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APPLIED STATISTICS AND EXPERIMENTAL DESIGN

Elements of Statistics

TITLE AND CONTENT SLIDE

Elements of Statistics

III. Statistics – parameter estimation

- 3.1. Introduction to Statistics
- 3.2. Parameter estimation
- 3.3. Hypothesis testing

3.1. Introduction to Statistics

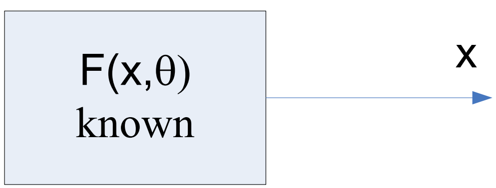
- Definitions
  - Probability – building an abstract models and its conclutions are deductions based on system of axioms
  - Statistics – applications of theory to real world problems and its conclutions are inferences based on observations.
    - Statistics: analysis + design
    - Analysis – mathematical statistics – involving repeated trials and events the probability of which close to 0 or 1.
    - Design –applied statistics deals with data collection and construction of experiments that can be adequately described by probabilistic models
    - Scope of study: mathematical statistics

3.1. Introduction to Statistics

- Probabilistic concepts and reality
  - Probability of event A:
    - P(A) is estimated by $P(A) \approx N_A/N$
    - This empirical formula is used for relative frequency interpretation of all probabilistic concepts
  - Example:
    - the mean η of a r.v can be estimated by
    - $\eta^{\wedge} = (1/n)\sum x_i$, where x_i are observed value of a r.v X.
    - Distribution function $F_X(x)$ can be estimated by
    - $F_X^{\wedge}(x) = n_x/n$, where n_x is number of $x_i \leq x$.
    - The relationship are empirical point estimates of the parameter η and $F_X(x)$ and a major objective of statistics is to give them an exact interpretation

3.1. Introduction to Statistics

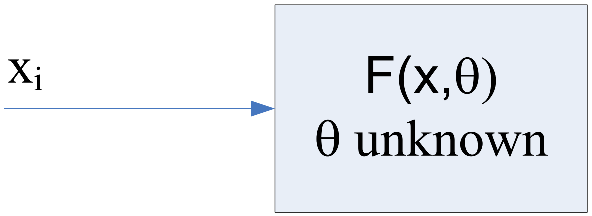
- Problems of statistics:
  - First class of problems:
    - Predict future observations when probabilistic model is known.
    - We proceed from models to observations.
    - Example:
      - Distribution functtion $F_X(x)$ of X is known, and we wish to predict average $\bar{X}$ of its n future samples
      - Probability P of an event A is known, and we wish to predict number of occurences of A in n future trials



Predict x

3.1. Introduction to Statistics

- Second class of problems:
  - One or more parameters θ_i of the model are unknown
  - Estimate values of parameters (parameter estimation)
  - Decide whether θ_i is a set of known constants θ_{0i} (hypothesis testing).
  - We proceed from observations to models



Estimate θ

3.1. Introduction to Statistics

- Example:
  - A coin is tossed 1000 times and heads show 475 times.

    + Estimate value of the probability of heads or

    + Decide whether the coin is fair.
  - The values x_i of r.v X are observed.

    + Estimate mean η of X or

    + Decide whether to accept the hypothesis that $\eta=5.3$.

3.1. Introduction to Statistics

- Prediction problems
  - Given r.v X with known $F_X(x)$
    - Predict its value at future trials
  - Point prediction of X:
    - Determine constant c: (X-c) min.
    - If criterion of selecting *c* is to minimize the MS error $E\{(X-c)^2\}$, then $c=E\{X\}$ MSE -mean square error
      - $MSE = E\{(X-c)^2\} = E\{ X^2 - 2Xc + c^2 \} = \psi(c)$ min.
      - $d\psi(c)/dc = 0$: $E\{2c - 2X\} = 0 \Rightarrow E\{ c-X\} = 0 \Rightarrow$

      $c - E\{X\} = 0$: $c = E\{X\}$.

