_2 \leqslant \cdots \leqslant C_m)$ and $C' = (C'_1 \leqslant C'_2 \leqslant \cdots \leqslant C'_n)$ we have $\mathcal{L}(C) = \mathcal{L}(C')$ exactly when $(C_1)^{-1}C_m = (C'_1)^{-1}C'_n$. Thus, we can label 1–simplices in K_{W_w} with group elements $x \in [1,w]^W$, we can label 2–simplices with factorisations of group elements in $[1,w]^W$ in to two parts (with the first part also in $[1,w]^W$) and so on. We denote an n–simplex $[x_1|x_2|\cdots|x_n]$ as in [PS21, Definition 2.8]. This notation also gives the gluing of the faces of $[x_1|x_2|\cdots|x_n]$ in the following way. A codimention 1 face of $[x_1|x_2|\cdots|x_n]$ is a subchain of $x_1 \leqslant x_1x_2 \leqslant \cdots \leqslant x_1x_2\ldots x_n$ consisting of $n-1$ elements. There are three ways to obtain such a subchain.

1. Remove the first element of the chain to get $x_2 \leqslant x_2x_3 \leqslant \cdots \leqslant x_2x_3\ldots x_n$
2. Remove the last element of the chain to get $x_1 \leqslant x_1x_2 \leqslant \cdots \leqslant x_1x_2\ldots x_{n-1}$.
3. Multiply two adjacent elements x_i and x_{i+1} to get the chain $x_1 \leqslant \cdots \leqslant x_1\ldots x_{i-1} \leqslant x_1\ldots x_{i-1}x_ix_{i+1} \leqslant \cdots \leqslant x_1\cdots x_n$

So the n–simplex $[x_1|x_2|\cdots|x_n]$ glues to $[x_2|x_3|\cdots|x_n]$, $[x_1|x_2|\cdots|x_{n-1}]$ and $[x_1|\cdots|x_ix_{i+1}|\cdots|x_n]$ for all $i < n$.

The particular poset group intervals $[1,w]^W$ we will consider will be *balanced.* A balanced group interval is such that $x \in [1,w]^W$ iff $l(g^{-1}x) + l(x) = l(g)$. I.e. all minimal factorisation of $x \in [1,w]^W$ also appear as a suffix in a minimal factorisation of w and all suffixes also appear as a prefix.

Where the interval is balanced, any such symbol $[x_1|x_2|\cdots|x_n]$ corresponds to an n–simplex in K_{W_w} given it satisfies the following [PS21]:

i) $x_i \neq 1$ for all i.

ii) $x_1x_2\cdots x_n \in [1,w]^W$

iii) $l(x_1x_2\cdots x_n) = l(x_1) + l(x_2) + \cdots + l(x_n)$

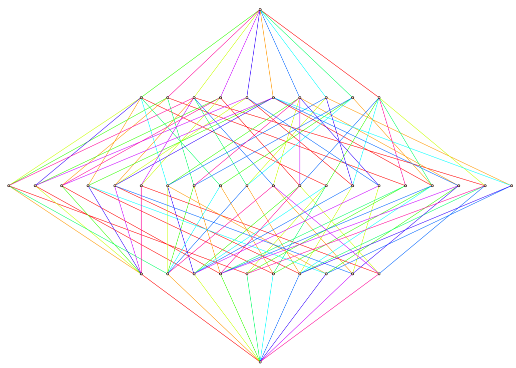


Figure 2.4: The interval $[1,(1,2)(2,3)(3,4)(4,5)]^{A_4}$ considering A_4 generated by all reflections, which label the edges by colour. Generated using `Sage` and `GAP` [Sag20]; [GAP22]

Hopefully the first two requirements are obvious. The third is because we require the chain $x_1 \leqslant x_1x_2 \leqslant \cdots \leqslant x_1x_2\ldots x_n$ to be contained in $[1,w]^W$ which translates to every subword of $x_1\cdots x_n$ also being in $[1,w]^W$. By ii and iii we have that there is some y such that $x_1\cdots x_ny = w$ and there is a minimal factorisation of w that respects the factors in $x_1\cdots x_ny$. We can take prefixes of this factorisation (and thus prefixes of $x_1\cdots x_ny$) and stay within $[1,w]^W$. We can use the balanced condition to move the suffix $x_2\cdots x_ny$ to the front. I.e. there exists y_2 such that $x_2\cdots x_nyy_2 = w$. We can then repeat these steps to show every subword of $x_1\cdots x_n$ is in $[1,w]^W$.

Definition 2.3.5 (Dual Artin group)**.** For a Coxeter group W generated by all reflections R with Coxeter element $w \in W$. Define $[1,w]^W$ as above. The *dual Artin group* W_w is the poset group $G([1,w]^W)$ with G defined as in Definition 2.1.3.

It should be clear the fundamental group of K_{W_w} is W_w. Furthermore, it is a classifying space.

Remark 2.3.6. The fundamental group of . See the end of Section 2.2.

Theorem 2.3.7 ([PS21, Theorems 2.9 and 2.14])**.** For a finite Coxeter group W, the interval complex is a $K(G_W,1)$ space.

Theorem 2.3.8. For a finite [Bes03] or affine [MS17] Coxeter group W and Coxeter element $w \in W$, the dual Artin group is isomorphic to the Artin group G_W (and thus that it does not depend on the choice of w).

