# A REVIEW OF
# *FREE PRODUCTS AND THE ISOMORPHISM BETWEEN STANDARD AND DUAL ARTIN GROUPS – RESTEGHINI*

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## 1. Introduction

In [Res24] *Resteghini* explores the dual artin group $A^* = A^*(W, S, h)$ for some Coxeter system $(W, S)$ and Coxeter element $h \in W$. He shows that certain conditions on the Coxeter group $W$ and Artin group $A$ imply the natural surjective homomorphism $A \to A^*$ is an isomorphism. In this review we will abstract a construction that gives (as a specific case) the dual Artin group $A^*$. We will then prove a general result about this construction and show that [Res24, Proposition 3.9] can be re–stated (and re–proved) in this general framework.

1.1. **Some facts about the objects involved.** Here we will state some facts and try to point to the relevant place to read up on them.

Throughout, let $W = (W, S)$ be some Coxeter system. Let $\{\{\}s_i\}_{1\leq i\leq n}$ be some ordering on $S$ and let $h = s_1 s_2 \cdots s_n$ be a choice of Coxeter element resulting from the ordering on $S$. From here forward, fix $W = (W, S)$ and $h$. Let $T = S^W$ be the set of reflections in $W$. This is all conjugations of $S$ in $W$. We define the dual Artin group $A^* = A^*(W, S, h)$, as in [Res24] or [PS21].

Let $B_n$ denote the braid group on $n$ strands generated by $\sigma_1, \sigma_2, \ldots \sigma_{n-1}$ in the usual way. Let $x^y = yxy^{-1}$ denote conjugation in groups.

**Definition 1.1** (The Hurwitz action)**.** *Let $X$ be some set on which we can define conjugates which lie within $X$. Define a group action of $B_n$ on $X^n$ in the following way.*

$$\sigma_i \cdot (t_1, \ldots t_n) = \left(t_1, \ldots t_{i-1}, t_{i+1}^{t_i}, t_i, t_{i+2} \ldots t_n\right).$$

Any group $G$ is a fine candidate for $X$. A subset that is closed under conjugation will also work, for example the set of reflections $T \subseteq W$ as defined above.

**Theorem 1.2.** *We can map $T^n$ to $W$ by multiplying all the elements in the tuple in order. Considering this map, the action of $B_n$ is transitive on minimal $T$–factorisations of $h$.*

We will use the symbol $h$ as to refer to both a Coxeter element $h \in W$, and a specific factorisation of $h$, realised as a tuple, $h \in T^n \subseteq W^n$. Let $T_0 := T \cap (B_n \cdot h)$ denote all reflections that occur in any position in a minimal $T$–factorisation of $h$.

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**Theorem 1.3.** *For each $r \in T_0$, and for each $i \in \{1, \dots n\}$ there exists a minimal $T$–factorisation $(t_1, \dots t_n) \in (B_n \cdot h)$ such that $t_i = r$. So every reflection that appears, appears in every possible position.*

**Theorem 1.4.** *The above theorem also applies to subwords. For each $(r_1, \dots r_n) \in (B_n \cdot h)$, each $i \leq n$ and each $j \leq n - i$, there exists a $(t_1, \dots t_n) \in (B_n \cdot h)$ such that $(t_j, t_{j+1}, \dots, t_{j+i}) = (r_1, \dots, r_i)$.*

**Theorem 1.5** ([MS17, Proposition 10.1])**.** *The generating $W$ generating set $S$ is contained in $T_0$ and this is also a generating set for $A^*$. The natural inclusion of $\overline{S} \subseteq A$ in to $A^*$ induces a surjective homomorphism.*

**Theorem 1.6** ( [Bes04, Lemma 7.11] )**.** *Let $[t]$ denote an abstract generator coming from some $t \in T \cap (B_n \cdot h)$. The dual Artin group has the following presentation.*

$$A^*(W, S, h) \cong \langle T_0 \mid \{[tt't][t][t']^{-1}[t]^{-1} \mid (t, t', x_3, \dots x_n) \in B_n \cdot h\} \rangle.$$

We call such $[tt't][t][t']^{-1}[t]^{-1}$ *dual braid relations.*

The following is a basic group theoretic fact, but it will be useful to explicitly state and prove it. For some set $S$, let $F_S$ be the free group generated by $S$.

