

# Fourier series

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October 23, 2020

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- 2 Fourier expansion of  $2l$ - periodic function
- 3 Fourier expansion of a function defined on  $[a; b]$ 
  - General idea
  - Half range Fourier sine or cosine series

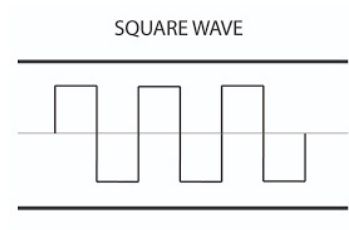
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## Recall: Taylor series

Expand an **infinitely differentiable** function  $f(x)$  in a **neighborhood of**  $x_0$  into power series of  $x - x_0$ .

Can we relax the smoothness of the function?

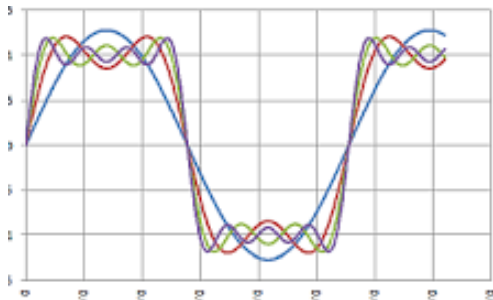
In physics, electrical engineering, one deals with periodic phenomenon.



# What is Fourier series?

Fourier series is a mathematical way to express a **nontrigonometric periodic** function in terms of trigonometric functions, if  $f(x)$  is periodic with period  $T$ , we want to expand

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega x + b_n \sin n\omega x), \quad \omega = \frac{2\pi}{T}.$$



# Trigonometric series

## Definition

**Trigonometric series** has the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

## Proposition (Sufficient condition for convergence)

*If  $\sum_{n=1}^{\infty} |a_n|, \sum_{n=1}^{\infty} |b_n|$  converge then the series above converges absolutely and uniformly on  $\mathbb{R}$ .*

Proof.

$$|a_n \cos nx + b_n \sin nx| \leq |a_n| + |b_n| \quad \forall n \in \mathbb{N}, \quad \forall x \in \mathbb{R}.$$

# Fourier series of a function $f(x)$

Let  $f(x)$  be a periodic function with period  $2\pi$  which is integrable over  $[-\pi, \pi]$  and assume that we expand  $f(x)$  into a trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

- Determine  $a_n, b_n, n = 0, 1, 2, \dots$ ?
- When does the series converge to  $f(x)$ ?

# Orthogonality properties for the sine and cosine functions

## Lemma

For  $m, n \in \mathbb{N}^*$ , we have

$$\int_{-\pi}^{\pi} \sin mx dx = 0, \quad \int_{-\pi}^{\pi} \cos mx dx = 0,$$

$$\int_{-\pi}^{\pi} \sin mx \cos nx dx = 0,$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx dx = \begin{cases} 0 & \text{if } m \neq n, \\ \pi & \text{if } m = n. \end{cases}$$

$$\int_{-\pi}^{\pi} \cos mx \cos nx dx = \begin{cases} 0 & \text{if } m \neq n, \\ \pi & \text{if } m = n. \end{cases}$$



If  $m = n \neq 0$ :

$$\begin{aligned}\int_{-\pi}^{\pi} \sin^2 mx dx &= \frac{1}{2} \int_{-\pi}^{\pi} (1 - \cos 2mx) dx \\ &= \frac{1}{2} \left( x - \frac{\sin 2mx}{2m} \right) \Big|_{-\pi}^{\pi} = \pi.\end{aligned}$$

If  $m \neq n$ :

$$\begin{aligned}\int_{-\pi}^{\pi} \sin mx \sin nx dx &= \frac{1}{2} \int_{-\pi}^{\pi} (\cos(m-n)x - \cos(m+n)x) dx \\ &= \frac{1}{2} \left( \frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n} \right) \Big|_{-\pi}^{\pi} = 0.\end{aligned}$$

Assume that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \quad (1)$$

we obtain (formally)

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) dx &= \int_{-\pi}^{\pi} \frac{a_0}{2} dx + \sum_{n=1}^{\infty} \left( \int_{-\pi}^{\pi} a_n \cos nx dx + \int_{-\pi}^{\pi} b_n \sin nx dx \right) \\ \Rightarrow a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx. \end{aligned}$$

Multiplying (1) by  $\sin mx$  then integrating over  $[-\pi, \pi]$ , we get

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \sin mx dx &= \int_{-\pi}^{\pi} \frac{a_0}{2} \sin mx dx + \sum_{n=1}^{\infty} \left( \int_{-\pi}^{\pi} a_n \cos nx \sin mx dx \right. \\ &\quad \left. + \int_{-\pi}^{\pi} b_n \sin nx \sin mx dx \right) \end{aligned}$$

$$\text{Hence, } b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx dx. \quad a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx dx.$$

# Fourier series

## Definition

Let  $f(x)$  be a  $2\pi$ -periodic function and be integrable over  $(-\pi, \pi)$ . The **Fourier series** or **Fourier expansion** corresponding to  $f(x)$  is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad (2)$$

where the **Fourier coefficients**  $a_n, b_n$  are

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \quad n \geq 0$$
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx, \quad n \geq 1.$$

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## Fourier expansion of $2l$ -periodic function

Assume that  $f(x)$  is periodic with period  $2l$ .

