Chapter 2
RANDOM VARIABLES AND PROBABILITY DISTRIBUTIONS

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Introduction

- Chapter 1 defines a probability model.
- In this chapter and for most of the remainder of the course, we will examine probability models that assign numbers to the outcomes in the sample space.
- We shall use a capital letter, say X, to denote a random variable and its corresponding small letter, x in this case, for one of its values.
- The set of possible values of X is the range of X: S_X.
- A probability model always begins with an experiment. Each random variable is related directly to this experiment. There are three types of relationships.
  1. The random variable is the **observation**.
  2. The random variable is a **function of the observation**.
  3. The random variable is a **function of another random variable**.

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2.1.1 Random Variable

Definition 1.

A **random variable** is a function that associates a real number with each element in the sample space.

Example 1.

Here are some random variables:

1. X, the number of students asleep in the next probability lecture.
2. Y, the number of phone calls you answer in the next hour.
3. Z, the number of minutes you wait until you next answer the phone.

Note

1. Random variables X and Y are **discrete** random variables. The possible values of these random variables form a **countable set**. The underlying experiments have sample spaces that are discrete.
2. The random variable Z can be any nonnegative real number. It is a **continuous** random variable. Its experiment has a continuous sample space.

2.1.2 Discrete Random Variable

Definition 2.

If a sample space contains a finite number of possibilities or an unending sequence with as many elements as there are whole numbers, it is called a **discrete sample space**.

Definition 3.

X is a discrete random variable if the range of X is a countable set

$$S_X = \{x_1, x_2, \dots\}.$$

Definition 4.

X is a **finite** random variable if the range is a finite set

$$S_X = \{x_1, x_2, \dots, x_n\}.$$

Note

A random variable whose set of possible values is an entire interval of numbers is not discrete.

2.1.3 Continuous Random Variable

Definition 5.

If a sample space contains an infinite number of possibilities equal to the number of points on a line segment, it is called a **continuous sample space**.

Definition 6.

When a random variable can take on values on a continuous scale, it is called a **continuous random variable**.

Example 2.

Let Y be the random variable defined by the waiting time, in hours, between successive speeders spotted by a radar unit. The random variable Y takes on all values y for which $y \geq 0$. Y is a continuous random variable.

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2.2.1 Probability Function/ Probability Mass Function

Definition 7.
The set of ordered pairs $(x, P_X(x))$ is a **probability function**, or **probability mass function** (PMF) of the discrete random variable X if, for each possible outcome x,

1. $P_X(x) = P[X = x]$.
2. $P_X(x) \geq 0$.
3. $\sum_{x \in S_X} P_X(x) = 1$.

Remark 1.
Note that $[X = x]$ is an event consisting of all outcomes s of the underlying experiment for which $X(s) = x$. On the other hand, $P_X(x)$ is a function ranging over all real numbers x. For any value of x, the function $P_X(x)$ is the probability of the event $[X = x]$.

2.2.1 Probability Function/ Probability Mass Function

Example 3.

Suppose we observe three calls at a telephone switch where voice calls (V) and data calls (D) are equally likely. Let X denote the number of voice calls, Y the number of data calls, and let $R = XY$. The sample space of the experiment and the corresponding values of the random variables X, Y, and R are

	Outcomes	DDD	DDV	DVD	DVV	VDD	VDV	VVD	VVV
	P[.]	1/8	1/8	1/8	1/8	1/8	1/8	1/8	1/8
random variable	X	0	1	1	2	1	2	2	3
random variable	Y	3	2	2	1	2	1	1	0
random variable	R	0	2	2	2	2	2	2	0

What is the probability mass function of R?

2.2.1 Probability Function/ Probability Mass Function

Solution. We see that $R = 0$ if either outcome, DDD or VVV, occurs so that

$$P[R = 0] = P[DDD] + P[VVV] = \frac{1}{4}.$$

For the other six outcomes of the experiment, $R = 2$ so that $P[R = 2] = 6/8$. The PMF of R is

$$P_R(r) = \begin{cases} 1/4, & r = 0, \\ 3/4, & r = 2, \\ 0, & \text{otherwise.} \end{cases}$$

2.2.1 Probability Function/ Probability Mass Function

Definition 8.

The **probability distribution** for a discrete random variable X is a formula, table, or graph that gives the possible values of X, and the probability associated with each value of X.

X	x_1	x_2	...
$P[X = x_i]$	$P[X = x_1]$	$P[X = x_2]$	...

