ĐẠI HỌC BÁCH KHOA HÀ NỘI
VIỆN CÔNG NGHỆ THÔNG TIN VÀ TRUYỀN THÔNG

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APPLIED STATISTICS AND EXPERIMENTAL DESIGN

INTRODUCTION

Applied Statistics and Experimental Design

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II. Basics of Probability theory

- 2.1. Probability.
- 2.2. Random variables.
- 2.3. Law of large number
- 2.4. Normal probability distribution (Gaussian distribution).

2.1. Probability

- Probability
  - Laplace classic definition of probability:

$$P(A) = \frac{\text{Number of outcomes favorable to } A}{\text{Total number of possible outcomes}},$$

  - Relative frequency definition of probability:

$$P(A) = \lim_{n \to \infty} \frac{n_A}{n}$$

2.1. Probability

- Kolmogorov axiomatic formulation
  - Ω: sample space: set of all experimental outcomes

    $$\Omega = \{ \xi_1, \xi_2, \dots \xi_k, \dots, \xi_n, \dots \}$$

  - Event – any subset of Ω. Number of subset of sample space : 2^n if $n < \infty$.
  - σ-field *F* of subsets of Ω
  - P: a probability measure on the sets in *F*
    - A – any event
    - 3 axiom of probability

$$\text{(i)} \quad P(A) \geq 0 \quad \text{(Probability is a nonnegative number)}$$

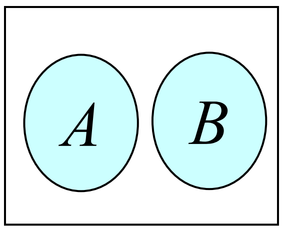
$$\text{(ii)} \quad P(\Omega) = 1 \quad \text{(Probability of the whole set is unity)}$$

$$\text{(iii)} \quad \text{If } A \cap B = \phi, \quad \text{then} \quad P(A \cup B) = P(A) + P(B).$$

  - $< \Omega, F, P >$: probability model

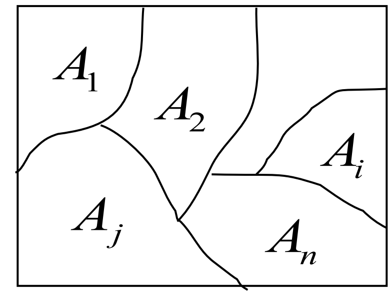
2.1. Probability

- Events: A and B
  - Mutualy exclusive events: $A \cap B = \varnothing$
  - Partition of Ω:



$A \cap B = \phi$

$$A_i \cap A_j = \phi, \quad \text{and} \quad \bigcup_{i=1}^{n} A_i = \Omega$$



  - Example: experiment of tossing two coins simultaneously
    - Elementary events:

$$\xi_1 = (H,H), \quad \xi_2 = (H,T), \quad \xi_3 = (T,H), \quad \xi_4 = (T,T)$$

    - The subset $A = \{ \xi_1, \xi_2, \xi_3 \}$

2.1. Probability

- Conditional probability and independence
  - N independent trials,
  - N_A, N_B, N_{AB} : the number of times events A, B and AB occur.
  - For large N $\quad P(A) \approx \dfrac{N_A}{N}, \quad P(B) \approx \dfrac{N_B}{N}, \quad P(AB) \approx \dfrac{N_{AB}}{N}.$
  - Conditional probability: P(A|B)

$$P(A \mid B) = \frac{N_{AB}}{N_B} = \frac{N_{AB} / N}{N_B / N} = \frac{P(AB)}{P(B)}$$

2.1. Probability

- Properties of conditional probability
  - P(A|B) is nonnegative:

$$P(A \mid B) = \frac{P(AB) \geq 0}{P(B) > 0} \geq 0,$$

  - P(Ω|B) = 1

$$P(\Omega \mid B) = \frac{P(\Omega B)}{P(B)} = \frac{P(B)}{P(B)} = 1,$$

  - If A∩C = ∅ ,

$$P(A \cup C \mid B) = P(A \mid B) + P(C \mid B),$$

2.1. Probability

- If B $\subset$ A then P(A|B) = 1
- If A $\subset$ B then P(A|B) > P(A)
- Let, A_1, A_2, …, A_n are pair wise disjoint and their union is Ω:

$$A_i \cap A_j = \varnothing, \qquad \bigcup_{i=1}^{n} A_i = \Omega.$$

  - B is an event

$$P(B) = \sum_{i=1}^{n} P(BA_i) = \sum_{i=1}^{n} P(B \mid A_i) P(A_i).$$

2.1. Probability

- Independence: *A* and *B* events

  P(AB) = P(A) P(B)

