Algorithms and Data Structures
Lecture slides: Asymptotic notations and growth rate of functions, Brassard Chap. 3

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Topics outline

Running time

In the previous lecture, the running time of an algorithm was obtained by figuring out the worst case instance and then counting the number of times the most frequent basic operations is executed

We did agree this was not an exact measurement of the execution time of an algorithm because

1. we count as "basic" pseudo-code operations that may take quite different numbers of machine operations compared to other pseudo-code operations
2. we don't count all the basic operations of an algorithm but only the one that is executed the most often

Further abstraction of running time

When counting the number of time a basic operation is executed within an algorithm, we end up with an expression like $an^2 + bn + c$ which again give us how many time a basic instruction has been executed

Here we move even further in abstracting measurements from real execution time by dropping all the constants and lower terms from the expression like the one above

So we end up with expression like n^2 in place of $an^2 + bn + c$

Clearly n^2 does not describe the number of time an instruction is executed, so what it is ?

From running time to growth rate ("orders")

The dominant term (n^2) in an expression describing the number of times an instruction is executed (such as this one : $an^2 + bn + c$) is called the *rate of growth*, or *order of growth*, of the running time.

The growth rate is an indication on how fast the running time increases, when the input size n of an algorithm increases.

The growth rate is also a bound on the running time of an algorithm, if we put a sufficiently large constant d in front of n^2 then $dn^2 > an^2 + bn + c$ because $n^2 > n$ and of course dn^2 bigger than any constant c for a sufficiently large d.

So for now on our main task will be to find the growth rate of an algorithm and to compare the running time of different algorithms using their respective growth rate.

Frequent “orders” or ”growth rates”

Some orders occur so frequently that we give them a name.

- Logarithmic algorithm : An algorithm never executes more than $c \log n$ basic operations, its run time is in the order of $\log n$ or logarithmic time.
- Linear algorithm : An algorithm never executes more than cn basic operations, its run time is in the order of n or linear time.
- Quadratic algorithm : An algorithm never executes more than cn^2 basic operations, its run time in the order of n^2 or quadratic time.
- Cubic, polynomial or exponential algorithms : For algorithms that have run time in the order of n^3, n^k or c^n.

Basic exercises

1. Give the order of the running time for selection sort
2. What is the order of the running time for insertion sort ?

Order notations

When the running time of an algorithm is bound above by a function like $\log n$, $n \log n$, n^2, etc, we denote the order of the corresponding algorithm as $O(\log n)$, $O(n \log n)$ or $O(n^2)$ which we pronounce as big O of some function.

Similarly, when the running time of an algorithm is bound below by some function like like $\log n$, $n \log n$, n^2, i.e. the number of time the basic instruction is executed is at least like $\log n$, $n \log n$, n^2, etc, we denote the running time of the algorithm as *Omega* of some function such as $\Omega(\log n)$, $\Omega(n \log n)$ or $\Omega(n^2)$

Finally, if the upper bound and the lower of the running time of an algorithm are the same function then the running time of the algorithm is denoted *Theta* of that function such as $\Theta(\log n)$, $\Theta(n \log n)$ or $\Theta(n^2)$

Order notations

These notations O, Ω and Θ are called asymptotic notations because they refer to the size of the input *in the limit*, i.e. as the size increases to infinity.

The asymptotic notations are defined precisely, we now introduce these definitions.

O-Notation : An Asymptotic Upper Bound I

Definition (Big O notation)

Let $g(n)$ be a function from $\mathbb{N}$ to $\mathbb{R}$. Denote

$$O(g(n)) = \{f(n) : \text{there exist positive constant } c \text{ and } n_0 \text{ such that } 0 \leq f(n) \leq c\, g(n) \text{ for all } n \geq n_0\}$$

the set of functions defined on natural numbers which are bounded above by a positive real multiple of $g(n)$ for sufficiently large n.

O-Notation : An Asymptotic Upper Bound II

Example : Let $f(n) = 27n^2 + \frac{355}{113}n + 12$ be the number of basic operations performed by an algorithm in the worst case, giving an input of size n. We would like to find a simple function $g(n)$ such that $f(x) \in O(g(n))$.

