Mean value theorems and applications

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Linear approximation

Linear approximation:

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x.$$

Lagrange's theorem:

$$f(x) = f(x_0) + f'(c)(x - x_0), c \text{ lies between } x \text{ and } x_0.$$

Aim: approximate $f(x)$ when x is near x_0 by polynomial

$$P_n(x) = c_0 + c_1(x - x_0) + \ldots + c_n(x - x_0)^n,$$

such that $f(x)$ and its first n derivatives have the same values at x_0 as $P_n(x)$ and its first n derivatives, respectively.

We obtain $c_k = \dfrac{f^{(k)}(x_0)}{k!}$, $0 \leq k \leq n$.

Taylor polynomials

Definition

$$P_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

is called the nth degree Taylor polynomial of $f(x)$ centered at x_0.

Taylor and Maclaurin expansions

Theorem (Taylor expansion)

Let $f(x)$ be $(n+1)$ times differentiable on (a,b), with $f^{(n)}$ continuous on $[a,b]$. Then for all $x_0 \in (a,b)$, we have

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{f^{(n+1)}(\bar{x}_0)}{(n+1)!}(x-x_0)^{n+1},$$

for some real number $\bar{x}_0$ between x and x_0.

The Taylor expansion at $x_0 = 0$ is called the Maclaurin expansion of $f(x)$.

Important Maclaurin expansions

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \frac{e^{\bar{x}_0}}{(n+1)!}x^{n+1}$$

$$\sin x = x - \frac{x^3}{3!} + \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \frac{\sin(\bar{x}_0 + (2n+2)\frac{\pi}{2})}{(2n+2)!}x^{2n+2}$$

$$\cos x = 1 - \frac{x^2}{2!} + \ldots + \frac{(-1)^n x^{2n}}{(2n)!} + \frac{\cos(\bar{x}_0 + (2n+1)\frac{\pi}{2})}{(2n+1)!}x^{2n+1}$$

where $\bar{x}_0$ lies between 0 and x.

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \ldots + \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}x^n + \\ + \frac{\alpha(\alpha-1)\ldots(\alpha-n)}{(n+1)!}(1+\bar{x}_0)^{\alpha-n-1}x^{n+1},$$

$$\ln(1+x) = x - \frac{x^2}{2} + \ldots + (-1)^{n-1}\frac{x^n}{n} + \frac{(-1)^n}{(n+1)(1+\bar{x}_0)^{n+1}}x^{n+1},$$

where $\bar{x}_0$ lies between 0 and x.

Corollary

If $|f^{(n+1)}(x)| \leq M$*, for all* $x \in (a, b)$*, then*

$$|R_n(x)| = |f(x) - P_n(x)| \leq \frac{M}{(n+1)!}|x - x_0|^{n+1}.$$

Finite expansion

$$f(x) = P_n(x - x_0) + o((x - x_0)^n), \text{ as } x \to x_0,$$

$o((x - x_0)^n)$ is an infinitesimal of higher order than $(x - x_0)^n$.

Example (Finite expansion of essential functions)

As $x \to 0$:

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + o(x^n)$$

$$\sin x = x - \frac{x^3}{3!} + \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + o(x^{2n+2})$$

$$\cos x = 1 - \frac{x^2}{2!} + \ldots + \frac{(-1)^n x^{2n}}{(2n)!} + o(x^{2n+1}).$$

Example (Finite expansion of essential functions)

As $x \to 0$:

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \ldots + \\ + \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}x^n + o(x^n),$$

$$\ln(1+x) = x - \frac{x^2}{2} + \ldots + (-1)^{n-1}\frac{x^n}{n} + o(x^n).$$

Example

Write the expansion to the given order of the following functions

1. $f(x) = \sin 2x + x\cos 3x^2,\ x_0 = 0,\ n = 5.$
2. $g(x) = \dfrac{1}{\sqrt{x}},\ x_0 = 1,\ n = 3.$ Apply to approximate the value $f(1,1)$, and estimate the error.

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Evaluating indeterminate forms of types $\frac{0}{0}$, $\frac{\infty}{\infty}$

Theorem (L'Hospital's rule)

Suppose $f(x)$ and $g(x)$ are differentiable, $g'(x) \neq 0$ near x_0, possibly except at x_0. Assume that

$$\lim_{x \to x_0} f(x) = 0 \text{ and } \lim_{x \to x_0} g(x) = 0$$

or

$$\lim_{x \to x_0} f(x) = \infty \text{ and } \lim_{x \to x_0} g(x) = \infty,$$

and there exists $\lim_{x \to x_0} \frac{f'(x)}{g'(x)} \in \bar{\mathbb{R}}$. *Then*

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}.$$

Remark

- The rule holds for one sided limits, limit at infinity.
- We can apply L'Hospital rule successively several times.
- This rule is a sufficient condition to evaluate the indeterminate forms, consider $\lim_{x \to +\infty} \dfrac{x - \cos x}{x + \cos x}$.

