Combinatoric and poset structures for the $K(\pi, 1)$ conjecture

by

Sean O'Brien

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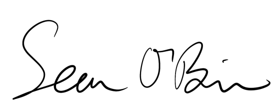

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Abstract This thesis concerns a very important topic.

Acknowledgement I acknowledge with thanks the inspiration of my supervisor.

Contents

Chapter 1

Introduction

A classifying space for a group G is a space X such that the fundamental group of X is G and all higher homotopy groups of X are trivial. In this paper we will be concerned with the $K(\pi, 1)$ conjecture for Artin groups, which states that the configuration space Y_W for any Coxeter group W is a classifying space for the Artin group G_W. This conjecture emerges as a generalisation of the result for Coxeter groups of type A_n and is originally attributed to Arnol'd, Pham and Thom in [vdL83]. See also [CD95] for a good overview of the history of the conjecture.

We will focus on the work of Paolini and Salvetti, *Proof of the* $K(\pi, 1)$ *conjecture for affine Artin groups* [PS21]. We will review some theorems therein and provide relevant background. Their main Theorem relies on proving a chain of homotopy equivalences (1), detailing these homotopy equivalences is the aim of this work. A strong theme will be the involvement of posets and related structures, hence this work's title. We will begin by providing a birds eye view of the conjecture and the main results of [PS21].

1.1 The conjecture and the objects involved

Coxeter groups emerge as generalisations of reflection groups. A Coxeter group is defined by a particular group presentation. The data of this presentation is typically encoded by a labelled graph. The group W, coupled with the data of its presentation is called a Coxeter system, denoted (W, S) where S is the generating set of W. Given a Coxeter system (W, S), we can form a different group G_W, called the Artin group associated to W.

For affine Coxeter groups W, the configuration space Y_W can be derived from a geometric realisation of W as a subgroup of $\mathrm{Isom}(\mathbb{E})$, the group of isometries on a Euclidean space $\mathbb{E}$. We will consider $\mathbb{E}$ as $\mathbb{R}^n$ without the notion of origin. Specifically, W is realised as a subgroup generated by a finite set of affine reflections S. Within W, we consider the set of all reflections R (not necessarily finite). To each reflection $r_i \in R$ there is a corresponding codimension–1 space $H_i \subset \mathbb{E}$ that is the plane of reflection of r_i. We call such spaces hyperplanes. Note there is no requirement that these hyperplanes be subspaces of $\mathbb{R}^n$.

The configuration space is realised as the complement of the complexification of all such hyperplanes H_i. It is known by work of Brieskorn [Bri71] that the fundamental group of Y_W is G_W. Thus proving the $K(\pi, 1)$ involves showing that the higher homotopy groups of Y_W are trivial. By previous work by Salvetti [Sal87, Sal94], there is a CW–complex X_W called the Salvetti complex that is homotopy equivalent to Y_W. Showing homotopy equivalence to X_W thus shows homotopy equivalence to Y_W. Because of this, the Salvetti complex is the starting point in a chain of homotopy equivalences reviewed in this work.

The finishing point of this chain is the interval complex K_W. This is a space realised using a certain poset structure on subsets of W. To this poset structure there is an associated group called the dual Artin group, denoted W_w. It was already known (by a now standard construction due to Garside [Gar69], extended by other authors, see [CMW02]) that K_W was a classifying space for the dual Artin group for finite W. In [PS21], the authors extend this result to affine W. Thus, considering the result of Brieskorn, showing $Y_W \simeq K_W$ for affine W shows that (for affine W) the higher homotopy groups of Y_W are trivial and that the dual Artin group associated to W is isomorphic to the Artin group associated to W.

In the following section, we will identify the intermediate spaces used in proving $X_W \simeq K_W$.

1.2 Proof overview

We will compile several main results from [PS21] in to two theorems. The concern of this work is Theorem 1.1 which proves that the *Salvetti complex* X_W is homotopy equivalent to the *interval complex* K_W. A *Coxeter element* is a product of all the elements of S in no particular order. Forming an interval complex associated to (W, S) involves making a choice of Coxeter element $w \in W$.

For a subset $T \subset S$, the *parabolic subgroup* W_T is the subgroup of W generated only by elements of T and only with relations explicitly involving elements of T. A parabolic Coxeter element w_T is a product of all elements of T that respects the order of multiplication in a Coxeter element $w \in W$. The space X'_W is a subspace of K_W associated to parabolic Coxeter elements associated to $T \subset S$ such that W_T is finite. Cells in X_W also correspond to such subsets, which is used in proving $X_W \simeq X'_W$.

The space K'_W is also a subspace of K_W. Connected components of preimages $\eta^{-1}(d)$ of a certain poset map $\eta \colon K_W \to \mathbb{N}$ have a linear structure as subposets of K_W. For each element $x \in \eta^{-1}(d)$, whether x is in K'_W or not is determined based on whether x comes in between two elements of X'_W in the linear structure of $\eta^{-1}(d)$.

Theorem 1.1 ([PS21])**.** *Given an affine Coxeter system (W, S), the configuration space Y_W is homotopy equivalent to the order complex K_W.*

Proof. By Theorem 2.21 the Salvetti complex X_W is homotopy equivalent to the configuration space Y_W. Therefore, we need only show $K_W \simeq X_W$. This is done through a composition of homotopy equivalences

$$X_W \overset{\text{(a)}}{\simeq} X'_W \overset{\text{(b)}}{\simeq} K'_W \overset{\text{(c)}}{\simeq} K_W \tag{1}$$

Where the results are gathered from the following sources:

(a) Theorem 3.15 [PS21, Theorem 5.5]

(b) [PS21, Theorem 8.14]

(c) Theorem 4.9 [PS21, Theorem 7.9] □

Furthermore, in the same paper another main result is shown.

Theorem 1.2 ([PS21, Theorem 6.6])**.** *Given an affine Coxeter system (W, S), corresponding affine type Artin group G_W and Coxeter element $w \in W$, the complex K_W is a classifying space for the dual Artin group W_w.*

By work of Brieskorn $\pi_1(Y_W) \cong G_W$. Thus considering $\pi_1(Y_W)$ and combining Theorems 1.1 and 1.2 gives

$$Y_W \simeq K(G_W, 1)$$

$$G_W \cong W_w$$

for affine G_W.

This proves the $K(\pi, 1)$ conjecture for affine Artin groups and provides a new proof than an affine Artin group is naturally isomorphic to its dual, which was already known for spherical or finite type Artin groups [Bes03] and affine Artin groups [MS17].

Alternatives to [Bri71] proving $\pi_1(Y_W) \cong G_W$ for affine Coxeter groups and Coxeter groups of type A_n are [VD83] and [FN62b] respectively.

1.3 Coxeter groups and Artin groups

In this section we will cover the constructions and properties of Coxeter groups and Artin groups. Coxeter groups are a generalisation of *reflection groups*, which are subgroups of $GL_n(\mathbb{R})$ generated by a finite set of reflections. Although the definition of Coxeter groups is tied to an abstract group presentation, we may also think of them as groups acting on a space by compositions of reflections. For example, finite Coxeter groups can be realised as reflection groups on spheres and affine Coxeter groups can be realised as groups generated by affine reflections in $\mathbb{R}^n$ (with plane of reflection not necessarily passing through the origin). Note that the realisation of a Coxeter group as a group generated by reflections is not unique and that some Coxeter groups cannot be realised as a subgroup of $GL_n(\mathbb{R})$.

Definition 1.3. Given a finite set S, let $m\colon S \times S \to \mathbb{N} \cup \{\infty\}$ be a symmetric matrix indexed over S such that $m(s, s) = 1$ for all $s \in S$ and $m(s, t)$ takes values in $\{2, 3, \dots\} \cup \{\infty\}$ for all $s \neq t$. The *Coxeter group* associated to m is the group with the following presentation.

$$W = \langle S \mid (st)^{m(s,t)} = 1 \quad \forall m(s, t) \neq \infty \rangle$$

The data of m and S along with the associated Coxeter group W is denoted (W, S) and called a *Coxeter system.*

The pairs (s, t) such that $m(s, t) = \infty$ are airs of generators that have no explicit relations. Since $m(s, s) = 1$ for all $s \in S$, all generators have order 2. Note that the data of S and m is not uniquely determined by the isomorphism class of W, hence the need to distinguish a Coxeter system, not just a Coxeter group. The set $R := \{wsw^{-1} \mid w \in W, s \in S\}$ is the set of *reflections* in W. Sometimes S is referred to as the set of *basic reflections*, or that a choice of S is a choice of basic reflections.

A labelled graph, called the *Coxeter diagram*, is often used to encode the data of the matrix m and its corresponding Coxeter group. In this graph, each element of S is a node and relations between pairs in S correspond to labelled edges. There are two conventions for this labelling: The *classical labelling*, where edges with $m(s,t) = 2$ are not drawn, edges with $m(s,t) = 3$ are drawn but not labelled and all other edges are drawn with the value of $m(s,t)$ as their label. And the *modern labelling*, where edges with $m(s,t) = \infty$ are not drawn, edges with $m(s,t) = 2$ are drawn but not labelled and all other edges are drawn and labelled. An example highlighting these different conventions is given in Fig. 1. We will only use the classical labelling here, but awareness of the modern labelling is useful.

Figure 1: Coxeter diagram for a certain Coxeter group with classical labelling (left) and modern labelling (right).

In the classical labelling, if the diagram has multiple connected components then W is a direct product of the groups corresponding to those components. Similarly, in the modern labelling, connected components are factors in a free product. Other topological properties of these diagrams can be used, for example, in work of Huang [Hua23] which proves the $K(\pi, 1)$ conjecture for certain W with diagrams being trees or containing cycles. The property of Coxeter groups that allows us to make this graph construction is that every relation in a Coxeter group only involves two generators and is encoded by a number.

To each Coxeter system (W, S) there is an associated Artin group G_W defined as follows.

Definition 1.4. For group elements s and t, let $\Pi(s, t\,;n)$ be the alternating product of s and t starting with s with total length n, e.g. $\Pi(s, t\,;3) = sts$. Given a Coxeter system (W, S) with associated matrix m, the associated *Artin group* is

$$G_W := \langle S \mid \Pi(s, t\,; m(s,t)) = \Pi(t, s\,; m(s,t)) \;\; \forall s \neq t \text{ and } m(s,t) \neq \infty \rangle.$$

Note that $m(s,s) = 1$ now carries no meaning in the presentation of G_W and that if we add the relations $s^2 = 1$ for all $s \in S$ we retrieve the original Coxeter group. The Coxeter diagram for W also encodes the data of G_W and the connected components of the diagram correspond to factors of G_W as a direct product or as a free product as with W.

Our notation for Artin groups (as with much of the notation here) is from [PS21]. Another common notation is W_Γ and A_Γ for the Coxeter and Artin groups corresponding to the Coxeter diagram Γ. When classifying Artin groups, it is common to inherit properties from the corresponding Coxeter group such that "`property` (type) Artin groups" describes a family of Artin groups to which their corresponding Coxeter groups are `property`.

In particular, spherical or finite type Artin groups have associated spherical or finite Coxeter groups. Similarly, affine Artin groups have associated Coxeter groups which are affine.

1.4 Configuration space

This section contains the definition of the configuration space Y_W for a given Coxeter group W. We also introduce the A_n family of Coxeter groups for which the space Y_W is the space of configurations of $n+1$ labelled points in $\mathbb{C}$.

For some finite or affine Coxeter group W acting on $\mathbb{R}^n$, the set of reflections $R \in W$ acts on $\mathbb{R}^n$ by reflection through hyperplanes, one for each $r \in R$. For some $r \in R$, denote its hyperplane by $H(r) \subseteq \mathbb{R}^n$. Denote the union of all hyperplanes by $\mathcal{H} := \bigcup_{r \in R} H(r)$. Consider the tensor product $\mathbb{R}^n \otimes \mathbb{C}$. This is isomorphic to $\mathbb{C}^n$ under the natural isomorphism $x \otimes \lambda \mapsto x\lambda$. We can extend the action $W \circlearrowleft \mathbb{R}^n$ to $W \circlearrowleft (\mathbb{R}^n \otimes \mathbb{C})$ via $w \cdot (x \otimes \lambda) = (w \cdot x) \otimes \lambda$. We call this act of transporting objects related to $\mathbb{R}^n$ over to $\mathbb{R}^n \times \mathbb{C}$ (which is isomorphic to $\mathbb{C}^n$) via the tensor product *complexification.*

Remark 1.5. Since R generates W, the action of $W \circlearrowleft \mathbb{R}^n$ fixes $\mathcal{H}$ and $w \cdot x \in \mathcal{H} \implies w \in \mathcal{H}$.

Definition 1.6. For an affine Coxeter group W and associated hyperplane system $\mathcal{H}$ as above, we define

$$Y := (\mathbb{R}^n \otimes \mathbb{C}) \setminus (\mathcal{H} \otimes \mathbb{C})$$

and define the *configuration space* Y_W to be the quotient Y/W with the action of W defined above. This action is well–defined by Remark 1.5.

Note that the importance of $\mathbb{C}$ is that it is 2 dimensional. When one takes the complement of a co-dimension 1 object, you typically will not get any interesting topology. By complexifying the hyperplanes and then taking the complement within $\mathbb{R}^n \otimes \mathbb{C}$, we are effectively taking the complement of a codimension–2 object, and there is much more room for interesting topologies. The same construction can be achieved using $\mathbb{R}^{2n}$ and

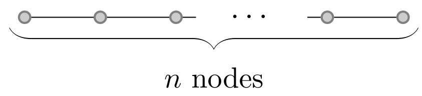


Figure 2: The classical Coxeter diagram for the Coxeter group of type A_n.

$\mathcal{H} \times \mathcal{H}$. A more general construction of Y_W for all Coxeter groups using the *Tits cone* can be found in [Par14]. We will not go in to the details of this construction, but will assume Y_W to be defined for all Coxeter groups W.

For a concrete example concerning Y_W, we will introduce the A_n family of Coxeter groups and show that the space Y_W for these groups is the space of configurations of $n+1$ points in $\mathbb{C}$, thus explaining the name *configuration space* for general Y_W.

The family A_n all have Coxeter diagrams of the form as in Fig. 2 and a specific A_n will have presentation.

$$A_n = \left\langle \sigma_1, \sigma_2, \ldots, \sigma_n \;\middle|\; \begin{array}{rl} \sigma_i^2 = 1 & \forall i \\ (\sigma_i \sigma_j)^2 = 1 & \forall (i+1 < j \leq n) \\ (\sigma_i \sigma_{i+1})^3 = 1 & \forall (i < n) \end{array} \right\rangle \tag{2}$$

This is well known to be a presentation for the symmetric group S_{n+1} with generators being adjacent transpositions [BB10, Proposition 1.5.4]. Accordingly, we will use the associated cycle notation for symmetric groups to talk about elements of A_n.

