

Lecture 6.2: Introduction to Optimization

Learning Objectives

- Recognize decision-making situations which that may benefit from an optimization modeling approach.
- Formulate algebraic models for linear programming problems.
- Develop spreadsheet models for linear programming problems.
- Use Excel's Solver Add-In to solve linear programming problems.
- Interpret the results of models and perform basic sensitivity analysis.

Optimization: Basic Ideas

- Major field within the discipline of Data Analytics, Operations Research and Management Science.
- Optimization Problem Components:
 - o Decision Variables.
 - o Objective Function (to maximize or minimize).
 - o Constraints (requirements or limitations).
- Basic Idea:
 - o Find the values of the decision variables that maximize (minimize) the objective function value, while staying within the constraints.

Linear Programming (LP)

- If the objective function and all constraints are linear functions of the decision variables (e.g., no squared terms, trigonometric functions, ratios of variables), then the problem is called a Linear Programming (LP) problem. LPs are much easier to solve by computer than problems involving nonlinear functions.
- Real-world LPs are solved which contain hundreds of thousands to millions of variables (with specialized software!). Our problems are obviously much smaller, but the basic concepts are much the same.

Real-World Examples

- Dynamic and Customized Pricing.
- Product Mix.
- Scheduling/Allocation.
- Routing/Logistics.
- Supply Chain Optimization.
- Facility Location.
- Financial Planning/Asset Management.
- Etc.

Solving Optimization Problems

- Understand the problem; draw a diagram.
- Write a problem formulation in words:
- Write the algebraic formulation of the problem:
 - o Define the decision variables.
 - o Write the objective function in terms of the decision variables.
 - o Write the constraints in terms of the decision variables.
- Develop Spreadsheet Model.
- Setup the Solver settings and solve the problem.
- Examine results, make corrections to model and/or Solver settings.

- Interpret the results and draw insights.

Solver

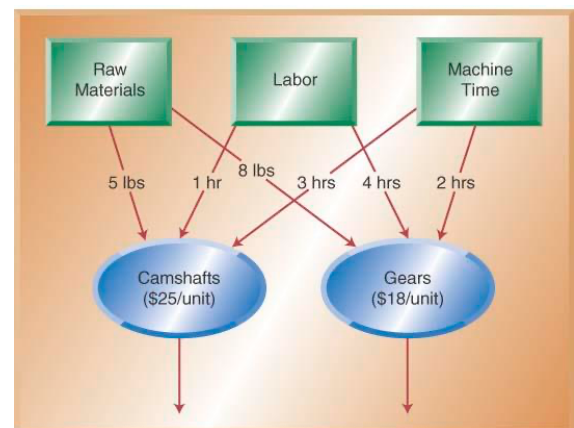
- We will use Solver as simple tool to illustrate the example, an Excel add-in, to solve Linear Programming problems.
- Add-In: An additional piece of software that Excel loads into memory when needed.

Example: Product Mix Decision

- DJJ Enterprises makes automotive parts, Camshafts & Gears.
- Unit Profit: Camshafts \$25/unit, Gears \$18/unit.
- Resources needed: Steel, Labor, Machine Time. In total, 5000 lbs. steel available, 1500 hours labor, and 1000 hours machine time.
- Camshafts need 5 lbs. steel, 1 hour labor, 3 hours machine time.
- Gears need 8 lbs. steel, 4 hours labor, 2 hours machine time.
- How many camshafts & gears to make to maximize profit?

Understanding the Problem

- Text-Based Formulation:
 - o Decision Variables: Number of camshafts to make, number of gears to make.
 - o Objective Function: Maximize profit.
 - o Constraints: Don't exceed amounts available of steel, labor, and machine time.



Algebraic Formulation

- Decision Variables:
 - o C = number of camshafts to make.
 - o G = number of gears to make.
- Objective Function:
 - o Maximize $25C + 18G$ (profit in \$).
- Constraints:
 - o $5C + 8G \leq 5000$ (steel in lbs.)
 - o $1C + 4G \leq 1500$ (labor in hours)
 - o $3C + 2G \leq 1000$ (machine time in hours)
 - o $C \geq 0, G \geq 0$ (non-negativity)

Important Concepts

- Linear Program: The objective function and constraint are linear functions of the decision variables. Therefore, this is a Linear Program.
- Feasibility:
 - o Feasible Solution. A solution is feasible for an LP if all constraints are satisfied.
 - o Infeasible Solution. A solution is infeasible if one or more constraints is violated.
 - o Check the solutions $C=75, G=200$; and $C=300, G=200$ for feasibility.
- Optimal Solution:
 - o The optimal solution is the feasible solution with the largest (for a max problem) objective value (smallest for a min problem).

