

A REVIEW OF FREE PRODUCTS AND THE ISOMORPHISM BETWEEN STANDARD AND DUAL ARTIN GROUPS – RESTEGHINI

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1. INTRODUCTION

In [Res24] *Resteghini* explores the dual Artin group $A^\vee = A^\vee(W, S, h)$ for some Coxeter system (W, S) and Coxeter element $h \in W$. He shows that certain conditions on the Coxeter group W and Artin group A imply the natural surjective homomorphism $A \rightarrow A^\vee$ is an isomorphism. In this review we will abstract a construction that gives (as a specific case) the dual Artin group A^\vee . We will then prove a general result about this construction and show that [Res24, Proposition 3.9] can be re-stated (and re-proved) in this general framework.

1.1. Some facts about the objects involved. Here we will state some facts and try to point to the relevant place to read up on them.

Throughout, let $W = (W, S)$ be some Coxeter system. Let $\{\{s_i\}_{1 \leq i \leq n}\}$ be some ordering on S and let $h = s_1 s_2 \cdots s_n$ be a choice of Coxeter element resulting from the ordering on S . From here forward, fix $W = (W, S)$ and h . Let $T = S^W$ be the set of reflections in W . This is all conjugations of S in W . We define the dual Artin group $A^\vee = A^\vee(W, S, h)$, as in [Res24] or [PS21].

Let B_n denote the braid group on n strands generated by $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$ in the usual way. Let $x^y = yxy^{-1}$ denote conjugation in groups.

Definition 1.1 (The Hurwitz action). *Let X be some set on which we can define conjugates which lie within X . Define a group action of B_n on X^n in the following way.*

$$\sigma_i \cdot (t_1, \dots, t_n) = (t_1, \dots, t_{i-1}, t_{i+1}^{t_i}, t_i, t_{i+2} \dots t_n).$$

Any group G is a fine candidate for X . A subset that is closed under conjugation will also work, for example the set of reflections $T \subseteq W$ as defined above.

Theorem 1.2. *We can map T^n to W by multiplying all the elements in the tuple in order. Considering this map, the action of B_n is transitive on minimal T -factorisations of h .*

We will use the symbol h as to refer to both a Coxeter element $h \in W$, and a specific factorisation of h , realised as a tuple, $h \in T^n \subseteq W^n$. Let $T_0 := T \cap (B_n \cdot h)$ denote all reflections that occur in any position in a minimal T -factorisation of h .

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Theorem 1.3. *For each $r \in T_0$, and for each $i \in \{1, \dots, n\}$ there exists a minimal T -factorisation $(t_1, \dots, t_n) \in (B_n \cdot h)$ such that $t_i = r$. So every reflection that appears, appears in every possible position.*

Theorem 1.4. *The above theorem also applies to subwords. For each $(r_1, \dots, r_n) \in (B_n \cdot h)$, each $i \leq n$ and each $j \leq n - i$, there exists a $(t_1, \dots, t_n) \in (B_n \cdot h)$ such that $(t_j, t_{j+1}, \dots, t_{j+i}) = (r_1, \dots, r_i)$.*

Theorem 1.5 ([MS17, Proposition 10.1]). *The generating W generating set S is contained in T_0 and this is also a generating set for A^\vee . The natural inclusion of $\overline{S} \subseteq A$ into A^\vee induces a surjective homomorphism.*

Theorem 1.6 ([Bes04, Lemma 7.11]). *Let $[t]$ denote an abstract generator coming from some $t \in T \cap (B_n \cdot h)$. The dual Artin group has the following presentation.*

$$A^\vee(W, S, h) \cong \langle T_0 \mid \{[tt't][t][t']^{-1}[t]^{-1} \mid (t, t', x_3, \dots, x_n) \in B_n \cdot h\} \rangle.$$

We call such $[tt't][t][t']^{-1}[t]^{-1}$ *dual braid relations*.

The following is a basic group theoretic fact, but it will be useful to explicitly state and prove it. For some set S , let F_S be the free group generated by S .

