Integration of functions of single variable

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The area problem

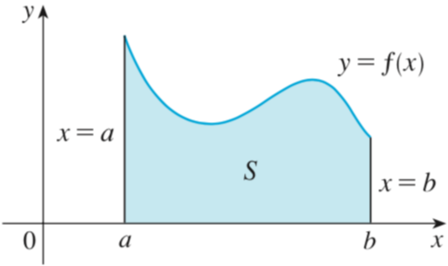
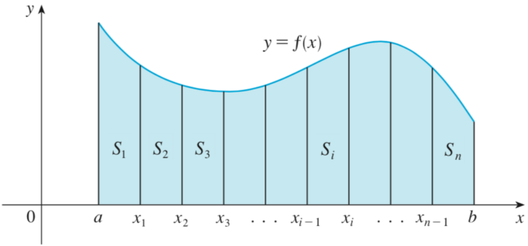


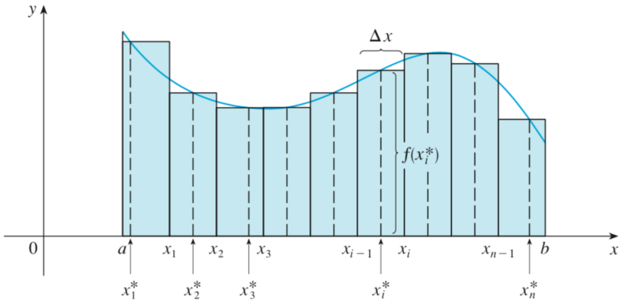
Figure: $S = \{(x, y) \mid a \leq x \leq b, 0 \leq y \leq f(x)\}$.

Problem: Find the area of the region S under the curve $y = f(x)$ from a to b, $(f(x) \geq 0)$.



A partition of $[a, b]$: $a \equiv x_0 < x_1 < \ldots < x_n \equiv b$, $\Delta x_i = x_i - x_{i-1}$.

$S = S_1 + S_2 + \ldots + S_n = \sum_{i=1}^{n} S_i$.



Pick a sample point $x_i^* \in [x_{i-1}, x_i]$, $1 \leq i \leq n$; $S_i \approx f(x_i^*)\Delta x_i$.

$$S \approx \sum_{i=1}^{n} f(x_i^*)\Delta x_i.$$

$S := \lim\limits_{\substack{n\to\infty \\ \lambda\to 0}} \sum\limits_{i=1}^{n} f(x_i^*)\Delta x_i$ if the limit exists, $\lambda = \max\limits_{1\leq i\leq n} \Delta x_i$.

Definition

Let $f(x)$ be defined on $[a, b]$. If $\lim\limits_{\substack{n\to\infty \\ \lambda\to 0}} \sum\limits_{i=1}^{n} f(x_i^*)\Delta x_i = I$ exists then I is called the definite integral of $f(x)$ on $[a, b]$.

Notation $I = \displaystyle\int_a^b f(x)dx$.

$f(x)$: integrand, a: lower limit, b: upper limit.

$f(x)$ is said to be integrable on $[a, b]$

The area of the region under the graph of $f(x) \geq 0$ from a to b is $S = \int_a^b f(x)dx$.

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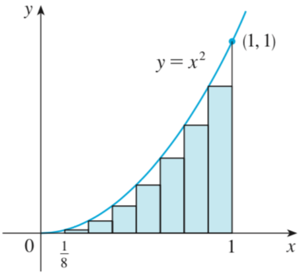
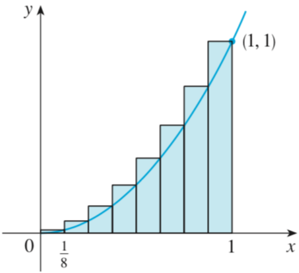
Theorem

$y = f(x)$ is integrable on $[a,b]$ iff $\lim\limits_{\substack{n\to\infty \\ \lambda\to 0}} (S - s) = 0$, where

$$S = \sum_{i=1}^{n} M_i \Delta x_i, \qquad s = \sum_{i=1}^{n} m_i \Delta x_i,$$

$$M_i = \max_{[x_{i-1},x_i]} f(x), m_i = \min_{[x_{i-1},x_i]} f(x).$$





Theorem

If $f(x)$ is continuous on $[a, b]$ then it is integrable on that interval.

