Combinatoric and poset structures for the $K(\pi, 1)$ conjecture

by

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Abstract This work reviews some central results from a paper by Paolini and Salvetti [PS21] where the authors prove the $K(\pi, 1)$ conjecture for affine Coxeter groups. This conjecture concerns the homotopy groups of the configuration space Y_W constructed from a Coxeter group W. We provide background for readers unfamiliar with the relevant groups or with posets and their applications to topology, including some new proofs. Using this background we introduce the interval complex K_W. We introduce the subspaces $X'_W, K'_W \subseteq K_W$ and introduce the Salvetti complex X_W which is known to be homotopy equivalent to Y_W. We show, as in [PS21], that $X_W \simeq X'_W$ and that $K'_W \simeq K_W$. We conclude by exploring a new construction, *fibre doubling*, and how it might be utilised to could show $K'_W \simeq X'_W$ for some classes of W.

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Contents

Chapter 1

Introduction

A classifying space for a group G is a space X such that the fundamental group of X is G and all higher homotopy groups of X are trivial. In this paper we will be concerned with the $K(\pi, 1)$ conjecture for Artin groups, which states that the configuration space Y_W for any Coxeter group W is a classifying space for the Artin group G_W. This conjecture emerges as a generalisation of the result for Coxeter groups of type A_n and is originally attributed to Arnol'd, Brieskorn, Pham and Thom in [Par14]. See also [CD95] for a good overview of the history of the conjecture.

We will focus on the work of Paolini and Salvetti, *Proof of the $K(\pi, 1)$ conjecture for affine Artin groups* [PS21]. We will review some theorems therein and provide relevant background. The theorem which is the namesake of [PS21] relies on proving a chain of homotopy equivalences (1). Detailing two of these homotopy equivalences is the aim of this work. A strong theme will be the involvement of posets and related structures, hence this work's title. We will begin by providing a birds eye view of the conjecture and the main results of [PS21].

The main results of this work are Theorems 2.10, 3.11 and 4.12. Theorem 2.10 is stated by McCammond in [McC05]. It is not proven there, but it is alluded to being widely accepted as true. Its proof here and the lemmas leading to that proof are the author's own work. Theorem 3.11 is proven by Paolini and Salvetti in [PS21]. The proof here follows that in [PS21] but fills in significant details. The proof of Theorem 4.12 largely follows that in [PS21]. The main original contribution to the proof of Theorem 4.12 here is the proof of Lemma 4.10, which is omitted in [PS21]. Fibre doubling and the concepts explored in Chapter 5 are the authors own work.

1.1 The conjecture and the objects involved

Coxeter groups emerge as generalisations of reflection groups. A Coxeter group is defined by a particular group presentation. The data of this presentation is typically encoded by a labelled graph. The group W, coupled with the data of its presentation is called a Coxeter system, denoted (W, S) where S is the generating set of W. Given a Coxeter system (W, S), we can construct a different group G_W, called the Artin group associated to W.

For affine Coxeter groups W, the configuration space Y_W can be derived from a geometric realisation of W as a subgroup of $\mathrm{Isom}(\mathbb{E})$, the group of isometries on a Euclidean space $\mathbb{E}$. We will consider $\mathbb{E}$ as $\mathbb{R}^n$ without the notion of origin. Specifically, W is realised as a subgroup generated by a finite set of affine reflections S. Within W, we consider the set of all reflections R (not necessarily finite). To each reflection $r \in R$ there is a corresponding codimension–1 space $H_r \subset \mathbb{E}$ that is the plane of reflection of r. We call such spaces hyperplanes. Note there is no requirement that these hyperplanes be subspaces of $\mathbb{R}^n$.

The configuration space is realised as the complement of the complexification of all such hyperplanes H_r. It is known by work of Brieskorn [Bri71] that the fundamental group of Y_W is G_W. Thus, proving the $K(\pi, 1)$ involves showing that the higher homotopy groups of Y_W are trivial. By previous work by Salvetti [Sal87, Sal94], there is a CW–complex X_W called the Salvetti complex that is homotopy equivalent to Y_W. Showing homotopy equivalence to X_W thus shows homotopy equivalence to Y_W. Because of this, the Salvetti complex is the starting point in a chain of homotopy equivalences reviewed in this work.

The finishing point of this chain is the interval complex K_W. This is a space realised using a certain poset structure on subsets of W. To this poset structure there is an associated group called the dual Artin group, denoted W_w. It was already known (by a now standard construction due to Garside [Gar69], extended by other authors, see [CMW02]) that K_W was a classifying space for the dual Artin group for finite W. In [PS21], the authors extend this result to affine W. Thus, showing $Y_W \simeq K_W$ for affine W shows that (for affine W) the higher homotopy groups of Y_W are trivial and that $W_w \cong G_W$.

In the following section, we will identify the intermediate spaces used in proving $X_W \simeq K_W$.

1.2 Proof overview

Here we will compile several main results from [PS21] in to two theorems. The concern of this work is Theorem 1.1 which proves that the *Salvetti complex* X_W is homotopy equivalent to the *interval complex* K_W. A *Coxeter element* is a non–repeating product of all the elements of S. A choice of order on S corresponds to a choice of Coxeter element. Constructing an interval complex associated to (W, S) involves making such a choice of Coxeter element $w \in W$.

For a subset $T \subseteq S$, the *parabolic subgroup* W_T is the subgroup of W generated only by elements of T and only with relations explicitly involving elements of T. A *parabolic Coxeter element* w_T is a product of all elements of T that respects the order of multiplication in a Coxeter element $w \in W$. The space X'_W is a subspace of K_W associated to parabolic Coxeter elements w_T with $T \subseteq S$ such that W_T is finite. Cells in X_W also correspond to such subsets, which is used in proving $X_W \simeq X'_W$.

The space K'_W is also a subspace of K_W. Given a CW–complex X, we can encode some information of how cells of X attach to each other in a poset called the *face poset* of X, denoted $\mathcal{F}(X)$. Connected components of preimages $\eta^{-1}(d)$ of a certain poset map $\eta\colon K_W \to \mathbb{N}$ have a linear structure as subposets of $\mathcal{F}(K_W)$. For each element $x \in \eta^{-1}(d)$, whether x is in K'_W or not is determined based on whether x comes in between two elements of X'_W in the linear structure of $\eta^{-1}(d)$.

Theorem 1.1 ([PS21])**.** *Given an affine Coxeter system (W, S), the configuration space Y_W is homotopy equivalent to the order complex K_W.*

Proof. By Theorem 2.23 the Salvetti complex X_W is homotopy equivalent to the configuration space Y_W. Therefore, we need only show $K_W \simeq X_W$. This is done through a composition of homotopy equivalences

$$X_W \overset{(a)}{\simeq} X'_W \overset{(b)}{\simeq} K'_W \overset{(c)}{\simeq} K_W \tag{1}$$

Where the results are gathered from the following sources:

(a) Theorem 3.11 [PS21, Theorem 5.5]

(b) Theorem 4.13 [PS21, Theorem 8.14]

(c) Theorem 4.12 [PS21, Theorem 7.9] □

In [PS21], another main result is shown.

Theorem 1.2 ([PS21, Theorem 6.6]). *Given an affine Coxeter system (W, S), corresponding affine type Artin group G_W and Coxeter element $w \in W$, the complex K_W is a classifying space for the dual Artin group W_w.*

We have that $\pi_1(Y_W) \cong G_W$ by [Bri71]. Thus, considering $\pi_1(Y_W)$ and combining Theorems 1.1 and 1.2 gives

$$Y_W \simeq K(G_W, 1)$$
$$G_W \cong W_w$$

for affine G_W.

This proves the $K(\pi, 1)$ conjecture for affine Artin groups and provides a new proof than an affine Artin group is naturally isomorphic to its dual, which was already known for spherical or finite type Artin groups [Bes03] and affine Artin groups [MS17].

Non–complete alternatives to [Bri71] are [VD83] and [FN62b], which show $\pi_1(Y_W) \cong G_W$ for affine Coxeter groups and Coxeter groups of type A_n respectively.

1.3 Coxeter groups and Artin groups

In this section we will cover the constructions and some properties of Coxeter groups and Artin groups. Coxeter groups are a generalisation of *reflection groups*, which are subgroups of $GL_n(\mathbb{R})$ generated by a finite set of reflections. Although the definition of Coxeter groups is tied to an abstract group presentation, we may also think of them as groups acting on a space by compositions of reflections. For example, finite Coxeter groups can be realised as reflection groups on spheres and affine Coxeter groups can be realised as groups generated by affine reflections in $\mathbb{R}^n$ (with plane of reflection not necessarily passing through the origin). Note that the realisation of a Coxeter group as a group generated by reflections is not unique and that some Coxeter groups cannot be realised as a subgroup of $GL_n(\mathbb{R})$. The ability to realise affine W as groups generated by reflections in $\mathbb{R}^n$ is a key tool in [PS21].

Definition 1.3. Given a finite set S, let $m\colon S \times S \to \mathbb{N} \cup \{\infty\}$ be a symmetric matrix indexed over S such that $m(s, s) = 1$ for all $s \in S$ and $m(s, t)$ takes values in $\{2, 3, \ldots\} \cup \{\infty\}$ for all $s \neq t$. The *Coxeter group* W associated to m is the group with the following presentation.

$$W = \langle\, S \mid (st)^{m(s,t)} = 1 \quad \forall m(s, t) \neq \infty \,\rangle$$

The data of m and S along with the associated Coxeter group W is denoted (W, S) and called a *Coxeter system.*

Throughout this work we will use 1 to denote the identity element in a group and $\{1\}$ to denote the trivial group. The pairs (s,t) such that $m(s,t)=\infty$ are pairs of generators that have no explicit relations. Since $m(s,s)=1$ for all $s\in S$, all generators have order 2 and $S=S^{-1}$. Note that the data of m is not uniquely determined by the isomorphism class of W, hence the need to distinguish a Coxeter system, not just a Coxeter group. The set $R := \{wsw^{-1} \mid w\in W, s\in S\}$ is the set of *reflections* in W. Sometimes S is referred to as the set of *basic reflections*, or that a choice of S is a choice of basic reflections.

A labelled graph, called the *Coxeter diagram*, is often used to encode the data of the matrix m and its corresponding Coxeter group. In this graph, each element of S is a node and relations between pairs in S correspond to labelled edges. There are two conventions for this labelling. The *classical labelling*, where edges with $m(s,t)=2$ are not drawn, edges with $m(s,t)=3$ are drawn but not labelled and all other edges are drawn with the value of $m(s,t)$ as their label. And the *modern labelling*, where edges with $m(s,t)=\infty$ are not drawn, edges with $m(s,t)=2$ are drawn but not labelled and all other edges are drawn and labelled. An example highlighting these different conventions is given in Fig. 1. We will only use the classical labelling here, but awareness of the modern labelling is useful.

Figure 1: Coxeter diagram for a certain Coxeter group with classical labelling (left) and modern labelling (right).

In the classical labelling, if the diagram has multiple connected components then W is a direct product of the Coxeter groups corresponding to those components. Similarly, in the modern labelling, connected components correspond to factors in a free product. Other topological properties of these diagrams can be used, for example, in work of Huang [Hua23] which proves the $K(\pi,1)$ conjecture for certain W with diagrams being trees or containing cycles. The property of Coxeter groups that allows us to make this graph construction is that every relation in a Coxeter group only involves two generators and is encoded by a number.

To each Coxeter system (W,S) there is an associated Artin group G_W defined as follows.

Definition 1.4. For group elements s and t, let $\Pi(s,t\,;n)$ be the alternating product of s and t starting with s with total length n, e.g. $\Pi(s,t\,;3)=sts$. Given a Coxeter system

(W, S) with associated matrix m, the associated *Artin group* is

$$G_W := \langle S \mid \Pi(s,t\,;\,m(s,t)) = \Pi(t,s\,;\,m(s,t)) \;\; \forall s \neq t \text{ and } m(s,t) \neq \infty \rangle.$$

Note that $m(s,s) = 1$ now carries no meaning in the presentation of G_W and that if we add the relations $s^2 = 1$ for all $s \in S$ we retrieve the original Coxeter group. The Coxeter diagram for W also encodes the data of G_W and the connected components of the diagram correspond to factors of G_W as a direct product or as a free product as with W.

Our notation for Artin groups (as with much of the notation here) is from [PS21]. Another common notation is W_Γ and A_Γ for the Coxeter and Artin groups corresponding to the Coxeter diagram Γ. When classifying Artin groups, it is common to inherit properties from the corresponding Coxeter group such that "`property` (type) Artin groups" describes a family of Artin groups to which their corresponding Coxeter groups are `property`.

In particular, spherical or finite type Artin groups have associated spherical or finite Coxeter groups. Similarly, affine Artin groups have associated Coxeter groups which are affine.

1.4 The configuration space

This section contains the definition of the configuration space Y_W for a given Coxeter group W. We also introduce the A_n family of Coxeter groups for which the space Y_W is the space of configurations of $n+1$ labelled points in $\mathbb{C}$.

For some finite or affine Coxeter group W acting on $\mathbb{R}^n$, the set of reflections $R \in W$ acts on $\mathbb{R}^n$ by reflection through hyperplanes, one for each $r \in R$. For some $r \in R$, denote its hyperplane by $H_r \subseteq \mathbb{R}^n$. Denote the union of all hyperplanes by $\mathcal{H} := \bigcup_{r \in R} H_r$. Consider the tensor product $\mathbb{R}^n \otimes \mathbb{C}$. This is isomorphic to $\mathbb{C}^n$ under the natural isomorphism $x \otimes \lambda \mapsto x\lambda$. We can extend the action $W \circlearrowleft \mathbb{R}^n$ to $W \circlearrowleft (\mathbb{R}^n \otimes \mathbb{C})$ via $w \cdot (x \otimes \lambda) = (w \cdot x) \otimes \lambda$. We call this act of transporting objects related to $\mathbb{R}^n$ over to $\mathbb{R}^n \times \mathbb{C}$ (which is isomorphic to $\mathbb{C}^n$) via the tensor product *complexification.*

Remark 1.5. Since R generates W, the action $W \circlearrowleft \mathbb{R}^n$ fixes $\mathcal{H}$ and by the nature of the action $R \circlearrowleft \mathbb{R}^n$, we have that $w \cdot x \in \mathcal{H} \implies x \in \mathcal{H}$.

Definition 1.6. For an affine Coxeter group W and associated hyperplane system $\mathcal{H}$ as above, we define

$$Y := (\mathbb{R}^n \otimes \mathbb{C}) \setminus (\mathcal{H} \otimes \mathbb{C})$$

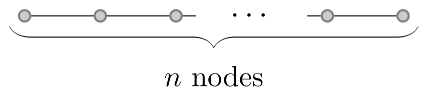


Figure 2: The classical Coxeter diagram for the Coxeter group of type A_n.

and define the *configuration space* Y_W to be the quotient Y/W with the action of W defined above. This action is well–defined by Remark 1.5.

Note that the importance of $\mathbb{C}$ is that it is 2–dimensional. When one takes the complement of a codimension–1 object, typically the resulting topology is not very interesting. By complexifying the hyperplanes and then taking the complement within $\mathbb{R}^n \otimes \mathbb{C}$, we are effectively taking the complement of a codimension–2 object, and there is much more room for interesting topologies. The same construction can be achieved using $\mathbb{R}^{2n}$ and $\mathcal{H} \times \mathcal{H}$. A more general construction of Y_W for all Coxeter groups using the *Tits cone* can be found in [Par14]. We will not go in to the details of this construction, but will assume Y_W to be defined for all Coxeter groups W, not just affine W as in Definition 1.6.

For a concrete example concerning Y_W, we will introduce the A_n family of Coxeter groups and show that the space Y_W for these groups is the space of configurations of $n+1$ points in $\mathbb{C}$, thus explaining the name *configuration space* for general Y_W.

The family A_n all have Coxeter diagrams of the form as in Fig. 2 and a specific A_n will have presentation.

$$A_n = \left\langle \sigma_1, \sigma_2, \ldots, \sigma_n \;\middle|\; \begin{array}{rl} \sigma_i^2 = 1 & \forall i \\ (\sigma_i\sigma_j)^2 = 1 & \forall (i+1 < j \leq n) \\ (\sigma_i\sigma_{i+1})^3 = 1 & \forall (i < n) \end{array} \right\rangle \tag{2}$$

This is well known to be a presentation for the symmetric group S_{n+1} with generators being adjacent transpositions [BB10, Proposition 1.5.4]. Accordingly, we will use the associated cycle notation for symmetric groups to talk about elements of A_n.

The action of A_n as a reflection group is realised on the space $\mathbb{R}^{n+1}$ with basis $\{e_i\}$, where A_n acts on $\mathbb{R}^{n+1}$ by permuting components with respect to that basis. The set of reflections R of A_n is all conjugations of the n adjacent generating transpositions $(l, l+1)$. So, R is the set of all transpositions (l, k). Some $(l, k) \in R$ acts on $\mathbb{R}^{n+1}$ as reflection through the plane $\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_l = x_k\}$. Thus, taking the complement of the complexification of all such planes, we have $Y = \{(\mu_1, \ldots, \mu_{n+1}) \in \mathbb{C}^{n+1} \mid \forall i, j \;\; \mu_i \neq \mu_j\}$ (here Y is as in Definition 1.6). We can think of this as the space of $n+1$ distinct, labelled points in $\mathbb{C}$, denoted $\mathrm{Conf}_{n+1}(\mathbb{C})$. The action of A_n on $\mathbb{R}^n \otimes \mathbb{C}$ is also by permutation of components, so $Y_W = Y/A_n$ is the space of $n+1$ *unlabelled* points in $\mathbb{C}$, denoted $\overline{\mathrm{Conf}}_n(\mathbb{C})$

Historically, Artin [Art47] originally defined the braid group on n strands B_n to be $\pi_1(\overline{\mathrm{Conf}_n}(\mathbb{C}))$. He then proved the well known presentation of the braid group.

$$B_n = \left\langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{cc} \sigma_i\sigma_j = \sigma_j\sigma_i & \forall(i+1 < j \le n) \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \forall(i < n) \end{array} \right\rangle$$

In this context, showing the validity of the presentation immediately proves $B_{n+1} \cong G_W$ and thus that $\pi_1(Y_W) \cong G_W$. See the work of Fox and Neuwirth [FN62b] for an alternative proof of the presentation.

1.5 The $K(\pi, 1)$ conjecture

Given a group G and natural number n, an *Eilenberg-MacLane space* [EM45] is a space X such that $\pi_n(X) = G$ and $\pi_i(X) = 0$ for all $1 \le i \ne n$. We call such an X a *$K(G, n)$ space.* We will also use the terminology *classifying space for G* to mean that X is a $K(G, 1)$ space.

Conjecture 1.7 ($K(\pi, 1)$ Conjecture)**.** *For all Coxeter groups W, the space Y_W is a $K(G_W, 1)$ space.*

Admittedly, the use of π in the name of the conjecture is confusing. An equivalent formulation of the conjecture is that the universal cover of Y_W is contractible. These statements are equivalent since a cover $p\colon \widetilde{X} \to X$, induces an isomorphism $p_*\colon \pi_n(\widetilde{X}, \widetilde{x_0}) \to \pi_n(X, x_0)$ for all $n \ge 2$ [Hat01, Proposition 4.1].

