Extreme values

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Content

Content

Definition

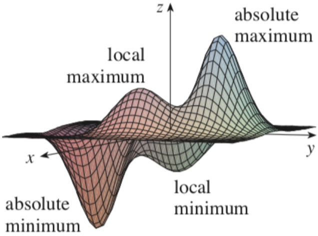
Given $z(x, y)$: $D \subset \mathbb{R}^2$. $M_0(x_0, y_0)$ is an interior point of D. We say that

- $z(x, y)$ has **a local/relative maximum at** M_0 if $z(M) - z(M_0) < 0$ for all $M \in B(M_0; \varepsilon) \setminus \{M_0\}$ for a certain $\varepsilon > 0$.
- $z(x, y)$ has **a local/relative minimum at** M_0 if $z(M) - z(M_0) > 0$ for all $M \in B(M_0; \varepsilon) \setminus \{M_0\}$ for a certain $\varepsilon > 0$.

Definition

Given $z = z(x, y)$: $D \subset \mathbb{R}^2$. $M_0(x_0, y_0) \in D$. We say that

- $z(x, y)$ has **an absolute/global maximum at** M_0 if $z(M) - z(M_0) \leq 0$ for all $M \in D$.
- $z(x, y)$ has **an absolute/global minimum at** M_0 if $z(M) - z(M_0) \geq 0$ for all $M \in D$.



In the figure: 2 local maxima and 2 local minima.

Theorem

If $z(x, y)$ has a local maximum or local minimum at M_0 and there exists $z'_x(M_0), z'_y(M_0)$, then $z'_x(M_0) = z'_y(M_0) = 0$.

- M_0 is a local extreme point of $z(x, y)$, in particular x_0 is a local extreme point of $z(x, y_0)$.
- By Fermat's theorem $z'_x(M_0) = 0$. Similarly, $z'_y(M_0) = 0$.

M_0: critical point.

A sufficient condition

Theorem

Suppose the second order partial derivatives of $z(x, y)$ are continuous in $B(M_0; \varepsilon)$, and $z'_x(M_0) = z'_y(M_0) = 0$. Denote

$$A = z''_{xx}(M_0), B = z''_{xy}(M_0), C = z''_{yy}(M_0), \Delta = B^2 - AC.$$

- *If $\Delta < 0$:*
  - *$A > 0$ then z attains a local minimum at M_0,*
  - *$A < 0$ then z attains a local maximum at M_0.*
- *If $\Delta > 0$, $z(x, y)$ does not attain a local extreme value at M_0 (M_0 is a saddle point).*
- *If $\Delta = 0$: no conclusion can be drawn.*

Sketch of Proof

By Taylor's theorem,

$$\begin{aligned}\Delta z = z(M) - z(M_0) &= z(x_0 + h, y_0 + k) - z(x_0, y_0)\\ &= dz(x_0, y_0) + \frac{1}{2} d^2 z(x_0, y_0) + R_3(h, k)\\ &= \frac{1}{2}\left[z''_{xx}(x_0, y_0)h^2 + 2z''_{xy}(x_0, y_0)hk + z''_{yy}(x_0, y_0)k^2\right]\\ &\quad + R_3(h, k).\end{aligned}$$

where $R(h, k)$ is an infinitesimal of higher order than $h^2 + k^2$.
Therefore, Δz is of the same sign as $Ah^2 + 2Bhk + Ck^2$.
We obtain the conclusion.

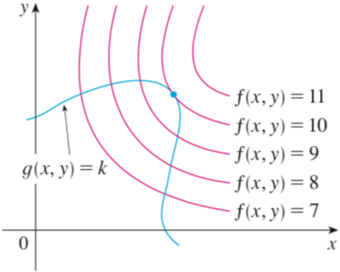
Example

Find the local maximum and minimum values of the functions

1. $z = x^2 - 3xy + x + y^3 - 2.$
2. $z = e^{-y}(3x - x^3 - y).$
3. $z = x^4 + y^4 + (x + y)^3.$

Content

Aim: Find extreme values of the function $z = z(x, y)$ subject to the constraint $g(x, y) = 0$.