Lemma 1.7. *Suppose we have the groups G_1 and G_2 such that G_1 has a presentation $G_1 \cong \langle S \mid R \rangle$ where R is a set of words in $S \cup S^{-1}$ that are the identity in G_1 . Further, suppose we are given some map $f: S \rightarrow G_2$. For all $s \in S$, define $f(s^{-1})$ to be $f(s)^{-1}$. If for each word $s_1 s_2 \dots s_n \in R$ we have that $f(s_1)f(s_2) \dots f(s_n) = 1$ in G_2 then we can extend f to a homomorphism $h: G_1 \rightarrow G_2$ where $h|_S = f$ considering S as a subset of G_1 .*

Proof. We can define a homomorphism $f': F_S \rightarrow G_2$ recursively by setting $f'(1) = 1$, $f'(s^{\pm 1}) = f(s)^{\pm 1}$ for all $s \in S$ and then setting $f'(s_1 s_2 \dots s_n) = f'(s_1)f'(s_2) \dots f'(s_n)$ for all words $s_1 s_2 \dots s_n \in \{S \cup S^{-1}\}^*$. Let N be the minimal normal subgroup in F_S which contains the elements of R such that $G_1 \cong F_S/N$. By assumption, for each $r \in R$ we have $f'(r) = 1$. So $R \subseteq \ker(f')$. So $N \subseteq \ker(f')$. By the universal property of the quotient we have the following commutative diagram where the unique map h is the required homomorphism.

$$\begin{array}{ccc} F_S & \xrightarrow{f'} & G_2 \\ q \downarrow & \nearrow \exists! h & \\ F_S/N \cong G_1 & & \end{array}$$

□

Note that if $\langle \text{Im}(f) \rangle = G_2$ then h is a surjection.

2. A PROOF IN [RES24]

We will overview a proof in [Res24] while trying to avoid unnecessary detail. The generators $S = \{s_1, \dots, s_n\}$ have a natural bijection with generators of A . Let $\overline{s_i}$ denote the generator in A corresponding to $s_i \in S \subseteq W$. We have an action of B_n on A^n defined in the exact same way as in Definition 1.1. We will consider the orbit of $\overline{h} = (\overline{s_1}, \dots, \overline{s_n})$ under this action.

Definition 2.1. Let G be a group. Let $\rho_n: G^n \rightarrow G$ be projection on to coordinate n . We define the map $- \circ -: B_n \times G^n \rightarrow G$ as follows

$$\tau \circ (g_1, \dots, g_n) = \rho_n(\tau \cdot (g_1, \dots, g_n)).$$

Remark 2.2. $G^n \neq G$ so this is not a group action!

Definition 2.3. Define the equivalence relation \sim on B_n by

$$\tau \sim \tau' \iff \tau \circ h = \tau' \circ h.$$

Definition 2.4. Define the equivalence relation $\dot{\sim}$ on B_n by

$$\tau \dot{\sim} \tau' \iff \tau \circ \bar{h} = \tau' \circ \bar{h}.$$

Theorem 2.5 ([Res24, Propsition 3.9]). The B_n relations \sim and $\dot{\sim}$ are the same iff A^\vee is isomorphic to A .

Proof. We will only concentrate on the \implies direction. Furthermore, it is easy enough to see that $\tau \dot{\sim} \tau' \implies \tau \sim \tau'$. As such, we will only prove that if $\tau \sim \tau' \implies \tau \dot{\sim} \tau'$ for all $\tau, \tau' \in B_n$, then $A \cong A^\vee$.

Define $\psi: T_0 \rightarrow A$ in the following way. For $t \in T_0$, there exists a $\tau \in B_n$ such that $\tau \circ h = t$. Define $\psi(t) = \tau \circ \bar{h}$. We will show this does not depend on our choice τ . Suppose that $\tau' \circ h$ is also t . Then $\tau \sim \tau'$. Then, by assumption $\tau \dot{\sim} \tau'$, so $\tau' \circ \bar{h} = \tau \circ \bar{h}$.

