# Limit of a function

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# Two equivalent definitions of limit of a function

## Definition

Let $f(x)$ be a function defined on $(a, b)$, possibly except at $x_0 \in (a, b)$. The limit of $f(x)$ is a finite number $L$ as $x$ tends to $x_0$, denoted by $\lim\limits_{x \to x_0} f(x) = L$, iff

$$\forall \{x_n\} \subset (a, b) \setminus \{x_0\} : \lim_{n\to\infty} x_n = x_0 \Rightarrow \lim_{n\to\infty} f(x_n) = L.$$

$$\Leftrightarrow \forall \varepsilon > 0, \exists \delta(\varepsilon, x_0) > 0 \,\Big|\, 0 < |x - x_0| < \delta : |f(x) - L| < \varepsilon.$$

## Example

- $\lim\limits_{x\to 1}(2x+3) = 5$.
- There does not exist $\lim\limits_{x\to 0} \sin\frac{1}{x}$.
  Prove of nonexistence of $\lim\limits_{x\to x_0} f(x)$: there exist two sequences $\{x_n\}$ and $\{x'_n\}$ such that $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} x'_n = x_0$
  BUT $\lim\limits_{n\to\infty} f(x_n) \neq \lim\limits_{n\to\infty} f(x'_n)$.

# One sided limits

**Definition**

Right hand limit at $x_0$:

$$\lim_{x \to x_0^+} f(x) = L \Leftrightarrow \forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 \mid 0 < x - x_0 < \delta : |f(x) - L| < \varepsilon.$$

Left hand limit at $x_0$:

$$\lim_{x \to x_0^-} f(x) = L \Leftrightarrow \forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 \mid 0 < x_0 - x < \delta : |f(x) - L| < \varepsilon.$$

We write: $f(x_0^+) = \lim_{x \to x_0^+} f(x)$, $f(x_0^-) = \lim_{x \to x_0^-} f(x)$.

Note: $\lim_{x \to x_0} f(x) = L \Leftrightarrow \lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = L$.

# Limit at infinity

## Definition

$$\lim_{x \to \infty} f(x) = L \Leftrightarrow \forall \varepsilon > 0, \exists N(\varepsilon) > 0 \mid |x| > N(\varepsilon) : |f(x) - L| < \varepsilon.$$

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# Properties

1. The limit $\lim\limits_{x \to x_0} f(x)$, if exists, is **unique**.
2. Squeeze theorem: assume that $f(x) \leq g(x) \leq h(x)$ for all $x \in (a, b) \setminus \{x_0\}$, and

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = L.$$

Then $\lim\limits_{x \to x_0} g(x) = L$.

## Example

$$\lim_{x \to 0} x \cos \frac{1}{x} = 0.$$

# Limit laws

## Theorem

*Assume that* $\lim_{x\to x_0} f(x) = a$, $\lim_{x\to x_0} g(x) = b$. *Then*

- $\lim_{x\to x_0} [f(x) \pm g(x)] = a \pm b.$
- $\lim_{x\to x_0} f(x).g(x) = a.b.$
- $\lim_{x\to x_0} \dfrac{f(x)}{g(x)} = \dfrac{a}{b}$ $(b \neq 0,\ g(x) \neq 0\ \forall x_0 - \varepsilon < x < x_0 + \varepsilon).$

Note: the theorem excludes the case $a = \infty$ or $b = \infty$.
$x_0$ stands for $x_0^{\pm}, \infty$. In such cases, we work with indeterminate forms $\infty - \infty$, $0.\infty$, $\frac{0}{0}$, $\frac{\infty}{\infty}$.

# Evaluating indeterminate forms

## Example

Find the following limits

$$a)\ \lim_{x\to 1} \frac{x^3 - 3x^2 + 2}{x^2 - 1} \qquad b)\ \lim_{x\to 1} \frac{\sqrt[3]{x} - 1}{\sqrt{x+3} - 2}$$

$$c)\ \lim_{x\to 2} (2 - x) \tan \frac{\pi x}{4}$$

# Limit of composite functions

**Theorem**

*Assume that* $\lim\limits_{x \to x_0} u(x) = u_0$, $\lim\limits_{u \to u_0} v(u) = v_0$. *Then*

$$\lim_{x \to x_0} v \circ u(x) = v_0.$$

**Example**

$$\lim_{x \to x_0} [f(x)]^{g(x)} = \lim_{x \to x_0} \exp\left(g(x) \ln f(x)\right) = \exp\left[\lim_{x \to x_0} \left(g(x) \ln f(x)\right)\right].$$

# Infinite limits

## Definition

$$\lim_{x \to x_0} f(x) = \infty \Leftrightarrow \forall\, M > 0, \exists \delta(\varepsilon) > 0 \,\Big|\, |x - x_0| < \delta(\varepsilon) : |f(x)| > M.$$

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# Definition

## Definition

A quantity $\alpha(x)$ is called an infinitesimal when $x \to x_0$ if $\lim_{x \to x_0} \alpha(x) = 0$.

## Example

- $x^m$, $m > 0$, is an infinitesimal as $x \to 0$.
- $e^{\sqrt{x}} - 1$ is an infinitesimal as $x \to 0^+$.
- $\cot x$, $\cos x$ are infinitesimals as $x \to \frac{\pi}{2}$.