(1)

Note

Requirements for discrete probability distribution:

1. The probability of each value of the discrete random variable is between 0 and 1, inclusive $(0 \leq P[X = x_i] \leq 1,\ i = 1, 2, \ldots)$.
2. The sum of all the probabilities is 1, that is $\sum_i P[X = x_i] = 1$.

2.2.1 Probability Function/ Probability Mass Function

Example 4.

A shipment of 20 similar laptop computers to a retail outlet contains 3 that are defective. If a school makes a random purchase of 2 of these computers, find the probability distribution for the number of defectives.

Solution. Let X be a random variable whose values x are the possible numbers of defective computers purchased by the school. Then X can only take the numbers 0, 1, and 2. Now

$$P[X=0] = \frac{C_3^0 C_{17}^2}{C_{20}^2} = \frac{68}{95}; \quad P[X=1] = \frac{51}{190}; \quad P[X=2] = \frac{3}{190}.$$

Thus, the probability distribution of X is

X	0	1	2
$P[X=x_i]$	$\frac{68}{95}$	$\frac{51}{190}$	$\frac{3}{190}$

2.2.1 Probability Function/ Probability Mass Function

Theorem 1.

For a discrete random variable X with PMF $P_X(x)$ and range S_X. If $B \subset S_X$, the probability that X is in the set B is

$$\boxed{P[B] = \sum_{x \in B} P_X(x)} \qquad (2)$$

Proof. Since the events $[X = x]$ and $[X = y]$ are disjoint when $x \neq y$, B can be written as the union of disjoint events $B = \cup_{x \in B}[X = x]$. Thus,

$$P[B] = \sum_{x \in B} P[X = x] = \sum_{x \in B} P_X(x).$$

2.2.2 Cumulative Distribution Function

Definition 9.
The cumulative distribution function (CDF) $F_X(x)$ of a discrete random variable X with probability distribution $P_X(x)$ is

$$\boxed{F_X(x) = P[X < x] = \sum_{t<x} P_X(t), \quad \textbf{for } -\infty < x < \infty} \tag{3}$$

Theorem 2.
For any discrete random variable X with range $S_X = \{x_1, x_2, \dots\}$ satisfying $x_1 \le x_2 \le \dots$,

1. $0 \le F_X(x) \le 1$.
2. *For all $x_1 < x_2$, $F_X(x_1) \le F_X(x_2)$ and $\lim_{x\to a^-} F_X(x) = F_X(a)$ for all $a \in \mathbb{R}$.*
3. *$F_X(-\infty) = 0$ and $F_X(+\infty) = 1$.*
4. *For $x_i \in S$ and ε, an arbitrarily small positive number, $F_X(x_i + \varepsilon) - F_X(x_i) = P_X(x_i)$.*
5. *$F_X(x) = F_X(x_i)$ for all x such that $x_i < x \le x_{i+1}$.*

2.2.2 Cumulative Distribution Function

Theorem 3.
For all $b \geq a$,

$$\boxed{F_X(b) - F_X(a) = P[a \leq X < b]}$$

Proof. To prove this theorem, express the event $Eab = \{a < X \leq b\}$ as a part of a union of disjoint events. Start with the event $Eb = \{X < b\}$. Note that Eb can be written as the union

$$Eb = \{X < b\} = \{X < a\} \cup \{a \leq X < b\} = Ea \cup Eab.$$

Note also that Ea and Eab are disjoint so that $P[Eb] = P[Ea] + P[Eab]$. Since $P[Eb] = F_X(b)$ and $P[Ea] = F_X(a)$, we can write $F_X(b) = F_X(a) + P[a \leq X < b]$. Therefore

$$P[a \leq X < b] = F_X(b) - F_X(a).$$

2.2.2 Cumulative Distribution Function

Example 5.

In Example 3, we found that random variable R has PMF

$$P_R(r) = \begin{cases} 1/4, & r = 0, \\ 3/4, & r = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Find and sketch the CDF of random variable R.

Solution. From the PMF $P_R(r)$, random variable R has CDF

$$F_R(r) = P[R < r] = \begin{cases} 0, & r \le 0, \\ 1/4, & 0 < r \le 2, \\ 1, & r > 2. \end{cases}$$

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2.3.1 Cumulative Distribution Function

Definition 10.
The **cumulative distribution function** (CDF) of random variable X is

$$\boxed{F_X(x) = P[X < x], \quad x \in \mathbb{R}} \tag{4}$$

Remark 2.
The key properties of the CDF, described in Theorem 2 and Theorem 3, apply to all random variables.