  - If A and B are independent events:

$$P(A \mid B) = P(A)$$

- Bayes theorem

$$P(A \mid B) = \frac{P(B \mid A)}{P(B)} \cdot P(A)$$

- Generalized Bayes theorem:

$$P(A_i \mid B) = \frac{P(B \mid A_i)P(A_i)}{P(B)} = \frac{P(B \mid A_i)P(A_i)}{\sum_{i=1}^{n} P(B \mid A_i)P(A_i)},$$

2.1. Probability

- Bayes' theorem interpretation:
  - *P*(*A*) represents the a-priori probability of the event *A*.
  - The event *B* is new knowledge obtained from an experiment.
  - Conditional probability *P(A|B)* of A given B –
  a-posteriori probability
  - The new information should be used to improve knowledge of *A*.

2.1. Probability

- Example:
  - In a box: 6 white and 4 black balls.
  - Remove two balls randomly without replacement.
  - P{the first one is white and the second one is black} = ?
  - Question: are events W_1 and B_2 independent ?
  - Solution:
    - W1 = “first ball removed is white”
    - *B*2 = “second ball removed is black”
    - $P(W_1 \cap B_2)$ = ?

2.1. Probability

- Example: Two boxes $B1$ and $B2$ contain 100 and 200 light bulbs respectively. The first box ($B1$) has 15 defective bulbs and the second - 5. Suppose a box is selected at random and one bulb is picked out.
  - (a) What is the probability that it is defective?
  - (b) Suppose we test the bulb and it is found to be defective. What is the probability that it came from box 1?

2.1. Probability

- Repeated trials, Bernoulli trials
  - Consider n independent experiments with models (Ω_1, F_1, P_1), (Ω_2, F_2, P_2), ..., (Ω_n, F_n, P_n).
    - Let $\xi_1 \in \Omega_1$, $\xi_2 \in \Omega_2$,..., $\xi_n \in \Omega_2$: elementary events.
    - A joint performance of the n experiments produces an elementary events $\omega = (\xi_1, \xi_2, ..., \xi_n)$.
    - Consider space $\Omega = \Omega_1 \times \Omega_2 \times ... \times \Omega_n : \xi_1 \in \Omega_1, ..., \xi_n \in \Omega_n,$
    - Events in combined space Ω are of the form $A_1 \times A_2 \times ... \times A_n$.
    - If n experiments are independent, then $P(A_1 \times A_2 \times ... \times A_n) = P(A_1) \times ... \times P(A_n)$

2.1. Probability

- An event *A* has probability *p* of occurring in a single trial. Find the probability that *A* occurs exactly *k* times in determined location, $k \le n$ in *n* trials.

$$P_0(\omega) = P(\{\xi_{i_1}, \xi_{i_2}, \cdots, \xi_{i_k}, \cdots, \xi_{i_n}\}) =$$
$$= P(\{\xi_{i_1}\})P(\{\xi_{i_2}\})\cdots P(\{\xi_{i_k}\})\cdots P(\{\xi_{i_n}\}) =$$
$$= \underbrace{P(A)P(A)\cdots P(A)}_{k}\ \underbrace{P(\overline{A})P(\overline{A})\cdots P(\overline{A})}_{n-k} = p^k q^{n-k}.$$

- P{ A occurs exactly *k* time in *n* trials } = $C^k_n p^k q^{n-k}$
  - Bernoulli formula.

2.1. Probability

- De Moivre – Laplace Theorem
  - Guess that: $n \to \infty$ with fixed p.
  - *k* is in the neighborhood $\sqrt{npq}$ of *np*.
  - Bernoulli probability estimation:

$$C_n^k p^k q^{n-k} \approx \frac{1}{\sqrt{2\pi npq}} e^{-(k-np)^2/2npq}.$$

- Stirling formula for n! approximation:

$$n! \sim \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n}$$

2.1. Probability

- Estimation of Bernoulli formula

$$\binom{n}{k} p^k q^{n-k} = \frac{n!}{(n-k)!k!} p^k q^{n-k},$$

$$\binom{n}{k} p^k q^{n-k} > c_1 \sqrt{\frac{n}{2\pi(n-k)k}} \left(\frac{np}{k}\right)^k \left(\frac{nq}{n-k}\right)^{n-k}$$

$$\binom{n}{k} p^k q^{n-k} < c_2 \sqrt{\frac{n}{2\pi(n-k)k}} \left(\frac{np}{k}\right)^k \left(\frac{nq}{n-k}\right)^{n-k}$$

- Các hằng số c_1 và c_2 khá gần nhau.

$$c_1 = e^{\left\{\frac{1}{12n+1} - \frac{1}{12(n-k)} - \frac{1}{12k}\right\}} \qquad c_2 = e^{\left\{\frac{1}{12n} - \frac{1}{12(n-k)+1} - \frac{1}{12k+1}\right\}}.$$

2.1. Probability

- Example:
  - Toss a coin *n* times. Probability of getting *k* heads in *n* trials = ?
  - Consider rolling a fair dice eight times. Find the probability that either 3 or 4 shows up five times.
  - Suppose 5,000 components are ordered. The probability that a part is defective equals 0.1. What is the probability that the total number of defective parts does not exceed 400 ?