In general, it is not known whether $G_W \cong G_w$, whether the isomorphism class of G_w depends on w or even if the isomorphism class of $[1,w]^W$ depends on w.

2.4 Salvetti Complex

Here we will define the *Salvetti Complex* for a Coxeter group W generated by S, which is homotopy equivalent to Y_W. First we must define a notion on subsets of S. For some subset $T \subseteq S$ define the *parabolic subgroup* of W with respect to T, W_T, to be the subgroup of W generated by T with all relations for W containing only elements of T. If Γ is the Coxeter diagram for W, then W_T is the Coxeter group corresponding to the complete subgraph of Γ containing the vertices T. From here we follow [Pao17, Section 2.3], with notation from [PS21].

Definition 2.4.1. For a Coxeter group W generated by S, define Δ_W to be the family of subsets $T \subseteq S$ such that W_T is finite.

For some $T \subseteq S$, we say some $w \in W$ is T–minimal if w is the unique element of minimum length (with respect to S) in the coset wT. Uniqueness is shown in [Bou08]. Define an order on the set $W \times \Delta_W$ by the following: $(u, X) \leqslant (v, Y)$ iff $X \subseteq Y$, $v^{-1}u \in W_Y$ and $v^{-1}u$ is X–minimal.

Definition 2.4.2 (Pre-Salvetti Complex [Pao17, Definition 2.19])**.** For a Coxeter group W, define $\mathrm{Sal}(W)$ to be $\Delta(W \times \Delta_W)$ under the order $\leqslant$ prescribed above, where the first Δ is as in Definition 2.2.1.

The Salvetti complex was originally defined (and refined) in [Sal87]; [Sal94]. In the latter of these papers, the Salvetti complex was defined to be the quotient of a space related to the action o W on a vector space. In [Par14, Theorem 3.3] it was shown that the definition we give generates space homeomorphic to that in the original definition. Let us quote some results that allow us to interpret the definition.

Lemma 2.4.3 ([Pao17, Lemma 2.18])**.** Consider both of these objects both as geometric simplicial complexes inside $\mathrm{Sal}(W)$.

$$C(v, Y) := \{(u, X) \in W \times \Delta_W \mid (u, X) \leqslant (v, Y)\}$$
$$\partial C(v, Y) := \{(u, X) \in W \times \Delta_W \mid (u, X) < (v, Y)\}$$

There is a homeomorphism $C(v, Y) \to D^n$ that restricts to a homeomorphism $\partial C(v, Y) \to S^{n-1}$ where $n = |\,Y\,|$.

This allows us to construct a CW complex for $\mathrm{Sal}(W)$ where each $C(w, X)$ is a $|\,X\,|$–cell for each $X \in \Delta_W$. Let us see what these cells look like. Note that the cells of the CW-complex and the simplices in $\Delta(W \times \Delta_W)$ as in Definition 2.4.2 comprise a completely different cell structure for $\mathrm{Sal}(W)$. We define $\langle\varnothing\rangle := \{1\}$ to give $W_\varnothing$ meaning as the trivial subgroup inside W.

Each $C(w, \varnothing)$ is a 0–cell. We will denote these cells w as a shorthand. In general, we have that $(u, X) \leqslant (v, X) \implies (u, X) = (v, X)$ since we require $v^{-1}u \in X$ we have $v^{-1}uX = X$. So if $v^{-1}u$ is minimal in $v^{-1}uX$ then $v^{-1}u = 1$.

In particular, there is no $(u, X) < (w, \varnothing)$, so these $w = C(w, \varnothing)$ are 0–simplices in $\Delta(W \times \Delta_W)$ as well.

Now consider each 1–cell $C(w, \{s\})$. Since $W_{\{s\}} = \{1, s\} \cong \mathbb{Z}/2$ we have $\{s\} \in \Delta_W$ for all $s \in S$. For some (u, X) to be less than $(w, \{s\})$, recall we require $w^{-1}u \in W_{\{s\}}$. So we have $u \in \{w, ws\}$. Locally, the Hasse diagram and (since we only have 1–chains here) $\Delta(W \times \Delta_W)$ both look like as in Fig. 2.5 (left). In the CW complex there would be only one 1–cell, labelled $C(w, \{s\})$ oriented from w to ws. Note that $C(ws, \{s\})$ also connects these two vertices, but is a different 1–cell. This doubling up will be inconsequential after we define the Salvetti complex, which will quotient away any such doubling.

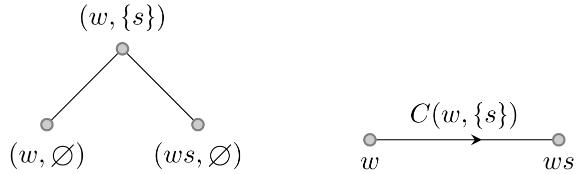


Figure 2.5: A local picture of $\mathrm{Sal}(W)$ as $\Delta(W \times \Delta_W)$ (which also resembles the Hasse diagram) (left). The corresponding 1–cell in the CW complex for $\mathrm{Sal}(W)$ (right).