The action of A_n as a reflection group is realised on the space $\mathbb{R}^{n+1}$ with basis $\{e_i\}$, where $A_n \circlearrowright \mathbb{R}^{n+1}$ by permuting components with respect to that basis. The set of reflections R of A_n is all conjugations of the n adjacent generating transpositions $(l, l+1)$. So, R is the set of all transpositions (l, k). Some $(l, k) \in R$ acts on $\mathbb{R}^{n+1}$ as reflection through the plane $\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_l = x_k\}$. Thus, taking the complement of the complexification of all such planes, we have $Y = \{(\mu_1, \ldots \mu_{n+1}) \in \mathbb{C}^{n+1} \mid \forall i, j \;\; \mu_i \neq \mu_j\}$ (here Y is as in Definition 1.6). We can think of this as the space of $n+1$ distinct, labelled points in $\mathbb{C}$, denoted $\mathrm{Conf}_{n+1}(\mathbb{C})$.

Historically, Artin [Art47] originally defined the braid group on n strands B_n to be $\pi_1(\mathrm{Conf}_n(\mathbb{R}^2))$. He then proved the well known presentation of the braid group.

$$B_n = \left\langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{rl} \sigma_i \sigma_j = \sigma_j \sigma_i & \forall (i+1 < j \leq n) \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & \forall (i < n) \end{array} \right\rangle$$

In this context, showing the validity of the presentation immediately proves $B_{n+1} \cong$

G_W and thus that $\pi_1(Y_W) \cong G_W$. See the work of Fox and Neuwirth [FN62b] for an alternative proof of the presentation.

1.5 The $K(\pi, 1)$ conjecture

Given a group G and natural number n, an *Eilenberg-MacLane space* [EM45] is a space X such that $\pi_n(X) = G$ and $\pi_i(X) = 0$ for all $1 \leq i \neq n$. We call such an X a *$K(G, n)$ space.* We will also use the terminology *classifying space for G* to mean that X is a $K(G, 1)$ space.

Conjecture 1.7 ($K(\pi, 1)$ Conjecture)**.** *For all Coxeter groups W, the space Y_W is a $K(G_W, 1)$ space.*

Admittedly, the use of π in the name of the conjecture is confusing. An equivalent formulation of the conjecture is that the universal cover of Y_W is contractible. These statements are equivalent since a cover $p\colon \widetilde{X} \to X$, induces an isomorphism $p_*\colon \pi_n(\widetilde{X}, \widetilde{x_0}) \to \pi_n(X, x_0)$ for all $n \geq 2$ [Hat01, Proposition 4.1].

In the previous section we focused on Coxeter groups of type A_n. We quoted the result that $\pi_1(Y_W) \cong G_W$ for these groups. We now prove the $K(\pi, 1)$ conjecture for this same family of Coxeter groups. To do so, we need to verify that the higher homotopy groups of Y_W are trivial.

Lemma 1.8. *For all $k > 1$, $\pi_k(\mathrm{Conf}_n(\mathbb{C}))$ is trivial.*

Proof. We use that $\mathrm{Conf}_n(\mathbb{C})$ is a fibre bundle over $\mathrm{Conf}_{n-1}(\mathbb{C})$ with projection p forgetting a point and fibres homeomorphic to $\mathbb{C} \setminus \{\text{n distinct points}\}$ [FN62a, Theorem 3].

The space $\mathbb{C} \setminus \{\text{n distinct points}\}$ is homotopy equivalent to $\bigvee_n S^1$, so we have the fibration $\bigvee_n S^1 \hookrightarrow \mathrm{Conf}_n(\mathbb{C}) \to \mathrm{Conf}_{n-1}(\mathbb{C})$ and the corresponding short exact sequence

$$\pi_k(\bigvee_{n-1} S^1) \longhookrightarrow \pi_k(\mathrm{Conf}_n(\mathbb{C})) \xrightarrow{\ p\ } \pi_k(\mathrm{Conf}_{n-1}(\mathbb{C})) \tag{3}$$

for all k. We prove that $\pi_k(\mathrm{Conf}_n \mathbb{C}) = 0$ for all k by induction on n. We know $\pi_k(\bigvee_n S^1) = 0$ for all $k > 1$ and for all n. So the leftmost term in (3) is always trivial. Our base case is $n = 2$. The rightmost term in (3) is $\mathrm{Conf}_1(\mathbb{C}) \cong \mathbb{C}$, which has trivial higher homotopy. The leftmost term is $\bigvee_1 S^1 \cong S^1$ which also has trivial higher homotopy. Thus, $\mathrm{Conf}_2(\mathbb{C})$ has trivial higher homotopy. Assuming now that $\pi_k(\mathrm{Conf}_{n-1}(\mathbb{C})) = 0$ for all $k > 1$, the inductive step follows immediately from the short exact sequence. $\square$

Note that so far we have only proved that Y (from Definition 1.6) has trivial higher homotopy, not Y_W.

Theorem 1.9. *The $K(\pi, 1)$ conjecture holds for Coxeter groups of type A_n.*

Proof. The action $A_n \circlearrowleft Y$ is by permutation of points, therefore Y_W is the space of configurations of $n+1$ *unlabelled* points. Let $q\colon \widetilde{Y} \to Y$ be the universal cover for Y. Let $r\colon Y \to Y_W$ be the quotient map induced by the group action. We see that r is a covering map such that $r^{-1}(z)$ is finite for all $z \in Y_W$. Thus, by [Mun00, Exercise 53.4] ○● Unfortunately I couldn't find a better (non-exercise) reference for this. The proof is OK, but not that short and a bit of a detour in my opinion ●○ the composition $r \circ q$ is also a covering map so $\widetilde{Y}$ is also the universal cover of Y_W and so Y_W also has trivial higher homotopy by Lemma 1.8. $\square$

A vital result due to Deligne expands on this result.

Theorem 1.10 ([Del72])**.** *The $K(\pi, 1)$ conjecture holds for all finite Coxeter groups W.*

The paper of interest to us, [PS21], proves the $K(\pi, 1)$ conjecture for affine Coxeter groups.

Chapter 2

Geometric realisations of poset structures

In Section 1.4 we used the realisation of the Coxeter group W as a reflection group on an affine space V. We considered the planes of the defining reflections of W as affine subspaces of V and used these to define Y_W, the configuration space. To progress with the $K(\pi, 1)$ conjecture we must develop some theory to explore the homotopic properties of Y_W, To do this, we will first construct a new space X_W, the *Salvetti complex*, which is homotopy equivalent to Y_W. This was originally defined [Sal87, Sal94] similarly using the realisation of W on a space. However, the Salvetti complex turns out to have another formulation based on algebraic properties of W and its defining relations. We will relate the Salvetti complex to another structure, K_W which is described by a partial order on W. With this in mind, we start with some definitions.

2.1 Posets

A partially ordered set or *poset* $(P, \leq)$ is a set P with a relation $\leq$ on pairs in P which encodes the topology of $\mathbb{R}$. The textbook [Grä11] provides a good introduction. An important note is that there is no requirement for every pair to be related, hence the name *partial order*. We will use P as shorthand for $(P, \leq)$ where possible.

In a poset P we define the *interval* between two elements $[x, y]$ as $[x, y] \coloneqq \{u \in P \mid x \leq u \leq y\}$, which is itself a poset. For convenience, we define $[-\infty, w] \coloneqq \{u \in P \mid u \leq w\}$ and equivalently for $[w, \infty]$. A *chain* is a subset $C \subseteq P$ that is a totally ordered, i.e. every pair in $(u, v) \in C \times C$ satisfies $u \leq v$ or $v \leq u$. The *covering relations* of P, denoted $\mathcal{E}(P)$

are defined as follows.

$$\mathcal{E}(P) = \{(x, y) \in P \times P \mid x \leq y \text{ and } [x, y] = \{x, y\}\}$$

These are strictly ordered pairs $x < y$ such that there does not exist any $z \in P$ such that $x < z < y$. If $(x, y) \in \mathcal{E}(P)$, we write $x \lessdot y$. We will call a chain C *saturated* if for all $x, y \in C$ such that $x < y$, there exists $z \in C$ such that $x \lessdot z$, i.e. there are no 'gaps' in the chain.

By transitivity, the covering relations encode the whole poset structure, which can in turn be drawn in a diagram.

Definition 2.1. Given a poset P, the *Hasse Diagram* is the directed graph encoding $\mathcal{E}(P)$ in the following way: For each element $x \in P$ draw a vertex. For each pair $(x, y) \in \mathcal{E}(P)$ draw a directed edge from x to y.

As is typical, we will draw Hasse diagrams such that for each edge $x \lessdot y$, the vertex x will be at a lower position on the page than y. Thus, we will not need to draw arrows to show direction. In this work, we will only with deal with graded and bounded posets. *Bounded* means that there are minimal and maximal elements, denoted $\hat{0}$ and $\hat{1}$ such that $\hat{0} \leq x \leq \hat{1}$ for all $x \in P$, and *graded* means that every saturated chain from $\hat{0}$ to $\hat{1}$ has the same (finite) length. In the Hasse diagram for a bounded, graded poset, we will draw $\hat{0}$ at the bottom, $\hat{1}$ at the top, and put all other elements in discrete vertical levels between these based on the position in the saturated chains between $\hat{0}$ and $\hat{1}$ where each element occurs. See Fig. 7 for an example. Graded posets have a natural notion of a *rank function* $\mathrm{rk}\colon P \to \mathbb{N}$ that encodes the height above $\hat{0}$ at which an element $p \in P$ occurs in the Hasse diagram. Rank is also well–defined for posets with multiple minimal or maximal elements, so long as all saturated chains from any minimal element to any maximal element have the same length.

Definition 2.2. We define an *edge labelled poset* to be a triple $(P, \leq, l)$ where $(P, \leq)$ is a poset and the function $l : \mathcal{E}(P) \to A$ is a labelling of covering relations with alphabet A.

We will use P as a shorthand for $(P, \leq, l)$ where possible. Given an edge labelled poset P, we can construct a group encoded by its labelling the and geometry of its Hasse diagram.

The *word corresponding to a saturated chain* is the word of the labels traversed in the Hasse diagram while tracing out that saturated chain.

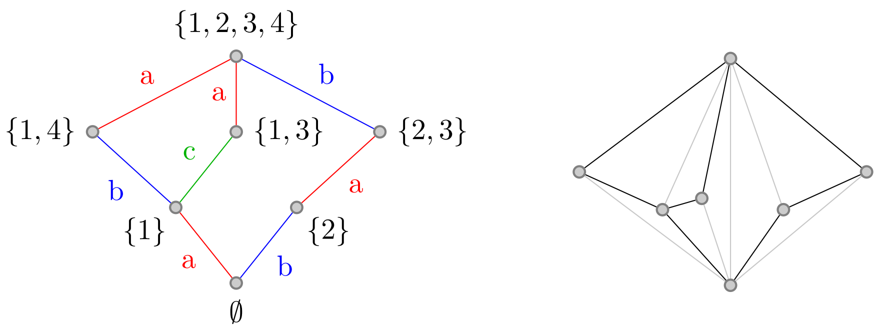


Figure 3: A simple example of a bounded and graded edge labelled poset where we have taken $\leq$ to be $\subseteq$ and $A = \{a, b, c\}$ (left). The same poset with all non-covering 1–chains drawn in faint lines to aid visualising $\Delta(P)$ introduced in Section 2.2 (right).

Definition 2.3. Given some edge labelled poset $(P, \leq, l\colon \mathcal{E}(P) \to A)$, define the *poset group* $G(P)$ to be the group generated by $\mathrm{Im}(l)$ with relations equating words corresponding to saturated chains in the Hasse diagram of P which start and end at the same vertices.

In the example given in Fig. 3, the poset group is $G(P) = \langle a, b, c \mid aba = bab,\, ba = ca\rangle$.

2.2 Poset complex

For some edge labelled poset P, we can construct a cell complex $K(P)$ from P such that $\pi_1(K(P))$ is $G(P)$. First we make some definitions. An *abstract simplicial complex* is a family of sets that is closed under taking arbitrary subsets.

Definition 2.4. Given a finite abstract simplicial complex X, the *geometric realisation* of that simplicial complex is defined as follows: For each single element set in X assign a point. For each two element set assign an open edge between the two vertices it contains. For each three element set assign an open triangle, the interior of the three edges of its three subsets of size two. In this way, continue constructing simplices of dimension n for each $n + 1$ size set in X.

The set of all chains in a poset P is an abstract simplicial complex. We define $\Delta(P)$ to be the geometric simplicial complex corresponding to the set of all chains in P where each n–simplex is an n–chain of P. Note that as in [MS17, Definition 1.7], we define an n–chain to have $n - 1$ elements, e.g. $(\{1\} \subseteq \{1, 2\})$ is a 1–chain.

For example, in Fig. 3, $\Delta(P)$ would be three 3–simplices all sharing an edge (a 1–simplex) corresponding to the 1–chain $(\emptyset \subseteq \{1, 2, 3, 4\})$. Two of the 3–simplices would share a face corresponding to the 2–chain ($\emptyset \subseteq \{1\} \subseteq \{1,2,3,4\}$). We also assign an orientation on edges in $\Delta(P)$ such that the edge corresponding to the 1–chain ($x \leq y$) points from x to y. For a two–dimensional example, consider the following poset P and corresponding $\Delta(P)$. Here we forget about edge labelling in P for a moment.

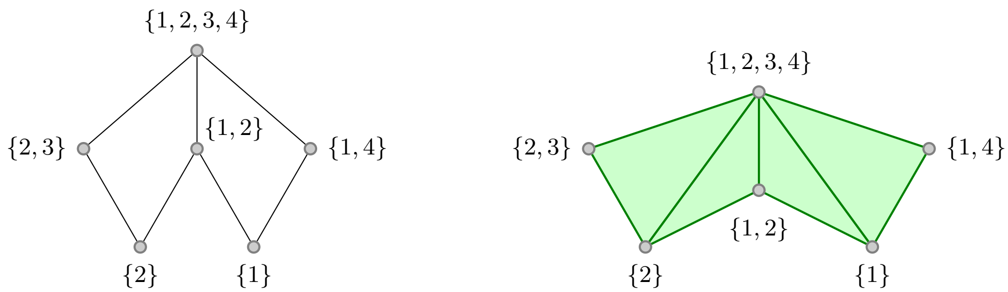


Figure 4: An example poset P (left) and corresponding $\Delta(P)$ (right).