2.1. Probability

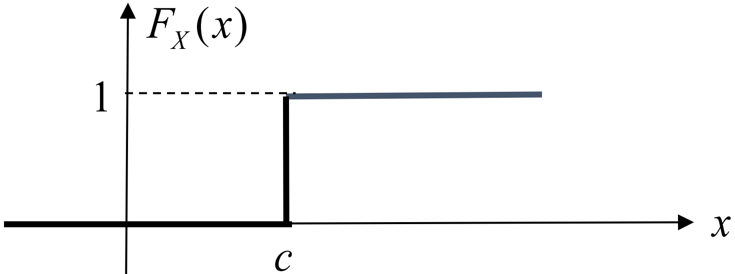
- Homeworks
  - For each of the following, explain why the events A and B are or are not independent.
    - Consider the population of HUST undergraduate students from which one student is selected at random. Let A denote the event that the student is female and let B denote the event that the student is concentrating in education.
    - Consider the population of registered voters, from which one voter is selected at random. Let A denote the event that the voter belongs to a country club and let B denote the event that the voter is a Republican

2.2. Random variables

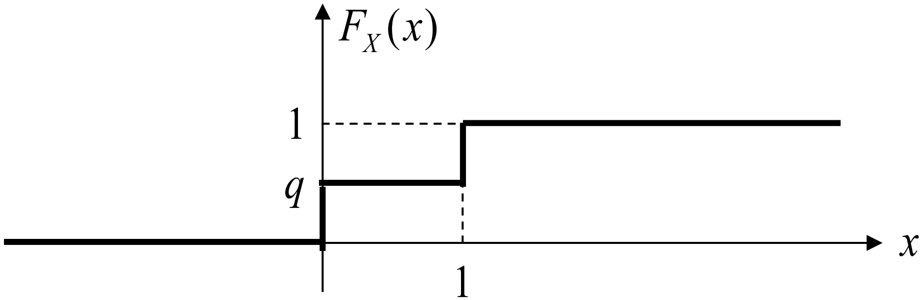
- Definition
  - (Ω, F, P) - probability model for an experiment,
  - X - a function that maps every $\xi \in \Omega$ to a unique point $x \in R$
  - **Random Variable (r.v)**: a finite single valued function that maps the set of all experimental outcomes Ω into the set of real numbers R is said to be a r.v, if the set $A=\{\xi | X(\xi) \leq x\}$ is an event $A \subset F$ for every x in R.

2.2. Random variables

- Probability distribution function
  - Probability distribution function of random variable X: (pdf)
  - $F_X(x)=P\{\xi|X(\xi)\leq x\} \geq 0$
  - Examples
    - X – r.v: $X(\xi) = c$, $\xi\in\Omega$, $F_X(x) = ?$



    - X – r.v: coin tossing, $\Omega = \{H, T\}$. X(T)=0, X(H)=1. $F_X(x) = ?$



2.2. Random variables

- Properties of probability distribution function
  - $F_X(x)$ is a distribution function:
    - $F_X(-\infty) = 0$; $F_X(+\infty) = 1$
    - Pdf is nondecreasing function:

      if $x_1 \leq x_2$ then $F_X(x_1) \leq F_X(x_2)$
    - Pdf is right continuous: $F_X(x^+) = F_X(x)\ \forall x$