Now consider the 2–cells in the CW complex. We have that $W_{\{s,t\}} \cong D_{2m(s,t)}$, the dihedral group of corresponding to the $m(s,t)$–gon (recall $m(s,t)$ from Definition 1.1.1). Thus, $\{s, t\} \in \Delta_W$ iff $m(s,t) \neq \infty$. We have $(u, \{s\})$ or $(v, \{t\})$ are less than $(w, \{s,t\})$ only when $u = wd$ for some $d \in W_{\{s,t\}}$, similarly for v. The second requirement then is that d is ($\{s\}$ or $\{t\}$)–minimal. In the general case, if $(u, X) \leqslant (v, Y)$ then $X \subseteq Y$ so $W_X \subseteq W_Y$. So the coset $v^{-1}uW_X \subseteq W_Y$. Due to the nature of Definition 1.1.1, only relations relevant to W_Y could have relevance to the word length of elements in W_Y. Thus, to determine if $v^{-1}u$ is X minimal we need only consider everything within W_Y, not the entire Coxeter group W. In particular, to tell if $(wd, \{s\}) \leqslant (w, \{s,t\})$ for some $d \in W_{\{s,t\}}$, we need only consider if d is $\{s\}$–minimal in the dihedral group $W_{\{s,t\}}$.

Considering for a moment s and t as letters only, a normal form comprising minimal length words for $W_{\{s,t\}}$ is

$$\{\Pi(s,t,n) \mid n \leqslant m(s,t)\} \cup \{\Pi(s,t,n) \mid n \leqslant m(s,t)\}$$

recalling the meaning of $\Pi(s,t,n)$ from Definition 1.1.2. Note that $\Pi(t, s, m(s,t))$ is also a minimal length word but is not included for the above to be a normal form. Thus, any $sts \cdots s$ is $\{t\}$–minimal if the total length of $sts \cdots s$ is strictly less than $m(s,t)$. Similarly, $sts \cdots t$ is $\{s\}$–minimal if the total length of $sts \cdots t$ is strictly less than $m(s,t)$, with equivalent results for $tst \cdots s$ and $tst \cdots t$ depending on the last letter in the word. A local picture of the Hasse diagram for the cell $C(w, \{s,t\})$ where $m(s,t) = 3$ is shown in Fig. 2.6. The CW cell itself has been drawn in Fig. 2.7.

There is a natural action $W \circlearrowleft \mathrm{Sal}(W)$ with $w \cdot (u, T) \coloneqq (wu, T)$. We can now define the following

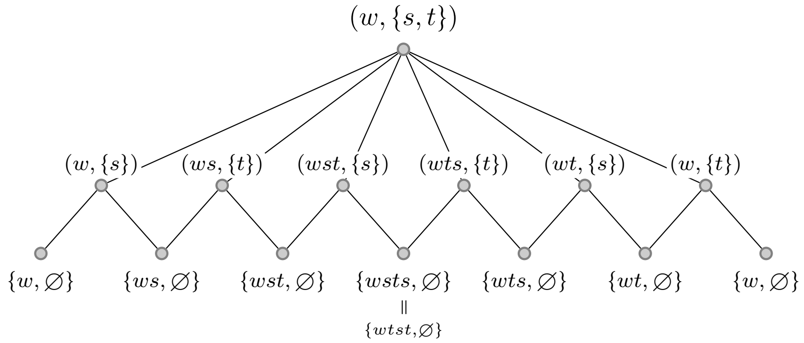


Figure 2.6: The local Hasse Diagram corresponding to the CW 2–cell $C(w, \{s,t\})$ where $m(s,t) = 3$. Note that w has been drawn twice for clarity in the picture. C.f. Fig. 2.7

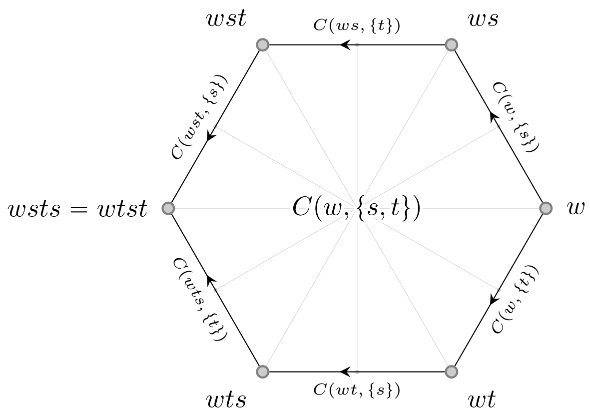


Figure 2.7: The 2–cell $C(w, \{s,t\})$ in the CW complex for $\mathrm{Sal}(W)$ where $m(s,t) = 3$. The faint lines are the simplices from $\Delta(W \times \Delta_W)$, which have all been incorporated in to one CW cell.

Definition 2.4.4 (Salvetti Complex)**.** For a Coxeter group W define the *Salvetti Complex* X_W to be $\mathrm{Sal}(W)/W$ under the action specified above.

The action is cellular, thus we have a CW structure for X_W as well. We now quote the following important result.

Theorem 2.4.5 ([Par14, Corollary 3.4][Sal87])**.** For a Coxeter group W, the Salvetti complex X_W is homotopy equivalent to the configuration space Y_W.

