SAMPLE TITLE

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My mum.

ABSTRACT. Abstract nonsense!

CONTENTS

1. INTRODUCTION

A classifying space for a group G is a space X such that the fundamental group of X is G and all higher homotopy groups of X are trivial. In this paper we will be concerned with the $K(\pi, 1)$ conjecture for Artin groups, which states that the configuration space Y_W for any Coxeter group W is a classifying space for the Artin group G_W. This conjecture emerges as a generalisation of the result for Coxeter groups of type A_n and is originally attributed to Arnol'd, Brieskorn, Pham and Thom in [**?**]. See also [**?**] for a good overview of the history of the conjecture.

We will focus on the work of Paolini and Salvetti, *Proof of the* $K(\pi, 1)$ *conjecture for affine Artin groups* [**?**]. We will review some theorems therein and provide relevant background. The theorem which is the namesake of [**?**] relies on proving a chain of homotopy equivalences (1). Detailing two of these homotopy equivalences is the aim of this work. A strong theme will be the involvement of posets and related structures, hence this work's title. We will begin by providing a birds eye view of the conjecture and the main results of [**?**].

The main results of this work are Theorem 1.1. Theorem 1.1 is stated by McCammond in [**?**]. It is not proven there, but it is alluded to being widely accepted as true. Its proof here and the lemmas leading to that proof are the author's own work. Theorem 1.1 is proven by Paolini and Salvetti in [**?**]. The proof here follows that in [**?**] but fills in significant details. The proof of Theorem 1.1 largely follows that in [**?**]. The main original contribution to the proof of Theorem 1.1 here is the proof of Theorem 1.3, which is omitted in [**?**]. Fibre doubling and the concepts explored in Section 1.1 are the authors own work.

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For the purpose of creating a functional template, here are some theorems to be referenced in later parts of this document.

Theorem 1.1 ([**?**, Theorem 5.5])**.** $1 = 1$

Theorem 1.2 ([**?**, Theorem 5.5])**.** $2 = 2$

1.1. **The conjecture and the objects involved.** Coxeter groups emerge as generalisations of reflection groups. A Coxeter group is defined by a particular group presentation. The data of this presentation is typically encoded by a labelled graph. The group W, coupled with the data of its presentation is called a Coxeter system, denoted (W, S) where S is the generating set of W. Given a Coxeter system (W, S), we can construct a different group G_W, called the Artin group associated to W.

For affine Coxeter groups W, the configuration space Y_W can be derived from a geometric realisation of W as a subgroup of $\mathrm{Isom}(\mathbb{E})$, the group of isometries on a Euclidean space $\mathbb{E}$. We will consider $\mathbb{E}$ as $\mathbb{R}^n$ without the notion of origin. Specifically, W is realised as a subgroup generated by a finite set of affine reflections S. Within W, we consider the set of all reflections R (not necessarily finite). To each reflection $r \in R$ there is a corresponding codimension–1 space $H_r \subset \mathbb{E}$ that is the plane of reflection of r. We call such spaces hyperplanes. Note there is no requirement that these hyperplanes be subspaces of $\mathbb{R}^n$.

The configuration space is realised as the complement of the complexification of all such hyperplanes H_r. It is known by work of Brieskorn [**?**] that the fundamental group of Y_W is G_W. Thus, proving the $K(\pi, 1)$ involves showing that the higher homotopy groups of Y_W are trivial. By previous work by Salvetti [**?**, **?**], there is a CW–complex X_W called the Salvetti complex that is homotopy equivalent to Y_W. Showing homotopy equivalence to X_W thus shows homotopy equivalence to Y_W. Because of this, the Salvetti complex is the starting point in a chain of homotopy equivalences reviewed in this work.

The finishing point of this chain is the interval complex K_W. This is a space realised using a certain poset structure on subsets of W. To this poset structure there is an associated group called the dual Artin group, denoted W_w. It was already known (by a now standard construction due to Garside [**?**], extended by other authors, see [**?**]) that K_W was a classifying space for the dual Artin group for finite W. In [**?**], the authors extend this result to affine W. Thus, showing $Y_W \simeq K_W$ for affine W shows that (for affine W) the higher homotopy groups of Y_W are trivial and that $W_w \cong G_W$.

In the following section, we will identify the intermediate spaces used in proving $X_W \simeq K_W$.

1.2. **Proof overview.** Here we will compile several main results from [**?**] in to two theorems. The concern of this work is Theorem 1.3 which proves that the *Salvetti complex* X_W is homotopy equivalent to the *interval complex* K_W. A *Coxeter element* is a non–repeating product of all the elements of S. A choice of order on S corresponds to a choice of Coxeter element. Constructing an interval complex associated to (W, S) involves making such a choice of Coxeter element $w \in W$.

For a subset $T \subseteq S$, the *parabolic subgroup* W_T is the subgroup of W generated only by elements of T and only with relations explicitly involving elements of T. A *parabolic Coxeter element* w_T is a product of all elements of T that respects the order of multiplication in a Coxeter element $w \in W$. The space X'_W is a subspace of K_W associated to parabolic Coxeter elements w_T with $T \subseteq S$ such that W_T

is finite. Cells in X_W also correspond to such subsets, which is used in proving $X_W \simeq X'_W$.

The space K'_W is also a subspace of K_W. Given a CW–complex X, we can encode some information of how cells of X attach to each other in a poset called the *face poset* of X, denoted $\mathcal{F}(X)$. Connected components of preimages $\eta^{-1}(d)$ of a certain poset map $\eta\colon K_W \to \mathbb{N}$ have a linear structure as subposets of $\mathcal{F}(K_W)$. For each element $x \in \eta^{-1}(d)$, whether x is in K'_W or not is determined based on whether x comes in between two elements of X'_W in the linear structure of $\eta^{-1}(d)$.

Theorem 1.3 ([**?**])**.** *Given an affine Coxeter system (W, S), the configuration space Y_W is homotopy equivalent to the order complex K_W.*

Proof. By Theorem 1.1 the Salvetti complex X_W is homotopy equivalent to the configuration space Y_W. Therefore, we need only show $K_W \simeq X_W$. This is done through a composition of homotopy equivalences

$$X_W \overset{\text{(a)}}{\simeq} X'_W \overset{\text{(b)}}{\simeq} K'_W \overset{\text{(c)}}{\simeq} K_W \tag{1}$$

Where the results are gathered from the following sources:

(a) Theorem 1.1 [**?**, Theorem 5.5]
(b) Theorem 1.2 [**?**, Theorem 8.14]
(c) Theorem 1.1 [**?**, Theorem 7.9]

□

In [**?**], another main result is shown.

Theorem 1.4 ([**?**, Theorem 6.6])**.** *Given an affine Coxeter system (W, S), corresponding affine type Artin group G_W and Coxeter element $w \in W$, the complex K_W is a classifying space for the dual Artin group W_w.*

We have that $\pi_1(Y_W) \cong G_W$ by [**?**]. Thus, considering $\pi_1(Y_W)$ and combining Theorems 1.3 and 1.4 gives

$$Y_W \simeq K(G_W, 1)$$
$$G_W \cong W_w$$

for affine G_W.

This proves the $K(\pi, 1)$ conjecture for affine Artin groups and provides a new proof than an affine Artin group is naturally isomorphic to its dual, which was already known for spherical or finite type Artin groups [**?**] and affine Artin groups [**?**].

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