We can guess $g(n) = n^2$. Thus,

$$\begin{aligned} f(n) &= 27n^2 + \frac{355}{113}n + 12 \\ &\leq 27n^2 + \frac{355}{113}n^2 + 12n^2 \\ &\leq 42\frac{16}{113}n^2 = 42\frac{16}{113}g(n) \end{aligned}$$

So instead of saying that an algorithm takes $27n^2 + \frac{355}{113}n + 12$ elementary operations to solve an instance of size n, we can say that the time of the algorithm is in order of n^2, or write the algorithm is in $O(n^2)$.

O-Notation : An Asymptotic Upper Bound III

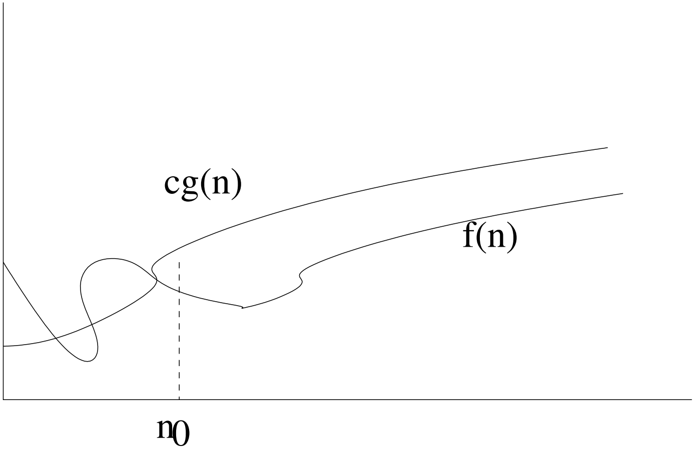
Terminologies : Let f and g be non-negative valued functions $\mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$:

1. We say that $f(n)$ is in the order of $g(n)$ if $f(n) \in O(g(n))$.
2. We say that "$f(n)$ is big-O of $g(n)$". For convenience, we also write $f(n) = O(g(n))$.
3. As n increases, $f(n)$ grows no faster than $g(n)$. In other words, $g(n)$ is an asymptotic upper bound of $f(n)$.

Graphic Example of O-notation

- $f(n) \in O(g(n))$ if there are constants c and n_0 such that

$$0 \leq f(n) \leq c\,g(n) \text{ for all } n_0 \leq n$$



How to find $g(n)$

Given that we have a function f that gives the exact number of elementary operations performed by an algorithm in the worst case, we need to find :

1. the simplest and slowest-growing function g such that $f(n) \in O(g(n))$
2. prove the relation $f(n) \in O(g(n))$ is true, i.e. show $\exists\ c$ and n_0 such that $f(n) \leq cg(n)$ for all $n \geq n_0$.

Strategy for finding $g(n)$

Assume $f(n) = 3n^2 + 2n$:

- Throw away the multiplicative constants : $3n^2 + 2n$ is replaced by $n^2 + n$
  - Also if you have $2^{n+1} = 2 \times 2^n$ can be replaced by 2^n.
  - If you have logs, throw away the bases since the log properties says that for any two bases a and b, $\log_b n = c \times \log_a n$ for some multiplicative constant c.
- Once $f(n)$ has been simplified, the fastest growing term in $f(n)$ is your $g(n)$.
- $f(n) = 3n^2 + 2n = O(n^2)$.

OK, this is a ways to find $g(n)$, but this is not a proof that $f(n) \in O(g(n))$

Prove that $f(n) \in O(g(n))$

Often the easiest way to prove that $f(n) \in O(g(n))$ is to take c to be the sum of the positive coefficients of $f(n)$.

Example : Prove $5n^2 + 3n + 20 \in O(n^2)$

- We pick $c = 5 + 3 + 20 = 28$. Then if $n \geq n_0 = 1$,

$$5\,n^2 + 3\,n + 20 \leq 5\,n^2 + 3\,n^2 + 20\,n^2 = 28\,n^2,$$

  thus $5n^2 + 3n + 20 \in O(n^2)$.
- We can also guess other values for c and then find n_0 that work.

