Inverse Method fo R.V Simulation

This piece of code is the implementation of the inversion method for simulating the Exercice 1 density function: $f(x) = \begin{cases} \frac{2}{3}, \text{ if } x \in [1,2] \\ -\frac{2}{3}(x-3), \text{ if } x \in [2,3] \\ 0, \text{otherwise} \end{cases}$

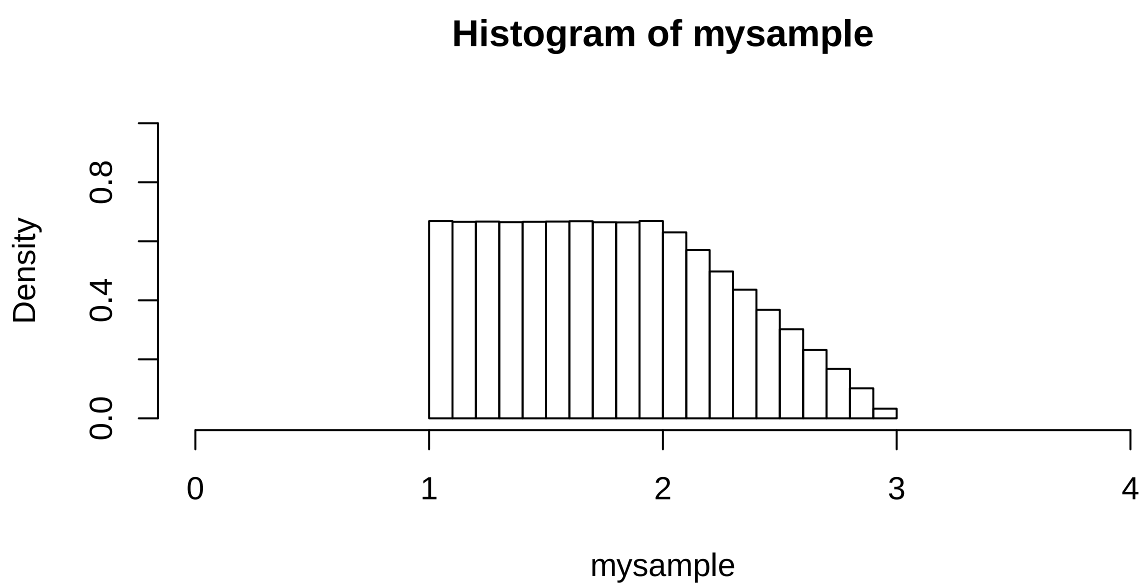
The inverse distribution function is:

$F^{-1}(y) = \begin{cases} \frac{3}{2}y + 1, \text{ if } y \in [0, \frac{2}{3}] \\ 3 - \sqrt{3-3y}, \text{ if } y \in [\frac{2}{3}, 1] \\ 0, \text{otherwise} \end{cases}$

```
f.inverse <- function(x) {
  if (x < 2/3) {
    result <- (3/2 * x) +1
  } else {
    result <- 3 - sqrt(3 - 3*x)
  }
  return(result)
}

mydensity <- function(n){
  x <- runif(n)
  y <- sapply(x, f.inverse)
  return(y)
}

mysample <- mydensity(1000000)
hist(mysample, freq = FALSE, xlim=c(0,4), ylim = c(0,1))
```



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MAHER SEBAI (A18)

Exercice 1:

$$f(x) = \begin{cases} a & \text{if } x \in [1,2] \\ -a\,(x-3) & \text{if } x \in [2,3] \\ 0 & \text{otherwise.} \end{cases}$$

a/ what value for a to have a density function.

f is a density function if $- \forall x \in \mathbb{R},\ f(x) \geq 0$ (1)

$- \int_{-\infty}^{+\infty} f(x)\,dx = 1.$ (2)

(1) => $a \geq 0$

(2) => $\int_{-\infty}^{+\infty} f(x)\,dx = \int_1^2 a\,dx + \int_2^3 -a(x-3)\,dx = 1$

<=> $a - a\left[\frac{x^2}{2} - 3x\right]_2^3 = a - a\left[\left(\frac{9}{2} - 9\right) - \left(\frac{4}{2} - 6\right)\right] = 1$

<=> $a - a\left[-\frac{9}{2} + 4\right] = a - a\left[-\frac{1}{2}\right] = a + \frac{a}{2} = \frac{3}{2}a = 1 \Rightarrow \boxed{a = \frac{2}{3}}$

b/ Let's compute $E[X]$ and $V[X]$

* $E[X] = \int_{-\infty}^{+\infty} x f(x)\,dx = \int_1^2 \frac{2}{3}x\,dx - \frac{2}{3}\int_2^3 x(x-3)\,dx$