We continue, now using an edge labelling on P (Fig. 5 (left)).

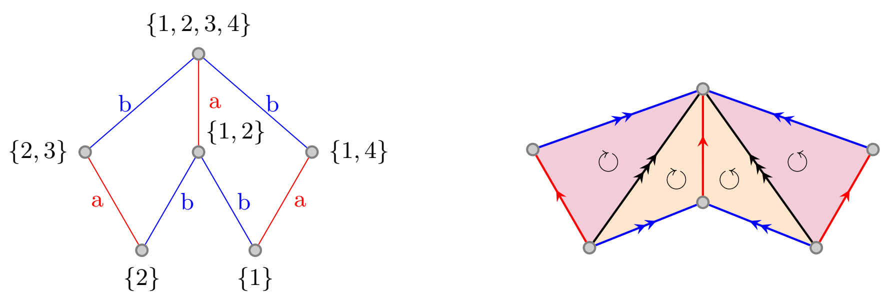


Figure 5: The poset in Fig. 4 with edge labelling (left) and the corresponding space $K(P)$ (right).

To construct $K(P)$, first we define a labelling on 1–chains in P which extends the edge labelling in P. Let A^* denote the set of all words corresponding to the alphabet A. ○● In the last draft I got this completely wrong. The extended labelling only extends to 1–chains. Not n–chains. This is important for how we identify n–simplices in $K(P)$, which has also changed since the last draft. ●○

Definition 2.5. Given some edge–labelled poset $(P, \leq, l\colon \mathcal{E}(P) \to A)$ and some 1–chain $(x \leq y)$ in P, the *extended label* $\mathcal{L}(\sigma) \subseteq A^*$ is the language of all words corresponding to all saturated chains that start at x and end at y.

For example, consider the chain $(\{2\} \subseteq \{1,2,3,4\})$ in the context of Fig. 5 (left). There are two corresponding saturated chains, $(\{2\} \subseteq \{1,2\} \subseteq \{1,2,3,4\})$ and $(\{2\} \subseteq \{2,3\} \subseteq \{1,2,3,4\})$, which respectively correspond to the words ba and ab. So $\mathcal{L}(\{2\} \subseteq \{1,2,3,4\}) = \{ba, ab\}$. Here are some illustrative examples:

- $\mathcal{L}(\{1\} \subseteq \{1,2\}) = \mathcal{L}(\{2\} \subseteq \{1,2\}) = \{b\}$.
- $\mathcal{L}(\{1\} \subseteq \{1,2,3,4\}) = \mathcal{L}(\{2\} \subseteq \{1,2,3,4\}) = \{ba, ab\}$.
- $\mathcal{L}(\{1\}) = \mathcal{L}(\{2\}) = \mathcal{L}(\{1,2\}) = \cdots = \emptyset$.

We use this extended labelling to form a quotient space of $\Delta(P)$. Recall the orientation of the edge $(x \leq y)$ is from x to y. Consider two closed n–simplices e^n_α and e^n_β in $\Delta(P)$. We form a relation $\sim$ on n–simplices such that $e^n_\alpha \sim e^n_\beta$ iff there exists a continuous edge–orientation and extended label preserving map $f\colon e^n_\alpha \to e^n_\beta$.

Definition 2.6 (Poset Complex [Mcc05, Definition 1.6])**.** Given some finite height edge–labelled poset P, the poset complex $K(P)$ is the quotient space $\Delta(P)/\approx$ where $\approx$ identifies all n–simplices (for any n) using the map $f\colon e^n_\alpha \to e^n_\beta$ corresponding to $\sim$ defined above.

In the example in Fig. 5, three red edges are identified, four blue edges are identified, two black edges are identified, two orange triangles are identified and two purple triangles are identified. The orientation of the triangles is indicated by a $\circlearrowleft$ symbol.

We see that this space is homeomorphic to a torus, which has fundamental group $\mathbb{Z}^2 \cong \langle a, b \mid ab = ba \rangle$, which is also the $G(P)$ for this edge–labelled poset. This fact holds in general, which we now begin to prove.

Lemma 2.7. *Given an edge labelled poset $(P, \leq, l\colon \mathcal{E}(P) \to A)$, the group $\pi_1(K(P))$ is generated by a set of loops in bijection with* $\mathrm{Im}(l)$.

Proof. Each vertex $p \in \Delta(P)$ has extended label $\emptyset$ and so there is only one vertex in $K(P)$, denote this point p_0. The 1–skeleton of $K(P)$ will be a wedge of circles, one for each extended label in $\{\mathcal{L}(C) \mid C \text{ is a 1–chain}\}$. The following is the setup of our proof.

1. Let Σ denote 1–chains in P.
2. Let Ω denote paths between vertices in $\Delta(P)$ along 1–simplices in $\Delta(P)$ such that the direction along the edge corresponding to $(x \leq y)$ is from x to y.
3. Let Λ denote loops along 1–simplices in $K(P)$ that start and end at p_0. If $\lambda \in \Lambda$, then $[\lambda] \in \pi_1(K(P), p_0)$.

Let α, be the map from 1–chains to corresponding paths in $\Delta(P)$ and $\beta\colon \Delta(P) \to K(P)$ be the quotient map as in the definition of $K(P)$. We have the following diagram.

$$\Sigma \xrightarrow{\alpha} \Omega \xrightarrow{\beta} \Lambda$$

By our remarks on the 1–skeleton of $K(P)$, we see that $\pi_1(K(P), p_0)$ is generated by the homotopy classes of $\mathrm{Im}(\beta \circ \alpha)$. Let $\sigma = (x < y) \in \Sigma$ be a 1–chain and let λ be such that $\beta(\alpha(\sigma)) = \lambda$. Let concatenation of loops λ and λ' be denoted $\lambda\lambda'$ such that $[\lambda][\lambda'] = [\lambda\lambda']$. We will show that λ has a factorisation where one of the factors is $\beta(\alpha(x \lessdot z))$ for some $z \in P$.

There exists $z \in P$ such that $x \lessdot z \leq y$. If $z \neq y$, there is a 2–simplex in $\Delta(P)$ corresponding to the 2–chain $(x \lessdot z < y)$. One of the edges of this 2–simplex corresponds to the 1–chain $(x \lessdot z)$. The path $\alpha(x \leq y)$ is homotopic (through the 2–simplex $(x \lessdot z < y)$) to the path $\alpha(x \lessdot z)\alpha(z < y)$. Let H witness this homotopy. We have that $\beta \circ H$ is well–defined (and continuous), so the loop λ is homotopic to the loop $\beta(\alpha(x \lessdot z)\alpha(z < y)) = \beta(\alpha(x \lessdot z))\beta(\alpha(z < y))$. We repeat the process replacing σ with $(z < y)$. Eventually this must stop since our poset is of finite height. After this, we achieve a factorisation of λ, entirely in factors of the form $\beta(\alpha(r \lessdot s))$.

Consider $\mathcal{E}(P)$ as a subset of all 1–chains in P. We see that $\pi_1(K(P), p_0)$ is generated by the homotopy classes of loops in $\mathrm{Im}(\beta \circ \alpha|_{\mathcal{E}(P)})$. Each covering relation $(r \lessdot s)$ has extended label $\{l(r \lessdot s)\}$, therefore $\mathrm{Im}(\beta \circ \alpha|_{\mathcal{E}(P)})$ is in bijection with $\mathrm{Im}(l)$. $\square$

Lemma 2.8. *For an edge-labelled poset P, there exists a surjective homomorphism $\varphi \colon G(P) \to \pi_1(K(P), p_0)$ where p_0 is as in the previous lemma.*

Proof. Let us follow from the notation in the proof of Lemma 2.7 and let $\theta \colon \mathrm{Im}(l) \to \pi_1(K(P), p_0)$ be the bijection at the end of that proof such that we have $\langle\, \mathrm{Im}(\theta \circ l)\, \rangle = \pi_1(K(P), p_0)$.

Let $\sigma = (x_1 \lessdot \cdots \lessdot x_i)$ and $\sigma' = (x'_1 \lessdot \cdots \lessdot x'_j)$ be two saturated chains such that $x_1 = x'_1$ and $x_i = x'_j$ with corresponding words $w = w_1 \cdots w_{i-1}$, and $w' = w'_1 \cdots w'_{j-1}$ in A^* such that $l(x_k \lessdot x_{k+1}) = w_k$ and similarly for σ'. Recall that w and w' are words that are identified by the relations in the defining presentation for $G(P)$. We want to show there exists a homotopy between the loop $\theta(w_1) \cdots \theta(w_i)$ and the loop $\theta(w'_1) \cdots \theta(w'_j)$.

By doing the process in the proof of Lemma 2.7 in reverse, we get a homotopy between $\theta(w_1) \cdots \theta(w_i)$ and $\beta(\alpha(x_1 \leq x_i))$ through the two skeleton of $K(P)$. By the same argument, we get a homotopy between $\theta(w'_1) \cdots \theta(w'_j)$ and $\beta(\alpha(x'_1 \leq x'_j))$. Since $x_1 = x'_1$ and $x_i = x'_j$, we have $\theta(w_1) \cdots \theta(w_i) \sim \theta(w'_1) \cdots \theta(w'_j)$.

We have shown that $\pi_1(K(P), p_0)$ has all necessary relations to extend θ to a surjective homomorphism $\varphi\colon G(P) \to \pi_1(K(P), p_0)$. □

To finish our proof, it is necessary to provide an inverse to φ, which we now complete

Theorem 2.9. *Given an edge labelled poset $(P, \leq, l\colon \mathcal{E}(P) \to A)$, we have $\pi_1(K(P)) \cong G(P)$.*

Proof. We again follow the notation from Lemma 2.7. In the proof of Lemma 2.7 we remarked that $\pi_1(K(P), p_0)$ is generated by a set of loops in bijection with

$$\mathcal{L}(\Sigma) := \{\mathcal{L}(C) \mid C \text{ is a 1–chain}\}.$$

Let $\chi\colon \mathcal{L}(\Sigma) \to \pi_1(K(P), p_0)$ denote this bijection. We now think of this presentation abstractly, and work to give a set of relations R such that we obtain a group $\langle \mathcal{L}(\Sigma) \mid R \rangle \cong \pi_1(K(P), p_0)$ where the isomorphism is an extension of χ.

By the structure of the 1–skeleton of $K(P)$ remarked in the proof of Lemma 2.7, there is a set of relations R in bijection with the set of 2–simplices in $K(P)$. Let e^2 be a 2–simplex with edges e^1_i for $i \in \{1, 2, 3\}$. We have that the edges e^1_i are oriented acyclically as in Fig. 6.

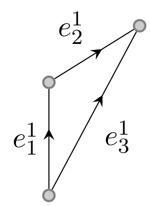


Figure 6: The orientation of all simplices in $\Delta(P)$ and $K(P)$.

All simplices are oriented this way because if $x \leq y$ and $y \leq z$ then $x \leq z$. Let the attaching map for each e^1_i trace the loop $\lambda_i \in \Lambda$ and choose extended labels ℓ_i such that $\chi(\ell_i) = \lambda_i$. We see that the 2–simplex e^2 corresponds to the relation $\ell_1\ell_2\ell_3^{-1} = 0$. Here multiplication of extended labels is formal. As are inverses, denoted ℓ^{-1}. Define R to be the following set of relations for $\langle\, \mathcal{L}(\Sigma)\, \rangle$.

$$R := \left\{ \ell_1\ell_2\ell_3^{-1} = 0 \;\middle|\; \exists \text{ 2–chain } x < y < z \text{ where } \begin{array}{l} \mathcal{L}(x < y) = \ell_1 \\ \mathcal{L}(y < z) = \ell_2 \\ \mathcal{L}(x < z) = \ell_3 \end{array} \right\}$$

Following from the definition of $K(P)$, these are exactly the relations we need to extend χ to an isomorphism witnessing $\langle \mathcal{L}(\Sigma) \mid R \rangle \cong \pi_1(K(P), p_0)$.

We now make some geometric observations about R. To each relation $r \in R$, there is a corresponding 2–simplex $e^2(r) \subseteq \Delta(P)$. Let $\omega_i \in \Omega$ be the paths going along the edges e_i^1 in either cyclic direction around $e^2(r)$ and choose ℓ_i such that $\chi(\ell_i) = \beta \circ \omega_i \in \pi_1(K(P), p_0)$. The two sides of any equation resulting from the relation r correspond to two paths along the edges of $e^2(r)$ that start and end at the same point e.g. $\ell_1 \ell_2 = \ell_3$, $\ell_1 = \ell_3 \ell_2^{-1}$, $\ell_1 \ell_2 \ell_3^{-1} = 0$, where 0 corresponds to the trivial path.

This also holds for repeated application of any number of relations in R. Suppose we are given some $x_i, y_i \in \mathcal{L}(\Sigma)$ such that we have the equation $x_1 x_2 \cdots x_m = y_1 y_2 \cdots y_n$ in $\langle \mathcal{L}(\Sigma) \mid R \rangle$. We see there must be two edge paths $s = s_1, s_2, \ldots, s_{m-1}$ and $t = t_1, t_2, \ldots, t_{n-1}$ where s_i, t_i are vertices in $\Delta(P)$ such that $\mathcal{L}(s_i \leq s_{i+1}) = x_i$ and $\mathcal{L}(t_i \leq t_{i+1}) = y_i$ with $s_1 = t_1$ and $s_m = t_n$.

Recall that $\pi_1(K(P), p_0)$ is generated by loops corresponding to covering relations, so the subgroup $\langle \mathcal{L}(\mathcal{E}(P)) \rangle$ is all of $\langle\, \mathcal{L}(\Sigma) \mid R \,\rangle$. Now consider an equation as above where all the x_i and y_i are single set labels $\{a\} \in \mathcal{L}(\mathcal{E}(P))$ for some $a \in A$. The paths s and t must consist of covering relations, i.e. $s_i \lessdot s_{i+1}$ and $t_i \lessdot t_{i+1}$ for all i. Edges in $\Delta(P)$ of the form $(r \lessdot s)$ correspond to edges in the Hasse diagram of P. So such an edge path in $\Delta(P)$ where all edges are covering relations corresponds to a path in the Hasse diagram of P. Therefore, any equation relating formal words in $\mathcal{L}(\mathcal{E}(P))$ corresponds to two paths in the Hasse diagram of P that start and end in the same point.

