Derivatives

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Invariance property of the first order differential

Given a differentiable function $y = f(x)$, we have

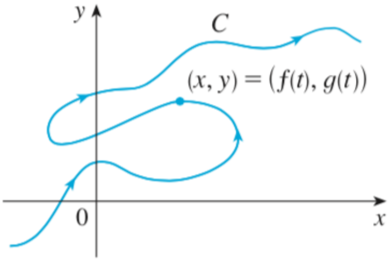
$$dy = f'(x)dx.$$

Assume that x is a dependent variable, namely $x = g(t)$
$\Rightarrow y = f(g(t))$.
We write in terms of t

$$dy = (f \circ g)'(t)dt = f'(g(t))g'(t)dt = f'(x)dx.$$

The differential $dy = f'(x)dx$ is invariant, whether x is an independent or a dependent variable.

Parametric curves



It is impossible to describe this curve by an equation $y = y(x)$. Assume that $f(t), g(t)$ are functions of a third variable, the parameter t. For each t, we determine a point $M(f(t), g(t))$. When t varies, M also varies and traces out a parametric curve C.

Tangents to parametric curves

Assume that $f'(t) \neq 0 \, \forall \, t \in (a, b)$, then x has an inverse: $t = f^{-1}(x)$, we can rewrite $y = g(f^{-1}(x))$.
It is obvious that $y = y(x)$ is differentiable and if $x'(t)$

$$y'(x) = \frac{dy}{dx} = \frac{g'(t)dt}{f'(t)dt} = \frac{g'(t)}{f'(t)}.$$

If $\dfrac{dx}{dt} = 0$, $\left(\dfrac{dy}{dt} \neq 0\right)$, the tangent is vertical.
If $\dfrac{dy}{dt} = 0$, $\left(\dfrac{dx}{dt} \neq 0\right)$, the tangent is horizontal.

Example

Compute the derivative $y'(x)$ of a function given by

a) $\begin{cases} x = e^{t^2}, \\ y = te^{t^2}. \end{cases}$ b) $\begin{cases} x = t - e^{t}, \\ y = 2t + e^{-t^2}. \end{cases}$

Rules

Theorem

- $d(u \pm v) = du \pm dv.$
- $d(u.v) = vdu + udv.$
- $d\left(\dfrac{u}{v}\right) = \dfrac{vdu - udv}{v^2},\ v \neq 0.$

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Definition

$$f^{(n)}(x) = \left(f^{(n-1)}(x)\right)'.$$

Example

- $(e^{ax})^{(n)} = a^n e^{ax}$.
- $(\sin x)^{(n)} = \sin\left(x + n\frac{\pi}{2}\right)$.
- $(\cos x)^{(n)} = \cos\left(x + n\frac{\pi}{2}\right)$.
- $\left((ax+b)^{\alpha}\right)^{(n)} = a^n.\alpha(\alpha-1)\dots(\alpha-n+1)(ax+b)^{\alpha-n}$.

Theorem (Linearity)

$$\Big(Af(x) + Bg(x)\Big)^{(n)} = Af^{(n)}(x) + Bg^{(n)}(x),$$

A, B are constants.

Theorem (Leibniz rule)

$$\big(f(x).g(x)\big)^{(n)} = \sum_{k=0}^{n} C_n^k f^{(k)}(x) g^{(n-k)}(x).$$

Note: Leibniz rule should be applied when a factor is a polynomial.

Example

Compute the n-th derivatives of the functions

1. $f(x) = \sin x \sin 3x$.
2. $f(x) = \dfrac{1}{x^2 - 3x + 2}$.
3. $f(x) = x^2 \sin(3x)$.
4. $f(x) = e^x \sin x$.

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Definition

$$d^n f(x) = f^{(n)}(x)(dx)^n.$$

Note: Higher order differentials depend on whether x is a dependent or an independent variable.

Example

1. $y = (2x+3)\cos(2x)$. Calculate $d^{10}y(0)$.
2. $y = \dfrac{x^3}{1-x}$. Calculate $d^n y,\ n \geq 4$.

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