Theorem 4.
For any random variable X,

(a) $F_X(-\infty) = 0$.

(b) $F_X(+\infty) = 1$.

(c) $P[a \leq X < b] = F_X(b) - F_X(a)$.

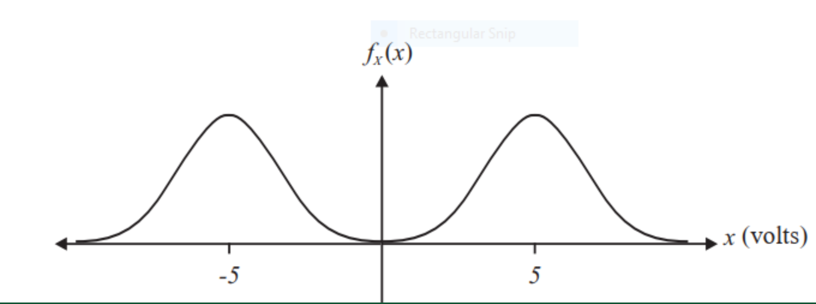
Definition 11.
X is a continuous random variable if the CDF $F_X(x)$ is a continuous function.

2.3.2 Probability Density Function

Definition 12.

The **probability density function** (PDF) of a continuous random variable X is

$$\boxed{f_X(x) = \frac{dF_X(x)}{dx}} \tag{5}$$



The PDF of the modem receiver voltage X

2.3.2 Probability Density Function

Theorem 5.
For a continuous random variable X with PDF $f_X(x)$,

(a) $f_X(x) \geq 0$ *for all* x,

(b) $\displaystyle\int_{-\infty}^{+\infty} f_X(x)dx = 1$,

(c) $\displaystyle F_X(x) = \int_{-\infty}^{x} f_X(u)du$.

Proof.

- The first statement is true because $F_X(x)$ is a nondecreasing function of x and therefore its derivative, $f_X(x)$, is nonnegative.
- The second statement follows from the second one and Theorem 4(b).
- The third fact follows directly from the definition of $f_X(x)$ and the fact that $F_X(-\infty) = 0$.

2.3.2 Probability Density Function

Theorem 6.

$$\boxed{P[a \leq X < b] = \int\limits_a^b f_X(x)dx} \tag{6}$$

Proof. From Theorem 5(b) and Theorem 4,

$$P[a \leq X < b] = P[X < b] - P[X < a] = F_X(b) - F_X(a) = \int\limits_a^b f_X(x)dx.$$

2.3.2 Probability Density Function

Remark 3.

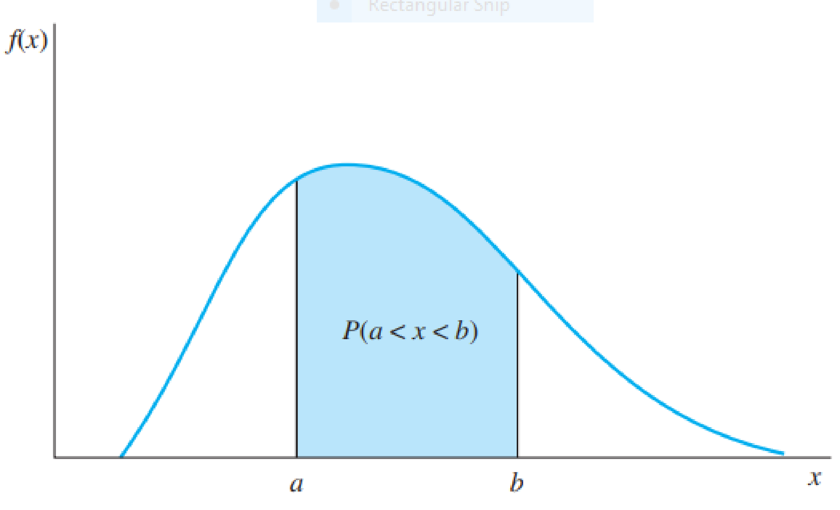
(a) $P[X = a] = 0$ for continuous random variables. This implies that

$$P[X \geq a] = P[X > a] \quad \text{and} \quad P[X \leq a] = P[X < a].$$

This is not true in general for discrete random variables.