Indeterminate forms of types $\dfrac{0}{0}, \dfrac{\infty}{\infty}$

Example

Find the limits

1. $\displaystyle\lim_{x\to 0}\frac{e^x\sin x - x(1+x)}{x^3}$
2. $\displaystyle\lim_{x\to 1^-}\frac{\tan\frac{\pi}{2}x}{\ln(1-x)}$
3. $\displaystyle\lim_{x\to 0}\frac{\sqrt[m]{1+\alpha x}-\sqrt[2m]{1+2\alpha x}}{x^2}$.

Indeterminate forms of other types

Example

Find the limits

1. $\lim_{x \to \infty} x^3 . e^{-x^2}$
2. $\lim_{x \to 0} (3^x + 4^x - 5^x)^{\frac{1}{3^x + 4^x - 2.5^x}}$
3. $\lim_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right)$

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Recall

Definition

The function $f(x)$ is said to be **strictly increasing** on $[a, b]$ if for all $x_1, x_2 \in [a, b]$, $x_1 < x_2$ then $f(x_1) < f(x_2)$.
The function $f(x)$ is said to be **strictly decreasing** on $[a, b]$ if for all $x_1, x_2 \in [a, b]$, $x_1 < x_2$ then $f(x_1) > f(x_2)$.

Theorem (Increasing/Decreasing Test)

- *If $f'(x) > 0$ on an interval then $f(x)$ is increasing on that interval.*
- *If $f'(x) < 0$ on an interval then $f(x)$ is decreasing on that interval.*

Example

1. Prove that $2x \arctan x \geq \ln(1 + x^2)$ for all $x \in \mathbb{R}$.
2. Prove that for all $x \geq y > 0$: $\arctan x^4 - \arctan y^4 \leq \ln \frac{x^2}{y^2}$.

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Recall:

Definition (Local extreme values)

$f(x)$ **attains a local extreme value at** $c \in (a, b)$ if there exists a neighborhood $U_\varepsilon(c) \subset (a, b)$ such that $f(x) - f(c)$ keeps its sign for all $x \in U_\varepsilon(c)$.

- c is a **local maximum** if $f(x) \leq f(c)$ in $U_\varepsilon(c)$.
- c is a **local minimum** if $f(x) \geq f(c)$ in $U_\varepsilon(c)$.

Theorem

Let $f(x)$ be defined on (a, b) and attain a local maximum or minimum at $c \in (a, b)$. If there exists $f'(c)$, then $f'(c) = 0$.

Therefore, we can determine the set of possible extreme points.

Definition

A critical number of f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Theorem (First Derivative Test)

Suppose that c is a critical number of a continuous function f.

- *If $f'(x)$ changes from positive to negative at c, then f has a local maximum at c.*
- *If $f'(x)$ changes from negative to positive at c, then f has a local minimum at c.*
- *If $f'(x)$ does not change sign at c, then f has no local extremum at c.*

Theorem (Second Derivative Test)

Suppose that $f''(x)$ is continuous at c.

- *If $f'(c) = 0$ and $f''(c) > 0$ then f has a local minimum at c.*
- *If $f'(c) = 0$ and $f''(c) < 0$ then f has a local maximum at c.*

Theorem

Suppose that f is n times differentiable in $(c-\varepsilon, c+\varepsilon)$ and $f'(c) = f''(c) = \ldots = f^{(n-1)}(c) = 0$; $f^{(n)}(c) \neq 0$.

- *In case n is even: if $f^{(n)}(c) > 0$ then f has a local minimum at c; if $f^{(n)}(c) < 0$ then f has a local maximum at c.*
- *In case n is odd then f has no local extremum at c.*

Proof.

Example

Find the local extreme values of the following functions

1. $y = (x\sqrt[3]{8+x})^2$
2. $y = \frac{3x^2+4x+4}{x^2+x+1}$

Definition (Global extreme values)

- $f(x)$ attains its maximum on $[a, b]$, $\max\limits_{[a,b]} f(x) = M$ if $f(x) \leq M$ for all $x \in [a, b]$ and there exists $x_0 \in [a, b]$ such that $f(x_0) = M$.
- $f(x)$ attains its minimum on $[a, b]$, $\min\limits_{[a,b]} f(x) = m$ if $f(x) \geq m$ for all $x \in [a, b]$ and there exists $x_1 \in [a, b]$ such that $f(x_1) = m$.