Let $\psi \colon \mathcal{L}(\mathcal{E}(P)) \to G(P)$ act as $\psi(\{a\}) = a$. By the previous remarks, we can extend ψ to a homomorphism $\psi \colon \langle\, \mathcal{L}(\mathcal{E}(P)) \,\rangle \to K(P)$ and $\psi \circ \chi^{-1}$ is inverse to φ from Lemma 2.8. $\square$

Note that our proof of the above was ambivalent to the identification of n–simplices for $n \geq 2$. Indeed, these identifications do not affect the fundamental group, but they do ensure that higher homotopy groups are trivial. We can see if we did not identify the 2–simplices in Fig. 5, $\pi_2(K(P))$ would be non-trivial.

2.3 Interval Complex

Starting from a Coxeter group W generated by S, we wish to give W a labelled–poset structure and use the constructions from the previous section. The edge labelled Hasse diagram for W will embed in to the Cayley graph $\mathrm{Cay}(W, S)$, and it is useful to be able to swap between these two objects, as we will do. First, given a group G, we must define a partial order on G.

Definition 2.10. For a group G generated by S, the word length with respect to S is the function $l_S : G \to \mathbb{Z}$ where

$$l_S(g) \coloneqq \min\{k \mid s_1 s_2 \ldots s_k = g\, ,\, s_i \in S \cup S^{-1}\}.$$

We will often omit the S in l_S where it is obvious from context.

Definition 2.11. For a group G generated by S, we define the partial order $x \leq y \iff l(x) + l(x^{-1}y) = l(y)$.

It can be readily checked that this does indeed define a partial order on G. This order encodes closeness to $e \in G$ along geodesics in $\mathrm{Cay}(G, S)$. We have $x \leq y$ precisely when there exists a geodesic in $\mathrm{Cay}(G, S)$ from e to y with x as an intermediate vertex, or to put it another way, when a minimal factorisation of x in to elements of S is a prefix of a minimal factorisation of y. The covering relations for this partial order are of the form $(g \lessdot gs)$ where $g \in G$ and $s \in S$. For some $w \in W$ we define the poset $[1, w]^W$ to be the interval in W up to w with respect to this order. We give this poset an edge labelling such that the edge between w and ws is labelled s for some $s \in S$. The Hasse diagram thus embeds in to $\mathrm{Cay}(W, S)$.

Definition 2.12 (Coxeter element)**.** For some Coxeter group W generated by S, we define a *Coxeter element* $w \in W$ to be any product of all the elements of S without repetition.

These Coxeter elements are what we will use as the upper bound of our interval. We will also need to consider W as the group generated by R, the set of all reflections, rather than just the set of simple reflections S. See Fig. 7 for an example of such a poset. In principle there are many choices of Coxeter element depending on what order we multiply the elements of S. However, we will see in Theorem 2.16 that some structures resulting from $[1, w]^W$ are independent of that choice.

We apply the steps from Section 2.2 to $[1, w]^W$ to form a space.

Definition 2.13 (Interval Complex)**.** For a Coxeter group W generated by all reflections R with Coxeter element $w \in W$, we call $K_W \coloneqq K([1, w]^W)$ the *interval complex.*

If W is infinite, then R is infinite and so K_W may have an infinite number of cells. We will later show that K_W deformation retracts to a finite subcomplex. Note that, as in [PS21], we have dropped w from our notation K_W even though it depends on w. This is eventually justified (Theorem 1.1) since the homotopy type of K_W is independent of w.

Certain properties of the poset permit a simplified notation for the simplices within K_W. In this context two n–simplices corresponding to the n–chains $C = (x_1 \leq x_2 \leq \cdots \leq x_n)$ and $C' = (x'_1 \leq x'_2 \leq \cdots \leq x'_n)$ will be identified in K_W exactly when $x_i^{-1}x_{i+1} = (x'_i)^{-1}x'_{i+1}$ for all $1 \leq i < n$. Thus, we can label 1–simplices in K_W with group elements $x \in [1, w]^W$, we can label 2–simplices with factorisations of group elements in $[1, w]^W$ in to two parts (with the first part also in $[1, w]^W$) and so on. As in [PS21, Definition 2.8], we denote an n–simplex corresponding to the n–chain $1 \leq x_1 \leq x_1x_2 \leq \cdots \leq x_1x_2 \cdots x_n$ as $[x_1|x_2| \cdots |x_n]$. There are multiple choices of n–chain corresponding to an n–simplex. The n–chain $1 \leq x_1 \leq x_1x_2 \leq \cdots \leq x_1x_2 \cdots x_n$ where the first element is 1 is called the *canonical chain* for $[x_1|x_2| \cdots |x_n]$. This notation also gives the gluing of the faces of $[x_1|x_2| \cdots |x_n]$ in the following way. A codimention 1 face of $[x_1|x_2| \cdots |x_n]$ is a $(n-1)$–subchain of $1 \leq x_1 \leq x_1x_2 \leq \cdots \leq x_1x_2 \ldots x_n$ (which will consist of n elements). There are three ways to obtain such a subchain.

1. Remove the first element of the chain to get $x_1 \leq x_1x_2 \leq \cdots \leq x_1x_2 \ldots x_n$ which has canonical chain $1 \leq x_2 \leq x_2x_3 \leq \cdots \leq x_2x_3 \ldots x_n$ and corresponds to the $(n-1)$–simplex $[x_2|x_3| \cdots |x_n]$.

2. Remove the last element of the chain to get $1 \leq x_1 \leq x_1x_2 \leq \cdots \leq x_1x_2 \ldots x_{n-1}$ which corresponds to the $(n-1)$–simplex $[x_1|x_2| \cdots |x_{n-1}]$.

3. Multiply two adjacent elements x_i and x_{i+1} to get the chain
$$1 \leq \cdots \leq x_1 \ldots x_{i-1} \leq x_1 \ldots x_{i-1}x_ix_{i+1} \leq \cdots \leq x_1 \cdots x_n$$
which corresponds to the $(n-1)$–simplex $[x_1| \cdots |x_ix_{i+1}| \cdots |x_n]$.

So the n–simplex $[x_1|x_2| \cdots |x_n]$ glues to $[x_2|x_3| \cdots |x_n]$, $[x_1|x_2| \cdots |x_{n-1}]$ and $[x_1| \cdots |x_ix_{i+1}| \cdots |x_n]$ for all $i < n$.

The particular poset group intervals $[1, w]^W$ we will consider will be *balanced.* A balanced group interval is such that $x \in [1, w]^W$ iff $l(g^{-1}x) + l(x) = l(g)$, i.e. all minimal factorisation of $x \in [1, w]^W$ also appear as a suffix in a minimal factorisation of w and all suffixes also appear as a prefix.

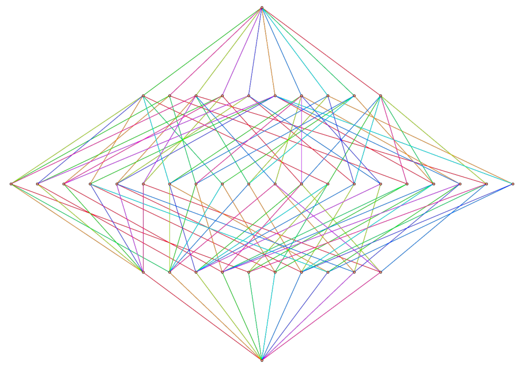


Figure 7: The interval $[1,(1,2)(2,3)(3,4)(4,5)]^{A_4}$ considering A_4 generated by all reflections, which label the edges by colour. Generated using `Sage` and `GAP` [Sag20, GAP22]

Where the interval is balanced, any such symbol $[x_1|x_2|\cdots|x_n]$ corresponds to an n–simplex in K_W given it satisfies the following [PS21, Definition 2.8]:

i) $x_i \neq 1$ for all i.

ii) $x_1x_2\cdots x_n \in [1,w]^W$

iii) $l(x_1x_2\cdots x_n) = l(x_1) + l(x_2) + \cdots + l(x_n)$

Hopefully the first two requirements are obvious. The third is because we require the chain $1 \leq x_1 \leq x_1x_2 \leq \cdots \leq x_1x_2\ldots x_n$ to be contained in $[1,w]^W$ which translates to every subword (without gaps) of $x_1\cdots x_n$ also being in $[1,w]^W$. By (ii) and (iii) there is some y such that $x_1\cdots x_ny = w$ and there is a minimal factorisation of w that respects the factors in $x_1\cdots x_ny$. We can take prefixes of this factorisation (and thus prefixes of $x_1\cdots x_ny$) and stay within $[1,w]^W$. We can use the balanced condition to move the suffix $x_2\cdots x_ny$ to the front, i.e. there exists y_2 such that $x_2\cdots x_nyy_2 = w$ and there is a minimal factorisation of w that respects these factors and conclude $x_2\cdots x_n \in [1,w]^W$. These steps can be repeated to show every subword of $x_1\cdots x_n$ is in $[1,w]^W$.

Definition 2.14. Given a Coxeter group W generated by all reflections R with Coxeter element $w \in W$. Define $[1,w]^W$ as above. The *dual Artin group* W_w is the poset group $G([1,w]^W)$ with G defined as in Definition 2.3.

By Theorem 2.9, the fundamental group of K_W is W_w. Furthermore, for finite W it is a classifying space.

Theorem 2.15 ([PS21, Theorems 2.9 and 2.14])**.** *Given a finite Coxeter group W, the interval complex is a $K(W_w,1)$ space.*

The same theorem for affine type is proved in [PS21] (quoted here as Theorem 1.2). It is also known that in certain cases the dual Artin group is isomorphic to the Artin group.

Theorem 2.16. *Given a finite [Bes03] or affine [MS17] Coxeter group W and Coxeter element $w \in W$, the dual Artin group is isomorphic to the Artin group G_W (and thus it does not depend on the choice of w).*

In general, it is not known whether $G_W \cong W_w$, whether the isomorphism class of W_w depends on w or even if the isomorphism class of $[1, w]^W$ depends on w.

2.4 Salvetti Complex

Here we will define the *Salvetti complex* for a Coxeter group W generated by S, which is a CW–complex homotopy equivalent to Y_W introduced by Salvetti in [Sal94, Sal94]. To define the Salvetti complex, we must first define a notion on subsets of S. For some subset $T \subseteq S$ define the *parabolic subgroup* of W with respect to T, W_T, to be the subgroup of W generated by T with all relations for W containing only elements of T. ○● I agree it is non–trivial that $\langle T\rangle \cong W_T$, but I don't think I have assumed this. Should I include it as a fact? ●○ If Γ is the Coxeter diagram for W, then W_T is the Coxeter group corresponding to the complete (or *full*) subgraph of Γ containing the vertices T. From here we follow [Pao17, Section 2.3], with notation mostly following [PS21].

Definition 2.17. For a Coxeter group W generated by S, define $\mathcal{S}_W$ to be the family of subsets $T \subseteq S$ such that W_T is finite.

For some $T \subseteq S$, we say some $w \in W$ is T–minimal if w is the unique element of minimum length (with respect to S) in the coset wW_T. Uniqueness is shown in [Bou08]. Define an order on the set $W \times \mathcal{S}_W$ by the following: $(u, X) \leq (v, Y)$ iff $X \subseteq Y$, $v^{-1}u \in W_Y$ and $v^{-1}u$ is X–minimal.

Definition 2.18 (Pre-Salvetti complex [Pao17, Definition 2.19])**.** For a Coxeter group W, define $\mathrm{Sal}(W)$ to be $\Delta(W \times \mathcal{S}_W)$ under the order $\leq$ prescribed above, where Δ is as in Definition 2.4.

The Salvetti complex was defined in [Sal94] to be the quotient of a space related to the action of W on a vector space. In [Par14, Theorem 3.3] it is shown that the definition we have given generates a space homeomorphic to that in the original definition by Salvetti. Let us quote some results that help us to interpret the Salvetti complex.

Lemma 2.19 ([Pao17, Lemma 2.18]). *Consider the following objects as geometric simplicial complexes inside* $\mathrm{Sal}(W)$.

$$C(v,Y) := \{(u,X) \in W \times \Delta_W \mid (u,X) \leq (v,Y)\}$$
$$\partial C(v,Y) := \{(u,X) \in W \times \Delta_W \mid (u,X) < (v,Y)\}$$

Let $n = |\,Y\,|$. *There is a homeomorphism* $C(v,Y) \to D^n$ *that restricts to a homeomorphism* $\partial C(v,Y) \to S^{n-1}$.

This allows us to exhibit $\mathrm{Sal}(W)$ as a CW–complex where each $C(w,X)$ is a $|\,X\,|$–cell for each $X \in \mathcal{S}_W$. Let us see what these cells look like. Note that the cells of the CW-complex and the simplices in $\Delta(W \times \mathcal{S}_W)$ as in Definition 2.18 comprise a completely different cell structure for $\mathrm{Sal}(W)$. We define $\langle \emptyset \rangle := \{1\}$ to give $W_\emptyset$ meaning as the trivial subgroup inside W. So $\emptyset \in \mathcal{S}(W)$.

Each $C(w,\emptyset)$ is a 0–cell. We will denote these cells w as a shorthand. In general, we have that $(u,X) \leq (v,X) \implies (u,X) = (v,X)$ since we require $v^{-1}u \in W_X$ we have $v^{-1}uW_X = W_X$. So if $v^{-1}u$ is minimal in $v^{-1}uW_X$ then $v^{-1}u = 1$. In particular, there is no $(u,X) < (w,\emptyset)$, so indeed each $w = C(w,\emptyset)$ is a 0–cell.

Now consider each 1–cell $C(w,\{s\})$. Since $W_{\{s\}} = \{1,s\} \cong \mathbb{Z}/2$ we have $\{s\} \in \mathcal{S}_W$ for all $s \in S$. Recall that if $(u,X) \leq (w,\{s\})$ we require $w^{-1}u \in W_{\{s\}}$. So we have $u \in \{w, ws\}$. Locally, the Hasse diagram and (since we only have 1–chains here) $\Delta(W \times \mathcal{S}_W)$ both look like Fig. 8 (left). In the CW–complex there would be only one 1–cell, labelled $C(w,\{s\})$ oriented from w to ws. Note that $C(ws,\{s\})$ also connects these two vertices, but in the opposite orientation.

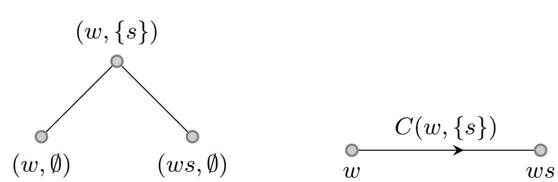


Figure 8: A picture of $[-\infty,(w,\{s\})]$ within $\mathrm{Sal}(W)$ as the geometric simplicial complex $\Delta(W \times \Delta_W)$ (which also resembles the Hasse diagram) (left). The corresponding 1–cell in the CW–complex for $\mathrm{Sal}(W)$ (right).

