Fourier series

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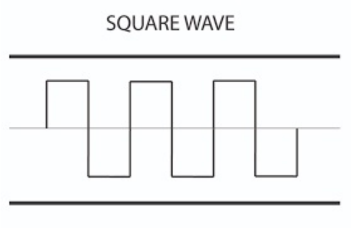
Content

Expand an **infinitely differentiable** function $f(x)$ in a **neighborhood of** x_0 into power series of $x - x_0$.
Can we relax the smoothness of the function?
In physics, electrical engineering, one deals with periodic phenomenon.



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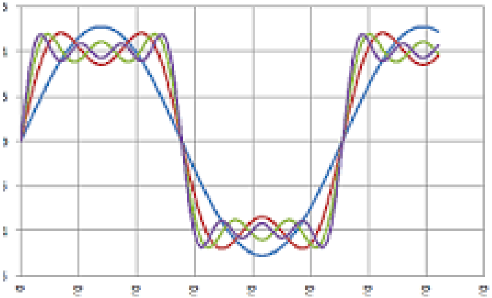
Fourier series: basic concepts
Fourier expansion of $2l-$ periodic function
Fourier expansion of a function defined on $[a;b]$

Fourier series is a mathematical way to express a **nontrigonometric periodic** function in terms of trigonometric functions, if $f(x)$ is periodic with period T, we want to expand

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos n\omega x+b_n\sin n\omega x\right),\ \omega=\frac{2\pi}{T}.$$



Definition

Trigonometric series has the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Proposition (Sufficient condition for convergence)

If $\sum\limits_{n=1}^{\infty} |a_n|, \sum\limits_{n=1}^{\infty} |b_n|$ converge then the series above converges absolutely and uniformly on $\mathbb{R}$.

Proof.

$|a_n \cos nx + b_n \sin nx| \leq |a_n| + |b_n| \forall n \in \mathbb{N}, \forall x \in \mathbb{R}.$

Let $f(x)$ be a periodic function with period 2π which is integrable over $[-\pi, \pi]$ and assume that we expand $f(x)$ into a trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right).$$

- Determine $a_n, b_n,\ n = 0, 1, 2 \ldots$?
- When does the series converge to $f(x)$?

Lemma

For $m, n \in \mathbb{N}^*$, we have

$$\int_{-\pi}^{\pi} \sin mx dx = 0, \qquad \int_{-\pi}^{\pi} \cos mx dx = 0,$$

$$\int_{-\pi}^{\pi} \sin mx \cos nx dx = 0,$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx dx = \begin{cases} 0 & \text{if } m \neq n, \\ \pi & \text{if } m = n. \end{cases}$$

$$\int_{-\pi}^{\pi} \cos mx \cos nx dx = \begin{cases} 0 & \text{if } m \neq n, \\ \pi & \text{if } m = n. \end{cases}$$

If $m = n \neq 0$:

$$\int_{-\pi}^{\pi} \sin^2 mx dx = \frac{1}{2}\int_{-\pi}^{\pi}(1-\cos 2mx)dx$$
$$= \frac{1}{2}\left(x - \frac{\sin 2mx}{2m}\right)\Big|_{-\pi}^{\pi} = \pi.$$

If $m \neq n$:

$$\int_{-\pi}^{\pi} \sin mx \sin nx dx = \frac{1}{2}\int_{-\pi}^{\pi}(\cos(m-n)x - \cos(m+n)x)dx$$
$$= \frac{1}{2}\left(\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right)\Big|_{-\pi}^{\pi} = 0.$$

Assume that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx). \qquad (1)$$

we obtain (formally)

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} \frac{a_0}{2}dx + \sum_{n=1}^{\infty}\left(\int_{-\pi}^{\pi} a_n \cos nxdx + \int_{-\pi}^{\pi} b_n \sin nxdx\right)$$

$$\Rightarrow a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx.$$

Multiplying (1) by $\sin mx$ then integrating over $[-\pi, \pi]$, we get

$$\int_{-\pi}^{\pi} f(x)\sin mxdx = \int_{-\pi}^{\pi} \frac{a_0}{2}\sin mxdx + \sum_{n=1}^{\infty}\left(\int_{-\pi}^{\pi} a_n \cos nx \sin mxdx + \int_{-\pi}^{\pi} b_n \sin nx \sin mxdx\right)$$

Hence, $b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mxdx$. $a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mxdx$.