So we have a map from the generating set of A^\vee to A . Recall the presentation of A^\vee from Theorem 1.6. Let R be the set of relations (words equal to the identity) from that presentation. As in Lemma 1.7, if we can show that the natural extension of ψ to R , maps all of R to the identity in A , then we can extend ψ to a homomorphism $\psi: A^\vee \rightarrow A$.

Let t, t' and $tt't$ be as in Theorem 1.6. Let τB_n be such that

$$\tau \cdot h = (x_1, x_2, \dots, x_{n-2}, t, t').$$

Let $\tau \cdot \bar{h} = (z_1, \dots, z_n)$. Thus, we have the following.

$$\begin{aligned} \psi(t') &= \tau \cdot \bar{h} = z_n \\ \psi(t) &= (\sigma_{n-1}\tau) \cdot \bar{h} = z_{n-1} \\ \psi(tt't) &= (\sigma_{n-1}^2\tau) \cdot \bar{h} = z_{n-1}z_nz_{n-1}^{-1} \end{aligned}$$

Thus, $\psi([tt't][t][t']^{-1}[t]^{-1}) = 1$ and ψ extends to a homomorphism $\psi: A^\vee \rightarrow A$.

Let φ denote the homomorphism in Theorem 1.5. Note that $\psi \circ \varphi|_{\bar{S}} = \text{Id}_{\bar{S}}$, thus $\psi \circ \varphi$ is the identity ψ is an isomorphism. \square

This proof gives the distinct impression of “what just happened”. Most striking, is how in showing ψ was a homomorphism, at no point did anything to do with A play a part. All we needed was that A was a group, and it all popped out. In fact, the key step was the existence of a map ψ , we got homomorphism for free. This is due to the structure of the relations in Theorem 1.6. We wish to now abstract the construction of the dual braid group emerging in Theorem 1.6 and develop that in to a reconstruction of the above proof.

3. MASKING GROUP RELATIONS USING LANGUAGES

Let $F: \mathbf{Set} \rightarrow \mathbf{Grp}$ be the free functor and denote its action on objects as $F_S := F(S)$ and its action on morphisms as $f^* := F(f: S \rightarrow T)$. To each quotient G of F_S we associate the surjective homomorphism $\pi_{(S,G)}: F_S \rightarrow G$ which is projection. In this context, we can frame a basic group theoretic fact. We write $G \cong \langle S | R \rangle$ if $G \cong F_S/N(R)$ where $N(R)$ is the minimal normal subgroup in F_S that contains R . We write $G \cong \langle S \rangle$ if G is isomorphic to some quotient of F_S .

Theorem 3.1. *Suppose we have two groups G_1 and G_2 . Suppose that $G_1 \cong \langle S | R \rangle$ where R is a subset of F_S for which $\pi_{(S,G_1)}(R) = \{1\}$. Given a map $f: S \rightarrow G_2$, if $\pi_{(G_2,G_2)} \circ f^*(R) = \{1\} \subseteq G_2$, then f defines a homomorphism $h: G_2 \rightarrow G_2$.*

Proof. We have $G_1 \cong F_S/N(R)$. By assumption, $R \subseteq \ker(\pi_{(G_2,G_2)} \circ f^*)$, which is normal, thus $N(R) \subseteq \ker(\pi_{(G_2,G_2)} \circ f^*)$. By the universal property of the quotient we have the following commutative diagram where the unique map h is the required homomorphism.

$$\begin{array}{ccc} F_S & \xrightarrow{\pi_{(G_2,G_2)} \circ f^*} & G_2 \\ \pi_{(S,G_1)} \downarrow & \nearrow \exists! h & \\ F_S/N(R) \cong G_1 & & \end{array}$$

□

The above theorem defines a homomorphism $h: \langle S | R \rangle \cong G_1 \rightarrow G_2$ when a map $f: S \rightarrow G_2$ is compatible with the relations in G_1 and G_2 . If we also had knowledge of a generating set for G_2 , we could construct homomorphisms in a different way.