3.1. Introduction to Statistics

- Interval prediction of X:
  - Determine constants c_1 and c_2 :

    $$P\{c_1<X<c_2\} = \gamma = 1 - \delta \qquad (1)$$

    - Where γ is a given constant called the confidence coefficient. γ100%
  - Predict: $x_i \in (c_1, c_2)$,
    - Correct prediction in 100γ% of the cases.
  - Interval prediction: find c_1, c_2: $(c_2 – c_1)$ min and (1)

3.1. Introduction to Statistics

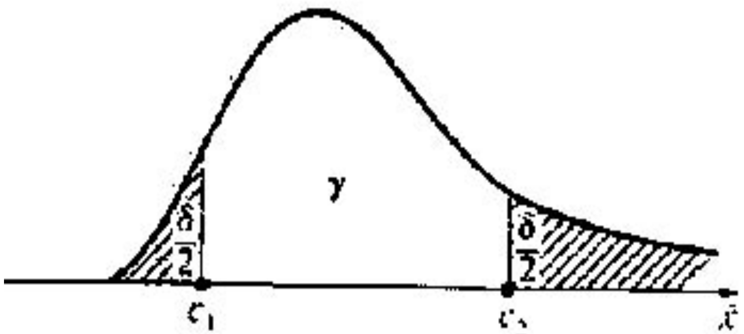
- Selection of γ :
  - If $\gamma \approx 1$, $X \in (c_1, c_2)$ is reliable, but $(c_2 - c_1)$ is large.
  - If $\gamma \downarrow \Rightarrow (c_2 - c_1) \downarrow$, the estimation is less reliable.
  - For optimum prediction: assign value to γ and determine c_1, c_2: $(c_2 - c_1)$ min and (1).
  - If $f_X(x)$ has single maximum $(c_2 - c_1)$ is minimum if $f_X(c_1) = f_X(c_2)$

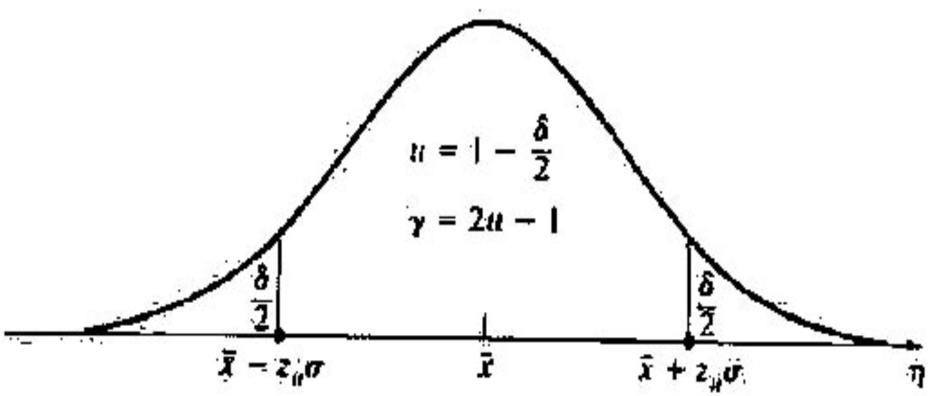
3.1. Introduction to statistics

- Suboptimal solution: if we determine c_1 and c_2:

$$P\{X < c_1\} = \delta/2 \text{ and } P\{X > c_2\} = \delta/2$$

- $\Rightarrow c_1 = x_{\delta/2}$ and $c_2 = x_{1-\delta/2}$
- This solution is optimum if the $f_X(x)$ is symmetrical about its mean η
- If X is normal: $X_u = \eta + z_u\sigma$





3.2. Parameter estimation

- Parameter estimation problem
  - X – r.v with p.d.f $F_X(x, \theta)$ of known form which is depends on parameter θ.
  - θ - scalar or vector
  - Estimate parameter θ.
  - Repeat experiment *n* time and x_i is observed value of x.
  - Based on observed value, find point estimate and interval estimate of θ.