Definition

Let $f(x)$ be a $2\pi-$periodic function and be integrable over $(-\pi, \pi)$. The Fourier series or Fourier expansion corresponding to $f(x)$ is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{2}$$

where the Fourier coefficients $\mathbf{a_n}, \mathbf{b_n}$ are

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \ n \geq 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx, \ n \geq 1.$$

Content

Assume that $f(x)$ is periodic with period $2l$.

Set $x' = \frac{\pi}{l}x$ and $f(x) = f\left(\frac{l}{\pi}x'\right) =: g(x')$.

It is obvious

$$g(x'+2\pi) = f\left(\frac{l}{\pi}(x'+2\pi)\right) = f\left(\frac{l}{\pi}x'+2l\right) = g(x'),$$

so $g(x')$ is periodic with period 2π.

Expand $g(x')$ into Fourier series, the Fourier coefficients of g are

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x')dx' = \frac{1}{l}\int_{-l}^{l} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x')\cos nx'dx' = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx.$$

f is periodic with period $2l$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right).$$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx$$

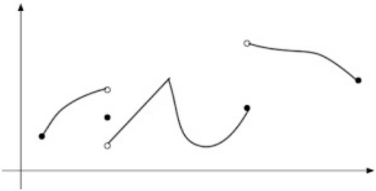
$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx.$$

Theorem (Dirichlet conditions)

Let $f(x)$ be a periodic function with period $2l$ which is defined except possibly at a finite number of points in $(-l; l)$, and $f(x)$, $f'(x)$ are piecewise continuous in $(-l; l)$. Then the Fourier series corresponding to $f(x)$ converges to

- *$f(x)$ if x is a point of continuity,*
- $\dfrac{f(x+0)+f(x-0)}{2}$ *if x is a point of discontinuity,*

where $f(x+0) = \lim\limits_{y \to x^+} f(y)$ and $f(x-0) = \lim\limits_{y \to x^-} f(y)$.



f is piecewise continuous in $[a, b]$ if there exist $a = x_0 < x_1 < \ldots < x_n = b$ such that $f(x)$ continuous in $[x_{i-1}, x_i]$ and x_i is a point of discontinuity of the first type.

Example

Expand into Fourier series the periodic function $f(x)$ with period 2π and $f(x) = x$, $-\pi \le x < \pi$.

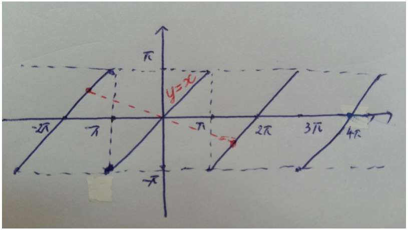
Fourier coefficients are

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x dx = 0; \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cos nx dx = 0.$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx dx \\ &= \frac{2}{\pi}\int_{0}^{\pi} x \frac{d(-\cos nx)}{n} = \frac{2}{\pi}\left[-\frac{x\cos nx}{n}\Big|_0^{\pi} + \int_0^{\pi}\frac{\cos nx}{n}dx\right] \\ &= \frac{2}{\pi}\left[\frac{(-1)^{n+1}\pi}{n} + \frac{\sin nx}{n^2}\Big|_0^{\pi}\right] = \frac{2.(-1)^{n+1}}{n}. \end{aligned}$$

So $f(x) = \sum_{n=1}^{\infty} \frac{2.(-1)^{n+1}}{n} \sin nx, x \neq (2k+1)\pi, k \in \mathbb{Z}$.

The graph of $f(x)$.



At $x = \pi$, $f(\pi + 0) = -\pi; f(\pi - 0) = \pi$, the series converges to 0.
At $x = (2k + 1)\pi$, the series converges to 0.