**Lemma 1.7.** *Suppose we have the groups $G_1$ and $G_2$ such that $G_1$ has a presentation $G_1 \cong \langle S \mid R \rangle$ where $R$ is a set of words in $S \cup S^{-1}$ that are the identity in $G_1$. Further suppose we are given some map $f \colon S \to G_2$. For all $s \in S$, define $f(s^{-1})$ to be $f(s)^{-1}$. If for each word $s_1 s_2 \cdots s_n \in R$ we have that $f(s_1)f(s_1) \cdots f(s_n) = 1$ in $G_2$ then we can extend $f$ to a homomorphism $h \colon G_1 \to G_2$ where $h|_S = f$ considering $S$ as a subset of $G_1$.*

*Proof.* We can define a homomorphism $f' \colon F_S \to G_2$ recursively by setting $f'(1) = 1$, $f'(s^{\pm 1}) = f(s)^{\pm 1}$ for all $s \in S$ and then setting $f'(s_1 s_2 \cdots s_n) = f'(s_1) f'(s_2 \cdots s_n)$ for all words $s_1 s_2 \cdots s_n \in \{S \cup S^{-1}\}^*$. Let $N$ be the minimal normal subgroup in $F_S$ which contains the elements of $R$ such that $G_1 \cong F_S/N$. By assumption, for each $r \in R$ we have $f'(r) = 1$. So $R \subseteq \ker(f')$. So $N \subseteq \ker(f')$. By the universal property of the quotient we have the following commutative diagram where the unique map $h$ is the required homomorphism.

$$\begin{tikzcd}
F_S \arrow[r, "f'"] \arrow[d, "q"'] & G_2 \\
F_S/N \cong G_1 \arrow[ur, "\exists ! h"'] &
\end{tikzcd}$$

□

Note that if $\langle \mathrm{Im}(f) \rangle = G_2$ then $h$ is a surjection.

## 2. A proof in [Res24]

We will overview a proof in [Res24] while trying to avoid unnecessary detail. The generators $S = \{s_1, \dots, s_n\}$ have a natural bijection with generators of $A$. Let $\overline{s_i}$ denote the generator in $A$ corresponding to $s_i \in S \subseteq W$. We have an action of $B_n$ on $A^n$ defined in the exact same way as in Definition 1.1. We will consider the orbit of $\overline{h} = (\overline{s_1}, \dots, \overline{s_n})$ under this action.

**Definition 2.1.** *Let $G$ be a group. Let $\rho_n\colon G^n \to G$ be projection on to coordinate $n$. We define the map $-\odot-\colon B_n \times G^n \to G$ as follows*

$$\tau \odot (g_1, \dots, g_n) = \rho_n(\tau \cdot (g_1, \dots, g_n)).$$

*Remark* 2.2. $G^n \neq G$ so this is not a group action!

**Definition 2.3.** *Define the equivalence relation $\sim$ on $B_n$ by*

$$\tau \sim \tau' \iff \tau \odot h = \tau' \odot h.$$

**Definition 2.4.** *Define the equivalence relation $\dot\sim$ on $B_n$ by*

$$\tau \dot\sim \tau' \iff \tau \odot \overline{h} = \tau' \odot \overline{h}.$$

**Theorem 2.5** ([Res24, Propsition 3.9])**.** *The $B_n$ relations $\sim$ and $\dot\sim$ are the same iff $A^*$ is isomorphic to $A$.*

*Proof.* We will only concentrate on the $\implies$ direction. Furthermore, it is easy enough to see that $\tau \dot\sim \tau' \implies \tau \sim \tau'$. As such, we will only prove that if $\tau \sim \tau' \implies \tau \dot\sim \tau'$ for all $\tau, \tau' \in B_n$, then $A \cong A^*$.

Define $\psi\colon T_0 \to A$ in the following way. For $t \in T_0$, there exists a $\tau \in B_n$ such that $\tau \odot h = t$. Define $\psi(t) = \tau \odot \overline{h}$. We will show this does not depend on our choice $\tau$. Suppose that $\tau' \odot h$ is also $t$. Then $\tau \sim \tau'$. Then, by assumption $\tau \dot\sim \tau'$, so $\tau' \odot \overline{h} = \tau \odot \overline{h}$.

So we have a map from the generating set of $A^*$ to $A$. Recall the presentation of $A^*$ from Theorem 1.6. Let $R$ be the set of relations (words equal to the identity) from that presentation. As in Lemma 1.7, if we can show that the natural extension of $\psi$ to $R$, maps all of $R$ to the identity in $A$, then we can extend $\psi$ to a homomorphism $\psi\colon A^* \to A$.

