Infinite series

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Example

We know

$$\begin{aligned} 0,(3) =& 0,3333\dots \\ =& \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \dots \\ =& \frac{3}{10}\left[1 + \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \dots\right] \\ =& \frac{3}{10}\frac{1}{1 - \frac{1}{10}} = \frac{1}{3}. \end{aligned}$$

Definition

Given a sequence $\{a_n\}_{n\geq 1}$. The formal sum

$$a_1 + a_2 + \ldots + a_n + \ldots$$

is called an infinite series, denote by $\sum\limits_{n=1}^{\infty} a_n$.

- a_n: general term.
- $S_n = a_1 + a_2 + \ldots + a_n$: $n-$th partial sum.
- If there exists $\lim\limits_{n\to\infty} S_n = S < \infty$, we say that the series $\sum\limits_{n=1}^{\infty} a_n$ converges, and its sum is S.

  Otherwise, we say that the series $\sum\limits_{n=1}^{\infty} a_n$ diverges.

Example (Geometric series)

Test for convergence and find the sum of the following series

$$\sum_{n=0}^{\infty} aq^n = a + aq + aq^2 + \ldots + aq^n + \ldots, a \neq 0.$$

- The $n-$**th partial sum** is

$$S_n = a + aq + aq^2 + \ldots + aq^{n-1} = \sum_{k=0}^{n-1} aq^k = a\frac{1-q^n}{1-q}.$$

- Passing to the limit as $n \to \infty$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} a\frac{1-q^n}{1-q} = \frac{a}{1-q} - \lim_{n\to\infty} a\frac{q^n}{1-q}$$

- $\sum\limits_{n=0}^{\infty} aq^n$ **converges** $\Leftrightarrow |q| < 1$, $S = \dfrac{a}{1-q}$.

Example

Test for convergence and find the sum of the following series

$$\sum_{n=2}^{\infty} \frac{1}{n(n+1)}$$

- The $n-$th partial sum is

$$S_n = \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{(n+1)(n+2)} = \frac{1}{2} - \frac{1}{n+2}.$$

- Passing to the limit $\lim_{n\to\infty} S_n = \frac{1}{2}$.
- The series is convergent and its sum is $S = \frac{1}{2}$.

Properties

Proposition

1. *If* $\sum_{n=1}^{\infty} a_n = S_1$, *then* $\sum_{n=1}^{\infty} \alpha a_n = \alpha S_1$.

   In particular, $\alpha = -1$: $\sum_{n=1}^{\infty} (-a_n) = -\sum_{n=1}^{\infty} a_n$.

2. *If* $\sum_{n=1}^{\infty} a_n = S_1$ *and* $\sum_{n=1}^{\infty} b_n = S_2$, *then* $\sum_{n=1}^{\infty} (a_n + b_n) = S_1 + S_2$.

3. *The two series* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=n_0}^{\infty} a_n$ *are either both convergent or both divergent. Their sums differ by* $\sum_{k=1}^{n_0-1} a_k$.

4. *If the series* $\sum_{n=1}^{\infty} a_n$ *converges then* $\lim_{n\to\infty} a_n = 0$.

Test for divergence

Corollary

If $\nexists \lim\limits_{n\to\infty} a_n$ or $\lim\limits_{n\to\infty} a_n \neq 0$, then the series $\sum\limits_{n=1}^{\infty} a_n$ diverges.

Remark

The converse is not necessarily true.

$\lim\limits_{n\to\infty} \dfrac{1}{n^2} = \lim\limits_{n\to\infty} \dfrac{1}{n} = 0$. But $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$ converges and $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ diverges.

Example

The following series are divergent

$$\text{a)} \sum_{n=1}^{\infty} 1 \qquad \text{b)} \sum_{n=1}^{\infty} \sin n$$

Fact: $\nexists \lim_{n\to\infty} \sin n, \nexists \lim_{n\to\infty} \cos n.$

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Comparison test I

Theorem

Let $\sum a_n$, $\sum b_n$ be infinite series and $0 \leq a_n \leq b_n$ for all $n \geq N$.
If $\sum b_n$ converges, then $\sum a_n$ converges.
If $\sum a_n$ diverges, then $\sum b_n$ diverges.

Proof

Fact: a bounded, monotone increasing sequence $\{S_n\}$ owns a limit.

$$S_n = a_1 + a_2 + \ldots + a_n \leq b_1 + b_2 + \ldots + b_n = T_n.$$

Example

Test for convergence

$$a)\sum_{n=1}^{\infty}\frac{1}{2^n+3} \qquad b)\sum_{n=1}^{\infty}\frac{1}{\ln n}$$

Quotient test

Theorem

Let $\sum a_n$, $\sum b_n$ be infinite series, $0 \le a_n, b_n$, $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = k$.
If $0 < k < \infty$, then the series $\sum a_n$, $\sum b_n$ either both converge or both diverge.

Remark

- If $k = 0$, $\sum b_n$ converges, then $\sum a_n$ converges.
- If $k = \infty$, $\sum b_n$ diverges, then $\sum a_n$ diverges.

Example

Test for convergence

$$\text{a)}\ \sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n^2+2} \qquad\qquad \text{b)}\ \sum_{n=1}^{\infty} \sin\frac{1}{2^n}$$

Integral test

Theorem

Assume that $f(x)$ is a positive, continuous and monotone decreasing function on $[1;+\infty)$ and $f(n)=a_n$. Then the series $\sum\limits_{n=1}^{\infty} a_n$ and the improper integral $\int\limits_{1}^{\infty} f(x)dx$ are either both convergent or both divergent.