3.2. Parameter estimation

- Point estimate
  - Point estimate: $\hat{\theta} = g(X)$ X=[x_1, ..., x_n] – observation vector;
  - R.v $\hat{\theta} = g(X)$ - point estimator of θ;
  - Any function of vector X=[x_1, ..., x_n] – statistic;
  - Point estimator is statistic.
  - $\hat{\theta}$ - unbias estimator of parameter θ if $E\{\hat{\theta}\} = \theta$
  - Otherwise – biased estimator with bias $b = E\{\hat{\theta}\} - \theta$
  - If *g(X)* is properly selected, the estimation error ↓ when *n*↑.
  - If estimation error $\hat{\theta} - \theta$ $\hat{\theta} - \theta \to 0$ when $n \to \infty$ then $\hat{\theta}$ is called consistent estimator

3.2. Parameter estimation

- Example: sample mean $\bar{X}$ of X
  - $\bar{X}$ is unbiased estimator of η_X
  - Its variance $\sigma^2/n \to 0$ when $n \to \infty$
  - $\bar{X} \to \eta_X$ in MS sense, also in probability.
  - $\bar{X}$ is consistent estimator of η_X
- The best estimator: $\hat{\theta} = g(X)$ MS error min

$$e = E\{[g(X) - \theta]^2\} = \int_R [g(X) - \theta]^2 f(X, \theta) dX$$

  - g(X) is usualy selected empirically.

3.2. Parameter estimation

- Empirically determination of g(X)
  - Suppose θ is the mean θ =E{q(X)} of some function q(X) of X.
  - Sample mean of q(X) is consistent estimator of θ

$$\hat{\theta} = \frac{1}{n}\sum_i q(x_i)$$

  - If sample mean of q(X) is used as the point estimator of θ, the estimate will be satisfactory at least for n large

3.2. Parameter estimation

- Interval Estimates
  - Definition
    - Interval estimate of parameter θ is an interval (θ_1, θ_2), the end points of which are function $\theta_1=g_1(X)$, $\theta_2=g_2(X)$ of the observation vector X.
    - Random interval (θ_1, θ_2) is an interval estimator of θ.
    - If $$P\{\theta_1 < \theta < \theta_2\} < \gamma, \qquad (2)$$
      (θ_1, θ_2) is γ confidence interval of θ.
    - The constant γ - confidence interval of the estimate and the difference $\delta = 1 - \gamma$ is confidence level
    - The objective of interval estimation is determination of functions $g_1(X)$ and $g_2(X)$ so as to minimize the length $(\theta_2 - \theta_1)$ subject to constrain (2)

3.2. Parameter estimation

- Mean estimate
  - R.v X with mean value η
  - The point estimate of mean value
  - The interval estimator of mean value
    - Normality assumption of $\bar{X}$.
  - Known variance
    - Suppose that the variance σ^2 of x is known.
    - z_u – the u percentile of the standard normal density,
    we have:

$$\bar{x} = \frac{1}{n}\sum_i x_i$$

$$P\{\eta - z_{1-\delta/2}\frac{\sigma}{\sqrt{n}} < \bar{x} < \eta + z_{1-\delta/2}\frac{\sigma}{\sqrt{n}}\} = G(z_{1-\delta/2}) - G(-z_{1-\delta/2}) = 1 - \frac{\delta}{2} - \frac{\delta}{2}$$

3.2. Parameter estimation

- Confidence coefficient γ:
  - η is in the interval $\bar{x} \pm z_{1-\delta/2}\sigma/\sqrt{n}$

$$P\{\bar{x} - z_{1-\delta/2}\frac{\sigma}{\sqrt{n}} < \eta < \bar{x} + z_{1-\delta/2}\frac{\sigma}{\sqrt{n}}\} = 1 - \delta = \gamma$$

- Determination of a confidence coefficient for η:
  - Observe the sample x_i của x
  - Form their average $\bar{x}$.
  - Select a number γ=1-δ
  - Find the standard percentile z_u for u=1-δ/2.
  - Form the interval $\bar{x} \pm z_u\sigma/\sqrt{n}$.
- If the discrete type r.v provided that n is large, this also holds.

