# Integration of functions of single variable

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# Motivation



- Given the velocity of a function, one wishes to know its position at a given time.
- Find a function whose derivative is known.

# Antiderivatives



## Definition

A function $F(x)$ is called an antiderivative of $f(x)$ on an interval $I$ if $F'(x) = f(x)$ for all $x \in I$.

## Example

- $\left(\ln(1+x^2)\right)' = \frac{2x}{1+x^2}$ so $\ln(1+x^2)$ is an antiderivative of $\frac{2x}{1+x^2}$.
- $x^4 + 2$ is an antiderivative of $4x^3$.

## Theorem

*If $F(x)$ is an antiderivative of $f(x)$ on $I$. Then the family of all antiderivatives of $f(x)$ is $F(x) + C$.*

## Definition

The family of all antiderivatives is called the indefinite integral of $f(x)$.
Denote $\int f(x)dx = F(x) + C$, where $F(x)$ is a known antiderivative.

$$\int \frac{2x}{1+x^2}dx = \ln(1+x^2) + C, \qquad \int 4x^3 dx = x^4 + C.$$

# Linearity

Let $F(x)$, $G(x)$ be antiderivatives of $f(x)$, $g(x)$ respectively. Then

$$\int (Af(x) + Bg(x))dx = AF(x) + BG(x) + C.$$

**Theorem**

*A function $f(x)$ which is continuous on $[a, b]$ has an antiderivative on that interval.*

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# Table of indefinite integrals



$$\int x^\alpha dx = \begin{cases} \frac{x^{\alpha+1}}{\alpha+1} + C & \text{if } \alpha \neq -1, \\ \ln|x| + C & \text{if } \alpha = -1. \end{cases}$$

$$\int a^x dx = \frac{a^x}{\ln a} + C, 0 < a \neq 1$$

$$\int \sin x dx = -\cos x + C, \qquad \int \cos x dx = \sin x + C$$

$$\int \frac{dx}{\sin^2 x} = -\cot x + C, \qquad \int \frac{dx}{\cos^2 x} = \tan x + C$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan \frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + C.$$

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# The Substitution Rule



## Theorem

*If $u = g(x)$ is a differentiable function whose range is an interval $I$ and $f$ is continuous on $I$, then*

$$\int f(g(x))g'(x)dx = \int f(u)du.$$

## Example

Evaluate the following integrals

1. $\displaystyle\int \frac{x^4 dx}{x^{10}+1}$
2. $\displaystyle\int \frac{dx}{e^x - e^{-x}}$
3. $\displaystyle\int \frac{dx}{\sqrt{x^2+4}}$

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# Integration by parts



Assume that $u(x)$, $v(x)$ are continuously differentiable functions. We have

$$\int u dv = uv - \int v du.$$

## Example

Evaluate the integrals

1. $\displaystyle\int \frac{x}{\cos^2 x} dx$
2. $\displaystyle\int e^x \cos 2x dx$
3. $\displaystyle\int \sqrt{x^2 + \alpha} dx$

# Table of indefinite integrals



$$\int \frac{dx}{\sqrt{x^2+\alpha}} = \ln\left|x+\sqrt{x^2+\alpha}\right| + C$$

$$\int \sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C$$

$$\int \sqrt{x^2+\alpha}dx = \frac{x}{2}\sqrt{x^2+\alpha} + \frac{\alpha}{2}\ln\left|x+\sqrt{x^2+\alpha}\right| + C.$$

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Aim: Evaluate $\int R(x)dx$, where

$$R(x) = \frac{b_0 + b_1x + \ldots + b_mx^m}{a_0 + a_1x + \ldots + a_nx^n}, a_i, b_i \in \mathbb{R}, a_n, b_m \neq 0,$$

is a rational function.

Method: expressing $R(x)$ as a sum of partial fractions.

1. Performing a polynomial division to reduce to proper rational function.
2. Factorizing the denominator into factors $(x-a)^k, (x^2+px+q)^k$, where $q - \frac{p^2}{4} > 0$.
3. Writing $R(x)$ as the sum of following functions

$$\int \frac{A_l dx}{(x-a)^l}, \quad \int \frac{B_l x + C_l}{(x^2+px+q)^l} dx, l = 1, 2, \ldots, k.$$

## Example

Evaluate the integrals

1. $$\int \frac{x\,dx}{(x+2)^2(x-1)}$$
2. $$\int \frac{x\,dx}{x^4+3x^2+2}$$

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# Trigonometric integrals $\int \mathcal{R}(\sin x, \cos x)dx$



Evaluate $\int \mathcal{R}(\sin x, \cos x)dx$, where $\mathcal{R}(\sin x, \cos x)$ is a rational function of the variables $u = \sin x$, $v = \cos x$.

General substitution $t = \tan \frac{x}{2}$, $t \in (-\pi, \pi)$. We obtain

$$\int \mathcal{R}\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{2dt}{1+t^2}.$$

## Example

Evaluate $\displaystyle\int \frac{dx}{2\sin x - \cos x + 5}$.