Solving Linear Programming Problems

- Trial and error:
 - o Possible for very small problems.
 - o Virtually impossible for large problems.
- Graphical approach:
 - o It is possible to solve a 2-variable problem graphically to find the optimal solution (not shown).
- Simplex Method:
 - o This is a mathematical approach developed by George Dantzig.
 - o Can solve small problems by hand.
- Computer Software:
 - o Most optimization software uses the Simplex Method to solve the problems.
 - o Excel's Solver Add-In is an example of such software.
- Solver can solve LPs of up to 200 variables.

Spreadsheet Model Development

- Develop correct, flexible, documented model.
- Sections for decision variables, objective function, and constraints.
- Use algebraic formulation and natural structure of the problem to guide structure of the spreadsheet.
- Use one cell for each decision variable.
- Store objective function coefficients in separate cells and use another cell to compute the objective function value.
- Store constraint coefficients in cells, compute the LHS value of each constraint, for comparison to the RHS value.

Spreadsheet Model

- C=75, G=200 (cells B5, C5) entered as trial values. This is a feasible, but not the optimal, solution.
- Note close relationship to algebraic formulation.
- Note that only one distinct formula needed to be entered; once entered in Cell D8, it was copied to Cells D11:D13.
- This is possible because the coefficients were stored separately in a specific structure.

	A	B	C	D	E	F	G
1	Example B.1						
2	DJJ Enterprises Production Planning						
3							
4	Decision Variables	Camshafts	Gears				
5	Units to Make	75	200				
6							
7	Objective			Total			
8	Profit	\$25	\$18	\$5,475			
9							
10	Constraints			Used		Available	
11	Steel (lbs)	5	8	1975	<=	5000	
12	Labor (hrs)	1	4	875	<=	1500	
13	Machine Time (hrs)	3	2	625	<=	1000	

D8: =B8*B\$5+C8*C\$5
(copied to D11:D13)

Solver Basics

- Don't even think about using Solver until you have a working, flexible spreadsheet model that you can use as a "what if" tool!
- Solver Settings
 - o Specify Objective Cell (objective function).
 - o Specify Changing Cells (decision variables).
 - o Specify Constraints.
 - o Specify Solver Settings
- Solve Problem to find Optimal Solution.

Solver Settings: Target Cell and Changing Cells

- Specify Target Cell: D8.
 - o Equal to: Max.
- Changing Cells (decision variables)
 - o B5, C5.

Solver Settings: Constraints

- Click "Add" to add constraints.
- Select LHS Cell (D11), relationship (\leq), and RHS Cell (F11).
- LHS Cell should contain a formula which computes the LHS Value of the constraint.
- Typically, RHS Cell should contain a fixed value, but this is not absolutely required.
- Repeat for the other two constraints (for labor and machine time).
- **OR use the range for the LHS and the RHS** (see next slide).

Solver Options

- Select Simplex LP"
- Tells Solver to use the Simplex Method, which is faster and is Solver's default optimization method.
- Check "Make unconstrained Variables Non-negative".
- Tells Solver that the decision variables (B5:C5, representing the number of Camshafts and Gears) must be ≥ 0 in any feasible solution.
- Leave other settings at their defaults.

Completed Solver Box

- Click "Solve" to tell Solver to find the Optimal Solution.

Solved Spreadsheet (Optimal Solution)

- Optimal Solution: Make 100 camshafts, 350 gears.
- Optimal Objective Value: \$8800 profit.
- Both pieces of information are important. Knowing the optimal objective value is useless without knowing how that value can be attained.

	A	B	C	D	E	F
1	Example B.1					
2	DJJ Enterprises Production Planning					
3						
4	Decision Variables	Camshafts	Gears			
5	Units to Make	100	350			
6						
7	Objective			Total		
8	Profit	\$25	\$18	\$8,800		
9						
10	Constraints			Used		Available
11	Steel (lbs)	5	8	3300	\leq	5000
12	Labor (hrs)	1	4	1500	\leq	1500
13	Machine Time (hrs)	3	2	1000	\leq	1000

Interpreting the Solution

- Which constraints are actively holding us back from making even more profit?
 - o Binding Constraints: Constraints at the optimal solution with the LHS equal to the RHS; equivalently for a resource, a binding constraint is one in which all the resource is used up.
 - o Labor (all 1500 hours used), Machine Time (all 1000 hours used).
- Non-Binding Constraints:
 - o Steel: Have 5000 lbs. available, but only need 3300 lbs. for this solution.
- What-If Analysis. Once the Spreadsheet and Solver model is set up, it is easy to change one or more input values and re-solve the problem.
 - o This interactive use can be a very powerful way to use optimization.