Now let us consider the cell structure of X_W. There is one $|T|$–cell for each $T \in \Delta_W$, in particular, there is one 0–cell corresponding to the trivial group $W_\varnothing$. Attached to this is a 1–cell for each $s \in S$ forming $\bigvee_{s\in S} S^1$ as the 1–skeleton. Then to this wedge is attached 2–cells following the procedure above for each $m(s,t) \neq \infty$. Each two cell corresponding to $\{s,t\}$ is attached to two 1–cells corresponding to $\{s\}$ and $\{t\}$. From examining this 2–skeleton it should be clear that $\pi_1(X_W) \cong G_W$. Thus combining with the previous theorem we have re-proved Theorem 1.2.2.

3 Implementing the CW Complexes

We will now begin to bridge the gap between some of the objects we have defined. Ultimately, we wish to show a homotopy equivalence between the space K_{W_w} and the Salvetti complex X_W, which is already known to be homotopy equivalent to Y_W. For now, we will define a subspace $K'_{W_w} \subseteq K_{W_w}$, inspired by our definition of the Salvetti complex, using subsets $T \subseteq S$ such that W_T is finite.

3.1 The subcomplex X'_W

The definition of K_{W_w} depends on the data of some Coxeter element w and thus implicitly some generating set S. For some $w = s_1 s_2 \cdots s_n$ and some $T \subseteq S$, define $w_T \in W$ to correspond subword of w consisting of elements that are in T, respecting the original order in w.

Definition 3.1.1 (The subcomplex)**.** Define X'_W to be the finite subcomplex of K_{W_w} consisting only of simplices $[x_1|\cdots|x_n]$ such that $x_1x_2\cdots x_n \in [1, w_T]^W$ for some $T \in \Delta_W$. Recalling Δ_W from Definition 2.4.1.

Note that we have dropped the w index. That will be justified in this section as we will show that, up to homotopy equivalence, X'_W has no w dependence.

Lemma 3.1.2 ([PS21, Lemma 5.2])**.** Q As far as I can tell, what I write here is all we need, but there they seem to go in to more detail unnecessarily For a Coxeter group W and any parabolic subgroup W_T, with sets of reflections R_W and R_{W_T} respectively. For some Coxeter element

$w \in W$ and corresponding $w_T \in W_T$, all minimal factorisations of w_T in R_W consist of only elements from R_{W_T}.

An immediate consequence of this is that the intervals $[1, w_T]^W$ and $[1, w_T]^{W_T}$ agree. This allows us to decompose X'_W in a useful way. For each $T \in \Delta_W$ and corresponding W_T, the space corresponding to the whole interval $[1, w_T]^W = [1, w_T]^{W_T}$ is a subspace inside X'_W. This subspace is exactly X'_{W_T} (with respect to w_T). Thus, we can think of X'_W as some union of all X'_{W_T} for $T \in \Delta_W$.

Each X'_{W_T} is exactly the same as its interval complex K_{W_T} since all subgroups of T generate finite Coxeter groups. Thus, X'_{W_T} is a classifying space for the dual Artin group W_w by Theorem 2.3.7. Furthermore, G_w is isomorphic to the Artin group G_{W_T} by Theorem 2.3.8.

In a very similar way, the Salvetti complex consists of subspaces corresponding to elements of Δ_W. For each $T \in \Delta_W$, the Salvetti complex X_{W_T} is a $|T|$–cell attached to all cells corresponding to $R \subseteq T$ is the appropriate way. This is a cellular subspace of X_W, and since W_T is finite, by Theorem 1.3.1, $Y_{W_T} \simeq X_{W_T}$ is a $K(G_{W_T}, 1)$. The following remark summarises.

Remark 3.1.3. The Salvetti complex X_W decomposes in to cellular subspaces X_{W_T} which are $K(G_{W_T}, 1)$ spaces. These subspaces are in bijection with cellular subspaces X'_{W_T} of X'_W, which are also $K(G_{W_T}, 1)$ spaces.

The following theorems will help us to exploit this similarity to show that $X_W \simeq X'_W$.

Lemma 3.1.4. *A null homotopic map $\rho \colon S^n \to X$ can be extended to a map $\sigma \colon D^{n+1} \to X$.*

Proof. Let $H \colon S^n \times I \to X$ witness the null homotopy with $H|_{S^n \times \{1\}} \colon S^n \to \{x_0\}$. We have that H factors uniquely through $(S^n \times I)/(S^n \times \{1\}) \cong D^{n+1}$. With σ being the necessary map as below.

$$\begin{tikzcd}
S^n \times I \arrow[r, "H"] \arrow[d, "q"] & X \\
(S^n \times I)/(S^n \times \{1\}) \arrow[ur, dotted, "\exists ! \sigma"'] &
\end{tikzcd}$$

□

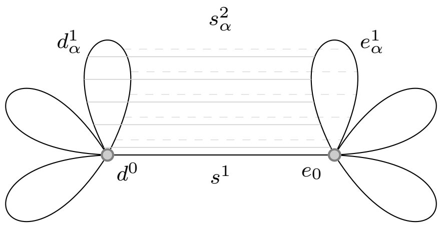
Theorem 3.1.5 ([Hat01, Proposition 1B.9])**.** *Let Y be a $K(G, 1)$ space and X a finite dimensional CW complex consisting of one 0–cell, the point x_0. Any homomorphism $\phi \colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is induced by a map $\tilde{\phi} \colon X \to Y$ where $\tilde{\phi}$ is unique up to homotopy fixing x_0.*

Proof. Clearly we must have $\tilde{\phi}(x_0) = y_0$. The 1–skeleton X^1 will be a wedge of circles and there is thus a presentation of $\pi_1(X, x_0)$ with each cell e^1_α corresponding to a generator $[e^1_\alpha] \in \pi_1(X, x_0)$. We can choose $\tilde{\phi}(e_\alpha)$ to trace to a path corresponding to $\phi([e^1_\alpha]) \in \pi_1(Y, y_0)$ for each $e^1_\alpha \in X^1$.

