Power series

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Definition

A power series (centered at x_0) is a **function series** of the form

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + \ldots + a_n(x-x_0)^n + \ldots$$

where a_n are constants, x is the variable.

Consider $x - x_0$ as X, in the following we consider power series of the form $\sum\limits_{n=0}^{\infty} a_n x^n$.

Example

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots = \frac{1}{1-x}, \ |x| < 1.$$

Theorem (Abel Theorem)

If the series $\sum_{n=0}^{\infty} a_n x^n$ *converges at* $x_0 \neq 0$ *then the series converges absolutely at all* x *that* $|x| < |x_0|$.
If the series $\sum_{n=0}^{\infty} a_n x^n$ *diverges at* $x_1 \neq 0$ *then the series diverges at all* x *that* $|x| > |x_1|$.

Proof.

The series $\sum_{n=0}^{\infty} a_n x^n$ always converges at $x = 0$.
$\exists\, R > 0$ such that the power series converges absolutely in $(-R, R)$ and diverges in $(-\infty, -R) \cup (R, \infty)$.
At the end points $x = \pm R$, the series may converge or diverge.

Definition

R is called the radius of convergence of the series.
$(-R; R)$ is called the interval of convergence of the series.

Theorem

Radius of convergence of the series $\sum_{n=1}^{\infty} a_n x^n$ is determined by

$$R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} \text{ or } R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{|a_n|}}.$$

Example

Find the domain of convergence

$$\text{a) } \sum_{n=1}^{\infty} \frac{x^n}{n+2} \qquad \text{b) } \sum_{n=1}^{\infty} n!x^n \qquad \text{c) } \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

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Proposition

Assume that $\sum_{n=0}^{\infty} a_n x^n = S(x)$ *has the radius of convergence* $R \neq 0$. *Then*

1. $\sum_{n=0}^{\infty} a_n x^n$ *converges uniformly on* $[a; b] \subset (-R; R)$.
2. $S(x)$ *is continuous on* $(-R, R)$.
3. $S(x)$ *is integrable on* $[a, b] \subset (-R, R)$.

$$\int \Big(\sum_{n=0}^{\infty} a_n x^n\Big) dx = \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1} + C.$$

4. $S(x)$ *is differentiable on* $(a, b) \subset (-R, R)$.

$$\Big(\sum_{n=0}^{\infty} a_n x^n\Big)' = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Remark

These series have the same radius of convergence R. But their domains of convergence might be different, because of the convergence at the endpoints $x = \pm R$.

Example

Find the sum

1. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$
2. $\displaystyle\sum_{n=0}^{\infty} (3n+1)x^n.$

Example

Find the sum $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(3n+1)}{8^n}.$

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Definition

Let $f(x)$ be an infinitely differentiable function at x_0.
The Taylor series of $f(x)$ at x_0 is the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

If $x_0 = 0$, the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

is called the Maclaurin series of $f(x)$.

Example

Consider $f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

$f(x)$ has derivatives of all orders and $f^{(n)}(0) = 0$, the Taylor series of $f(x)$ is 0.

Remark

The Taylor series of $f(x)$ at x_0 may converge or diverge. In case it converges, the sum may not equal $f(x)$.

Theorem

Let $f(x)$ have the derivatives of all orders in $I = (x_0 - R; x_0 + R)$. If there is $M > 0$ such that $|f^{(n)}(x)| \leq M$ for all $x \in I$, $n \in \mathbb{N}$. Then the Taylor series $\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$ converges to $f(x)$ in $(x_0 - R; x_0 + R)$.

Example

Expand $f(x) = e^x$ into power series.

- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \, x \in \mathbb{R}.$
- $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \, x \in \mathbb{R}.$
- $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \, x \in \mathbb{R}.$
- $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, |x| < 1.$
- $(1+x)^\alpha = \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!} x^n, |x| < 1.$

Example

Expand the following functions into Maclaurin series

1. $f(x) = \dfrac{1}{x^2 - 3x + 2}$.
2. $f(x) = \ln(1 + x)$.
3. $f(x) = \arctan x$.
4. $f(x) = \dfrac{1}{(1 - x)^2}$.

Example

Expand $f(x) = \ln x$ into Taylor series near $x = 1$.