Second order ordinary differential equations

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Second order ODEs

A **second order** differential equation has the form

- $F(x, y, y', y'') = 0$.
- $y''(x) = G(x, y, y')$.

A second order **linear** differential equation has the form $y'' = f(x) - q(x)y - p(x)y'$.

$$\boxed{y'' + p(x)y' + q(x)y = f(x)}$$

$p(x), q(x)$: coefficients, $f(x)$: right hand side of the equation.
We restrict $x \in I \subset \mathbb{R}$ in which $p(x), q(x), f(x)$ are continuous.

Theorem (Existence and Uniqueness theorem)

Consider the initial value problem (IVP)

$$\begin{cases} y'' = f(x, y, y'), x \in U_\varepsilon(x_0), \\ y(x_0) = y_0, y'(x_0) = y'_0. \end{cases}$$

Assume that the function $f(x, y, y')\colon D \subset \mathbb{R}^3 \to \mathbb{R}$ and its partial derivatives $f'_y(x, y, y'), f'_{y'}(x, y, y')$ are **continuous** *on D, $(x_0, y_0, y'_0) \in D$. Then there exists a unique solution $y(x)$ in a vinicity $(x_0 - \varepsilon, x_0 + \varepsilon)$.*

Under the assumption on continuity of $p(x), q(x), f(x)$, the IVP

$$y'' + p(x)y' + q(x)y = f(x), y(x_0) = y_0, y'(x_0) = y'_0$$

has exactly one solution on I.

Definition

1. General solution $y = \varphi(x, C_1, C_2)$ such that
   - $\varphi(x, C_1, C_2)$ satisfies the equation for all C_1, C_2.
   - Given an initial value $(x_0, y_0, y_0') \in D$, we can find C_1^0, C_2^0 such that $\varphi(x, C_1^0, C_2^0)$ solves the IVP.
2. Particular solution $y = \varphi(x, C_1^0, C_2^0)$ which is obtained from the **general solution**.

Example

Spring-mass system $kx'' + mx = 0$ has general solution $x(t) = C_1 \cos \omega t + C_2 \sin \omega t$, $\omega = \sqrt{\dfrac{k}{m}}$. $x(t) = A \cos \omega t$ is a particular which satisfies the initial values $x(0) = A, x'(0) = 0,\ C_1 = A, C_2 = 0$.

General form of second order ODEs

$$F(x, y, y', y'') = 0.$$

1. Equations without y: $F(x, y', y'') = 0$.
2. Equations without x: $F(y, y', y'') = 0$.

Equations without y

1. Equations without y: $F(x, y', y'') = 0$. (Reduction of order)
   Set $y' = p$, then $y'' = p'$, we obtain $F(x, p, p') = 0$
   $\Rightarrow y' = p = \varphi(x, C_1) \Rightarrow y = \Phi(x, C_1, C_2)$.

Example

Solve the equation $xy'' + 2y' = 12x$.

Equations without x

2. Equations without x: $F(y, y', y'') = 0$.
   Set $y' = p$, $p = p(y)$.

$$y''(x) = p'(x) = p'(y)y'(x) = p'(y).p.$$

   The equation reads as $G\left(y, p, p'\right) = 0$ (first order ODE).
   $\Rightarrow p = \varphi(y, C_1) \Leftrightarrow y' = \varphi(y, C_1)$ (first order equation).

Example

Solve the equation $y''(1 + y) = y'^2 + y'$.

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$$\boxed{y'' + p(x)y' + q(x)y = f(x)} \tag{1}$$

where $p(x), q(x), f(x)$ are continuous on I.
$f(x) \equiv 0$: corresponding homogeneous equation.
$f(x) \neq 0$: inhomogeneous equation.
$Ly = y'' + p(x)y' + q(x)y$ then $L(y_1 + y_2) = Ly_1 + Ly_2$.

Theorem (Superposition of solutions)

Assume that

- y_1 *is a solution of the equation* $y'' + p(x)y' + q(x)y = f_1(x)$,
- y_2 *is a solution of the equation* $y'' + p(x)y' + q(x)y = f_2(x)$.

Then $y_1 + y_2$ *solves the equation*

$$y'' + p(x)y' + q(x)y = f_1(x) + f_2(x).$$

Theorem (Structure of the solution)

The general solution of the inhomogeneous equation has the form

$$y = \bar{y} + y^*,$$

- $\bar{y}$ *is the general solution of the corresponding homogeneous equation* $y'' + p(x)y' + q(x)y = 0$.
- y^* *is a particular solution of the inhomogeneous equation* (1).

$$y'' + p(x)y' + q(x)y = 0. \tag{2}$$

Theorem

If $y_1(x), y_2(x)$ are solutions of (2) then $y(x) = C_1y_1(x) + C_2y_2(x)$, $C_1, C_2 \in \mathbb{R}$ is also a solution of (2).

