Fundamentals of Optimization

Modelling

Pham Quang Dung

dungpq@soict.hust.edu.vn

Department of Computer Science

Content

- Optimization problems
- Modelling overview
- Examples

Optimization problems

- Maximize or minimize some function relative to some set (range of choices)
- The function represents the quality of the choice, indicating which is the “best”

Modelling overview

- Modelling consists of specifying
  - Decision variables
  - Constraints
  - Objective functions
- A problem can be modelled in different ways (how to define variables)
- Take into account the modelling languages of software tools
  - Constraint Programming solvers: constraints can be stated in flexible ways

Constraint linearization

- Motivation
  - Linear Programming solvers are very efficient
- Examples
  - How to model $X = \min\{x_1, x_2\}$?

Solution: define an auxiliary binary variable y, use big constant M

- $x_1 \geq X$
- $x_2 \geq X$
- $X \geq x_1 - M(1-y)$
- $X \geq x_2 - My$

Constraint linearization

- Examples
  - How to model $(x = 1) \Rightarrow (z \geq y)$ where x is binary variable, y and z are real variables ?

Solution: use big constant M

- $M(x-1) + y \leq z$

Balanced Course Assignment Problem

Description

- At the beginning of the semester, the head of a computer science department D have to assign courses to teachers in a balanced way. The department D has m teachers $T=\{1, 2, ..., m\}$ and n courses $C=\{1, 2,..., n\}$.
  - Each teacher $t \in T$ has a preference list which is a list of courses he/she can teach depending on his/her specialization. The preference information is represented by a 0-1 matrix $A_{m \times n}$ in which $A(t,c) = 1$ indicates that teacher t can teach the course c and $A(t,c) = 0$, otherwise
  - We know a set B of pairs of conflicting two courses that cannot be assigned to the same teacher as these courses have been already scheduled in the same slot of the timetable.
  - The load of a teacher is the number of courses assigned to her/him. How to assign n courses to m teacher such that each course assigned to a teacher is in his/her preference list, no two conflicting courses are assigned to the same teacher, and the maximal load among teachers is minimal.

Balanced Course Assignment Problem

Description

- Example

Course	0	1	2	3	4	5	6	7	8	9	10	11	12
credits	3	3	4	3	4	3	3	3	4	3	3	4	4

Teachers	Preference Courses
0	0, 2, 3, 4, 8, 10
1	0, 1, 3, 5, 6, 7, 8
2	1, 2, 3, 7, 9, 11, 12

Conflicting courses

0	2
0	4
0	8
1	4
1	10
3	7
3	9
5	11
5	12
6	8
6	12

Balanced Course Assignment Problem
Description

- Example

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Conflicting courses

0	2
0	4
0	8
1	4
1	10
3	7
3	9
5	11
5	12
6	8
6	12

Teacher	Assigned courses	Load
0	2, 4, 8, 10	15
1	0, 1, 3, 5, 6	15
2	7, 9, 11, 12	14

Balanced Course Assignment Problem

Constraint Programming (CP) model

- Decision variables
  - $X(i)$: teacher assigned to course i, $\forall i \in C$, domain $D(X(i)) = \{t \in T \mid A(t,i) = 1\}$
  - $Y(i)$: load of teacher i, domain $D(Y(i)) = \{0,1,\ldots,n\}$
  - Z: maximum load among teachers
- Constraints
  - $X(i) \neq X(j)$, $\forall (i,j) \in B$
  - $Y(i) = \sum_{j \in C} (X(j) = i)$, $\forall i \in T$
  - $Z \geq Y(i)$, $\forall i \in T$
- Objective function to be minimized: Z

Balanced Course Assignment Problem

Integer Linear Programming (ILP) model

- Decision variables
  - $X(i,j) = 1$: teacher i is assigned to course j, and $X(i,j) = 0$, otherwise, $\forall\ i \in T, j \in C$, domain $D(X(i,j)) = \{0,1\}$
  - $Y(i)$: load of teacher i, domain $D(Y(i)) = \{0,1,\ldots,n\}$
  - Z: maximum load among teachers
- Constraints
  - $\sum_{i \in T} X(i,j) = 1,\ ,\ \forall j \in C$
  - $X(t,i) + X(t,j) \leq 1,\ \forall (i,j) \in B,\ t \in T$
  - $Y(i) = \sum_{j \in C} X(i,j),\ \forall i \in T$
  - $Z \geq Y(i),\ \forall i \in T$
- Objective function to be minimized: Z

Travelling Salesman Problem

Description

- A salesman departs from point 0, visiting N points 1, 2, …, N and comes back to the point 0. The travelling distance from point i and point j is $d(i,j)$, $i,j = 0,1,\ldots,N$. Compute the route of minimal total travelling distance

Travelling Salesman Problem

- Decision variables
  - Binary variable $X(i,j) = 1$ if the route traverses from point i to point j, and $X(i,j) = 0$, otherwise.
- Constraints
  - $\sum_{j=1}^{N} X(i,j) = \sum_{j=1}^{N} X(j,i) = 1, \ \forall i \in \{1,2,\ldots,N\}$
  - $\sum_{(i,j)\in S} X(i,j) \leq |S| - 1, \ \forall\ S \subseteq \{1,2,\ldots,N\}$ and $|S| < N$
- Objective function to be minimized

