Curve sketching

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Content

Content

General scheme

1. Domain.
2. Monotonicity.
3. Local extremum.
4. Concavity, inflection point.
5. Asymptotes.
6. Table of variation.
7. Sketch of the graph.

Monotonicity

Theorem (Increasing/Decreasing Test)

- *If* $f'(x) > 0$ *on an interval then* $f(x)$ *is increasing on that interval.*
- *If* $f'(x) < 0$ *on an interval then* $f(x)$ *is decreasing on that interval.*

Local extremum

Theorem (First Derivative Test)

Suppose that c is a critical number of a continuous function f.

- *If $f'(x)$ changes from positive to negative at c, then f has a local maximum at c.*
- *If $f'(x)$ changes from negative to positive at c, then f has a local minimum at c.*
- *If $f'(x)$ does not change sign at c, then f has no local extremum at c.*

Concavity

Definition

A point $I(c, f(c))$ is called an inflection point of $y = f(x)$ iff $f''(x)$ changes its sign when crossing $x = c$.

Theorem (Second Derivative Test)

Suppose that $f(x)$ is twice differentiable.

- *If $f''(x) > 0$ on (a, b) then f is concave upward on that interval.*
- *If $f''(x) < 0$ on (a, b) then f is concave downward on that interval.*

Asymptotes

1. $x = a$ is called a vertical asymptote of the curve $y = f(x)$ iff $\lim_{x \to a} y = \infty$.
2. $y = b$ is called a horizontal asymptote of the curve $y = f(x)$ iff $\lim_{x \to \pm\infty} y = b$.
3. $y = ax + b$ is called a slant asymptote of the curve $y = f(x)$ iff $\lim_{x \to \pm\infty} [y - ax - b] = 0$. We have $a = \lim_{x \to \pm\infty} \frac{y}{x}$, $b = \lim_{x \to \pm\infty} [y - ax]$.

1. Find the asymptotes of the curve $y = xe^{2/x} + 1$.
2. Sketch the curve $y = \sqrt{\frac{x^3}{x-1}}$
   Note: Oddness and evenness.

Content

Parametric curves

Assume that $f(t), g(t)$ are functions of the third variable, the parameter t. For each t, we determine a point $M(f(t), g(t))$.When t varies, M also varies and traces out a parametric curve C.

Sketching parametric curves

1. Domain of defnition.
2. Monotonicity of $x(t)$, $y(t)$ w.r.t. t.
3. Asymptotes.
4. Table of variation of $x(t)$, $y(t)$ w.r.t. t.
5. Sketch of the curve.

Asymptotes

Determine t_0 such that as $t \to t_0$ either x or y or both tend to ∞ .
Set $\lim_{t \to t_0} x(t) = l_1, \lim_{t \to t_0} y(t) = l_2$.

- If $l_1 = a, l_2 = \infty$ then $x = a$ is a vertical asymptote.
- If $l_1 = \infty, l_2 = b$ then $y = b$ is a horizontal asymptote.
- If $l_1 = \infty, l_2 = \infty, \lim_{t \to t_0} \frac{y(t)}{x(t)} = 0, \lim_{t \to t_0} [y(t) - ax(t)] = b$ then $y = ax + b$ is a slant asymptote.

Example

Determine the asymtotes of the following curves

$$a)\begin{cases} x(t) = \frac{3t}{1+t^3} \\ y(t) = \frac{2t^2}{1+t^3} \end{cases} \qquad b)\begin{cases} x(t) = \frac{3t^3}{1+t^2} \\ y(t) = \frac{t^2}{1-t^2} \end{cases}$$

Example

Sketching the curve $\begin{cases} x = a\cos^3 t \\ y = a\sin^3 t. \end{cases}$

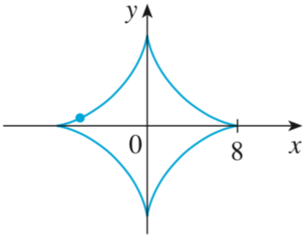


Figure: Astroid, $a = 8$.

In Cartesian coordinates: $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

Example

Sketching the curve $\begin{cases} x = \dfrac{at^2}{t^3+1} \\ y = \dfrac{at}{t^3+1}. \end{cases}$

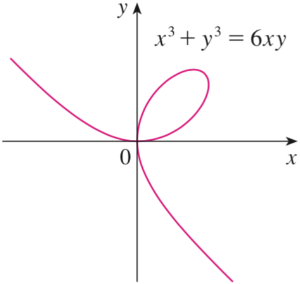


Figure: Folium of Descartes, $a = 6$

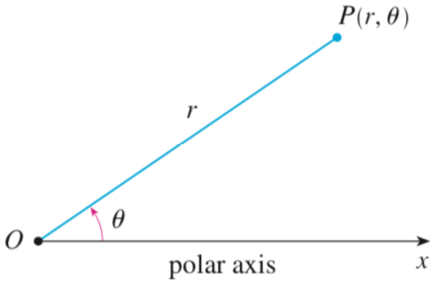
In Cartesian coordinates: $x^3 + y^3 = axy$.