Theorem 3.2. *Suppose $G_1 \cong \langle S_1 | R_1 \rangle$ and $G_2 \cong \langle S_2 \rangle$. Given a map $f: S_1 \rightarrow S_2$, if there exists a map h such that the following diagram commutes, then h is a homomorphism.*

$$(1) \quad \begin{array}{ccccc} S_1 & \xrightarrow{i_{S_1}} & F_{S_1} & \xrightarrow{\pi_{(S_1,G_1)}} & G_1 \\ \downarrow f & & \downarrow f^* & & \downarrow h \\ S_2 & \xrightarrow{i_{S_2}} & F_{S_2} & \xrightarrow{\pi_{(S_2,G_2)}} & G_2 \end{array}$$

Proof. Consider some $a, b \in G_1$. Since $\pi_{(S_1,G_1)}$ is surjective, we can choose some $\bar{a} \in \pi_{(S_1,G_1)}^{-1}(a)$ and $\bar{b} \in \pi_{(S_1,G_1)}^{-1}(b)$. Since $\pi_{(S_1,G_1)}$ is a homomorphism, we have $\overline{ab} \in \pi_{(S_1,G_1)}^{-1}(ab)$. So $h(ab) = h \circ \pi_{(S_1,G_1)}(\overline{ab})$. By the commutative diagram, this is $\pi_{(S_2,G_2)} \circ f^*(\overline{ab})$. Since $\pi_{(S_2,G_2)} \circ f^*$ is a homomorphism, this is

$$\begin{aligned} \pi_{(S_2,G_2)} \circ f^*(\bar{a}) \pi_{(S_2,G_2)} \circ f^*(\bar{b}) &= \\ h \circ \pi_{(S_1,G_1)}(\bar{a}) h \circ \pi_{(S_1,G_1)}(\bar{b}) &= \\ h(a)h(b) & \end{aligned}$$

□

Corrolary 3.3. *A map $h: G_1 \rightarrow G_2$ is a homomorphism if and only if we can construct the relevant commuting diagram (1).*

Proof. Theorem 3.2 gives us the \Leftarrow direction. If we consider $G \cong \langle G_2 \rangle$ we can construct the \Rightarrow direction. \square

Remark 3.4. If we construct a homomorphism using Theorem 3.1, then we get the middle commuting square of (1) automatically.

Now consider (1), but replace F_{S_1} with some subset $Q \subseteq F_{S_1}$. We will now construct a group, which we will denote G^Q , where such a diagram defines a homomorphism from G^Q to G_2 .

Definition 3.5 (Group with relations visible in Q). *Suppose we have a group $G \cong \langle S \rangle$. Fix some $Q \subseteq F_S$ such that $S \subseteq Q$. Let $\pi := \pi_{(S,G)}$. We have the following maps.*

$$S \xrightarrow{i} Q \xrightarrow{\pi} \pi(Q).$$

Define the group G with relations visible in Q , to be

$$G^Q := \langle \pi(Q) \mid \{ \pi(q) = (\pi \circ i)^*(q) \mid q \in Q \} \rangle.$$

Remark 3.6. Do not think of the generators $\pi(Q)$ as elements of G . They are abstract generators. Specifically, G^Q is a quotient of $F_{\pi(Q)}$. Thus, our relations should be equations in $F_{\pi(Q)}$, which they are.

Remark 3.7. G^Q is generated by $\pi(S) \subseteq \pi(Q)$. G is a quotient of G^Q .

Lemma 3.8. *Let $G \cong \langle S \rangle$ be a group and consider the following setup.*

$$S \xrightarrow{i_S} F_S \xrightarrow{\pi_{(S,G)}} G \xrightarrow{i_G} F_G \xrightarrow{\pi_{(G,G)}} G$$

The following maps are equal.

$$\pi_{(G,G)} \circ i_G \circ \pi_{(S,G)} = \pi_{(G,G)} \circ (\pi_{(S,G)} \circ i_S)^*.$$

