Chapter 3

Important Probability Distributions

3.1 Some Discrete Probability Distributions

3.1.1 Binomial Distribution $\mathcal{B}(n, p)$

(a) The Bernoulli Process

Example 3.1 Consider the following experiments:

1. Flip a coin and let it land on a table. Observe whether the side facing up is heads or tails. Let X be the number of heads observed.
2. Select a student at random and find out her telephone number. Let $X = 0$ if the last digit is even. Otherwise, let $X = 1$.
3. Observe one bit transmitted by a modem that is downloading a file from the Internet. Let X be the value of the bit (0 or 1).

   All three experiments lead to the probability mass function

$$P_X(x) = \begin{cases} 1/2, & x = 0, \\ 1/2, & x = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Remark 3.1 Because all three experiments lead to the same probability mass function, they can all be analyzed the same way. The PMF in Example 3.1 is a member of the family of Bernoulli random variables.

Definition 3.1 (Bernoulli Random Variable) X is a Bernoulli random variable if the PMF of X has the form

$$P_X(x) = \begin{cases} 1-p, & x = 0, \\ p, & x = 1, \\ 0, & \text{otherwise,} \end{cases}$$

where the parameter p is in the range $0 < p < 1$.

Example 3.2 Suppose you test one circuit. With probability p, the circuit is rejected. Let X be the number of rejected circuits in one test. What is $P_X(x)$?

Solution. Because there are only two outcomes in the sample space, $X = 1$ with probability p and $X = 0$ with probability $1 - p$.

$$P_X(x) = \begin{cases} 1-p, & x = 0, \\ p, & x = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, the number of circuits rejected in one test is a Bernoulli (p) random variable.

Theorem 3.1 *The mean and variance of the Bernoulli random variable X are*

$$\mu = E[X] = p \quad \text{and} \quad \sigma^2 = Var[X] = p(1-p).$$

Proof. $E[X] = (0)P_X(0) + (1)P_X(1) = (0)(1-p) + (1)(p) = p.$

Remark 3.2 (The Bernoulli Process) Strictly speaking, the Bernoulli process must possess the following properties:

1. The experiment consists of repeated trials. Each trial results in an outcome that may be classified as a success or a failure.

2. The probability of success, denoted by p, remains constant from trial to trial.

3. The repeated trials are independent.

(b) Binomial Distribution $\mathcal{B}(n, p)$

The number X of successes in n Bernoulli trials is called a **binomial random variable**.

Definition 3.2 (Binomial random variable) X is a binomial random variable if the PMF of X has the form

$$P_X(x) = \begin{cases} C_n^x p^x (1-p)^{n-x}, & \text{for } x = 0, 1, 2, \ldots, n, \\ 0, & \text{otherwise,} \end{cases}$$

where $0 < p < 1$ and n is an integer such that $n \geq 1$.

The probability distribution of this discrete random variable is called the **binomial distribution**, and is denoted by $\mathcal{B}(n, p)$ (or $X \sim \mathcal{B}(n, p)$) since they depend on the number of trials and the probability of a success on a given trial.

Example 3.3 Suppose we test n circuits and each circuit is rejected with probability p independent of the results of other tests. Let K equal the number of rejects in the n tests. Find the PMF $P_K(k)$.

Solution. Adopting the vocabulary of Section 1.6, we call each discovery of a defective circuit a success, and each test is an independent trial with success probability p. The event $K = k$ corresponds to k successes in n trials, which we have already found, in Equation (1.34), to be the binomial probability

$$P_K(k) = C_n^k(p^k)(1-p)^{n-k}, \quad \text{for } k = 0, 1, 2, \ldots, n.$$

K is an example of a binomial random variable.

If there is a 0.2 probability of a reject and we perform 10 tests,

$$P_K(k) = C_{10}^k(0.2)^k(0.8)^{10-k}, \quad \text{for } k = 0, 1, 2, \ldots, n.$$

Example 3.4 The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease, what is the probability that (a) at least 10 survive, (b) from 3 to 8 survive, and (c) exactly 5 survive?