In the previous section we focused on Coxeter groups of type A_n. We gave context to and quoted the result that $\pi_1(Y_W) \cong G_W$ for these Coxeter groups. We now prove the $K(\pi, 1)$ conjecture for this same family of Coxeter groups. To do so, we need to verify that the higher homotopy groups of Y_W are trivial.

Lemma 1.8. *For all $k > 1$, $\pi_k(\mathrm{Conf}_n(\mathbb{C}))$ is trivial.*

Proof. We use a result of Fadell and Neuwirth [FN62a, Theorem 3] which tells us that $\mathrm{Conf}_n(\mathbb{C})$ is a fibre bundle over $\mathrm{Conf}_{n-1}(\mathbb{C})$ with projection p forgetting a point and fibres homeomorphic to $\mathbb{C} \setminus \{\text{n distinct points}\}$.

The space $\mathbb{C} \setminus \{\text{n distinct points}\}$ is homotopy equivalent to $\bigvee_n S^1$, so we have the fibration $\mathbb{C} \setminus \{\text{n distinct points}\} \hookrightarrow \mathrm{Conf}_n(\mathbb{C}) \to \mathrm{Conf}_{n-1}(\mathbb{C})$ and the corresponding short exact sequence

$$\pi_k(\textstyle\bigvee_{n-1} S^1) \longrightarrow \pi_k(\mathrm{Conf}_n(\mathbb{C})) \xrightarrow{p} \pi_k(\mathrm{Conf}_{n-1}(\mathbb{C})) \tag{3}$$

for all k. We prove that $\pi_k(\mathrm{Conf}_n \mathbb{C}) = 0$ for all $k > 1$ by induction on n. The base case is $n = 1$, where trivially $\mathrm{Conf}_n(\mathbb{C}) \cong \mathbb{C}$ which is contractible. For the inductive step, we know $\pi_k(\bigvee_n S^1) = 0$ for all $k > 1$ and for all n. So the leftmost term in (3) is always trivial. Assuming now that $\pi_k(\mathrm{Conf}_{n-1}(\mathbb{C})) = 0$ for all $k > 1$, the inductive step follows immediately from the short exact sequence. $\square$

Note that so far we have only proved that Y (from Definition 1.6) has trivial higher homotopy, not Y_W (which in this case is $\overline{\mathrm{Conf}}_n(\mathbb{C})$). We now prove this.

Theorem 1.9. *The $K(\pi, 1)$ conjecture holds for Coxeter groups of type A_n.*

Proof. Denote $\mathrm{Conf}_n(\mathbb{C})$ and $\overline{\mathrm{Conf}}_n(\mathbb{C})$ by Y and Y_W respectively. Let $p\colon \widetilde{Y} \to Y$ be the universal cover for Y. Let $q\colon Y \to Y_W$ be the quotient map induced by the action $A_n \circlearrowleft Y$ as in Definition 1.6. We see that q is a covering map such that $q^{-1}(z)$ is finite for all $z \in Y_W$. Thus, by [Mun00, Exercise 53.4] the composition $q \circ p$ is also a covering map so $\widetilde{Y}$ is also the universal cover of Y_W and so Y_W also has trivial higher homotopy by Lemma 1.8 and [Hat01, Proposition 4.1]. $\square$

A vital result due to Deligne expands on this.

Theorem 1.10 ([Del72])**.** *The $K(\pi, 1)$ conjecture holds for all finite Coxeter groups W.*

The paper of interest to us, [PS21], proves the $K(\pi, 1)$ conjecture for affine Coxeter groups.

Chapter 2

Geometric realisations of poset structures

In Section 1.4 we used the realisation of the Coxeter group W as a reflection group on an affine space V. We considered the planes of the defining reflections of W as affine subspaces of V and used these to define Y_W, the configuration space. To progress with the $K(\pi, 1)$ conjecture we must develop some theory to explore the homotopic properties of Y_W. To do this, we will first construct a new space X_W, the *Salvetti complex*, which is homotopy equivalent to Y_W. This was originally defined [Sal87, Sal94] similarly using the realisation of W as symmetries of a space. However, the Salvetti complex turns out to have another formulation based on purely algebraic properties of (W, S) and the poset structure of S with the partial order induced by $\subseteq$. We will eventually relate the Salvetti complex to another space, K_W which is described by a partial order on W. With this in mind, we start by introducing posets.

2.1 Posets

A partially ordered set or *poset* $(P, \leq)$ is a set P with a relation $\leq$ on pairs in P with the same restrictions we would expect considering $\leq$ as a relator on $\mathbb{R}$. The textbook [Grä11] provides a good introduction. An important note is that there is no requirement for every pair to be related, i.e. there may exist pairs (x, y) such that neither $x \leq y$ nor $y \leq x$ are true, hence the name *partial order*. We will use P as shorthand for $(P, \leq)$ where possible.

In a poset P we define the *interval* between two elements $[x, y]$ as $[x, y] \coloneqq \{u \in P \mid x \leq u \leq y\}$, which is itself a poset. For convenience, we define $[-\infty, w] \coloneqq \{u \in P \mid u \leq w\}$ and equivalently for $[w, \infty]$. A *chain* is a subset $C \subseteq P$ that is a totally ordered, i.e. every pair in $(u, v) \in C \times C$ satisfies $u \leq v$ or $v \leq u$. The *covering relations* of P, denoted $\mathcal{E}(P)$

are defined as follows.

$$\mathcal{E}(P) = \{(x, y) \in P \times P \mid x \leq y \text{ and } [x, y] = \{x, y\}\}$$

These are strictly ordered pairs $x < y$ such that there does not exist any $z \in P$ such that $x < z < y$. If $(x, y) \in \mathcal{E}(P)$, we write $x \lessdot y$. We will call a chain C *saturated* if for all $x, y \in C$ such that $x < y$, there exists $z \in C$ such that $x \lessdot z$, i.e. there are no 'gaps' in the chain.

By transitivity, the covering relations encode the whole poset structure, which can in turn be drawn in a diagram.

Definition 2.1. Given a poset P, the *Hasse Diagram* is the directed graph encoding $\mathcal{E}(P)$ in the following way. For each element $x \in P$ draw a vertex. For each pair $(x, y) \in \mathcal{E}(P)$ draw a directed edge from x to y.

As is typical, we will draw Hasse diagrams such that for each edge $x \lessdot y$, the vertex x will be at a lower position on the page than y. Thus, we will not need to draw arrows to show direction. In this work, we will typically deal with graded and bounded posets. *Bounded* means that there are minimal and maximal elements, denoted $\hat{0}$ and $\hat{1}$ such that $\hat{0} \leq x \leq \hat{1}$ for all $x \in P$, and *graded* means that every saturated chain from $\hat{0}$ to $\hat{1}$ has the same (finite) length. In the Hasse diagram for a bounded, graded poset, we will draw $\hat{0}$ at the bottom, $\hat{1}$ at the top, and put all other elements in discrete vertical levels between these based on the position in the saturated chains between $\hat{0}$ and $\hat{1}$ where each element occurs. See Fig. 7 for an example. Graded posets have a natural notion of a *rank function* $\mathrm{rk}\colon P \to \mathbb{N}$ that encodes the height above $\hat{0}$ at which an element $p \in P$ occurs in the Hasse diagram. Rank is also well–defined for posets with multiple minimal or maximal elements, so long as all saturated chains from any minimal element to any maximal element have the same length.

Definition 2.2. We define an *edge labelled poset* to be a triple $(P, \leq, l)$ where $(P, \leq)$ is a poset and the function $l : \mathcal{E}(P) \to A$ is a labelling of covering relations with alphabet A. For simplicity of notation, we require that l be surjective.

We will use P as a shorthand for $(P, \leq, l)$ where possible. The labelling l also gives a labelling on the edges of the Hasse Diagram for P.

Let A^* denote all formal words in A. The *word corresponding to a saturated chain* is the word in A^* of the labels traversed in the Hasse diagram while tracing out that saturated chain. Given an edge labelled poset P, we can construct a group encoded by its labelling the and geometry of its Hasse diagram in the following way.

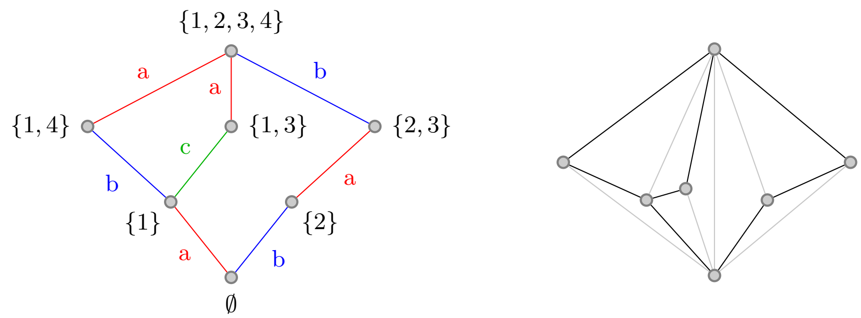


Figure 3: A simple example of a bounded and graded edge labelled poset where we have taken $\leq$ to be $\subseteq$ and $A = \{a, b, c\}$ (left). The same poset with all non-covering 1–chains drawn in faint lines to aid visualising $\Delta(P)$ introduced in Section 2.2 (right).

Definition 2.3. Given some finite–height edge labelled poset $(P, \leq, l\colon \mathcal{E}(P) \to A)$, define the *poset group* $G(P)$ to be the group generated by A with relations equating words corresponding to saturated chains in the Hasse diagram of P which start and end at the same elements.

In the example given in Fig. 3, the poset group is $G(P) = \langle a, b, c \mid aba = bab,\, ba = ca \rangle$.

2.2 The poset complex

Given some edge labelled poset P, we can construct a cell complex $K(P)$ from P such that $\pi_1(K(P))$ is $G(P)$. First we make some definitions. An *abstract simplicial complex* is a family of sets that is closed under taking arbitrary subsets.

Definition 2.4. Given a finite abstract simplicial complex X, the *geometric realisation* of that simplicial complex is defined as follows. For each single element set in X assign a point. For each two element set assign an open 1–simplex between the two vertices it contains. For each three element set assign an open 2–simplex, which is the interior of the three 1–simplices corresponding to its three subsets of size two. In this way, continue constructing simplices of dimension n for each $n + 1$ size set in X.

The set of all chains in a poset P is an abstract simplicial complex. We define $\Delta(P)$ to be the geometric simplicial complex corresponding to the set of all chains in P where each n–simplex is an n–chain of P. Note that as in [MS17, Definition 1.7], we define an n–chain to have $n - 1$ elements, e.g. $(\{1\} \subseteq \{1, 2\})$ is a 1–chain.

For example, in Fig. 3, $\Delta(P)$ would be three 3–simplices all sharing an edge (a 1–simplex) corresponding to the 1–chain $(\emptyset \subseteq \{1, 2, 3, 4\})$. Two of the 3–simplices would share a face corresponding to the 2–chain $(\emptyset \subseteq \{1\} \subseteq \{1, 2, 3, 4\})$.

We also assign an orientation on edges in $\Delta(P)$ such that the edge corresponding to the 1–chain $(x \le y)$ points from x to y. For a two–dimensional example, consider the following poset P and corresponding $\Delta(P)$. Here we forget about edge labelling in P for a moment.

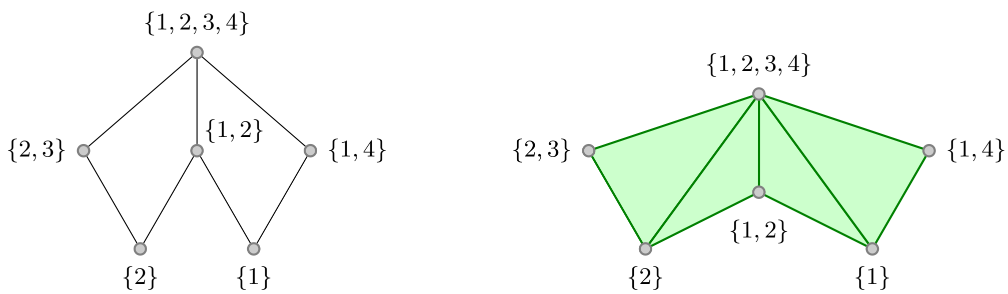


Figure 4: An example poset P (left) and corresponding $\Delta(P)$ (right).

We continue, now using an edge labelling on P (Fig. 5 (left)).

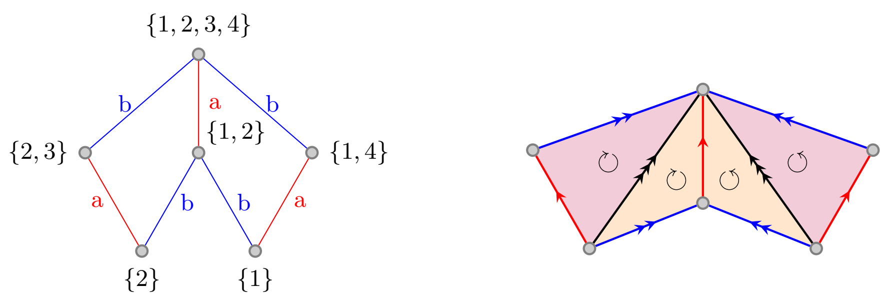


Figure 5: The poset in Fig. 4 with edge labelling (left) and the corresponding space $K(P)$ (right).

To construct $K(P)$, first we define a labelling on 1–chains in P (which are edges in $\Delta(P)$) which extends the edge labelling l.

Definition 2.5. Given some edge–labelled poset $(P, \le, l\colon \mathcal{E}(P) \to A)$ and some 1–chain $\sigma = (x \le y)$ in P, the *extended label* $\mathcal{L}(\sigma) \subseteq A^*$ is the language of all words corresponding to all saturated chains that start at x and end at y. For any element $p \in P$ (a 0–chain), define $\mathcal{L}(p)$ to be $\emptyset$.

For example, consider the chain $(\{2\} \subseteq \{1,2,3,4\})$ in the context of Fig. 5 (left). There are two corresponding saturated chains, $(\{2\} \subseteq \{1,2\} \subseteq \{1,2,3,4\})$ and $(\{2\} \subseteq \{2,3\} \subseteq \{1,2,3,4\})$, which respectively correspond to the words ba and ab. So $\mathcal{L}(\{2\} \subseteq \{1,2,3,4\}) = \{ba, ab\}$. Here are some illustrative examples:

- $\mathcal{L}(\{1\} \subseteq \{1,2\}) = \mathcal{L}(\{2\} \subseteq \{1,2\}) = \{b\}$.
- $\mathcal{L}(\{1\} \subseteq \{1,2,3,4\}) = \mathcal{L}(\{2\} \subseteq \{1,2,3,4\}) = \{ba, ab\}$.

- $\mathcal{L}(\{1\}) = \mathcal{L}(\{2\}) = \mathcal{L}(\{1,2\}) = \cdots = \emptyset$.

We use this extended labelling to form a quotient space of $\Delta(P)$. Recall the orientation of the edge $(x \leq y)$ is from x to y.

Definition 2.6 (Poset Complex [McC05, Definition 1.6])**.** Define the following relation $\sim$ on $\Delta(P)$. For any n, consider two n–simplices e^n_α and e^n_β. Denote the closure of these simplices a and b respectively, i.e. a and b are the open simplices along with all edges (with extended labels) and faces. If there exists an edge–orientation and extended label preserving homeomorphism $f\colon a \to b$, then we identify the points of $e^n_\alpha \subseteq a$ and $e^n_\beta \subseteq b$ under $\sim$ using the map f, i.e. $a \sim f(a)$. Increasing n from 0, we list every pair of n–simplices and identify pairs of points under $\sim$ as in the above construction. Finally, we take the transitive closure of $\sim$ once this construction is complete. Define the *poset complex* $K(P)$ to be the quotient space $\Delta(P)/\sim$.

In the example in Fig. 5, three red edges are identified, four blue edges are identified, two black edges are identified, two orange triangles are identified and two purple triangles are identified. The orientation of the triangles is indicated by a $\circlearrowleft$ symbol.

We see that this space is homeomorphic to a torus, which has fundamental group $\mathbb{Z}^2 \cong \langle a, b \mid ab = ba \rangle$, which is also $G(P)$ for this edge–labelled poset. This fact holds in general, which we now begin to prove.

Here, and in several other places in this work, we will compute the fundamental group of a CW–complex with only one 0–cell. Given such a CW–complex X, denote its set of 1–cells $\{e^1_\alpha\}_{\alpha \in A}$ and set of 2–cells $\{e^2_\beta\}_{\beta \in B}$. For each $\beta \in B$, let $\psi_b\colon S^1 \to X^1$ denote the attaching map for the 2–cell e^2_β. For each $\alpha \in A$, let λ_α denote a loop (in any orientation) going around the 1–cell e^1_α. For each $\beta \in B$, let r_β be a factorisation of $[\Psi_\beta]$ in the set $\{[\lambda_\alpha]\}_{\alpha \in A} \cup \{[\lambda_\alpha]^{-1}\}_{\alpha \in A}$, which is always possible. The fundamental group of X will have the following useful presentation using only this data [Hat01, Proposition 1.26].

$$\pi_1(X) = \langle\, \{[\lambda_\alpha]\}_{\alpha \in A} \mid \{r_\beta\}_{\beta \in B} \,\rangle \tag{4}$$

Each relation r_β should be thought of as the equation $r_\beta = 1$. Accepting that the fundamental group of X^1 is the free group generated by $\{[\lambda_\alpha]\}_{\alpha \in A}$, we get this result by repeated use of Van–Kampen's theorem, inductively attaching each 2–cell in $\{e^2_\beta\}_{\beta \in B}$. An immediate corollary of this is that the fundamental group of such a CW–complex depends only on its 2–skeleton. This generalises to all CW–complexes and for higher homotopy groups [Hat01, Corollary 4.12].

The following is a basic group theoretic fact, but it will be useful to explicitly state and prove it. For some set S, let F_S be the free group generated by S.

Lemma 2.7. *Suppose we have the groups G_1 and G_2 such that G_1 has a presentation $G_1 \cong \langle S \mid R \rangle$ where R is a set of words in $S \cup S^{-1}$ that are the identity in G_1. Further suppose we are given some map $f\colon S \to G_2$. For all $s \in S$, define $f(s^{-1})$ to be $f(s)^{-1}$. If for each word $s_1 s_2 \cdots s_n \in R$ we have that $f(s_1)f(s_1)\cdots f(s_n) = 1$ in G_2 then we can extend f to a homomorphism $h\colon G_1 \to G_2$ where $h|_S = f$ considering S as a subset of G_1.*

Proof. We can define a homomorphism $f'\colon F_S \to G_2$ recursively by setting $f'(1) = 1$, $f'(s^{\pm 1}) = f(s)^{\pm 1}$ for all $s \in S$ and then setting $f'(s_1 s_2 \cdots s_n) = f'(s_1)f'(s_2 \cdots s_n)$ for all words $s_1 s_2 \cdots s_n \in \{S \cup S^{-1}\}^*$. Let N be the minimal normal subgroup in F_S which contains the elements of R such that $G_1 \cong F_S/N$. By assumption, for each $r \in R$ we have $f'(r) = 1$. So $R \subseteq \ker(f')$. So $N \subseteq \ker(f')$. By the universal property of the quotient we have the following commutative diagram where the unique map h is the required homomorphism.

$$\begin{tikzcd} F_S \arrow[r, "f'"] \arrow[d, "q"'] & G_2 \\ F_S/N \cong G_1 \arrow[ur, "\exists! h"'] & \end{tikzcd}$$

□

Note that if $\langle \mathrm{Im}(f) \rangle = G_2$ then h is a surjection.