$= \frac{2}{3}\left[\frac{x^2}{2}\right]_1^2 - \frac{2}{3}\left[\frac{x^3}{3} - 3\frac{x^2}{2}\right]_2^3 = \frac{2}{3}\left[\frac{4}{2} - \frac{1}{2}\right] - \frac{2}{3}\left[\left(9 - \frac{3^3}{2}\right) - \left(\frac{8}{3} - \frac{12}{2}\right)\right]$

$= 1 - \frac{2}{3}\left[\left(-\frac{9}{2}\right) + \frac{10}{3}\right] = \frac{16}{9} \Rightarrow \boxed{E[X] = \frac{16}{9}}$

* $E[X^2] = \int_{-\infty}^{+\infty} x^2 f(x)\,dx = \int_1^2 \frac{2}{3}x^2\,dx - \frac{2}{3}\int_1^2 x^2(x-3)\,dx.$

$= \frac{2}{3}\left[\frac{x^3}{3}\right]_1^2 - \frac{2}{3}\left[\frac{x^4}{4} - x^3\right]_2^3$

$= \frac{2}{9}[8-1] - \frac{2}{3}\left[\left(\frac{3^4}{4} - 3^3\right) - \left(\frac{2^4}{4} - 2^3\right)\right].$

$= \frac{14}{9} - \frac{2}{3}\left[-\frac{27}{4} + 4\right] = \frac{61}{18} \Rightarrow V(X) = E[X^2] - (E[X])^2$

$= \frac{61}{18} - \frac{16}{9} = \frac{29}{18}$

$$\boxed{V[X] = \frac{29}{18}}$$

c/ Let's Determine Distribution function.

$F(y) = P(X \leq y) = \int_{-\infty}^{y} f(x)\,dx$

- if $y \leq 1$: $F(y) = 0$

if $y \in [1,2]$: $F(y) = \int_{-\infty}^{y} f(x)\,dx = \int_{1}^{y} \frac{2}{3}\,dx = \frac{2}{3}[x]_{1}^{y} = \frac{2}{3}(y-1)$

if $y \in [2,3]$: $F(y) = \int_{-\infty}^{y} f(x)\,dx = \int_{1}^{2} \frac{2}{3}\,dx + \int_{2}^{y} -\frac{2}{3}(x-3)\,dx$

$= \frac{2}{3} - \frac{2}{3}\left[\frac{x^2}{2} - 3x\right]_{2}^{y} = \frac{2}{3} - \frac{2}{3}\left[\left(\frac{y^2}{2} - 3y\right) + 4\right]$

$= \frac{2}{3}\left[1 - \frac{y^2}{2} + 3y - 4\right] = \frac{2}{3}\left[3y - \frac{y^2}{2} - 3\right] = \frac{1}{3}\left[6y - y^2 - 6\right]$

$= \frac{1}{3}\left[-(y-3)^2 + 9 - 6\right] = \frac{1}{3}\left[3 - (3-y)^2\right]$

if $x \geq 3$: $F(y) = 1$

$$\Rightarrow F(y) = \begin{cases} 0 \text{ if } y \leq 1 \\ \frac{2}{3}(y-1) \text{ if } y \in [1,2] \\ \frac{1}{3}\left[3-(3-y)^2\right] \text{ if } y \in [2,3] \\ 1 \text{ if } y \geq 3 \end{cases}$$

d/ Simulation of X:

we can use the distribution inverse Method to be able to make a simulation of X.

let's compute it : F^{-1}

for $x \in [1,2]$ $y = F(x) = \frac{2}{3}(x-1) \Rightarrow x = \frac{3}{2}y + 1$

for $x \in [2,3]$ $y = F(x) = \frac{1}{3}\left[3-(3-x)^2\right] \Rightarrow x = 3 - \sqrt{3-3y}$

- see pdf for R simulation

Exercice 2

Y exponential random variable with parameter $\lambda > 0$

$\Rightarrow f_Y(y) = \lambda e^{-\lambda y}, \ \forall y \geq 0$

CDF of Y $F_Y(y) = 1 - e^{-\lambda y}$ if $x \geq 0$; 0 if $x < 0$

$Z = 2Y + 3 \Rightarrow F_Z(x) = P(Z \leq x) = P(2Y+3 \leq x)$

$= P(Y \leq \frac{x-3}{2}) = F_Y(\frac{x-3}{2})$

by derivation $F_Z(x) = \begin{cases} 0 \text{ if } \frac{x-3}{2} < 0 \\ 1 - \exp(-\lambda(\frac{x-3}{2})) \text{ if } \frac{x-3}{2} \geq 0 \end{cases}$

