EXERCISES - CALCULUS 3 - MI1131E

Chapter 1

Series

1.1 Number series

Exercise 1.1. Test for convergence and find the sum (if exists):

a) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

b) $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \dots$

c) $\sum_{n=1}^{\infty} (\sin n + 1 - \sin n)$

d) $\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right)$

e) $\sum_{n=1}^{\infty} \left(\frac{9}{10^n} - \frac{2}{5^n}\right)$

f) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}.3^n}{10^{n+2}}$

g) $\sum_{n=1}^{\infty} \arctan \frac{1}{n^2+n+1}$

Exercise 1.2. Test for convergence:

1. Divergence test

a) $\sum_{n=1}^{\infty} \frac{(-1)^n.n}{n+1}$

b) $\sum_{n=1}^{\infty} \frac{2n+3}{6n-1}$

c) $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n^2}\right)$

d) $\sum_{n=1}^{\infty} \left(\frac{n+1}{n+2}\right)^n$

e) $\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{1}{n}\right)$

2. Comparison tests

a) $\sum_{n=1}^{\infty} \frac{\sqrt{n^2+n+1}}{n^2\sqrt{n}+2}$

b) $\sum_{n=1}^{\infty} \frac{\sqrt{n+2}-\sqrt{n}}{2n+1}$

c) $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right)$

d) $\sum_{n=1}^{\infty} \frac{\sqrt[n]{e}-1}{n}$

e) $\sum_{n=2}^{\infty} \arctan(2^{-n})$

f) $\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$

g) $\sum_{n=1}^{\infty} \frac{2}{\ln(2n+1)}$

h) $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \sin\frac{1}{\sqrt{n}}\right)$

i) $\sum_{n=1}^{\infty} \frac{4+\cos n}{n^2(1+e^{-n})}$

3. Ratio test

a) $\sum_{n=1}^{\infty} \frac{2019^n}{n!}$
c) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n+1)!}$
e) $\sum_{n=1}^{\infty} \frac{n^n}{4^n.n!}$
b) $\sum_{n=2}^{\infty} \frac{1}{3^n} \frac{(2n+1)!}{n^2-1}$
d) $\sum_{n=1}^{\infty} \frac{n!}{3^{n^2}}$
f) $\sum_{n=2}^{\infty} \frac{e^n n!}{n^n}$

4. Root test

a) $\sum_{n=1}^{\infty} \left(\frac{3n+1}{3n+2}\right)^{n^2}$
c) $\sum_{n=2}^{\infty} \left(\frac{n}{n+2}\right)^{n^2-1}$
e) $\sum_{n=2}^{\infty} \left(\cos\frac{1}{n}\right)^{n^3}$
b) $\sum_{n=2}^{\infty} \frac{1}{4^n}\left(1-\frac{1}{n}\right)^{n^2}$
d) $\sum_{n=1}^{\infty} \frac{1}{3^n}\left(\frac{n-2}{n}\right)^{n^2+1}$

5. Integral test

a) $\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$
c) $\sum_{n=4}^{\infty} \frac{1}{n\ln n\ln(\ln n)}$
e) $\sum_{n=2}^{\infty} \frac{1}{\ln(n!)}$
b) $\sum_{n=2}^{\infty} \frac{\ln n}{n}$
d) $\sum_{n=1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}}$

6. Series with sign-changing terms

a) $\sum_{n=1}^{\infty} \frac{\cos n}{\sqrt{n^3+1}}$
d) $\sum_{n=1}^{\infty} \left(\frac{3-2n}{2^n+5}\right)^{n^2}$
g) $\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n+\cos n}$
b) $\sum_{n=1}^{\infty} \frac{(-1)^n\sqrt{n}}{n+1}$
e) $\sum_{n=1}^{\infty} \frac{n\cos(n\pi)}{2n^2+1}$
h) $\sum_{n=2}^{\infty} \frac{(-1)^n+\cos n}{n\ln^2 n}$
c) $\sum_{n=1}^{\infty} \frac{(-1)^n.n^3}{2^n-1}$
f) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}.n^3}{(n^2+1)^{\frac{4}{3}}}$
i) $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}\sin\frac{1}{\sqrt{n}}$

