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PRESENTATION TITLE

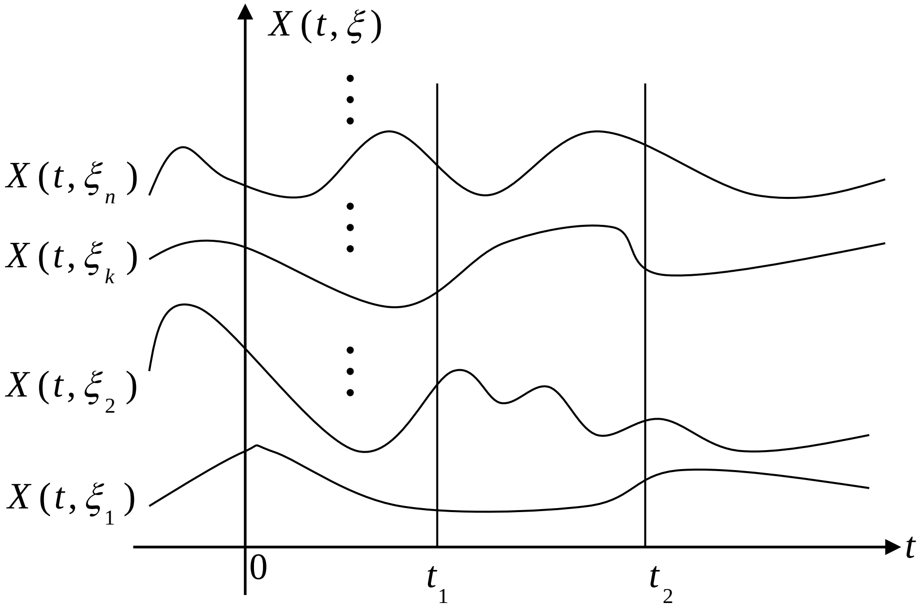
Presentation Subtitle

IV. Stochastic processes

- Definitions
- Stationary processes
- Spectral density function
- Ergodicity of stochastic processes

IV. Stochastic processes
4.1. Definitions

- Stochastic processes
  - Let ξ denote the random outcome of an experiment. To every such outcome suppose a waveform X(t, ξ) is assigned.
  - The collection of such waveforms form a stochastic process.



IV. Stochastic processes
4.1. Definitions

- The set of $\{\xi_k\}$ and the time index *t* can be continuous or discrete (countably infinite or finite) as well.
- For fixed $\xi_i \in S$ (the set of all experimental outcomes), $X(t, \xi_i)$ is a specific time function.
- For fixed *t*, $X_1=X(t_1, \xi_i)$ is a random variable. The ensemble of all such realizations $X(t, \xi)$ over time represents the stochastic process $X(t)$.

IV. Stochastic processes
4.1. Definitions

- Example
  - $X(t)=A\cos(\omega_0 t+\varphi)$
    - Where φ is a uniformly distributed random variable in $(0, 2\pi)$ represents a stochastic process.
- Some stochastic processes
  - Brownian motion,
  - Stock market fluctuations,
  - Various queuing systems.

IV. Stochastic processes
4.1. Definitions

- If *X*(*t*) is a stochastic process, then for fixed *t*, *X*(*t*) represents a random variable.
- Its distribution function is given by

$$F_X(x,t) = P\{X(t) \le x\}$$

- Notice that $F_X(x, t)$ depends on *t*, since for a different *t*, we obtain a different random variable.

$$f_X(x,t) \triangleq \frac{dF_X(x,t)}{dx}$$

- Derivative of $F_X(x, t)$ represents the first-order probability density function of the process *X*(*t*).

IV. Stochastic processes
4.1. Definitions

- For $t = t_1$ and $t = t_2$,
  - $X(t)$ represents two different random variables $X_1 = X(t_1)$ and $X_2 = X(t_2)$ respectively.
  - Their joint distribution is given by

$$F_X(x_1, x_2, t_1, t_2) = P\{X(t_1) \le x_1, X(t_2) \le x_2\}$$

  - Their joint density function is:

$$f_X(x_1, x_2, t_1, t_2) \triangleq \frac{\partial^2 F_X(x_1, x_2, t_1, t_2)}{\partial x_1 \, \partial x_2}$$

  - And represents the second-order density function of the process $X(t)$.

