# Series of functions

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March 24, 2020

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## Definition

Given a **sequence of functions** $\{u_n(x)\}_{n\geq 1}$ defined on a set $X$. Series of functions is the sum

$$\sum_{n=1}^{\infty} u_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots$$

The $n-$th partial sum is

$$S_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x).$$

# Domain of convergence

## Definition

$\sum_{n=1}^{\infty} u_n(x)$ converges at $x_0$ if $\sum_{n=1}^{\infty} u_n(x_0)$ **converges**.

$\sum_{n=1}^{\infty} u_n(x)$ diverges at $x_0$ if $\sum_{n=1}^{\infty} u_n(x_0)$ **diverges**.

The set of all $x_0$ at which the series of functions $\sum_{n=1}^{\infty} u_n(x)$ converges is called the domain of convergence of the series.

For $x$ in the domain of convergence: $\sum_{n=1}^{\infty} u_n(x) = S(x)$, $S(x)$ is called the sum of the series.

$S(x) = \lim_{n\to\infty} S_n(x)$. Similarly, we define absolute convergence and conditional convergence at a point.

## Example

Find the domain of convergence

$$a)\ \sum_{n=1}^{\infty} x^n \qquad b)\ \sum_{n=1}^{\infty} n^x \qquad c)\ \sum_{n=1}^{\infty} \frac{1}{1+x^n} \qquad d)\ \sum_{n=1}^{\infty} \frac{x^n}{n}$$

a) The geometric series $\sum aq^n$ converges $\Leftrightarrow |q| < 1$, so the domain of convergence of the functional series $\sum x^n$ is $(-1, 1)$.

b) The series $\sum \frac{1}{n^p}$ converges $\Leftrightarrow p > 1$, so the functional series $\sum n^x$ for $-x > 1$, hence the domain of convergence is $(-\infty, -1)$.

c) Neccessary condition for convergence:
$\lim_{n\to\infty} \frac{1}{1+x^n} = 0 \Rightarrow \lim_{n\to\infty} x^n = \infty \Rightarrow |x| > 1$. If $|x| > 1$:

$$\left|\frac{1}{1+x^n}\right| = \frac{1}{|1+x^n|} \sim \frac{1}{|x^n|} \text{ as } n \to \infty.$$

As $\frac{1}{|x|} < 1$, the $\sum \frac{1}{|x|^n}$ converges, so the given series also converges.
The series converges iff $|x| > 1$.

d) We have

$$D = \lim_{n\to\infty} \left| \frac{x^{n+1}}{n+1} \frac{n}{x^n} \right| = |x| \lim_{n\to\infty} \frac{n}{n+1} = |x|.$$

$|x| < 1 \Rightarrow D < 1$: the series converges.
$|x| > 1 \Rightarrow D > 1$: the series diverges.
$x = 1$, the series becomes $\sum \frac{1}{n}$ which diverges.
$x = -1$, the series becomes $\sum \frac{(-1)^n}{n}$ which converges.
The domain of convergence is $[-1, 1)$.

- Find $D = \lim\limits_{n\to\infty}\left|\dfrac{u_{n+1}(x)}{u_n(x)}\right|$ or $C = \lim\limits_{n\to\infty}\sqrt[n]{|u_n(x)|}$.
- Find $x$ such that $D < 1$ or $C < 1$, the series converges.
- Test for convergence at **endpoints**.
  At these points $D = 1$ (or $C = 1$), we have to use other criteria.

# Content

# Uniform convergence

$$\sum_{n=1}^{\infty} u_n(x) = S(x) \Leftrightarrow \lim_{n\to\infty} S_n(x) = S(x) \Leftrightarrow$$

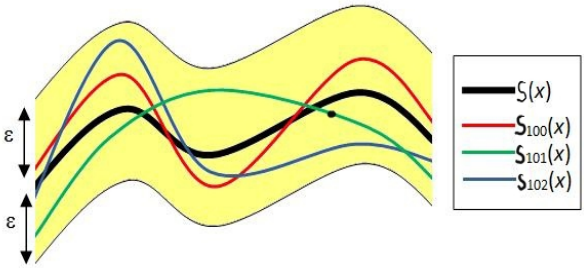
$$\forall\, \varepsilon > 0, \exists\, N_0(\varepsilon, x) \in \mathbb{N} : \forall\, n \geq N_0 : |S_n(x) - S(x)| < \varepsilon.$$

## Definition

The series of functions $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly to $S(x)$ **on the set** $X$ if

$$\forall\, \varepsilon > 0, \exists\, N_0(\varepsilon) \in \mathbb{N} \,\Big|\, \forall\, n \geq N_0 : |S_n(x) - S(x)| < \varepsilon, \forall x \in X.$$

# Illustration



# Weierstrass test

## Proposition

*If*

- $|u_n(x)| \leq a_n, \forall n \in \mathbb{N}, \forall x \in X,\ a_n \in \mathbb{R},$
- *the number series* $\sum\limits_{n=1}^{\infty} a_n$ *converges,*

*then the series* $\sum\limits_{n=1}^{\infty} u_n(x)$ *converges uniformly on the set* $X$*.*

## Example

Test for uniform convergence.