Now consider the 2–cells in the CW–complex. We have that $W_{\{s,t\}} \cong D_{2m(s,t)}$, the dihedral group of corresponding to the $m(s,t)$–gon (recall $m(s,t)$ from Definition 1.3). Thus, $\{s,t\} \in \Delta_W$ iff $m(s,t) \neq \infty$. We have $(u,\{s\}) \leq (w,\{s,t\})$ only when $u = wd$ for some $d \in W_{\{s,t\}}$, similarly for $(v,\{t\}) \leq (w,\{s,t\})$. The second requirement is that

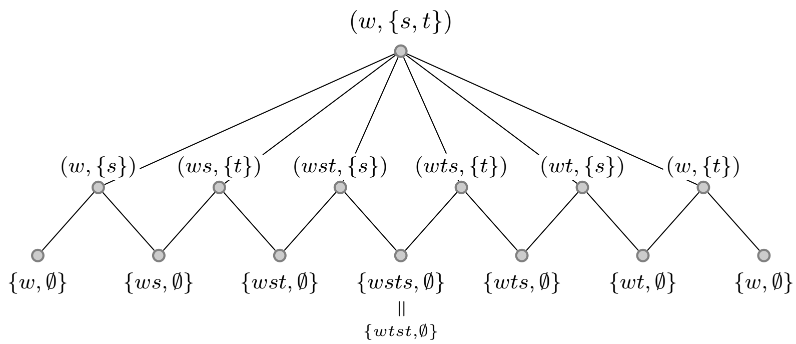


Figure 9: The local Hasse Diagram corresponding to the CW 2–cell $C(w, \{s, t\})$ where $m(s, t) = 3$. Note that $\{w, \emptyset\}$ has been drawn twice for clarity in the picture. C.f. Fig. 10

d is ($\{s\}$ or $\{t\}$)–minimal. Generally, if $(u, X) \leq (v, Y)$ then $X \subseteq Y$ so $W_X \subseteq W_Y$. We also require $v^{-1}u \in W_Y$ so the coset $v^{-1}uW_X$ lies inside W_Y. Due to the nature of Definition 1.3, only relations in W_Y could have relevance to the word length of an element $x \in W_Y$ (even if x is considered an element of the ambient Coxeter group W). Thus, to determine if $v^{-1}u$ is X minimal we need only consider $v^{-1}uW_X$ within W_Y, not the entire Coxeter group W. In particular, to tell if $(wd, \{s\}) \leq (w, \{s, t\})$ for some $d \in W_{\{s,t\}}$, we need only consider if d is $\{s\}$–minimal in the dihedral group $W_{\{s,t\}}$.

Considering for a moment s and t as letters only, a normal form comprising minimal length words for $W_{\{s,t\}}$ is

$$\{\Pi(s, t\ ;\ n) \mid n \leq m(s, t)\} \cup \{\Pi(t, s\ ;\ n) \mid n < m(s, t)\}$$

recalling the meaning of $\Pi(s, t\ ;\ n)$ from Definition 1.4. Note that $\Pi(t, s, m(s, t))$ is also a minimal length word but is not included for the above to be a normal form. Thus, any $sts \cdots s$ is $\{t\}$–minimal if the total length of $sts \cdots s$ is strictly less than $m(s, t)$. Similarly, $sts \cdots t$ is $\{s\}$–minimal if the total length of $sts \cdots t$ is strictly less than $m(s, t)$, with equivalent results for $tst \cdots s$ and $tst \cdots t$ depending on the last letter in the word. A picture of the Hasse diagram for the interval $[-\infty, (w, \{s, t\})]$ corresponding to the cell $C(w, \{s, t\})$ where $m(s, t) = 3$ is shown in Fig. 9. The CW cell itself has been drawn in Fig. 10.

There is a natural action $W \circlearrowleft \mathrm{Sal}(W)$ with $w \cdot (u, T) := (wu, T)$. We can now define the following:

Definition 2.20 (Salvetti Complex)**.** For a Coxeter group W define the *Salvetti Complex* X_W to be the quotient space $\mathrm{Sal}(W)/W$ under the action specified above.

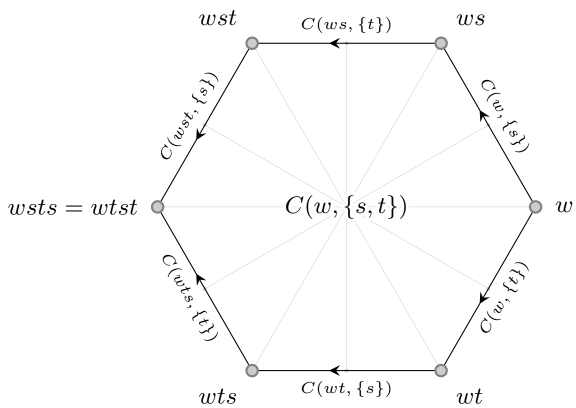


Figure 10: The 2–cell $C(w, \{s, t\})$ in the CW–complex for $\mathrm{Sal}(W)$ where $m(s, t) = 3$. The faint lines are the simplices from $\Delta(W \times \Delta_W)$, which have all been incorporated in to one CW cell.

The action is cellular, thus we have a CW structure for X_W as well. We now quote the following important result.

Theorem 2.21 ([Par14, Corollary 3.4] [Sal87])**.** *For a Coxeter group W, the Salvetti complex X_W is homotopy equivalent to the configuration space Y_W.*

Now let us consider the cell structure of X_W. There is one $|T|$–cell for each $T \in \Delta_W$, in particular, there is one 0–cell corresponding to the trivial group $W_\emptyset$. Attached to this is a 1–cell for each $s \in S$ forming $\bigvee_{s \in S} S^1$ as the 1–skeleton. Then to this wedge is attached 2–cells following the procedure above for each $m(s, t) \neq \infty$. Each two cell corresponding to $\{s, t\}$ is attached to two 1–cells corresponding to $\{s\}$ and $\{t\}$. From examining this 2–skeleton it should be clear that $\pi_1(X_W) \cong G_W$. Thus combining with the previous theorem we have re-proved $\pi_1(Y_W) \cong G_W$.

Chapter 3

An adjunction homotopy equivalence

We will now begin to bridge the gap between some of the objects we have defined. Ultimately, we wish to show a homotopy equivalence between the space K_W and the Salvetti complex X_W, which is already known to be homotopy equivalent to Y_W. For now, we will define a subspace $K'_W \subseteq K_W$, inspired by our definition of the Salvetti complex, using subsets $T \subseteq S$ such that W_T is finite.

3.1 The subcomplex X'_W

The definition of K_W depends on the data of some Coxeter element w and thus implicitly on some generating set S. For some $w = s_1 s_2 \cdots s_n$ and some $T \subseteq S$, define $w_T \in W$ to be the (non–consecutive) subword of w consisting of elements that are in T, respecting the original order in w.

Definition 3.1. Define X'_W to be the finite subcomplex of K_W consisting only of simplices $[x_1|\cdots|x_n]$ such that $x_1 x_2 \cdots x_n \in [1, w_T]^W$ for some $T \in \mathcal{S}_W$. (Recalling $\mathcal{S}_W$ from Definition 2.17).

Note again the absence of w from the notation of X'_W. This will be justified in Section 3.3 where we will show that the homotopy type of X'_W has no dependence on w.

Lemma 3.2 ([PS21, Lemma 5.2])**.** *Fix a Coxeter group W and any parabolic subgroup W_T, with sets of reflections R_W and R_{W_T} respectively. Given some Coxeter element $w \in W$*

and corresponding $w_T \in W_T$, all minimal factorisations of w_T by factors in R_W consist only of factors in R_{W_T}.

An immediate consequence of this is that the intervals $[1, w_T]^W$ and $[1, w_T]^{W_T}$ agree. This allows us to decompose X'_W in a useful way. For each $T \in \mathcal{S}_W$ and corresponding W_T, the space corresponding to the whole interval $[1, w_T]^W = [1, w_T]^{W_T}$ is a subspace inside X'_W. This subspace is exactly X'_{W_T} (with respect to w_T). The union of all such $[1, w_T]^{W_T}$ contains all the simplices of X'_W by Definition 3.1, thus, we can think of X'_W as the union (with appropriate gluing) of all X'_{W_T} for $T \in \mathcal{S}_W$.

For $T \in \mathcal{S}_W$, each X'_{W_T} is exactly the same as its interval complex K_{W_T} since all subgroups of T would also generate finite Coxeter groups. Thus, using known results for finite Coxeter groups, X'_{W_T} is a classifying space for the dual Artin group W_{w_T} by Theorem 2.15. Furthermore, W_{w_T} is isomorphic to the Artin group G_{W_T} by Theorem 2.16.

In a very similar way, the Salvetti complex consists of subspaces corresponding to elements of $\mathcal{S}_W$. For each $T \in \mathcal{S}_W$, the Salvetti complex X_{W_T} is a $|\,T\,|$–cell attached to all cells corresponding to $R \subseteq T$ is the appropriate way. This is a cellular subspace of X_W, and since W_T is finite, by Theorem 1.10, $Y_{W_T} \simeq X_{W_T}$ is a $K(G_{W_T}, 1)$. The following remark summarises these observations.

Remark 3.3. The Salvetti complex X_W decomposes in to cellular subspaces X_{W_T} which are $K(G_{W_T}, 1)$ spaces. These subspaces are in bijection with cellular subspaces X'_{W_T} of X'_W, which are also $K(G_{W_T}, 1)$ spaces.

We now develop some theory to help us exploit this similarity and eventually show that $X_W \simeq X'_W$, which is an intermediate step in (1).

3.2 Maps in to classifying spaces

Here we will show a fundamental link between homomorphisms in to groups G and certain classifying spaces for G. This result will show that two classifying spaces, with a certain CW–structure, for a group G are necessarily homotopy equivalent.

Lemma 3.4. *A null homotopic map $\rho\colon S^n \to X$ can be extended to a map $\sigma\colon D^{n+1} \to X$.*

Proof. Let $H\colon S^n \times I \to X$ witness the null homotopy with $H|_{S^n \times \{1\}}\colon S^n \to \{x_0\} \in X$. We have that H factors uniquely through $(S^n \times I)/(S^n \times \{1\}) \cong D^{n+1}$. With σ being the unique induced map as below.

$$\begin{tikzcd} S^n \times I \arrow[r, "H"] \arrow[d, "q"] & X \\ (S^n \times I)/(S^n \times \{1\}) \arrow[ur, dotted, "\exists! \sigma"'] & \end{tikzcd}$$

□

Theorem 3.5 ([Hat01, Proposition 1B.9]). *Let Y be a $K(G,1)$ space and X a finite dimensional CW–complex with only one 0–cell, the point x_0. Any homomorphism $\varphi\colon \pi_1(X,x_0) \to \pi_1(Y,y_0)$ is induced by a map $\overline{\varphi}\colon X \to Y$ where $\overline{\varphi}$ is unique up to homotopy fixing x_0.*

Proof. We must have $\overline{\varphi}(x_0) = y_0$. The 1–skeleton X^1 will be a wedge of circles. Let each λ_α be a loop going around the 1–cell e^1_α in a particular direction. Let each r_β be a factorisation in the set $\{[\lambda_\alpha]\}$ of the homotopy class of the attaching map $\psi_\beta\colon S^1 \to X^1$ for a 2–cell e^2_β. We have the following presentation for $\pi_1(X,x_0)$.

$$\pi_1(X,x_0) \cong \langle [\lambda_\alpha] \mid r_\beta \rangle$$

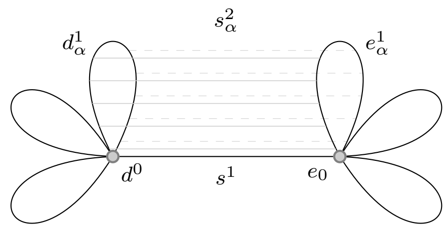
We can choose $\overline{\varphi}(e^1_\alpha)$ to trace out any path in the homotopy class $\varphi([\lambda_\alpha]) \in \pi_1(Y,y_0)$ for each $e^1_\alpha \in X^1$. ∘• I disagree that this requires any specific structure on Y. In Hatcher, Y is an arbitrary space and thus doesn't necessarily have a 1–skeleton. My previous statement here wasn't very clear and hopefully that's been amended. •∘ The induced map $(\overline{\varphi})_*$ is determined by its action on the generators $[\lambda_\alpha]$. We see that our construction ensures that $(\overline{\varphi})_* = \varphi$.

Let $\psi_\beta\colon S^1 \to X^1$ be an attaching map for a 2–cell $e^2_\beta \subseteq X$. Let $i\colon X^1 \hookrightarrow X$ be the inclusion. We have that i_* is the surjection from the free group generated by each e^1_α to $\pi_1(X,x_0)$. The attaching of the 2–cell e^2_β provides a null homotopy for the path traced by ψ_β. In the presentation of $\pi_1(X,x_0)$ as above, each relation corresponds to the path traced out by some ψ_β. Thus, $i_*([\psi_\beta]) = 0$ and so $\overline{\varphi}_*([\psi_\beta]) = \varphi \circ i_*([\psi_\beta]) = 0$. Thus, $\overline{\varphi} \circ \psi_\beta$ is null homotopic and so can be extended over all of the closure of e^2_β by Lemma 3.4. This is an extension of $\overline{\varphi}$ and repeating this allows us to extend $\overline{\varphi}$ over all of X^2.

To extend $\overline{\varphi}$ over some e^3_γ we use that $\pi_2(Y,y_0) = 0$. Let $\psi_\gamma\colon S^2 \to X^2$ be the attaching map for e^3_γ. Since $\pi_2(Y,y_0) = 0$, we have that $\overline{\varphi} \circ \psi_\gamma\colon S^2 \to X^2 \to Y$ is nullhomotopic and so can be extended over all of the closure of e^3_γ. This same argument applies for any e^n_δ for $n \geq 3$ since $\pi_n(Y,y_0) = 0$ for all $n \geq 2$. We can thus extend $\overline{\varphi}$ over the 3–cells and proceeding inductively on the n–skeletons, over all of X.