# Properties

1. If $\lim_{x\to x_0} f(x) = a$, we can write $f(x) = a + \alpha(x)$, where $\alpha(x)$ is an infinitesimal as $x \to x_0$.
2. If $\alpha(x), \beta(x)$ are infinitesimals as $x \to x_0$ then $c_1\alpha(x) + c_2\beta(x)$, $c_1, c_2$ are constants, is also an infinitesimal as $x \to x_0$.
3. If $\alpha(x)$ is an infinitesimal as $x \to x_0$, and $f(x)$ is **bounded** in $(x_0 - \varepsilon, x_0 + \varepsilon)$ then $\alpha(x)f(x)$ is also an infinitesimal as $x \to x_0$.

**Example**

$3x \sin \frac{1}{x} + 5 \ln(1 - 2x)$ is an infinitesimal as $x \to 0$.

# Comparison of infinitesimals

## Definition

Let $\alpha(x)$, $\beta(x)$ be infinitesimals as $x \to x_0$ and $\lim\limits_{\mathbf{x}\to\mathbf{x_0}} \frac{\alpha(\mathbf{x})}{\beta(\mathbf{x})} = \mathbf{k}$.

- If $k = 0$, $\alpha(x)$ is of higher order than $\beta(x)$, written as $\boldsymbol{\alpha(x) = o(\beta(x))}$. Or $\beta(x)$ is of lower order than $\alpha(x)$.
- If $k \neq 0, \infty$, $\alpha(x)$ and $\beta(x)$ are of the same order, written as $\boldsymbol{\alpha(x) = O(\beta(x))}$.

  In particular, $k = 1$, $\alpha(x)$ and $\beta(x)$ are said to be equivalent, written as $\boldsymbol{\alpha(x) \sim \beta(x)}$.

Q: If $k = \infty$ $\beta(x) = o(\alpha(x))$.
For $m, n > 0$: $x^m$ is of higher order than $x^n$ as $x \to 0$ iff $m > n$.

**Example (Fundametal equivalent pairs of infinitesimals)**

As $x \to 0$:

- $x \sim \sin x \sim \tan x \sim \arcsin x \sim \arctan x$.
- $\ln(1+x) \sim x \sim e^x - 1$.
- $1 - \cos x \sim \frac{x^2}{2}$.
- $(1+x)^\alpha - 1 \sim \alpha x,\ \alpha \in \mathbb{R}$.

# Evaluating indeterminate forms

In the process $x \to x_0$

## Proposition

1. $\alpha(x) + o(\alpha(x)) \sim \alpha(x)$.
2. *If* $\alpha(x) \sim \widetilde{\alpha}(x)$, $\beta(x) \sim \widetilde{\beta}(x)$, *then*

$$\lim_{x \to x_0} \frac{\alpha(x)}{\beta(x)} = \lim_{x \to x_0} \frac{\widetilde{\alpha}(x)}{\widetilde{\beta}(x)}.$$

## Example

Find the following limits

1. $\lim\limits_{x\to 0} \dfrac{(2^x - 1)\ln(1+2x)}{\sin^2 x}$
2. $\lim\limits_{x\to 0} \dfrac{\sqrt[4]{1+5x} - \sqrt[3]{1-2x}}{x}$
3. $\lim\limits_{x\to 0} \dfrac{x + \sin^3 x - 2\tan^4 x}{e^{2x} - \cos x}$

# Definition

**Definition**

A quantity $A(x)$ is called an infinity as $x \to x_0$ if $\lim_{x \to x_0} A(x) = \infty$.

**Example**

- $x^n$, $n > 0$, are infinities as $x \to \infty$.
- $x^n$, $n < 0$, are infinities as $x \to 0$.
- $\tan x$ is an infinity as $x \to \frac{\pi}{2}$.

Note: $A(x)$ is an infinity as $x \to x_0$ iff $\frac{1}{A(x)}$ is an infinitesimal as $x \to x_0$.

# Comparison of infinities

## Definition

Let $A(x)$, $B(x)$ be two infinities as $x \to x_0$ and $\lim\limits_{x \to x_0} \frac{A(x)}{B(x)} = k$.

- If $k = 0$, $A(x)$ is said to be of lower order than $B(x)$ as $x \to x_0$, ($B(x)$ is an infinity of higher order than $A(x)$).
- If $k \neq 0, \infty,$ we say $A(x)$ and $B(x)$ are of the same order.

  In particular, $k = 1$, we say $A(x)$ and $B(x)$ are equivalent, written as $A(x) \sim B(x)$.

Q: $k = \infty$?

# Evaluating indeterminate forms

As $x \to x_0$:

**Proposition**

1. *If $\overline{A}(x)$ is an infinity of lower order than $A(x)$. Then $A(x) + \overline{A}(x) \sim A(x)$.*
2. *If $A(x) \sim \overline{A}(x)$, $B(x) \sim \overline{B}(x)$, then*

$$\lim_{x \to x_0} \frac{A(x)}{B(x)} = \lim_{x \to x_0} \frac{\overline{A}(x)}{\overline{B}(x)}.$$

**Example**

$$\lim_{x \to \infty} \frac{x + 2x^2 + 4x^5}{2014x^2}$$

PAY ATTENTION

Be careful when substituting equivalent infinitesimals or infinities in a sum or a difference.

Ex: $\sin x \sim \tan x$ khi $x \to 0$.

BUT $\sin x - x \sim -\frac{x^3}{3!}$ and $\tan x - x \sim \frac{x^3}{3!}$ are not equivalent as $x \to 0$.