2.2. Random variables

- If $F_X(x_0) = 0$ for some x_0,

  then $F_X(x) = 0 \; \forall x \leq x_0$

- $P\{ X(\xi) > x \} = 1 - F_X(x)$

- If $x_2 > x_1$, then

  $P\{ x_1 < X(\xi) \leq x_2 \} = F_X(x_2) - F_X(x_1)$

- $P\{ X(\xi) = x \} = F_X(x) - F_X(x^-)$

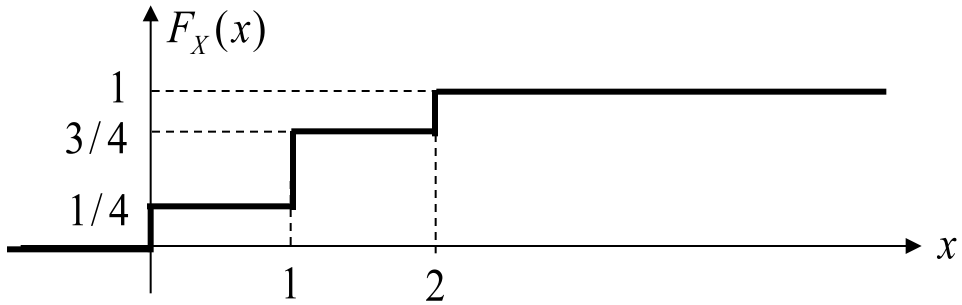
2.2. Random variables

- Continuous and discrete random variables
  - X – continuous r.v if $F_X(x)$ is continuous
    - For continuous r.v, $F_X(x^-) = F_X(x)$ and $P\{X=x\} = 0$
  - If $F_X(x)$ = const, except for a finite number of jump discontinuities, then *X* is said to be a discrete-type r.v.
    - If x_i is such a discontinuity point, then

$$p_i = P\{ X = x_i \} = F_X(x) - F_X(x^-)$$

2.2. Random variables

- Examples
  - A fair coin is tossed twice, and let the r.v *X* represent the number of heads. Find $F_X(x)$



2.2. Random variables

- Distribution density function
  - Derivative of distribution function $F_X(x)$ is density function $f_X(x)$ of the r.v.

$$f_X(x) = \frac{dF_X(x)}{dx}.$$

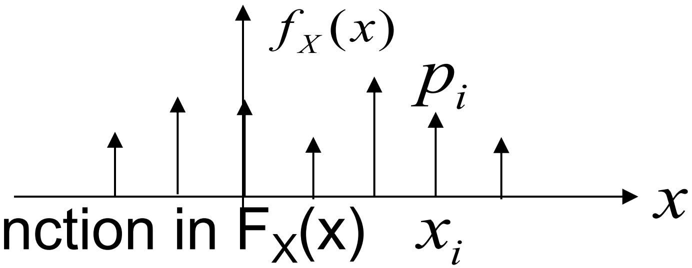
  - $F_X(x)$ is monoton nondecreasing function, so that

$$f_X(x) = \frac{dF_X(x)}{dx} = \lim_{\Delta x \to 0} \frac{F_X(x+\Delta x) - F_X(x)}{\Delta x} \geq 0,$$

2.2. Random variables

- If r.v is continuous then $f_X(x)$ is continuous.
- For discrete r.v

$$f_X(x) = \sum_i p_i \delta(x - x_i),$$



- x_i is jump-discontinuity point function in $F_X(x)$
-

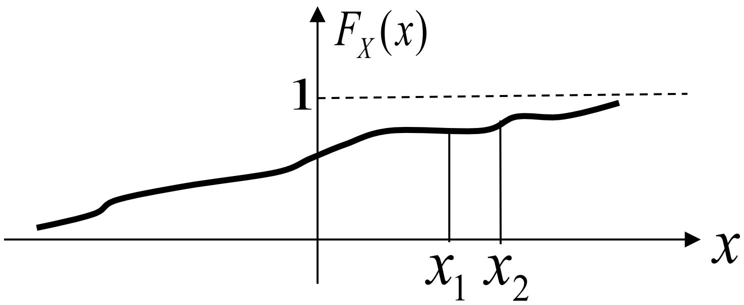
$$F_X(x) = \int_{-\infty}^{x} f_x(u)du.$$

$$F_X(+\infty) = 1, \qquad \int_{-\infty}^{+\infty} f_x(x)dx = 1,$$

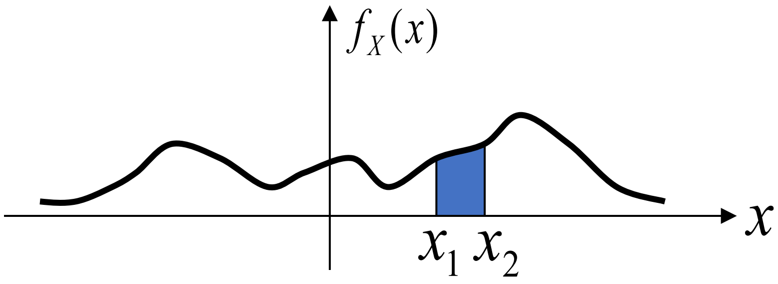
$$P\{ x_1 < X(\xi) \le x_2 \} = F_X(x_2) - F_X(x_1) = \int_{x_1}^{x_2} f_X(x)dx.$$