Now we turn to the uniqueness of $\overline{\varphi}$ up to homotopy. Let φ be some homomorphism and $\overline{\varphi}_0$ and $\overline{\varphi}_1$ be any such maps constructed as above. We have that $\overline{\varphi}_0(x_0) = \overline{\varphi}_1(x_0)$ and

$\overline{\varphi}_0|_{X^1} \sim \overline{\varphi}_1|_{X^1}$ ∘• This is homotopy not equality, since there is a choice of loop "*in the homotopy class of* $\varphi([e^1_\alpha]) \in \pi_1(Y, y_0)$ *for each* $e^1_\alpha \in X^1$" •∘ by the restrictions of our construction. Let H witness this homotopy. Give $X \times I$ the following CW structure: Let $X \times \{0\}$ and $X \times \{1\}$ both have the same cell structure as X with cells notated d^n_α and e^n_α respectively. Connect d^0 to e^0 with a 1–cell s^1, called the *spine*. Connect a 2–cell s^2_α along d^1_α, then s^1 then e^1_α then back along s^1 with opposite orientations on d^1_α and e^1_α such that $d^0 \cup e^0 \cup s^1 \cup d^1_\alpha \cup e^1_\alpha \cup s^2_\alpha \cong S^1 \times I$. The spine now consists of $s_1 \cup s^2_\alpha$. Repeat this for each 1–cell in X and then repeat for each 2–cell and so on, attaching an s^n_β along d^{n-1}_β, e^{n-1}_β and s^{n-1}_β, inductively building up the spine. A picture of this CW–complex completed for one s^2_α is below.



We can now extend the domain of H from $X^1 \times I$ to all of $X \times I$ using this cell structure. Note that now we have two 0–cells, but this does not cause any issues. Let H have domain $X^1 \times I \subseteq X \times I$. Now extend H to have domain $X^1 \times I \cup X \times \{0\} \cup X \times \{1\}$ such that $H|_{X \times \{0\}}$ agrees with $\overline{\varphi}_0$ and $H|_{X \times \{1\}}$ agrees with $\overline{\varphi}_1$. This is possible because H is a homotopy between restrictions of these maps. Note that now H is defined on the whole 2–skeleton of $X \times I$. We can extend H to all the higher dimensional cells by the exact same argument as before, using that $\pi_n(Y, y_0) = 0$ for $n \geq 2$. Thus, we have a continuous function $H \colon X \times I \to Y$ witnessing the homotopy $\overline{\varphi}_0 \sim \overline{\varphi}_1$. □

Corrolary 3.6. *Let X and Y both be $K(G, 1)$ spaces. If both spaces are CW–complexes with only one 0–cell, then any isomorphism $\varphi \colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ induces a homotopy equivalence witnessing $X \simeq Y$.*

Proof. We have maps $\overline{\varphi} \colon X \to Y$ and $\overline{(\varphi^{-1})} \colon Y \to X$ with $(\overline{\varphi} \circ \overline{(\varphi^{-1})})_* = \mathrm{Id}_{\pi_1(Y, y_0)}$. Thus, since the homotopy class of such maps is determined by the induced action on their fundamental groups $\overline{\varphi} \circ \overline{(\varphi^{-1})} \sim \mathrm{Id}_Y$. Similarly, $\overline{(\varphi^{-1})} \circ \overline{\varphi} \sim \mathrm{Id}_X$. □

To prove the next lemma we need to define some abstract properties of groups. Let G be a finite group. By an elementary counting argument, we have that any surjection $\varphi \colon G \to G$ is also an isomorphism. This does not always hold for infinite groups.

Definition 3.7. A group G is said to be *Hopfian* if all surjective maps $\varphi\colon G \to G$ are isomorphisms. Equivalently, a group G is Hopfian if all proper quotients G/N are not isomorphic to G.

By the previous remark, all finite groups are Hopfian. For an example of a non-Hopfian group consider

$$G = \mathbb{Z} \times \mathbb{Z} \times \cdots$$

and the surjection $(x_1, x_2, \ldots) \mapsto (x_2, x_3, \ldots)$ which is not an isomorphism. This is an example of an infinitely generated non-Hopfian group but note that there are also finitely generated non-Hopfian groups. If G is finitely generated, we can prove that G is Hopfian given a certain a property of homomorphisms out of G.

Definition 3.8. A group G is said to be *residually finite* if for all $x \in G \setminus \{1\}$ there exists a homomorphism $h_x\colon G \to K$ to a finite group K such that $h_x(x) \neq 1 \in K$.

Lemma 3.9. *All finitely generated and residually finite groups are Hopfian.*

Proof. Let G be such a group, generated by some finite S. Given a finite group K, there are finitely many homomorphisms $\varphi\colon G \to K$ since any such φ is determined by the map $\varphi|_S$, which is a map between finite sets, of which there are finitely many possibilities.

We now complete our proof by contradiction. Suppose there exists some surjection $\varphi\colon G \to G$ with $g \in \ker(\varphi) \setminus \{1\}$. Since G is residually finite, there exists some finite group K and a homomorphism $\rho\colon G \to K$ such that $\rho(g) \neq 1$. We will now show that $\rho \circ \varphi^m \neq \rho \circ \varphi^n$ for $m < n$. Since φ is surjective, φ^m is also surjective. So there exists some $h \in G$ such that $\varphi^m(h) = g$ and thus that $\rho \circ \varphi^m(h) \neq 1$. Since $n > m$ we have that $\varphi^n(h) = \varphi^{n-m} \circ \varphi^m(h) = 1$ so $\rho \circ \varphi^n(h) = 1$ and so $\rho \circ \varphi^m$ and $\rho \circ \varphi^n$ are distinct. We have constructed infinitely many distinct homomorphisms $\rho \circ \varphi^n\colon G \to K$. This is a contradiction, so given a surjection $\varphi\colon G \to G$, we have $\ker(\varphi) \setminus \{1\} = \emptyset$. □

Lemma 3.10. *All finite type Artin groups are Hopfian.*

Proof. All Artin groups are finitely generated. A result of Cohen and Wales [CW02] shows that all finite type Artin groups are linear, i.e. can be realised as a subgroup of $GL_n(\mathbb{K})$ for some field $\mathbb{K}$. A classical result of Mal'cev [Mal65] shows that finitely generated linear groups are residually finite. Combining these with Lemma 3.9 completes our proof. □

Lemma 3.11. *Let $T \in \mathcal{S}_W \setminus \emptyset$. Let $\varphi\colon \bigcup_{Q\subsetneq T} X_{W_Q} \to \bigcup_{Q\subsetneq T} X'_{W_Q}$ be a homotopy equivalence. We can extend φ to a homotopy equivalence $\psi\colon X_{W_T} \to X'_{W_T}$ such that the following diagram commutes.*

$$\begin{tikzcd} \bigcup_{Q\subsetneq T} X_{W_Q} \arrow[r, "\varphi"] \arrow[d, hook, "i"'] & \bigcup_{Q\subsetneq T} X'_{W_Q} \arrow[d, hook, "j"] \\ X_{W_T} \arrow[r, "\psi"] & X'_{W_T} \end{tikzcd} \tag{4}$$

Proof. We prove this by cases. By Remark 3.3, X_{W_T} and X'_{W_T} are classifying spaces.

i) If $|T| = 1$ then any $Q \subsetneq T$ is uniquely $\emptyset$ ◦• I disagree that Q is $\{\emptyset\}$. $\mathcal{S}_W$ is a collection of sets, $T \in \mathcal{S}_W$ is a set, so is a subset $Q \subsetneq T$. •◦ . We have that $X_{W_T} \cong X'_{W_T} \cong S^1$. Let ψ be any map witnessing $X_{W_T} \simeq X'_{W_T}$ that restricts to the map $\psi|_{X_{W_\emptyset}} : X_{W_\emptyset} \to X'_{W_\emptyset}$.

ii) Let $T = \{u, v\}$. We have a homotopy equivalence $\varphi\colon X_{W_{\{u\}}} \cup X_{W_{\{v\}}} \to X'_{W_{\{u\}}} \cup X'_{W_{\{v\}}}$. By the same argument as in the proof of Lemma 3.11, which allows us to extend $j \circ \varphi$ to a map $\psi\colon X_{W_T} \to X'_{W_T}$. We briefly recall this argument: Let ω the attaching map of the 2–cell e^2 corresponding to T. We have that $\theta_*([\omega]) = 0$, therefore we can extend θ over the closure of the 2–cell e^2.

We now have a commutative diagram of maps (ψ is not a–priori a homotopy equivalence) as in (4) and the following induced commutative diagram on homotopy groups.

$$\begin{tikzcd} \pi_1(\bigcup_{Q\subsetneq T} X_{W_Q}, X_{W_\emptyset}) \arrow[r, "\cong"] \arrow[d, "i_*"'] & \pi_1(\bigcup_{Q\subsetneq T} X'_{W_Q}, X'_{W_\emptyset}) \arrow[d, "j_*"] \\ \pi_1(X_{W_T}, X_{W_\emptyset}) \arrow[r, "\psi_*"] & \pi_1(X'_{W_T}, X'_{W_\emptyset}) \end{tikzcd} \tag{5}$$

We have that j_* is the surjection from the free group on the generators of $\pi_1(X'_{W_T}, X'_{W_\emptyset})$ on to $\pi_1(X'_{W_T}, X'_{W_\emptyset})$ itself. Thus, ψ_* is also a surjection. Since by Remark 3.3 both $\pi_1(X_{W_T}, X_{W_\emptyset})$ and $\pi_1(X'_{W_T}, X'_{W_\emptyset})$ are isomorphic to the finite type Artin group G_{W_T}, by Lemma 3.10 ψ_* is an isomorphism. Both X_{W_T}, and X'_{W_T} are CW–complexes with only one 0–cell so by Corollary 3.6 ψ is a homotopy equivalence.

iii) If $|T| \geq 3$ then we can extend $j \circ \varphi$ to some map ψ using the same methods as in the proof of Theorem 3.5, utilising that $\pi_n(X'_{W_T}, X'_{W_\emptyset}) = 0$ for all $n \geq 2$. In this case, $\bigcup_{Q\subsetneq T} X_{W_Q}$ contains the 2–skeleton of X_T and similarly for $\bigcup_{Q\subsetneq T} X'_{W_Q}$ and X'_T. So the induced map on the inclusion $i_*\colon \pi_1(\bigcup_{Q\subsetneq T} X_{W_Q}, X_{W_\emptyset}) \to \pi_1(X_T, X_{W_\emptyset})$ is an isomorphism and similarly for X'_T [Hat01, Corollary 4.12]. By assumption φ is a homotopy equivalence and so φ_* is an isomorphism. Therefore, ψ_* is an isomorphism. Both X_{W_T}, and X'_{W_T} are CW–complexes with only one 0–cell so by Corollary 3.6 ψ is a homotopy equivalence. □

3.3 Adjunction spaces and proving $X_W \simeq X'_W$

Here we will develop a formalism for gluing together two spaces in to what is called an *adjunction space*. Adjunction spaces inherit homotopic properties from their two parent spaces in a useful way, that will help us to exploit Remark 3.3.

Definition 3.12 (Adjunction Space)**.** Given two spaces X and U, with a continuous map $f\colon A \to U$ for some subspace $A \subseteq X$, the *adjunction space* $X \sqcup_f U$ is the space formed by gluing X and U via the map f.

$$X \sqcup_f U := (X \sqcup Y)/(a \sim f(a))$$

An adjunction space is associated to the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"] & U \arrow[d, "\widetilde{i}"] \\
X \arrow[r, "\widetilde{f}"] & X \sqcup_f U
\end{tikzcd} \tag{6}$$

where i is inclusion of A in to X and $\widetilde{i}$ is inclusion of U in to $X \sqcup_f U$. Suppose we also have the adjunction space $Y \sqcup_g V$ with $g\colon B \to V$ and $B \subseteq Y$. Suppose further that we have maps $\varphi_1\colon X \to Y$, $\varphi_2\colon A \to B$ and $\varphi_3\colon U \to V$ such that the following diagram commutes.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "\varphi_2"] \arrow[dd, "i"'] & & U \arrow[dr, "\varphi_3"] \arrow[dd, "\widetilde{i}" near start] & \\
& B \arrow[rr, "g" near start, crossing over] & & V \arrow[dd, "\widetilde{j}"] \\
X \arrow[rr, "\widetilde{f}" near end] \arrow[dr, "\varphi_1"'] & & X \sqcup_f U \arrow[dr, "\varphi"] & \\
& Y \arrow[rr, "\widetilde{g}"'] \arrow[from=uu, "j" near start, crossing over] & & Y \sqcup_g V
\end{tikzcd} \tag{7}$$

If all φ_1, φ_2 and φ_3 are homotopy equivalences then the following lemma tells us that φ is also a homotopy equivalence.

Lemma 3.13 ([Bro06, Theorem 7.5.7])**.** *Consider a commutative diagram as in* (7) *where the front and back faces define an adjunction space as in* (6)*. If i and j are closed cofibrations and φ_1, φ_2 and φ_3 are homotopy equivalences, then the φ as determined by the diagram is also a homotopy equivalence.*

In the cases important to us, i and j will be cellular inclusions in to finite CW–complexes, and thus closed cofibrations. See [Bro06] for more details on adjunction spaces.

To use Lemma 3.13 we must be able to construct X_W and X'_W as a sequence of adjunction spaces. Consider X_W in the following example.

Example 3.14. To clean our notation in this example let X and X_T be shorthand for X_W and X_{W_T} respectively. Accordingly, X^n is the n–skeleton for X_W and X^n_T is the n–skeleton for X_{W_T}. Suppose W is generated by $S = \{s, t.u\}$ such suppose $\mathcal{S}_W = \{Q \subsetneq S\}$. For convenience define $\mathcal{S}^n_W := \{T \in \mathcal{S}_W \mid |\,T\,| = n\}$. When denoting adjunction spaces, we will use $\sqcup_f$ to denote a specific adjunction space that is currently being constructed. We use the $\cup$ symbol to denote some adjunction space or space resulting from a sequence of adjunctions without specifying the map f. We have that $X = \bigcup_{T\in\mathcal{S}^2_W} X_T$. Suppose we had the 1–skeleton $X^1 \subsetneq X$, and some ordering on $\mathcal{S}^2_W = (\{s,t\},\{s,u\},\{t,u\})$. To construct X, we would first glue $X_{\{s,t\}}$ to X^1 as an adjunction space in the following way.

$$\begin{tikzcd} X_{\{s\}} \cup X_{\{t\}} \arrow[r, "f"] \arrow[d, "i_1"] & X_{\{s,t\}} \arrow[d, "\widetilde{i_1}"] \\ X^1 \arrow[r, "\widetilde{f}"] & X^1 \sqcup_f X_{\{s,t\}} \end{tikzcd} \tag{8}$$

Where f is inclusion of the 1–cells $X_{\{s\}}$ and $X_{\{t\}}$ in to $X_{\{s,t\}}$ and i_1 is the inclusion in to X^1, which in this case makes $X^1 \sqcup_f X_{\{s,t\}} \cong X_{\{s,t\}}$. When using this construction in the proof of Theorem 3.15 we inductively assume to have already constructed $X_{\{s\}} \cup X_{\{t\}}$. We can now add $X_{\{t,u\}}$ to the preceding adjunction space in the following way.

$$\begin{tikzcd} X_{\{t\}} \cup X_{\{u\}} \arrow[r, "g"] \arrow[d, "i_2"] & X_{\{t,u\}} \arrow[d, "\widetilde{i_2}"] \\ X^1 \sqcup_f X_{\{s,t\}} \arrow[r, "\widetilde{g}"] & (X^1 \sqcup_f X_{\{s,t\}}) \sqcup_g X_{\{t,u\}} \end{tikzcd} \tag{9}$$

Again, g and i_2 are just inclusions. After this step, we would continue with $\{u, v\}$ in the same manner. In the final space, $X_{\{u,v\}}$ is glued to $(X^1_W \sqcup_f X_{\{s,t\}}) \sqcup_g X_{\{t,u\}}$ along

$X_{\{u\}} \cup X_{\{v\}}$. Given the $(n-1)$–skeleton, we can always choose an order on $\{X_T \mid T \in \mathcal{S}_W^n\}$ and attach each such X_T to the previous adjunction space along the $(n-1)$–skeleton of X_T (which is always a subspace of the adjunction space of the previous step).

