Functions of multivariables: basic concepts

Nguyen Thu Huong

School of Applied Mathematics and Informatics
Hanoi University of Science and Technology

January 11, 2021

Content

Content

The space $\mathbb{R}^n$

Consider the set

$$\mathbb{R}^n = \{x = (x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}, i = 1, 2, \ldots, n\}.$$

Vector space structure:

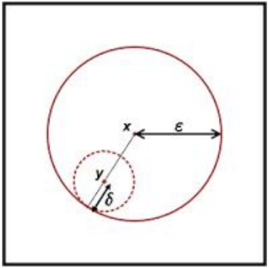
$$\begin{aligned} x + y &= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n), \\ kx &= (kx_1, kx_2, \ldots, kx_n), \qquad k \in \mathbb{R}, x, y \in \mathbb{R}^n. \end{aligned}$$

Euclidean space $\mathbb{R}^n$ with the distance: for $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$:

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_n - y_n)^2}.$$

Example

- $n = 1$, $d(x, y) = |x - y|$.
- $n = 2$, $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$.
- $n = 3$, $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$.



Example

- **Closed ball** $\overline{B}(x, \varepsilon) = \{M \in \mathbb{R}^n \mid d(M, x) \leq \varepsilon\}$.
- **Open ball** $B(y, \delta) = \{M \in \mathbb{R}^n \mid d(M, y) < \delta\}$.

Definition

Let $X \subset \mathbb{R}^n$.

- M_0 is an interior point of X if there exists $\varepsilon > 0$ such that $B(M_0, \varepsilon) \subset X$.
- M_0 is a boundary point of X if for all $\varepsilon > 0$: each $B(M_0, \varepsilon) \cap X \neq \emptyset$ and $B(M_0, \varepsilon) \setminus X \neq \emptyset$.

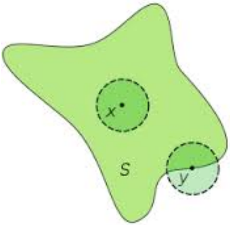
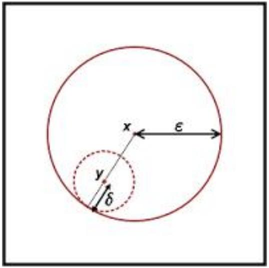


Figure: Interior and boundary points

Definition

Let $X \subset \mathbb{R}^n$.

- X is closed if it contains all boundary points.
- X is open if all points of X are interior points.
- A set X is bounded if there exists $R > 0$ such that $X \subset B(0; R)$.



$B(M_0, \varepsilon)$ is open, $\bar{B}(M_0, \varepsilon)$ is closed.

Content

Definition

Let $D \subset \mathbb{R}^n, D \neq \emptyset$. A function $f: D \to \mathbb{R}$ is a rule that assigns $\mathbf{x}$ to a unique value $f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n)$. D is called the domain of f.

We have three ways of looking at f:

- as a function of n variables $(x_1, x_2, \ldots, x_n)$.
- as a function of a single point $M(x_1, x_2, \ldots, x_n)$.
- as a function of a vector variable $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.

2D: $z = f(x, y)$, x, y: independent variables, z: dependent variable.

Example

Find the domains of the following functions:

- $z = z(x, y) = \frac{x}{\sqrt{1-x^2-y^2}}$.
- $z = z(x, y) = \arcsin \frac{y}{x-1}$.
- $f(x, y, z) = \arccos\big(\ln(x + y - z^2 + 1)\big)$.

Visualization

- The graph of $z = f(x_1, x_2, \ldots, x_n) \colon D \subset \mathbb{R}^n \to \mathbb{R}$ is the set $\Gamma(f) = \{(x_1, x_2, \ldots, x_n, f(x_1, x_2, \ldots, x_n)), x \in D\} \subset \mathbb{R}^{n+1}$.
- The graph of $z = f(x, y)$ is a surface in $\mathbb{R}^3$.

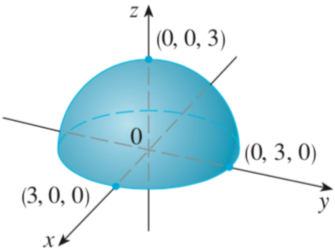
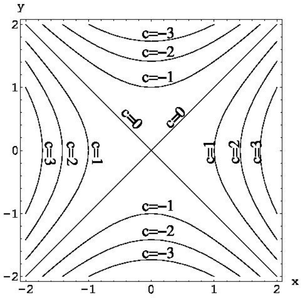
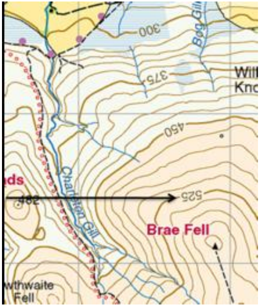


Figure: $z = \sqrt{9 - x^2 - y^2}$

The level curves of a function f of two variables are the curves with equations $f(x, y) = k$ where k is a constant (in the range of f).





