## Integration of functions of single variable

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#### Indefinite integrals

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## Motivation

#### Definition. **Properties**

- Given the velocity of a function, one wishes to know its position at a given time.
- Find a function whose derivative is known.

Definition.

Properties

## Antiderivatives

#### Definition

A function F(x) is called an antiderivative of f(x) on an interval I if F'(x) = f(x) for all  $x \in I$ .

#### Example

- $\left(\ln(1+x^2)\right)' = \frac{2x}{1+x^2}$  so  $\ln(1+x^2)$  is an antiderivative of  $\frac{2x}{1+x^2}$ .
- $x^4 + 2$  is an antiderivative of  $4x^3$ .

## Theorem

If F(x) is an antiderivative of f(x) on I. Then the family of all antiderivatives of f(x) is F(x) + C.

#### Definition

The family of all antiderivatives is called the indefinite integral of f(x).

Denote  $\int f(x)dx = F(x) + C$ , where F(x) is a known antiderivative.

$$\int \frac{2x}{1+x^2} dx = \ln(1+x^2) + C, \qquad \int 4x^3 dx = x^4 + C.$$

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# Linearity

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Let F(x), G(x) be antiderivatives of f(x), g(x) respectively. Then

$$\int (Af(x) + Bg(x))dx = AF(x) + BG(x) + C.$$

#### Theorem

A function f(x) which is continuous on [a, b] has an antiderivative on that interval.

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$$\int x^{\alpha} dx = \begin{cases} \frac{x^{\alpha+1}}{\alpha+1} + C & \text{if } \alpha \neq -1, \\ \ln|x| + C & \text{if } \alpha = -1. \end{cases}$$

$$\int a^{x} dx = \frac{a^{x}}{\ln a} + C, 0 < a \neq 1$$

$$\int \sin x dx = -\cos x + C, \qquad \int \cos x dx = \sin x + C$$

$$\int \frac{dx}{\sin^{2} x} = -\cot x + C, \qquad \int \frac{dx}{\cos^{2} x} = \tan x + C$$

$$\int \frac{dx}{a^{2} + x^{2}} = \frac{1}{a} \arctan \frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{a^{2} - x^{2}}} = \arcsin \frac{x}{a} + C.$$

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Substitution Rule

## The Substitution Rule

#### Theorem

If u = g(x) is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)dx = \int f(u)du.$$

#### Example

Evaluate the following integrals

$$\int \frac{x^4 dx}{x^{10} + 1}$$

$$\int \frac{dx}{e^x - e^{-x}}$$

$$\int \frac{dx}{\sqrt{x^2+4}}$$

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## Integration by parts

Assume that u(x), v(x) are continuously differentiable functions. We have

$$\int u dv = uv - \int v du.$$

#### Example

Evaluate the integrals

$$\int \frac{x}{\cos^2 x} dx$$

$$\int \sqrt{x^2 + \alpha} dx$$

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$$\int \frac{dx}{\sqrt{x^2 + \alpha}} = \ln|x + \sqrt{x^2 + \alpha}| + C$$

$$\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a} + C$$

$$\int \sqrt{x^2 + \alpha} dx = \frac{x}{2} \sqrt{x^2 + \alpha} + \frac{\alpha}{2} \ln|x + \sqrt{x^2 + \alpha}| + C.$$

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Trigonometric integrals Rationalizing substitutions Aim: Evaluate  $\int R(x)dx$ , where

$$R(x) = \frac{b_0 + b_1 x + \ldots + b_m x^m}{a_0 + a_1 x + \ldots + a_n x^n}, a_i, b_i \in \mathbb{R}, a_n, b_m \neq 0,$$

is a rational function.

Method: expressing R(x) as a sum of partial fractions.

- Performing a polynomial division to reduce to proper rational function.
- 2 Factorizing the denominator into factors  $(x-a)^k$ ,  $(x^2+px+q)^k$ , where  $q-\frac{p^2}{4}>0$ .
- 3 Writing R(x) as the sum of following functions

$$\int \frac{A_l dx}{(x-a)^l}, \quad \int \frac{B_l x + C_l}{(x^2 + px + q)^l} dx, l = 1, 2, \dots, k.$$

#### Integrals of rational

#### Example

#### Evaluate the integrals

$$\int \frac{xdx}{(x+2)^2(x-1)}$$

1 
$$\int \frac{xdx}{(x+2)^2(x-1)}$$
  
2  $\int \frac{xdx}{x^4+3x^2+2}$ 

$$\int \frac{xdx}{x^4 + 3x^2 + 2}$$

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## Trigonometric integrals $\int \mathcal{R}(\sin x, \cos x) dx$

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Trigonometric integrals Rationalizing substitutions Examples of Euler substitutions Evaluate  $\int \mathcal{R}(\sin x, \cos x) dx$ , where  $\mathcal{R}(\sin x, \cos x)$  is a rational function of the variables  $u = \sin x$ ,  $v = \cos x$ .

General substitution  $t = \tan \frac{x}{2}$ ,  $t \in (-\pi, \pi)$ . We obtain

$$\int \mathcal{R}\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right) \frac{2dt}{1+t^2}.$$

#### Example

Evaluate 
$$\int \frac{dx}{2\sin x - \cos x + 5}.$$

## Special cases

## Indefinite integrals

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Rationalizing substitutions Examples of Eule substitutions ■ If  $\mathcal{R}(-\sin x, -\cos x) = \mathcal{R}(\sin x, \cos x)$ , set  $t = \tan x$  or  $t = \cot x$ . Examples:  $\int \frac{dx}{\sin^2 x \cos^4 x}$ ,  $\int \frac{\tan x dx}{1 + \cos^2 x}$ .