To maximize $f(x, y)$ subject to the constraint $g(x, y) = 0$ is to find the largest value of c such that the level curve $f(x, y) = c$ intersects $g(x, y) = 0$.

Theorem

Assume $f(x,y)$ attains a local extreme value at $M_0(x_0,y_0)$ subject to the constraint $g(x,y)=0$. If in a certain ball $B(M_0,\varepsilon)$ the functions $f(x,y)$, $g(x,y)$ have continuous first order partial derivatives and $(g'_x,g'_y)\neq(0,0)$, then at M_0 the following holds

$$\begin{vmatrix} f'_x & f'_y \\ g'_x & g'_y \end{vmatrix} = 0.$$

Corollary

There exists $\lambda_0\neq 0$ such that $\begin{cases} f'_x(x_0,y_0)+\lambda_0 g'_x(x_0,y_0)=0, \\ f'_y(x_0,y_0)+\lambda_0 g'_y(x_0,y_0)=0. \end{cases}$

Method of Lagrange multipliers

- Consider $L(x, y, \lambda) = f(x, y) + \lambda g(x, y)$.
  The previous system reads as $L'_x(x_0, y_0, \lambda_0) = L'_y(x_0, y_0, \lambda_0) = 0$,

  the constraint reads as $L'_\lambda(x_0, y_0, \lambda_0) = 0$.

  Hence, we find all critical points (x_0, y_0, λ_0) of $L(x, y, \lambda)$.
- General case: By definition, we consider the sign of the increment $\Delta f = f(x_0 + h, y_0 + k) - f(x_0, y_0)$ at points subject to the constraint, i.e. $g(x_0, y_0) = g(x_0 + h, y_0 + k) = 0$.
- Special case: If (x_0, y_0) is an extreme point of $L(x, y, \lambda_0)$ then (x_0, y_0) is an extreme point of $f(x, y)$ subject to the condition $g(x, y) = 0$.

  Indeed, for (x, y) close to (x_0, y_0): if $L(x_0, y_0, \lambda_0) < L(x, y, \lambda_0)$, then it holds in particular for (x, y) such that $g(x, y) = 0$. The inequality becomes $f(x_0, y_0) < f(x, y)$.

  Therefore, we consider $d^2L(M_0)$ with $\lambda = \lambda_0$.

  If M_0 is **not** an extreme point of $L(x, y, \lambda_0)$, we cannot conclude whether (x_0, y_0) is a constrained extreme point of $f(x, y)$.

In practice, we look for global extreme values under certain constraints. Assume the absolute extreme values exist and $(g'_x, g'_y) \neq (0,0)$. Evaluate f at all (x, y) from the first step. The largest of these values is the maximum value of f, the smallest is the minimum value of f.

Example

Find the extreme values of $z = x^2 + y^2$ on the line $\frac{x}{2} + \frac{y}{3} = 1$.

Consider $L(x, y, \lambda) = x^2 + y^2 + \lambda(\frac{x}{2} + \frac{y}{3} - 1)$.
Find the critical points of $L(x, y, \lambda)$

$$\begin{cases} L'_x = 2x + \frac{\lambda}{2} = 0 \\ L'_y = 2y + \frac{\lambda}{3} = 0 \\ \frac{x}{2} + \frac{y}{3} = 1 \end{cases} \quad \Rightarrow \lambda = -\frac{72}{13}, x = \frac{18}{13}, y = \frac{12}{13}$$

$d^2L = 2(dx)^2 + 2(dy)^2 > 0$ so $\left(\frac{18}{13}, \frac{12}{13}\right)$ is a minimum of z.

Problem: Find the extreme values of $u = f(x, y, z)$ subject to the constraint $g(x, y, z) = 0$.