IV. Stochastic processes 4.1. Definitions

- Similarly $f_x(x_1, ..., x_n, t_1, ..., t_n)$ represents the n^{th} order density function of the process $X(t)$.
- Complete specification of the stochastic process $X(t)$ requires the knowledge of $f_x(x_1, ..., x_n, t_1, ..., t_n)$ for all t_i, $i = 1, ..., n$ and for all n. (an almost impossible task in reality).

IV. Stochastic processes
4.1. Definitions

- Characterisctics of stochastic processes
  - Mean of a stochastic process

$$\mu(t) \triangleq E\{X(t)\} = \int_{-\infty}^{+\infty} x f_X(x,t)dx$$

    - μ(t) represents the mean value of a process *X*(*t*). In general, the mean of a process can depend on the time index *t*.
  - Autocorrelation function of a process *X*(*t*) is defined as

$$R_{XX}(t_1,t_2) \triangleq E\{X(t_1)X^*(t_2)\} = \int\int x_1 x_2^* f_X(x_1,x_2,t_1,t_2)dx_1dx_2$$

    - It represents the interrelationship between the random variables *X*1 = *X*(*t*1) and *X*2 = *X*(*t*2) generated from the process *X*(*t*).

IV. Stochastic processes
4.1. Definitions

- Properties of autocorrelation function

1. $R_{XX}(t_1, t_2) = R^*_{XX}(t_2, t_1) = [E\{X(t_2)X^*(t_1)\}]^*$

2. $R_{XX}(t, t) = E\{|X(t)|^2\} \geq 0.$

(Average instantaneous power)

- 3. $R_{xx}(t_1, t_2)$ represents a nonnegative definite function, i.e., for *any* set of constants $\{a_i\}^n_{i=1}$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j^* R_{XX}(t_i, t_j) \geq 0.$$

  - This equation follows by noticing that: $E\{|Y|^2\} \geq 0$ and

$$Y = \sum_{i=1}^{n} a_i X(t_i).$$

IV. Stochastic processes
4.1. Definitions

- The **autocovariance** function of the process *X*(*t*).

$$C_{XX}(t_1,t_2) = R_{XX}(t_1,t_2) - \mu_X(t_1)\mu_X^*(t_2)$$

- Examples
  - Given

$$z = \int_{-T}^{T} X(t)dt.$$

$$E[|z|^2] = \int_{-T}^{T}\int_{-T}^{T} E\{X(t_1)X^*(t_2)\}dt_1 dt_2$$

$$= \int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1,t_2)dt_1 dt_2$$

IV. Stochastic processes
4.1. Definitions

- Consider process X(t)

$$X(t) = a\cos(\omega_0 t + \varphi), \qquad \varphi \sim U(0,2\pi).$$

$$\mu_X(t) = E\{X(t)\} = aE\{\cos(\omega_0 t + \varphi)\}$$

$$= a\cos\omega_0 t\, E\{\cos\varphi\} - a\sin\omega_0 t\, E\{\sin\varphi\} = 0,$$

$$\text{since} \quad E\{\cos\varphi\} = \tfrac{1}{2\pi}\int_0^{2\pi}\cos\varphi\, d\varphi = 0 = E\{\sin\varphi\}.$$

$$R_{XX}(t_1,t_2) = a^2 E\{\cos(\omega_0 t_1 + \varphi)\cos(\omega_0 t_2 + \varphi)\}$$

$$= \frac{a^2}{2} E\{\cos\omega_0(t_1 - t_2) + \cos(\omega_0(t_1 + t_2) + 2\varphi)\}$$

$$= \frac{a^2}{2}\cos\omega_0(t_1 - t_2).$$

IV. Stochastic processes
4.2. Stationary processes

- Strict sense and wide sense stationarity
  - Stationarity
    - Stationary processes exhibit statistical properties that are invariant to shift in the time index.
    - Thus, for example, second-order stationarity implies that the statistical properties of the pairs $\{X(t_1), X(t_2)\}$ and $\{X(t_1+c), X(t_2+c)\}$ are the same for *any* c.
    - Similarly first-order stationarity implies that the statistical properties of $X(t_i)$ and $X(t_i+c)$ are the same for any c.