2.2. Random variables

$$P\{\,x_1 < X(\xi) \le x_2\,\} = F_X(x_2) - F_X(x_1) = \int_{x_1}^{x_2} f_X(x)\,dx.$$



(a)

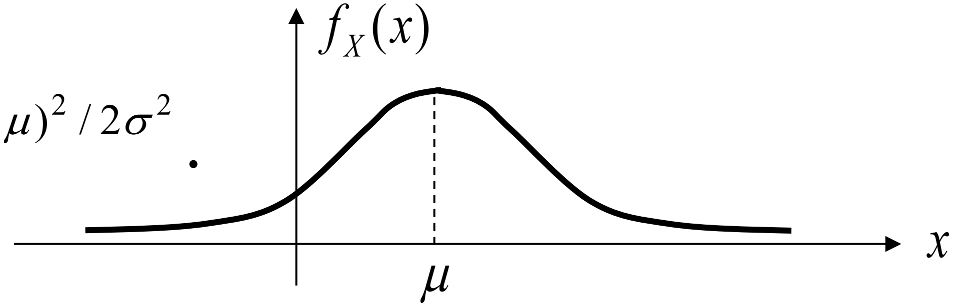


(b)

2.2. Random variables

- Some continuous random variables
  - Normal (Gaussian) random variables $X \sim N(\mu, \sigma^2)$
    - Density function

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}.$$



    - Bell shape curve
    - Distribution function

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(y-\mu)^2/2\sigma^2} dy = G\left(\frac{x-\mu}{\sigma}\right),$$

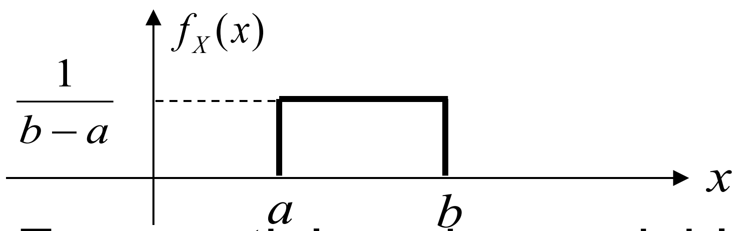
$$G(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$$

2.2. Random variables

- Uniform random variables:
  - Density function

$$X \sim U(a,b),\ a < b,$$

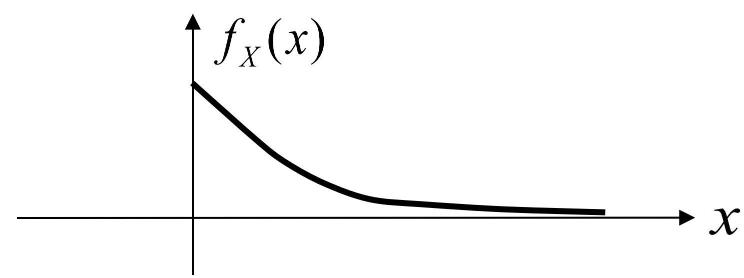
$$f_X(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x \le b, \\ 0, & \text{otherwise.} \end{cases}$$



- Exponential random variables
  - Density function

$$X \sim \varepsilon(\lambda)$$

$$f_X(x) = \begin{cases} \dfrac{1}{\lambda} e^{-x/\lambda}, & x \ge 0, \\ 0, & \text{otherwise.} \end{cases}$$



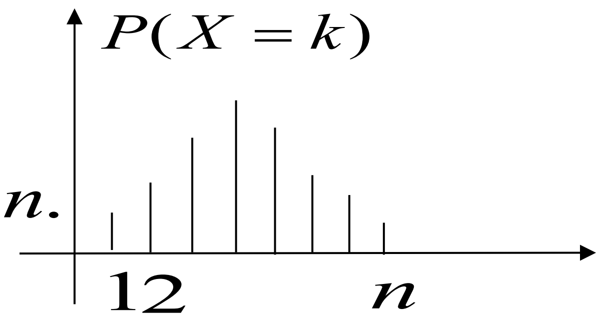
2.2. Random variables

- Some important discrete random variables
  - Bernoulli r.v: X takes values 0, 1

$$P(X=0)=q, \quad P(X=1)=p.$$

  - Binomial r.v $\qquad X \sim B(n,p),$

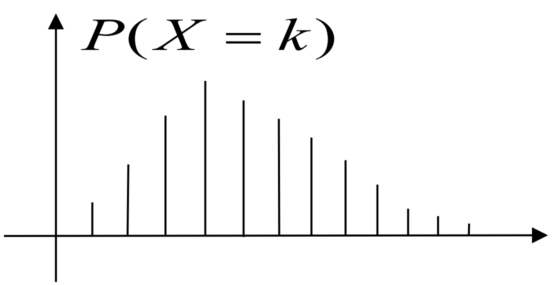
$$P(X=k)=\binom{n}{k}p^k q^{n-k},\ k=0,1,\cdots,n.$$