Example

The series $\sum\limits_{n=2}^{\infty} \dfrac{1}{n^p}$ if and only if $p>1$.

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Infinite series

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Absolute and conditional convergence

Definition

- $\sum_{n=1}^{\infty} a_n$ is said to converge absolutely $\Leftrightarrow \sum_{n=1}^{\infty} |a_n|$ converges.
- $\sum_{n=1}^{\infty} a_n$ is said to converge conditionally $\Leftrightarrow \sum_{n=1}^{\infty} |a_n|$ diverges and $\sum_{n=1}^{\infty} a_n$ converges.

Proposition

If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n$ converges.

Example

Test for convergence

$$\text{a)}\ \sum_{n=1}^{\infty} \frac{\sin n^2}{\sqrt{n^3}} \qquad\qquad \text{b)}\ \sum_{n=1}^{\infty} \cos n^2$$

Ratio test

Theorem

Assume that $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = D.$

- *If* $D < 1$, *then the series converges (absolutely).*
- *If* $D > 1$, *then the series diverges.*

Remark

If $D = 1$, the test fails.

Example: $\sum \frac{1}{n^p}$ converges iff $p > 1$, $D = 1$.

Proof

a) $D < 1$. Take $0 < \varepsilon < 1 - D$, then $\forall n \geq N_0$

$$\left| \left| \frac{a_{n+1}}{a_n} \right| - D \right| < \varepsilon \Rightarrow \left| \frac{a_{n+1}}{a_n} \right| < D + \varepsilon < 1$$
$$\Rightarrow |a_{n+1}| < (D+\varepsilon)^{n+1-N_0} |a_{N_0}|$$

By comparison test: the series $\sum\limits_{n=N_0}^{\infty} |a_n|$ converges, hence the given series converges (absolutely).

b) $D > 1$. Take $0 < \varepsilon < D - 1$, $\forall n \geq N_0$:

$$\left| \left| \frac{a_{n+1}}{a_n} \right| - D \right| < \varepsilon \Rightarrow |a_{n+1}| > (D - \varepsilon)|a_n| > |a_n|,$$

hence $\lim\limits_{n\to\infty} a_n \neq 0$, the series diverges.

$n-$th root test

Theorem

Assume that $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = C$*.*

- *If* $C < 1$*, the series converges (absolutely).*
- *If* $C > 1$*, the series diverges.*

Remark

- If $C = 1$ the test fails.
- $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$, $\lim\limits_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^n = e$.

Example

a) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{2n+1}{3n+1}\right)^{2n}$

b) $\displaystyle\sum_{n=1}^{\infty}\frac{3^n}{(2n-1)!}$

c) $\displaystyle\sum_{n=1}^{\infty}\frac{(2n)!!}{n^n}$

d) $\displaystyle\sum_{n=1}^{\infty}\frac{2n^3-2n+1}{2^n(n+1)}\sin\frac{1}{n}$

Alternating series

Definition

Alternating series is the one whose successive terms are alternately positive and negative, namely it is of the form

$$-a_1 + a_2 - a_3 + \ldots + a_{2n} - a_{2n+1} + \ldots = \sum_{n=1}^{\infty} (-1)^n a_n$$

or

$$a_1 - a_2 + a_3 - \ldots + a_{2n-1} - a_{2n} + \ldots = \sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

where $\mathbf{a_n > 0}$.

Alternating series test

Theorem (Leibniz test)

If $\lim_{n\to\infty} a_n = 0$ *and* $a_{n+1} \le a_n, \forall\, n \ge N$*, then the alternating series* $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ *converges. Its sum satisfies* $|S| \le a_1$*.*

Proof.

- The sequence $\{S_{2m}\}$ is increasing and bounded from above, $\lim_{m\to\infty} S_{2m} = S$.
- The sequence $\{S_{2m+1}\}$ is decreasing and bounded from below, $\lim_{m\to\infty} S_{2m+1} = S'$.
- $S_{2m+1} = a_{2m+1} + S_{2m}$, passing to the limit $m \to \infty$, then $S = S'$.

Example

Test for convergence:

$$\text{a)}\ \sum_{n=1}^{\infty}\frac{\cos n\pi}{\sqrt{n}} \quad \text{b)}\ \sum_{n=1}^{\infty}(-1)^n\frac{\ln n}{n} \quad \text{c)}\ \sum_{n=1}^{\infty}\frac{(-1)^{n^2}.n}{\sqrt{2n^2+1}} \quad \text{d)}\ \sum_{n=1}^{\infty}\frac{(-1)^n}{n\sqrt{n}}$$

- Commutativity and associativity hold for finite sums.

Example

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n} = \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \ldots > 0$$

But

$$\frac{1}{2} - \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{7} - \frac{1}{9} + \ldots < 0$$

- Commutativity and associativity hold for **absolutely convergent series**.

Properties of absolutely convergent series

Proposition

1. *The terms of an absolutely convergent series can be rearranged in any order or grouped without changing the sum.*
2. *The terms of a conditionally convergent series can be suitably rearranged or grouped to result a series which may diverge or converge to any desired sum.*