IV. Stochastic processes
4.2. Stationary processes

- Strict sense stationarity
  - In strict terms, the statistical properties of a stochastic process are governed by the joint probability density function.
  - A process is n^{th}-order **Strict-Sense Stationary (S.S.S)** if

$$f_X(x_1, x_2, \cdots, x_n; t_1, t_2, \cdots, t_n) = f_X(x_1, x_2, \cdots, x_n; t_1 + c, t_2 + c, \cdots, t_n + c)$$

    for *any c*, where the left side represents the joint density function of the random variables $X_1=X(t_1)$, ..., $X_n=X(t_n)$ and the right side corresponds to the joint density function of the random variables $X'_1=X(t_1+c)$, ..., $X'_n=X(t_n+c)$.
  - A process *X*(*t*) is said to be **strict-sense stationary** if equation above is true for all t_i, i=1, ..., n; n=1, 2, ... and any c.

IV. Stochastic processes
4.2. Stationary processes

- First order strict sense stationary process
  - For any c

$$f_X(x,t) \equiv f_X(x,t+c)$$

  - In particular, if c = -t, then

$$f_X(x,t) = f_X(x)$$

  - That means, the first-order density of *X*(*t*) is independent of *t*. In that case

$$E[X(t)] = \int_{-\infty}^{+\infty} x f(x)dx = \mu, \ \ a \ constant. \qquad (1)$$

IV. Stochastic processes
4.2. Stationary processes

- Second order strict sense stochastic process
  - From definition, we have:

$$f_X(x_1, x_2,\ t_1, t_2) \equiv f_X(x_1, x_2,\ t_1 + c, t_2 + c)$$

  - for any c. If c is choosen so as c = $-t_2$, we get

$$f_X(x_1, x_2,\ t_1, t_2) \equiv f_X(x_1, x_2,\ t_1 - t_2)$$

  - The second order density function of a strict sense stationary process depends only on the difference of the time indices $t_1 - t_2 = \tau$.

IV. Stochastic processes
4.2. Stationary processes

- The autocorrelation function is given by

$$
\begin{aligned}
R_{XX}(t_1, t_2) &\triangleq E\{X(t_1)X^*(t_2)\} \\
&= \int\int x_1 x_2^* f_X(x_1, x_2, \tau = t_1 - t_2) dx_1 dx_2 \\
&= R_{XX}(t_1 - t_2) \triangleq R_{XX}(\tau) = R_{XX}^*(-\tau), \qquad (2)
\end{aligned}
$$

- The autocorrelation function of a second order strict-sense stationary process depends only on the difference of the time indices t_2-t_1=τ

IV. Stochastic processes
4.2. Stationary processes

- Wide sense stationarity
  - A process *X*(*t*) is said to be **Wide-Sense Stationary** if:
  - $E\{X(t)\} = \mu$ and $E\{X(t_1)X^*(t_2)\} = R_{XX}(t_1 - t_2),$
  - Since these equations follow from (1) and (2), strict-sense stationarity always implies wide-sense stationarity.
  - In general, the converse is *not true*
    - Exception: the Gaussian process (normal process).
    - This follows, since if *X*(*t*) is a Gaussian process, then by definition $X_1=X(t_1)$, ..., $X_n=X(t_n)$ are jointly Gaussian random variables for any $t_1,..., t_n$ whose joint characteristic function is given by

$$\phi_X(\omega_1, \omega_2, \cdots, \omega_n) = e^{j\sum_{k=1}^{n}\mu(t_k)\omega_k - \sum_{l,k}^{n}\sum C_{xx}(t_i, t_k)\omega_i\omega_k / 2}$$