■ If  $\mathcal{R}(-\sin x, \cos x) = -\mathcal{R}(\sin x, \cos x)$ , set  $t = \cos x$ . Examples  $\int \frac{dx}{(2+\cos x)\sin x}$ 

If  $\mathcal{R}(\sin x, -\cos x) = -\mathcal{R}(\sin x, \cos x)$ , set  $t = \sin x$ .

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$$\int \frac{a_1 \cos x + b_1 \sin x}{a \sin x + b \cos x} dx = \int \left( A + B \frac{a \cos x - b \sin x}{a \sin x + b \cos x} \right) dx$$
$$= Ax + B \ln|a \sin x + b \cos x| + C.$$

#### Example

$$\int \frac{\sin x - \cos x}{\sin x + 2\cos x} dx, \quad \int \frac{\sin x}{\sin x - 3\cos x} dx$$

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$$\int \mathcal{R}(x, \sqrt{a^2 - x^2}) dx.$$
Set  $x = a \sin t$ ,  $t \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right]$ , or  $x = a \cos t$ ,  $t \in [0, \pi]$ .

$$\int \mathcal{R}(x, \sqrt{x^2 - a^2}) dx.$$
Set  $x = \frac{a}{\sin t}$ ,  $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ ,  $t \neq 0$ , or  $x = \frac{a}{\cos t}$ ,  $t \in (0, \pi)$ ,  $t \neq \frac{\pi}{2}$ .

$$\int \mathcal{R}(x, \sqrt{a^2 + x^2}) dx.$$
Set  $x = a \tan t$ ,  $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ , or  $x = a \cot t$ ,  $t \in (0, \pi)$ .

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#### Example

#### Evaluate

$$\int \frac{x^3 dx}{\sqrt{1-x^2}}$$

$$\int \frac{dx}{x\sqrt{x^2 + 2x + 2}}$$

$$\int x\sqrt{-x^2+4x-3}dx$$

$$\int \mathcal{R}(x,\sqrt{Ax^2+Bx+C})dx,\ A>0$$

Rule: A > 0, set  $\sqrt{Ax^2 + Bx + C} = t \pm \sqrt{Ax}$ .

#### Example

Evaluate 
$$I = \int \frac{dx}{x + \sqrt{x^2 + x + 1}}$$

Set  $\sqrt{x^2 + x + 1} = t - x$ . Then  $x = \frac{t^2 - 1}{2t + 1}$ ,  $dx = \frac{2(t^2 + t + 1)}{(2t + 1)^2} dt$ .

$$I = \int \frac{2(t^2 + t + 1)}{t(2t+1)^2} dt = \int \left(\frac{2}{t} - \frac{3}{2t+1} - \frac{3}{(2t+1)^2}\right) dt$$

$$= 2\ln|t| - \frac{3}{2}\ln|2t+1| + \frac{3}{2(2t+1)} + C$$

$$= 2\ln|\sqrt{x^2 + x + 1} + x| - \frac{3}{2}\ln|2(\sqrt{x^2 + x + 1} + x) + 1| + \frac{3}{4(\sqrt{x^2 + x + 1} + x) + 2} + C$$

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$$\int \mathcal{R}(x,\sqrt{Ax^2+Bx+C})dx$$
,  $C>0$ 

Rule: C > 0, set  $\sqrt{Ax^2 + Bx + C} = xt \pm \sqrt{C}$ .

#### Example

Evaluate  $I = \int \frac{dx}{1+\sqrt{1-2x-x^2}}$ Set  $\sqrt{1-2x-x^2} = tx-1 \Rightarrow x = \frac{2(t-1)}{t^2+1}$ ,  $dx = \frac{2(3t^2-2t+1)}{(t^2+1)^2}dt$ .

$$I = \int \frac{3t^2 - 2t + 1}{t(t - 1)^2(t^2 + 1)} dt = \int \left( -\frac{1}{t} + \frac{1}{t - 1} + \frac{2}{t^2 + 1} \right) dt$$

$$= -\ln|t| + \ln|t - 1| + 2\arctan t + C$$

$$= -\ln\left|\frac{\sqrt{1 - 2x - x^2} + 1}{x}\right| + \ln\left|\frac{\sqrt{1 - 2x - x^2} + 1}{x} - 1\right| + 2\arctan\frac{\sqrt{1 - 2x - x^2} + 1}{x} + C$$

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$$\int \mathcal{R}(x,\sqrt{A(x-\alpha)(x-\beta)})dx$$

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Rule: If 
$$Ax^2 + B + C = A(x - \alpha)(x - \beta)$$
, set  $\sqrt{Ax^2 + Bx + C} = t(x - \alpha)$  or  $\sqrt{Ax^2 + Bx + C} = t(x - \beta)$ .

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Examples of Euler substitutions Fact: there are functions that  $\int f(x)dx$  is not an elementary function. We cannot evaluate their integrals in terms of the functions we know, for instance

$$\int e^{x^2} dx, \int \frac{\sin x}{x} dx, \int \frac{1}{\ln x} dx \dots$$