Let $\psi_\beta \colon S^1 \to X^1$ be an attaching map for a 2–cell $e^2_\beta \in X$. Let $i \colon X^1 \hookrightarrow X$ be the inclusion. We have that i_* is the surjetcion from the free group generated by each e^1_α to $\pi_1(X, x_0)$. The attaching of the 2–cell e^2_β provides a null homotopy for the path traced by ψ_β. In the presentation of $\pi_1(X, x_0)$ as above, each relation corresponds to the path of a ψ_β. Thus, $i_*([\psi_\beta]) = 0$ and so $\tilde{\phi}_*([\psi_\beta]) = \phi \circ i_*([\psi_\beta]) = 0$. Thus, $\tilde{\phi} \circ \psi_\beta$ is null homotopic and so can be extended over all of the closure of e^2_β by Lemma 3.1.4. This is an extension of $\tilde{\phi}$ and repeating this allows us to extend $\tilde{\phi}$ over all of X^2.

To extend $\tilde{\phi}$ over e^3_γ we use that S^2 is simply connected (as for any S^n with $n \geqslant 2$) and so for the attaching map $\psi_\gamma \colon S^2 \to X^2$ we have that $\tilde{\phi} \circ \psi_\gamma$ lifts to the universal cover of Y, which is contractible since Y is a $K(G, 1)$, so $\tilde{\phi} \circ \psi_\gamma$ is null homotopic. This same argument applies for any e^n_δ for $n \geqslant 3$. We can thus extend $\tilde{\phi}$ over the 3–cells and proceeding inductively, over all of X.

Now we turn to the uniqueness of $\tilde{\phi}$ up to homotopy. Let ϕ be some homomorphism and $\tilde{\phi}_0$ and $\tilde{\phi}_1$ be any such maps constructed as above. Clearly $\tilde{\phi}_0(x_0) = \tilde{\phi}_1(x_0)$ and $\tilde{\phi}_0|_{X^1} \sim \tilde{\phi}_1|_{X^1}$ by the restrictions of our construction. Let H witness this homotopy. Give $X \times I$ the following CW structure: Let $X \times \{0\}$ and $X \times \{1\}$ both have the same cell structure as X with cells notated d^n_α and e^n_α respectively. Connect d^0 to e^0 with a 1–cell s^1, called *the spine*. Connect a 2–cell s^2_α along d^1_α, then s^1 then e^1_α then back along s^1 with opposite orientations on d^1_α and e^1_α such that $d^0 \cup e^0 \cup s^1 \cup d^1_\alpha \cup e^1_\alpha \cup s^2_\alpha \cong S^1 \times I$. The spine now consists of $s_1 \cup s^2_\alpha$. Repeat this for each 1–cell in X and then repeat for each 2–cell and so on, attaching an s^n_β along d^{n-1}_β, e^{n-1}_β and s^{n-1}_β, inductively building up the spine. A picture of this CW complex completed for one s^2_α is below.



We can now extend the domain of H from $X^1 \times I$ to all of $X \times I$ using this cell structure. Note that now we have two 0–cells, but this does not cause any issues. Let H have domain $X^1 \times I \subseteq X \times I$. Now extend H such that $H|_{X \times \{0\}}$ agrees with $\tilde{\phi}_0$ and $H|_{X \times \{1\}}$ agrees with $\tilde{\phi}_1$. This is possible because H is a homotopy between restrictions of these maps. Note that now H is defined on the whole 2–skeleton of $X \times I$. We can extend H to all the higher dimensional cells by the exact same argument as before, using the contractability of the universal cover of Y. Thus, we have a continuous function $H\colon X \times I \to Y$ witnessing the homotopy $\tilde{\phi}_0 \sim \tilde{\phi}_1$. $\square$

Corrolary 3.1.6. Let X and Y both be $K(G,1)$ spaces. Any isomorphism $\phi\colon \pi_1(X,x_0) \to \pi_1(Y,y_0)$ induces a homotopy equivalence witnessing $X \simeq Y$.