Let $t$, $t'$ and $tt't$ be as in Theorem 1.6. Let $\tau B_n$ be such that

$$\tau \cdot h = (x_1, x_2, \dots, x_{n-2}, t, t').$$

Let $\tau \cdot \overline{h} = (z_1, \dots z_n)$. Thus, we have the following.

$$\begin{aligned} \psi(t') &= \tau \cdot \overline{h} = z_n \\ \psi(t) &= (\sigma_{n-1}\tau) \cdot \overline{h} = z_{n-1} \\ \psi(tt't) &= (\sigma_{n-1}^2\tau) \cdot \overline{h} = z_{n-1}z_n z_{n-1}^{-1} \end{aligned}$$

Thus, $\psi([tt't][t][t']^{-1}[t]^{-1}) = 1$ and $\psi$ extends to a homomorphism $\psi\colon A^* \to A$.

Let $\varphi$ denote the homomorphism in Theorem 1.5. Note that $\psi \circ \varphi|_{\overline{S}} = \mathrm{Id}_{\overline{S}}$, thus $\psi \circ \varphi$ is the identity $\psi$ is an isomorphism. □

This proof gives the distinct impression of "what just happened". Most striking, is how in showing $\psi$ was a homomorphism, at no point did anything to do with $A$ play a part. All we needed was that $A$ was a group, and it all popped out. In fact, the key step was the existence of a map $\psi$, we got homomorphism for free. This is due to the structure of the relations in Theorem 1.6. We wish to now abstract the construction of the dual braid group emerging in Theorem 1.6 and develop that in to a reconstruction of the above proof.

## 3. Masking group relations using languages

For a set of symbols $\mathcal{S}$, define $\mathcal{S}^\vee$ to be the language of all words in $\mathcal{S}$. Consider the following and compare with Lemma 1.7.

**Lemma 3.1.** *Let $G$ and $G'$ be groups generated by $S$ and $S'$ respectively. Let $\pi_G\colon (S \cup S^{-1})^\vee \to G$ denote multiplication of words in the group, and similarly for $\pi_{G'}$. Suppose we have a map $\theta\colon S \to S'$. We can extend $\theta$ to a function $\theta^\vee\colon (S \cup S^{-1})^\vee \to (S' \cup (S')^{-1})^\vee$ in the obvious way. Consider the following diagram.*

$$\begin{tikzcd}
S \arrow[r, hook] \arrow[d, "\theta"] & (S \cup S^{-1})^\vee \arrow[r, "\pi_G"] \arrow[d, "\theta^\vee"] & G \\
S' \arrow[r, hook] & (S' \cup (S')^{-1})^\vee \arrow[r, "\pi_{G'}"] & G'
\end{tikzcd}.$$

*The left square commutes. If there exists a* map *$f\colon G \to G'$ that makes the right square commute, then $f$ is a homomorphism.*

*Remark* 3.2. Note that if $G \cong \langle S \mid R \rangle$ then such a map (and thus homomorphism) $f$ exists exactly when $\pi_{G'} \circ \theta^\vee(R) = 1$. The above lemma frames up the construction in Lemma 1.7, but does not provide any useful way to detect when such a homomorphism is possible.

**Definition 3.3** (Group with relations visibile in Q)**.** *Let $G \cong \langle S \mid R \rangle$. Let $Q$ be some language in $S \cup S^{-1}$ such that, considering $S$ as one-letter words, $S \subseteq Q$. We define the group with relations visible in $Q$, $G^Q$ to be*

$$G^Q \coloneqq \langle \pi_G(Q) \mid \{\pi_G(q_1 q_2 \cdots q_n) = \pi_G(q_1)\pi_G(q_2) \cdots \pi_G(q_n) \mid q_1 \cdots q_n \in Q\} \rangle.$$

*Note that the generators $\pi_G(Q)$ are abstract generators. They do not necessarily inherit anything from $G$. The only equations that are necessarily true in $G^Q$ (which are also tautologically true in $G$) are those spelled out by the language $Q$.*

*Remark* 3.4. $G^Q$ is generated by the generators $\pi_G(S)$. $G$ is a quotient of $G^Q$.

**Theorem 3.5.** *Let $G \cong \langle S \mid R \rangle$, let $G'$ be generated by $S'$ and let $Q$ be a language in $S \cup S^{-1}$ such that $S \subseteq Q$ as in Definition 3.3. Suppose we have a map $\theta\colon S \to S'$ and define $\theta^\vee$ as in Lemma 3.1. For brevity, let $\overline{Q} \coloneqq \pi_G(Q)$. Consider the following diagram.*

$$\begin{tikzcd}
S \arrow[r, hook] \arrow[d, "\theta"] & Q \arrow[r, "\pi_G"] \arrow[d, "\theta^\vee"] & \overline{Q} \arrow[r, hook] \arrow[d, "\exists f?"] & (\overline{Q} \cup \overline{Q}^{-1})^\vee \arrow[d, "\pi_{G^Q}"] \\
S' \arrow[r, hook] & (S' \cup (S')^{-1})^\vee \arrow[r, "\pi_{G'}"] & G' & G^Q \arrow[l, "\implies \exists g"']
\end{tikzcd}.$$