Lemma 2.8. *Given an edge labelled poset $(P, \leq, l\colon \mathcal{E}(P) \to A)$, the group $\pi_1(K(P))$ is generated by a set of loops in bijection with A.*

Proof. Each 0–simplex $p \in \Delta(P)$ has extended label $\emptyset$ and so there is only one 0–simplex in $K(P)$, denote this point p_0. The 1–skeleton of $K(P)$ will be a wedge of circles, one for each extended label in $\{\mathcal{L}(C) \mid C \text{ is a 1–chain}\}$. The fundamental group will have a presentation as in (4). The following is the setup of our proof.

1. Let Σ denote the set of all 1–chains in P.
2. Let Ω denote the set of paths between 0–simplices of $\Delta(P)$ along a single 1–simplex of $\Delta(P)$ such that the direction along the 1–simplex corresponding to $(x \leq y)$ is from x to y.
3. Let Λ denote the set of loops along 1–simplices in $K(P)$ based at p_0. If $\lambda \in \Lambda$, then $[\lambda] \in \pi_1(K(P), p_0)$.

Let α, be the map from 1–chains to corresponding paths in $\Delta(P)$ and $\beta\colon \Delta(P) \to K(P)$ be the quotient map as in the definition of $K(P)$. Furthermore, let β_* be the push forward map on paths. We have the following diagram.

$$\Sigma \xrightarrow{\alpha} \Omega \xrightarrow{\beta_*} \Lambda$$

By our remarks on the 1–skeleton of $K(P)$, we see that $\pi_1(K(P), p_0)$ is generated by the homotopy classes of $\mathrm{Im}(\beta_* \circ \alpha)$. Let $\sigma = (x < y) \in \Sigma$ be a 1–chain and let λ be such that $\beta_* \circ \alpha(\sigma) = \lambda$. Let concatenation of loops λ_1 and λ_2 be denoted $\lambda_1\lambda_2$ such that $[\lambda_1][\lambda_2] = [\lambda_1\lambda_2]$. We similarly denote concatenation of paths in Ω. We will show that $[\lambda]$ has a factorisation where one of the factors is $[\beta_* \circ \alpha(x \lessdot z)]$ for some $z \in P$.

There exists $z \in P$ such that $x \lessdot z \leq y$. If $z \neq y$, there is a 2–simplex in $\Delta(P)$ corresponding to the 2–chain $(x \lessdot z < y)$. One of the edges of this 2–simplex corresponds to the 1–chain $(x \lessdot z)$. The path $\alpha(x \leq y)$ is homotopic (through the 2–simplex $(x \lessdot z < y)$) to the path $\alpha(x \lessdot z)\alpha(z < y)$. Let H witness this homotopy such that $\beta \circ H$ factors through $I/(0 \sim 1) \times I \cong S^1 \times I$ as in the following diagram.

$$\begin{tikzcd}
I \times I \arrow[r, "H"] \arrow[d] & \Delta(P) \arrow[r, "\beta"] & K(P) \\
S^1 \times I \arrow[urr, "\overline{H}"'] & &
\end{tikzcd}$$

The map $\overline{H}$ witnesses the following homotopy of loops.

$$\lambda = \beta_*(\alpha(\sigma)) \sim \beta_*(\alpha(x \lessdot z)\alpha(z < y)) = (\beta_* \circ \alpha(x \lessdot z))(\beta_* \circ \alpha(z < y))$$

So we have $[\lambda] = [\beta_* \circ \alpha(x \lessdot z)][\beta_* \circ \alpha(z < y)]$. We then repeat the process, replacing σ with $(z < y)$. Eventually this must stop since our poset is of finite height. After this, we achieve a factorisation of $[\lambda]$, entirely in factors of the form $[\beta_* \circ \alpha(r \lessdot s)]$.

Consider $\mathcal{E}(P)$ as a subset of Σ. We see that $\pi_1(K(P), p_0)$ is generated by the homotopy classes of loops in $\mathrm{Im}(\beta_* \circ \alpha|_{\mathcal{E}(P)})$. Each covering relation $(r \lessdot s)$ has extended label $\{l(r \lessdot s)\}$. For each $a \in A$ there is exactly one edge in $K(P)$ with label $\{a\}$. For each a, let $\sigma_a = (x \lessdot y) \in \mathcal{E}(P)$ be a 1–chain such that $\mathcal{L}(\sigma_a) = \{a\}$. We have that $\theta\colon A \to \pi_1(K(P), p_0)$ defined as $\theta(a) = [\beta_* \circ \alpha(\sigma_a)]$ is our required bijection. Note that θ does not depend on the map $a \mapsto \sigma_a$ (for which there is, in principle, some choice to be made). □

Lemma 2.9. *For an edge-labelled poset P, there exists a surjective homomorphism $\varphi\colon G(P) \to \pi_1(K(P), p_0)$ where p_0 is as in the previous lemma.*

Proof. Let us follow from the notation in the proof of Lemma 2.8 and let $\theta\colon A \to \pi_1(K(P), p_0)$ be the bijection at the end of that proof such that we have $\langle \mathrm{Im}(\theta \circ l) \rangle = \pi_1(K(P), p_0)$.

Let $\sigma = (x_1 \lessdot \cdots \lessdot x_i)$ and $\sigma' = (x'_1 \lessdot \cdots \lessdot x'_j)$ be two saturated chains such that $x_1 = x'_1$ and $x_i = x'_j$ with corresponding words $w = w_1 \cdots w_{i-1}$, and $w' = w'_1 \cdots w'_{j-1}$ in

A^* such that $l(x_k \lessdot x_{k+1}) = w_k$ and similarly for σ'. Recall that w and w' are words that are identified by the relations in the defining presentation for $G(P)$. We want to show there exists a homotopy between representatives of $\theta(w_1)\cdots\theta(w_i)$ and representatives of $\theta(w_1')\cdots\theta(w_j')$.

By doing the process in the proof of Lemma 2.8 in reverse, we get a homotopy between a representative of $\theta(w_1)\cdots\theta(w_i)$ and $\beta_* \circ \alpha(x_1 \leq x_i)$ through the two skeleton of $K(P)$. By the same argument, we get a homotopy between a representative of $\theta(w_1')\cdots\theta(w_j')$ and $\beta_* \circ \alpha(x_1' \leq x_j')$. Since $x_1 = x_1'$ and $x_i = x_j'$, we have $\theta(w_1)\cdots\theta(w_i) = \theta(w_1')\cdots\theta(w_j')$.

We have shown that $\pi_1(K(P), p_0)$ has all necessary relations to extend θ to a surjective homomorphism $\varphi\colon G(P) \to \pi_1(K(P), p_0)$ by Lemma 2.7. □

Theorem 2.10. *Given an edge labelled poset* $(P, \leq, l\colon \mathcal{E}(P) \to A)$*, we have* $\pi_1(K(P)) \cong G(P)$*.*

Proof. We again follow the notation from Lemma 2.8. At the start of the proof of Lemma 2.8 we remarked that by the structure of the 1–skeleton of $K(P)$ that $\pi_1(K(P), p_0)$ is generated by a set of loops in bijection with

$$\mathcal{L}(\Sigma) \coloneqq \{\mathcal{L}(C) \mid C \text{ is a 1–chain}\}.$$

Of course, Lemma 2.8 tells us that this is not a minimal generating set. Let $\chi\colon \mathcal{L}(\Sigma) \to \pi_1(K(P), p_0)$ denote this bijection. We now think of this presentation abstractly, and work to give a set of relations R such that we obtain a group $\langle \mathcal{L}(\Sigma) \mid R\,\rangle \cong \pi_1(K(P), p_0)$ where the isomorphism is an extension of χ.

Considering the 2–cells attached to this 1–skeleton, (as in (4)) there is such a set of relations R in bijection with the set of 2–simplices in $K(P)$. Let e^2 be a 2–simplex with edges e_i^1 for $i \in \{1, 2, 3\}$. We have that the edges e_i^1 are oriented acyclically as in Fig. 6.

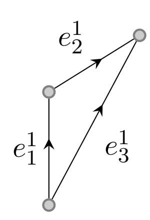


Figure 6: The orientation of all simplices in $\Delta(P)$ and $K(P)$.

All simplices are oriented this way because if $x \leq y$ and $y \leq z$ then $x \leq z$. Define loops $\lambda_i \in \Lambda$ such that each λ_i follows the oriented edge e_i. Choose extended labels ℓ_i such that $\chi(\ell_i) = [\lambda_i]$. We see that the 2–simplex e^2 corresponds to the relation $\ell_1\ell_2\ell_3^{-1} = 1$. Here multiplication of extended labels is formal, as are inverses, denoted ℓ^{-1}. Define R to be the following set of relations for $\langle\, \mathcal{L}(\Sigma)\,\rangle$.

$$R \coloneqq \left\{ \ell_1\ell_2\ell_3^{-1} = 1 \;\middle|\; \exists \text{ 2–chain } x < y < z \text{ where } \begin{array}{r} \mathcal{L}(x < y) = \ell_1 \\ \mathcal{L}(y < z) = \ell_2 \\ \mathcal{L}(x < z) = \ell_3 \end{array} \right\} \tag{5}$$

Following from the definition of $K(P)$, and considering its 2–skeleton, these are exactly the relations we need to extend χ to an isomorphism witnessing $\langle \mathcal{L}(\Sigma) \mid R \rangle \cong \pi_1(K(P), p_0)$.

We can choose (using axiom of choice if necessary) an element of ℓ for all $\ell \in \mathcal{L}(\Sigma)$. We encode this choice in the function $f\colon \mathcal{L}(\Sigma) \to A^*$ such that $f(\ell) \in \ell$ for all $\ell \in \mathcal{L}(\Sigma)$. Considering A as generators of $G(P)$ with $\{A \cup A^{-1}\}^*$ corresponding to elements of $G(P)$, we now work to show that f extends to a homomorphism $h\colon \langle \mathcal{L}(\Sigma) \mid R \rangle \to G(P)$.

Recall the definition of $\mathcal{L}$ and the underlying geometry of relations in R. Consider the relation $\ell_1\ell_2\ell_3^{-1} = 1$, as in (5). Let x, y and z be such that the relevant 2–chain corresponding to the relation $\ell_1\ell_2\ell_3^{-1} = 1$ is $x < y < z$. There exists a saturated chain $C_1 = (x_1 \lessdot \cdots \lessdot x_m)$ such that the word corresponding to C_1 is $f(\ell_1)$ with $x_1 = x$ and $x_m = y$. There also exists a chain $C_2 = (y_1 \lessdot \cdots \lessdot y_n)$ such that the word corresponding to C_2 is $f(\ell_2)$ with $y_1 = y$ and $y_n = z$. Finally, there is a chain $C_3 = (z_1 \lessdot \cdots \lessdot z_k)$ such that the word corresponding to C_3 is $f(\ell_3)$ with $z_1 = x$ and $z_k = z$. The chains $(x = x_1 \lessdot \cdots \lessdot x_m = y = y_1 \lessdot \cdots \lessdot y_n = z)$ and $(x = z_1 \lessdot \cdots \lessdot z_k = z)$ correspond to paths in the Hasse diagram of P that start and end at the same place. Thus, they correspond to the $G(P)$ relation $f(\ell_1)f(\ell_2) = f(\ell_3)$. Thus, by Lemma 2.7 we can extend f to a homomorphism $h\colon \langle \mathcal{L}(\Sigma) \mid R \rangle \to G(P)$.

Recall φ from Lemma 2.9. We have that $h(\{a\}) = a$, thus $h \circ \chi^{-1} \circ \varphi\colon G(P) \to G(P)$ is the identity on the generating set $A \subseteq G(P)$. Thus, $h \circ \chi^{-1} \circ \varphi$ is the identity and φ is an isomorphism. $\square$

Note that our proof of the above was ambivalent to the identification of n–simplices for $n \geq 2$. Indeed, these identifications do not affect the fundamental group, but they do ensure that higher homotopy groups are trivial. We can see if we did not identify the 2–simplices in Fig. 5, $\pi_2(K(P))$ would be non-trivial.

2.3 The interval complex

Starting from a Coxeter group W generated by S, we wish to give W a labelled–poset structure and use the constructions from the previous section. The edge labelled Hasse diagram for W will embed in to the Cayley graph $\mathrm{Cay}(W, S)$, and it is useful to be able to swap between these two objects, as we will do. First, given a group G, we must define a partial order on G.

Definition 2.11. For a group G generated by S, the word length with respect to S is the function $l_S : G \to \mathbb{N}$ where

$$l_S(g) \coloneqq \min\{k \mid s_1 s_2 \dots s_k = g\ ,\ s_i \in S \cup S^{-1}\}.$$

We will often omit the S in l_S where it is obvious from context.

Definition 2.12. For a group G generated by S, we define the partial order $x \leq y \iff l(x) + l(x^{-1}y) = l(y)$.

It can be readily checked that this does indeed define a partial order on G. This order encodes closeness to $e \in G$ along geodesics in $\mathrm{Cay}(G, S)$. We have $x \leq y$ precisely when there exists a geodesic in $\mathrm{Cay}(G, S)$ from e to y with x as an intermediate vertex, or to put it another way, when a minimal factorisation of x in to elements of $S \cup S^{-1}$ is a prefix of a minimal factorisation of y. The covering relations for this partial order are of the form $(g \lessdot gs)$ where $g \in G$ and $s \in S$.

Given a Coxeter group W and some $w \in W$, we define the poset $[1, w]^W$ to be the interval in W up to w with respect to this order. We give this poset an edge labelling such that the edge $(w \lessdot ws)$ is labelled s. Thus, the (edge–labelled) Hasse diagram of $[1, w]^W$ embeds in to $\mathrm{Cay}(W, S)$.

Definition 2.13 (Coxeter element)**.** For some Coxeter group W generated by S, we define a *Coxeter element* $w \in W$ to be any product of all the elements of S without repetition.

A Coxeter elements is what we will use as the upper bound of our interval $[1, w]^W$ in the following construction. We will also need to consider W as the group generated by R, the set of all reflections, rather than just the set of simple reflections S. See Fig. 7 for an example of such a poset. In principle there are many choices of Coxeter element depending on what order we multiply the elements of S. However, we will see in Theorem 2.17 that some structures resulting from $[1, w]^W$ are independent of that choice.

We apply the steps from Section 2.2 to $[1, w]^W$ to form a space.

Definition 2.14 (Interval Complex)**.** Given a Coxeter group W, consider the set of all reflections $R \subseteq W$ as the generating set of W. Thus, the partial order $\leq$ on W is defined using l_R. Fix a Coxeter element $w \in W$. We define the *interval complex* for W and w to be the space $K_W \coloneqq K([1, w]^W)$ (recall the function K from Definition 2.6).

If W is infinite, then R is infinite and so K_W may have an infinite number of cells. We will later show that K_W deformation retracts to a finite subcomplex. Note that, as in [PS21], we have dropped w from our notation K_W even though it depends on w. This is justified (Theorem 1.1) since the homotopy type of K_W is independent of w.

Certain properties of the poset $[1,w]^W$ permit a simplified notation for the simplices within K_W. In this context two n–simplices corresponding to the n–chains $C = (\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n)$ and $C' = (\alpha'_1 \leq \alpha'_2 \leq \cdots \leq \alpha'_n)$ will be identified in K_W exactly when $\alpha_i^{-1}\alpha_{i+1} = (\alpha'_i)^{-1}\alpha'_{i+1}$ for all $1 \leq i < n$. Thus, we can label 1–simplices in K_W with group elements $x \in [1,w]^W$, we can label 2–simplices with factorisations of group elements in $[1,w]^W$ in to two parts (with the first part also in $[1,w]^W$) and so on. As in [PS21, Definition 2.8], we denote an n–simplex corresponding to the n–chain $1 \leq x_1 \leq x_1x_2 \leq \cdots \leq x_1x_2\cdots x_n$ as $[x_1|x_2|\cdots|x_n]$. There are multiple choices of n–chain corresponding to an n–simplex. The n–chain $1 \leq x_1 \leq x_1x_2 \leq \cdots \leq x_1x_2\cdots x_n$ where the first element is 1 is called the *canonical chain* for $[x_1|x_2|\cdots|x_n]$.

This notation also gives the gluing of the faces of $[x_1|x_2|\cdots|x_n]$ in the following way. A codimention 1 face of $[x_1|x_2|\cdots|x_n]$ is a $(n-1)$–subchain of $1 \leq x_1 \leq x_1x_2 \leq \cdots \leq x_1x_2\ldots x_n$ (which will consist of n elements). There are three ways to obtain such a subchain.

1. Remove the first element of the chain to get $x_1 \leq x_1x_2 \leq \cdots \leq x_1x_2\ldots x_n$ which has canonical chain $1 \leq x_2 \leq x_2x_3 \leq \cdots \leq x_2x_3\ldots x_n$ and corresponds to the $(n-1)$–simplex $[x_2|x_3|\cdots|x_n]$.

2. Remove the last element of the chain to get $1 \leq x_1 \leq x_1x_2 \leq \cdots \leq x_1x_2\ldots x_{n-1}$ which corresponds to the $(n-1)$–simplex $[x_1|x_2|\cdots|x_{n-1}]$.

3. Multiply two adjacent elements x_i and x_{i+1} to get the chain
$$1 \leq \cdots \leq x_1\ldots x_{i-1} \leq x_1\ldots x_{i-1}x_ix_{i+1} \leq \cdots \leq x_1\cdots x_n$$
which corresponds to the $(n-1)$–simplex $[x_1|\cdots|x_ix_{i+1}|\cdots|x_n]$.

So the n–simplex $[x_1|x_2|\cdots|x_n]$ glues to $[x_2|x_3|\cdots|x_n]$, $[x_1|x_2|\cdots|x_{n-1}]$ and $[x_1|\cdots|x_ix_{i+1}|\cdots|x_n]$ for all $i < n$.

The particular intervals $[1,w]^W$ we will consider will be *balanced*. A balanced group interval is such that $x \in [1,w]^W \iff l(g^{-1}x) + l(x) = l(g)$, i.e. all minimal factorisation of $x \in [1,w]^W$ also appear as a suffix in a minimal factorisation of w and all suffixes also appear as a prefix.

Where the interval is balanced, any such symbol $[x_1|x_2|\cdots|x_n]$ corresponds to an n–simplex in K_W given it satisfies the following [PS21, Definition 2.8].