3.2. Parameter estimation

- The choice of the confidence interval γ is dictated by two conflicting requirements:
  - If $\gamma \approx 1$, the estimate is reliable but the size $2z_u\sigma/\sqrt{n}$ of the confidence interval is large.
  - If γ is reduced, z_u is reduced, but the estimate is less reliable.
  - The final choice is a compromise bazed on the applications.

3.2. Parameter estimation

- Tchebycheff inequality
  - Suppose that the distribution of $\overline{X}$ is unknown.
  - From Tchebycheff inequality:

$$P\{|x-\eta| \geq \varepsilon\} \leq \sigma^2/\varepsilon^2$$

    - Substitute X by $\overline{X}$, σ by $\sigma/\sqrt{n}$ and set $\varepsilon = \sigma/n\delta$
    - We have:

$$P\{\overline{x} - \frac{\sigma}{\sqrt{n\delta}} < \eta < \overline{x} + \frac{\sigma}{\sqrt{n}}\} > 1 - \delta = \gamma$$

    - This shows that, the exact γ confidence interval of η is contained in the interval ($\overline{X} \pm \sigma\sqrt{n\delta}$).

3.2. Parameter estimation

- Unknown variance σ^2
  - To estimate η:
    - Sample variance is unbiased estimator of variance σ^2.
    - It tens to σ^2 when $n \to \infty$

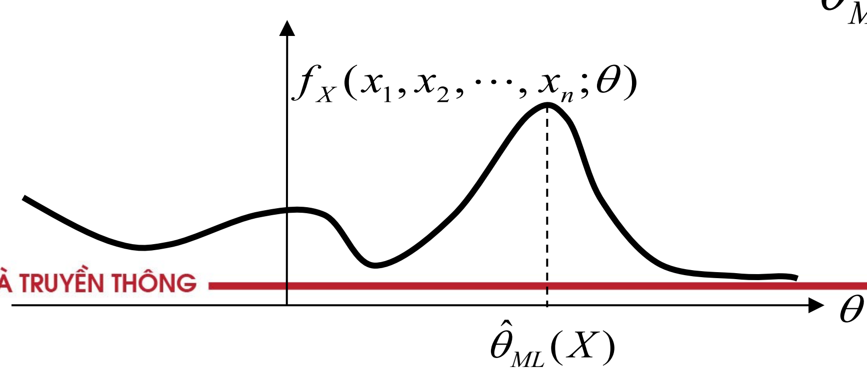
$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

    - For n large, we can use approximation $s \approx \sigma$
    - Confidence interval:

$$\bar{x} - z_{1-\delta/2}\frac{s}{\sqrt{n}} < \eta < \bar{x} + z_{1-\delta/2}\frac{s}{\sqrt{n}}$$

3.2. Parameter estimation

- Maximum likelihood estimation
  - R.v X has density f(x, θ).
  - Estimate θ in terms of a single observation of the r.v X.
  - Assume that the joint p.d.f of $X_1, ..., X_n$ given by $f_X(x_1, ..., x_n; \theta)$ depends on θ.
  - Observations $x_1, ..., x_n$ are given. *T*he value of θ that maximizes f_X is the most likely value for θ.
  - This value is chosen as the ML estimate $\hat{\theta}_{ML}(X)$ for θ