Theorem 3.9. *Suppose we have two groups $G_1 \cong \langle S_1 \rangle$ and $G_2 \cong \langle S_2 \rangle$. Fix some $Q \subseteq F_{S_1}$ such that $S \subseteq Q$. Let $\bar{Q} := \pi_{(S_1,G_1)}(Q)$. If in the following diagram (2), if there exists a map f that makes the diagram commute, then there is a homomorphism $h: (G_1)^Q \rightarrow G_2$.*

$$(2) \quad \begin{array}{ccccc} S_1 & \xrightarrow{i_{S_1}} & Q & \xrightarrow{\pi_{(S_1,G_1)}} & \bar{Q} \\ \downarrow \theta & & \downarrow \theta^* & & \downarrow \exists f? \\ S_2 & \xrightarrow{i_{S_2}} & F_{S_2} & \xrightarrow{\pi_{(S_2,G_2)}} & G_2 \end{array}$$

Proof. Suppose we have such a map f . Note that \bar{Q} is a generating set for G^Q , thus we have the setup of Theorem 3.1. We use the following commutative diagram to define some inclusion maps and as a reference for the setup.

$$(3) \quad \begin{array}{ccccccc} S_1 & \xrightarrow{i_{S_1}} & Q & \xrightarrow{\pi_{(S_1,G_1)}} & \bar{Q} & \xrightarrow{i_{\bar{Q}}} & F_{\bar{Q}} \\ \downarrow \theta & & \downarrow \theta^* & & \downarrow f & & \downarrow f^* \\ S_2 & \xrightarrow{i_{S_2}} & F_{S_2} & \xrightarrow{\pi_{(S_2,G_2)}} & G_2 & \xrightarrow{i_{G_2}} & F_{G_2} \xrightarrow{\pi_{(G_2,G_2)}} G_2 \end{array}$$

We can construct a homomorphism if $\pi_{(G_2, G_2)} \circ f^*(R) = \{1\}$, where R are the relations for G^Q , defined in Definition 3.5. For some $q \in Q$, the corresponding element in R equalling the identity is $i_{\overline{Q}} \circ \pi_{(S_1, G_1)}(q)(\pi_{(S_1, G_1)} \circ i_1)^*(q)^{-1}$. Using that $\pi_{(G_2, G_2)} \circ f^*$ is a homomorphism, we have

$$\begin{aligned} & \pi_{(G_2, G_2)} \circ f^* \left(i_{\overline{Q}} \circ \pi_{(S_1, G_1)}(q)(\pi_{(S_1, G_1)} \circ i_1)^*(q)^{-1} \right) = \\ & \left(\pi_{(G_2, G_2)} \circ f^* \circ i_{\overline{Q}} \circ \pi_{(S_1, G_1)}(q) \right) \left(\pi_{(G_2, G_2)} \circ f^* \circ (\pi_{(S_1, G_1)} \circ i_1)^*(q)^{-1} \right). \end{aligned}$$

We will concentrate on each factor separately.

First we consider the first factor. Since $\pi_{(S_1, G_1)}(q) \in \overline{Q}$, by the rightmost commuting square of (3), we have $f^* \circ i_{\overline{Q}} \circ \pi_{(S_1, G_1)}(q) = i_{G_2} \circ f \circ \pi_{(S_1, G_1)}(q)$. This gives us

$$\pi_{(G_2, G_2)} \circ f^* \circ i_{\overline{Q}} \circ \pi_{(S_1, G_1)}(q) = \pi_{(G_2, G_2)} \circ i_{G_2} \circ f \circ \pi_{(S_1, G_1)}(q).$$

We then use the middle commuting square of (3) to give us

$$\pi_{(G_2, G_2)} \circ i_{G_2} \circ f \circ \pi_{(S_1, G_1)}(q) = \pi_{(G_2, G_2)} \circ i_{G_2} \circ \pi_{(S_2, G_2)} \circ \theta^*(q).$$

We then use Lemma 3.8 and functoriality, giving us

$$\begin{aligned} \pi_{(G_2, G_2)} \circ i_{G_2} \circ \pi_{(S_2, G_2)} \circ \theta^*(q) &= \pi_{(G_2, G_2)} \circ (\pi_{(S_2, G_2)} \circ i_{S_2})^* \circ \theta^*(q) \\ &= \pi_{(G_2, G_2)} \circ (\pi_{(S_2, G_2)} \circ i_{S_2} \circ \theta)^*(q) \end{aligned}$$