The exact same construction works for X'_W. Note that none of the maps in (8) or (9) are the attaching maps of cells as in the CW–complex. It is possible to construct CW–complexes as a sequence of adjunction spaces, but that is not the construction presented in Example 3.14. In this construction we assume to already have fully-constructed $|\,T\,|$–dimensional CW–complexes X_T, then we just glue these on to the $(|\,T\,|-1)$–skeleton which is a subspace of the adjunction space formed in the previous step. All the internal structure of the CW–complexes X_T is ignored. This construction may seem too abstracted to be useful, but we will soon see otherwise.

The following proof uses induction on the steps presented in Example 3.14. Note that a step is the gluing of a single X_T. In principle, to construct the n–skeleton from the $(n-1)$–skeleton takes multiple steps.

Theorem 3.15 ([PS21, Theorem 5.5])**.** *For a Coxeter group W, the space X'_W as in Definition 3.1 is homotopy equivalent to the Salvetti complex X_W.*

Proof. We achieve this by induction on adjunction gluing steps. To tidy our notation, as in Example 3.14, we drop W so that X, X', X_T and X'_T correspond to X_W, X'_W, X_{W_T} and X'_{W_T} respectively. Let $\mathcal{S}_W^n$ be as in Example 3.14.

Our inductive hypotheses are as follows:

1. We have a homotopy equivalence between $(n-1)$–skeletons $\alpha\colon X^{n-1} \to (X')^{n-1}$.

2. For any subset $\mathcal{T} \subseteq \mathcal{S}_W^{n-1}$ we have that α restricts to a homotopy equivalence $\alpha'\colon \bigcup_{Q\in\mathcal{T}} X_Q \to \bigcup_{Q\in\mathcal{T}} X'_Q$ such that the following diagram commutes.

$$\begin{tikzcd} \bigcup_{Q\in\mathcal{T}} X_Q \arrow[r, "\alpha'"] \arrow[d, hook] & \bigcup_{Q\in\mathcal{T}} X'_Q \arrow[d, hook] \\ X^{n-1} \arrow[r, "\alpha"] & (X')^{n-1} \end{tikzcd}$$

3. We have a homotopy equivalence $\beta\colon Y \to Y'$ where $Y \subseteq X$ and $Y' \subseteq X'$ are intermediate steps in constructing X^n and $(X')^n$ respectively. For example, Y could be the upper-right term in (9) for $n = 2$. This homotopy equivalence β restricts to the homotopy equivalence α on X^{n-1}.

The base case is $X_\emptyset \simeq X'_\emptyset \simeq \{\bullet\}$. Fix an ordering $(T_1, T_2, \dots, T_k)$ on $\mathcal{S}^n_W$. For each T_i we can extend the map $\varphi_i \colon \bigcup_{Q \subsetneq T_i} X_Q \to \bigcup_{Q \subsetneq T_i} X'_Q$ to $\psi_i \colon X_{T_i} \to X'_{T_i}$ such that we have the following commutative diagram.

$$\begin{tikzcd}
\bigcup_{Q \subsetneq T_i} X_Q \arrow[r, "\varphi_i"] \arrow[d, hook] & \bigcup_{Q \subsetneq T_i} X'_Q \arrow[d, hook] \\
X_{T_i} \arrow[r, "\psi_i"] & X'_{T_i}
\end{tikzcd}$$

This is possible by Lemma 3.11. Fix each ψ_i.

• Case 1: Suppose $Y = \bigcup_{m<i} X_{T_m}$ and $Y' = \bigcup_{m<i} X'_{T_m}$ and $i \leq k-1$, i.e. in the next step, we will *not* complete $X^n \simeq (X')^n$.

In this step, we will glue X_{T_i} to Y and X'_{T_i} to Y' such that the inductive hypotheses remain for these resulting adjunction spaces. By combining inductive hypothesis 3 and 2, we have the following commutative diagram.

$$\begin{tikzcd}
\bigcup_{Q \subsetneq T_i} X_Q \arrow[r, "\varphi_i"] \arrow[d, hook] & \bigcup_{Q \subsetneq T_i} X'_Q \arrow[d, hook] \\
Y \arrow[r, "\beta"] & Y'
\end{tikzcd}$$

Where φ_i and β are homotopy equivalences and the vertical maps are inclusions. We now have the following commutative diagram.

$$\begin{tikzcd}
\bigcup_{Q \subsetneq T_i} X_Q \arrow[rr, hook, "f"] \arrow[dd, hook] \arrow[dr, "\varphi_i"] & & X_{T_i} \arrow[dr, "\psi_i"] \arrow[dd, hook] & \\
& \bigcup_{Q \subsetneq T_i} X'_Q \arrow[rr, hook, "g"] \arrow[dd, hook] & & X'_{T_i} \arrow[dd, hook] \\
Y \arrow[rr, hook, "\widetilde{f}"'] \arrow[dr, "\beta"] & & Y \sqcup_f X_{T_i} \arrow[dr, "\sigma"] & \\
& Y' \arrow[rr, hook, "\widetilde{g}"'] & & Y' \sqcup_g X'_{T_i}
\end{tikzcd} \tag{10}$$

Where all maps not coming out of the plane of the page are inclusions and front and back faces determine the adjunction spaces $Y \sqcup_f X_{T_i}$ and $Y' \sqcup_g X'_{T_i}$ respectively. By Lemma 3.13, the map σ determined by the maps β, φ_i and ψ_i is a homotopy equivalence. In the next step, we will still be constructing $X^n \simeq (X')^n$ and will be gluing $X_{T_{i+1}}$ to Y and $X'_{T_{i+1}}$ to Y'. From (10), we see that σ restricts to β and so by induction also restricts to $\alpha\colon X^{n-1} \to (X')^{n-1}$. We replace Y with $Y \sqcup_f X_{T_i}$ and Y' with $Y' \sqcup_g X'_{T_i}$. Accordingly, we replace β with σ.

With these replacements, we maintain all our inductive hypotheses and can continue.

- Case 2: Suppose $Y = \bigcup_{m<i} X_{T_m}$ and $Y' = \bigcup_{m<i} X'_{T_m}$ and $i = k$, i.e. in the next step, we *will* complete $X^n \simeq (X')^n$.

The steps up to creating the commutative diagram (10) are exactly the same as in Case 1. However, we must make a further argument to ensure the inductive hypotheses continue to be true for the next inductive step. Suppose we have completed the adjunction spaces $Y \sqcup_f X_{T_i}$ and $Y' \sqcup_g X'_{T_i}$ and have a homotopy equivalence σ between them. In this case, $Y \sqcup_f X_{T_i} \cong X^n$ and $Y' \sqcup_g X'_{T_i} \cong (X')^n$, so σ is a homotopy equivalence $\sigma\colon X^n \to (X')^n$. For the next step we will replace both α and β with σ. We immediately achieve inductive hypotheses 1 and 3. But 2 is not obvious.

Recall that we had fixed an ordering $(T_1, T_2, \ldots, T_K)$ for $\mathcal{S}_W^n$. We see that in the intermediate steps between constructing the $(n-1)$–skeletons and the n–skeletons, we constructed all $\bigcup_{m\leq l} X_{T_m}$ and $\bigcup_{m\leq l} X_{T_m}$ for each $l \leq k$. Furthermore, we constructed homotopy equivalences between $\bigcup_{m\leq l} X_{T_m}$ and $\bigcup_{m\leq l} X_{T_m}$ that are restrictions of our homotopy equivalence between X^n and $(X')^n$. Therefore, we have inductive hypothesis 2, but only for subsets of $\mathcal{S}_W^n$ that correspond to a prefix of our ordering $(T_1, T_2, \ldots, T_K)$. Clearly this is not all subsets of $\mathcal{S}_W^n$.

However, suppose we are given an arbitrary subset $\mathcal{T} \subseteq \mathcal{S}_W^n$. We could have chosen an ordering $\chi = (T_{a_1}, T_{a_2}, \ldots, T_{a_k})$ of $\mathcal{S}_W^n$ such that $\mathcal{T}$ is some prefix of χ. We would have continued through the same steps and obtained a (potentially different) homotopy equivalence $\sigma_\chi\colon X^n \to (X')^n$. We observe from (10) that σ_χ must restrict to the homotopy equivalence $\psi_i\colon X_{T_i} \to X'_{T_i}$ for all $1 \leq i \leq k$. These ψ_i are each fixed. Since $\bigcup_{1\leq m\leq k} X_{T_m} = X^n$, as a map, these ψ_i define σ (or σ_χ), so $\sigma = \sigma_\chi$. Thus, σ is independent of our choice of ordering and the restriction of σ to $\bigcup_{Q\in\mathcal{T}} X_Q$ for *any* $\mathcal{T} \subseteq \mathcal{S}_W^n$ is the necessary homotopy equivalence to satisfy inductive hypothesis 2.

Our proof concludes by observing that there is a minimal n such that $\mathcal{S}_W^{n+1} = \emptyset$, thus giving $X^n = X$ and $(X')^n = X'$. The inductive process outlined above eventually ends with a homotopy equivalence witnessing $X \simeq X'$. □

Chapter 4

Discrete Morse Theory

In this section we will prove the next homotopy equivalence along the chain in (1). This will again involve the use of posets and their combinatorics. Morse theory for smooth manifolds gives a way to infer topological properties of manifolds from analytical properties of certain smooth functions on that manifold. Discrete Morse theory is a CW (non–smooth) analogue. Certain functions on the (discrete) set of cells of a CW–complex can tell us topological facts about the CW–complex. Here we will only give a brief introduction to the main results of this theory that are relevant to us.

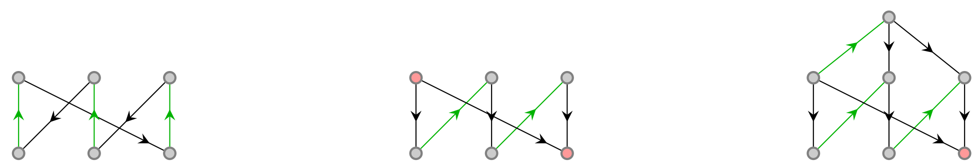
4.1 The Face Poset and Acyclic Matchings

In previous sections we gave constructions that formed spaces from posets, we now give a construction in the opposite direction. Given a CW–complex X denote the set of open cells as X^*.

Definition 4.1. Given a CW–complex X, the *face poset* $\mathcal{F}(X)$ is an ordering on X^* where $\tau \leq \sigma$ when $\bar{\tau} \subseteq \bar{\sigma}$.

For a finite dimensional and connected CW–complex, $\mathcal{F}(X)$ is a bounded and graded poset with rank function $\mathrm{rk}(\sigma) = \dim(\sigma)$. Let P denote $\mathcal{F}(X)$. Consider some subset of the covering relations $\mathcal{M} \subseteq \mathcal{E}(P)$. We consider this as a set of edges in the Hasse diagram for P, which is denoted H. From $\mathcal{M}$ we define an ordering on the graph H such that $p \lessdot q$ is oriented from p to q if $(p \lessdot q) \in \mathcal{M}$, and otherwise in the opposite direction. We denote this oriented graph $H_{\mathcal{M}}$. We call $\mathcal{M}$ a *matching* if for all $p \in P$, at most one $m \in \mathcal{M}$ contains p. A matching is *acyclic* if $H_{\mathcal{M}}$ contains no directed cycles. Furthermore, a matching is *proper* if for all $p \in P$, the set of all nodes in $H_{\mathcal{M}}$ reachable by a directed path from p is finite. Fig. 11 gives some (non)examples of matchings.

We observe that the requirement of being a matching means that any path through $H_{\mathcal{M}}$ will never consecutively go through two edges in $\mathcal{M}$. A cycle in $H_{\mathcal{M}}$ must clearly start and end at the same rank. Since edges in $\mathcal{M}$ increase rank and edges in $\mathcal{E}(P) \setminus \mathcal{M}$ decrease rank, a cycle must therefore be (cyclically) alternating between edges in $\mathcal{M}$ and edges in $\mathcal{E}(P) \setminus \mathcal{M}$. Therefore, if a cycle is to start at $p \in P$, it must completely occur in $\{q \in P \mid \operatorname{rk}(q) - \operatorname{rk}(p) \in \{0, 1\}\}$ or completely in $\{q \in P \mid \operatorname{rk}(q) - \operatorname{rk}(p) \in \{0, -1\}\}$. I.e. the horizontal bands above or below p in H. Since it is alternating, a cycle must also comprise an even number of edges.



(a) A cyclic matching.

(b) An acyclic matching with invalid subcomplex.

(c) An acyclic matching with valid subcomplex.

Figure 11: Directed Hasse diagrams corresponding to face posets and choices of $\mathcal{M}$. Green edges are those in $\mathcal{M}$ and red nodes are critical cells.

We call σ a *face* of τ if $\sigma \lessdot \tau$. Consider Φ, the characteristic map of some n–cell τ with $\Phi \colon D^n \to X$. We call σ a *regular face* of τ if it is a face and the following hold.

1) $\Phi|_{\Phi^{-1}(\sigma)} \colon \Phi^{-1}(\sigma) \to \sigma$ is a homeomorphism.

2) $\overline{\Phi^{-1}(\sigma)}$ is homeomorphic to D^{n-1} as a subset of D^n.