$\Leftrightarrow F_Z(x) = \begin{cases} 0 \text{ if } x < 3 \\ 1 - \exp(-\lambda(\frac{x-3}{2})) \text{ if } x \geq 3 \end{cases}$

Exercice 3 :

$d(t) = \begin{cases} a \cdot \frac{x_0^a}{t^{a+1}} \text{ if } t \geq x_0 \\ 0 \quad \text{otherwise} \end{cases}$

a) d is a density function

d is a density function if $d(t) \geq 0$ and $\int_{-\infty}^{+\infty} d(t)\,dt = 1$

$\Rightarrow a \geq 0$; $x_0 \geq 0$

$\Rightarrow \int_{-\infty}^{+\infty} d(t)\,dt = \int_{x_0}^{+\infty} a \frac{x_0^a}{t^{a+1}}\,dt = a x_0^a \left[\frac{1}{-a t^a}\right]_{x_0}^{+\infty}$

$= -\frac{a}{a}\left[\left(\frac{x_0}{t}\right)^a\right]_{x_0}^{+\infty} = -1\left[0 - \left(\frac{x_0}{x_0}\right)^a\right] = -1[0-1] = 1$

$\underbrace{}_{\leq 1}$

$\Rightarrow \int_{-\infty}^{+\infty} d(t)\,dt = 1 \Rightarrow d$ is a density function

b) Let's compute $E[X]$ and $V[X]$

$E[X] = \int t\,d(t)\,dt = \int_{x_0}^{+\infty} a \frac{x_0^a}{t^a}\,dt = a x_0^a \left[\frac{1}{(1-a)t^{a-1}}\right]_{x_0}^{+\infty}$

$= \frac{a}{1-a} x_0 \left[\left(\frac{x_0}{t}\right)^{a-1}\right]_{x_0}^{+\infty} = \boxed{\frac{a}{a-1} x_0}$

$$E[X^2] = \int t^2 d(t)\,dt = \int_{x_0}^{+\infty} a\frac{x_0^a}{t^{a-1}}\,dt = \left[\frac{a}{2-a} x_0^a \frac{1}{t^{a-2}}\right]_{x_0}^{+\infty}$$

$$= \frac{a}{a-2} x_0^2$$

$$\Rightarrow V[X] = E[X^2] - (E[X])^2 = \frac{a}{a-1} x_0 - \frac{a}{a-2} x_0^2$$

$$= \boxed{a\left(\frac{x_0}{a-1} - \frac{x_0^2}{a-2}\right)}$$

c/ let's determine Distribution of X

$$F(y) = P(X \leq y) = \int_{x_0}^{y} a\frac{x_0^a}{t^{a+1}}\,dt$$

$$= \frac{a x_0^a}{-a}\left[\frac{1}{t^a}\right]_{x_0}^{y} = -x_0^a\left[\frac{1}{y^a} - \frac{1}{x_0^a}\right] = 1 - \left(\frac{x_0}{y}\right)^a$$

$$\Rightarrow F(y) = \begin{cases} 1 - \left(\frac{x_0}{y}\right)^a & \text{if } y \geq x_0 \\ 0, & \text{otherwise.} \end{cases}$$

d/ $\hat{x}_0$ with Method of Moments

$$\left.\begin{array}{l} - \mu = E[X] = \frac{a}{a-1} x_0 \\ - \hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i = \overline{X}_n \end{array}\right\} \Rightarrow \hat{x}_0 = \frac{a-1}{a}\overline{X}_n$$

e/ Another estimator of x_0 with $Z_n = \min(X_1, \dots, X_n)$.

first, let's compute the CDF of Z_n.