Prove that $f(n) \in O(g(n))$

Another way is to assume $c = 1$ and find for which n_0 $f(n) \leq g(n)$

Example : Show that $\frac{1}{2}n^2 + 3n \in O(n^2)$

Proof : The dominant term is $\frac{1}{2}n^2$, so $g(n) = n^2$. Therefore we need to find c and n_0 such that

$$0 \leq \frac{1}{2}n^2 + 3n \leq c\,n^2 \text{ for all } n \geq n_0.$$

Since we decided to fix $c = 1$, we have

$$\frac{1}{2}n^2 + 3n \leq n^2 \Leftrightarrow 3n \leq \frac{1}{2}n^2 \Leftrightarrow 6 \leq n$$

Thus, we pick $n_0 = 6$.

We have just shown that if $c = 1$, and $n_0 = 6$, then
$0 \leq \frac{1}{2}n^2 + 3n \leq c\,n^2$ for all $n \geq n_0$, i.e. $\frac{1}{2}n^2 + 3n \in O(n^2)$.

Some properties for Big O notation

1. Reflexivity : $f(n) \in O(f(n))$.
2. Scalar rule : Let f be a non-negative valued functions defined on $\mathbb{N}$ and c be a positive constant. Then

$$O(cf(n)) \in O(f(n)).$$

   Example : $6n^2 \in O(n^2)$
3. Maximum rule : Let f, g be non-negative functions. Then

$$O(f(n) + g(n)) \in O(\max\{f(n), g(n)\}).$$

Exercises on Big O I

1. Given the following algorithm written in pseudo code :

   $t := 0$;
   for $i := 1$ **to** n **do**
       for $j := 1$ **to** n **do**
           $t := t + i + j$;
   return t.

   1.1 Which instruction can be used as elementary operation ?
   1.2 Express the running time of this algorithm in terms of the number of times your selected elementary operation is executed ?
   1.3 Give (without proof) a big $\mathcal{O}$ estimate for the running time of the algorithm.
   1.4 What is computed and returned by this algorithm ?

Exercises on Big O I

2. Find $g(n)$ for each of the following functions $f_i(n)$ such that $f_i(n) = O(g(n))$.
   - $f_1 = 3n\log_2 n + 9n - 3\log_2 n - 3$
   - $f_2 = 2n^2 + n\log_3 n - 15$
   - $f_3 = 100n + (n+1)(n+5)/2 + n^{3/2}$
   - $f_4 = 1,000n^2 + 2^n + 36n\log n + \left(\frac{3}{2}\right)^{n+1}$

Exercises on Big O II

3. Which of the following statements are true ?

- $n^2 \in O(n^3)$
- $2^n \in O(3^n)$
- $3^n \in O(2^n)$
- $n \log n \in O(n^{\frac{3}{2}})$
- $n^{\frac{3}{2}} \in O(n \log n)$
- $2^{n+1} \in O(2^n)$
- $O(2^{n+1}) = O(2^n)$
- $O(2^n) = O(3^n)$

Exercises on Big O III

4. Give an upper bound on the worst-case asymptotic time complexity of the following function used to find the Kth smallest integer in an unordered array of integers. Justify your result. You do not need to find the closed form of summations.

```
int selectkth( int A[ ], int k, int n )
  int i, j, mini, tmp ;
  for ( i = 0 ; i < k ; i++ )
    mini = i ;
    for ( j = i + 1 ; j < n ; j++ )
      if ( A[j] < A[mini] )
        mini = j ;
        tmp = A[i] ;
        A[i] = A[mini] ;
        A[mini] = tmp ;
  return A[k-1] ;
```

Exercises on Big O IV

5. How $n \log n$ compares with $n^{1.\epsilon}$ for $0 < \epsilon < 1$?

   Answer : Note that $n \log n = n \times (\log n)$ and $n^{1.\epsilon} = n \times n^{\epsilon}$.

   The grow rate of $\log n$ is slower than n^{ϵ} for any value of $\epsilon > 0$

   Eventually n^{ϵ} catch up with $\log n$ for some value of $n > n_0$, depending on how small is ϵ.

   Therefore, $n \log n \in O(n^{1.\epsilon})$ for $0 < \epsilon$.