(b) The probability that X will fall into a particular interval say, from a to b is equal to the area under the curve between the two points a and b. This is the shaded area in Figure 24.

2.3.2 Probability Density Function



The probability distribution $f_X(x)$; $P[a < X < b]$ is equal to the shaded area under the curve

2.3.2 Probability Density Function

Example 6.

Consider an experiment that consists of spinning the pointer three times and observing Y meters, the maximum value of X in the three spins. The CDF of Y is

$$F_Y(y) = \begin{cases} 0, & y \leq 0, \\ y^3, & 0 < y \leq 1, \\ 1, & y > 1. \end{cases}$$

Find the PDF of Y and the probability that Y is between $1/4$ and $3/4$.

Solution. Applying Definition 12, $f_Y(y) = \begin{cases} 3y^2, & 0 < y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$

Theorem 4 or Theorem 6 can be used to calculate the probability of observing Y between $1/4$ and $3/4$:

$$P[1/4 < Y < 3/4] = F_Y(3/4) - F_Y(1/4) = (3/4)^3 - (1/4)^3 = 13/32,$$

and equivalently, $P[1/4 < Y < 3/4] = \int_{1/4}^{3/4} f_Y(y)dy = \int_{1/4}^{3/4} 3y^2 dy = 13/32.$

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2.4.1 Expected/Mean of a Random Variable

Statisticians work with several kinds of averages. The ones that are used the most are the mean, the median, and the mode.

Example 7.

For one quiz, 10 students have the following grades (on a scale of 0 to 10):

$$9, 5, 10, 8, 4, 7, 5, 5, 8, 7$$

Find the mean, the median, and the mode.

Solution.

- The sum of the ten grades is 68. The mean value is $68/10 = 6,8$.
- The median is 7 since there are four scores below 7 and four scores above 7.
- The mode is 5 since that score occurs more often than any other. It occurs three times.

2.4.1 Expected/Mean of a Random Variable

Definition 13.
A **mode** of random variable X is a number x_{mod} satisfying

$$\boxed{P_X(x_{mod}) \geq P_X(x) \quad \textbf{for all} \quad x} \tag{7}$$

Definition 14.
A **median**, x_{med}, of random variable X is a number that satisfies

$$\boxed{P[X < x_{med}] = P[X \geq x_{med}]} \tag{8}$$

2.4.1 Expected/Mean of a Random Variable

Example 8.

The probability density function of the continuous random variable X is

$$f_X(x) = \begin{cases} \frac{3}{4}x(2-x), & 0 \le x \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

What is x_{mod}? What is x_{med}?

Solution. Applying Theorem 5(c), $F_X(x) = \begin{cases} 0, & x \le 0, \\ \dfrac{3}{4}\left(x^2 - \dfrac{x^3}{3}\right), & 0 < x \le 2, \\ 1, & x > 2. \end{cases}$

So x_{med} is a solution of the equation $F_X(x) = \dfrac{1}{2}$, or $x^3 - 3x^2 + 2 = 0$ with $0 < x \le 2$. Hence $x_{med} = 1$.

Taking the derivative of the PDF $f_X(x)$, $g(x) := f'_X(x) = \begin{cases} 0, & x \le 0, \\ \dfrac{3}{2}(1-x), & 0 < x < 2, \\ 0, & \text{otherwise.} \end{cases}$

We can see the function $g(x)$ reaches maximum at $x = 1$, so $x_{mod} = 1$.

2.4.1 Expected/Mean of a Random Variable

Definition 15.
Let X be a random variable with probability distribution $P_X(x)$, or $f_X(x)$. The **mean value**, or **expected value**, of X is

$$\boxed{\mu_X = E[X] = \sum_{x \in S_X} x P_X(x) \quad \textbf{if } X \textbf{ is discrete}} \tag{9}$$

and

$$\boxed{\mu_X = E[X] = \int_{-\infty}^{+\infty} x f_X(x) dx \quad \textbf{if } X \textbf{ is continuous}} \tag{10}$$

2.4.1 Expected/Mean of a Random Variable

Example 9.

Random variable R in Example 3 has PMF

$$P_R(r) = \begin{cases} 1/4, & r = 0, \\ 3/4, & r = 2, \\ 0. & \text{otherwise.} \end{cases}$$

What is $E[R]$?

Solution.

$$E[R] = \mu_R = (0)P_R(0) + (2)P_R(2) = (0)\left(\frac{1}{4}\right) + (2)\left(\frac{3}{4}\right) = \frac{3}{2}.$$

Example 10.