$$P(Z_n \geq x) = P(\min(X_1, \dots, X_n) \geq x) = P(X_1 \geq x \wedge X_2 \geq x \wedge \dots \wedge X_n \geq x)$$

$$= \prod_{i=1}^{n} P(X_i \geq x) \quad (\text{because } X_i \text{ are i.i.d}).$$

$$= \prod_{i=1}^{n} (1 - F(x)) = (1 - F(x))^n$$

$$\Rightarrow F_{Z_n}(y) = 1 - [1 - F(x)]^n = 1 - \left[1 - \left(1 - \left(\frac{x_0}{y}\right)^a\right)\right]^n$$

$$= 1 - \left(\frac{x_0}{y}\right)^{an}$$

so the PDF of Z_n is

$$f_{Z_n}(y) = F'_{Z_n}(y) = \frac{an\, x_0^{an}}{y^{an+1}}$$

$$E[Z_n] = \int_{x_0}^{+\infty} a_n x_0^{an} \frac{y}{y^{an+1}}\,dy = \int_{x_0}^{+\infty} a_n x_0^{an}\frac{1}{y^{an}}\,dy = a_n x_0^{an}\left[\frac{y^{-an+1}}{-an+1}\right]_{x_0}^{+\infty}$$

$$= \frac{an\, x_0}{1-an}\left[\left(\frac{x_0}{y}\right)^{an-1}\right]_{x_0}^{+\infty} = \frac{an\, x_0}{an-1}$$

$\Rightarrow E\left[\frac{a_m - 1}{a_m} Z_m\right] = \alpha_0 \Rightarrow \boxed{\hat{\alpha}_0 = \frac{a_m-1}{a_m} Z_m}$

$\Rightarrow \hat{\alpha}_0 = \frac{a_m - 1}{a_m} Z_m$ is an unbiased estimation of α_0

Exercice 4:

a/ the average number of calls is 3 calls per minute

$\Rightarrow$ so for every second the success probability to have a call is $\frac{3}{60}$ $= 0{,}05$

so for 4 minutes $= 4 \times 60 = 240$ seconds we have a binomial distribution of parameter $n = 240$ and p = rate of success $= \frac{3}{60} = 0{,}05$

b/ Poisson Approximation : for a binomial (n, p)

the poisson Approximation holds if n large, p small $\leq 0{,}1$ and $np(p-1) \simeq 10$

here we have $n = 240$

$p = 0{,}05 < 0{,}1$

$np(1-p) = 11{,}4 \simeq 10$

$\Rightarrow$ Poisson Approximation of binomial (n, p) is valid.

with $\lambda = n \cdot p = 240 \times 0{,}05 = 12$

c/ confidence interval for the Expectation.

for poisson distribution the mean and the variance are equal λ

as the theoretical variance is known we can approximate the mean with Normal Distribution thanks to central limit theorem

so for a threshold α and standard error $\sqrt{\frac{\lambda}{n}}$ (n being the number of Experiment of collecting number of calls during 4 minutes)

the confidence interval is $\left[\lambda \pm z_{\alpha/2} \times \sqrt{\frac{\lambda}{n}}\right]$

for 95% C.I $\Rightarrow \left[\lambda \pm 1{,}96 \sqrt{\frac{\lambda}{n}}\right]$

Example: for $n = 100$ C.I $= [12 \pm 0{,}68] = \left[12 \pm 1{,}96 \sqrt{\frac{12}{100}}\right]$

Exercicio 5

$X \sim \mathcal{N}(\mu = 2\ ;\ \sigma = 0{,}1) \Rightarrow \sigma^2 = 0{,}01$

a). $P(X \leq 2{,}1) = P\left(\frac{X-\mu}{\sigma} \leq \frac{2{,}1-\mu}{\sigma}\right)$

$= P\left(\frac{X-\mu}{\sigma} \leq \frac{2{,}1-2}{0{,}1}\right) = P\left(\frac{X-\mu}{\sigma} \leq 1\right) = 0{,}8413$

b). $P(X > 1{,}75) = 1 - P(X \leq 1{,}75) = 1 - P\left(\frac{X-\mu}{\sigma} \leq \frac{1{,}75-2}{0{,}1}\right)$

$= 1 - P\left(\frac{X-\mu}{\sigma} \leq -2{,}5\right) = 1 - 0{,}0062$

$= 0{,}9938$

c). $P(1{,}99 < X < 2{,}02) = P\left(\frac{1{,}99-2}{0{,}1} < \frac{X-\mu}{\sigma} < \frac{2{,}02-2}{0{,}1}\right)$

$= P\left(-0{,}1 < \frac{X-\mu}{\sigma} < 0{,}2\right)$

$= P\left(\frac{X-\mu}{\sigma} < 0{,}2\right) - P\left(\frac{X-\mu}{\sigma} \leq -0{,}1\right)$

$= 0{,}5793 - 0{,}4602$

$= 0{,}1191$