Exercises on Big O V

6. Find the appropriate "Big-Oh" relationship between the functions $n \log n$ and $5n$ and find the constants c and n_0
   Answer : $5n \in O(n \log n)$. Looking for c and n_0 such that $0 \leq 5n \leq cn \log n$

$$\begin{aligned} 5n &\leq cn \log n \\ &\leq c \log n \\ 5 &\leq \log 32 \end{aligned}$$

As $2^5 = 32$ and for $c = 1$, we have $5n \leq cn \log n$

Exercises on Big O VI

7. Give the polynomial expression describing the running time of the code below. Provide the asymptotic time complexity of this code using the "Big-Oh" notation.

   for $(i = 0; i < n; i++)\{$
       for $(j = 0; j < 2 * n; j++)$
         $sum = sum + A[i] * A[j]$
       for $(j = 0; j < n * n; j++)$
         $sum = sum + A[i] + A[j]$
   $\}$

Big Omega (Ω) : An Asymptotic Lower Bound

Given a non-negative valued function $g(n)$. Denote

$$\Omega(g(n)) = \{f(n) : \text{there exist positive constant } \; c \; \text{ and } \; n_0 \text{ such that } f(n) \geq c\, g(n) \text{ for all } n \geq n_0\}$$

Definition

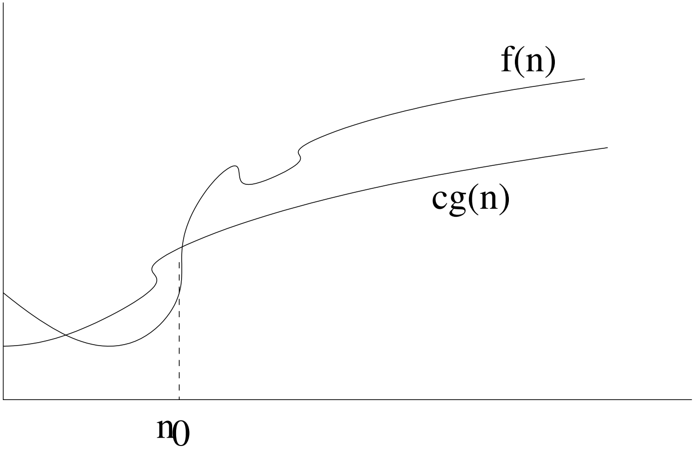
Let f and g be non-negative valued functions $\mathbb{N} \to \mathbb{R}^{\geq 0}$:

1. We say that $f(n)$ is in omega of $g(n)$ if $f(n) \in \Omega(g(n))$.
2. As n increases, $f(n)$ grows no slower than $g(n)$. In other words, $g(n)$ is an asymptotic lower bound of $f(n)$.

Graphic Example of Ω-notation

- $f(n) = \Omega(g(n))$ if there are constants c and n_0 such that

$$0 \leq c\,g(n) \leq f(n) \text{ for all } n \geq n_0.$$



Big Omega : Examples

1. $f(n) = 3n^2 + n + 12$ is $\Omega(n^2)$ and also $\Omega(n)$, but not $\Omega(n^3)$.

2. $n^3 - 4n^2 \in \Omega(n^2)$.

   Proof : Let $c = 1$. Then we must have

$$
\begin{aligned}
cn^2 &\leq n^3 - 4n^2 \\
1 &\leq n - 4
\end{aligned}
$$

   which is true when $n \geq 5$, therefore $n_0 = 5$
   so

$$0 \leq n^2 \leq n^2\,(n-4) = n^3 - 4\,n^2$$

Ω Proofs : How to choose c and n_0

To prove that $f(n) \in \Omega(g(n))$, we must find positive values of c and n_0 that make $c \cdot g(n) \leq f(n)$ for all $n > n_0$.

- You can assume that $c < 1$, pick a n_0 such that $f(n)$ is larger than $c \cdot g(n)$ and then find the exact constant c for n_0, OR
- Choose c to be some positive constant less than the multiplicative constant of the fastest growing term of $f(n)$, then find n_0 that works with the chosen c.

Example 1

For this example we assume that $c < 1$ and find an appropriate n_0

Show that $(n \log n - 2\,n + 13) \in \Omega(n \log n)$

Proof : We need to show that there exist positive constants c and n_0 such that

$$0 \le c\,n \log n \le n \log n - 2\,n + 13 \text{ for all } n \ge n_0.$$

Since $\quad n \log n - 2\,n \le n \log n - 2\,n + 13,$

we will instead show that

$$c\,n \log n \le n \log n - 2\,n,$$

Example 1 continue

$$
\begin{aligned}
c\,n\log n &\le n\log n - 2\,n \\
c &\le \frac{n\log n}{n\log n} - \frac{2n}{n\log n} \\
c &\le 1 - \frac{2}{\log n}
\end{aligned}
$$

so, $c \le 1 - \frac{2}{\log n}$, when $n > 1$.