In Example 6, find the expected value of the maximum stopping point Y of the three spins:

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y) dy = \int_0^1 y(3y^2) dy = 3/4 \text{ meter.}$$

2.4.2 Functions of a Random Variable

Definition 16.
Each sample value y of a derived random variable Y is a mathematical function $g(x)$ of a sample value x of another random variable X. We adopt the notation $Y = g(X)$ to describe the relationship of the two random variables.

Example 11.
The random variable X is the number of pages in a facsimile transmission. Based on experience, you have a probability model $P_X(x)$ for the number of pages in each fax you send. The phone company offers you a new charging plan for faxes: \$0.10 for the first page, \$0.09 for the second page, etc., down to \$0.06 for the fifth page. For all faxes between 6 and 10 pages, the phone company will charge \$0.50 per fax. (It will not accept faxes longer than ten pages.) Find a function $Y = g(X)$ for the charge in cents for sending one fax.

Solution. The following function corresponds to the new charging plan.

$$Y = g(X) = \begin{cases} 10.5X - 0.5X^2, & 1 \leq X \leq 5, \\ 50, & 6 \leq X \leq 10. \end{cases}$$

2.4.2 Functions of a Random Variable

Theorem 7.
For a discrete random variable X, the PMF of $Y = g(X)$ is

$$\boxed{P_Y(y) = \sum_{x:g(x)=y} P_X(x)} \tag{11}$$

Theorem 8.
Let X be a random variable with probability distribution $P_X(x)$, or $f_X(x)$. The expected value of the random variable $Y = g(X)$ is

$$\boxed{\mu_Y = E[g(X)] = \sum_{x \in S_X} g(x)P_X(x) \quad \textbf{if } X \textbf{ is discrete}} \tag{12}$$

and

$$\boxed{\mu_Y = E[g(X)] = \int_{-\infty}^{+\infty} g(x)f_X(x)dx \quad \textbf{if } X \textbf{ is continuous}} \tag{13}$$

2.4.2 Functions of a Random Variable

Example 12.

In Example 11, suppose all your faxes contain $1, 2, 3$, or 4 pages with equal probability. Find the PMF and expected value of Y, the charge for a fax.

Solution. From the problem statement, the number of pages X has PMF

$$P_X(x) = \begin{cases} 1/4, & x = 1, 2, 3, 4, \\ 0, & \text{otherwise.} \end{cases}$$

The charge for the fax, Y, has range $S_Y = \{10, 19, 27, 34\}$ corresponding to $S_X = \{1, 2, 3, 4\}$. Here each value of Y results in a unique value of X. Hence,

$$P_Y(y) = \begin{cases} 1/4, & x = 10, 19, 27, 34, \\ 0, & \text{otherwise.} \end{cases}$$

The expected fax bill is $E[Y] = \dfrac{1}{4}(10 + 19 + 27 + 34) = 22.5$ cents.

2.4.2 Functions of a Random Variable

Example 13.

In Example 12, $P_X(x) = \begin{cases} 1/4, & x = 1, 2, 3, 4, \\ 0, & \text{otherwise} \end{cases}$ and

$$Y = g(X) = \begin{cases} 10.5X - 0.5X^2, & 1 \le X \le 5, \\ 50, & 6 \le X \le 10. \end{cases}$$

What is $E[Y]$?

Solution. Applying Theorem 8 we have

$$\begin{aligned} E[Y] &= \sum_{x=1}^{4} P_X(x)g(x) \\ &= \frac{1}{4}[(10.5)(1) - (0.5)(1)^2] + \frac{1}{4}[(10.5)(2) - (0.5)(2)^2] \\ &\quad + \frac{1}{4}[(10.5)(3) - (0.5)(3)^2] + \frac{1}{4}[(10.5)(4) - (0.5)(4)^2] \\ &= \frac{1}{4}[10 + 19 + 27 + 34] = 22.5 \text{ cents.} \end{aligned}$$

2.4.2 Functions of a Random Variable

Theorem 9.

For any random variable X,

(i) $E[X - \mu_X] = 0$.

(ii) $E[aX + b] = aE[X] + b$.

Corollary 1.

(i) *Setting* $a = 0$, *we see that* $E[b] = b$.

(ii) *Setting* $b = 0$, *we see that* $E[aX] = aE[X]$.