IV. Stochastic processes
4.2. Stationary processes

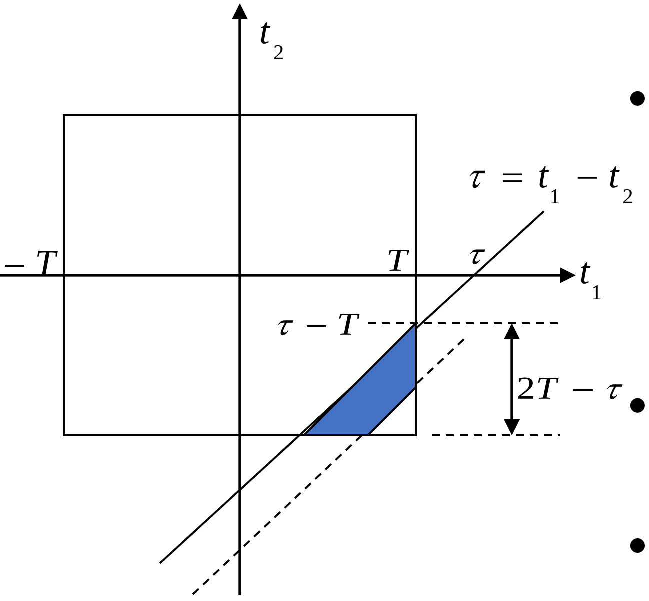
- Examples
  - The process:

$$X(t) = a\cos(\omega_0 t + \varphi), \qquad \varphi \sim U(0,2\pi).$$

    - This process is wide sense stationary but not strict sense stationary.
  - If the process X(t) has zero mean, then σ^2_z is reduced to:

$$z = \int_{-T}^{T} X(t)dt.$$

$$\sigma_z^2 = E\{|z|^2\} = \int_{-T}^{T}\int_{-T}^{T} R_{xx}(t_1 - t_2)dt_1 dt_2.$$



  - As t_1 and t_2 varies from –T to T, so $\tau=t_2-t_1$ varies from -2T to 2T.
  - $R_{xx}(\tau)$ is a constant over the shaded region in the figure on the left.

IV. Stochastic processes
4.3. Power spectrum

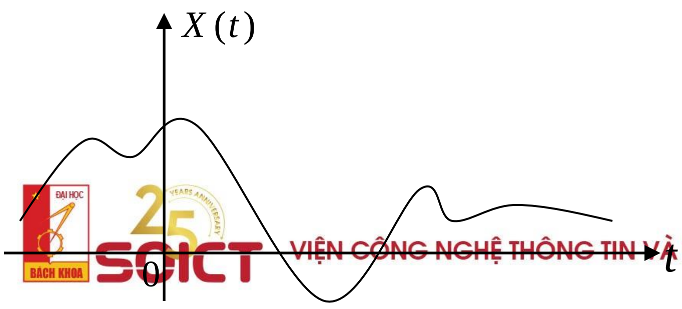
- Power spectrum
  - For a deterministic signal *x*(t)
    - The spectrum is well defined: If X(ω) represents its Fourier transform,

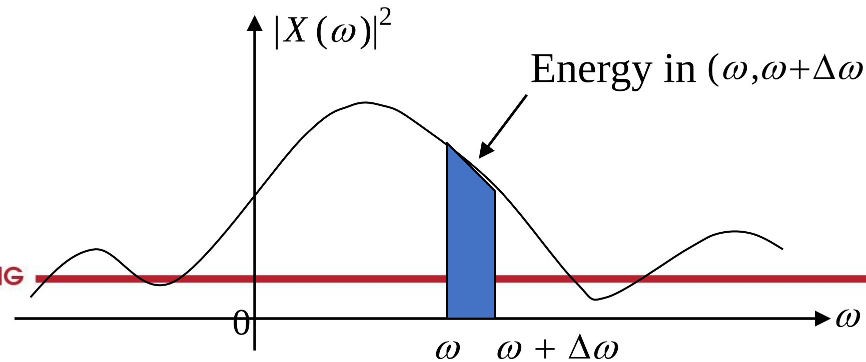
$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt,$$

