Euler equations and systems of ODEs

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December 9, 2020

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2. Power series solutions
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Euler equations

An Euler equation has the form

$$x^2y'' + axy' + by = f(x), \qquad a, b \in \mathbb{R}.$$

Set $|x| = e^t \Rightarrow t = \ln|x|$.

$$\begin{aligned} y'(x) &= \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = e^{-t}\frac{dy}{dt} \\ y''(x) &= \frac{d}{dx}\Big(\frac{dy}{dx}\Big) = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dx}\Big(\frac{dy}{dt}\Big) \\ &= -e^{-2t}\frac{dy}{dt} + e^{-2t}\frac{d}{dt}\Big(\frac{dy}{dt}\Big). \end{aligned}$$

The given equation reads as

$$y''(t) + (a-1)y'(t) + by(t) = g(t).$$

(A linear ODE with constant coefficients. Look for $y(t)$, transform back to $y(x)$).

Example

Solve the following ODEs

1. $x^2y'' - 9xy' + 21y = x\ln x$.
2. $x^2y'' - 2xy' + 2y = 3x^2$.

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1 Euler equations

2 Power series solutions

3 Systems of ODEs

Consider a homogeneous linear equation of second order

$$a(x)y'' + b(x)y' + c(x) = 0,\ a(x) \neq 0, x \in I.$$

Recall:

- Let the power series $y = \sum_{n=0}^{\infty} a_n x^n$ converge in $(-R, R) \Rightarrow y$ is infinitely differentiable in this interval.

$$y' = a_1 + 2a_2 x + \ldots + n a_n x^{n-1} + \ldots$$

- $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n \Leftrightarrow a_n = b_n, \forall n \geq 0.$

Power series solutions

Example

Solve the ODE $(x^2+1)y'' - 4xy' + 6y = 0$.

We look for a solution of the form $y = \sum\limits_{n=0}^{\infty} a_n x^n$. Then,

$$\begin{array}{c|l} \times 6 & y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots \\ \times(-4x) & y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots \\ \times(x^2+1) & y'' = 2a_2 + 6a_3 x + 12a_4 x^2 + 30a_5 x^3 + \dots \end{array}$$

The equation becomes

$$(6a_0 + 2a_2) + (6a_1 - 4a_1 + 6a_3)x + (6a_2 - 8a_2 + 2a_2 + 12a_4)x^2 + \\ (6a_3 - 12a_3 + 6a_3 + 30a_5)x^3 + \dots = 0.$$

Or $(x^2+1)\sum\limits_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 4x\sum\limits_{n=1}^{\infty} n a_n x^{n-1} + 6\sum\limits_{n=0}^{\infty} a_n x^n = 0.$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - 4\sum_{n=1}^{\infty} n a_n x^n + 6\sum_{n=0}^{\infty} a_n x^n = 0.$$

Identifying the coefficients of $x^n, n \geq 0$, in two sides, we get: $a_2 = -3a_0$, $a_3 = -\frac{1}{3}a_1$, $a_4 = 0$, $a_5 = 0$.

For $n \geq 4$, the coefficients of x^n are:

$n(n-1)a_n + (n+2)(n+1)a_{n+2} - 4na_n + 6a_n = 0$.

$\Rightarrow a_{n+2} = -\frac{n^2 - 5n + 6}{(n+2)(n+1)}a_n$. As $a_4 = a_5 = 0$, $a_n = 0$ for all $n \geq 4$.

The general solution of the equation is $y = a_0(1 - 3x^2) + a_1\big(x - \frac{x^3}{3}\big)$.

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Predator - Prey system

Consider a habitat which contains two species: the *prey* has an ample food supply and the *predator* which feeds on the prey.
$R(t)$: the number of prey, $W(t)$: the number of predators.
Absence of predators, the ample food supply support the exponential growth of the prey: $R'(t) = kR, k > 0$.
Absence of prey, the predator population decays $W'(t) = -rW, r > 0$.

Both species present, assume

- principal cause of death among the prey is being eaten by a predator,
- the birth and survival rates of the predators depend on the prey,
- the two species encouter each other at a rate that is proportional to both population,

$$\begin{cases} R'(t) &= kR - aRW, \\ y'(t) &= -rW + bRW, \quad a, b > 0. \end{cases}$$

System of first order ODEs

System of first order ODEs

$$\begin{cases} y_1' & = f_1(x, y_1, y_2, \ldots, y_n) \\ y_2' & = f_2(x, y_1, y_2, \ldots, y_n) \\ \ldots & \\ y_n' & = f_n(x, y_1, y_2, \ldots, y_n), \end{cases} \tag{1}$$

where x is the variable, $y_1, y_2, \ldots, y_n$ are unknown functions.

The Cauchy problem

Cauchy problem: the system and intial data $y_i(x_0) = y_i^0, 1 \leq i \leq n$.

Theorem (Existence and uniqueness theorem)

Suppose f_i and the partial derivatives $\frac{\partial f_i}{\partial y_j}$, $1 \leq i, j \leq n$, are continuous on $D \subset \mathbb{R}^{n+1}$. Let $(x_0, y_1^0, y_2^0, \ldots, y_n^0) \in D$. There exists a neighborhood $U_\varepsilon(x_0)$ such that the system (1) has a unique solution $(y_1, y_2, \ldots, y_n)$ which satisfies $y_i(x_0) = y_i^0$, $1 \leq i \leq n$.

Definition

The general solution is a set of n functions $(y_1, y_2, \dots, y_n)$, $y_i = y_i(x, C_1, C_2, \dots, C_n)$, $1 \le i \le n$, where $C_1, C_2, \dots, C_n \in \mathbb{R}$ are parameters, which satisfies

- $(y_1, y_2, \dots, y_n)$ satisfy the system for all $C_1, C_2, \dots, C_n$.
- given $(x_0, y_1^0, y_2^0, \dots, y_n^0) \in D \subset \mathbb{R}^{n+1}$, there are $C_1^0, C_2^0, \dots, C_n^0$ such that $y_i = y_i(x, C_1^0, C_2^0, \dots, C_n^0)$ satisfies the initial data $y_i(x_0, C_1^0, C_2^0, \dots, C_n^0) = y_i^0$, $1 \le i \le n$.

Definition

A particular solution is obtained from the general solution by letting $C_i = C_i^0$, $1 \le i \le n$.

Converting a higher order equation to a system of first order ODEs

Given the equation

$$y^{(n)} = f(x, y, y', y'', \dots, y^{(n-1)}).$$

Set $y = y_1, y' = y_2, \dots, y^{(n-1)} = y_n$, we obtain the following system

$$\begin{cases} y_1' &= y_2 \\ y_2' &= y_3 \\ \dots & \\ y_n' &= f(x, y_1, y_2, \dots, y_n). \end{cases}$$

Substitution - converting a system to a higher order equation

Example

Solve the following systems

a) $\begin{cases} y' = 5y + 4z \\ z' = 4y + 5z \end{cases}$ b) $\begin{cases} y' = y + 5z \\ z' = -y - 3z \end{cases}$