SAMPLE TITLE

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My mum.

Abstract.

Contents

Introduction

A classifying space for a group G is a space X such that the fundamental group of X is G and all higher homotopy groups of X are trivial. In this paper we will be concerned with the $K(\pi, 1)$ conjecture for Artin groups, which states that the configuration space Y_W for any Coxeter group W is a classifying space for the Artin group G_W. This conjecture emerges as a generalisation of the result for Coxeter groups of type A_n and is originally attributed to Arnol'd, Brieskorn, Pham and Thom in [**?**]. See also [**?**] for a good overview of the history of the conjecture.

We will focus on the work of Paolini and Salvetti, *Proof of the* $K(\pi, 1)$ *conjecture for affine Artin groups* [**?**]. We will review some theorems therein and provide relevant background. The theorem which is the namesake of [**?**] relies on proving a chain of homotopy equivalences (1). Detailing two of these homotopy equivalences is the aim of this work. A strong theme will be the involvement of posets and related structures, hence this work's title. We will begin by providing a birds eye view of the conjecture and the main results of [**?**].

The main results of this work are thm:fund$_g roup_p oset_c omplex_p oset_g roup, thm :$ $salvetti_c x_e quiv_{X_p} rime, thm : subcomplex_{K_p} rime_h om_e quiv_K .thm : fund_g roup_p oset_c omplex_p oset_g roupissta$

1. The conjecture and the objects involved

Coxeter groups emerge as generalisations of reflection groups. A Coxeter group is defined by a particular group presentation. The data of this presentation is typically encoded by a labelled graph. The group W, coupled with the data of its presentation is called a Coxeter system, denoted (W, S) where S is the generating set of W. Given a Coxeter system (W, S), we can construct a different group G_W, called the Artin group associated to W.

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For affine Coxeter groups W, the configuration space Y_W can be derived from a geometric realisation of W as a subgroup of (), the group of isometries on a Euclidean space . We will consider as ^{n} without the notion of origin. Specifically, W is realised as a subgroup generated by a finite set of affine reflections S. Within W, we consider the set of all reflections R (not necessarily finite). To each reflection $r \in R$ there is a corresponding codimension–1 space $H_r \subset$ that is the plane of reflection of r. We call such spaces hyperplanes. Note there is no requirement that these hyperplanes be subspaces of ^{n}.

The configuration space is realised as the complement of the complexification of all such hyperplanes H_r. It is known by work of Brieskorn [**?**] that the fundamental group of Y_W is G_W. Thus, proving the $K(\pi, 1)$ involves showing that the higher homotopy groups of Y_W are trivial. By previous work by Salvetti [**?**, **?**], there is a CW–complex X_W called the Salvetti complex that is homotopy equivalent to Y_W. Showing homotopy equivalence to X_W thus shows homotopy equivalence to Y_W. Because of this, the Salvetti complex is the starting point in a chain of homotopy equivalences reviewed in this work.

The finishing point of this chain is the interval complex K_W. This is a space realised using a certain poset structure on subsets of W. To this poset structure there is an associated group called the dual Artin group, denoted W_w. It was already known (by a now standard construction due to Garside [**?**], extended by other authors, see [**?**]) that K_W was a classifying space for the dual Artin group for finite W. In [**?**], the authors extend this result to affine W. Thus, showing $Y_W \simeq K_W$ for affine W shows that (for affine W) the higher homotopy groups of Y_W are trivial and that $W_w \cong G_W$.

In the following section, we will identify the intermediate spaces used in proving $X_W \simeq K_W$.

2. Proof overview

Here we will compile several main results from [**?**] in to two theorems. The concern of this work is thm:proof$_o$*verviewwhichprovesthattheSalvetti complex*X_W is homotopy equivalent to the *interval complex* K_W. A *Coxeter element* is a non–repeating product of all the elements of S. A choice of order on S corresponds to a choice of Coxeter element. Constructing an interval complex associated to (W, S) involves making such a choice of Coxeter element $w \in W$.

For a subset $T \subseteq S$, the *parabolic subgroup* W_T is the subgroup of W generated only by elements of T and only with relations explicitly involving elements of T. A *parabolic Coxeter element* w_T is a product of all elements of T that respects the order of multiplication in a Coxeter element $w \in W$. The space X'_W is a subspace of K_W associated to parabolic Coxeter elements w_T with $T \subseteq S$ such that W_T is finite. Cells in X_W also correspond to such subsets, which is used in proving $X_W \simeq X'_W$.

The space K'_W is also a subspace of K_W. Given a CW–complex X, we can encode some information of how cells of X attach to each other in a poset called the *face poset* of X, denoted $\mathcal{F}(X)$. Connected components of preimages $\eta^{-1}(d)$ of a certain poset map $\eta\colon K_W \to$ have a linear structure as subposets of $\mathcal{F}(K_W)$. For each element $x \in \eta^{-1}(d)$, whether x is in K'_W or not is determined based on whether x comes in between two elements of X'_W in the linear structure of $\eta^{-1}(d)$.

[[**?**]] Given an affine Coxeter system (W, S), the configuration space Y_W is homotopy equivalent to the order complex K_W.

Proof. By thm:salvetti$_h om_e quiv_c onfig the Salvetti complex$ X_W is homotopy equivalent to the configuration space Y_W. Therefore, we need only show $K_W \simeq X_W$. This is done through a composition of homotopy equivalences

(1)

$$X_W \simeq eqmid : salvetti_s alvettiprime X'_W \simeq eqmid : salvettiprime_i ntervalcomplexprime K'_W \simeq eqmid : inte$$

Where the results are gathered from the following sources:

[(a)]thm:salvetti$_c x_e quiv_{X_p} rime[\mathbf{?}, Theorem 5.5] \quad thm : K_p rime_h om_e quiv_{X_p} rime[\mathbf{?}, Theorem 8.14] \quad thm$ □

In [**?**], another main result is shown.

[[**?**, Theorem 6.6]] Given an affine Coxeter system (W, S), corresponding affine type Artin group G_W and Coxeter element $w \in W$, the complex K_W is a classifying space for the dual Artin group W_w.

We have that $\pi_1(Y_W) \cong G_W$ by [**?**]. Thus, considering $\pi_1(Y_W)$ and combining thm:proof$_o verview, thm : KW_c lassifying spacegives for affine \mathrm{G}_W$.

This proves the $K(\pi, 1)$ conjecture for affine Artin groups and provides a new proof than an affine Artin group is naturally isomorphic to its dual, which was already known for spherical or finite type Artin groups [**?**] and affine Artin groups [**?**].

Non–complete alternatives to [**?**] are [**?**] and [**?**], which show $\pi_1(Y_W) \cong G_W$ for affine Coxeter groups and Coxeter groups of type A_n respectively.

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