Laplace transform and applications to ODEs

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   - Derivatives on $s-$domain
   - Integrals on $s-$ domain
   - Convolution

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Integrals on $t-$domain

Common assumptions: $f(t)$ is piecewise continuous on $[0, \infty)$ and exponentially bounded.

Theorem

$$\mathcal{L}\Big\{\int_0^t f(v)dv\Big\}(s) = \frac{F(s)}{s}.$$

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\}(t) = \int_0^t f(v)dv.$$

Example

Calculate

1. $\mathcal{L}^{-1}\{\frac{1}{s(s^2+4)}\}(t)$.
2. $\mathcal{L}^{-1}\{\frac{1}{s^2(s+4)}\}(t)$.

Derivatives on $s-$domain

Theorem

$$\mathcal{L}\{-tf(t)\}(s) = F'(s).$$

$$\mathcal{L}^{-1}\{F'(s)\}(t) = -tf(t).$$

Proof.

Generally,

$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n F^{(n)}(s).$$

Example

Find the Laplace and inverse Laplace transform

1. $\mathcal{L}\{t \sin kt\}(s)$,
2. $\mathcal{L}\left((t - e^{2t})^2\right)(s)$,
3. $\mathcal{L}^{-1}\left(\ln \dfrac{s^2+4}{s^2+9}\right)$.

Example (Application to linear ODEs)

Solve the following ODEs:

1. $tx'' + (3t-1)x' + 3x = 0, x(0) = 0.$
2. $tx'' - tx' + x = 2, x(0) = 2.$

Integrals on s-domain

Theorem

Assume there exists $\lim\limits_{t\to 0^+} \frac{f(t)}{t}$. *Then*

$$\mathcal{L}\{\frac{f(t)}{t}\}(s) = \int_s^\infty F(u)du.$$

$$\mathcal{L}^{-1}\{\int_s^\infty F(u)du\}(t) = \frac{f(t)}{t}.$$

Example

Calculate $\mathcal{L}\{\frac{\sin t}{t}\}(s)$.

Linearity

$$\mathcal{L}\{f+g\}(s) = \mathcal{L}\{f\}(s) + \mathcal{L}\{g\}(s).$$

Does it hold for a product?

$$\mathcal{L}\{f.g\}(s) = \mathcal{L}\{f\}(s).\mathcal{L}\{g\}(s).$$

WRONG!!!

Convolution

Definition

Let f, g be piecewise continuous functions. Convolution of f and g is

$$(f * g)(t) = \int_0^t f(u)g(t-u)du.$$

Example

$\cos t * \sin t = \frac{1}{2} t \sin t.$

Properties

1. Commutativity $g * f = f * g$.
2. Associativity $f * (g * h) = (f * g) * h$.
3. Distributivity $f * (g + h) = f * g + f * h$.

Theorem

$$\mathcal{L}\{f * g\}(s) = F(s).G(s).$$

$$\mathcal{L}^{-1}\{F(s).G(s)\}(t) = (f * g)(t).$$

Example

Find the inverse Laplace transform

1. $\mathcal{L}^{-1}\left\{\dfrac{2sk}{(s^2+k^2)^2}\right\}(t)$.
2. $\mathcal{L}^{-1}\left\{\dfrac{1}{(s^2+k^2)^2}\right\}(t)$.