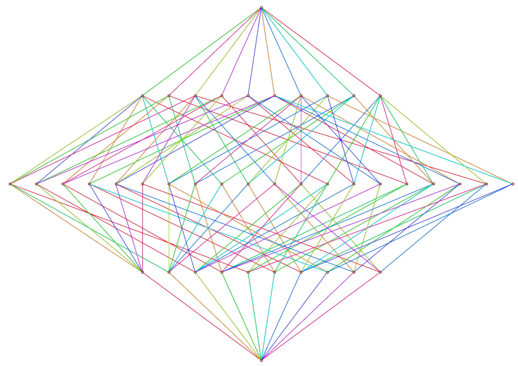


Figure 7: The interval $[1,(1,2)(2,3)(3,4)(4,5)]^{A_4}$ considering A_4 generated by all reflections, which label the edges by colour. Generated using `Sage` and `GAP` [Sag20, GAP22]

i) $x_i \neq 1$ for all i.

ii) $x_1x_2 \cdots x_n \in [1,w]^W$

iii) $l(x_1x_2\cdots x_n) = l(x_1) + l(x_2) + \cdots + l(x_n)$

Hopefully the first two requirements are obvious. The third is because we require the chain $1 \leq x_1 \leq x_1x_2 \leq \cdots \leq x_1x_2\dots x_n$ to be contained in $[1,w]^W$ which translates to every subword (without gaps) of $x_1\cdots x_n$ also being in $[1,w]^W$. By (ii) and (iii) there is some y_1 such that $x_1\cdots x_ny_1 = w$ and there is a minimal factorisation of w that respects the factors in $x_1\cdots x_ny_1$. We can take prefixes of this factorisation (and thus prefixes of $x_1\cdots x_ny_1$) and stay within $[1,w]^W$. We can use the balanced condition to move the suffix $x_2\cdots x_ny_1$ to the front, i.e. there exists y_2 such that $x_2\cdots x_ny_1y_2 = w$ and there is a minimal factorisation of w that respects these factors and conclude $x_2\cdots x_n \in [1,w]^W$. These steps can be repeated to show every subword of $x_1\cdots x_n$ is in $[1,w]^W$.

Definition 2.15. Given a Coxeter group W generated by all reflections R with Coxeter element $w \in W$. Define $[1,w]^W$ as above. The *dual Artin group* W_w is the poset group $G([1,w]^W)$ with G defined as in Definition 2.3.

In the presentation resulting from Definition 2.15, W_w is generated by the set of reflections appearing in the interval $R_0 \coloneqq R \cap [1,w]^W$. By Theorem 2.10, the fundamental group of K_W is W_w. By the following, for finite W, K_W is a classifying space for W_w.

Theorem 2.16 ([PS21, Theorems 2.9 and 2.14])**.** *Given a finite Coxeter group W, the interval complex K_W is a $K(W_w,1)$ space.*

The same theorem for affine W is proved in [PS21] (quoted here as Theorem 1.2). The set R_0 always contains the set of simple reflections S. Thus, there is natural map from S to W_w. In certain cases, this extends to a group isomorphism.

Theorem 2.17. *Given a finite [Bes03, Theorem 2.2.5] or affine [MS17, Theorem C] Coxeter system (W, S) and Coxeter element $w \in W$, the inclusion $S \hookrightarrow W_w$ extends to an isomorphism witnessing $G_W \cong W_w$.*

Corrolary 2.18. *Given a finite or affine Coxeter group W and Coxeter element w, the isomorphism class of the dual Artin group W_w is independent of w.*

In general, it is not known whether $G_W \cong W_w$, whether the isomorphism class of W_w depends on w or even if the isomorphism class of $[1, w]^W$ depends on w.

2.4 The Salvetti complex

Here we will define the *Salvetti complex* for a Coxeter group W generated by S, which is a CW–complex homotopy equivalent to Y_W introduced by Salvetti in [Sal94, Sal94]. To define the Salvetti complex, we must first define a notion on subsets of S. For some subset $T \subseteq S$ define the *parabolic subgroup* of W with respect to T to be the group generated by T, inheriting relations from W that exclusively concern elements of T. We denote this by W_T. If Γ is the Coxeter diagram for W, then W_T is the Coxeter group corresponding to the complete (or *full*) subgraph of Γ containing the vertices T. It is known that W_T is isomorphic to the subgroup $\langle T \rangle \subseteq W$ via a natural homomorphism induced by the inclusion $T \hookrightarrow \langle T \rangle$ [Hum90, Section 5.5]. Therefore, we can think of W_T as a subgroup of W in a natural way. From here we follow [Pao17, Section 2.3], with notation mostly following [PS21].

Definition 2.19. For a Coxeter group W generated by S, define $\mathcal{S}_W$ to be the family of subsets $T \subseteq S$ such that W_T is finite.

For some $T \subseteq S$, we say some $w \in W$ is T–minimal if w is the unique element of minimum length (with respect to S) in the coset wW_T. Uniqueness is shown in [Bou08]. Define a partial order on the set $W \times \mathcal{S}_W$ by the following: $(u, X) \leq (v, Y)$ iff $X \subseteq Y$, $v^{-1}u \in W_Y$ and $v^{-1}u$ is X–minimal.

Definition 2.20 (Pre-Salvetti complex [Pao17, Definition 2.19])**.** For a Coxeter group W, define the *pre–Salvetti Complex* $\mathrm{Sal}(W)$ to be $\Delta(W \times \mathcal{S}_W)$ under the partial order $\leq$ prescribed above (recall Δ from Definition 2.4).

Let us quote some results that help us to interpret $\mathrm{Sal}(W)$.

Lemma 2.21 ([Pao17, Lemma 2.18])**.** *Consider the following objects as cellular subspaces* *of* $\mathrm{Sal}(W)$.

$$C(v, Y) \coloneqq [-\infty, (v, Y)]$$
$$\partial C(v, Y) \coloneqq [-\infty, (v, Y)] \setminus \{(v, Y)\}$$

Let $n = |Y|$. *There is a homeomorphism* $C(v, Y) \to D^n$ *that restricts to a homeomorphism* $\partial C(v, Y) \to S^{n-1}$.

This allows us to exhibit $\mathrm{Sal}(W)$ as a CW–complex where $C(w, X)$ is a $|X|$–cell for each $X \in \mathcal{S}_W$. Let us see what these cells look like. Note that the cells of the CW-complex and the simplices in $\Delta(W \times \mathcal{S}_W)$ as in Definition 2.20 comprise a completely different cell structure for $\mathrm{Sal}(W)$. We define $\langle \emptyset \rangle \coloneqq \{1\}$ to give $W_\emptyset$ meaning as the trivial subgroup inside W. So $\emptyset \in \mathcal{S}_W$.

In general, we have that $(u, X) \leq (v, X) \implies u = v$. This is because $(u, X) \leq (v, X) \implies v^{-1}u \in W_X$, so we have $v^{-1}uW_X = W_X$, so if $v^{-1}u$ is minimal in $v^{-1}uW_X$ then $v^{-1}u = 1$. In particular, there is no $(u, X) < (w, \emptyset)$. So indeed each $C(w, \emptyset)$ is a 0–cell. We will denote these cells w as a shorthand.

Now consider each 1–cell $C(w, \{s\})$. Since $W_{\{s\}} = \{1, s\} \cong \mathbb{Z}/2$ we have $\{s\} \in \mathcal{S}_W$ for all $s \in S$. Recall that if $(u, X) \leq (w, \{s\})$ we require $w^{-1}u \in W_{\{s\}}$. So we have $u \in \{w, ws\}$. Locally, the Hasse diagram and (since we only have saturated 1–chains here) $\Delta(W \times \mathcal{S}_W)$ both look like Fig. 8 (left). In the CW–complex there would be only one 1–cell, labelled $C(w, \{s\})$ oriented from w to ws. Note that $C(ws, \{s\})$ also connects these two vertices, but in the opposite orientation.

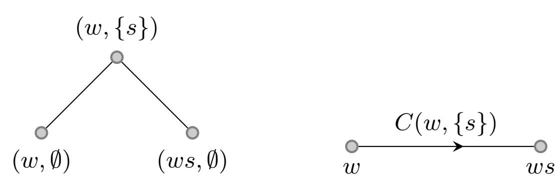


Figure 8: A picture of $[-\infty, (w, \{s\})]$ within $\mathrm{Sal}(W)$ as the geometric simplicial complex $\Delta(W \times \Delta_W)$ (which also resembles the Hasse diagram) (left). The corresponding 1–cell in the CW–complex for $\mathrm{Sal}(W)$ (right).

Now consider the 2–cells in the CW–complex. We have that $W_{\{s,t\}} \cong D_{2m(s,t)}$, the dihedral group corresponding to the $m(s, t)$–gon (recall $m(s, t)$ from Definition 1.3). Thus, $\{s, t\} \in \Delta_W$ iff $m(s, t) \neq \infty$. We have $(u, \{s\}) \leq (w, \{s, t\})$ only when $u = wd$ for some $d \in W_{\{s,t\}}$, similarly for $(v, \{t\}) \leq (w, \{s, t\})$. The second requirement is that d is ($\{s\}$ or $\{t\}$)–minimal. Generally, if $(u, X) \leq (v, Y)$ then $X \subseteq Y$ so $W_X \subseteq W_Y$. We also require $v^{-1}u \in W_Y$ so the coset $v^{-1}uW_X$ lies inside W_Y. Due to the nature of Definition 1.3, only

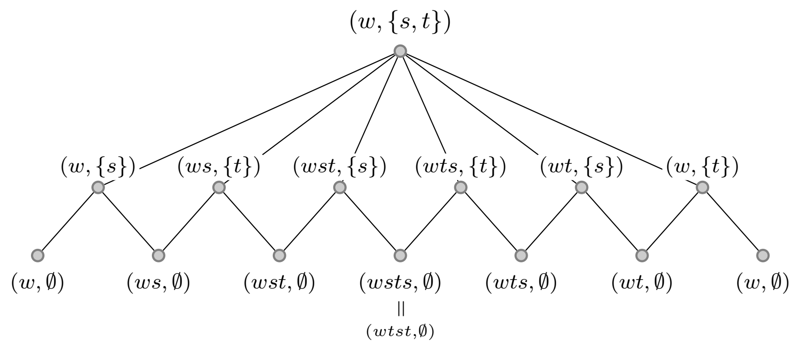


Figure 9: The Hasse Diagram corresponding to the CW 2–cell $C(w, \{s,t\})$ where $m(s,t) = 3$. Note that $(w, \emptyset)$ has been drawn twice for clarity in the picture. C.f. Fig. 10

relations in W_Y could have relevance to the word length of an element $x \in W_Y$ (even if x is considered an element of the ambient Coxeter group W). Thus, to determine if $v^{-1}u$ is X minimal we need only consider $v^{-1}uW_X$ within W_Y, not the entire Coxeter group W. In particular, to tell if $(wd, \{s\}) \leq (w, \{s,t\})$ for some $d \in W_{\{s,t\}}$, we need only consider if d is $\{s\}$–minimal in the dihedral group $W_{\{s,t\}}$.

Considering for a moment s and t as letters only, a normal form comprising minimal length words for $W_{\{s,t\}}$ is the following.

$$\{\Pi(s,t\,;n) \mid n \leq m(s,t)\} \cup \{\Pi(t,s\,;n) \mid n < m(s,t)\}$$

Recall the meaning of $\Pi(s,t\,;n)$ from Definition 1.4. Note that $\Pi(t,s,m(s,t))$ is also a minimal length word but is not included for the above to be a normal form. Thus, any $sts\cdots s$ is $\{t\}$–minimal if the total length of $sts\cdots s$ is strictly less than $m(s,t)$. Similarly, $sts\cdots t$ is $\{s\}$–minimal if the total length of $sts\cdots t$ is strictly less than $m(s,t)$, with equivalent results for $tst\cdots s$ and $tst\cdots t$ depending on the last letter in the word. A picture of the Hasse diagram for the interval $[-\infty, (w, \{s,t\})]$ corresponding to the cell $C(w, \{s,t\})$ where $m(s,t) = 3$ is shown in Fig. 9. The CW–cell itself has been drawn in Fig. 10.

There is a natural action $W \circlearrowleft \mathrm{Sal}(W)$ with $w \cdot (u,T) := (wu,T)$. We can now define the following:

Definition 2.22 (Salvetti Complex)**.** For a Coxeter group W define the *Salvetti Complex* X_W to be the quotient space $\mathrm{Sal}(W)/W$ under the action specified above.

The action is cellular, thus we have a CW structure for X_W as well. The Salvetti complex was defined in [Sal94] to be the quotient of a space related to the action of W on a vector space. In [Par14, Theorem 3.3] it is shown that the definition we have given

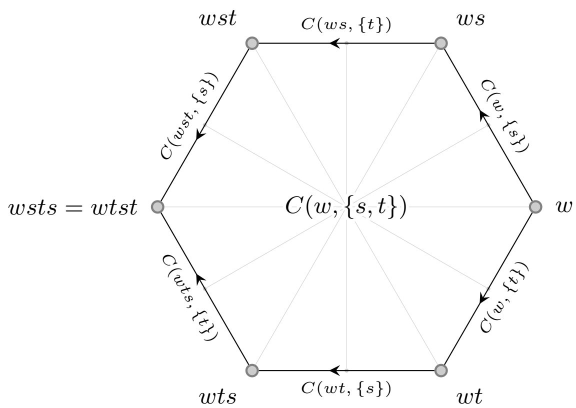


Figure 10: The 2–cell $C(w, \{s, t\})$ in the CW–complex for $\mathrm{Sal}(W)$ where $m(s, t) = 3$. The faint lines are the simplices from $\Delta(W \times \Delta_W)$, which have all been incorporated in to one CW–cell.

generates a space homeomorphic to that in the original definition by Salvetti. We now quote the following important result.

Theorem 2.23 ([Par14, Corollary 3.4] [Sal87])**.** *For a Coxeter group W, the Salvetti complex X_W is homotopy equivalent to the configuration space Y_W.*

Now let us consider the cell structure of X_W. There is one $|T|$–cell for each $T \in \Delta_W$, in particular, there is one 0–cell corresponding to the trivial group $W_\emptyset$. Attached to this is a 1–cell for each $s \in S$ forming $\bigvee_{s \in S} S^1$ as the 1–skeleton. Then to this wedge is attached 2–cells following the procedure above for each $m(s, t) \neq \infty$. Each 2–cell corresponding to $\{s, t\}$ is attached to two 1–cells corresponding to $\{s\}$ and $\{t\}$.

Remark 2.24. By considering the 2–skeleton of X_W and the presentation for X_W as in (4), and by generalising the picture in Fig. 10 to all finite values of $m(s, t)$, we have that $\pi_1(X_W) \cong G_W$.

Thus, combining with Theorem 2.23, we have re-proved $\pi_1(Y_W) \cong G_W$.

Chapter 3

An adjunction homotopy equivalence

We will now begin to bridge the gap between some of the objects we have defined. Ultimately, we wish to show a homotopy equivalence between the space K_W and the Salvetti complex X_W, which is already known to be homotopy equivalent to Y_W. For now, we will define a subspace $X'_W \subseteq K_W$, inspired by our definition of the Salvetti complex, using subsets $T \subseteq S$ and $\mathcal{S}_W$. We will then prove $X_W \simeq X'_W$.

3.1 The subcomplex X'_W

The definition of K_W depends on the data of some Coxeter element w and thus implicitly on some generating set S. For some $w = s_1 s_2 \cdots s_n$ and some $T \subseteq S$, define $w_T \in W$ to be the (potentially non–consecutive) subword of w consisting of elements that are in T, respecting the original order in w. We call such a w_T a *parabolic Coxeter element.*

Definition 3.1. Define X'_W to be the finite subcomplex of K_W consisting only of simplices $[x_1|\cdots|x_n]$ such that $x_1 x_2 \cdots x_n \in [1, w_T]^W$ for some $T \in \mathcal{S}_W$ (recall $\mathcal{S}_W$ from Definition 2.19).

Note again the absence of w from the notation of X'_W. This will be justified in Section 3.3 where we will show that the homotopy type of X'_W has no dependence on w.

Lemma 3.2 ([PS21, Lemma 5.2])**.** *Fix a Coxeter group W and any parabolic subgroup W_T, with sets of reflections R_W and R_{W_T} respectively. Given some Coxeter element $w \in W$ and corresponding $w_T \in W_T$, all minimal factorisations of w_T by factors in R_W consist only of factors in R_{W_T}.*

An immediate consequence of this is that the intervals $[1, w_T]^W$ and $[1, w_T]^{W_T}$ agree. This allows us to decompose X'_W in a useful way. For each $T \in \mathcal{S}_W$ and corresponding W_T, the interval $[1, w_T]^{W_T}$ is a subposet of $[1, w]^W$. So, the space corresponding to the interval $[1, w_T]^{W_T}$ is a subspace of X'_W. This subspace is exactly X'_{W_T} (with respect to w_T). The union of all such $[1, w_T]^{W_T}$ contains all the simplices of X'_W by Definition 3.1, thus, we can think of X'_W as the union (with appropriate gluing) of all X'_{W_T} for $T \in \mathcal{S}_W$.

For $T \in \mathcal{S}_W$, each X'_{W_T} is exactly the same as its interval complex K_{W_T} since all subgroups of T would also generate finite Coxeter groups. By construction W_T is finite, thus $X'_{W_T} = K_{W_T}$ is a classifying space for the dual Artin group W_{w_T} by Theorem 2.16 and W_{w_T} is isomorphic to the Artin group G_{W_T} by Theorem 2.17.

In a very similar way, the Salvetti complex consists of subspaces corresponding to elements of $\mathcal{S}_W$. For each $T \in \mathcal{S}_W$, the Salvetti complex X_{W_T} is a $|T|$–cell attached to all cells corresponding to $V \subsetneq T$ is the appropriate way. This is a cellular subspace of X_W, and since W_T is finite, by Theorem 1.10, $Y_{W_T} \simeq X_{W_T}$ is a $K(G_{W_T}, 1)$. The following remark summarises these observations.

Remark 3.3. The Salvetti complex X_W decomposes in to cellular subspaces X_{W_T} which are $K(G_{W_T}, 1)$ spaces. These subspaces are in bijection with cellular subspaces X'_{W_T} of X'_W, which are also $K(G_{W_T}, 1)$ spaces.

We now develop some theory to help us exploit this similarity and eventually show that $X_W \simeq X'_W$, which is an intermediate step in (1).

3.2 Maps in to classifying spaces

Here we will show a fundamental link between homomorphisms in to groups G and maps between certain classifying spaces for G.