Solution Reports

- Solver can generate three solution reports:
 - o Answer Report.
 - o Sensitivity Report.
 - o Limits Report: Not covered here.
- The Answer Report presents in a standard format the Solver Settings and the optimal solution.
- The Sensitivity Report shows what will happen if certain problem parameters are changed from their current values.

Answer Report

- Tightening a binding constraint can only worsen the objective function value, and loosening a binding constraint can only improve the objective function value. As such, once an optimal solution is found, managers can seek to improve that solution by finding ways to relax binding constraints.
- Three sections: Objective Cell, Changing Cells, Constraints.
- “Final Value” indicates optimal solution.
- For each constraint, binding/non-binding status, and “slack” (difference between LHS and RHS), slack is zero for binding constraints.
- Also note that the Solver Settings for Objective Cell, Changing Cells, and Constraints are reported here. This can be a useful debugging tool.

Objective Cell (Max)					
Cell	Name	Original Value	Final Value		
\$D\$8	Profit Total	\$0	\$8,800		
Variable Cells					
Cell	Name	Original Value	Final Value	Integer	
\$B\$5	Units to Make Camshafts	0	100	Contin	
\$C\$5	Units to Make Gears	0	350	Contin	
Constraints					
Cell	Name	Cell Value	Formula	Status	Slack
\$D\$11	Steel (lbs) Used	3300	\$D\$11<=\$F\$11	Not Binding	1700
\$D\$12	Labor (hrs) Used	1500	\$D\$12<=\$F\$12	Binding	0
\$D\$13	Machine Time (hrs) Used	1000	\$D\$13<=\$F\$13	Binding	0

Sensitivity Report

Variable Cells						
Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
\$B\$5	Units to Make Camshafts	100	0	25	2	20.5
\$C\$5	Units to Make Gears	350	0	18	82	1.333333333
Constraints						
Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
\$D\$11	Steel (lbs) Used	3300	0	5000	1E+30	1700
\$D\$12	Labor (hrs) Used	1500	0.4	1500	500	1166.666667
\$D\$13	Machine Time (hrs) Used	1000	8.2	1000	1416.666667	250

- Note two sections:
 - o Variable Cells.
 - o Constraints.
- **Constraint Section**
 - o Shadow Price: This is the amount the optimal objective value will change by if the RHS of the constraint is increased by one unit.
 - o Units of shadow price: (objective function units/constraint units). Example:
 - For the steel constraint, the units are \$/lb.
 - For the labor constraint, the units are \$/hour.

- Allowable Increase/Decrease: Provides the increase/decrease of the RHS of the constraint for which the shadow price stays the same; that is, the effect on the objective value stays the same in this range.
- Example:
 - Shadow Price of Machine Time is \$8.20/hour, valid for an increase of 1416 hours or a decrease of 250 hours.
 - If we make 200 additional machine time hours available, profit can be increased by (200 hours) (\$8.20/hour) = \$1640.
- What is the shadow price of a non-binding constraint? Why? Will this always be the case?
- **Variable Cells Section**
- Allowable Increase/Decrease: This is the amount the objective coefficient for a decision variable can be increased/decreased without changing the optimal solution.
- Example
 - Gears currently have a profit of \$18/unit (objective coefficient).
 - Allowable Increase/Decrease is \$82 and \$1.33, respectively.
 - Interpretation: If the unit profit for gears is between \$16.67 and \$100, the optimal solution (production plan) will be to make 100 camshafts and 350 gears.
 - Note: The optimal solution (production plan) stays the same within this range, but the optimal objective value (profit) changes since the unit profit is changing. So, if the profit on gears goes up to \$20/unit, profit will increase by (\$20-\$18) (350) = \$700.

Sensitivity Report: Caveat

- Effects identified in the Sensitivity Report need to be carefully interpreted.
- Specifically, the effect of the change of one parameter (e.g., a RHS value or an objective coefficient) assume that all other parameters of the model stay at their base case values.
- For example, the Sensitivity Report does not tell us what happens if additional quantities of both Labor and Machine Time become available. In this scenario, we would need to enter the new values and re-solve the model.

Highlights

- People use informal “optimization” to make decisions almost every day.
- Organizations use formal optimization methods to address problems across the organization, from optimal pricing to locating a new facility.
- The algebraic formulation of an LP comprises the definitions of the decision variables, an algebraic statement of the objective function, and algebraic statements of the constraints.
- The spreadsheet model for an optimization problem should be guided by the algebraic formulation.
- Solver, an Excel Add-In, can solve both linear and nonlinear problems. This lecture focuses on solving linear problems.
- After solving an LP, you must interpret the results to see if they make sense, fix problems with the model, and find the insights useful for management.
- Solver can generate the Answer and Sensitivity Reports. The Sensitivity Report provides additional information about what happens to the solution when certain coefficients of the problem are changed.

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