- Find all critical points of $L(x, y, z, \lambda) = f(x, y, z) + \lambda g(x, y, z)$.
- Consider the sign of the increment $\Delta u = f(x_0 + h, y_0 + k, z_0 + l) - f(x_0, y_0, z_0)$, where

$$g(x_0 + h, y_0 + k, z_0 + l) = g(x_0, y_0, z_0) = 0,$$

  then make the conclusion.

Example

Find the extreme values of $u(x, y, z) = x - 2y + 2z$ subject to the constraint $x^2 + y^2 + z^2 = 9$.

Consider the Lagrange function
$L = x - 2y + 2z + \lambda(x^2 + y^2 + z^2 - 9)$.
We solve the system

$$\begin{cases} 1 + 2\lambda x = 0 \\ -2 + 2\lambda y = 0 \\ 2 + 2\lambda z = 0 \\ x^2 + y^2 + z^2 = 9 \end{cases} \Rightarrow M_0\left(-1, 2, -2, \frac{1}{2}\right), M_1\left(1, -2, 2, -\frac{1}{2}\right)$$

1. Pick $(-1+h, 2+k, -2+l)$ close to $(-1,2,-2)$.
   $\Delta u = (-1+h) - 2(2+k) + 2(-2+l) - (-9) = h - 2k + 2l$.
   where $(-1+h)^2 + (2+k)^2 + (-2+l)^2 = 9$.
   Therefore, $\Delta u = h - 2k + 2l = \frac{h^2+k^2+l^2}{2} > 0$.
   Hence, $(-1,2,-2)$ is a local minimum point under the constraint. $u_{loc.min.} = -9$.
2. Pick $(1+h, -2+k, 2+l)$ close to $(1,-2,2)$.
   $\Delta u = (1+h) - 2(-2+k) + 2(2+l) - 9 = h - 2k + 2l$.
   where $(1+h)^2 + (-2+k)^2 + (2+l)^2 = 9$.
   Therefore, $\Delta u = h - 2k + 2l = -\frac{h^2+k^2+l^2}{2} < 0$.
   Hence, $(1,-2,2)$ is a local maximum point under the constraint. $u_{loc.max.} = 9$.
   Second way: Fix $\lambda_0 = -\frac{1}{2}$.
   $d^2L = 2\lambda_0((dx)^2 + (dy)^2 + (dz)^2) = -((dx)^2 + (dy)^2 + (dz)^2) < 0$.
   Hence, $(1,-2,2)$ is a local maximum point of f under the constraint.

Content

Theorem (Extreme value theorem)

A continuous function $f : D \subset \mathbb{R}^n \to \mathbb{R}$ on a closed, bounded domain D attains its absolute maximum and minimum values on that domain.

1. Evaluate the values of f at critical points in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of these values is the absolute maximum value of f, the smallest of these values is the absolute minimum value of f.

Example

Find the absolute maximum and minimum values of the following functions

1. $z = x^2 + 2xy - 4x + 8y$ on the rectangle enclosed by $x = 0$, $x = 1$, $y = 0$, $y = 2$.
2. $z = x^2 - y^2$ on the domain $x^2 + y^2 \leq 4$.
3. $z = xy(3 - x - y)$ on the domain $0 \leq x \leq 2, 0 \leq y \leq 2$.

1) Solve $\begin{cases} y(3-2x-y)=0 \\ x(3-x-2y)=0, \\ 0<x,y<2 \end{cases} \Rightarrow M_0(1;1).$

2) Four pieces of boundary, x, y have equal roles, therefore, we consider $x=0$ and $x=2$.

$x=0$, $z=0$.

$x=2$, $0 \le y \le 2$: $z=2y(1-y)$, critical point $y=\frac{1}{2}$; endpoints $y=0, y=2$.

3) We compare the values of z at $(1;1)$, $(2;\frac{1}{2})$, $(2;0)$, $(2;2)$, $(\frac{1}{2},2)$, $(0;2)$, $(0;0)$.

Maximum value: $z_{max}=1$ which is attained at $(1;1)$

Minimum value: $z_{min}=-4$ which is attained at $(2;2)$.