Second order ordinary differential equations

Nguyen Thu Huong

ĐẠI HỌC
BÁCH KHOA

School of Applied Mathematics and Informatics
Hanoi University of Science and Technology

December 2, 2020

Content

Content

Content

Theorem

The general solution to the inhomogeneous equation

$$y'' + p(x)y' + q(x)y = f(x) \tag{1}$$

has the form $y = \bar{y} + y^*$, *where*

- $\bar{y}$ *is the general solution to the corresponding homogeneous equation* $y'' + p(x)y' + q(x)y = 0$.
- y^* *is a particular solution to* (1).

Aim: Find y^*.

Variation of parameters

The general solution of the homogeneous equation $y'' + p(x)y' + q(x)y = 0$ is:

$$\bar{y}(x) = C_1 y_1(x) + C_2 y_2(x).$$

We look for a particular solution of the inhomogeneous equation in the form:

$$y^*(x) = C_1(x) y_1(x) + C_2(x) y_2(x),$$

Substituting in the equation

$$\begin{aligned} y^* &= C_1 y_1 + C_2 y_2 \\ (y^*)' &= C_1 y_1' + C_2 y_2' + \underbrace{C_1' y_1 + C_2' y_2}_{0} \\ (y^*)'' &= C_1 y_1'' + C_2 y_2'' + C_1' y_1' + C_2' y_2' \\ L y^* &= C_1 L y_1 + C_2 L y_2 + C_1' y_1' + C_2' y_2' = f(x) \Rightarrow C_1' y_1' + C_2' y_2' = f(x). \end{aligned}$$

A particular solution of the inhomogeneous equation is found in the form:

$$y^*(x) = C_1(x)y_1(x) + C_2(x)y_2(x),$$

where $C_1'(x)$, $C_2'(x)$ satisfy the system

$$\begin{cases} C_1'(x)y_1(x) + C_2'(x)y_2(x) = 0, \\ C_1'(x)y_1'(x) + C_2'(x)y_2'(x) = f(x). \end{cases}$$

We obtain $\begin{cases} C_1(x) = \varphi_1(x) + K_1, \\ C_2(x) = \varphi_2(x) + K_2, K_1, K_2 \in \mathbb{R}, \end{cases}$

The general solution is given by

$y = (\varphi_1(x) + K_1)\, y_1(x) + (\varphi_2(x) + K_2)\, y_2(x).$

Solve the equation $y'' + p(x)y' + q(x) = f(x)$

- Solve the homogeneous equation $y'' + p(x)y' + q(x) = 0$. If a solution is given, we find another one by Liouville formula. Then we obtain $\bar{y}$.
- Find a particular solution of the inhomogeneous by the method of variation of parameters.

Example

Solve the equation $y'' + \frac{x}{1-x}y' - \frac{1}{1-x}y = x - 1$ that the corresponding homogeneous equation has a solution $y_1 = e^x$.

Example

Solve the ODE $y'' + \frac{x}{1-x}y' - \frac{1}{1-x}y = x - 1$ given a particular solution $y_1 = e^x$ to the corresponding homogeneous linear equation.

Example

Solve the ODE $y'' - 2y' + y = \frac{e^x}{x^2+1}$.

Inhomogeneous equations with constant coefficients

Form $y'' + py' + qy = f(x)$.
General RHS: Variation of parameters.
Special RHS $f(x) \Rightarrow$ special form of y^*:

- $f(x) = e^{\alpha x} P_n(x)$,
- $f(x) = e^{\alpha x}(P_n(x) \cos \beta x + Q_m(x) \sin \beta x)$,

where $P_n(x)$, $Q_m(x)$ are polynomials of x.

The case $f(x) = e^{\alpha x} P_n(x)$

Consider the RHS of the form

$$f(x) = e^{\alpha x} P_n(x),$$

where $P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$ is a polynomial of degree n.

Compare α with the root of the characteristic equation. We look for a particular solution y^* to the **inhomogeneous equation** of the form

- if α is not a root: $y^* = e^{\alpha x} Q_n(x)$,
- if α is a single root, $y^* = x e^{\alpha x} Q_n(x)$,
- if α is a double root, $y^* = x^2 e^{\alpha x} Q_n(x)$,

where $Q_n(x)$ is a polynomial of degree n.

Plug y^* in the equation to find the coefficients of $Q_n(x)$.

Example

Solve the following equations:

- $y'' + 4y = (2x+1)e^x$.
- $y'' - 3y' + 2y = (x-1)e^{2x}$.
- $y'' - 4y' + 4y = 3e^{2x}$.

The case $f(x) = e^{\alpha x} P_n(x) \cos \beta x$ or $f(x) = e^{\alpha x} P_n(x) \sin \beta x$

Consider RHS of the form

$$f(x) = e^{\boldsymbol{\alpha} x} P_n(x) \cos \boldsymbol{\beta} x \text{ or } f(x) = e^{\boldsymbol{\alpha} x} P_n(x) \sin \boldsymbol{\beta} x,$$

where $P_n(x)$ is a polynomial of degree n.
We look for y^*

- if $\alpha \pm i\beta$ is **not a root** of the characteristic equation, $y^* = e^{\alpha x}\Big(Q_n(x)\cos \beta x + R_n(x) \sin \beta x\Big)$,
- if $\alpha \pm i\beta$ is **a single root** of the characteristic equation, $y^* = x e^{\alpha x}\Big(Q_n(x)\cos \beta x + R_n(x) \sin \beta x\Big)$,

where $Q_n(x), R_n(x)$ are polynomial of degree n.
We plug y^* in the equation to find the coefficients of $Q_n(x), R_n(x)$.

Example

Solve the following ODEs:

1. $y'' + 2y' + 5y = 17\cos 2x$.
2. $y'' + y = 2\cos x$.
3. $y'' + y = \sin x + 2e^x$.