Theorem

If $f(x)$ is bounded on $[a, b]$ and has finitely many jump discontinuities then it is integrable on that interval.

Theorem

If $f(x)$ is integrable on $[a, b]$, then $f(x)$ is bounded.

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Let $f(x), g(x)$ be two integrable functions on $[a, b]$.

1. $\displaystyle\int_b^a f(x)dx = -\int_a^b f(x)dx.$
2. Linearity: $A, B \in \mathbb{R}$
$$\int_a^b (Af(x) + Bg(x))dx = A\int_a^b f(x)dx + B\int_a^b g(x)dx.$$
3. Let $c \in (a, b)$
$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$
4. If $f(x) \geq 0$ for $x \in [a, b]$. Then
$$\int_a^b f(x)dx \geq 0.$$

Order preserving

Corollary

If $f(x) \leq g(x)$ for $x \in [a, b]$, then

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

In particular,

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx.$$

Mean value theorem

Theorem (Mean value theorem)

If $m \leq f(x) \leq M$ *on* $[a, b]$, *then*

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$

There exists $\mu \in [m, M]$ *such that*

$$\int_a^b f(x)dx = \mu(b-a).$$

Moreover, if $f(x)$ *is continuous on* $[a, b]$ *then there exists* $c \in [a, b]$ *such that*

$$\int_a^b f(x)dx = f(c)(b-a).$$

Example

Express the following limits in terms of definite integrals

1. $\lim_{n\to\infty} \frac{1}{n}\left(1 + \sqrt{1+\frac{2}{n}} + \ldots + \sqrt{1+\frac{n-1}{n}}\right)$
2. $\lim_{n\to\infty} \frac{1 + 2^p + \ldots + n^p}{n^{p+1}}$, $p > 1$.

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Theorem

If $f(x)$ is continuous on $[a, b]$, then

$$F(x) = \int_a^x f(t)dt, a \leq x \leq b,$$

is an antiderivative of $f(x)$ on (a, b), namely

$$\left(\int_a^x f(t)dt\right)' = f(x).$$

Corollary

Let $\alpha(x), \beta(x)$, $f(x)$ be continuous on $[a, b]$. Then

$$\frac{d}{dx}\left(\int_{\alpha(x)}^{\beta(x)} f(t)dt\right) = f(\beta(x))\beta'(x) - f(\alpha(x))\alpha'(x).$$

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Theorem (Newton-Leibniz formula)

If $f(x)$ is continuous on $[a, b]$ and $F(x)$ is an antiderivative of $f(x)$, then

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a).$$

Example

Compute the limits

1. $$\lim_{x\to 0} \frac{\int_0^{\sin x} t^2 \sin t dt}{x^4}$$
2. $$\lim_{x\to 0} \frac{\int_0^{\sin^2 x} (e^{-t^2} - 1) dt}{\int_0^{x^2} \ln(1+3t^2) dt}$$

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Theorem

If $u = g(x)\colon [a, b] \to [g(a), g(b)]$ *is continuously differentiable on* $[a, b]$, f *is continuous on* $[g(a), g(b)]$, *then*

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

Example

Evaluate the integrals

1. $$\int_1^e \frac{dx}{x(\ln^2 x + 3\ln x + 2)}$$
2. $$\int_0^1 \frac{dx}{\sqrt{1 + 2e^{-x} + 5e^{-2x}}}$$
3. $$\int_0^{\frac{3\pi}{2}} \frac{dx}{2 + \cos x}.$$

Integrals of symmetric functions

Example

1. If $f(x)$ is an even function on $[-l, l]$, then

$$\int_{-l}^{l} f(x)dx = 2\int_{0}^{l} f(x)dx.$$

2. If $f(x)$ is an odd function on $[-l, l]$, then

$$\int_{-l}^{l} f(x)dx = 0.$$

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Theorem

Let $u(x), v(x)$ be continuously differentiable on $[a, b]$. We have

$$\int_a^b u\,dv = uv \Big|_a^b - \int_a^b v\,du.$$

Example

Evaluate the integrals

1. $\displaystyle\int_0^1 x \arctan x\,dx$
2. $\displaystyle\int_0^1 \arccos^2 x\,dx$.