    - then $|X(\omega)|^2$ represents its energy spectrum. This follows from Parseval's theorem since the signal energy is given by

$$\int_{-\infty}^{+\infty} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(\omega)|^2 \, d\omega = E.$$

    - Thus $|X(\omega)|^2 \Delta\omega$ represents the signal energy in the band $(\omega, \omega+\Delta\omega)$





IV. Stochastic processes
4.3. Power spectrum

- For stochastic processes,
  - A direct application of Fourier transform generates a sequence of random variables for every ω
  - For a stochastic process, $E\{|X(t)|^2\}$ represents the ensemble average power (instantaneous energy) at the instant *t*.
  - Partial Fourier transform of a process $X(t)$ based on $(-T, T)$ is given by

$$X_T(\omega) = \int_{-T}^{T} X(t) e^{-j\omega t} dt$$

  - The power distribution associated with that realization based on $(-T, T)$ is represented by

$$\frac{|X_T(\omega)|^2}{2T} = \frac{1}{2T} \left| \int_{-T}^{T} X(t) e^{-j\omega t} dt \right|^2$$

IV. Stochastic processes
4.3. Power spectrum

- The average power distribution based on (– *T*, *T*) is ensemble average of power distribution for ω

$$P_T(\omega) = E\left\{\frac{|X_T(\omega)|^2}{2T}\right\} = \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} E\{X(t_1)X^*(t_2)\}e^{-j\omega(t_1-t_2)}dt_1dt_2$$

$$= \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1,t_2)e^{-j\omega(t_1-t_2)}dt_1dt_2$$

- We have this represents the power distribution of *X*(*t*) based on (– *T*, *T*).
- If *X*(*t*) is assumed to be w.s.s, then
- and we have $R_{XX}(t_1,t_2) = R_{XX}(t_1-t_2)$

$$P_T(\omega) = \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1-t_2)e^{-j\omega(t_1-t_2)}dt_1dt_2.$$

IV. Stochastic processes
4.3. Power spectrum

- Let $\tau = t_2 - t_1$, we obtain

$$P_T(\omega) = \frac{1}{2T}\int_{-2T}^{2T} R_{XX}(\tau)e^{-j\omega\tau}(2T - |\tau|)d\tau$$

$$= \int_{-2T}^{2T} R_{XX}(\tau)e^{-j\omega\tau}\left(1 - \tfrac{|\tau|}{2T}\right)d\tau \ \geq \ 0$$

- This is the power distribution of the w.s.s. process $X(t)$ based on $(-T, T)$.
- Leting $T \to \infty$, we obtain

$$S_{XX}(\omega) = \lim_{T\to\infty} P_T(\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau)e^{-j\omega\tau}d\tau \geq 0$$

- $S_{xx}(\omega)$ is the *power spectral density* of the w.s.s process $X(t)$.

IV. Stochastic processes
4.3. Power spectrum

- Khinchin-Wiener theorem
  - The autocorrelation function and the power spectrum of a w.s.s process form a Fourier transform pair.

$$R_{XX}(\omega) \xleftrightarrow{\text{F·T}} S_{XX}(\omega) \geq 0.$$

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega) e^{j\omega\tau} d\omega$$