Proof. We have maps $\tilde{\phi}\colon X \to Y$ and $\widetilde{(\phi^{-1})}\colon Y \to X$ with $(\tilde{\phi} \circ \widetilde{(\phi^{-1})})_* = \mathrm{Id}_{\pi_1(Y,y_0)}$. Thus, since the homotopy class of such maps is determined by the induced action on their fundamental groups $\tilde{\phi} \circ \widetilde{(\phi^{-1})} \sim \mathrm{Id}_Y$. Similarly, $\widetilde{(\phi^{-1})} \circ \tilde{\phi} \sim \mathrm{Id}_X$. $\square$

Lemma 3.1.7. Let $T \in \Delta_W \backslash \varnothing$. Let $\phi\colon \bigcup_{Q \subsetneq T} X_{W_Q} \to \bigcup_{Q \subsetneq T} X'_{W_Q}$ be a homotopy equivalence. We can extend ϕ to $\psi\colon X_{W_T} \to X'_{W_T}$ such that the following diagram commutes.

$$\begin{tikzcd}
\bigcup_{Q \subsetneq T} X_{W_Q} \arrow[r, "\phi"] \arrow[d, hook] & \bigcup_{Q \subsetneq T} X'_{W_Q} \arrow[d, hook] \\
X_{W_T} \arrow[r, "\psi"] & X'_{W_T}
\end{tikzcd}$$

Proof. We prove this by cases. By Remark 3.1.3, X_{W_T} and X'_{W_T} are classifying spaces.

i) If $|\,T\,| = 1$ then Q is uniquely $\varnothing$. Let ψ be any map witnessing $X_{W_T} \simeq X'_{W_T}$ that fixes the point corresponding to X_{W_Q}.

ii) If $|\,T\,| = 2$ then we can extend ϕ to ψ such that $\psi_*\colon \pi_1(X_{W_T}, X_\varnothing) \to \pi_1(X'_{W_T}, X'_\varnothing)$ is an isomorphism using the same argument as in the proof of Theorem 3.1.5. This map is a homotopy equivalence by Corollary 3.1.6.

iii) If $|\,T\,| \geqslant 3$ then we can extend ϕ to some map ψ using the same methods as in Theorem 3.1.5. In this case, $\bigcup_{Q \subsetneq T} X_{W_Q}$ contains the 2–skeleton of X_T and similarly for $\bigcup_{Q \subsetneq T} X'_{W_Q}$ and X'_T. So $\pi_1(\bigcup_{Q \subsetneq T} X_{W_Q}, X_{W_\varnothing}) = \pi_1(X_T, X_{W_\varnothing})$ and similarly for X'_T. By assumption ϕ is a homotopy equivalence and so ϕ_* is an isomorphism. Therefore ψ_* is the same isomorphism and thus ψ a homotopy equivalence by Corollary 3.1.6.

. $\square$

Definition 3.1.8 (Adjunction Space)**.** For two spaces X and U, with a continuous map $f\colon A \to U$ for some subspace $A \subseteq X$. The *adjunction space* $X \sqcup_f U$ is the space formed by gluing X and U via the map f.

$$X \sqcup_f U \coloneqq (X \sqcup Y)/(a \sim f(a))$$

An adjunction space is associated to the commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"] & U \arrow[d, "\bar{i}"] \\ X \arrow[r, "\bar{f}"] & X \sqcup_f U \end{tikzcd} \tag{3.1}$$

where i is inclusion of A in to X and $\bar{i}$ is inclusion of U in to $X \sqcup_f U$. Suppose we also have the adjunction space $Y \sqcup_g V$ with $g\colon B \to V$ and $B \subseteq Y$. Suppose further that we have maps $\phi_1\colon X \to Y$, $\phi_2\colon A \to B$ and $\phi_3\colon U \to V$ such that the following diagram commutes.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\phi_2"'] \arrow[dd, "i"'] & & U \arrow[dr, "\phi_3"] \arrow[dd, "\bar{i}" near start] & \\ & B \arrow[rr, "g" near start] \arrow[dd, "j" near start] & & V \arrow[dd, "\bar{j}"] \\ X \arrow[rr, "\bar{f}"' near end] \arrow[dr, "\phi_1"'] & & X \sqcup_f U \arrow[dr, "\phi"] & \\ & Y \arrow[rr, "\bar{g}"'] & & Y \sqcup_g V \end{tikzcd} \tag{3.2}$$

If all ϕ_1, ϕ_2 and ϕ_3 are homotopy equivalences then the following lemma tells us that ϕ is also a homotopy equivalence.

Lemma 3.1.9 ([Bro06, Theorem 7.5.7])**.** Consider a commutative diagram as in (3.2) where the front and back faces define an adjunction space as in (3.1). If i and j are closed cofibrations and ϕ_1, ϕ_2 and ϕ_3 are homotopy equivalences, then the ϕ as determined by the diagram is also a homotopy equivalence.

The restriction of i and j being closed cofibrations is quite mild. In the cases important to us, i and j will be cellular inclusions in to finite CW complexes, and thus closed cofibrations. See [Bro06] for more details on pushout squares and adjunction spaces.

To use Lemma 3.1.9 we must be able to construct X_W and X'_W as a sequence of adjunction spaces. Consider X_W in the following toy example.