*The left square commutes.*

*Finally, the theorem is as follows. If there exists a map $f\colon \pi_G(Q) \to G'$ that makes the middle square commute, then there is a homomorphism $g\colon G^Q \to G'$ such that the right square commutes.*

*Proof.* Suppose we have such a diagram and map $f$. Note that, in $f$, we have a map from the generating set of $G^Q$ to $G'$, which can be considered a generating set for $G'$. Accordingly, replace $\theta$ with $f$ in Lemma 3.1.

$$\begin{tikzcd}
\overline{Q} \arrow[r, hook] \arrow[d, "f"] & (\overline{Q} \cup \overline{Q}^{-1})^{\vee} \arrow[r, "\pi_{G^Q}"] \arrow[d, "f^{\vee}"] & G^Q \\
G' \arrow[r, hook] & (G' \cup (G')^{-1})^{\vee} \arrow[r, "\pi_{G'}"] & G'
\end{tikzcd}.$$

If we can show we have a map $h\colon G^Q \to G'$ such that the right square commutes, then by Lemma 3.1 this is a homomorphism. This will also satisfy the necessary commutation relations.

We can use the defining presentation for $G^Q \cong \langle \pi_G(Q) \mid R \rangle$ from Definition 3.3 and Remark 3.2 to show that $g$ exists. Let $R$ be our relations from Definition 3.3. Showing $\pi_{G'} \circ f^{\vee}(R) = 1$ will finish our proof.

Let $(q_1 q_2 \cdots q_n) = q \in Q$.bullet This corresponds to the relation $\pi_G(q_1 \cdots q_n) = \pi_G(q_1) \cdots \pi_G(q_n)$ in $G^Q$. Each of $\pi_G(q)$ and the $\pi_G(q_i)$ are in $\pi_G(Q)$, so $f^{\vee}(\pi_G(q)) = f(\pi_G(q))$ and $f^{\vee}(\pi_G(q_i)) = f(\pi_G(q_i))$. By our assumed commuting diagram, $f(\pi_G(q)) = \pi_{G'}(\theta^{\vee}(q))$, so we have the following.

$$\begin{aligned}
f^{\vee}(\pi_G(q)) &= f(\pi_G(q)) \\
&= \pi_{G'}(\theta^{\vee}(q)) \\
&= \pi_{G'}(\theta^{\vee}(q_1 \cdots q_n)) \\
&= \pi_{G'}(\theta(q_1) \cdots \theta(q_n)) \\
&= \pi_{G'}(\theta(q_1)) \cdots \pi_{G'}(\theta(q_n)) \\
&= f(\pi_G(q_1)) \cdots f(\pi_G(q_n)) \\
&= f^{\vee}(\pi_G(q_1)) \cdots f^{\vee}(\pi_G(q_n))
\end{aligned}$$

Thus also

$$\begin{aligned}
&\pi_{G'}(f^{\vee}(\pi_G(q)\pi_G(q_n)^{-1}\pi_G(q_{n-1})^{-1} \cdots \pi_G(q_1)^{-1})) = \\
&\pi_{G'}(f^{\vee}(\pi_G(q_1)) \cdots f^{\vee}(\pi_G(q_n))(f^{\vee}(\pi_G(q_n)))^{-1} \cdots (f^{\vee}(\pi_G(q_1)))^{-1}) = \\
&\pi_{G'}(1) = 1
\end{aligned}$$

□

*Remark* 3.6. Again, we note that the last part of the proof had no dependence on $G'$. Once we had the map $f$, we just chased the diagram and everything popped out. This is because of the very specific form of the relations in $G^Q$. I believe that $G^Q$ is defined by the diagram in Theorem 3.5, but I don't know enough category theory to explore that.

*Remark* 3.7. Note that in Theorem 3.5, c.f. Lemma 3.1 we replaced $(S \cup S^{-1})^{\vee}$ with $Q$.

*Remark* 3.8. If we set $Q = S$ (the minimum language allowed), then $G^Q$ is $F_S$, the free group generated by $S$. Then $f$ always exists (and is $\theta$) and Theorem 3.5 tells us the standard theorem about homomorphisms from the free group.