3.2. Parameter estimation

- Given $X_1 = x_1, ..., X_n = x_n$,
- The likelihood function $f_X(x_1, ..., x_n;\theta)$,
- Determination of the ML estimate by:

$$\sup_{\hat{\theta}_{ML}} f_X(x_1, x_2, \cdots, x_n; \theta)$$

- Or

$$L(x_1, x_2, \cdots, x_n; \theta) = \log f_X(x_1, x_2, \cdots, x_n; \theta).$$

- If $L(x_1, ..., x_n;\theta)$ is differentiable and a supremum $\theta\hat{}_{ML}$ exists, then following equation must be satisfied:

$$\left. \frac{\partial \log f_X(x_1, x_2, \cdots, x_n; \theta)}{\partial \theta} \right|_{\theta = \hat{\theta}_{ML}} = 0.$$

3.2. Parameter estimation

- Example
  - $X_i = \theta + w_i$, i=1,...,n: n observations
    - θ - unknown parameter
    - w_i – n independent norrmal r.v with μ_i=0 and variance σ^2.
    - ML estimate of θ ?
  - Solution:
    - Likelihood function
    - Each X_i is normal r.v with mean θ and variance σ^2.

$$f_X(x_1, x_2, \cdots, x_n; \theta) = \prod_{i=1}^{n} f_{X_i}(x_i; \theta).$$

$$f_{X_i}(x_i;\theta) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x_i-\theta)^2/2\sigma^2}.$$

3.2. Parameter estimation

- Likelihood function.

$$f_X(x_1, x_2, \cdots, x_n; \theta) = \frac{1}{(2\pi\sigma^2)^{n/2}} e^{-\sum_{i=1}^{n} (x_i - \theta)^2 / 2\sigma^2}.$$

- Log-likelihood function:

$$L(X;\theta) = \ln f_X(x_1, x_2, \cdots, x_n; \theta) = \frac{n}{2} \ln(2\pi\sigma^2) - \sum_{i=1}^{n} \frac{(x_i - \theta)^2}{2\sigma^2},$$

- ML requirement:

$$\left. \frac{\partial \ln f_X(x_1, x_2, \cdots, x_n; \theta)}{\partial \theta} \right|_{\theta = \hat{\theta}_{ML}} = 2 \sum_{i=1}^{n} \left. \frac{(x_i - \theta)}{2\sigma^2} \right|_{\theta = \hat{\theta}_{ML}} = 0,$$

- And we have

$$\hat{\theta}_{ML}(X) = \frac{1}{n} \sum_{i=1}^{n} X_i.$$

3.2. Parameter estimation

- ML estimator is a r.v with expected value:

$$E[\hat{\theta}_{ML}(x)] = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \theta,$$

- This estimator is unbiased estimator for θ.
- The variance of the estimator:

$$Var(\hat{\theta}_{ML}) = E[(\hat{\theta}_{ML} - \theta)^2] = \frac{1}{n^2} E\left\{\left(\sum_{i=1}^{n} X_i - \theta\right)^2\right\}$$

$$= \frac{1}{n^2}\left\{\sum_{i=1}^{n} E(X_i - \theta)^2 + \underbrace{\sum_{i=1}^{n}\sum_{j=1,\ i\neq j}^{n} E(X_i - \theta)(X_j - \theta)}_{0}\right\}$$

3.2. Parameter estimation

- We have

$$Var(\hat{\theta}_{ML}) = \frac{1}{n^2}\sum_{i=1}^{n} E(X_i - \theta)^2 = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) = \frac{1}{n^2} n\sigma^2 = \frac{\sigma^2}{n}$$

- When $n \to \infty$,

$$Var(\hat{\theta}_{ML}) \to 0$$

The estimator is consistent

3.2. Parameter estimation

- Best Unbiased Estimator:
  - From last example of estimating mean, we have an unbiased estimator for θ with variance σ^2/n.
  - It is possible that, for a given *n*, there may be other unbiased estimators to this problem with even lower variances.
  - Question: In a given scenario, is it possible to determine the lowest possible value for the variance of *any* unbiased estimator?
    - A theorem by Cramer and Rao (Rao 1945; Cramer 1948) gives a complete answer to this problem.