Content

Polar coordinate system

Choose in the plane a fixed point O called the pole.
Draw a ray starting at O called the polar axis. A point P has polar coordinates (r, θ) determined as follows:

- $r = |\overrightarrow{OP}|; 0 \leq r \leq \infty$ – polar radius,
- $\theta = (\overrightarrow{Ox}, \overrightarrow{OP}); 0 \leq \theta < 2\pi$ (rotating $\overrightarrow{Ox}$ counterclockwise until reaching $\overrightarrow{OP}$) – polar angle.



Converting to the Cartesian coordinates xOy:

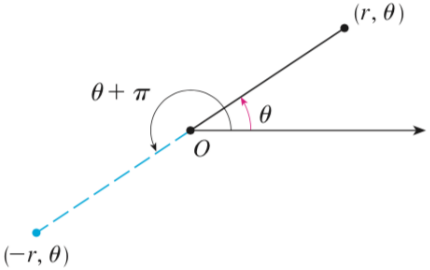
$$(r, \theta) \mapsto (x, y), x = r\cos\theta, y = r\sin\theta.$$
$$(x, y) \mapsto (r, \theta), r = \sqrt{x^2 + y^2}, \tan\theta = \frac{y}{x},$$

such that $\sin\theta$ and y are of the same sign.

Generalized polar coordinates

When mentioning a curve given in polar coordinates, one often means the generalized polar coordinates (r, θ), where $r \in \mathbb{R}, \theta \in \mathbb{R}$, which corresponds to the following point in polar coordinates:

- If $r \geq 0$ then $(r, \theta) = (r, \theta_0)$ where $\theta_0 \in [0, 2\pi)$ and $\theta - \theta_0 = 2k\pi, k \in \mathbb{Z}$.
- If $r < 0$ then $(r, \theta) = (-r, \theta + \pi)$.

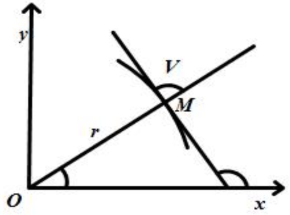


Scheme $r = r(\theta)$

- Domain of definition $r(\theta)$.
- Table of variation of $r(\theta)$.
- Special points of the curve.
- Sketching the curve.

Note: We use the tangent line at a point P to sketch the curve more precisely locally.

V: the angle between the polar radius OP and the tangent line.
α: the angle between the polar axis and the tangent line.



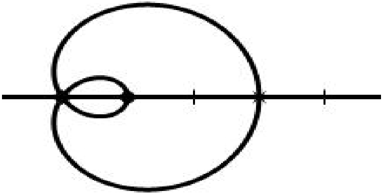
$$V = \alpha - \theta \Rightarrow \tan V = \frac{\tan\alpha - \tan\theta}{1 + \tan\alpha\tan\theta}$$

$$\tan\alpha = \frac{dy}{dx} = \frac{d(r\sin\theta)}{d(r\cos\theta)} = \frac{r'(\theta)\sin\theta + r(\theta)\cos\theta}{r'(\theta)\cos\theta - r(\theta)\sin\theta}$$

$$\Rightarrow \tan V = \frac{r}{r'}$$

Example

Sketching the curve $r = 1 + 2\cos\theta$.



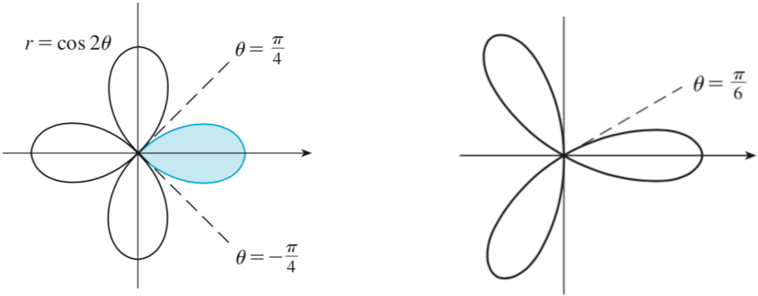


Figure: Four leaved rose $r = a\cos 2\theta$. Three leaved rose $r = a\cos 3\theta$, $(a > 0)$.