Lemma 3.4. *A null homotopic map $\rho\colon S^n \to X$ can be extended to a map $\sigma\colon D^{n+1} \to X$.*

Proof. Let $H\colon S^n \times I \to X$ witness the null homotopy with $H|_{S^n \times \{1\}}\colon S^n \to \{x_0\} \in X$. We have that H factors uniquely through $(S^n \times I)/(S^n \times \{1\}) \cong D^{n+1}$. With σ being the unique induced map as below.

$$\begin{tikzcd} S^n \times I \arrow[r, "H"] \arrow[d, "q"'] & X \\ (S^n \times I)/(S^n \times \{1\}) \arrow[ur, dotted, "\exists! \sigma"'] & \end{tikzcd}$$

□

Theorem 3.5 ([Hat01, Proposition 1B.9]). *Let Y be a $K(G,1)$ space and X a finite dimensional CW–complex with only one 0–cell, the point x_0. Any homomorphism $\varphi\colon \pi_1(X,x_0) \to \pi_1(Y,y_0)$ is induced by a map $\overline{\varphi}\colon X \to Y$ where $\overline{\varphi}$ is unique up to homotopy fixing x_0.*

Proof. We construct the map $\overline{\varphi}$ by inductively constructing it on the n–skeletons of X. We must have $\overline{\varphi}(x_0) = y_0$. The 1–skeleton X^1 will be a wedge of circles. Let A be an indexing set for the 1–cells of X. For each $\alpha \in A$, let each λ_α be a loop going around the closure of the 1–cell e^1_α in either orientation. We restrict that each λ_α be a homeomorphism witnessing $S^1 \cong e^1_\alpha \cup \{x_0\}$. Let B be an indexing set for the 2–cells of X. For each $\beta \in B$, let r_β be a factorisation in the set $\{[\lambda_\alpha]\}_{\alpha\in A} \cup \{[\lambda_\alpha]^{-1}\}_{\alpha\in A}$ of the homotopy class of the attaching map $\psi_\beta\colon S^1 \to X^1$ for a 2–cell e^2_β. We have the following presentation for $\pi_1(X,x_0)$, as in (4).

$$\pi_1(X,x_0) \cong \langle \{[\lambda_\alpha]\}_{\alpha\in A} \mid r_\beta \rangle \tag{6}$$

We can choose $\overline{\varphi}(e^1_\alpha)$ to trace out any path in the homotopy class $\varphi([\lambda_\alpha]) \in \pi_1(Y,y_0)$ for each $e^1_\alpha \in X^1$. Let $i\colon X^1 \hookrightarrow X$ be the inclusion. We have that i_* is the surjection from the free group generated by $\{[\lambda_\alpha]\}_{\alpha\in A}$ to $\pi_1(X,x_0)$. The induced map $(\overline{\varphi})_*$ is determined by its action on the generators $\{[\lambda_\alpha]\}_{\alpha\in A}$ and $(\overline{\varphi})_*([\lambda_\alpha]) = \varphi([\lambda_\alpha])$. Thus, our construction so far gives us the following commutative diagram.

$$\begin{tikzcd}
\pi_1(X^1,x_0) \arrow[rr, "(\overline{\varphi})_*"] \arrow[dr, "i_*"'] & & \pi_1(Y,y_0) \\
& \pi_1(X,x_0) \arrow[ur, "\varphi"'] &
\end{tikzcd} \tag{7}$$

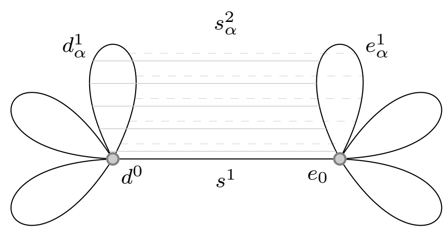
Let $\psi_\beta\colon S^1 \to X^1$ be an attaching map for a 2–cell $e^2_\beta \subseteq X$. We wish to show that $\overline{\varphi} \circ \psi_\beta$ is null homotopic. The attaching of the 2–cell e^2_β provides a null homotopy for the loop traced by ψ_β. In the presentation of $\pi_1(X,x_0)$ in (6), each relation corresponds to the path traced out by some ψ_β. Thus, $i_*([\psi_\beta]) = 1$ and so $\overline{\varphi}_*([\psi_\beta]) = \varphi \circ i_*([\psi_\beta]) = 1$ by (7). Thus, $\overline{\varphi} \circ \psi_\beta$ is null homotopic and so can be extended over all of the closure of e^2_β by Lemma 3.4. This is an extension of $\overline{\varphi}$ and repeating this for each 2–cell allows us to extend $\overline{\varphi}$ over all of X^2.

To extend $\overline{\varphi}$ over some e^3_γ we use that $\pi_2(Y,y_0) = \{1\}$. Let $\psi_\gamma\colon S^2 \to X^2$ be the attaching map for e^3_γ. Since $\pi_2(Y,y_0) = \{1\}$, we have that $\overline{\varphi} \circ \psi_\gamma\colon S^2 \to X^2 \to Y$ is nullhomotopic and so can be extended over all of the closure of e^3_γ. This same argument applies for any e^n_δ for $n \geq 3$ since $\pi_n(Y,y_0) = \{1\}$ for all $n \geq 2$. We can thus extend $\overline{\varphi}$ over the 3–cells and proceeding inductively on the n–skeletons, over all of X.

Now we turn to the uniqueness of $\overline{\varphi}$ up to homotopy. Let $\varphi\colon \pi_1(X,x_0) \to \pi_1(Y,y_0)$ be some homomorphism and $\overline{\varphi}_0$ and $\overline{\varphi}_1$ be any maps from X to Y such that $(\overline{\varphi}_0)_* = (\overline{\varphi}_1)_* =$

φ. We have that $\overline{\varphi}_0(x_0) = \overline{\varphi}_1(x_0)$. To agree as maps, $(\overline{\varphi}_0)_*$ and $(\overline{\varphi}_1)_*$ must agree on generators $\{[\lambda_\alpha]\}_{\alpha \in A}$. Thus, we have $\overline{\varphi}_0 \circ \lambda_\alpha \sim \overline{\varphi}_1 \circ \lambda_\alpha$ for all $\alpha \in A$. Since we restricted each λ_α to be a homeomorphism, this gives us that $\overline{\varphi}_0|_{\mathrm{Im}(\lambda_\alpha)} \sim \overline{\varphi}_1|_{\mathrm{Im}(\lambda_\alpha)}$. Let H_α witness this homotopy. Each $\mathrm{Im}(\lambda_\alpha)$ is the closure of the 1–cell e^1_α, thus $\bigcup_{\alpha \in A} \mathrm{Im}(\lambda_\alpha) = X^1$ and the union $H \coloneqq \bigcup_{\alpha \in A} H_\alpha$ witnesses the homotopy $\overline{\varphi}_0|_{X^1} \sim \overline{\varphi}_1|_{X^1}$.

Give $X \times I$ the following CW structure. Let $X \times \{0\}$ and $X \times \{1\}$ both have the same cell structure as X with cells denoted d^n_α and e^n_α respectively. Connect d^0 to e^0 with a 1–cell s^1, called the *spine*. Connect a 2–cell s^2_α along d^1_α, then s^1 then e^1_α then back along s^1 with opposite orientations on d^1_α and e^1_α such that $d^0 \cup e^0 \cup s^1 \cup d^1_\alpha \cup e^1_\alpha \cup s^2_\alpha \cong S^1 \times I$. The spine now consists of $s_1 \cup s^2_\alpha$. Repeat this for each 1–cell in X and then repeat for each 2–cell and so on, attaching an s^n_β along d^{n-1}_β, e^{n-1}_β and s^{n-1}_β, inductively building up the spine. A picture of this CW–complex completed for one s^2_α is below.



We can now extend the domain of H from $X^1 \times I$ to all of $X \times I$ using this cell structure in the following way. Note that now we have two 0–cells, but this does not cause any issues. Let H have domain $X^1 \times I \subseteq X \times I$. Now extend H to have domain $X^1 \times I \cup X \times \{0\} \cup X \times \{1\}$ such that $H|_{X \times \{0\}}$ agrees with $\overline{\varphi}_0$ and $H|_{X \times \{1\}}$ agrees with $\overline{\varphi}_1$. This is possible because H is a homotopy between restrictions of these maps. Note that now H is defined on the whole 2–skeleton of $X \times I$. We can extend H to all the higher dimensional cells by the exact same argument as before, using that $\pi_n(Y, y_0) = \{1\}$ for $n \geq 2$. Thus, we have a continuous function $H\colon X \times I \to Y$ witnessing the homotopy $\overline{\varphi}_0 \sim \overline{\varphi}_1$. $\square$

Corrolary 3.6 ([Hat01, Theorem 1B.8]). *Let X and Y both be $K(G,1)$ spaces. If both spaces are CW–complexes with only one 0–cell, then any isomorphism $\varphi\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ induces a homotopy equivalence witnessing $X \simeq Y$.*

Proof. We have maps $\overline{\varphi}\colon X \to Y$ and $\overline{(\varphi^{-1})}\colon Y \to X$ with $(\overline{\varphi} \circ \overline{(\varphi^{-1})})_* = \mathrm{Id}_{\pi_1(Y,y_0)}$. Thus, since the homotopy class of such maps is determined by the induced action on their fundamental groups $\overline{\varphi} \circ \overline{(\varphi^{-1})} \sim \mathrm{Id}_Y$. Similarly, $\overline{(\varphi^{-1})} \circ \overline{\varphi} \sim \mathrm{Id}_X$. $\square$

The following is a technical lemma that uses arguments very similar to those in the proof of Lemma 3.4 to give extensions of certain homotopy equivalences between unions of X_{W_Q} and unions of X'_{W_Q}.

Lemma 3.7. *Let $T \in \mathcal{S}_W \setminus \emptyset$. Let $\varphi\colon \bigcup_{Q \subsetneq T} X_{W_Q} \to \bigcup_{Q \subsetneq T} X'_{W_Q}$ be a homotopy equivalence. And $i\colon \bigcup_{Q \subsetneq T} X_{W_Q} \hookrightarrow X_{W_T}$ and $j\colon \bigcup_{Q \subsetneq T} X'_{W_Q} \hookrightarrow X'_{W_T}$ be inclusions. If:*

1. *$|T| \in \{1, 3, 4, 5, \dots\}$ or*
2. *$|T| = 2$ with $T = \{u, v\}$ and $\varphi\colon X_{W_{\{u\}}} \cup X_{W_{\{v\}}} \to\colon X'_{W_{\{u\}}} \cup X'_{W_{\{v\}}}$ restricts to the homotopy equivalences $\varphi_{X_{W_{\{u\}}}} : X_{W_{\{u\}}} \to X'_{W_{\{u\}}}$ and $\varphi_{X_{W_{\{v\}}}} : X_{W_{\{v\}}} \to X'_{W_{\{v\}}}$*

then we can extend φ to a homotopy equivalence $\psi\colon X_{W_T} \to X'_{W_T}$ such that the following diagram commutes.

$$\begin{tikzcd} \bigcup_{Q \subsetneq T} X_{W_Q} \arrow[r, "\varphi"] \arrow[d, hook, "i"'] & \bigcup_{Q \subsetneq T} X'_{W_Q} \arrow[d, hook, "j"] \\ X_{W_T} \arrow[r, "\psi"] & X'_{W_T} \end{tikzcd} \tag{8}$$

Proof. We prove this by cases. By Remark 3.3, X_{W_T} and X'_{W_T} are classifying spaces.

i) If $|T| = 1$ then any $Q \subsetneq T$ is uniquely $\emptyset$. We have that $X_{W_T} \cong X'_{W_T} \cong S^1$. Let ψ be any map witnessing $X_{W_T} \simeq X'_{W_T}$ that restricts to the map $\psi|_{X_{W_\emptyset}} : X_{W_\emptyset} \to X'_{W_\emptyset}$.

ii) Suppose $|T| = 2$ with $T = \{u, v\}$. By Remark 2.24, there is an isomorphism $\sigma\colon \pi_1(X_{W_T}, X_{W_\emptyset}) \to G_{W_T}$ that sends the loops going around $X_{W_{\{u\}}}$ and $X_{W_{\{v\}}}$ to the two generators of G_{W_T}. Let $\chi\colon G_{W_T} \to W_{w_T}$ be the isomorphism from Theorem 2.17 that restricts to the inclusion $T \hookrightarrow W_{w_T}$. Furthermore, let $\tau\colon W_{w_T} \to \pi_1(X'_{W_T}, X_{W_\emptyset})$ be the isomorphism given to us by Theorem 2.10.

   Let $[\lambda_u]$ and $[\lambda_v]$ be generators of $\pi_1(X_{W_{\{u\}}}, X_{W_\emptyset})$ and $\pi_1(X_{W_{\{v\}}}, X_{W_\emptyset})$ respectively. By assumption, φ witnesses the homotopy equivalences $X_{W_{\{u\}}} \simeq X'_{W_{\{u\}}}$ and $X_{W_{\{v\}}} \simeq X'_{W_{\{v\}}}$. Let $u_\pm \circlearrowleft \pi_1(X_{W_T}, X_{W_\emptyset})$ be the automorphisms sending $[\lambda_u]$ to either $[\lambda_u]$ or $[\lambda_u]^{-1}$ and similarly define $v_\pm$. With an appropriate choice for $u_\pm$, we see that

$$(j \circ \varphi)_*([\lambda_u]) = (\tau \circ \chi \circ \sigma \circ u_\pm \circ i_*)([\lambda_u])$$

   and similarly for $[\lambda_v]$. The use of $u_\pm$ and $v_\pm$ is necessary as we have no control over possible orientation flips in the homotopy equivalences $X_{W_{\{u\}}} \simeq X'_{W_{\{u\}}}$ and $X_{W_{\{v\}}} \simeq X'_{W_{\{v\}}}$ witnessed by φ. Together, $[\lambda_u]$ and $[\lambda_v]$ generate $\pi_1(X_{W_{\{u\}}} \cup X_{W_{\{v\}}}, X_{W_\emptyset})$, so we have completely specified $(j \circ \varphi)_*$. Considering all of these maps together, we have the following commutative diagram.

$$\begin{tikzcd} \pi_1(X_{W_{\{u\}}} \cup X_{W_{\{v\}}}, X_{W_\emptyset}) \arrow[rrr, "(j\circ\varphi)_*"] \arrow[d, "i_*"'] & & & \pi_1(X'_{W_T}, X'_{W_\emptyset}) \\ \pi_1(X_{W_T}, X_{W_\emptyset}) \arrow[r, "u_\pm \circ v_\pm"] & \pi_1(X_{W_T}, X_{W_\emptyset}) \arrow[r, "\sigma"] & G_{W_T} \arrow[r, "\chi"] & W_{w_T} \arrow[u, "\tau"'] \end{tikzcd} \tag{9}$$

We compare this with (7) and continue as in the proof of Theorem 3.5. Let $\beta\colon S^1 \to X_{W_{\{u\}}} \cup X_{W_{\{v\}}}$ be the attaching map of the unique 2–cell e^2 in X_{W_T}. This 2–cell provides a null homotopy for the loop β in X_{W_T}, therefore $i_*([\beta]) = 1$. Thus, $(j \circ \varphi)_*([\beta]) = 1$ by (9) and so $j \circ \varphi \circ \beta$ can be extended to all of the closure of e^2, which is all of X_{W_T}. Let $\psi\colon X_{W_T} \to X'_{W_T}$ be this extension. As a map, ψ satisfies the commutative diagram in (8). The action of the map ψ_* is defined by the action of the map $(j \circ \varphi)_*$. We see from (9) that $\psi_* = \tau \circ \chi \circ \sigma \circ t_\pm \circ s_\pm$ which is an isomorphism. Both X_{W_T} and X'_{W_T} are CW–complexes with only one 0–cell so by Corollary 3.6 ψ is a homotopy equivalence.

iii) If $|T| \geq 3$ then we can extend $j \circ \varphi$ to some map ψ using the same methods as in the proof of Theorem 3.5, utilising that $\pi_n(X'_{W_T}, X'_{W_\emptyset}) = \{1\}$ for all $n \geq 2$. Now we show that ψ is a homotopy equivalence. In this case, $\bigcup_{Q \subsetneq T} X_{W_Q}$ contains the 2–skeleton of X_T and similarly for $\bigcup_{Q \subsetneq T} X'_{W_Q}$ and X'_T. So the induced map on the inclusion $i_*\colon \pi_1(\bigcup_{Q \subsetneq T} X_{W_Q}, X_{W_\emptyset}) \to \pi_1(X_T, X_{W_\emptyset})$ is an isomorphism and similarly for j_* and X'_T [Hat01, Corollary 4.12]. By assumption φ is a homotopy equivalence and so φ_* is an isomorphism. Therefore, ψ_* is an isomorphism. Both X_{W_T}, and X'_{W_T} are CW–complexes with only one 0–cell so by Corollary 3.6 ψ is a homotopy equivalence. □

3.3 Adjunction spaces and proving $X_W \simeq X'_W$

Here we will develop a formalism for gluing together two spaces in to what is called an *adjunction space*. Adjunction spaces inherit homotopic properties from their two parent spaces in a useful way, that will help us to exploit Remark 3.3.

Definition 3.8 (Adjunction Space)**.** Given two spaces X and U, with a continuous map $f\colon A \to U$ for some subspace $A \subseteq X$, the *adjunction space* $X \sqcup_f U$ is the space formed by gluing X and U via the map f.

$$X \sqcup_f U := (X \sqcup Y)/(a \sim f(a))$$

An adjunction space is associated to the commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"] & U \arrow[d, "\widetilde{i}"] \\ X \arrow[r, "\widetilde{f}"] & X \sqcup_f U \end{tikzcd} \tag{10}$$

where i is inclusion of A in to X and $\widetilde{i}$ is inclusion of U in to $X \sqcup_f U$. Suppose we also have the adjunction space $Y \sqcup_g V$ with $g\colon B \to V$ and $B \subseteq Y$. Suppose further that we have maps $\varphi_1\colon X \to Y$, $\varphi_2\colon A \to B$ and $\varphi_3\colon U \to V$ such that the following diagram commutes.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "\varphi_2"] \arrow[dd, "i"'] & & U \arrow[dr, "\varphi_3"] \arrow[dd, "\widetilde{i}"] & \\
& B \arrow[rr, "g"] \arrow[dd, "j"] & & V \arrow[dd, "\widetilde{j}"] \\
X \arrow[rr, "\widetilde{f}"] \arrow[dr, "\varphi_1"] & & X \sqcup_f U \arrow[dr, "\varphi"] & \\
& Y \arrow[rr, "\widetilde{g}"'] & & Y \sqcup_g V
\end{tikzcd} \tag{11}$$

If all φ_1, φ_2 and φ_3 are homotopy equivalences then the following lemma tells us that φ is also a homotopy equivalence.

Lemma 3.9 ([Bro06, Theorem 7.5.7])**.** *Consider a commutative diagram as in* (11) *where the front and back faces define an adjunction space as in* (10)*. If i and j are closed cofibrations and φ_1, φ_2 and φ_3 are homotopy equivalences, then the φ as determined by the diagram is also a homotopy equivalence.*

In the cases important to us, i and j will be cellular inclusions in to finite CW–complexes, and thus closed cofibrations. See [Bro06] for more details on adjunction spaces.

To use Lemma 3.9 we must be able to construct X_W and X'_W as a sequence of adjunction spaces. Consider X_W in the following example.