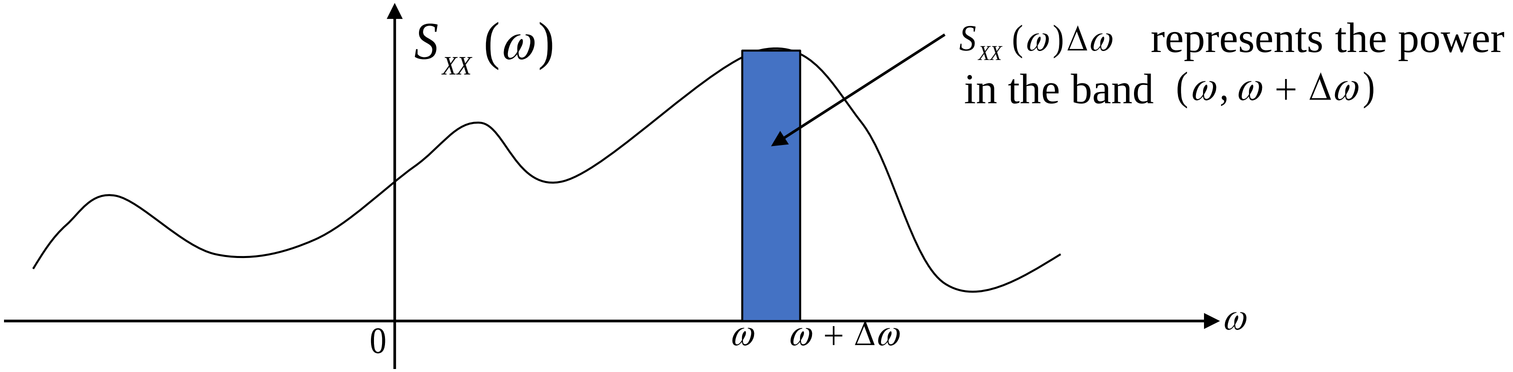
- For τ=0,

$$S_{XX}(\omega) = \lim_{T\to\infty} P_T(\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau \geq 0$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega) d\omega = R_{XX}(0) = E\{|X(t)|^2\} = P, \quad \text{the total power}.$$

IV. Stochastic processes
4.3. Power spectrum

- The area under $S_{xx}(\omega)$ represents the total power of the process $X(t)$, and hence $S_{xx}(\omega)$ truly represents the power spectrum.



- The nonnegative-definiteness property of the auto-correlation function translates into the "nonnegative" property for its Fourier transform (power spectrum).

$$R_{XX}(\tau) \text{ nonnegative-definite} \quad \Leftrightarrow \quad S_{XX}(\omega) \geq 0.$$

IV. Stochastic processes
4.3. Power spectrum

- If $X(t)$ is a real w.s.s process, then

$$R_{XX}(\tau) = R_{XX}(-\tau)$$

- So that

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

$$= \int_{-\infty}^{+\infty} R_{XX}(\tau) \cos \omega\tau d\tau$$

$$= 2\int_{0}^{\infty} R_{XX}(\tau) \cos \omega\tau d\tau = S_{XX}(-\omega) \geq 0$$

- The power spectrum is an even function, (in addition to being real and nonnegative).

IV. Stochastic processes
4.4. Ergodicity

- Time averages
  - Given wide-sense stationary process X(t).
  - Time averages
    - Mean

$$\mathrm{n} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathrm{x}(t)dt$$

    - Autocorrelation

$$\mathrm{R}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathrm{x}(t+\tau)\mathrm{x}(t)dt$$

    - These limits are random variables.
    - Problems:

$$\mathrm{n} \overset{?}{=} E\{\mathrm{x}(t)\} \qquad \mathrm{R}(\tau) \overset{?}{=} E\{\mathrm{x}(t+\tau)\mathrm{x}(t)\}$$

IV. Stochastic processes
4.4. Ergodicity

- Ergodicity
  - X(t) is ergodic if in the most general form if all its statistics can be determined from a single function $X(t, \zeta)$ of the process.
  - X(t) is ergodic if time averages equal ensemble averages (expected values)

IV. Stochastic processes
4.4. Ergodicity

- Ergodicity of the mean
  - Time average of a given process X(t)

$$n_T = \frac{1}{2T}\int_{-T}^{T} x(t)\,dt$$

  - n_T is a random variable.
  - Since E{X(t)} is a constant, we have

$$E\{n_T\} = E\{X(t)\} = \eta$$

  - The variance of n_T is given by:

$$\sigma_{n_T}^{\ 2} = \frac{1}{T}\int_{0}^{2T}\left(1 - \frac{\tau}{2T}\right)[R(\tau) - \eta^2]\,d\tau$$

  - $R(\tau)$ is the autocorrelation of X(t).
  - If this variance tends to zero with $T \to \infty$, then n_T tends to its expected value.