Exercise 1.3. Test for absolute and conditional convergence:

a) $\sum_{n=2}^{\infty} \frac{(-1)^n n}{n^2+1}$
c) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$
e) $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}+(-1)^n}$
b) $\sum_{n=1}^{\infty} \frac{(-1)^n\sqrt{n}}{n+100}$
d) $\sum_{n=1}^{\infty} (-1)^n\left(\frac{2n+100}{3n+1}\right)^n$

Exercise 1.4. Test for convergence

a) $\sum_{n=1}^{\infty} \frac{n+1}{(n^2+2)\ln(n+3)}$
c) $\sum_{n=1}^{\infty} \frac{n^5}{3^n+2^n}$
e) $\sum_{n=2}^{\infty} \left(e^{\frac{(-1)^n}{\sqrt{n}}}-1\right)$
b) $\sum_{n=1}^{\infty} \frac{2-n^2.3^{-n^2}}{n^2}$
d) $\sum_{n=1}^{\infty} \left(\cos\frac{1}{n+1}-\cos\frac{1}{n}\right)$
f) $\sum_{n=2}^{\infty} \frac{(-1)^n(n-1)}{n^2+1}$

1.2 Function series

Exercise 1.5. Determine the domain of convergence of the following functions series:

a) $\sum\limits_{n=1}^{\infty} \dfrac{1}{1+n^{-x}}$
d) $\sum\limits_{n=1}^{\infty} \dfrac{x^n}{x^{2n}+1}$
g) $\sum\limits_{n=1}^{\infty} \dfrac{\sqrt{n^3+1}}{(x^2+1)n^x}$

b) $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{n^x}$
e) $\sum\limits_{n=1}^{\infty} \dfrac{n^x+(-1)^n}{n}$
h) $\sum\limits_{n=1}^{\infty} n.e^{-nx}$

c) $\sum\limits_{n=1}^{\infty} \dfrac{1}{x^n+1}$
f) $\sum\limits_{n=1}^{\infty} \left(x+\dfrac{1}{n}\right)^n$
i) $\sum\limits_{n=1}^{\infty} \dfrac{(n!)^2}{(2n)!}x^n$

Exercise 1.6. Determine the domain of convergence of the following power series:

a) $\sum\limits_{n=1}^{\infty} \dfrac{x^{2n}}{n}$
d) $\sum\limits_{n=1}^{\infty} \dfrac{e^{nx}}{n^2+n+1}$
g) $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n.n}{\sqrt{n^3+1}}(1-3x)^n$

b) $\sum\limits_{n=1}^{\infty} \dfrac{(x+2)^n}{n\sqrt{n}}$
e) $\sum\limits_{n=1}^{\infty} \dfrac{n^2}{1+n^3}(2x+1)^n$
h) $\sum\limits_{n=1}^{\infty} \left(\dfrac{1-2n}{2n+3}\right)^n x^{2n+1}$

c) $\sum\limits_{n=1}^{\infty} \dfrac{n}{2n+1}(x-2)^n$
f) $\sum\limits_{n=1}^{\infty} \dfrac{x^n}{2^n+3^n}$

Exercise 1.7. Test for uniform convergence on the given set of the following series:

a) $\sum\limits_{n=1}^{\infty} \dfrac{\sin nx}{2x^2+n^2}$, on $\mathbb{R}$
e) $\sum\limits_{n=1}^{\infty} \dfrac{x}{1+n^4x^2}$, on $[0,\infty)$

b) $\sum\limits_{n=1}^{\infty} \dfrac{e^{-nx}+1}{n^2}$, on $[0,\infty)$
f) $\sum\limits_{n=1}^{\infty} \dfrac{x}{(1+(n-1)x)(1+nx)}$, on $(0,1]$

c) $\sum\limits_{n=1}^{\infty} \dfrac{x^n}{(4x^2+9)^n}$, $x\in\mathbb{R}$
g) $\sum\limits_{n=1}^{\infty} (1-x)x^n$, on $[0,1]$

d) $\sum\limits_{n=1}^{\infty} \dfrac{1}{2^n}\left(\dfrac{2x+1}{x+2}\right)^n$, $x\in[-1;1]$
h) $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{x^2+n+2}$, on $\mathbb{R}$.