Example 3.10. To clean our notation in this example let X and X_T be shorthand for X_W and X_{W_T} respectively. Accordingly, X^n is the n–skeleton for X_W and X^n_T is the n–skeleton for X_{W_T}. Suppose W is generated by $S = \{s,t,u\}$ and suppose $\mathcal{S}_W = \{Q \subsetneq S\}$. For convenience define $\mathcal{S}^n_W \coloneqq \{T \in \mathcal{S}_W \mid |T| = n\}$. When denoting adjunction spaces, we will use $\sqcup_f$ to denote a specific adjunction space that is currently being constructed. We use the $\cup$ symbol to denote some adjunction space or space resulting from a sequence of adjunctions without specifying the map f. We have that $X = \bigcup_{T \in \mathcal{S}^2_W} X_T$. Suppose we had the 1–skeleton $X^1 \subsetneq X$, and some ordering on $\mathcal{S}^2_W = (\{s,t\},\{s,u\},\{t,u\})$. To construct X, we would first glue $X_{\{s,t\}}$ to X^1 as an adjunction space in the following way.

$$\begin{tikzcd}
X_{\{s\}} \cup X_{\{t\}} \arrow[r, "f"] \arrow[d, "i_1"'] & X_{\{s,t\}} \arrow[d, "\widetilde{i_1}"] \\
X^1 \arrow[r, "\widetilde{f}"] & X^1 \sqcup_f X_{\{s,t\}}
\end{tikzcd} \tag{12}$$

Where f is inclusion of the 1–cells $X_{\{s\}}$ and $X_{\{t\}}$ in to $X_{\{s,t\}}$ and i_1 is the inclusion in to X^1, which in this case makes $X^1 \sqcup_f X_{\{s,t\}} \cong X_{\{s,t\}}$. If we were using this construction in the proof of Theorem 3.11 we would inductively assume to have already constructed $X_{\{s\}} \cup X_{\{t\}}$. We can now add $X_{\{t,u\}}$ to the preceding adjunction space in the following way.

$$\begin{tikzcd} X_{\{t\}} \cup X_{\{u\}} \arrow[r, "g"] \arrow[d, "i_2"'] & X_{\{t,u\}} \arrow[d, "\widetilde{i_2}"] \\ X^1 \sqcup_f X_{\{s,t\}} \arrow[r, "\widetilde{g}"] & (X^1 \sqcup_f X_{\{s,t\}}) \sqcup_g X_{\{t,u\}} \end{tikzcd} \tag{13}$$

Again, g and i_2 are just inclusions. After this step, we would continue with $\{u, v\}$ in the same manner. In the final space, $X_{\{u,v\}}$ is glued to $(X^1_W \sqcup_f X_{\{s,t\}}) \sqcup_g X_{\{t,u\}}$ along $X_{\{u\}} \cup X_{\{v\}}$. Given the $(n-1)$–skeleton, we can always choose an order on $\{X_T \mid T \in \mathcal{S}^n_W\}$ and attach each such X_T to the previous adjunction space along the $(n-1)$–skeleton of X_T (which is always a subspace of the adjunction space of the previous step).

The exact same construction works for X'_W. Note that none of the maps in (12) or (13) are the attaching maps of cells as in the CW–complex. It is possible to construct CW–complexes as a sequence of adjunction spaces, but that is not the construction presented in Example 3.10. In this construction we assume to already have fully-constructed $|T|$–dimensional CW–complexes X_T, then we just glue these on to the $(|T|-1)$–skeleton which is a subspace of the adjunction space formed in the previous step. All the internal structure of the CW–complexes X_T is ignored. This construction may seem too abstracted to be useful, but we will soon see otherwise.

The following proof uses induction on the steps presented in Example 3.10. Note that a step is the gluing of a single X_T. In principle, to construct the n–skeleton from the $(n-1)$–skeleton takes multiple steps.

Theorem 3.11 ([PS21, Theorem 5.5])**.** *For a Coxeter group W, the space X'_W as in Definition 3.1 is homotopy equivalent to the Salvetti complex X_W.*

Proof. We achieve this by induction on adjunction gluing steps. To tidy our notation, as in Example 3.10, we drop W so that X, X', X_T and X'_T correspond to X_W, X'_W, X_{W_T} and X'_{W_T} respectively. Let $\mathcal{S}^n_W$ be as in Example 3.10.

Our inductive hypotheses are as follows.

1. We have a homotopy equivalence between $(n-1)$–skeletons $\alpha\colon X^{n-1} \to (X')^{n-1}$.

2. For any subset $\mathcal{T} \subseteq \mathcal{S}^{n-1}_W$ we have that α restricts to a homotopy equivalence

$\alpha' \colon \bigcup_{Q \in \mathcal{T}} X_Q \to \bigcup_{Q \in \mathcal{T}} X'_Q$ such that the following diagram commutes.

$$\begin{tikzcd} \bigcup_{Q \in \mathcal{T}} X_Q \arrow[r, "\alpha'"] \arrow[d, hook] & \bigcup_{Q \in \mathcal{T}} X'_Q \arrow[d, hook] \\ X^{n-1} \arrow[r, "\alpha"] & (X')^{n-1} \end{tikzcd}$$

3. We have a homotopy equivalence $\beta \colon Y \to Y'$ where $X^{n-1} \subseteq Y \subseteq X^n$ and $(X')^{n-1} \subseteq Y' \subseteq (X')^n$ so Y and Y' are intermediate steps in constructing X^n and $(X')^n$ respectively. For example, Y could be the upper-right term in (13) for $n = 2$. This homotopy equivalence β restricts to the homotopy equivalence α on X^{n-1}.

The base case is $X_\emptyset \simeq X'_\emptyset \simeq \{\bullet\}$. Fix an ordering $(T_1, T_2, \ldots, T_k)$ on $\mathcal{S}^n_W$. For each T_i we can extend the map $\varphi_i \colon \bigcup_{Q \subsetneq T_i} X_Q \to \bigcup_{Q \subsetneq T_i} X'_Q$ to $\psi_i \colon X_{T_i} \to X'_{T_i}$ such that we have the following commutative diagram.

$$\begin{tikzcd} \bigcup_{Q \subsetneq T_i} X_Q \arrow[r, "\varphi_i"] \arrow[d, hook] & \bigcup_{Q \subsetneq T_i} X'_Q \arrow[d, hook] \\ X_{T_i} \arrow[r, "\psi_i"] & X'_{T_i} \end{tikzcd}$$

This is possible by Lemma 3.7. The extra axioms in the case where $|T| = 2$ are satisfied due to inductive hypothesis 2. Fix each ψ_i.

- Case 1: Suppose $Y = \bigcup_{m<i} X_{T_m}$ and $Y' = \bigcup_{m<i} X'_{T_m}$ and $i \leq k-1$, i.e. in the next step, we *will not* complete $X^n \simeq (X')^n$.

In this step, we will glue X_{T_i} to Y and X'_{T_i} to Y' such that the inductive hypotheses remain for these resulting adjunction spaces. By combining inductive hypothesis 3 and 2, we have the following commutative diagram.

$$\begin{tikzcd} \bigcup_{Q \subsetneq T_i} X_Q \arrow[r, "\varphi_i"] \arrow[d, hook] & \bigcup_{Q \subsetneq T_i} X'_Q \arrow[d, hook] \\ Y \arrow[r, "\beta"] & Y' \end{tikzcd}$$

Where φ_i and β are homotopy equivalences and the vertical maps are inclusions. We now have the following commutative diagram.

$$
\begin{tikzcd}
\bigcup_{Q\subsetneq T_i} X_Q \arrow[rr, hook, "f"] \arrow[dr, "\varphi_i"'] \arrow[dd, hook] & & X_{T_i} \arrow[dr, "\psi_i"] \arrow[dd, hook] & \\
& \bigcup_{Q\subsetneq T_i} X'_Q \arrow[rr, hook, "g"] \arrow[dd, hook] & & X'_{T_i} \arrow[dd, hook] \\
Y \arrow[rr, hook, "\widetilde{f}"'] \arrow[dr, "\beta"'] & & Y \sqcup_f X_{T_i} \arrow[dr, "\sigma"] & \\
& Y' \arrow[rr, hook, "\widetilde{g}"'] & & Y' \sqcup_g X'_{T_i}
\end{tikzcd} \tag{14}
$$

Where all maps not coming out of the plane of the page are inclusions and front and back faces determine the adjunction spaces $Y \sqcup_f X_{T_i}$ and $Y' \sqcup_g X'_{T_i}$ respectively. By Lemma 3.9, the map σ determined by the maps β, φ_i and ψ_i is a homotopy equivalence. In the next step, we will still be constructing $X^n \simeq (X')^n$ and will be gluing $X_{T_{i+1}}$ to Y and $X'_{T_{i+1}}$ to Y'. From (14), we see that σ restricts to β and so by induction also restricts to $\alpha\colon X^{n-1} \to (X')^{n-1}$. We replace Y with $Y \sqcup_f X_{T_i}$ and Y' with $Y' \sqcup_g X'_{T_i}$. Accordingly, we replace β with σ.

With these replacements, we maintain all our inductive hypotheses and can continue.

- Case 2: Suppose $Y = \bigcup_{m<i} X_{T_m}$ and $Y' = \bigcup_{m<i} X'_{T_m}$ and $i = k$, i.e. in the next step, we *will* complete $X^n \simeq (X')^n$.

The steps up to creating the commutative diagram (14) are exactly the same as in Case 1. However, we must make a further argument to ensure the inductive hypotheses continue to be true for the next inductive step. Suppose we have completed the adjunction spaces $Y \sqcup_f X_{T_i}$ and $Y' \sqcup_g X'_{T_i}$ and have a homotopy equivalence σ between them. In this case, $Y \sqcup_f X_{T_i} = X^n$ and $Y' \sqcup_g X'_{T_i} = (X')^n$, so σ is a homotopy equivalence $\sigma\colon X^n \to (X')^n$. For the next step we will replace both α and β with σ. We immediately achieve inductive hypotheses 1 and 3. But 2 is not obvious.

Recall that we had fixed an ordering $(T_1, T_2, \dots, T_k)$ for $\mathcal{S}^n_W$. We see that in the intermediate steps before constructing the n–skeletons, we constructed all $\bigcup_{m\le l} X_{T_m}$ and $\bigcup_{m\le l} X_{T_m}$ for each $l \le k$. Furthermore, we constructed homotopy equivalences between $\bigcup_{m\le l} X_{T_m}$ and $\bigcup_{m\le l} X_{T_m}$ that are restrictions of our homotopy equivalence $\sigma\colon X^n \to (X')^n$. Therefore, we have inductive hypothesis 2, but only for subsets of $\mathcal{S}^n_W$ that correspond to a prefix of our ordering $(T_1, T_2, \dots, T_k)$. Clearly this is not all subsets of $\mathcal{S}^n_W$.

However, suppose we are given an arbitrary subset $\mathcal{T} \subseteq \mathcal{S}^n_W$. We could have chosen an ordering $\chi = (T_{a_1}, T_{a_2}, \dots, T_{a_k})$ of $\mathcal{S}^n_W$ such that $\mathcal{T}$ is some prefix of χ. We would have continued through the same steps and obtained a (potentially different) homotopy equivalence $\sigma_\chi \colon X^n \to (X')^n$. We observe from (14) that σ_χ must restrict to the homotopy equivalence $\psi_i \colon X_{T_i} \to X'_{T_i}$ for all $1 \leq i \leq k$. These ψ_i are each fixed. Since $\bigcup_{1\leq m\leq k} X_{T_m} = X^n$, these ψ_i completely determine σ (or σ_χ) as a map, so $\sigma = \sigma_\chi$. Thus, σ is independent of our choice of ordering and the restriction of σ to $\bigcup_{Q\in\mathcal{T}} X_Q$ for *any* $\mathcal{T} \subseteq \mathcal{S}^n_W$ is the necessary homotopy equivalence to satisfy inductive hypothesis 2.

Our proof concludes by observing that there is a minimal n such that $\mathcal{S}^{n+1}_W = \emptyset$, thus giving $X^n = X$ and $(X')^n = X'$. The inductive process outlined above eventually ends with a homotopy equivalence witnessing $X \simeq X'$. $\square$

Chapter 4

Discrete Morse theory

In this section we will prove the rightmost homotopy equivalence in (1). This will again involve the use of posets and their combinatorics. Morse theory for smooth manifolds gives a way to infer topological properties of manifolds from analytical properties of certain smooth functions on that manifold. Discrete Morse theory is a CW–complex (non–smooth) analogue. Certain functions on the (discrete) set of cells of a CW–complex can tell us topological facts about the CW–complex. Here we will only give a brief introduction to a particular formulation of the main results of this theory that are relevant to us.

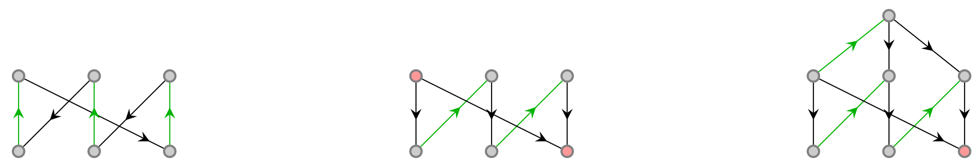
4.1 The face poset and acyclic matchings

In Section 2.2 we defined a construction that formed spaces from posets, we now define a construction in the opposite direction. Given a CW–complex X, denote its set of open cells as X^*. Given some $\sigma \in X^*$, denote its closure by $\overline{\sigma}$.

Definition 4.1. Given a CW–complex X, the *face poset* $\mathcal{F}(X)$ is a partial ordering on X^* where $\tau \leq \sigma$ when $\overline{\tau} \subseteq \overline{\sigma}$.

For a finite dimensional and connected CW–complex, $\mathcal{F}(X)$ is a bounded and graded poset with rank function $\mathrm{rk}(\sigma) = \dim(\sigma)$. Let P denote $\mathcal{F}(X)$. Consider some subset of the covering relations $\mathcal{M} \subseteq \mathcal{E}(P)$. We consider this as a set of edges in the Hasse diagram for P, which is denoted H. From $\mathcal{M}$ we define a new edge–orientation on the (already oriented) graph H such that $(p \lessdot q)$ is oriented from p to q if $(p \lessdot q) \in \mathcal{M}$, and otherwise in the opposite direction. We denote this oriented graph $H_{\mathcal{M}}$. We call $\mathcal{M}$ a *matching* if for all $p \in P$, at most one $m \in \mathcal{M}$ contains p. A matching is *acyclic* if $H_{\mathcal{M}}$ contains no directed cycles. Furthermore, a matching is *proper* if for all $p \in P$, the set of all nodes in $H_{\mathcal{M}}$ reachable by a directed path from p is finite. Fig. 11 gives some examples of matchings.

We observe that the requirement of being a matching means that any path through $H_{\mathcal{M}}$ will never consecutively go through two edges in $\mathcal{M}$. A cycle in $H_{\mathcal{M}}$ must start and end at the same rank. So, since edges in $\mathcal{M}$ increase rank and edges in $\mathcal{E}(P) \setminus \mathcal{M}$ decrease rank, a cycle must therefore be (cyclically) alternating between edges in $\mathcal{M}$ and edges in $\mathcal{E}(P) \setminus \mathcal{M}$. Therefore, if a cycle is to start at $p \in P$, it must completely occur in $\{q \in P \mid \operatorname{rk}(q) - \operatorname{rk}(p) \in \{0, 1\}\}$ or completely in $\{q \in P \mid \operatorname{rk}(q) - \operatorname{rk}(p) \in \{0, -1\}\}$, i.e. the horizontal bands above or below p in H. Since it is alternating, a cycle must also comprise an even number of edges.



(a) A cyclic matching.

(b) An acyclic matching with invalid subcomplex.

(c) An acyclic matching with valid subcomplex.

Figure 11: Directed Hasse diagrams corresponding to face posets and choices of $\mathcal{M}$. Green edges are those in $\mathcal{M}$ and red nodes are critical cells.

We call σ a *face* of τ if $\sigma \lessdot \tau$. Let $\Phi\colon D^n \to X$ be the characteristic map of some n–cell τ. We call σ a *regular face* of τ if it is a face and the following hold.

1) $\Phi|_{\Phi^{-1}(\sigma)}\colon \Phi^{-1}(\sigma) \to \sigma$ is a homeomorphism.

2) $\overline{\Phi^{-1}(\sigma)}$ as a subset of D^n is homeomorphic to D^{n-1}.

Remark 4.2. Given some natural number $n > 1$ and an n–cell $\sigma = [x_1|\cdots|x_n]$ in K_W, all the $n+1$ faces of σ are distinct. Thus, since K_W is a simplicial complex, every face of $[x_1|\cdots|x_n]$ is a regular face (considering K_W as a CW–complex). The only non–regular face of K_W is $[\,] \lessdot [w]$.

For a matching $\mathcal{M}$, any $p \in P$ that is disjoint from all of $\mathcal{M}$ is called *critical.* In this context, a *critical cell.* We can now state the version of discrete Morse theory we will use.

Theorem 4.3 ([PS21, Theorem 2.4])**.** *Consider a CW–complex X, a subcomplex $Y \subseteq X$, and a proper, acyclic matching $\mathcal{M}$ on X. If $\mathcal{F}(Y) \subseteq \mathcal{F}(X)$ is the set of critical cells in X with respect to $\mathcal{M}$ and if every σ is a regular face of τ for every $(\sigma \lessdot \tau) \in \mathcal{M}$, then X deformation retracts on to Y.*

You may notice that this seems to have no link to discrete functions $X^* \to \mathbb{N}$, as promised in the prologue of this section. Indeed, this statement is a reformulation of discrete Morse theory due [Cha00] and [Bat02]. The original formulation of discrete Morse theory is due to [For98]. The exact wording of Theorem 4.3 is important, we explore this in the following example.

Example 4.4. The Hasse diagrams in Fig. 11 correspond to the obvious CW–complex for a (hollow or filled) triangle. Figs. 11a and 11b are for a hollow 1–dimensional triangle and Fig. 11c is for a filled 2–dimensional triangle. We know that a hollow triangle cannot deformation retract on to any of its subcomplexes, thus the required construction for Theorem 4.3 should fail for Figs. 11a and 11b. We see that Fig. 11a is a cyclic matching, but we achieve an acyclic (vacuously proper) matching in Fig. 11b. Importantly, the space corresponding to the union of the critical cells, which are highlighted in the figure, is not a valid subcomplex, thus Theorem 4.3 does not apply. For a subset of cells $Y^* \subseteq X^*$ to correspond to a valid subcomplex $Y \subseteq X$, we require

$$\bigcup_{y \in \mathcal{F}(Y)} [-\infty, y] = \mathcal{F}(Y) \tag{15}$$

where $[-\infty, y]$ is taken within the poset $\mathcal{F}(X)$. For Fig. 11b, the left–hand side of (15) would include the bottom left cell in the Hasse diagram, which we see is not critical. In Fig. 11c, we have a valid subcomplex and the critical cell corresponds to a vertex in the 2–dimensional triangle, which is of course a valid deformation retract of the whole complex.

We will also require the following standard tool for forming acyclic matchings. For this we introduce the notion of a *poset map*, which a map between posets $P \to Q$ that respects the poset structure. Given such a map φ, we call preimages of single elements *fibres*.