Example 3.1.10. Let $\Delta_W = \{\varnothing, \{s\}, \{t\}, \{u\}, \{s,t\}, \{s,u\}, \{t,u\}\}$ and let $\Delta_W^n \coloneqq \{T \in \Delta_W \mid |T| = n\}$. Clearly we have that $X_W = \bigcup_{T \in \Delta_W^2} X_{W_T}$ with the appropriate gluing. Suppose we had the 1–skeleton $X_W^1 \subsetneq X_W$, and some ordering on $\Delta_W^2 = (\{s,t\}, \{s,u\}, \{t,u\})$. We would first glue

$X_{\{s,t\}}$ to X_W^1 as an adjunction space in the following way.

$$\begin{tikzcd} X_{\{s\}} \cup X_{\{t\}} \arrow[r, "f"] \arrow[d, "i_1"] & X_{\{s,t\}} \arrow[d, "\overline{i_1}"] \\ X_W^1 \arrow[r, "\bar{f}"] & X_W^1 \sqcup_f X_{\{s,t\}} \end{tikzcd}$$

Where f is inclusion of those 1–cells in to $X_{\{s,t\}}$, which in this case makes $X_W^1 \sqcup_f X_{\{s,t\}} \cong X_{\{s,t\}}$. Note that $X_{\{s\}} \cup X_{\{t\}}$ is really shorthand for another adjunction space. We can then add $X_{\{t,u\}}$ to the preceding adjunction space in the following way.

$$\begin{tikzcd} X_{\{t\}} \cup X_{\{u\}} \arrow[r, "g"] \arrow[d, "i_2"] & X_{\{t,u\}} \arrow[d, "\overline{i_2}"] \\ X_W^1 \sqcup_f X_{\{s,t\}} \arrow[r, "\bar{g}"] & (X_W^1 \sqcup_f X_{\{s,t\}}) \sqcup_g X_{\{t,u\}} \end{tikzcd} \tag{3.3}$$

After which we would continue with $\{u,v\}$ in the same manner. In the final space, $X_{\{s,t\}}$ is glued to $X_{\{t,u\}}$ along $X_{\{t\}}$. In general, for $T_1, T_2 \in \Delta_W^n$ are glued along $T_1 \cap T_2 \in \Delta_W^{n-1}$. We can always construct the n–skeleton from the $(n-1)$–skeleton in exactly this way.

The exact same construction works for X'_W.

Theorem 3.1.11 ([PS21, Theorem 5.5])**.** For a Coxeter group W, the space X'_W as in Definition 3.1.1 is homotopy equivalent to the Salvetti complex X_W.

Proof. We achieve this inductively. To tidy our notation, in this proof we drop W so that X, X', X_T and X'_T correspond to X_W, X'_W, X_{W_T} and X'_{W_T} respectively. For convenience define $\Delta_W^n := \{T \in \Delta_W \mid |T| = n\}$. Suppose we have the $(n-1)$–skeletons X^{n-1} and $(X')^{n-1}$ and a homotopy equivalence $\phi \colon X^{n-1} \to (X')^{n-1}$. We wish to show that we can extend ϕ to a homotopy equivalence for the respective n–skeletons. We construct the n–skeletons as adjunction spaces of the $(n-1)$–skeletons as in Example 3.1.10. To show this extension is a homotopy equivalence we use Lemma 3.1.9. Suppose we are gluing on the cells corresponding to some $T \in \Delta_W^n$. So any $Q \subsetneq T$ will correspond to a cell in X^{n-1}. Let Y and Y' be some intermediate steps in the adjunction gluing, such as the bottom–left term of (3.3). Suppose we have a homotopy equivalence $\phi_1 \colon Y \to Y'$ such that $\phi_1|_{X^{n-1}} = \phi$ so the following commutes.

$$\begin{tikzcd}
\bigcup_{Q \subsetneq T} X_Q \arrow[r, "\phi"] \arrow[d, hook] & \bigcup_{Q \subsetneq T} X'_Q \arrow[d, hook] \\
Y \arrow[r, "\phi_1"] & Y'
\end{tikzcd} \tag{3.4}$$

We wish to extend ϕ to a homotopy equivalence ψ such that the following commutes.

$$\begin{tikzcd}
X_T \arrow[r, "\psi"] & X'_T \\
\bigcup_{Q \subsetneq T} X_Q \arrow[r, "\phi"] \arrow[u, hook] & \bigcup_{Q \subsetneq T} X'_Q \arrow[u, hook]
\end{tikzcd}$$

This is possible by Lemma 3.1.7. Now we have the following commutative diagram.

$$\begin{tikzcd}
\bigcup_{Q \subsetneq T} X_Q \arrow[rr, "f"] \arrow[dr, "\phi"'] \arrow[dd, hook] & & X_T \arrow[dr, "\psi"] \arrow[dd, hook] & \\
& \bigcup_{Q \subsetneq T} X'_Q \arrow[rr, "g"] \arrow[dd, hook] & & X'_T \arrow[dd, hook] \\
Y \arrow[rr, "\bar{f}"'] \arrow[dr, "\phi_1"'] & & Y \sqcup_f X_T \arrow[dr, "\sigma"] & \\
& Y' \arrow[rr, "\bar{g}"'] & & Y' \sqcup_g X'_T
\end{tikzcd} \tag{3.5}$$

Where the induced map σ is a homotopy equivalence by Lemma 3.1.9. At the next inductive step, $Y \sqcup_f X_T$ and $Y' \sqcup_g X'_T$ replace Y and Y' respectively. Accordingly, σ replaces ϕ_1. Suppose we are next going to glue the cells corresponding to $\tilde{T} \in \Delta_W$. To proceed inductively, there are two possible outcomes:

1. We are still constructing $X^n \simeq (X')^n$ and $\tilde{T} \in \Delta_W^n$.
2. We completely constructed $X^n \simeq (X')^n$ in the previous step and $\tilde{T} \in \Delta_W^{n+1}$.