3.2. Parameter estimation

- **Cramer - Rao Bound**:
  - Variance of any unbiased estimator $\hat{\theta}$ based on observations $X_1 = x_1, \cdots, X_n = x_n$ for θ must satisfy the lower bound

$$Var(\hat{\theta}) \geq \frac{1}{E\left(\frac{\partial \ln f_X(x_1, x_2, \cdots, x_n; \theta)}{\partial \theta}\right)^2} = \frac{-1}{E\left(\frac{\partial^2 \ln f_X(x_1, x_2, \cdots, x_n; \theta)}{\partial \theta^2}\right)}.$$

  - This important result states that the right side of the inequality acts as a lower bound on the variance of *all* unbiased estimator for θ, provided their joint p.d.f satisfies certain regularity restrictions.

3.2. Parameter estimation

- Any unbiased estimator whose variance coincides with that in inequality above, must be the best.
- Such estimates are known as *efficient* estimators.
- Example
  - Let examine whether $\theta^{\wedge}{}_{ML}$ for mean represents an efficient estimator. We have:

$$\left(\frac{\partial \ln f_X(x_1, x_2, \cdots, x_n; \theta)}{\partial \theta}\right)^2 = \frac{1}{\sigma^4}\left(\sum_{i=1}^{n}(X_i - \theta)\right)^2;$$

- and

$$E\left(\frac{\partial \ln f_X(x_1, x_2, \cdots, x_n; \theta)}{\partial \theta}\right)^2 = \frac{1}{\sigma^4}\left\{\sum_{i=1}^{n} E[(X_i - \theta)^2] + \sum_{i=1}^{n}\sum_{j=1, i\neq j}^{n} E[(X_i - \theta)(X_j - \theta)]\right\}$$

$$= \frac{1}{\sigma^4}\sum_{i=1}^{n}\sigma^2 = \frac{n}{\sigma^2},$$

- After substitution this into Cramer-Rao inequality, we obtain the Cramer - Rao lower bound for this problem to be σ^2/n

3.3. Hypothesis Testing

- Hypothesis test
  - Statistical hypothesis:
    - Assumption about the values of parameters of a statistical models
    - Hypothesis testing is process for establishing the validity of a hypothesis.

3.3. Hypothesis Testing

- Problem
  - R.v X has known distribution function $F(x, \theta)$ depending on parameter θ.
  - Test assumption $\theta = \theta_0$ against $\theta \neq \theta_0$
  - Hypothesis $\theta = \theta_0$ – null hypothesis H_0
  - Hypothesis $\theta \neq \theta_0$ – alternative hypothesis H_1
  - $\theta \in \Theta_I$.
  - Simple hypothesis: Θ_I consists of single points
    - Otherwise – composite
  - Null hypothesis is simple in most cases.
- Hypohesis testing: whether observations reject null hypothesis ?

3.3. Hypothesis Testing

- Decision regions:
  - Based on observed sample x of X.
  - Suppose that under hypothesis H_0, the density $f(x, \theta_0)$ of the sample x is negligible in a certain region D_c of the sample space, taking significant values only in complement $\overline{D}_c$ of D_c.
  - This is resonable to reject H_0 if x in D_c and to accept H_0 if x is in $\overline{D}_c$.
  - The set D_c is called the critical region of the test and $\overline{D}_c$ is called the region of acceptance H_0.

3.3. Hypothesis Testing

- Example
  - Experiment of fair coin tossing.
    - Toss a coin n times
    - The heads show k times
    - If k << n/2, so the coin is not fair
    - If $k \approx n/2$, so we can accept H_0.

3.3. Hypothesis Testing

- Type of error, which may be occured depending on location of x.
  - First, suppose H_0 is true, if x $\in D_c$, we reject H_0 even though it is true.
    - Error type 1.
    - α - significance level of the test – the probability for the such an error

$$\alpha = P\{x \in D_c \mid H_0\}$$

    - The difference

$$1 - \alpha = P\{x \notin D_c \mid H_0\}$$

Equals the probability that we accept H_0 when true.