Algorithm to find extreme values of functions

Theorem

If $f(x)$ is continuous on $[a,b]$ then it attains its minimum and maximum on $[a,b]$.

1. Determine critical points $c_i \in (a,b), i = 1, 2, \ldots, n$.
2. $\max\limits_{x \in [a,b]} f = \max\{f(c_i), f(a), f(b)\}$
   $\min\limits_{x \in [a,b]} f = \min\{f(c_i), f(a), f(b)\}$.

Example

Determine the extreme values of the following functions

1. $f(x) = x - \ln(1+x)$ on $\left[-\frac{1}{2}, \frac{1}{2}\right]$.
2. $f(x) = x + \frac{4}{x}$ on $[1, 3]$.

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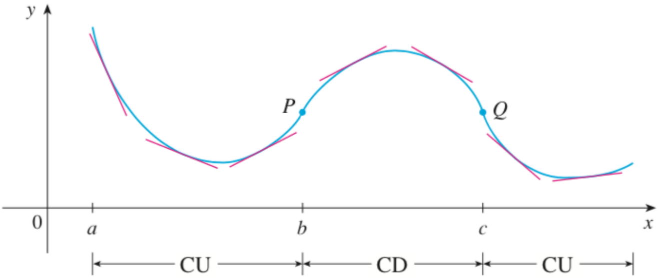
Definition

- If the graph of f lies **above** all its tangents on an interval I then it is called concave upward on the interval.
- If the graph of f lies **below** all its tangents on an interval I then it is called concave downward on the interval.



Another realization of concavity

- **Concave upward** on an interval I if $\forall a, b \in I$ and $t \in [0, 1]$:

$$tf(a) + (1-t)f(b) \geq f(ta + (1-t)b).$$

- **Concave downward** on an interval I if $\forall a, b \in I$ and $t \in [0, 1]$:

$$tf(a) + (1-t)f(b) \leq f(ta + (1-t)b).$$

Theorem (Second Derivative Test)

Suppose that $f(x)$ is twice differentiable.

- *If $f''(x) > 0$ on (a, b) then f is concave upward on that interval.*
- *If $f''(x) < 0$ on (a, b) then f is concave downward on that interval.*

Example

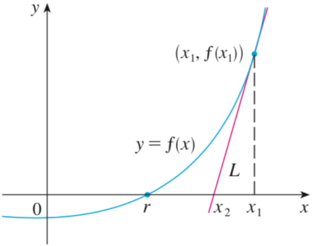
Prove that

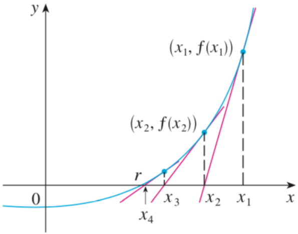
1. $\forall x > 0$ we have $3\arctan x + \arctan(x+2) < 4\arctan(x+1)$.
2. $\forall x$ we have $2\operatorname{arccot} x + \operatorname{arccot}(x+2) > 3\operatorname{arccot}(x+1)$.

Newton's method

Aim: Solve the equation $f(x) = 0$.

Idea: Using a sequence of approximate root. The tangent line is close to the curve, so its $x-$intercept is close to the $x-$intercept of the curve.





$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Failure analysis:

1. Bad starting points:
   - iteration point is stationary. Example: $f(x) = 1 - x^2, x_0 = 0$.
   - starting point enters a cycle. Example: $f(x) = x^3 - 2x + 2, x_0 = 0$.
2. Derivative issues: the derivative does not exist at root.

Proposition

Assume that the equation $f(x) = 0$ has a unique solution in $[a, b]$, and $f'(x), f''(x)$ are continuous and do not vanish and change their signs on (a, b). If x_0 is chosen such that $f(x_0).f''(x_0) > 0$ then the iterative process converges to the root of the equation.

Example

Starting with $x_1 = 2$, find the third approximation x_3 to the root of the equation $x^3 - 2x - 5 = 0$.

We apply the Newton's method with $f(x) = x^3 - 2x - 5$;
$f'(x) = 3x^2 - 2$.
$x_1 = 2$; $x_2 = 2,1$; $x_3 \approx 2,0946$.