Continuous functions

Nguyen Thu Huong

School of Applied Mathematics and Informatics
Hanoi University of Science and Technology

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Definition

Let $f(x)$ be defined on (a, b), $x_0 \in (a, b)$.

- $f(x)$ is said to be continuous at x_0 if $\lim_{x \to x_0} f(x) = f(x_0)$.
- If $f(x)$ is not continuous at x_0, it is said to be discontinuous at x_0.
- $f(x)$ is continuous on (a, b) iff $f(x)$ is continuous at $x \in (a, b)$.

Definition

- $f(x)$ is said to be continuous from the left at x_0 if

$$f(x_0^-) = \lim_{x \to x_0^-} f(x) = f(x_0).$$

- $f(x)$ is said to be continuous from the right at x_0 if

$$f(x_0^+) = \lim_{x \to x_0^+} f(x) = f(x_0).$$

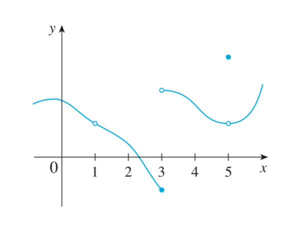
Note: f is continuous at $x_0 \Leftrightarrow \lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = f(x_0)$.

Example

1. Elementary functions are continuous on their domains.
2. Determine whether the following function is continuous or not at $x = 1$

$$g(x) = \begin{cases} e^x - 2 & \text{if } x \geq 1, \\ \cos \pi x & \text{if } x < 1. \end{cases}$$

3. Discontinuity from graph



Definition

$f(x)$ is continuous on $[a, b]$ iff $f(x)$ is continuous on (a, b), **continuous from the right at** a and **continuous from the left at** b.

Classification of discontinuities

Definition

Let x_0 be a discontinuity of the function $f(x)$. x_0 is called

- a jump discontinuity if there exist finite limits $\lim\limits_{x \to x_0^+} f(x)$ and $\lim\limits_{x \to x_0^-} f(x)$.
- a removable discontinuity if $\lim\limits_{x \to x_0^+} f(x) = \lim\limits_{x \to x_0^-} f(x) < \infty$.
- an infinite discontinuity otherwise.

Examples

Example

- $x = 0$ is a removable discontinuity of $f(x) = \frac{\sin x}{x}$.
- $x = 0$ is an infinite disconitnuity of $f(x) = e^{\frac{1}{x}}$.
- Classify the discontinuities of the function $y = \frac{2}{1-2^{\frac{x-1}{x}}}$.

Theorem

If $f(x)$ is continuous on $[a, b]$, then the graph of $f(x)$ has no break in it.

New continuous functions from old ones

Theorem

Let $f(x)$ and $g(x)$ be continuous at $x_0 \in (a,b)$.
Then $f(x) \pm g(x)$, $f(x).g(x)$ are also continuous at x_0.
If $g(x_0) \neq 0$ then $\dfrac{f(x)}{g(x)}$ is also continuous at x_0.

Theorem

Given $f(x)\colon (a,b) \to (c,d)$, $g(x)\colon (c,d) \to \mathbb{R}$. If $f(x)$ is continuous at $x_0 \in (a,b)$, $g(x)$ is continuous at $y_0 = f(x_0)$ then $g \circ f(x)$ is continuous at x_0.

Theorem (Weierstrass)

If $f(x)$ is continuous on $[a, b]$ then it is **bounded, and attains its minimum and maximum on** $[a, b]$.

Theorem

Let $f(x)$ be continuous on $[a, b]$. Then $f(x)$ attains all intermediate values between $f(a)$ and $f(b)$.

Corollary (Intermediate Value Theorem)

Assume that $f(x)$ *is* **continuous on the closed interval** $[a,b]$ *and* $f(a).f(b) < 0$. *Then the equation* $f(x) = 0$ *has at least one solution on* (a,b).

Example

Let $f(x)\colon [0,1] \to [0,1]$ be a continuous function. Prove that the equation $f(x) = x$ has at least one solution in $[0,1]$.

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Uniformly continuous functions

Recall: $f(x)$ is continuous at $x_0 \in X \Leftrightarrow \lim_{x \to x_0} f(x) = f(x_0)$,
$\forall \varepsilon > 0, \exists\, \delta(\varepsilon, x_0) > 0 : \forall\, 0 < |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$.

Definition

A function $f(x)$ is said to be uniformly continuous on X if

$$\forall\, \varepsilon > 0,\ \exists\, \delta(\varepsilon) > 0 : \forall x_1, x_2 \in X, |x_1 - x_2| < \delta : |f(x_1) - f(x_2)| < \varepsilon.$$

Note: If $f(x)$ is uniformly continuous on X, then it is also continuous on X.

Example

- $f(x) = \frac{1}{x}$ is uniformly continuous on $[1, 2]$.
- $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0, 1]$.

Theorem (Cantor)

A function $f(x)$ which is continuous on a **closed interval [a,b]** *is also uniformly continuous on that set.*