Partial derivatives and total differentials

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Definition

Let $f(x, y) : D \to \mathbb{R}$, $(x_0, y_0) \in D$, open set $D \subset \mathbb{R}^2$.
The partial derivative with respect to x is

$$f'_x(x_0, y_0) \frac{\partial f}{\partial x}(x_0, y_0) = \frac{d}{dx} f(x, y_0) \Big|_{x=x_0}$$
$$= \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

The partial derivative with respect to y is

$$f'_y(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = \frac{d}{dy} f(x_0, y) \Big|_{y=y_0}$$
$$= \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

Rule: when differentiating with respect to x, other variables are considered constants.

Example

Compute the partial derivatives of the following functions

1. $f(x, y) = x^y$ at $(2, 1)$.
2. $u(x, y, z) = z\sqrt{x^2 + y^2}$.
3. $f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$

Multivariable chain rule (case 1)

Theorem

Suppose that $z = f(x, y)$ has continuous partial derivatives, where $x = x(t)$ and $y(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$z'(t) = f'_x.x'(t) + f'_y.y'(t).$$

Multivariable chain rule (case 2)

Recall: The composition $g \circ f : X \subset \mathbb{R}^2 \xrightarrow{f} Y \subset \mathbb{R}^2 \xrightarrow{g} \mathbb{R}$ is

$$(x, y) \overset{f}{\mapsto} (u(x, y), v(x, y)) \overset{g}{\mapsto} \underbrace{g((u(x, y), v(x, y)))}_{F(x,y)}$$

Theorem

Assume that $g(u, v)$ has continuous partial derivatives in Y, u, v have partial derivatives in X. Then the composition function $F = g \circ f : X \to \mathbb{R}$ has partial derivatives in X and

$$F'_x = g'_u.u'_x + g'_v.v'_x, \qquad F'_y = g'_u.u'_y + g'_v.v'_y$$

The chain rule can be written in the matrix form as

$$\begin{cases} F'_x = g'_u.u'_x + g'_v.v'_x, \\ F'_y = g'_u.u'_y + g'_v.v'_y \end{cases} \Rightarrow \begin{pmatrix} F'_x & F'_y \end{pmatrix} = \begin{pmatrix} g'_u & g'_v \end{pmatrix} \begin{pmatrix} u'_x & u'_y \\ v'_x & v'_y \end{pmatrix},$$

where $\dfrac{D(u,v)}{D(x,y)} = \begin{pmatrix} u'_x & u'_y \\ v'_x & v'_y \end{pmatrix}$ is called the Jacobian matrix of f.

Example

Compute the partial derivatives of the following functions

1. $f(u,v) = \sin(u^2+v) - e^{2u-v},\ u = \ln(x^2+y^2), v = xy.$
2. $f(x,y) = x.e^{xy},\ x = \ln(2+t^2), y = t^2 - t + 1.$
3. $g(t) = \ln(t^3+1) + \cos(2t^2),\ t = x^2 + 2y.$

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Total differentials

Definition

Given $f(x, y)\colon D \subset \mathbb{R}^2 \to \mathbb{R}$, $M_0(x_0, y_0) \in D$. If we can express

$$\Delta f = f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0) = A\Delta x + B\Delta y + \alpha\Delta x + \beta\Delta y,$$

where the constants A, B depend only on x_0, y_0, and the infinitesimals α, β tend to 0 as $\Delta x, \Delta y \to 0$, we say that $f(x, y)$ is differentiable at M_0;

$df(x_0, y_0) = A\Delta x + B\Delta y$: the total diffenrential of f at M_0.

f is said to be differentiable on D if f is differentiable at all $M_0 \in D$.

Example

Is the function $z = 2x - y^2$ differentiable at $(1, 0)$?

Properties

- If f is differentiable at M_0 then f is continuous at M_0. Indeed, let $\Delta x, \Delta y \to 0$, then
  $f(x_0+\Delta x, y_0+\Delta y)-f(x_0, y_0) = A\Delta x+B\Delta y+\alpha\Delta x+\beta\Delta y \to 0.$
- $\Delta y = 0$, $\dfrac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} = A + \alpha.$
  Passing to the limit as $\Delta x \to 0$: $A = f'_x(x_0, y_0)$.
  Similarly, $B = f'_y(x_0, y_0)$. $df = f'_x\Delta x + f'_y\Delta y$.
  If f is differentiable at M_0 then f has partial derivatives at M_0. However, the converse is not necessary true.

Example

Is $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0), \end{cases}$ differentiable at $(0,0)$?

f is not differentiable at $(0,0)$ although $f'_x(0,0) = f'_y(0,0) = 0$.

Theorem

If $f'_x(x,y), f'_y(x,y)$ exist in $B(M_0,\varepsilon)$ and are **continuous** *at M_0. Then, $f(x,y)$ is differentiable at M_0 and*

$$df(M_0) = f'_x(M_0)\Delta x + f'_y(M_0)\Delta y.$$

Remark

$f(x,y)$ **discontinuous at** $M_0 \Rightarrow f$ **is not differentiable** at M_0.

Take $f(x,y) = x$, then $df = \Delta x = dx$.
Similarly, $\Delta y = dy$, $df = f'_x dx + f'_y dy$.

Example

Compute the total differential of the following functions

$$a)z = \frac{1}{2}(x^2 + y^2) \qquad b)z = x^y$$

$$c)z = \arctan xy \qquad d)u = \frac{z}{\sqrt{x^2 + y^2}}, du(1, 2, 3).$$

Approximations using total differentials

We have:

$$f(x_0+\Delta x, y_0+\Delta y) \approx f(x_0, y_0) + \underbrace{f'_x(x_0, y_0)\Delta x + f'_y(x_0, y_0)\Delta y}_{df(x_0,y_0)}.$$

Example

Approximate the following values

$$a)\, \ln(\sqrt[3]{1,02} + \sqrt[4]{0,98} - 1) \qquad b)\, \sqrt[3]{1,02^2 + 1,98^3 - 1}$$

Invariance of the first order total differential

Given $f(u, v)$. The total differential of f is

$$df = f'_u du + f'_v dv.$$

If we can write $u = u(x, y), v = v(x, y)$ then f can be rewritten as $F(x, y) := f(u(x, y), v(x, y))$. The total differential of the composition function is $dF = F'_x dx + F'_y dy$. By the chain rule

$$\begin{cases} F'_x = f'_u u'_x + f'_v v'_x, \\ F'_y = f'_u u'_y + f'_v v'_y. \end{cases}$$

Therefore,

$$\begin{aligned} dF &= (f'_u u'_x + f'_v v'_x)dx + (f'_u u'_y + f'_v v'_y)dy \\ &= f'_u(u'_x dx + u'_y dy) + f'_v(v'_x dx + v'_y dy) \\ &= f'_u du + f'_v dv = df. \end{aligned}$$