2.4.2 Functions of a Random Variable

Example 14.

Recall that in Examples 3 and 9, we found that R has PMF

$$P_R(r) = \begin{cases} 1/4, & r = 0, \\ 3/4, & r = 2, \\ 0 & \text{otherwise}, \end{cases}$$

and expected value $E[R] = 3/2$. What is the expected value of $V = g(R) = 4R + 7$?

Solution. From Theorem 9(ii),

$$E[V] = E[g(R)] = 4E[R] + 7 = 4 \times \frac{3}{2} + 7 = 13.$$

We can verify this result by applying Theorem 8. Using the PMF $P_R(r)$ given in Example 3, we can write

$$E[V] = g(0)P_R(0) + g(2)P_R(2) = 7 \times \frac{1}{4} + 15 \times \frac{3}{4} = 13.$$

2.4.2 Functions of a Random Variable

Theorem 10.
The expected value of the sum or difference of two or more functions of a random variable X is the sum or difference of the expected values of the functions. That is,

$$\boxed{E[g(X) \pm h(X)] = E[g(X)] \pm E[h(X)]} \tag{14}$$

2.4.2 Functions of a Random Variable

Example 15.

Let X be a random variable with probability distribution as follows:

X	0	1	2	3
$P_X(x)$	$1/3$	$1/2$	0	$1/6$

Find the expected value of $Y = (X-1)^2$.

Solution. Applying Theorem 10 to the function $Y = (X-1)^2$, we can write

$$E[(X-1)^2] = E[X^2] - 2E[X] + E[1].$$

From Corollary 1, $E[1] = 1$, and by direct computation,

$$E[X] = (0)\left(\frac{1}{3}\right) + (1)\left(\frac{1}{2}\right) + (2)(0) + (3)\left(\frac{1}{6}\right) = 1 \quad \text{and}$$

$$E[X^2] = (0)\left(\frac{1}{3}\right) + (1)\left(\frac{1}{2}\right) + (4)(0) + (9)\left(\frac{1}{6}\right) = 2.$$

Hence, $E[(X-1)^2] = 2 - (2)(1) + 1 = 1$.

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2.5.1 Definitions

Definition 17.

The **variance** of random variable X is

$$\boxed{Var[X] = E\left[(X - \mu_X)^2\right]} \tag{15}$$

Definition 18.

The **standard deviation** of random variable X is

$$\boxed{\sigma_X = Var[X]} \tag{16}$$

Remark 4.

- It is useful to take the square root of $Var[X]$ because σ_X has the same units (for example, exam points) as X. The units of the variance are squares of the units of the random variable (exam points squared). Thus σ_X can be compared directly with the expected value.
- We think of sample values within σ_X of the expected value, $x \in [\mu_X - \sigma_X, \mu_X + \sigma_X]$, as "typical" values of X and other values as "unusual."

2.5.1 Definitions

Note

Because $(X-\mu_X)^2$ is a function of X, $Var[X]$ can be computed according to Theorem 8.

$$\boxed{\sigma_X^2 = Var[X] = \sum_{x\in S_X}(x-\mu_X)^2 P_X(x) \quad \textbf{if } X \textbf{ is discrete}} \tag{17}$$

and

$$\boxed{\sigma_X^2 = Var[X] = \int_{-\infty}^{+\infty}(x-\mu_X)^2 f_X(x)dx \quad \textbf{if } X \textbf{ is continuous}} \tag{18}$$

2.5.2 Properties and Examples

Theorem 11.
$Var[X] = E[X^2] - \mu_X^2$ *where*

$$\boxed{E[X^2] = \sum_{x \in S_X} x^2 P_X(x) \quad \textbf{if } X \textbf{ is discrete}} \tag{19}$$

and

$$\boxed{E[X^2] = \int_{-\infty}^{+\infty} x^2 f_X(x)dx \quad \textbf{if } X \textbf{ is continuous}} \tag{20}$$

Definition 19.
For random variable X:

(a) The nth **moment** is $E[X^n]$.

(b) The nth **central moment** is $E[(X - \mu_X)^n]$.

2.5.2 Properties and Examples

Example 16.

In Example 3, we found that random variable R has PMF

$$P_R(r) = \begin{cases} 1/4, & r = 0, \\ 3/4, & r = 2, \\ 0 & \text{otherwise.} \end{cases}$$

In Example 9, we calculated $E[R] = \mu_R = 3/2$. What is the variance of R?