# Special cases



- If $\mathcal{R}(-\sin x, -\cos x) = \mathcal{R}(\sin x, \cos x)$, set $t = \tan x$ or $t = \cot x$. Examples: $\int \frac{dx}{\sin^2 x \cos^4 x}$, $\int \frac{\tan x dx}{1+\cos^2 x}$.
- If $\mathcal{R}(-\sin x, \cos x) = -\mathcal{R}(\sin x, \cos x)$, set $t = \cos x$. Examples $\int \frac{dx}{(2+\cos x)\sin x}$
- If $\mathcal{R}(\sin x, -\cos x) = -\mathcal{R}(\sin x, \cos x)$, set $t = \sin x$.

If we can write
$a_1 \cos x + b_1 \sin x = A(a \sin x + b \cos x) + B(a \cos x - b \sin x)$.
Then

$$\int \frac{a_1 \cos x + b_1 \sin x}{a \sin x + b \cos x} dx = \int \left(A + B\frac{a \cos x - b \sin x}{a \sin x + b \cos x}\right) dx$$
$$= Ax + B \ln |a \sin x + b \cos x| + C.$$

## Example

$$\int \frac{\sin x - \cos x}{\sin x + 2\cos x} dx, \quad \int \frac{\sin x}{\sin x - 3\cos x} dx$$

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- $\int \mathcal{R}(x, \sqrt{a^2 - x^2})dx$.
  Set $x = a\sin t,\ t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, or $x = a\cos t,\ t \in [0, \pi]$.
- $\int \mathcal{R}(x, \sqrt{x^2 - a^2})dx$.
  Set $x = \frac{a}{\sin t},\ t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), t \neq 0$, or $x = \frac{a}{\cos t}$, $t \in (0, \pi), t \neq \frac{\pi}{2}$.
- $\int \mathcal{R}(x, \sqrt{a^2 + x^2})dx$.
  Set $x = a\tan t,\ t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, or $x = a\cot t,\ t \in (0, \pi)$.

## Example

Evaluate

1. $$\int \frac{x^3 dx}{\sqrt{1-x^2}}$$
2. $$\int \frac{dx}{x\sqrt{x^2+2x+2}}$$
3. $$\int x\sqrt{-x^2+4x-3}dx$$

# $\int \mathcal{R}(x, \sqrt{Ax^2+Bx+C})dx,\ A>0$



Rule: $A>0$, set $\sqrt{Ax^2+Bx+C} = t \pm \sqrt{A}x$.

## Example

Evaluate $I = \int \frac{dx}{x+\sqrt{x^2+x+1}}$

Set $\sqrt{x^2+x+1} = t - x$. Then $x = \frac{t^2-1}{2t+1}$, $dx = \frac{2(t^2+t+1)}{(2t+1)^2}dt$.

$$I = \int \frac{2(t^2+t+1)}{t(2t+1)^2}dt = \int \left(\frac{2}{t} - \frac{3}{2t+1} - \frac{3}{(2t+1)^2}\right)dt$$

$$= 2\ln|t| - \frac{3}{2}\ln|2t+1| + \frac{3}{2(2t+1)} + C$$

$$= 2\ln|\sqrt{x^2+x+1}+x| - \frac{3}{2}\ln|2(\sqrt{x^2+x+1}+x)+1| +$$

$$+ \frac{3}{4(\sqrt{x^2+x+1}+x)+2} + C$$

# $\int \mathcal{R}(x, \sqrt{Ax^2+Bx+C})dx,\ C>0$



Rule: $C>0$, set $\sqrt{Ax^2+Bx+C} = xt \pm \sqrt{C}$.

## Example

Evaluate $I = \int \frac{dx}{1+\sqrt{1-2x-x^2}}$

Set $\sqrt{1-2x-x^2} = tx-1 \Rightarrow x = \frac{2(t-1)}{t^2+1}$, $dx = \frac{2(3t^2-2t+1)}{(t^2+1)^2}dt$.

$$I = \int \frac{3t^2-2t+1}{t(t-1)^2(t^2+1)}dt = \int \left(-\frac{1}{t} + \frac{1}{t-1} + \frac{2}{t^2+1}\right)dt$$
$$= -\ln|t| + \ln|t-1| + 2\arctan t + C$$
$$= -\ln\left|\frac{\sqrt{1-2x-x^2}+1}{x}\right| + \ln\left|\frac{\sqrt{1-2x-x^2}+1}{x} - 1\right| +$$
$$+ 2\arctan\frac{\sqrt{1-2x-x^2}+1}{x} + C$$

## $\int \mathcal{R}(x, \sqrt{A(x-\alpha)(x-\beta)})dx$



Rule: If $Ax^2 + B + C = A(x-\alpha)(x-\beta)$, set $\sqrt{Ax^2+Bx+C} = t(x-\alpha)$ or $\sqrt{Ax^2+Bx+C} = t(x-\beta)$.

Fact: there are functions that $\int f(x)dx$ is not an elementary function. We cannot evaluate their integrals in terms of the functions we know, for instance

$$\int e^{x^2}dx, \int \frac{\sin x}{x}dx, \int \frac{1}{\ln x}dx \ldots$$