  - Poisson r.v

$$X \sim P(\lambda),$$

$$P(X=k)=e^{-\lambda}\frac{\lambda^k}{k!},\ k=0,1,\cdots,\infty.$$



2.3 Some characteristics of random variables

- Mean value

$$\eta_X = \overline{X} = E(X) = \int_{-\infty}^{+\infty} x\, f_X(x)dx.$$

  - For discrete random variable

$$\eta_X = \overline{X} = E(X) = \int x \sum_i p_i \delta(x - x_i)dx = \sum_i x_i p_i \underbrace{\int \delta(x - x_i)dx}_{1}$$

$$= \sum_i x_i p_i = \sum_i x_i\, P(X = x_i)\,.$$

  - Example: uniform random variable
  - Example: exponential random variable

2.3 Some characteristics of random variables

- Variance
  - For r.v X with mean μ
  - Variance

$$\sigma_X^2 = E[(X-\mu)^2] > 0.$$

  - Or

$$\sigma_X^2 = \int_{-\infty}^{+\infty} (x-\mu)^2 f_X(x)dx > 0.$$

  - Standard deviation

$$\sigma_X = \sqrt{E(X-\mu)^2}$$

  - Variance anf means

$$Var(X) = \sigma_X^2 = E[(X-\mu)^2] = E[X^2 - 2X\mu + \mu^2] =$$

$$E[X^2] - E[2X\mu] + E[\mu^2] = E[X^2] - 2E[X]\mu + \mu^2$$

$$= E(X^2) - [E(X)]^2 = \overline{X^2} - \overline{X}^2.$$

2.3 Some characteristics of random variables

- Example: variance of Poisson r.v.
- Example: variance of Gaussian r.v.
- Example

2.3 Some characteristics of random variables

- Moments
  - X – random variables

$$m_n = \overline{X^n} = E(X^n), \quad n \geq 1$$

  - Central moments

$$\mu_n = E[(X-\mu)^n]$$

  - Relation between moments and central moments

$$\mu_n = E[(X-\mu)^n] = \sum_{k=0}^{n} C_n^k \ m_k \ (-\mu)^{n-k}.$$

  - Mean and variance:

$$\mu = m_1, \quad \sigma^2 = \mu_2.$$

2.3 Some characteristics of random variables

- Generalized moments of X about *a*

$$E[(X-a)^n]$$

- Absolute moments of X

$$E[|X|^n]$$

2.3 Some characteristics of random variables

- Characteristic function
  - Continuous r.v X

$$\Phi_X(\omega) \stackrel{\Delta}{=} E\left(e^{jX\omega}\right) = \int_{-\infty}^{+\infty} e^{jx\omega} f_X(x)dx.$$

    - We have:

$$\Phi_X(0) = 1,$$

    - And

$$\left|\Phi_X(\omega)\right| \le 1 \qquad \forall \omega$$

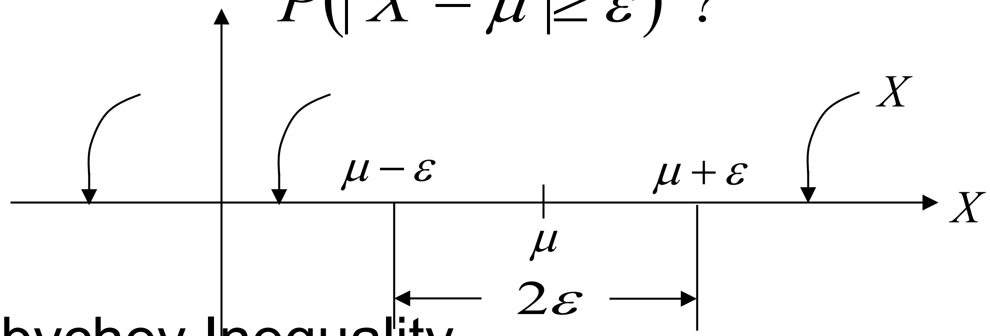
  - Discrete r.v X

$$\Phi_X(\omega) = \sum_k e^{jk\omega} P(X = k).$$

2.4 Chebychev Inequality and Law of Large numbers

- Chebychev Inequality
  - Consider an interval of width 2ε symmetrically centered around its mean μ

$$P(|X-\mu| \geq \varepsilon)\ ?$$



  - Chebychev Inequality

$$P(|X-\mu| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2},$$

2.4 Chebychev Inequality and Law of Large numbers

- Weak law and strong law of large numbers
  - X_i - independent, identically distributed Bernoulli random variables:

$$P(X_i) = p, \qquad P(X_i = 0) = 1 - p = q,$$

  - k = $X_1 + X_2 + ... + X_n$ – number of successes in *n* trials
  - Weak law of large numbers:

$$P\left\{ \left| \frac{k}{n} - p \right| > \varepsilon \right\} \leq \frac{pq}{n\varepsilon^2}.$$