If $n \ge 8$, then $2/(\log n) \le 2/3$, and picking $c = 1/3$ suffices.

Thus if $c = 1/3$ and $n_0 = 8$, then for all $n \ge n_0$, we have

$$0 \le c\,n\log n \le n\log n - 2\,n \le n\log n - 2\,n + 13.$$

Thus $(n\log n - 2\,n + 13) \in \Omega(n\log n)$.

Example 2

For this example we select c to be smaller than the constant of the fastest growing term in the expression describing the running time.

Prove that $f(n) = 3n^2 - 2n - 7 \in \Omega(n^2)$.

Proof : The fastest growing term of $f(n)$ is $3n^2$. Try $c = 1$, since $1 < 3$.

Then

$$1 \cdot n^2 \leq 3n^2 - 2n - 7 \quad \text{for all } n > n_0$$

is true only if (subtracting n^2 from both sides)

$$0 \leq 2n^2 - 2n - 7 \quad \text{for all } n > n_0$$

is also true.

Choose $n_0 = 3$, then the inequality above hold for any $n \geq 3$.

An Asymptotic Tight Bound Θ-notation

Let $g(n)$ be a non-negative valued function. Denote

$$\Theta(g(n)) = \{f(n) : \text{there exist positive constants } \; c_1, c_2 \; \text{ and } \; n_0 \text{ s.t.} \\ c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n) \text{ for all } n \geq n_0\}$$

Definition

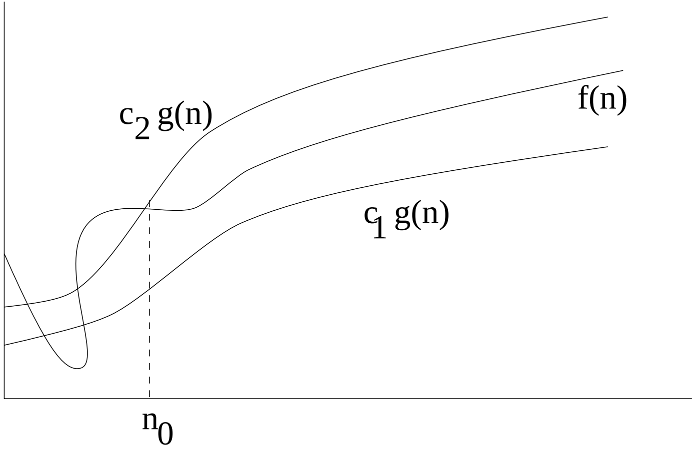
Let f and g be non-negative valued functions $\mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$:

1. We say that $f(n)$ is in the theta of $g(n)$ if $f(n) \in \Theta(g(n))$.
2. As n increases, $f(n)$ grows at the same rate as $g(n)$. In other words, $g(n)$ is an asymptotic tight bound of $f(n)$.

Graphic Example of Θ-notation

- $f(n) = \Theta(g(n))$ if there are constants c_1, c_2 and n_0 such that

$$0 \le c_1\, g(n) \le f(n) \le c_2\, g(n) \text{ for all } n_0 \le n$$



Θ-notation : Example

Prove that $n^2 - 5n + 7 \in \Theta(n^2)$.

Proof : Let $c_1 = \frac{1}{2}$, $c_2 = 1$, and $n_0 = 10$. Then $\frac{1}{2}n^2 \geq 5n$ and $-5n + 7 \leq 0$. Thus,

$$0 \leq \frac{1}{2}n^2 \leq n^2 - \frac{1}{2}n^2 \leq n^2 - 5n \leq n^2 - 5n + 7 \leq n^2$$

If $f(n)$ is $\Theta(g(n))$, then

- $f(n)$ is "sandwiched" between $c_1 g(n)$ and $c_2 g(n)$ for sufficiently large n;
- $g(n)$ is an asymptotically tight bound for $f(n)$;

Big Theta Proofs

The following theorem shows us that proving $f(n) \in \Theta(g(n))$ is nothing new :

- **Theorem :** $f(n) \in \Theta(g(n))$ if and only if $f(n) \in O(g(n))$ and $f(n) \in \Omega(g(n))$.
- Thus, we just apply the previous two strategies.