Content

Limit of a sequence

Definition

The sequence $\{M_n(x_n, y_n)\}_{n\in\mathbb{N}}$ approaches $M_0(x_0, y_0)$, written as $M_n \xrightarrow{n\to\infty} M_0$ iff $d(M_0, M_n) \to 0$ as $n \to \infty$.

$$d(M_n, M_0) = \sqrt{(x_n - x_0)^2 + (y_n - y_0)^2}.$$

Hence, $M_n \xrightarrow{n\to\infty} M_0 \Leftrightarrow \begin{cases} x_n \to x_0, \\ y_n \to y_0. \end{cases}$

Example

Determine the limit of the sequence of points $\left\{\left(\frac{2}{n}, \frac{2n^2-1}{n^2+1}\right)\right\}$ khi $n \to \infty$, $\left\{\left(-\frac{n}{n+1}, \frac{2}{n}, \frac{3n^2-1}{n^2+2}\right)\right\}$ khi $n \to \infty$.

Limit of a function

Definition

Let $f: D \to \mathbb{R}$, D contains points close to $M_0(x_0, y_0)$, may be $(x_0, y_0) \notin D$. We say that the limit of $f(x, y)$ is a when (x, y) tends to (x_0, y_0), written as $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = a$ iff

$$\Leftrightarrow \forall \{M_n\}, M_n \xrightarrow{n\to\infty} M_0 : \lim_{n\to\infty} f(M_n) = a.$$

Equivalent definition

$$\lim_{M\to M_0} f(x, y) = a \Leftrightarrow \forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 : d(M, M_0) < \delta$$

$$\Rightarrow |f(M) - a| < \varepsilon.$$

Limit of a function of multivariables has similar properties as those of limit of a function of single variable.

Theorem

Assume that $\lim_{M \to M_0} f(M)$, $\lim_{M \to M_0} g(M)$ *are finite.*

$$\lim_{M \to M_0} (f(M) \pm g(M)) = \lim_{M \to M_0} f(M) \pm \lim_{M \to M_0} g(M)$$
$$\lim_{M \to M_0} f(M).g(M) = \lim_{M \to M_0} f(M). \lim_{M \to M_0} g(M).$$

Theorem (Squeeze theorem)

If $f(M) \leq g(M) \leq h(M)$ *when* M *is close to* M_0, *then we have*

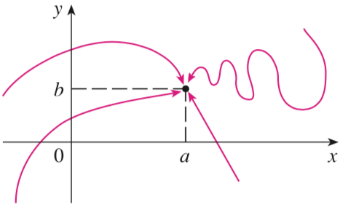
$$\lim_{M \to M_0} f(M) \leq \lim_{M \to M_0} g(M) \leq \lim_{M \to M_0} h(M).$$

Example

Compute the limits

1. $$\lim_{(x,y)\to(2,1)} \frac{x^3 - y^3}{x^2 + y^2}$$
2. $$\lim_{(x,y)\to(0,0)} \sqrt{x^2 + y^2} \cos \frac{1}{xy}$$
3. $$\lim_{(x,y)\to(0,0)} \frac{x^3 - y^3}{x^2 + y^2}$$

In the line, $x \to x_0$ either from the left or the right.



Note: If $f(x, y) \to a_1$ as $(x, y) \to (x_0, y_0)$ along the path $\mathcal{C}_1$, $f(x, y) \to a_2$ as $(x, y) \to (x_0, y_0)$ along the path $\mathcal{C}_2$; $a_1 \neq a_2$ then there does not exist $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y)$.

Example

Compute the limits

1. $$\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y}$$
2. $$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$$
3. $$\lim_{(x,y)\to(0,0)} \frac{x^4-y^2}{x^4+2y^2}$$

Continuous functions

Definition

Let $f(M)$ be defined on D, $M_0 \in D$. The function $f(M)$ is said to be continuous at M_0 iff $\lim\limits_{M \to M_0} f(M) = f(M_0)$.

$f(M)$ is said to be continuous on D if it is continuous at any $M_0 \in D$.

Example

Investigate the continuity at $(x, y) = (0, 0)$ of the following functions

1. $$f(x, y) = \begin{cases} \cos \dfrac{2xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$
2. $$g(x, y) = \begin{cases} \dfrac{x \ln(1 + y) - y \ln(1 + x)}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Example

Determine a such that the given function is continuous at $(x, y) = (0, 0)$

1. $$f(x, y) = \begin{cases} \dfrac{x \tan y - y \tan x}{x^2 + 2y^2} & \text{if } (x, y) \neq (0, 0), \\ a & \text{if } (x, y) = (0, 0). \end{cases}$$

2. $$g(x, y) = \begin{cases} \sin \dfrac{x - 2y}{2x + 3y} & \text{if } (x, y) \neq (0, 0), \\ a & \text{if } (x, y) = (0, 0). \end{cases}$$

Properties

Theorem

The graph of a continuous function has no hole in it.

Theorem

A continuous function on a **closed, bounded domain** *is bounded on it and attains its maximum, minimum on that domain.*

Uniform continuity

Definition

A function $f(M)$ is said to be uniformly continuous over a set X if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that: for all $M_1, M_2 \in X$, $d(M_1, M_2) < \delta$ then $|f(M_1) - f(M_2)| < \varepsilon$.

Uniform continuity implies continuity.

Theorem

A function $f(x)$, which is continuous on a **closed, bounded set** *X, is uniformly continuous on it.*