  - Strong law of large numbers:
    - The ratio *k/n* tends to *p* not only *in probability*, but *with probability* 1

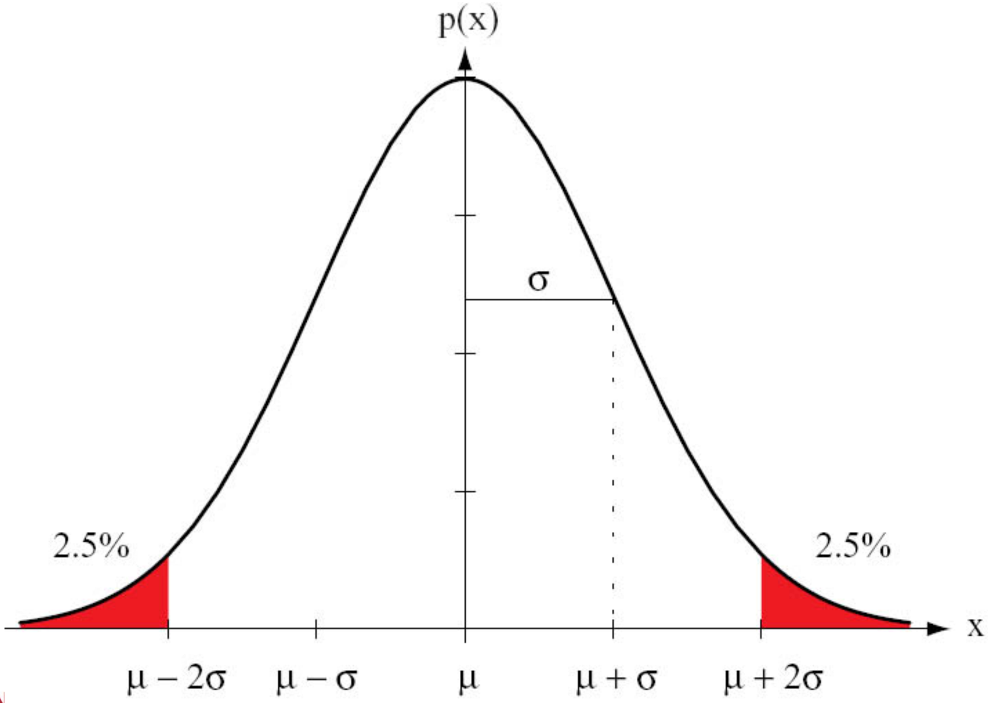
2.5. Gaussian distribution

- Gaussian distribution
  - One dimentional normal distribution

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2} \left(\frac{x-\mu}{\sigma} \right)^2 \right]$$

$$\mu \equiv E(x) = \int_{-\infty}^{\infty} x p(x) dx$$

$$\sigma^2 \equiv E((x-\mu)^2) = \int_{-\infty}^{\infty} (x-\mu)^2 p(x) dx$$



2.5. Gaussian distribution

- Multidimentional Gaussian distribution (*d* dimentional)

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}\left|\sum\right|^{1/2}} \exp\left[-\frac{1}{2}(\mathbf{x}-\mu)^T \sum{}^{-1} (\mathbf{x}-\mu)\right]$$

$$\mu \equiv E(\mathbf{x}) = \int \mathbf{x} p(\mathbf{x}) d\mathbf{x}$$

- Where, **x**, **μ** are *d*-dimentional vector
- ∑ covariance matrix - symmetric and semi-positive defined matrix.
- Consider cases, when ∑ positive defined.