Theorem 4.5 (Patchwork Theorem [Koz08, Theorem 11.10])**.** *Given a poset map $\varphi\colon P \to Q$, assume we have acyclic matchings on all fibres $\varphi^{-1}(q)$, where the matching need only be acyclic within the fibre itself. The union of all of these matchings is an acyclic matching on P.*

4.2 The subcomplex K'_W

We now introduce the particular poset map we will use. Recall that w is a choice of Coxeter element. Given the standard ordering of $\mathbb{N}$, the map $\eta\colon \mathcal{F}(K_W) \to \mathbb{N}$ with

$$\eta([x_1|x_2|\cdots|x_d]) \coloneqq \begin{cases} d & \text{if } x_1x_2\dots x_d = w \\ d+1 & \text{otherwise} \end{cases} \tag{16}$$

is a poset map. This can be readily checked. We call a connected component of $\eta^{-1}(d)$ a *d–fibre component*. We now investigate the form of these d–fibre components.

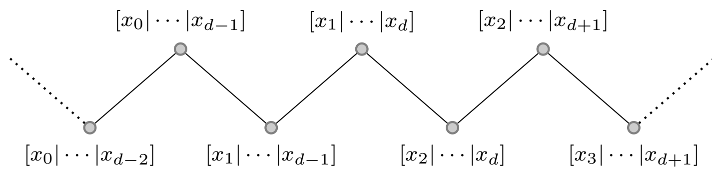


Figure 12: The general form of a connected d–fibre component.

Consider such a d–fibre component f such that $\sigma \coloneqq [x_1|x_2|\cdots|x_d] \in f$. We must have $x_1x_2\ldots x_d = w$ and so if $\sigma \lessdot \tau$ for some τ, then τ would be in $\eta^{-1}(d+1)$, not in f. The faces of σ are $[x_2|\cdots|x_d]$, $[x_1|\cdots|x_{d-1}]$ and $[x_1|\cdots|x_ix_{i+1}|\cdots|x_d]$ for all $i<d$. Only the first two of these faces will be in f since $\eta([x_1|\cdots|x_ix_{i+1}|\cdots|x_d]) = d-1$. Therefore, a d–fibre component will look like Fig. 12. We say that $[x_1|\cdots|x_d]$ satisfying the first case of (16) such that $x_1x_2\cdots x_d = w$ are in the *top row* of their fibre component. The cells satisfying the second case of (16) are in the *bottom row* of their fibre component.

To each fibre component there is an associated sequence $(x_i)_{i\in\mathbb{Z}}$ such that every product of d consecutive elements is equal to w and each corresponding cell from the sequence is in K_W. Denote the set of all such sequences S_d. For each fibre component, the choice of $s \in S_d$ is unique up translation of indices.

Remark 4.6. Consider some cell $[x_1|\cdots|x_{d-1}]$, i.e. a cell in the bottom row of Fig. 12. To form the whole fibre component, you may choose to move up and to the right along the Hasse diagram from to $[x_1|\cdots|x_d]$. This is always possible and completely deterministic, since $x_1x_2\ldots x_{d-1}$ is a prefix of a minimal factorisation of w and the product $x_1x_2\ldots x_d$ must equal w. Similarly, a movement up and leftward is always possible and deterministic using the balanced property of $[1,w]^W$. Thus, a fibre component does not have initial of final nodes and its Hasse diagram will either be homeomorphic to $\mathbb{R}$ or S^1. A finite fibre component will have a repeating sequence. If the sequence repeats every n, then the Hasse diagram of the fibre component will contain $2n$ nodes. An infinite fibre component will be non–repeating, though an individual group element x_i may be repeated in the sequence.

The previous remark shows that each sequence $s \in S_d$ is uniquely determined by any length $(d-1)$ subsequence. Furthermore, by the following remark, once we have fixed any length d subsequence we have an explicit formula for translating this subsequence to generate the whole sequence.

Remark 4.7. Let φ denote conjugation by w, i.e. $\varphi(x) = w^{-1}xw$. Then we have $\varphi(x_i) = x_{i+d}$ and $\varphi^{-1}(x_i) = x_{i-d}$ by the following factorisation.

$$\varphi(x_i) = (x_{i+d-1}^{-1}x_{i+d-2}^{-1}\ldots x_i^{-1})x_i(x_{i+1}x_{i+2}\ldots x_{i+d})$$

We will now define a subcomplex $K'_W \subseteq K_W$ based on these fibre components. Denote the power set of $\mathcal{F}(K_W)$ by $\mathcal{P}(\mathcal{F}(K))$. Let $F \subseteq \mathcal{P}(\mathcal{F}(K))$ be the set of all connected fibre components such that $\bigcup_{f\in F} f = \mathcal{F}(K_W)$. Recall X'_W from Definition 3.1.

Definition 4.8 (The subcomplex K'_W)**.** For each infinite $f \in F$ let f' be the elements of f in between and including the first and last elements of $f \cap X'_W$, where *first*, *last* and *in between* derive from the linear form of the fibre components as in (12). Define K'_W to be the union of all finite f and f' such that f is infinite.

For K' to be well–defined, we require that the construction of each f' be well–defined. This requires that $f \cap X'_W \neq \emptyset$ for each infinite f. This happens to be a detail for which the proof relies on W being of affine type.

Lemma 4.9 ([PS21, Lemma 7.6])**.** *Given a Coxeter group W of affine type and $f \in F$ a d–fibre component, there exists a simplex $\tau = [x_1|x_2|\cdots|x_{d-1}] \in f$ such that τ is also in X'_W.*

Note that for finite f, the Hasse diagram of f is homeomorphic to S^1. So, for finite f there is no notion of *in between*. Recall that X'_W is finite, so $f \cap X'_W$ is finite and *first* and *last* are well–defined.

The next property of K'_W we must prove is that K'_W is indeed a valid subcomplex of K_W, satisfying (15).

Lemma 4.10. *As defined above, K'_W is a valid subcomplex of K_W.*

Proof. We first concentrate on the infinite $f \in \eta^{-1}(d)$. We need to show that for any infinite $f \in F$, if $\tau \in f'$ and $\sigma \lessdot \tau$, then there exists some $g \in F$ such that $\sigma \in g'$. Let $s = (x_n)_{n\in\mathbb{Z}}$ be the sequence of group elements corresponding to f such that x_0 is the first group element in the symbol for τ (i.e. τ begins with $[x_0|\cdots)$. Let τ be between α and ω, both in $f \cap X'_W$. See Fig. 13. All of $[-\infty, \alpha]$ and $[-\infty, \omega]$ are also in X'_W, so we may assume that both α and ω are in the bottom row of the Hasse diagram of f' and accordingly each consist of $d-1$ group elements. Choose $a, z \in \mathbb{Z}$ such that $\alpha = [x_a|\cdots|x_{a+d-2}]$ and $\omega = [x_z|\cdots|x_{z+d-2}]$.

• First we will prove the case where τ is in the top row of f' such that $\tau = [x_0|x_1|\cdots|x_{d-1}]$ with $x_0x_1\cdots x_{d-1} = w$. The two faces $[x_1|\cdots|x_{d-1}]$ and $[x_0|\cdots|x_{d-2}]$ are both already in f', so we need only check the faces

$$\sigma^i = [x_0|\cdots|x_ix_{i+1}|\cdots|x_{d-1}] \lessdot \tau.$$

Each of the σ^i are in $\eta^{-1}(d-1)$. Given the sequence $s = (x_n)_{n\in\mathbb{Z}}$, we define the following.

$$s^i := (y^i_n)_{n\in\mathbb{Z}} = (\ldots, x_0, x_1, \ldots, x_ix_{i+1}, \ldots x_d, \ldots, x_{d+i}x_{d+i+1}, \ldots, x_{2d}, \ldots)$$

Where we multiply each adjacent pair x_j, x_{j+1} by removing the comma wherever $j \equiv i \mod d$. We see that every product of $(d-1)$ consecutive terms in s^i is w. Each s^i is the sequence corresponding to some connected component of $\eta^{-1}(d-1)$. Each face $\sigma^i \lessdot \tau$ is in the connected component associated to s^i. Denote this component g^i. We need to show that there exists $\alpha', \omega' \in g^i$ such that $\alpha' \leq \alpha$ and $\omega' \leq \omega$ and thus that $\alpha', \omega' \in X'_W$. We may choose α' and ω' to be any $[y^i_k|\cdots|y^i_{k+d-3}]$, i.e. a cell on the bottom row of g^i consisting of $(d-2)$ group elements. Let us concentrate on α'. There are three possibilities. Recall that $\alpha = [x_a|\cdots|x_{a+d-2}]$ and $\omega = [x_z|\cdots|x_{z+d-2}]$.

1. $a - i \equiv 1 \mod d$ in which case the relevant part of s^i looks like

$$\ldots, x_{a-2}, x_{a-1}x_a, x_{a+1}, \ldots, x_{a+d-2}, x_{a+d-1}x_{a+d}, x_{a+d+1}, \ldots.$$

   We choose $\alpha' = [x_{a+1}|\cdots|x_{a+d-2}]$.

2. $a - i \equiv 2 \mod d$ in which case the relevant part of s^i looks like

$$\ldots, x_{a-3}, x_{a-2}x_{a-1}, x_a, \ldots, x_{a+d-3}, x_{a+d-2}x_{a+d-1}, x_{a+d}, \ldots.$$

   We choose $\alpha' = [x_a|\cdots|x_{a+d-3}]$.

3. $a - i \equiv k \mod d$ with $3 \leq k \leq d$ in which case the relevant part of s^i looks like

$$\ldots, x_a, x_{a+1}, \ldots, x_{a+i}x_{a+i+1}, \ldots, x_{a+d-1}, x_{a+d}, \ldots.$$

   We choose $\alpha' = [x_a|\cdots|x_{a+i}x_{a+i+1}|\cdots|x_{a+d-2}]$.

In all cases α' is a cell in the bottom row of g' and in $[-\infty, \alpha]$. The same argument works for ω as well. We see that σ is indeed between two cells in X'_W within some fibre component.

- Now we consider the case where $\tau = [x_0|\cdots|x_{d-2}]$ is in the bottom row of f'. We can use the same methods as before. The two additional faces $[x_1|\cdots|x_{d-2}]$ and $[x_0|\cdots|x_{d-3}]$ to consider do not pose any extra difficulty. We can choose an appropriate s^i and proceed as before.

We now focus on the case where $\tau \in f$ and f is finite. By Remark 4.6 the sequence s associated to f will be repeating. We can again case–split whether τ is in the top or bottom row of the Hasse diagram for f. In either case, a face $\sigma \lessdot \tau$ will be in a component $g \in F$ associated to an appropriately chosen s^i. This s^i will also be repeating, thus the associated g will be finite. $\square$

Remark 4.11. Given some non–repeating sequence $s \in S_d$, repeated application of the map $s \mapsto s^i$ (for valid choices of i each time) must eventually result in a repeating sequence.

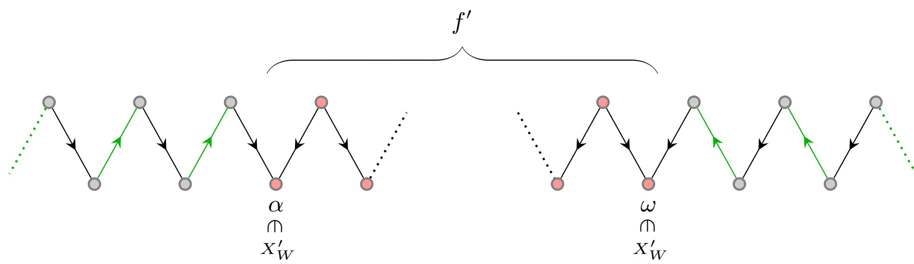


Figure 13: The unique acyclic matching $\mathcal{M}_f$ on f such that the critical cells correspond to exactly f'. Here α and ω are the relevant first and last elements in $f \cap X'_W$ as in the proof of Lemma 4.10. Critical cells are red and edges in $\mathcal{M}_f$ are green.

We now get to reap the benefits of working in this strange setting of face posets.

Theorem 4.12 ([PS21, Theorem 7.9, Lemma 7.11])**.** *The subcomplex K'_W is a deformation retract of K_W.*

Proof. We will utilise the patchwork theorem (Theorem 4.5). We need to show that for each $f \in F$, we have an acyclic matching $\mathcal{M}_f$ such that the critical vertices corresponding to $\mathcal{M}_f$ are exactly those contained in K'_W. For finite f, this is trivial. We choose $\mathcal{M}_f = \emptyset$. For infinite f, choosing such an $\mathcal{M}_f$ is very simple, as shown in Fig. 13. This choice of $\mathcal{M}_f$ is unique. The union of these $\mathcal{M}_f$ is an acyclic matching on $\mathcal{F}(K_W)$ by Theorem 4.5. Denote this matching $\mathcal{M}$. Denote the directed graph resulting from orientation of edges in the Hasse diagram of $\mathcal{F}(K_W)$ by this matching as $H_{\mathcal{M}}$. We have that $\mathcal{M}$ is also proper since any directed path in $H_{\mathcal{M}}$ can only travel inwards along fibre components, eventually reaching some element in $\mathcal{F}(K'_W)$. After reaching an element of $\mathcal{F}(K'_W)$, the path can only travel downwards. Since $\mathcal{F}(K_W)$ is finite–height, this path must eventually terminate. So by $\mathcal{M}$ is an acyclic, proper matching. By Theorem 4.3, this gives a deformation retract on to the union of the critical cells of $\mathcal{M}$, which is exactly K'_W. □

4.3 The final step

The remaining homotopy equivalence to be shown in (1) is $K'_W \simeq X'_W$. This is completed using discrete Morse theory in [PS21, Section 8]. We will not review this part of the proof in detail but instead provide a very brief overview.

At this stage, it is useful to remark that the only part of this work so far that has been specific to *affine* Coxeter groups has been Lemma 4.9. Everything else generalises to all Coxeter groups. However, the proof of $K'_W \simeq X'_W$ in [PS21] relies heavily on the geometry of affine Coxeter groups, as we will see.

The *rank* of a Coxeter group W is the maximum n such that $\mathcal{S}_W^n \neq \emptyset$. Given a realisation of an affine W as a reflection group on $\mathbb{R}^n$, where n is the rank of W, the *Coxeter axis* ℓ corresponding to a Coxeter element w is an affine subset of $\mathbb{R}^n$ associated to w. As the name suggests, the Coxeter axis is always a 1–dimensional affine subspace of $\mathbb{R}^n$. There is also an orientation on ℓ. See [PS21, Section 4] and [McC15, Section 7].

Given a Coxeter element w and corresponding interval $[1, w]^W$, we recall $R_0 \coloneqq R \cap [1, w]^W$. There exists a total ordering on R_0 using ℓ and its orientation [PS21, Definition 4.10]. Using properties of this ordering and the linear structure of fibre components, the authors define a function

$$\mu \colon \mathcal{F}(K'_W) \setminus \mathcal{F}(X'_W) \to \mathcal{F}(K'_W) \setminus \mathcal{F}(X'_W).$$

It is shown that μ is an involution, i.e. $\mu(\mu(\sigma)) = \mu(\sigma)$ and thus the authors define the matching

$$\mathcal{M} \coloneqq \{(\mu(\sigma), \sigma) \mid \sigma \in \mathcal{F}(K'_W) \setminus \mathcal{F}(X'_W) \text{ and } \mu(\sigma) \lessdot \sigma\}.$$

This matching is shown to be acyclic, proper and with critical cells exactly corresponding to X'_W. Thus, the proof concludes using Theorem 4.3.

Theorem 4.13 ([PS21, Theorem 8.14])**.** *Given an affine Coxeter group W, the space K'_W deformation retracts on to X'_W.*

Chapter 5

Ideas for future exploration and conclusions

The proof of $X'_W \simeq K'_W$ in [PS21, Section 8] constructs the necessary acyclic matching without using the patchwork theorem. Here we will suggest an alternative method that could be used to show $X'_W \simeq K'_W$ using the patchwork theorem similarly to how it was used in the proof of Theorem 4.12. There are obvious obstacles that would rule out this method as a replacement or generalisation of the work of Paolini and Salvetti, however there is the possibility that it could show $X'_W \simeq K'_W$ for some non–affine W (in principle K' is not well–defined for non-affine W, but this will be addressed).

First we must explore why the methods used in the proof of Theorem 4.12 are not suitable to prove $X'_W \simeq K'_W$.

5.1 Linear obstacles to matchings

Consider an element $\sigma \in \mathcal{F}(K'_W) \setminus \mathcal{F}(X'_W)$ in the context of a fibre component f. For affine W, we have that σ is necessarily in a section of f between two elements, β and ψ both in $\mathcal{F}(X'_W)$. As in the proof of Lemma 4.10 we can assume β and ψ to be in the bottom row of the Hasse diagram for f. We see in Fig. 14 that there is no matching on the fibre component f that corresponds to the contraction of K'_W on to X'_W.

We say that the matching on the section just to the right of β in Fig. 14 is *oriented towards* β. Similarly, the matching on the section just to the left of ψ is oriented towards ψ. In this terminology, to form the necessary matching on f, we need a way to *change orientation* on a matching, as in Fig. 15.

The simple form of the fibre components in Chapter 4 makes them easy to conceptualise, but we see that they lack enough structure to allow for the orientation change that

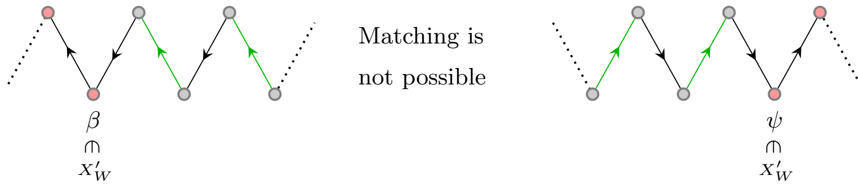


Figure 14: C.f. Fig. 13. The central section between β and ψ is contained within $\mathcal{F}(K'_W) \setminus \mathcal{F}(X'_W)$. We see there is no way to form a matching on this fibre component such that all the critical cells are in the complement of this central section.

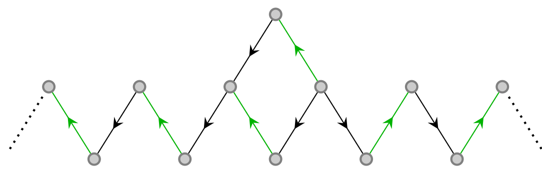


Figure 15: Certain structures, if added to the fibre component, allow us to change orientation in a matching. Note that this is only demonstrative and no such fibre component exists.

we have shown is necessary. In the following section we propose a conceptually simple change to the construction in Section 4.2, that changes the poset map $\eta\colon \mathcal{F}(K_W) \to \mathbb{N}$ and will hopefully give us fibre components with enough structure to allow for orientation changes.

5.2 Fibre doubling and junctions

We will now outline this idea for a proof of $K' \simeq X'_W$ different to that in [PS21, Section 8]. This will involve utilizing fibre components and the patchwork theorem, as in Section 4.2, but we will alter the poset map used to define the fibres.

Let $h\colon \mathbb{N} \to \mathbb{N}$ be the poset map such that $h(n) = 2 \cdot \lfloor n/2 \rfloor$ and recall η from (16). Consider the following map ν.

$$\nu : \mathcal{F}(K_W) \xrightarrow{\eta} \mathbb{N} \xrightarrow{h} \mathbb{N} \tag{17}$$

We have that ν is also a poset map. If d is odd then $\nu^{-1}(d) = \emptyset$. If d is even then $\nu^{-1}(d) = \eta^{-1}(d) \cup \eta^{-1}(d+1)$. We call the connected (non–empty) components of $\nu^{-1}(d)$ *double fibre components*. Let G denote the set of double fibre components and F denote the set of fibre components as in Section 4.2. To each $g \in G$ we associate a subset $D(g) \subseteq F$ that is the set of fibre components in F that comprise the double fibre component g.