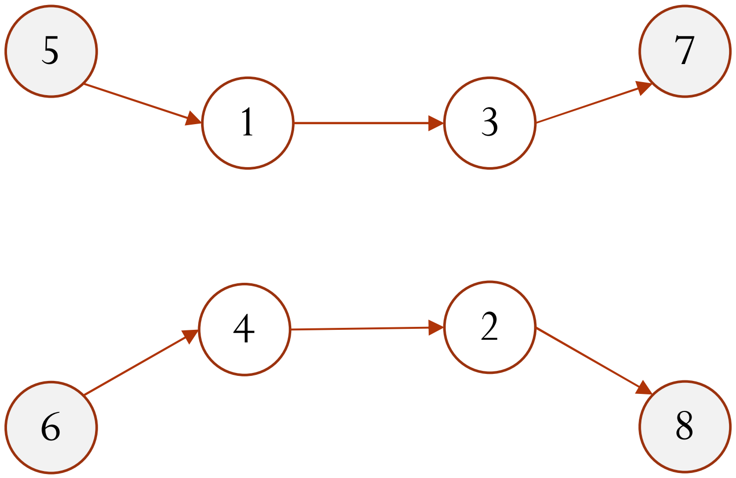
$$f(X) = \sum_{j=1}^{N} \sum_{i=1}^{N} d(i,j) X(i,j)$$

Capacitated Vehicle Routing Problem
Description

- A fleet of K trucks 1, 2, …, K must be scheduled to visit N customers 1, 2, …, N for collecting items
  - Customer i located at point i and requests to be collected $r(i)$ items, i = 1, 2, …, N
  - Truck k (k = 1,…, K)
    - Departs from point $N+k$ and terminates at point $N + K + k$ ($N+k$ and $N+K+k$ might refer to the central depot)
    - has capacity $c(k)$ which is the maximum number of items it can carry at a time
  - Travel distance from point i to point j is $d(i,j)$, $i,j = 1,\ldots,N + 2K$
- Compute the delivery solution such that the total travelling distance is minimal
  - Satisfy capacity constraints
  - Each customer is visited exactly once by exactly one truck

Capacitated Vehicle Routing Problem

Description



	1	2	3	4	5	6	7	8
1	**0**	**2**	**3**	**4**	**3**	**3**	**3**	**3**
2	**4**	**0**	**2**	**6**	**1**	**1**	**1**	**1**
3	**2**	**4**	**0**	**2**	**1**	**1**	**1**	**1**
4	**5**	**7**	**7**	**0**	**4**	**4**	**4**	**4**
5	**3**	**1**	**5**	**7**	**0**	**0**	**0**	**0**
6	**3**	**1**	**5**	**7**	**0**	**0**	**0**	**0**
7	**3**	**1**	**5**	**7**	**0**	**0**	**0**	**0**
8	**3**	**1**	**5**	**7**	**0**	**0**	**0**	**0**

	1	2	3	4
r	**4**	**3**	**6**	**5**

	1	2
c	**10**	**10**

Capacitated Vehicle Routing Problem

- Notations
  - $B = \{1,\ldots, N+2K\}$
  - $F_1 = \{(i, k+N) \mid i \in B, k \in \{1,\ldots,K\}\}$
  - $F_2 = \{(k+K+N, i) \mid i \in B, k \in \{1,\ldots,K\}\}$
  - $F_3 = \{(i, i) \mid i \in B\}$
  - $A = B^2 \setminus F_1 \setminus F_2 \setminus F_3$
  - $A^+(i) = \{ j \mid (i, j) \in \mathrm{A}\}$, $A^-(i) = \{ j \mid (j, i) \in \mathrm{A}\}$
- Decision variables
  - $X(k,i,j) = 1$ if truck k travel from point i to point j, $\forall k = 1,\ldots,K$, $(i,j) \in A$
  - $Y(k,i)$: number of items on truck k after leaving point i, $\forall k = 1,\ldots,K$, $\forall i = 1,\ldots,N+2K$
  - $Z(i)$: index of truck visiting point i, $\forall\ i = 1,2,\ldots, N+2K$

Capacitated Vehicle Routing Problem

- Constraints
  - $\sum_{k=1}^{K} \sum_{j \in A^{+}(i)} X(k,i,j) = \sum_{k=1}^{K} \sum_{j \in A^{-}(i)} X(k,j,i,) = 1, \forall\, i = 1,\ldots, N$
  - $\sum_{j \in A^{+}(i)} X(k,i,j) = \sum_{j \in A^{-}(i)} X(k,j,i), \forall i = 1,\ldots,N,\ k = 1,\ldots, K$
  - $\sum_{j=1}^{N} X(k,k+N,j) = \sum_{j=1}^{N} X(k,j,k+K+N) = 1, \forall k = 1,\ldots, K$
  - $M(1-X(k,i,j)) + Z(i) \geq Z(j), \forall\ (i,j) \in A, \forall k = 1,\ldots,K$
  - $M(1-X(k,i,j)) + Z(j) \geq Z(i), \forall\ (i,j) \in A, \forall k = 1,\ldots,K$
  - $M(1-X(k,i,j)) + Y(k,j) \geq Y(k,i) + r(j), \forall\ (i,j) \in A, \forall k = 1,\ldots,K$
  - $M(1-X(k,i,j)) + Y(k,i) + r(j) \geq Y(k,j), \forall\ (i,j) \in A, \forall k = 1,\ldots,K$
  - $Y(k,k+K+N) \leq c(k), \forall k = 1,\ldots,K$
  - $Y(k,k+N) = 0, \forall k = 1,\ldots,K$
  - $Z(k+N) = Z(k+K+N) = k, \forall k = 1,\ldots,K$

Capacitated Vehicle Routing Problem

- Objective function
  - $f(X,Y,Z) = \sum_{k=1}^{K} \sum_{(i,j)\in A} X(k,i,j)d(i,j)$ → min