3.3. Hypothesis Testing

- Second, suppose that H_0 is false.
  - If $x \notin D_c$, we accept H_0 even though it is false.
  - Error type 2.
  - The probability for such an error is denoted by function $\beta(\theta)$, where θ is called the operating characteristics of the test.

$$\beta(\theta) = P\{\mathrm{x} \notin D_c \mid H_1\}$$

  - The difference $1-\beta(\theta)$ is the probability that we reject hypothesis H_0 when false.
  - $P(\theta)$ – power of the test:

$$P(\theta) = 1 - \beta(\theta) = P\{\mathrm{x} \notin D_c \mid H_1\}$$

3.3. Hypothesis Testing

- Critical region
  - The region D_c is chosen so as to keep the probabilities of both types of errors are small.
  - The selection of the region D_c proceeds as follows:
    - Assign value to typer I error α and search for region D_c of the sample space so as to minimize type II error probability for specific θ.
      - If the resulting value $\beta(\theta)$ is too large, increase α to its tolerable value.
      - If $\beta(\theta)$ still too large, increase the number *n* of samples.

3.3. Hypothesis Testing

- The test is called most powerful if $\beta(\theta)$ is minimum.
  - In general, the critical region of a most powerful test depends on θ.
    - If it is the same for every $\theta \in \Theta_I$, the test is uniformly most powerful.
    - Such a test does not always exist.
    - The determination of the critical region of the most powerful test involve a search in the n-dimentional sample space.

3.3. Hypothesis Testing

- Test statistic
  - Prior to any experiment, we select a function q = g(X) of a sample vector X.
  - We find a set R_c of the real line where under the hypothesis H_0, the density of q is negligible.
  - We reject H_0 if the value q=g(X) of q is in R_c.
  - R_c is the critical region of the test.
  - The r.v q is the test statistic.
  - In the selection of function g(X), we are giuded by the point estimate of θ.

3.3. Hypothesis Testing

- Distribution
  - Hypothesis: the distribution function $F(x)$ of a r.v X equals a given function $F_0(x)$.
    - H_0: $F(x) \equiv F_0(x)$;
    - H_1: $F(X) \neq F_0(x)$.
  - Kolmogoroff-Smirnov test
  - Anderson-Zanderling test
  - Chi-square χ^2 test

3.3. Hypothesis Testing

- Kolmogoroff-Smirnov test
  - Form the random process $F^{\wedge}(x)$ as in the estimation problem and use as the test statistic the r.v

$$q = \max_x | F^{\wedge}(x) - F_0(x) | \qquad (1)$$

  - For a specific ζ, the function $F^{\wedge}(x)$ is the empirical estimate of F(x) and it tends to F(x) as n tends to ∞. From this, it follows that:

$E\{F^{\wedge}(x)\} = F(x)$ and

$$\hat{F}(x) \underset{n\to\infty}{\longrightarrow} F(x)$$

  - It shows that, for large n,
    - q is close to 0 if H_0 is true and
    - It is close to $F(x) - F_0(x)$ if H_1 is true

3.3. Hypothesis Testing

- Conclusion: we must reject H_0 if q is larger than some constant c.
  - Constant c is determined in terms of the significance level α=P{q>c|H_0} and the distribution q.
  - Under hypothesis H_0, the test statistic q equals r.v w in equation w=$\max_x$|F^(x)-F(x)|.
  - Using Kolmogoroff approximation, we obtain:

$$\alpha = P\{\mathrm{q} > c \mid H_0\} = 1 - e^{-2nc^2}$$

- K-S test procedure:
  - Form the empirical estimate F^(x) of F(x)
  - Determine q = $\max_x$|F^(x)-F_0(x)|
  - Accept H_0 if and only if

$$q > \sqrt{-\frac{1}{2n}\ln\left(\frac{\alpha}{2}\right)}$$

  - The resulting error type II error probability is reasonably small only if n large

3.3. Hypothesis Testing

- Chi-square χ^2 test

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Thank you for your attentions!