$$\Sigma = E((\mathbf{x}-\mu)(\mathbf{x}-\mu)^T) = \int (\mathbf{x}-\mu)(\mathbf{x}-\mu)^T p(\mathbf{x}) d\mathbf{x}$$

$$\mu_i = E(x_i)$$

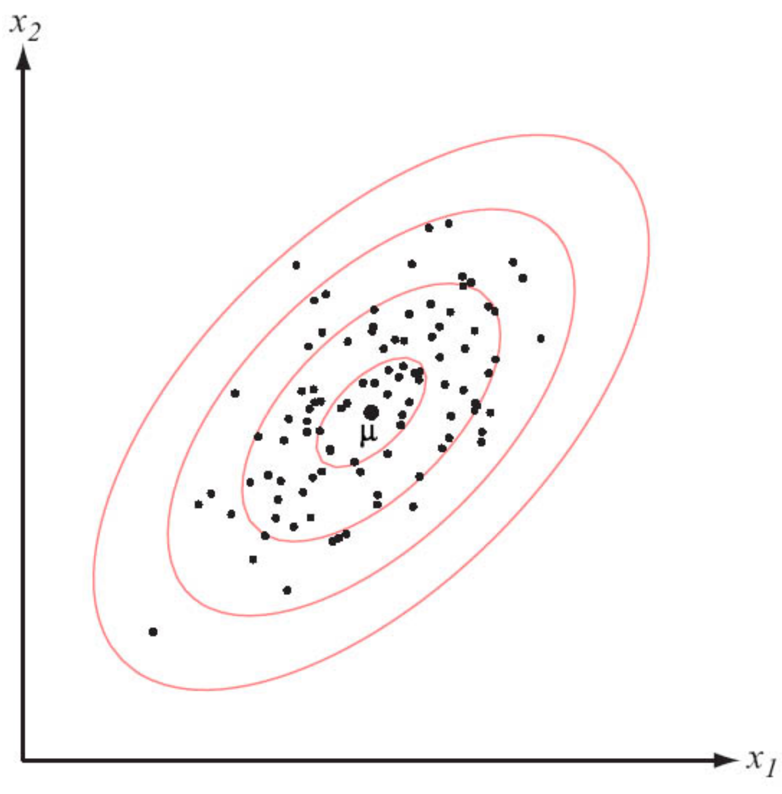
$$\sigma_{ij} = E((x_i - \mu_i)(x_j - \mu_j))$$

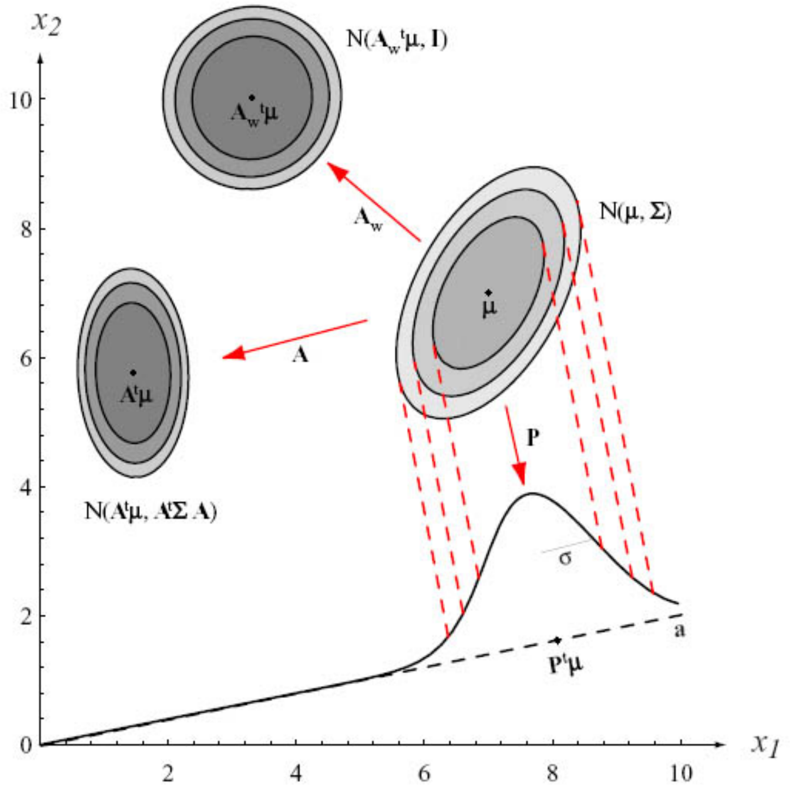
2.5. Gaussian distribution

- Covariance matrix

$$\Sigma = \begin{bmatrix} \sigma_{00} & & & \\ & \sigma_{11} & \sigma_{ij, i \neq j} & \\ & & \sigma_{22} & \\ & & & \sigma_{33} \end{bmatrix}$$

2.5. Gaussian distribution





2.5. Gaussian distribution

- Multidimentional Gaussian distribution is fully defined by d+d(d+1)/2 parameters of mean vector $\boldsymbol{\mu}$ and covariance matrix Σ.
- Mahalanobis distance:

$$r = (\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu})$$

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