Example

Expand into Fourier series the periodic function $f(x)$ with period 2π and $f(x) = \begin{cases} -1, & \text{if } -\pi \le x < 0, \\ 1, & \text{if } 0 \le x < \pi. \end{cases}$

Fourier coefficients are

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)dx + \int_0^{\pi} dx\right] = \frac{1}{\pi}[-\pi + \pi] = 0.$$

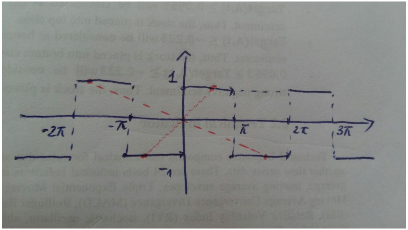
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\cos nx)dx + \int_0^{\pi} \cos nxdx\right]$$

$$= \frac{1}{n\pi}\left[-\sin nx\Big|_{-\pi}^{0} + \sin nx\Big|_{0}^{\pi}\right] = 0.$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\sin nx)dx + \int_0^{\pi} \sin nxdx\right] = \frac{2(1-(-1)^n)}{n\pi}.$$

Hence, $f(x) = \sum\limits_{n=1}^{\infty} \dfrac{2(1-(-1)^n)}{n\pi} \sin nx, x \neq k\pi, k \in \mathbb{Z}.$

Graph of $f(x)$



$f(x)$ is odd so $a_n = 0, \forall n \geq 0$ and

$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin nx dx = \frac{2}{\pi}\frac{-\cos(nx)}{n}\Big|_0^{\pi}$$

$$= \frac{2(1-(-1)^n)}{n\pi} = \begin{cases} 0 & \text{if } n = 2l, \\ \dfrac{4}{n\pi} & \text{if } n = 2l+1. \end{cases}$$

We get

$$f(x) = \sum_{n=1}^{\infty}\frac{2(1-(-1)^n)}{n\pi}\sin nx = \sum_{l=0}^{\infty}\frac{4}{(2l+1)\pi}\sin(2l+1)x, x \neq k\pi.$$

At $x = k\pi$, the series converges to 0.

- If $f(x)$ is an **odd function**: $a_0 = a_n = 0$,

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx, n \geq 1$$

Fourier expansion $\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$ consists of only sine functions.

- If $f(x)$ is an **even function**: $b_n = 0$,

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{L}dx, n \geq 0.$$

Fourier expansion $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$ consists of only cosine functions.

Content

Let $f(x)$ be a function defined on the interval $[a, b]$ that f and f' are piecewise continuous.
To expand $f(x)$ into Fourier series, we will construct a function $g(x)$ that satisfies

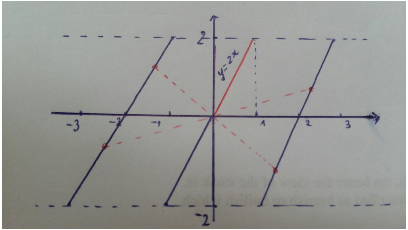
- $g(x) = f(x)$ for all $x \in [a, b]$.
- $g(x)$ is periodic (with period $T \geq b - a$).

Fourier series corresponding to $g(x)$ is the Fourier series corresponding to $f(x)$ for $x \in [a; b]$.

Example

Expand $f(x) = 2x$, $0 \le x < 1$, into Fourier sine series.

Consider a function $g_1(x)$ whose graph is as follows.



$g_1(x) = f(x) = 2x$ for $0 \le x < 1$; $g_1(x)$ is an odd function and periodic with period $T = 2, l = 1$.

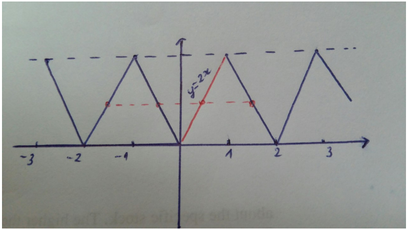
Fourier coefficients: $a_n = 0, b_n = 2\displaystyle\int_0^1 2x\sin(n\pi x)dx$, and

$f(x) = \displaystyle\sum_{n=1}^{\infty} b_n \sin(n\pi x), 0 \le x < 1$. HW: Work out the details.

Example

Expand $f(x) = 2x$, $0 \le x < 1$, into Fourier cosine series.

Consider $g_2(x)$:



$g_2(x) = f(x) = 2x$ for $0 \le x < 1$; $g_2(x)$ is an even function and periodic with $T = 2, l = 1$.

Fourier coefficients $b_n = 0, a_n = 2\int_0^1 2x\cos(n\pi x)dx$, and

$f(x) = \dfrac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos(n\pi x), 0 \le x < 1$. HW: Work out the details.