*Remark* 3.9. If we set $Q = (S \cup S^{-1})^{\vee}$ (the maximum language), then $G^Q$ is $G$, $f$ is $g$ and Theorem 3.5 tells us nothing more than Lemma 3.1.

## 4. Re–framing the proof of Theorem 2.5

We will work to re–frame the proof of Theorem 2.5 using the objects and tools developed in Section 3. To do so, let us first inspect exactly what kind of group elements are generated by the Hurwitz action, specifically, let us try to describe $B_n \odot h$ as a subset of $W$.

**Definition 4.1.** *For a group $G \cong \langle S \mid R \rangle$ let $\ell_S \colon G \to \mathbb{N}$ be the usual length function with respect to the generating set $S$. We will omit $S$ when it is obvious from context.*

**Definition 4.2.** *For some group $G \cong \langle S \mid R \rangle$ and $n \in \mathbb{N}$, we define the symbol $G|_n$ to be $\ell^{-1}(\{0, \dots, n\})$, i.e. the group elements that have length less than or equal to $n$.*

*Example* 4.3. Let $W$ be some rank–3 Coxeter group, let $h = (s_1, s_2, s_3)$ and let $B = B_3$. We can act on $h$ by generators of $B$ to get

$$\begin{aligned}
\sigma_1 \cdot h &= (s_2^{s_1}, s_1, s_3) \\
\sigma_2 \cdot h &= (s_1, s_3^{s_2}, s_2) \\
\sigma_1^{-1} \cdot h &= (s_2, s_1^{\left(s_2^{-1}\right)}, s_3) && (= (s_2, s_1^{s_2}, s_3)) \\
\sigma_2^{-1} \cdot h &= (s_1, s_3, s_2^{\left(s_3^{-1}\right)}) && (= (s_1, s_3, s_2^{s_3}))
\end{aligned}$$

We see that $B|_1 \odot h$ is a subset of $S \cup \{s_i^{s_j} \mid s_i, s_j \in S, s_i \neq s_j\}$A

**Definition 4.4.** *Let $(W, S)$ and $h = (s_1, \dots, s_n)$ be some Coxeter system and Coxeter element We will recursively define a sequence of sets Let $C_0 \coloneqq S$ Define $C_{i+1}$ to be*

$$C_{i+1} \coloneqq C_i \cup \{a^b \mid a, b \in C_i, a \neq b\}.$$

**Definition 4.5.** *We call an element of $C_i \setminus (C_{i-1} \cup \dots \cup C_0)$ an element of conjugation length $i$. We can use a compact symbol, which is best described recursively, to describe such an element of conjugation length $i$. Setting $[s_0] \coloneqq s_0$, we define*

$$[s_0, \dots, s_i] \coloneqq s_i^{[s_0, \dots, s_{i-1}]}.$$

*Remark* 4.6. $[x_0, \dots, x_i]^{[y_0, \dots, y_j]} = [x_0, \dots, x_i, y_0, \dots, y_j]$

This notation allows for a compact description of the Hurwitz action, where conjugation becomes concatenation. We can take this further by writing $i$ instead of $s_i$. See this MathOverflow post I made, asking if anyone knows what this game is.

*Example* 4.7. Let us continue as in Example 4.3. We have

$$\begin{aligned}
\sigma_1 \cdot h &= ([2,1],[1],[3]) \\
\sigma_2 \cdot h &= ([1],[3,2],[2]) \\
\sigma_1^{-1} \cdot h &= ([2],[1,2],[3]) \\
\sigma_2^{-1} \cdot h &= ([1],[3],[2,3]) \\
\sigma_1^{2} \cdot h &= ([1,2,1],[2,1],[3]) \\
\sigma_1\sigma_2 \cdot h &= ([3,2,1],[1],[2])
\end{aligned}$$

**Theorem 4.8.** $C_i = \{[x_0, \dots, x_j] \mid j \le i, \mathit{all}\, x_k \in S\}$

**Theorem 4.9.** *Given some Coxeter system* $(W, S)$ *and Coxeter element* $h$*, we have* $(B_n)|_j \odot h \subseteq C_j$*. Thus, for each* $t \in T \cap B_n \cdot h$*, there exists a* $j \in N$ *such that* $t \in C_j$

*Remark* 4.10. These $C_i$ grow in size much faster than $(B_n)|_i \cdot h$. Also, there will typically exist $[x_0, \dots, x_j] \notin T_0$. A–priori, the exact elements that appear in $T_0$, depend on combinatorics specific to $h$.

**Definition 4.11.** *Let* $C_i^\vee$ *be defined just like* $C_i$*, except its elements are words, rather than group elements We do not perform any multiplication. Similarly define* [

## References


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