Set  $x' = \frac{\pi}{l}x$  and  $f(x) = f\left(\frac{l}{\pi}x'\right) =: g(x')$ .

It is obvious

$$g(x' + 2\pi) = f\left(\frac{l}{\pi}(x' + 2\pi)\right) = f\left(\frac{l}{\pi}x' + 2l\right) = g(x'),$$

so  $g(x')$  is periodic with period  $2\pi$ .

Expand  $g(x')$  into Fourier series, the Fourier coefficients of  $g$  are

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x') dx' = \frac{1}{l} \int_{-l}^l f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x') \cos nx' dx' = \frac{1}{l} \int_{-l}^l f(x) \cos \frac{n\pi x}{l} dx.$$

$f$  is periodic with period  $2l$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right).$$

$$a_0 = \frac{1}{l} \int_{-l}^l f(x) dx$$

$$a_n = \frac{1}{l} \int_{-l}^l f(x) \cos \frac{n\pi x}{l} dx$$

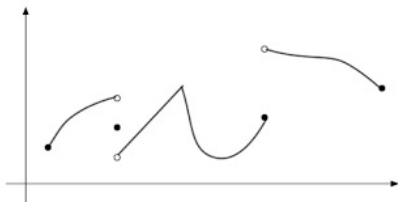
$$b_n = \frac{1}{l} \int_{-l}^l f(x) \sin \frac{n\pi x}{l} dx.$$

## Theorem (Dirichlet conditions)

Let  $f(x)$  be a periodic function with period  $2l$  which is defined except possibly at a finite number of points in  $(-l; l)$ , and  $f(x)$ ,  $f'(x)$  are piecewise continuous in  $(-l; l)$ . Then the Fourier series corresponding to  $f(x)$  converges to

- $f(x)$  if  $x$  is a point of continuity,
- $\frac{f(x+0) + f(x-0)}{2}$  if  $x$  is a point of discontinuity,

where  $f(x+0) = \lim_{y \rightarrow x^+} f(y)$  and  $f(x-0) = \lim_{y \rightarrow x^-} f(y)$ .



$f$  is piecewise continuous in  $[a, b]$  if there exist

$a = x_0 < x_1 < \dots < x_n = b$  such that  $f(x)$  is continuous in  $[x_{i-1}, x_i]$  and  $x_i$  is a point of discontinuity of the first type.

## Example

Expand into Fourier series the periodic function  $f(x)$  with period  $2\pi$  and  $f(x) = x$ ,  $-\pi \leq x < \pi$ .

Fourier coefficients are

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x dx = 0;$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos nx dx = 0.$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx dx$$

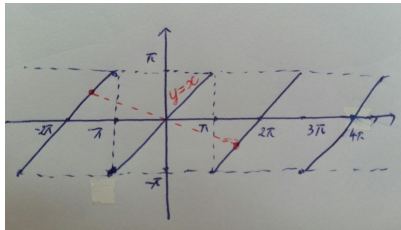
$$= \frac{2}{\pi} \int_0^{\pi} x \frac{d(-\cos nx)}{n} = \frac{2}{\pi} \left[ -\frac{x \cos nx}{n} \Big|_0^{\pi} + \int_0^{\pi} \frac{\cos nx}{n} dx \right]$$

$$= \frac{2}{\pi} \left[ \frac{(-1)^{n+1} \pi}{n} + \frac{\sin nx}{n^2} \Big|_0^{\pi} \right] = \frac{2 \cdot (-1)^{n+1}}{n}.$$

$$\text{So } f(x) = \sum_{n=1}^{\infty} \frac{2 \cdot (-1)^{n+1}}{n} \sin nx, x \neq (2k+1)\pi, k \in \mathbb{Z}.$$



The graph of  $f(x)$ .



At  $x = \pi$ ,  $f(\pi + 0) = -\pi$ ;  $f(\pi - 0) = \pi$ , the series converges to 0.

At  $x = (2k + 1)\pi$ , the series converges to 0.