We now concentrate on the second factor. Using functoriality, then the middle commuting square of (3), we get

$$\begin{aligned} \pi_{(G_2, G_2)} \circ f^* \circ (\pi_{(S_1, G_1)} \circ i_{S_1})^*(q)^{-1} &= \pi_{(G_2, G_2)} \circ (f \circ \pi_{(S_1, G_1)} \circ i_{S_1})^*(q)^{-1} \\ &= \pi_{(G_2, G_2)} \circ (\pi_{(S_2, G_2)} \circ \theta^* \circ i_{S_1})^*(q)^{-1}. \end{aligned}$$

Then, we use the leftmost commuting square of (3) to get

$$\pi_{(G_2, G_2)} \circ (\pi_{(S_2, G_2)} \circ \theta^* \circ i_{S_1})^*(q)^{-1} = \pi_{(G_2, G_2)} \circ (\pi_{(S_2, G_2)} \circ i_{S_2} \circ \theta)^*(q)^{-1}.$$

This is the inverse of the left factor. \square

Remark 3.10. If we set $Q = S_1 \subseteq F_{S_1}$ (the minimum subset allowed), then G^Q is F_{S_1} . In this case, f always exists (and is θ) and Theorem 3.9 tells us the standard theorem about homomorphisms from the free group.

Remark 3.11. If we set $Q = F_{S_1}$, then G^Q is G , f is g and Theorem 3.9 tells us nothing more than Theorem 3.2.

Corrolary 3.12. *The homomorphism h resulting from Theorem 3.9 makes the following diagram (4) commute. Thus, considering S_1 as a generating set for $(G_1)^Q$, h is an extension of θ , as in Theorem 3.2.*

$$(4) \quad \begin{array}{ccccccc} S_1 & \xrightarrow{i_{S_1}} & Q & \xrightarrow{\pi_{(S_1, G_1)}} & \overline{Q} & \xleftarrow{i_{\overline{Q}}} & F_{\overline{Q}} \xrightarrow{\pi_{(\overline{Q}, (G_1)^Q)}} (G_1)^Q \\ \downarrow \theta & & \downarrow \theta^* & & \downarrow f & & \downarrow f^* \\ S_2 & \xrightarrow{i_{S_2}} & F_{S_2} & \xrightarrow{\pi_{(S_2, G_2)}} & G_2 & \xleftarrow{i_{G_2}} & F_{G_2} \xrightarrow{\pi_{(G_2, G_2)}} G_2 \end{array}$$

Proof. By Remark 3.4, we get the rightmost square in (4). \square

Corrolary 3.13. *If $\theta: S_1 \rightarrow S_2$ is surjective, then the homomorphism $h: (G_1)^Q \rightarrow G_2$ resulting from Theorem 3.9 is surjective.*

We now consider altering Q .

Corrolary 3.14. *Suppose we have some group $G \cong \langle S \mid R \rangle$. Given some $Q, Q' \subseteq F_S$ such that $S \subseteq Q \subseteq Q'$, there exists a surjective homomorphism $a : G^Q \rightarrow G^{Q'}$.*

Proof. Apply Theorem 3.9 with $G_2 = G^{Q'}$ and $\theta = \text{Id}_S$. We get the required homomorphism if there exists an f such that the following diagram commutes.

$$\begin{array}{ccc} & & Q \\ & \swarrow & \\ \overline{Q} & \xleftarrow{\pi_{(S,G)}} & \end{array}$$

□

REFERENCES

- [Bes04] David Bessis. Topology of complex reflection arrangements, November 2004.
- [MS17] Jon McCammond and Robert Sulway. Artin groups of Euclidean type. *Inventiones mathematicae*, 210(1):231–282, October 2017.
- [PS21] Giovanni Paolini and Mario Salvetti. Proof of the $K(\pi, 1)$ conjecture for affine Artin groups. *Inventiones mathematicae*, 224(2):487–572, May 2021.
- [Res24] Sirio Resteghini. Free Products and the Isomorphism between Standard and Dual Artin Groups, October 2024.

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