Given some fibre component $f \in F$. Define $\Sigma(f)$ to be a sequence $(\sigma_i)_{i\in\mathbb{Z}}$ of elements in $\mathcal{F}(K_W)$ such that $\bigcup_{i\in\mathbb{Z}} \sigma_i = f$, and such that $(\sigma_i)_{i\in\mathbb{Z}}$ respects the order in f. This

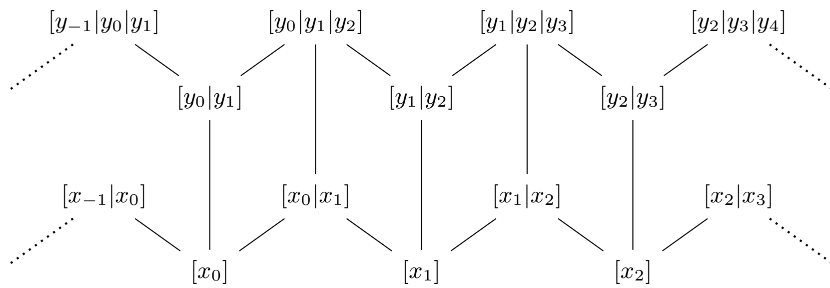


Figure 16: A length 5 junction starting at $([x_0], [y_0|y_1])$. This corresponds to a connected component of $\nu^{-1}(2)$. Here f is the 2–fibre component with sequence $(x_i)_{i\in\mathbb{Z}}$ and f' is the 3–fibre component with sequence $(y_i)_{i\in\mathbb{Z}}$ such that $y_1 y_2 = x_1$. This necessitates $y_0 = x_0$ and $y_3 = x_2$.

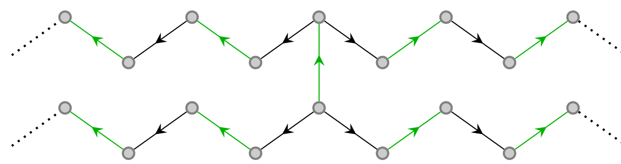


Figure 17: A demonstration of how a junction can be used to simultaneously change matching orientation on a pair of fibre components. In our case, we require one of the bridges of the junction to be between the upper rows of the respective fibre components.

sequence is unique up to translation of indices. Note this is different but related to the sequence of group elements defined in Section 4.2.

Definition 5.1. Given some $g \in G$, a connected component of $\nu^{-1}(d)$, let $f, f' \in D(g)$ be two distinct fibre components within g such that f is a d–fibre component and f' is a $(d+1)$–fibre component. Let $\Sigma(f) = (\sigma_i)_{i\in\mathbb{Z}}$ and $\Sigma(f') = (\sigma'_i)_{i\in\mathbb{Z}}$. Suppose there is a consecutive sequence of integers $(n, n+1, \ldots, n+j)$ and some constant of translation $m \in \mathbb{Z}$ such that $\sigma_i \lessdot \sigma'_{i+m}$ for all $i \in (n, n+1, \ldots, n+j)$. We say that

$$\bigcup_{n\le i\le n+j} \sigma_i \qquad \cup \qquad \bigcup_{n+m\le i\le n+j+m} \sigma'_i$$

is the *junction* between f and f' of length $j+1$, starting at $(\sigma_n, \sigma'_{n+m})$.

We say that the individual $\sigma_i \lessdot \sigma'_{i+m}$ corresponding to a junction are the *bridges* of that junction. See Fig. 16 for an example picture of a junction.

A useful property of junctions is that they enable us to change orientation in a matching, see Fig. 17. However, we see that this change of direction is necessarily paired with the other fibre component.

There are some immediate limitations to this technique. Let $d_{\max} \coloneqq \max\{d \in \mathbb{N} \mid \eta^{-1}(d) \neq \emptyset\}$. Recalling that $\nu^{-1}(d) \neq \emptyset$ only if d is even, we see that if $d_{\max}$ is even then $\nu^{-1}(d_{\max}) = \eta^{-1}(d_{\max})$ and connected components of $\nu^{-1}(d_{\max})$ will be simple and linear ($d_{\max}$)–fibre components, as in Fig. 12. So, the required matching would be is impossible on $\nu^{-1}(d_{\max})$

For a given Coxeter system (W, S) and Coxeter element w, the value of $d_{\max}$ is $|S|$. So this particular construction will not be useful for Coxeter systems for which $|S|$ is even. However, given $|S|$ is odd, we always have junctions between fibre components.

Lemma 5.2 ([PS21, Lemma 5.1])**.** *Fix a Coxeter system (W, S) with set of reflections $R \subseteq W$ and let $l_R\colon W \to \mathbb{N}$ be the length function on W with respect to the generating set R. For any Coxeter element $w \in W$, we have $l_R(w) = |S|$.*

Lemma 5.3. *Fix a Coxeter system (W, S) such that $|S| \geq 3$ is odd and fix a Coxeter element $w \in W$. For some $d \geq 1$, either $\nu^{-1}(d) = \emptyset$ or $\nu^{-1}(d)$ contains a junction.*

Proof. Denote the set of reflections in W by R. Suppose $\nu^{-1}(d) = \eta^{-1}(d) \cup \eta^{-1}(d+1) \neq \emptyset$. We have that d is even. Bear in mind that for any n, $\eta^{-1}(n) \neq \emptyset \iff 1 \leq n \leq |S|$. At least one of $\eta^{-1}(d)$ or $\eta^{-1}(d+1)$ are non–empty. Given our restrictions, $\eta^{-1}(d)$ is definitely non–empty, so $d \leq |S|$. Since d is even, $d \leq |S|$ and $|S|$ is odd, we have $d < |S|$.

Now let $\sigma = [x_1|\cdots|x_{d-1}]$ be in some d–fibre component such that $x_1x_2\cdots x_d = w$. There is a factorisation of w in R that respects the factors $x_1x_2\cdots x_d = w$. By Lemma 5.2, this factorisation contains $|S|$ factors. Since $|S| > d$, by pigeonhole principle there is at least one $x_i \in \{x_1, \ldots, x_d\}$ that non–trivially factors in R. Suppose $x_i = r_1r_2\cdots r_k$ is this factorisation, where the r_j are in R. We have that

$$\tau = [x_1|\cdots|x_{i-1}|r_1|r_2r_3\cdots r_k|x_{i+1}|\cdots|x_d] \in \eta^{-1}(d+1)$$

and $\sigma \lessdot \tau$. $\square$

Remark 5.4. The construction of this proof is doing the construction in the proof of Lemma 4.10 with s^i, but in the inverse order (i.e. the inverse of the map $s \mapsto s^i$ in the proof of Lemma 4.10). The above construction results in a junction of length at least $2d+1$. See Fig. 16 for an example of such a junction.

At this point it is worth noting why we have chosen to define ν as in (17). Suppose in (17) we had instead chosen h to be $h(n) = 2\lceil n/2\rceil$ instead of $h(n) = 2\lfloor n/2\rfloor$. The non–empty fibres of ν would instead look like $\eta^{-1}(d) \cup \eta^{-1}(d-1)$ and this construction could potentially work for (W, S) where $|S|$ is even. However, consider now $\eta^{-1}(1)$. This comprises two elements $[\,]$ and $[w]$. If we chose $h(n) = 2\lceil n/2\rceil$ then we would have $\nu^{-1}(2) = \eta^{-1}(2) \cup \eta^{-1}(1)$. There would be no way to use the junctions between $[w] \in \eta^{-1}(1)$ and all the 2–fibre components to construct a useful matching. The relevant picture would be to imagine Fig. 17 but with only one vertex in the bottom row, to be shared between all 2–fibre components. Hence, why we made our specific choice for h in defining ν.

Also note that $\nu^{-1}(0) = \eta^{-1}(1)$ is missed out by Lemma 5.3. This poses no issues since $\eta^{-1}(1)$ is always a subset of X'_W by our previous remarks and so no matching construction is necessary on this fibre.

At this point we should address how this construction could avoid using Lemma 4.9, which is only proven for affine Coxeter groups. Suppose we had some infinite fibre component f such that $f \cap X'_W = \emptyset$. It may seem possible that we could form a matching as in Fig. 18 which has no critical cells. However, the number of cells reachable by a directed path from any of the cells in this matching is infinite. So the matching is improper and is not useful for our construction.

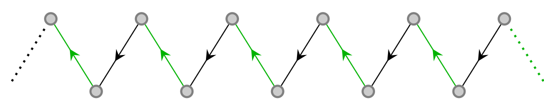


Figure 18: A matching on an infinite component with no critical cells. However, this matching is not proper.

However, consider the case where there is a junction j between f and another fibre component such that a bridge of j attaches to the bottom row of f, then we could construct a matching as in Fig. 19. This matching is proper on f and no critical cells are in f.

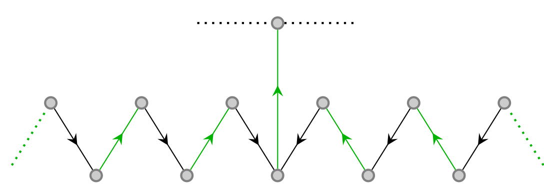


Figure 19: A proper matching on an infinite component f such that no critical cells are in f. This utilises a junction from the bottom row of f to some other fibre component.

5.3 How fibre doubling could be used

We will now outline more explicitly how fibre doubling could be used to show the homotopy equivalence $K'_W \simeq X'_W$.

Let F denote the set of fibre components of K_W as in Section 4.2. Alter Definition 4.8 such that all $f \in F$ with $f \cap X'_W = \emptyset$ are entirely included in K'_W (eliminating the need for Lemma 4.9). Denote by M the set of connected components of $f \cap (K'_W \setminus X'_W)$ as f ranges through F. Each $m \in M$ is a connected section of a fibre component in $K'_W \setminus X'_W$

bounded (not inclusively) at each end by an element of X'_W, as in Fig. 14. We have that $\bigcup_{m\in M} m = K'_W \setminus X'_W$. For each $f \in F$ we need to construct a matching on all $m \subseteq f$ with critical cells being exactly $f \cap X'_W$. Define ν as before and denote the set of double fibre components by G. For each $g \in G$, denote by $M(g)$ the set of $m \in M$ such that $m \subseteq g$. We must find a pairing $\Pi \subseteq M(g) \times M(g)$ such that

1. For each pair $(m, s) \in \Pi$, m is finite iff s is finite.
2. For each pair $(m, s) \in \Pi$ with m and s finite, there is a bridge of a junction $b((m, s))$ with one end of $b((m, s))$ contained in the top row of m and one end in top row of s.
3. For each pair $(m, s) \in \Pi$ with m and s infinite, there is a bridge of a junction $b((m, s))$ with one end of $b((m, s))$ contained in the bottom row of m and one end in the bottom row of s.
4. For all $p_1, p_2 \in \Pi$, the ends of $b(p_1)$ and $b(p_2)$ are distinct.
5. For all $m \in M(g)$, Π contains exactly one element containing m.

Given such a pairing Π for each $g \in G$, we conjecture that we can then form a matching on all the fibres of ν such that the critical cells of that matching are X'_W. We will only motivate this, without proof.

Fix some $g \in G$. Given such a pairing Π as above, every bridge $\{b(p) \mid p \in \Pi\}$ is included in a matching $\mathcal{M}_g$ on g. We continue along sections $m \in M$ such that $m \subseteq g$, alternately including edges not involved in a bridge in $\mathcal{M}_g$. For finite sections this looks like Fig. 17 and for infinite sections like Fig. 19. Given requirement 4, this is a valid matching. Given requirement 5, we switch orientation of the matching for each $m \in M(g)$, thus allowing us to create a matching with the required critical cells. We then complete this for all $g \in G$, obtaining a matching on all the fibres on ν. We would then conclude using the patchwork theorem.

These may seem like very difficult restrictions to meet, but given Remark 5.4, we can construct relatively long junctions very easily between certain pairs of fibres in F. Requirement 5 means that $M(g)$ must be even, which may be an impossible restriction to meet. If we are fortunate, in cases where it is possible, the exercise may be more in bookkeeping (ensuring requirement 4 and 5 mainly). At this point we can only pose the following question.

Question 5.5. *Is there a way to form pairings Π as above? If this is not possible for all Coxeter groups W, is there a non–empty class of Coxeter groups for which it is possible?*

Of course, even if this were true for some W, we would also need to prove that K_W is a $K(W_w, 1)$ or $K(G_W, 1)$ to prove the $K(\pi, 1)$ conjecture for those W.

5.4 Concluding remarks

Classifying spaces are useful tools in studying the properties of groups. There are several algebraic properties of Artin groups that can be verified from an understanding of their classifying spaces, see [Cha08]. For instance, proving the $K(\pi, 1)$ conjecture would give us that Artin groups are torsion free by [Hat01, Proposition 2.45]. However, the difficulty in using classifying spaces to understand groups is that we move from the discrete mathematics of algebra to (typically high dimensional) topology. Sometimes this leads to a beneficial geometric intuition, but sometimes it leads to seemingly intractable problems. In either case, the power of studying groups using topology is well established.

The methods we have presented here demonstrate ways to translate problems of topology back in to problems of discrete mathematics and combinatorics, allowing us to use tools such as induction or to leverage intuition of graphs. The particularly prominent role of posets is intriguing. We note the many roles they play in this work. In Section 2.2, we showed ways to construct spaces from posets such that the fundamental group of this space is encoded by the geometry of the poset. In Section 2.3 we showed that for certain Coxeter groups W, the isomorphism class of W is encoded by the geometry of a subset of its Cayley graph that behaves well as a poset. In Chapter 4 we formed a poset using the CW-structure of a space. We then proved facts about the space using combinatoric and geometric properties of that poset.

Even ignoring the role of discrete Morse theory in this work, it is clear of the importance of posets in understanding Coxeter groups and Artin groups. This is made even clearer now that the seemingly roundabout route taken by Paolini and Salvetti, going via dual Artin group and its defining poset, has proved so fruitful. A better understanding of the geometry of interval posets $[1, w]^W$ would be a valuable tool, and it seems very possible that a similar route may be used to show the $K(\pi, 1)$ conjecture in more general cases in the future.

References

[Art47] E. Artin. Theory of Braids. *Annals of Mathematics*, 48(1):101–126, 1947.

[Bat02] Ekkehard Batzies. Discrete Morse theory for cellular resolutions. *Journal für die reine und angewandte Mathematik*, 2002(543):147–168, January 2002.

[BB10] Anders Bjorner and Francesco Brenti. *Combinatorics of Coxeter Groups*. Springer Berlin Heidelberg, October 2010.

[Bes03] David Bessis. The dual braid monoid. *Annales Scientifiques de l'École Normale Supérieure*, 36(5):647–683, September 2003.

[Bou08] Nicolas Bourbaki. *Elements of Mathematics. Lie Groups and Lie Algebras Chapters 4 - 6*. Springer, Berlin, Heidelberg, 2008.

[Bri71] E. Brieskorn. Die Fundamentalgruppe des Raumes der regulären Orbits einer endlichen komplexen Spiegelungsgruppe. *Inventiones mathematicae*, 12(1):57–61, March 1971.

[Bro06] Ronald Brown. *Topology and Groupoids*. Booksurge, 2006.

[CD95] Ruth Charney and Michael W. Davis. The K(pi, 1)-Problem for Hyperplane Complements Associated to Infinite Reflection Groups. *Journal of the American Mathematical Society*, 8(3):597–627, 1995.

[Cha00] Manoj K. Chari. On discrete Morse functions and combinatorial decompositions. *Discrete Mathematics*, 217(1):101–113, April 2000.

[Cha08] Ruth Charney. Problems related to Artin groups, 2008.

[CMW02] Ruth Charney, John Meier, and Kim Whittlesey. Bestvina's normal form complex and the homology of Garside groups, March 2002.

[Del72] Pierre Deligne. Les immeubles des groupes de tresses générales. *Inventiones Mathematicae*, 17:273–302, 1972.

[EM45] Samuel Eilenberg and Saunders MacLane. Relations Between Homology and Homotopy Groups of Spaces. *Annals of Mathematics*, 46(3):480–509, 1945.

[FN62a] Edward Fadell and Lee Neuwirth. Configuration Spaces. *Mathematica Scandinavica*, 10:111–118, 1962.

[FN62b] R. Fox and L. Neuwirth. The Braid Groups. *Mathematica Scandinavica*, 10:119–126, 1962.

[For98] Robin Forman. Morse Theory for Cell Complexes. *Advances in Mathematics*, 134(1):90–145, March 1998.

[GAP22] GAP Group. GAP – Groups, Algorithms, and Programming, 2022.

[Gar69] Frank Arnold Garside. The braid group and other groups. *The Quarterly Journal of Mathematics*, 20(1):235–254, January 1969.

[Grä11] George Grätzer. *Lattice Theory: Foundation*. Springer, Basel, 2011.

[Hat01] Allen Hatcher. *Algebraic Topology*. Cambridge University Press, 2001.

[Hua23] Jingyin Huang. Labeled four cycles and the K(pi,1)-conjecture for Artin groups, May 2023.

[Hum90] James E. Humphreys. *Reflection Groups and Coxeter Groups*. Cambridge University Press, 1 edition, June 1990.

[Koz08] Dmitry Kozlov. *Combinatorial Algebraic Topology*. Number 21 in Algorithms and Computation in Mathematics. Springer, 2008.

[McC05] Jon McCammond. An introduction to Garside structures, 2005.

[McC15] Jon McCammond. Dual euclidean Artin groups and the failure of the lattice property. *Journal of Algebra*, 437:308–343, September 2015.

[MS17] Jon McCammond and Robert Sulway. Artin groups of Euclidean type. *Inventiones mathematicae*, 210(1):231–282, October 2017.

[Mun00] James R. Munkres. *Topology*. Prentice Hall, Inc, Upper Saddle River, NJ, 2nd ed edition, 2000.

[Pao17] Giovanni Paolini. On the classifying space of Artin monoids. *Communications in Algebra*, 45(11):4740–4757, November 2017.

[Par14] Luis Paris. K(pi, 1) conjecture for Artin groups. *Annales de la Faculté des sciences de Toulouse : Mathématiques*, 23(2):361–415, 2014.

[PS21] Giovanni Paolini and Mario Salvetti. Proof of the K(pi,1) conjecture for affine Artin groups. *Inventiones mathematicae*, 224(2):487–572, May 2021.

[Sag20] SageMath Developers. SageMath, October 2020.

[Sal87] M. Salvetti. Topology of the complement of real hyperplanes in $\mathbb{C}$N. *Inventiones mathematicae*, 88(3):603–618, October 1987.

[Sal94] Mario Salvetti. The Homotopy Type of Artin Groups. *Mathematical Research Letters*, 1(5):565–577, September 1994.

[VD83] Nguyêñ Viêt Dũng. The fundamental groups of the spaces of regular orbits of the affine weyl groups. *Topology*, 22(4):425–435, 1983.