Conversely, is any solution of (2) of this form?

Linear dependence VS linear independence

Definition

The functions $y_1(x), y_2(x)$ are called linearly independent on I if

$$k_1 y_1(x) + k_2 y_2(x) = 0 \,\forall\, x \in I \text{ implies that } k_1 = k_2 = 0.$$

$y_1(x), y_2(x)$ are called linearly dependent on I if there exist k_1, k_2 not both zero such that

$$k_1 y_1(x) + k_2 y_2(x) = 0 \,\forall\, x \in I.$$

Example

- e^x, e^{2x}.
- x, x^2.

Wronsky determinant

Definition

Wronsky determinant of the two solutions $y_1(x), y_2(x)$ is

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'.$$

When y_1, y_2 are clear, we denote by $W(x)$.

Theorem

If the solutions $y_1(x), y_2(x)$ are **linearly dependent** *on $[a, b]$ then $W(x) = 0$ for all $x \in [a, b]$.*

Theorem

Let y_1, y_2 be two solutions of the homogeneous equation on I. Then the following are equivalent

1. y_1, y_2 *are linearly independent.*
2. $W(y_1, y_2)(x_0) \neq 0$ *for some* $x_0 \in I$.
3. $W(y_1, y_2)(x) \neq 0$ *for all* $x \in I$.

Proof. $W(x)$ satisfies that $W' + p(x)W = y_1 Ly_2 - y_2 Ly_1 = 0$.

$$W(x) = W(x_0) e^{-\int_{x_0}^{x} p(t)dt}, \forall x \in I.$$

Structure of solutions to the homogeneous equations

Theorem

Assume that $y_1(x), y_2(x)$ *are two* **linearly independent solutions** *of the equation*

$$y'' + p(x)y' + q(x)y = 0.$$

Then the general solution is

$$\bar{y}(x) = C_1 y_1(x) + C_2 y_2(x).$$

Proof.

- y_1, y_2 are solutions then $C_1 y_1 + C_2 y_2$ is also a solution.
- Plugging the initial conditions $y(x_0) = y_0, y'(x_0) = y_0'$, we get
  $$\begin{cases} C_1 y_1(x_0) + C_2 y_2(x_0) = y_0 \\ C_1 y_1'(x_0) + C_2 y_2'(x_0) = y_0'. \end{cases}$$
  y_1, y_2 are linearly independent, so $W(y_1, y_2)(x_0) \neq 0 \Rightarrow$ we get unique solutions C_1^0 và C_2^0.

Reduction of order

Question: Given a particular solution to the homogeneous equation, find a second particular solution $y_2(x)$ which is linearly independent with $y_1(x)$.
We look for $y_2(x) = u(x).y_1(x)$, where $u(x) \not\equiv C$:

$$W(x) = \begin{vmatrix} y_1 & y_1 u \\ y_1' & (y_1 u)' \end{vmatrix} = -y_1^2 u' \neq 0.$$

Theorem (Liouville formula)

$y_2(x)$ can be found as

$$y_2(x) = y_1(x) \int \frac{e^{-\int p(x)dx}}{y_1^2(x)} dx.$$

Example

Solve the following differential equations

- $y'' + \dfrac{x}{1-x}y' - \dfrac{1}{1-x}y = 0$, given a particular solution $y_1 = e^x$.
- $x^2y'' + xy' - y = 0,\ y_1 = x$.
- $xy'' + 2y' + xy = 0,\ y_1 = \dfrac{\sin x}{x}$.

Homogeneous equations with constant coefficients

The corresponding homogeneous equation

$$y'' + py' + qy = 0.$$

Solve the characteristic equation

$$k^2 + pk + q = 0. \tag{3}$$

1. (3) has two distinct **real roots** $k_1 \neq k_2$, then $\bar{y} = C_1 e^{k_1 x} + C_2 e^{k_2 x}$.
2. (3) has a **double root** $k_1 = k_2 = k$, then $\bar{y} = (C_1 x + C_2) e^{kx}$.
3. (3) has two **complex conjugate roots** $k_{1,2} = \alpha \pm i\beta$, then $\bar{y} = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$.

Example

Solve the equations

- $y'' - 3y' + 2y = 0.$
- $y'' + 2y' + y = 0.$
- $y'' + 6y' + 10y = 0,\ y(0) = -1, y'(0) = 2.$