1. $\sum\limits_{n=1}^{\infty} \dfrac{\cos nx}{n^2 + x^2}$ on $\mathbb{R}$.
2. $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n x^{2n}}{n\sqrt{n}}$ on $(-1, 1)$.

1. We have

$$\left|\frac{\cos nx}{n^2 + x^2}\right| \leq \frac{1}{n^2 + x^2} \leq \frac{1}{n^2}, \text{ for all } x \in \mathbb{R}.$$

Moreover, the series $\sum \frac{1}{n^2}$ converges. Hence, the given series converges uniformly in $\mathbb{R}$.

2. We have

$$\left|\frac{(-1)^n x^{2n}}{n\sqrt{n}}\right| \leq \frac{1}{n\sqrt{n}}, \text{ for all } x \in (-1, 1).$$

The series $\sum \frac{1}{n\sqrt{n}}$ converges. Hence, the series converges uniformly in $(-1, 1)$.

$$S - S_m = \sum_{m+1}^{\infty} \frac{(-1)^n x^{2n}}{n\sqrt{n}}$$

Alternating series

$$|S - S_m| = \left| \sum_{m+1}^{\infty} \frac{(-1)^n x^{2n}}{n\sqrt{n}} \right|$$

$$\Rightarrow |S - S_m| \leq \frac{x^{2(m+1)}}{(m+1)\sqrt{m+1}} \leq \frac{1}{(m+1)\sqrt{m+1}}, \text{for all } x \in (-1,1).$$

For all $\varepsilon > 0$, choose $N_0(\varepsilon)$ to be the smallest integer such that $\dfrac{1}{N_0\sqrt{N_0}} < \varepsilon$. We have, for all $m+1 \geq N_0$

$$\Rightarrow |S - S_m| \leq \frac{1}{(m+1)\sqrt{m+1}} \leq \frac{1}{N_0\sqrt{N_0}} < \varepsilon, \text{for all } x \in (-1,1).$$

# Properties of uniformly convergent series of functions

**Theorem (Continuity)**

*If* $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly *to* $S(x)$ *on the set* $X$ *and* $u_n(x)$ *are continuous functions on* $X$*, then* $S(x)$ *is continuous on* $X$.

**Theorem (Integrability)**

*If* $\sum_{n=1}^{\infty} u_n(x)$ *converges uniformly to* $S(x)$ *on* $[a, b]$, $u_n(x)$ *are continuous functions on* $[a, b]$*. Then* $S(x)$ *is integrable on* $[a, b]$*. Moreover,*

$$\int_a^b S(x)dx = \int_a^b \Big(\sum_{n=1}^{\infty} u_n(x)\Big) dx = \sum_{n=1}^{\infty} \int_a^b u_n(x)dx.$$

## Theorem (Differentiability)

*If* $\sum_{n=1}^{\infty} u_n(x)$ *converges pointwise to* $S(x)$ *on* $(a, b)$, $u_n(x)$ *are continuously differentiable on* $(a, b)$, $\sum_{n=1}^{\infty} u_n'(x)$ *converges uniformly on* $(a, b)$ *then* $S(x)$ *is differentiable on* $(a, b)$. *Moreover,*

$$S'(x) = \left(\sum_{n=1}^{\infty} u_n(x)\right)' = \sum_{n=1}^{\infty} u_n'(x).$$

## Example

Find the sum $\sum\limits_{n=1}^{\infty} \frac{x^n}{n}$.

Denote $S(x) = \sum\limits_{n=1}^{\infty} \frac{x^n}{n}$, $x \in [-1, 1)$ (see example d-DoC).
For $x \in (-1, 1)$, we have

$$\begin{aligned} S'(x) &= \sum_{n=1}^{\infty} \left( \frac{x^n}{n} \right)' \\ &= \sum_{n=1}^{\infty} x^{n-1} = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}. \end{aligned}$$

Therefore, $S(x) = \int \frac{dx}{1-x} = -\ln(1-x) + C$.
$S(0) = 0 \Rightarrow C = 0$.
In conclusion, we obtain $\sum\limits_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x)$, $x \in (-1, 1)$.