Mean value theorems

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Content

Definition

Let $f(x)$ be defined on D. $f(x)$ has

- an absolute maximum at $c \in D$ if $f(c) \geq f(x) \forall x \in D$. $M = f(c)$ is called the maximum value of f on D.
- an absolute minimum at $d \in D$ if $f(d) \leq f(x) \forall x \in D$. $m = f(d)$ is called the minimum value of f on D.
- a local maximum at $c \in D$ if $f(c) \geq f(x)$, $\forall x \in (c - \varepsilon, c + \varepsilon)$. $f(c)$ is called a local maximum value of f on D.
- a local minimum at $d \in D$ if $f(d) \leq f(x)$, $\forall x \in (d - \varepsilon, d + \varepsilon)$. $f(d)$ is called a local minimum value of f on D.

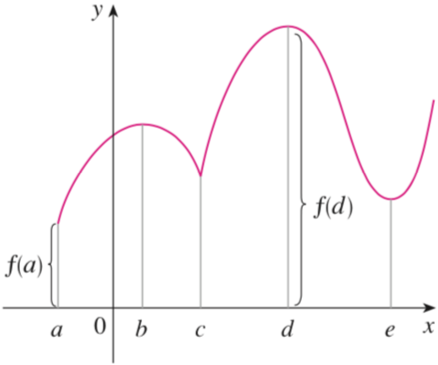


FIGURE 2

Abs min $f(a)$, abs max $f(d)$

loc min $f(c)$, $f(e)$, loc max $f(b)$, $f(d)$

Theorem

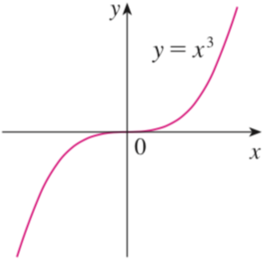
Let $f(x)$ be defined on (a, b) and attain a local maximum or minimum at $c \in (a, b)$. If there exists $f'(c)$, then $f'(c) = 0$.

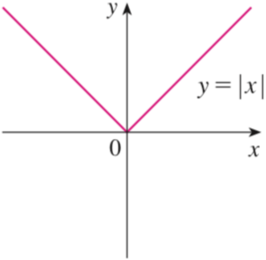
Without loss of generality, assume that c is a local minimum of $f(x)$.
Then

$$f'_{+}(c) = \lim_{x \to c^{+}} \frac{f(x) - f(c)}{x - c} \geq 0$$
$$f'_{-}(c) = \lim_{x \to c^{-}} \frac{f(x) - f(c)}{x - c} \leq 0$$

There exists $f'(c)$, hence, $f'_{+}(c) = f'_{-}(c) = 0$.

The converse of Fermat's theorem is not true.





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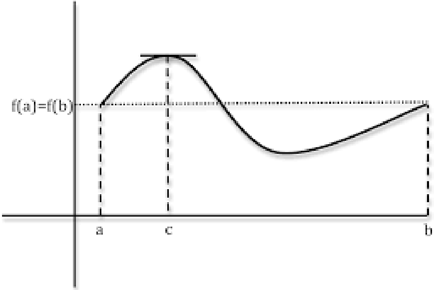
Theorem

Let $f(x)$ be defined and continuous on $[a, b]$, differentiable on (a, b). If $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Geometric illustration of Rolle's theorem



The graph of $y = f(x)$ has at least one horizontal tangent in the interval (a, b).

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Theorem

Let $f(x)$ be defined and continuous on $[a, b]$, differentiable on (a, b). Then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or equivalently

$$f(b) - f(a) = f'(c)(b - a).$$

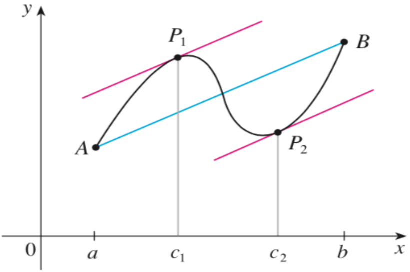

Figure: Geometric interpretation

Corollary

Let $f(x)$ be differentiable and $f'(x) = 0$ for all $x \in (a, b)$. Then $f(x)$ is constant on (a, b).

Example

1. Find a point in the graph of the function $y = x^3$ such that the tangent line at that point is parallel to the segment AB, where $A(-1, -1)$ and $B(2, 8)$.
2. Let a, b, c satisfy $a + b + c = 0$. Prove that the equation $ax^3 + 2bx + 2c = 0$ has a solution in $(0, 2)$.
3. Prove that for all $x, y \geq 1$, we have

$$|\arctan x - \arctan y| \leq \frac{1}{2}|x - y|.$$

Content

Fermat's theorem
Rolle's theorem
Lagrange's theorem
Cauchy's theorem

Theorem

Let $f(x)$, $g(x)$ be continuous on $[a, b]$, differentiable in (a, b), and $g'(x) \neq 0$ on (a, b). There exists $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Example

Is the Cauchy's theorem applicable to the functions $f(x) = x^2$ and $g(x) = x^3$ on $[-1, 3]$?

Example

Prove that for all $x \geq y > 0$, we have

$$\arctan x^2 - \arctan y^2 \leq \ln \frac{x}{y}.$$