Example

Show that $\frac{1}{2}n^2 - 3n \in \Theta(n^2)$

Proof :

- Find positive constants c_1, c_2, and n_0 such that

$$0 \leq c_1 n^2 \leq \frac{1}{2}n^2 - 3\,n \leq c_2 n^2 \text{ for all } n \geq n_0$$

- Dividing by n^2, we get $0 \leq c_1 \leq \frac{1}{2} - \frac{3}{n} \leq c_2$
- $c_1 \leq \frac{1}{2} - \frac{3}{n}$ holds for $n \geq 10$ and $c_1 = 1/5$
- $\frac{1}{2} - \frac{3}{n} \leq c_2$ holds for $n \geq 10$ and $c_2 = 1$.
- Thus, if $c_1 = 1/5$, $c_2 = 1$, and $n_0 = 10$, then for all $n \geq n_0$,

$$0 \leq c_1 n^2 \leq \frac{1}{2}n^2 - 3\,n \leq c_2 n^2 \text{ for all } n \geq n_0.$$

Thus we have shown that $\frac{1}{2}n^2 - 3n \in \Theta(n^2)$.

Cookbook for asymptotic notations

Theorem (Limit rule)

Given non-negative valued functions f *and* $g : \mathbb{N} \to \mathbb{R}^{\geq 0}$. *Then the following statements are true*

1. *if* $0 < \lim_{n\to\infty} \frac{f(n)}{g(n)} = L < \infty$, *then* $f(n) \in \Theta(g(n))$ *and consequently* $f(n) \in \mathcal{O}(g(n))$ *and* $f(n) \in \Omega(g(n))$.
2. *if* $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$, *then* $f(n) \in O(g(n))$.
3. *if* $\lim_{n\to\infty} \frac{f(n)}{g(n)} = +\infty$, *then* $f(n) \notin O(g(n))$ *but* $g(n) \in O(f(n))$ *and* $f(n) \in \Omega(g(n))$.

Exercises

1. Prove that $f(n) = n^3 + 20n + 1 \in O(n^3)$
2. Prove that $f(n) = n^3 + 20n + 1 \notin O(n^2)$
3. Prove that $f(n) = n^3 + 20n + 1 \in O(n^4)$.
4. Prove $f(n) = n^3 + 20n \in \Omega(n^2)$.
5. Prove $f(n) = \frac{1}{2}n^2 - 3n \in \Omega(n^2)$.
6. Prove that $f(n) = 5n^2 - 7n \in \Theta(n^2)$.
7. Prove that $f(n) = 23n^3 - 10n^2 \log n + 7n + 6 \in \Theta(n^3)$.
8. Find the appropriate Ω relationship between the functions n^3 and $3n^3 - 2n^2 + 2$ and find the constants c and n_0.

Exercises (continue)

9. Consider the following iterative procedure :

   for $(i = 0; i < n; i++)\{$
   $\quad$ **for** $(j = 0; j < 2 * n; j++)$
   $\quad\quad sum = sum + A[i] * A[j]$
   $\quad$ **for** $(j = 0; j < n * n; j++)$
   $\quad\quad sum = sum + A[i] + A[j]$
   $\}$

   9.1 Give a function f describing the computing time of this procedure in terms of the input size n.
   9.2 Bound above the running time of this code using the "Big-Oh" notation. Prove your result.
   9.3 Give a lower bound on the running time of this code using the "Ω" notation. Prove your result. Then argue, based on your two previous results about an exact time complexity of f

Exercises (continue)

10. To illustrate how the asymptotic notation can be used to rank the efficiency of algorithms, use the relation $\subset$ and $=$ to put the orders of the following functions into a sequence.

$$n^2,\ 1,\ n^{3/2},\ 2^n,\ \log n,\ n^n,\ 3^n,\ n,\ n^3,\ n\log n,\ \sqrt{n}, \log\log n, n!$$

11. Similar to previous question, order the following growth rate functions

$$n! \quad (n+1)! \quad 2^n \quad 2^{n+1} \quad 2^{2n} \quad n^n \quad n^{\sqrt{n}} \quad n^{\log n}.$$