## Example

Expand into Fourier series the periodic function  $f(x)$  with period  $2\pi$  and

$$f(x) = \begin{cases} -1, & \text{if } -\pi \leq x < 0, \\ 1, & \text{if } 0 \leq x < \pi. \end{cases}$$

Fourier coefficients are

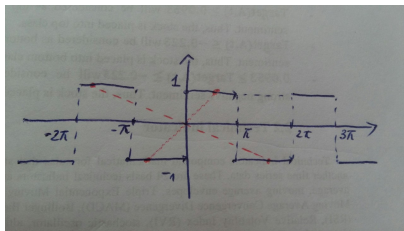
$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \left[ \int_{-\pi}^0 (-1) dx + \int_0^{\pi} 1 dx \right] = \frac{1}{\pi} [-\pi + \pi] = 0.$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \left[ \int_{-\pi}^0 (-\cos nx) dx + \int_0^{\pi} \cos nx dx \right] \\ &= \frac{1}{n\pi} \left[ -\sin nx \Big|_{-\pi}^0 + \sin nx \Big|_0^{\pi} \right] = 0. \end{aligned}$$

$$b_n = \frac{1}{\pi} \left[ \int_{-\pi}^0 (-\sin nx) dx + \int_0^{\pi} \sin nx dx \right] = \frac{2(1 - (-1)^n)}{n\pi}.$$

Hence, 
$$f(x) = \sum_{n=1}^{\infty} \frac{2(1 - (-1)^n)}{n\pi} \sin nx, \quad x \neq k\pi, \quad k \in \mathbb{Z}.$$

## Graph of $f(x)$



$f(x)$  is odd so  $a_n = 0, \forall n \geq 0$  and

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi} \sin nx dx = \frac{2}{\pi} \frac{2 - \cos(nx)}{n} \Big|_0^{\pi} \\ &= \frac{2(1 - (-1)^n)}{n\pi} = \begin{cases} 0 & \text{if } n = 2l, \\ \frac{4}{n\pi} & \text{if } n = 2l + 1. \end{cases} \end{aligned}$$

We get

$$f(x) = \sum_{n=1}^{\infty} \frac{2(1 - (-1)^n)}{n\pi} \sin nx = \sum_{l=0}^{\infty} \frac{4}{(2l+1)\pi} \sin(2l+1)x, x \neq k\pi.$$

At  $x = k\pi$ , the series converges to 0.

## Fourier expansion of odd and even functions

- If  $f(x)$  is an **odd function**:  $a_0 = a_n = 0$ ,

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx, n \geq 1$$

Fourier expansion  $\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$  consists of only sine functions.

- If  $f(x)$  is an **even function**:  $b_n = 0$ ,

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{L} dx, n \geq 0.$$

Fourier expansion  $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$  consists of only cosine functions.

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Let  $f(x)$  be a function defined on the interval  $[a, b]$  that  $f$  and  $f'$  are piecewise continuous.

To expand  $f(x)$  into Fourier series, we will construct a function  $g(x)$  that satisfies

- $g(x) = f(x)$  for all  $x \in [a, b]$ .
- $g(x)$  is periodic (with period  $T \geq b - a$ ).

Fourier series corresponding to  $g(x)$  is the Fourier series corresponding to  $f(x)$  for  $x \in [a; b]$ .

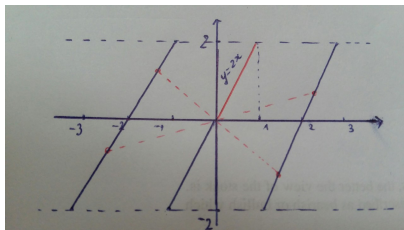
Assume that  $f(x)$  is defined on  $(0, L)$ . The Fourier coefficients:

- ① Fourier cosine series:  $b_n = 0, a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx.$
- ② Fourier sine series:  $a_n = 0, b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx.$

## Example

Expand  $f(x) = 2x$ ,  $0 \leq x < 1$ , into Fourier sine series.

Consider a function  $g_1(x)$  whose graph is as follows.



$g_1(x) = f(x) = 2x$  for  $0 \leq x < 1$ ;  $g_1(x)$  is an odd function and periodic with period  $T = 2$ ,  $l = 1$ .

Fourier coefficients:  $a_n = 0$ ,  $b_n = 2 \int_0^1 2x \sin(n\pi x) dx$ , and

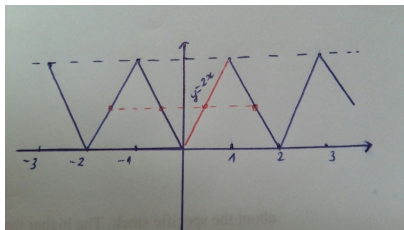
$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$ ,  $0 \leq x < 1$ . **HW: Work out the details.**



## Example

Expand  $f(x) = 2x$ ,  $0 \leq x < 1$ , into Fourier cosine series.

Consider  $g_2(x)$ :



$g_2(x) = f(x) = 2x$  for  $0 \leq x < 1$ ;  $g_2(x)$  is an even function and periodic with  $T = 2, l = 1$ .

Fourier coefficients  $b_n = 0, a_n = 2 \int_0^1 2x \cos(n\pi x) dx$ , and

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x), 0 \leq x < 1$ . **HW: Work out the details.**