In Case 1, we have that any $Q \subsetneq \tilde{T}$ corresponds to cells in X^{n-1}. By the inductive hypothesis, we can restrict $\phi_1 \colon Y \to Y'$ to X^{n-1}, and thus we can do the same for σ and so the restriction $\sigma|_{X^{n-1}}$ is well–defined, we can get (3.4) with the appropriate replacements and proceed inductively.

In Case 2, $Y \sqcup_f X_T$ and $Y' \sqcup_g X'_T$ are X^n and $(X')^n$ respectively. Some $Q \subsetneq \tilde{T}$ will correspond to cells in X^n, but σ is exactly the restriction $\sigma|_{X^n}$ so the restriction is well–defined. We get (3.4) with the appropriate replacements and proceed inductively.

The base case is $X_\varnothing \simeq X'_\varnothing \simeq \{\bullet\}$. □

3.2 Proof of theorem 5.5

4 Bin

4.1 Configuration space

4.2 Proof Overview

Here we compile many theorems from [PS21] in to one theorem.

> **Theorem 4.2.1** ([PS21])**.** Given an affine Coxeter group W, the configuration space Y_W is homotopy equivalent to the order complex K_W.

Proof. This is done through a composition of homotopy equivalences

$$Y_W \overset{\text{(a)}}{\simeq} X_W \overset{\text{(b)}}{\simeq} X'_W \overset{\text{(c)}}{\simeq} K'_W \overset{\text{(d)}}{\simeq} K_W \tag{4.1}$$

Where the results are gathered from the following sources:

(a): [Sal87, Theorem 1] (b): [PS21, Theorem 5.5] (c): [PS21, Theorem 8.14] (d): [PS21, Theorem 7.9] □

Furthermore, in the same paper another main result is shown.

> **Theorem 4.2.2** ([PS21, Theorem 6.6])**.** Given an affine Coxeter group W, corresponding affine Artin group G_W and Coxeter element $w \in W$, the complex K_W is a classifying space for the dual artin group W_w. I.e.
>
> $$K_W \simeq K(W_w, 1)$$

It was already known [Bri71] that $\pi_1(Y_W) = G_W$. Thus considering $\pi_1(Y_W)$ and combining Theorems 4.2.1 and 4.2.2 gives

$$Y_W \simeq K(G_W, 1) \tag{4.2}$$
$$G_W \cong W_w \tag{4.3}$$

for affine G_W.

This proves the $K(\pi, 1)$ conjecture for affine Artin groups and provides a new proof than an affine Artin group is naturally isomorphic to its dual, which was already known for finite [Bes03] and affine [MS17] cases.

The proof of $\pi_1(Y_W) \cong G_W$ for all W in [Bri71] is in German and only German or Russian translations are available. This result is fundamental and non-trivial. Alternative proofs for Coxeter groups of type A_n [FN62] or affine type [Viê83] are available in English. Here we will repeat roughly the proof in [FN62], revealing the reason for the choice of name *configuration space* for Y_W.

4.3 The Garside Complex K

4.4 Interval Groups

Starting from a Coxeter group W generated by S, we wish to give W a labelled–poset structure and use the constructions from the previous section. The edge labelled Hasse diagram for W will embed in to the Cayley graph $\mathrm{Cay}(W, S)$, and it is useful to be able to swap between these two objects, as we will do. First we must define an order on our group W.

> **Definition 4.4.1** (Word length in a group)**.** For a group G generated by S, the word length with respect to S is the function $l_S : G \to \mathbb{Z}$ where $l_S(g) = \min\{k \mid s_1 s_2 \ldots s_k = g\,,\, s_i \in S\}$.

We will often omit the S in l_S.

> **Definition 4.4.2** (Order on a group)**.** On a group G we define the order $x \leqslant y \iff l(x) + l(x^{-1}y) = l(y)$.

It can be readily checked that this does indeed define an order on G. This order encodes closeness to $e \in G$ along geodesics in $\mathrm{Cay}(G, S)$. We have $x \leqslant y$ precisely when there exists a geodesic in $\mathrm{Cay}(G, S)$ from e to y with x as an intermediate vertex.

For some $w \in W$, the poset $[1, w]^W$ (now no longer a group) is simply the interval $[1, w]$ with respect to this order. We now define the precise $w \in W$ for which we want to make this construction.

> **Definition 4.4.3** (Coxeter element)**.** For some Coxeter group W for which $R \subseteq W$ is all reflections in W, we define a Coxeter element $w \in W$ to be any product of all the elements of R.

These Coxeter elements are what we will use as the upper bound of our interval. In principle there are many choices of Coxeter element depending on what order we multiply the elements of R. However, we will see that in many cases these choices necessarily result in isomorphic $[1, w]^W$. We see that $S \subseteq R$, in particular R generates W, since in the standard presentation of Coxeter groups all of S are reflections.

The interval $[1, w]^W$ is given the obvious edge labelling that makes it a subgraph of $\mathrm{Cay}(R, S)$, so two connected vertices g and gs will be labelled by $s \in S$. With this edge labelling we now define a new group W_w constructed from the geometry of $[1, w]^W$.

Definition 4.4.4 (Interval group)**.** Given a Coxeter group W and Coxeter element $w \in W$, construct the edge–labelled poset $[1, w]^W$ as above. The interval group W_w is the poset group (as in Definition 2.1.3) corresponding to $[1, w]^W$.

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