IV. Stochastic processes
4.4. Ergodicity

- Ergodic theorem for E{X(t)}

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\mathrm{x}(t)dt = E\{\mathrm{x}(t)\} = \eta$$

$$iff$$

$$\lim_{T\to\infty}\frac{1}{T}\int_{0}^{2T}\left(1 - \frac{\tau}{2\mathrm{T}}\right)[R(\tau)-\eta^{2}]d\tau = 0$$

IV. Stochastic processes
4.4. Ergodicity

- Ergodicity of autocorrelation
  - We form the average:

$$\mathrm{R}_T(\lambda) = \frac{1}{2T}\int_{-T}^{T} \mathrm{x}(t+\lambda)\mathrm{x}(t)dt$$

  - We have

$$E\{\mathrm{R}_T(\lambda)\} = \frac{1}{2T}\int_{-T}^{T} E\{\mathrm{x}(t+\lambda)\mathrm{x}(t)\}dt = R(\lambda)$$

  - For a given λ, $R_T(\lambda)$ is the time average of the process $\Phi(t)=x(t+\lambda)x(t)$
  - The mean of the process $\Phi(t)$ is given by

$$E\{\Phi(t)\}=E\{x(t+\lambda)x(t)\} = R(\lambda)$$

IV. Stochastic processes
4.4. Ergodicity

- Its autocorrelation

$R_{\Phi\Phi}(\tau)=E\{x(t+\lambda+\tau)x(t+\tau)x(t+\lambda)x(t)\}$

- Hence with w(t) = Φ(t), we have
- Ergodicity theorem for autocorrelation
  - For a given λ

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\mathrm{x}(t+\lambda)\mathrm{x}(t)dt = E\{\mathrm{x}(t+\lambda)\mathrm{x}(t)\} = R(\lambda)$$

iff

$$\lim_{T\to\infty}\frac{1}{T}\int_{0}^{2T}\left(1-\frac{\tau}{2\mathrm{T}}\right)[R_{\Phi\Phi}(\tau)-R^2(\lambda)]d\tau = 0$$

IV. Stochastic processes
4.4. Ergodicity

- Ergodicity of the distribution function
  - We determine first order distribution F(x) = E{X(t)≤x} of a given process X(t) by a suitable time average.
  - Consider the process

$$\mathrm{y}(t)=\begin{cases}1 & \text{if } \mathrm{x}(t)\le x\\ 0 & \text{if } \mathrm{x}(t)>x\end{cases}$$

  - Its mean is given by
  - Its autocorrelation

$$E\{\mathrm{y}(t)\}=1.P\{\mathrm{x}(t)\le x\}=F(x)$$

  - Where F(x,x;τ) is the second order distribution of x(t)

$$E\{\mathrm{y}(t+\tau)\mathrm{y}(t)\}=1.P\{\mathrm{x}(t+\tau)\le x,\mathrm{x}(t)\le x\}=F(x,x;\tau)$$

IV. Stochastic processes
4.4. Ergodicity

- We form the time average

$$y_T = \frac{1}{2T}\int_{-T}^{T} y(t)dt$$

- We have

$$E\{y_T\} = E\{y(t)\} = F(x)$$

- The variance of y_T is given by

$$\frac{1}{T}\int_{0}^{2T}\left(1-\frac{\tau}{2T}\right)[R(\tau)-\eta^2]d\tau = 0$$

- Where $R(\tau)$ and η are replaced by $F(x, x;\tau)$ and $F(x)$

IV. Stochastic processes
4.4. Ergodicity

- Ergodic theorem for distribution function
  - For a given x,

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\mathrm{y}(t)dt = F(x)$$

$$iff$$

$$\lim_{T\to\infty}\frac{1}{T}\int_{0}^{2T}\left(1-\frac{\tau}{2\mathrm{T}}\right)[F(x,x;\tau)-F^{2}(x)]d\tau = 0$$

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Thank you for your attentions!