For a matching $\mathcal{M}$, any $p \in P$ that is disjoint from all of $\mathcal{M}$ is called *critical*. In this context, a *critical cell*. We can now state the version of discrete Morse theory we will use.

Theorem 4.2 ([PS21, Theorem 2.4])**.** *Consider a CW–complex X, a subcomplex $Y \subseteq X$, and a proper, acyclic matching $\mathcal{M}$ on X. If $\mathcal{F}(Y) \subseteq \mathcal{F}(X)$ is the set of critical cells in X with respect to $\mathcal{M}$ and every if σ is a regular face of τ for every $(\sigma \lessdot \tau) \in \mathcal{M}$, then X deformation retracts on to Y.*

You may notice that this seems to have no link to discrete functions $X^* \to \mathbb{N}$, as promised in the prologue of this section. Indeed, this statement is a reformulation of Discrete Morse Theory due [Cha00] and [Bat02]. The original formulation of Discrete

Morse Theory is due to [For98]. The exact wording of Theorem 4.2 is important, we explore this in the following example.

Example 4.3. The Hasse diagrams in Fig. 11 correspond to the obvious CW–complex for a triangle. Figs. 11a and 11b are for a hollow 1–dimensional triangle and Fig. 11c is for a filled 2–dimensional triangle. We know that a hollow triangle cannot deformation retract on to any of its subcomplexes, thus the required construction for Theorem 4.2 should fail for Figs. 11a and 11b. We see that Fig. 11a is a cyclic matching, but we achieve an acyclic (vacuously proper) matching in Fig. 11b. Importantly, the space corresponding to the union of the critical cells, which are highlighted in the figure, is not a valid subcomplex, thus Theorem 4.2 does not apply. For a subset of cells $Y^* \subseteq X^*$ to correspond to a valid subcomplex $Y \subseteq X$, we require

$$\bigcup_{y \in \mathcal{F}(Y)} [-\infty, y] = \mathcal{F}(Y) \tag{11}$$

where $[-\infty, y]$ is taken within the poset $\mathcal{F}(X)$. For Fig. 11b, the left–hand side of (11) would include the bottom left cell in the Hasse diagram, which we see is not critical. In Fig. 11c, we have a valid subcomplex and the critical cell corresponds to a vertex in the 2–dimensional triangle, which is of course a valid deformation retract of the whole complex.

We will also require the following standard tool for forming acyclic matchings. For this we introduce the notion of a *poset map*, which a map between posets $P \to Q$ that respects the poset structure. Given such a map φ, we call preimages of single elements *fibres*.

Theorem 4.4 (Patchwork Theorem [Koz08, Theorem 11.10])**.** *Given a poset map $\varphi\colon P \to Q$, assume we have acyclic matchings on all fibres $\varphi^{-1}(q)$, where the matching need only be acyclic within the fibre itself. The union of all of these matchings is an acyclic matching on P.*

4.2 A New Subcomplex, K'_W

We now introduce the particular poset map we will use. Recall that w is a choice of Coxeter element. Given the standard ordering of $\mathbb{N}$, the map $\eta\colon \mathcal{F}(K_W) \to \mathbb{N}$ with

$$\eta([x_1|x_2|\cdots|x_d]) := \begin{cases} d & \text{if } x_1x_2\dots x_d = w \\ d+1 & \text{otherwise} \end{cases}$$

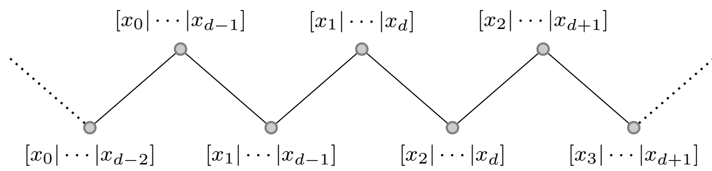


Figure 12: The general form of a connected d–fibre component.

is a poset map. This can be readily checked. We call a connected component of $\eta^{-1}(d)$ a d–*fibre component*. We now investigate the form of these d–fibre components. Again, denote $\mathcal{F}(K_W)$ by P. Consider such a d–fibre component C such that $x \coloneqq [x_1|x_2|\cdots|x_d] \in C$. Clearly $x_1x_2\ldots x_d = w$ and so x is not the face of anything in P. The faces of x are $[x_2|\cdots|x_d]$, $[x_1|\cdots|x_{d-1}]$ and $[x_1|\cdots|x_ix_{i+1}|\cdots|x_d]$ for all $i < d$. Only the first two faces will be in C since $\eta([x_1|\cdots|x_ix_{i+1}|\cdots|x_d]) = d-1$. Therefore, a d–fibre component will look like Fig. 12.

To each fibre component there is an associated sequence $(x_i)_{i\in\mathbb{Z}}$ such that every product of d consecutive elements is equal to w and each corresponding cell from the sequence is in K_W. Denote the set of all such sequences S_d. For each fibre component, the choice of $s \in S_d$ is unique up translation of indices.

Remark 4.5. Consider some cell $[x_1|\cdots|x_{d-1}]$, i.e. a cell in the bottom row of Fig. 12. To form the whole fibre component, you may choose to move up and to the right along the Hasse diagram from to $[x_1|\cdots|x_d]$. This is always possible and completely deterministic, since $x_1x_2\ldots x_{d-1}$ is a prefix of a minimal factorisation of w and the product $x_1x_2\ldots x_d$ must equal w. Similarly, a movement up and leftward is always possible and deterministic using the balanced property of K_W. Thus, a fibre component does not have initial of final nodes and its Hasse diagram will either be homeomorphic to $\mathbb{R}$ or S^1. A finite fibre component will have a repeating sequence. If the sequence repeats every n, then the Hasse diagram of the fibre component will contain $2n$ nodes. An infinite fibre component will be non–repeating, though an individual group element x_i may be repeated in the sequence.

The previous remark already shows that each sequence $s \in S_d$ is uniquely determined by any length $(d-1)$ subsequence. This is made clearer by the following observation. Let φ denote conjugation by w, i.e. $\varphi(x) = w^{-1}xw$. Then we have $\varphi(x_i) = x_{i+d}$ by the following factorisation.

$$\varphi(x_i) = (x_{i+d-1}^{-1}x_{i+d-2}^{-1}\ldots x_i^{-1})x_i(x_{i+1}x_{i+2}\ldots x_{i+d})$$

We will now define a subcomplex $K'_W \subseteq K_W$ based on these fibre components, after which we will prove the necessary properties of K'_W. Let $F \subseteq 2^{\mathcal{F}(K)}$ be the set of all connected fibre components such that $\bigcup_{f\in F} f = \eta^{-1}(\mathbb{N})$. Recall X'_W from Definition 3.1.

Definition 4.6 (The subcomplex K'_W)**.** For each infinite $f \in F$ let f' be the elements of f in between and including the first and last elements of $f \cap X'_W$, where *first*, *last* and *in between* derive from the linear form of the fibre components as in (12). Define K'_W to be the union of all finite f and f' such that f is infinite.

Note that since for finite f, the Hasse diagram of f is homeomorphic to S^1, so for finite f there is no notion of *in between*. Recall that X'_W is finite, so $f \cap X'_W$ is finite and *first* and *last* are well–defined. The first property of K'_W we must prove is that K'_W is indeed a valid subcomplex, satisfying (11).

Lemma 4.7. *As defined above, K'_W is a valid subcomplex of K_W.*

Proof. We first concentrate on the infinite $f \in \eta^{-1}(d)$. We need to show that some infinite $f \in F$, if $\tau \in f'$ and $\sigma \lessdot \tau$, then there exists some $g \in F$ such that $\sigma \in g'$. Let $s = (x_n)_{n\in\mathbb{Z}}$ be the sequence of group elements corresponding to f and let $\tau = [x_0|\cdots|x_{d-1}]$. Let τ be between α and ω both in $f \cap X'_W$. See Fig. 13. By Lemma 3.2 all of $[-\infty,\alpha]$ and $[-\infty,\omega]$ are also in X'_W, so we may assume that both α and ω are in the *bottom row* of the Hasse diagram of f' and accordingly consist of $d-1$ group elements. Choose $a, z \in \mathbb{Z}$ such that $\alpha = [x_a|\cdots|x_{a+d-2}]$ and $\omega = [x_z|\cdots|x_{z+d-2}]$.

• First we will prove the case where τ is in the top row of the Hasse diagram for f' such that $x_0x_1\ldots x_{d-1} = w$. The two faces $[x_1|\cdots|x_{d-1}]$ and $[x_0|\cdots|x_{d-2}]$ are both already in f', so we need only check the faces

$$\sigma^i = [x_0|\cdots|x_ix_{i+1}|\cdots|x_{d-1}] \lessdot \tau.$$

Each of the σ^i are in $\eta^{-1}(d-1)$. Given the sequence $s = (x_n)_{n\in\mathbb{Z}}$, we define the following.

$$s^i := (y^i_n)_{n\in\mathbb{Z}} = (\ldots, x_0, x_1, \ldots, x_ix_{i+1}, \ldots x_d, \ldots, x_{d+i}x_{d+i+1}, \ldots, x_{2d}, \ldots)$$

Where we multiply each adjacent pair x_j, x_{j+1} by removing the comma wherever $j \equiv i \mod d$. We see that every product of $(d-1)$ consecutive terms in s^i is w. Each s^i is the sequence corresponding to some connected component of $\eta^{-1}(d-1)$. Each face $\sigma^i \lessdot \tau$ is associated to the connected component associated to s^i. Denote this component g^i. We need to show that there exists $\alpha', \omega' \in g^i$ such that $\alpha' \leq \alpha$ and $\omega' \leq \omega$ and thus that $\alpha', \omega' \in X'_W$. We may choose α' and ω' to be any $[y^i_k|\cdots|y^i_{k+d-3}]$, i.e. a cell on the bottom row of g^i consisting of $(d-2)$ group elements. Let us concentrate on α'. There are three possibilities, remember that $\alpha = [x_a|\cdots|x_{a+d-2}]$ and $\omega = [x_z|\cdots|x_{z+d-2}]$.

1. $a - i \equiv 1 \mod d$ in which case the relevant part of s^i looks like

$$\ldots, x_{a-2}, x_{a-1}x_a, x_{a+1}, \ldots, x_{a+d-2}, x_{a+d-1}x_{a+d}, x_{a+d+1}, \ldots.$$

   We choose $\alpha' = [x_{a+1}|\cdots|x_{a+d-2}]$.

2. $a - i \equiv 2 \mod d$ in which case the relevant part of s^i looks like

$$\ldots, x_{a-3}, x_{a-2}x_{a-1}, x_a, \ldots, x_{a+d-3}, x_{a+d-2}x_{a+d-1}, x_{a+d}, \ldots.$$

   We choose $\alpha' = [x_a|\cdots|x_{a+d-3}]$.

3. $a - i \equiv k \mod d$ with $3 \leq k \leq d$ in which case the relevant part of s^i looks like

$$\ldots, x_a, x_{a+1}, \ldots, x_{a+i}x_{a+i+1}, \ldots, x_{a+d-1}, x_{a+d}, \ldots.$$

   We choose $\alpha' = [x_a|\cdots|x_{a+i}x_{a+i+1}|\cdots|x_{a+d-2}]$.

In all cases α' is a cell in the bottom row of g' and a face of α. The same argument works for ω as well. We see that σ is indeed in a fibre component between two cells in X'_W.

• For the case where τ is in the bottom row of f' we may set $\tau = [x_0|\cdots|x_{d-2}]$. We can use the same methods as before. The extra two faces $[x_1|\cdots|x_{d-2}]$ and $[x_0|\cdots|x_{d-3}]$ to consider do not pose any extra difficulty. We can choose an appropriate s^i and proceed as before.

We now focus on the case where $\tau \in f$ and f is finite. By Remark 4.5 the sequence s associated to f will be repeating. We can again case–split whether τ is in the top or bottom row of the Hasse diagram for f. In either case, a face $\sigma \lessdot \tau$ will be in a component $g \in F$ associated to an appropriately chosen s^i. This s^i will also be repeating, thus the associated g will be finite. $\square$

For the definition of K' to be well–defined, we require that the construction of each f' be well–defined. This requires that $f \cap X'_W \neq \emptyset$ for each infinite f. This happens to be a detail for which the proof relies on W being of affine type.

Lemma 4.8 ([PS21, Lemma 7.6])**.** *Given a Coxeter group W of affine type and $f \in F$ a d–fibre component, there exists a simplex $\tau = [x_1|x_2|\cdots|x_{d-1}] \in f$ such that τ is also in X'_W.*

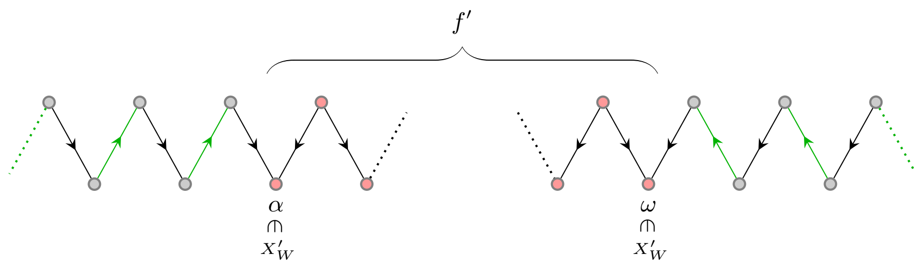


Figure 13: The unique acyclic matching $\mathcal{M}_f$ on f such that the critical cells correspond to exactly f'. Here α and ω are the relevant first and last elements in $f \cap X'_W$ as in the proof of Lemma 4.7. Critical cells are red and edges in $\mathcal{M}_f$ are green.

This covers all the details of Definition 4.6 and so K'_W is a valid subcomplex of K_W. We now get to reap the benefits of working in this strange setting of face posets.

Theorem 4.9 ([PS21, Theorem 7.9, Lemma 7.11])**.** *The subcomplex K'_W is a deformation retract of K_W.*

Proof. We need to show that for each $f \in F$, we have an acyclic matching $\mathcal{M}_f$ such that the critical vertices corresponding to $\mathcal{M}_f$ are exactly those contained in K'_W. For finite f, this is trivial. We choose $\mathcal{M}_f = \emptyset$. For infinite f, choosing such an $\mathcal{M}_f$ is very simple. As shown in Fig. 13. There is a unique choice of $\mathcal{M}_f$. The union of these $\mathcal{M}_f$ is an acyclic matching on $\mathcal{F}(K_W)$ by Theorem 4.4. This acyclic matching gives a deformation retract on to the union of critical cells, which is exactly K'_W, by Theorem 4.2. □

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