2.5.2 Properties and Examples

Solution. In order of increasing simplicity, we present three ways to compute $Var[R]$.

1. From Definition 17, define $W = (R - \mu_R)^2 = (R - 3/2)^2$. The PMF of W is

$$P_W(w) = \begin{cases} 1/4, & w = (0 - 3/2)^2 = 9/4, \\ 3/4, & w = (2 - 3/2)^2 = 1/4, \\ 0 & \text{otherwise.} \end{cases}$$

   Then $Var[R] = E[W] = (1/4)(9/4) + (3/4)(1/4) = 3/4$.

2. Recall that Theorem 8 produces the same result without requiring the derivation of $P_W(w)$.

$$\begin{aligned} Var[R] &= E[(R - \mu_R)^2] \\ &= (0 - 3/2)^2 P_R(0) + (2 - 3/2)^2 P_R(2) = 3/4. \end{aligned}$$

3. To apply Theorem 11, we find that

$$E[R^2] = 0^2 P_R(0) + 2^2 P_R(2) = 3.$$

   Thus Theorem 11 yields $Var[R] = E[R^2] - \mu_R^2 = 3 - (3/2)^2 = 3/4$.

2.5.2 Properties and Examples

Note

Note that $(X - \mu_X)^2 \geq 0$. Therefore, its expected value is also nonnegative. That is, for any random variable X

$$\boxed{Var[X] \geq 0} \tag{21}$$

The following theorem is related to Theorem $9(ii)$.

Theorem 12.

$Var[aX + b] = a^2 Var[X]$.

2.5.2 Properties and Examples

Example 17.

Find the variance and standard deviation of Y, the maximum pointer position after three spins, in Example 6.

Solution. We proceed as in Example **??**. We have $f_Y(y)$ from Example 6 and $E[Y] = 3/4$ from Example 10:

$$E[Y^2] = \int_{-\infty}^{+\infty} y^2 f_Y(y)dy = \int_0^1 y^2(3y^2)dy = 3/5.$$

Thus the variance is

$$Var[Y] = 3/5 - (3/4)^2 = 3/80 \text{ m}^2,$$

and the standard deviation is $\sigma[Y] = 0.194$ meters.

2.5.2 Properties and Examples

Theorem 13.

Let X be a random variable with probability distribution $P_X(x)$, or $f_X(x)$. The variance of the random variable $Y = g(X)$ is

$$\boxed{\sigma_Y^2 = E[g(X) - \mu_{g(X)}]^2 = \sum_{x \in S_X} [g(x) - \mu_{g(X)}]^2 P_X(x) \quad \textbf{if } X \textbf{ is discrete}} \tag{22}$$

and

$$\boxed{\sigma_Y^2 = E[g(X) - \mu_{g(X)}]^2 = \int_{-\infty}^{+\infty} [g(x) - \mu_{g(X)}]^2 f_X(x) dx \quad \textbf{if } X \textbf{ is continuous}} \tag{23}$$

2.5.2 Properties and Examples

Example 18.

Calculate the variance of $g(X) = 2X + 3$, where X is a random variable with probability distribution

X	0	1	2	3
$P_X(x)$	1/4	1/8	1/2	1/8

Solution. First, we find the mean of the random variable $2X + 3$. According to Theorem 8,

$$\mu_{2X+3} = E[2X+3] = \sum_{x=0}^{3}(2x+3)P_X(x) = 6.$$

Now, using Theorem 13, we have

$$\sigma^2_{2X+3} = E[(2X+3-6)^2] = E[(4X^2 - 12X + 9)] = \sum_{x=0}^{3}(4x^2 - 12x + 9)P_X(x) = 4.$$

2.5.2 Properties and Examples

Example 19.

Let X be a random variable with density function

$$f_X(x) = \begin{cases} \dfrac{x^2}{3}, & -1 < x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

Find the expected value and the variance of $g(X) = 4X + 3$.

Solution. By Theorem 8 we have

$$E[4X+3] = \int_{-1}^{2} \frac{(4x+3)x^2}{3}dx = \frac{1}{3}\int_{-1}^{2}(4x^3+3x^2)dx = 8.$$

Now, using Theorem 13,

$$\sigma^2_{4X+3} = E[(4X+3) - 8^2] = E[(4X-5)^2]$$

$$= \int_{-1}^{2}(4x-5)^2\frac{x^2}{3}dx = \frac{51}{5}.$$