Exercise 1.8. 1. Let $F(x)=\sum\limits_{n=1}^{\infty} \dfrac{\sin nx}{n^3}$. Prove that

(a) $F(x)$ is continuous for all x.

(b) $\lim\limits_{x\to 0} F(x)=0$.

(c) $F'(x)=\sum\limits_{n=1}^{\infty} \dfrac{\cos nx}{n^2}$

2. Prove that $\displaystyle\int_0^{\pi}\left(\frac{\cos 2x}{1.3}+\frac{\cos 4x}{3.5}+\frac{\cos 6x}{5.7}+\ldots\right)dx=0.$

3. Prove that $\displaystyle\lim_{n\to\infty}\int_0^1 \frac{dx}{(1+x/n)^n}=1-e^{-1}.$

Exercise 1.9. Find the sum of the following function series or number series:

a) $\sum\limits_{n=1}^{\infty} nx^n$, $x\in(-1;1)$
c) $\sum\limits_{n=1}^{\infty} (n^2+n)x^{n+1}, x\in(-1,1)$

b) $\sum\limits_{n=1}^{\infty} \dfrac{x^n}{n+1}, x\in(-1,1)$
d) $\sum\limits_{n=1}^{\infty} \dfrac{x^{n+1}}{n(n+1)}$, $x\in(-1;1)$

e) $\sum\limits_{n=1}^{\infty} \frac{x^{4n-3}}{4n-3}$, $x \in (-1;1)$

f) $\sum\limits_{n=1}^{\infty} \frac{(-1)^n.\pi^{2n+1}}{(2n+1)!}$

g) $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{(n+1).2^n}$

h) $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)3^n}$

i) $\sum\limits_{n=1}^{\infty} \frac{3n+1}{8^n}$

j) $\sum\limits_{n=1}^{\infty} \frac{1}{(2n)!!}$

Exercise 1.10. Find the Maclaurin series of the following functions:

a) $y = \sin^2 x \cos^2 x$

b) $y = \sin x \sin 3x$

c) $y = e^{2x} + 3x \cos x$

d) $y = \frac{2x+1}{x^2 - 3x + 2}$

e) $y = \frac{2x-1}{x^2+2x-3}$

f) $y = \frac{1}{x^2+x+1}$

g) $y = \frac{1}{\sqrt{4-x^2}}$

h) $y = \ln(1+2x)$

i) $y = x\ln(x+2)$

j) $y = \ln(1+x-2x^2)$

k) $y = \arcsin x$

Exercise 1.11. Find the Taylor series of y at the given point:

a) $y = \frac{1}{2x+3}$, $x_0 = 4$

b) $y = \sin \frac{\pi x}{3}$, $x_0 = 1$

c) $y = \sqrt{x}$, $x_0 = 4$

Exercise 1.12. Graph each of the following periodic functions and find their corresponding Fourier series

a) $y = x$, $x \in (-\pi, \pi)$, $T = 2\pi$

b) $y = |x|$, $x \in (-\pi, \pi)$, $T = 2\pi$

c) $y = \begin{cases} 4, & 0 < x < 2, \\ -4, & 2 < x < 4 \end{cases}$, $T = 4$

d) $y = \begin{cases} 2x, & 0 \le x < 3, \\ 0, & -3 < x < 0 \end{cases}$, $T = 6$

e) $y = 2x, 0 < x < 10$, $T = 10$

f) $y = \begin{cases} 2-x, & 0 < x < 4, \\ x-6, & 4 < x < 8 \end{cases}$, $T = 8$

In each part, find the points of discontinuity of the function. To what value does the series converge at those points?

Exercise 1.13. Expand the function in a Fourier series

a) $f(x) = x, x \in [0, \pi]$, $f(x)$ is an odd and periodic function of $T = 2\pi$.

b) $f(x) = 2 - x, x \in (0, 2)$, $f(x)$ is an even and periodic of $T = 4$.

c) $f(x) = x + 1$, $x \in [0, \pi)$.

d) $f(x) = x - 1$, $x \in (0, \pi)$ into a Fourier sine series.

e) $f(x) = x(\pi - x)$, $x \in [0, \pi]$ into a Fourier cosine series. Then prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$