Fundamentals of Optimization

Subgradient method

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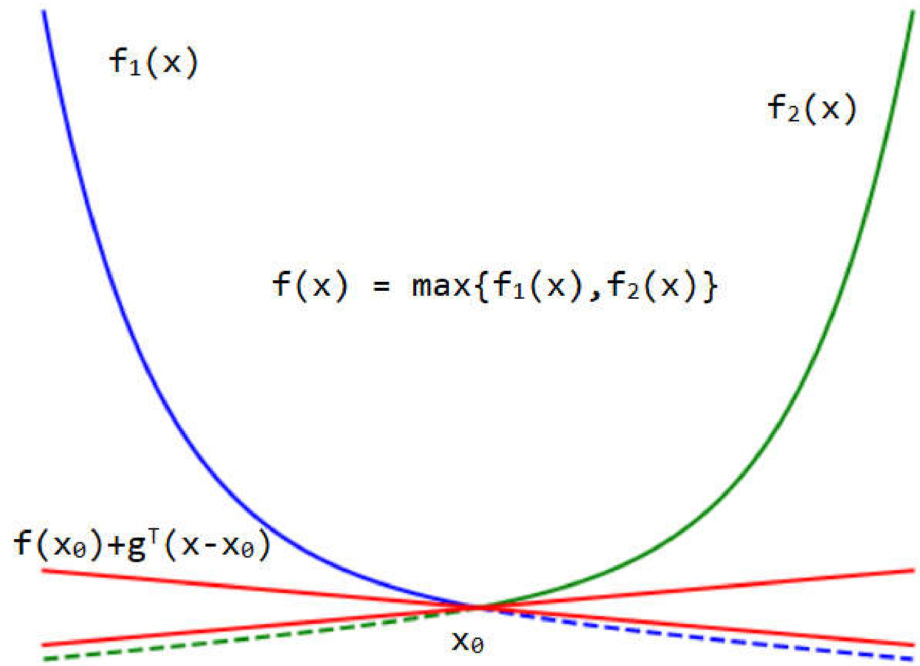
Subgradient method

- For minimize nondifferentiable convex function
- Subgradient method is not a descent method: the function value can increase

Subgradient method

- Subgradient of f at x
  - Any vector g such that $f(x') \geq f(x) + g^T(x'-x)$
  - If f is differentiable, only possible choice is $g^{(k)} = \nabla f(x^{(k)})$, → the subgradient method reduces to the gradient method



Basic subgradient method

$$x^{(k+1)} = x^{(k)} - \alpha_k g^{(k)}$$

- $x^{(k)}$: is at the k^{th} iteration
- $g^{(k)}$: any subgradient of f at $x^{(k)}$
- $\alpha_k > 0$ is the k^{th} step size
- Note: subgradient is not a descent method, thus $f_{best}^{(k)} = \min\{f(x^{(1)}), f(x^{(2)}), \ldots, f(x^{(k)})\}$

Convergence proof

- Notations: x^* is a minimizer of f
- Assumptions
  - Norm of the subgradients is bounded (with a constant G): $||g^{(k)}||_2 \leq G$ (this is the case if, for example, f satisfies the Lipschitz condition $|f(u) - f(v)| \leq G||u\text{-}v||_2$)
  - $||x^{(1)} - x^*||2 \leq R$ (with a known constant R)
- We have $||x^{(k+1)} - x^*||_2^2 = ||x^{(k)} - \alpha_k g^{(k)} - x^*||_2^2$

$= ||x^{(k)} - x^*||_2^2 - 2\alpha_k g^{(k)T}(x^{(k)} - x^*) + \alpha_k^2||g^{(k)}||_2^2$

$\leq ||x^{(k)} - x^*||_2^2 - 2\alpha_k(f(x^{(k)}) - f(x^*)) + \alpha_k^2||g^{(k)}||_2^2$ (due to the fact that $f(x^*) \geq f(x^{(k)}) + g^{(k)T}(x^* - x^{(k)})$) (1)

Convergence proof

- Apply the inequality (1) recursively, we have

$||x^{(k+1)} - x^*||_2^2 \le ||x^{(1)} - x^*||_2^2 - 2\sum_{i=1}^{k} \alpha_i (f(x^{(i)}) - f^*) + \sum_{i=1}^{k} \alpha_i^2 ||g^{(i)}||_2^2$ (where $f^* = f(x^*)$)

$\rightarrow 2\sum_{i=1}^{k} \alpha_i (f(x^{(i)}) - f^*) \le R^2 + \sum_{i=1}^{k} \alpha_i^2 ||g^{(i)}||_2^2$

$\rightarrow R^2 + \sum_{i=1}^{k} \alpha_i^2 ||g^{(i)}||_2^2 \ge 2\sum_{i=1}^{k} \alpha_i (f(x^{(i)}) - f^*) \ge 2(\sum_{i=1}^{k} \alpha_i) min_{i=1,\dots,k} (f(x^{(i)}) - f^*) = 2\sum_{i=1}^{k} \alpha_i (f_{best}^{(k)} - f^*)$

$$\rightarrow f_{best}^{(k)} - f^* \le \frac{R^2 + \sum_{i=1}^{k} \alpha_i^2 ||g^{(i)}||_2^2}{2\sum_{i=1}^{k} \alpha_i} \qquad (2)$$

Convergence proof

- Different cases
  - Constant step size $\alpha_k = \alpha$

$\rightarrow f_{best}^{(k)} - f^* \leq \dfrac{R^2 + G^2\alpha^2 k}{2\alpha k}$

$\rightarrow f_{best}^{(k)} - f^*$ converges to $G^2\alpha/2$ when $k \rightarrow \infty$

  - Constant step length $\alpha_k = \gamma / ||g^{(k)}||_2$

$\rightarrow f_{best}^{(k)} - f^* \leq \dfrac{R^2 + \gamma^2 k}{2\gamma k/G}$

$\rightarrow f_{best}^{(k)} - f^*$ converges to $G\gamma/2$ when $k \rightarrow \infty$

Convergence proof

- Different cases
  - Square summable but not summable

$||\alpha||_2^2 = \sum_{i=1}^{\infty} \alpha_i^2 < \infty$ and $\sum_{i=1}^{\infty} \alpha_i = \infty$

$$\rightarrow f_{best}^{(k)} - f^* \leq \frac{R^2 + G^2||\alpha||_2^2}{2\sum_{i=1}^{k} \alpha_i}$$

$\rightarrow f_{best}^{(k)} - f^*$ converges to 0 as $k \rightarrow \infty$

Example

$$\text{minimize } f(x) = \max_{i=1,\dots,m}(a_i^T x + b_i)$$

- Finding subgradient: given x, the index j for which

$$a_j^T x + b_j = \max_{i=1,\dots,m}(a_i^T x + b_i)$$

→ subgradient at x is $g = a_j$

Example

```
import numpy as np
def solve(A,b):
    f = lambda x: np.max(A.dot(x) + b)
    sg = lambda x: A[np.argmax(A.dot(x) + b)]
    x0 = [0,0,0,0]
    x = np.array(x0).T
    f_best = f(x)
    for i in range(100000):
        alpha = 2
        x = x - alpha*sg(x)
        if f_best > f(x):
            f_best = f(x)
    return f_best
```

Example

```
def main():
    A = np.array([[1,-2,3,-5],[2,-2,1,1],[-3,2,-2,7]],dtype='double’)
    b = np.array([3,4,5]).T
    rs = solve(A,b)
    print('rs = ',rs)

if __name__ == '__main__':
    main()
```