Implicit functions

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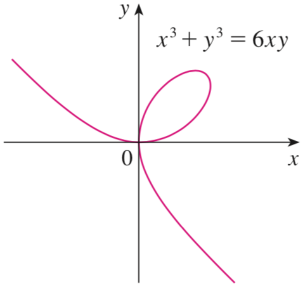
Implicit function $y = y(x)$ given by $f(x, y) = 0$

- Given $x^2 + y^2 - 1 = 0$, we solve for $y(x) = \pm\sqrt{1 - x^2}$.
- Given the equation $x^3 + y^3 - 3xy = 0$.



For all $x \in [0, \sqrt[3]{2}]$, the equation determines three y such that $f(x, y) = 0$. Hence, the equation $f(x, y) = 0$ determines one or more implicit functions $y = y(x)$.

Derivative of the function $y = y(x)$ given by $f(x, y) = 0$

Theorem (Existence)

Let $f(x, y)\colon D \subset \mathbb{R}^2 \to \mathbb{R}$ have continuous partial derivatives in a neighborhood of $M_0(x_0, y_0)$, $f(M_0) = 0$ and $f'_y(M_0) \neq 0$. Then, the equation $f(x, y) = 0$ determines in a neighborhood of x_0 a continuously differentiable implicit function $y = y(x)$ that $y(x_0) = y_0$. Moreover,

$$y'(x) = -\frac{f'_x(x, y)}{f'_y(x, y)}.$$

It holds $f(x, y) = 0 \Leftrightarrow y = y(x)$, or $f(x, y(x)) = 0$.

Implicit function $z = z(x, y)$ determined by $f(x, y, z) = 0$

Theorem (Existence)

Let $f(x, y, z)\colon D \subset \mathbb{R}^3 \to \mathbb{R}$ have continuous partial derivatives in a neighborhood of $M_0(x_0, y_0, z_0)$, $f(M_0) = 0$ and $f'_z(M_0) \neq 0$. Then, the equation $f(x, y, z) = 0$ determines in a neighborhood of (x_0, y_0) an implicit function $z = z(x, y)$ such that $z(x_0, y_0) = z_0$ and z has continuous partial derivatives. Moreover,

$$z'_x(x, y) = -\frac{f'_x(M)}{f'_z(M)}, \qquad z'_y(x, y) = -\frac{f'_y(M)}{f'_z(M)}.$$

Example

1. Assume that the function $y = y(x)$ is given by the equation $xe^y + x^3 + 2x^2y - 2 = 0$. Compute $y'(1)$.
2. Let $z = z(x, y)$ be determined by the equation $2x^3y - x^2 + xe^z + 2xyz = 2$. Compute $z(1; 1)$, $z'_x(1; 1)$, $z'_y(1; 1)$.
3. Let $y = y(x)$ be determined by the equation $xe^y + 2x^3 - 2x^2y + 3 = 0$. Approximate $y(-1, 01)$.
4. Let $z = z(x, y)$ be determined by the equation $z^3 + x^2y + xy^2 = 3xyz$.
   a) Compute $z(1; -1)$, $z'_x(1; -1)$, $z'_y(1; -1)$.
   b) Approximate $z(0, 99; -1, 01)$.

Theorem

Let $f(M_0) = g(M_0) = 0$, *where* f, g *have continuous partial derivatives near* $M_0(x_0, y_0, z_0, u_0, v_0)$ *and*

$$\frac{D(f,g)}{D(u,v)}(M_0) = \begin{vmatrix} f'_u & f'_v \\ g'_u & g'_v \end{vmatrix}(M_0) \neq 0.$$

Then, the system $\begin{cases} f(x,y,z,u,v) = 0 \\ g(x,y,z,u,v) = 0 \end{cases}$ *determines two implicit functions* $u = u(x,y,z)$, $v = v(x,y,z)$ *near* (x_0, y_0, z_0) *such that* $u(x_0, y_0, z_0) = u_0$, $v(x_0, y_0, z_0) = v_0$. *The functions* u, v *have continuous partial derivatives.*

Differentiating w.r.t. x both sides of the equations of the system

$$\begin{cases} f'_x + f'_u.u'_x + f'_v.v'_x = 0 \\ g'_x + g'_u.u'_x + g'_v.v'_x = 0 \end{cases}$$

We obtain

$$u'_x = -\frac{\frac{D(f,g)}{D(x,v)}}{\frac{D(f,g)}{D(u,v)}}, \qquad v'_x = -\frac{\frac{D(f,g)}{D(u,x)}}{\frac{D(f,g)}{D(u,v)}}.$$

Similarly, we can compute u'_y, u'_z, v'_y, v'_z.

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Definition

Second order partial derivatives of $f(x, y)$ are:

$$f''_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial x}) \qquad f''_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial y})$$

$$f''_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial x}) \qquad f''_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial y})$$

Example

1. Compute all the second order partial derivatives of the function $z = \arctan(x^2 y)$.
2. Compute the second order partial derivatives at $(0,0)$ of the function $f(x,y) = \begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

Schwarz theorem

Theorem

Assume that $z(x, y)$ has partial derivatives f''_{xy} và f''_{yx} near $M_0(x_0, y_0)$ which are continuous at M_0. Then,

$$f''_{xy}(M_0) = f''_{yx}(M_0).$$

Higher order total differentials

Definition

Higher order total differential of $z = f(x, y)$:

$$d^n z = d(d^{n-1} z),\ n \geq 2.$$

If x, y are independent variables, then $d^n z = \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy\right)^n f$

Example

Second order total differential
$d^2 z = \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy\right)^2 f = \frac{\partial^2 f}{\partial x^2}(dx)^2 + 2\frac{\partial^2 f}{\partial x \partial y} dxdy + \frac{\partial^2 f}{\partial y^2}(dy)^2.$

Note: Higher order total differentials do not fulfill the invariance property.

Example

1. Given $z = 2^x \sin y$. Compute the second order differential d^2z and $d^2z(0, \pi)$.
2. Given $z = \arctan(xy)$. Compute the second order differential d^2z and $d^2z(1, -1)$.

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Taylor theorem for multivariable functions

Theorem

Let $f(x, y)$ have continuous partial derivatives up to order $n+1$ in an open set $U \subset \mathbb{R}^2$, $M_0(x_0, y_0), M(x_0 + \Delta x, y + \Delta y) \in U$. There exists $0 < \theta < 1$ such that

$$f(x_0+\Delta x, y+\Delta y)-f(x_0, y_0) = df(x_0, y_0)+\frac{1}{2!}d^2 f(x_0, y_0)+\dots$$
$$+\frac{1}{n!}d^n f(x_0, y_0) + \frac{1}{(n+1)!}d^{n+1} f(x_0 + \theta\Delta x, y + \theta\Delta y).$$

Example

Compute the second degree Taylor polynomial of the function $f(x, y) = e^{xy}$ at $(0, 1)$.