Solution. Let X be the number of people who survive. $X \sim \mathcal{B}(15, 0.4)$ and the PMF of X is $P_X(x) = C_{15}^x(0.4)^x(0.6)^{15-x}$. Hence,

(a) $P[X \geq 10] = \sum_{x=10}^{x=15} P_X(x) = 0.0338.$

(b) $P[3 \leq X \leq 8] = \sum_{x=3}^{x=8} P_X(x) = 0.8779.$

(c) $P[X = 5] = C_{15}^5(0.4)^5(0.6)^{10} = 0.1859.$

Theorem 3.2 *The mean and variance of the binomial random variable X are*

$$\mu = E[X] = np \quad \textit{and} \quad \sigma^2 = Var[X] = npq, \quad q = 1-p.$$

Proof. Let the outcome on the jth trial be represented by a Bernoulli random variable I_j, which assumes the values 0 and 1 with probabilities q and p, respectively. Therefore, in a binomial experiment the number of successes can be written as the sum of the n independent indicator variables. Hence,

$$X = I_1 + I_2 + \cdots + I_n.$$

The mean of any I_j is $E[I_j] = (0)(q) + (1)(p) = p$. Therefore, the mean of the binomial distribution is

$$\mu = E[X] = E[I_1] + E[I_2] + \cdots + E[I_n] = p + p + \cdots + p = np.$$

The variance of any I_j is $\sigma_{I_j} = E[I_j^2] - (E[I_j])^2 = (0)^2(q) + (1)^2(p) - p^2 = p(1-p) = pq$. Extending the property of variance to the case of n independent Bernoulli variables gives the variance of the binomial distribution as

$$\sigma_X^2 = \sigma_{I_1}^2 + \sigma_{I_2}^2 + \cdots + \sigma_{I_n}^2 = pq + pq + \cdots + pq = npq.$$

3.1.2 Geometric Distribution $\mathcal{G}(p)$

(a) Geometric Distribution $\mathcal{G}(p)$

If repeated independent trials can result in a success with probability p and a failure with probability $q = 1 - p$, then the PMF of the random variable X, the number of the trial on which the first success occurs, is

$$P_X(x) = \begin{cases} p(1-p)^{x-1}, & x = 1, 2, \dots, \\ 0, & \text{otherwise}, \end{cases} \tag{3.1}$$

where the parameter p is in the range $0 < p < 1$.

Definition 3.3 (Geometric random variable) X is a geometric random variable if the PMF of X has the form (3.1).

The probability distribution of the geometric random variable is called the **geometric distribution**, and is denoted by $\mathcal{G}(p)$ (or $X \sim \mathcal{G}(p)$).

Example 3.5 In a test of integrated circuits there is a probability p that each circuit is rejected. Let Y equal the number of tests up to and including the first test that discovers a reject. What is the PMF of Y?

Solution. We see that $P[Y = 1] = p$, $P[Y = 2] = p(1-p)$, $P[Y = 3] = p(1-p)^2$, and, in general, $P[Y = y] = p(1-p)^{y-1}$. Therefore,

$$P_Y(y) = \begin{cases} p(1-p)^{y-1}, & y = 1, 2, \dots, \\ 0, & \text{otherwise}. \end{cases}$$

Y is referred to as a geometric random variable because the probabilities in the PMF constitute a geometric series.

If there is a 0.2 probability of a reject

$$P_Y(y) = \begin{cases} (0.2)(0.8)^{y-1}, & y = 1, 2, \dots, \\ 0, & \text{otherwise}. \end{cases}$$

Theorem 3.3 *The mean and variance of a random variable X following the geometric distribution are*

$$\mu = E[X] = \frac{1}{p} \quad \textit{and} \quad \sigma^2 = Var[X] = \frac{1-p}{p^2}.$$

Proof. We have

$$\begin{aligned} \mu = E[X] &= \sum_{i=1}^{\infty} ip(1-p)^{i-1} = p\sum_{i=1}^{\infty} i(1-p)^{i-1} = p\sum_{i=1}^{\infty}\Big((1-p)^i\Big)' \\ &= p\Big(\sum_{i=1}^{\infty}(1-p)^i\Big)' = p\Big(\frac{1-p}{p}\Big)' = \frac{p}{p^2} = \frac{1}{p}, \quad 0 < (1-p) < 1. \end{aligned}$$

(b) Hypergeometric Distribution $\mathcal{H}(N,n,k)$

We are interested in the probability of selecting x successes from the k items labeled successes and $n-x$ failures from the $N-k$ items labeled failures when a random sample of size n is selected from N items. This is known as a **hypergeometric experiment**, that is, one that possesses the following two properties:

1. A random sample of size n is selected without replacement from N items.
2. Of the N items, k may be classified as successes and $N-k$ are classified as failures.

Definition 3.4 (Hypergeometric random variable) The number X of successes of a hypergeometric experiment is called a hypergeometric random variable.

The PMF of the hypergeometric random variable X, the number of successes in a random sample of size n selected from N items of which k are labeled success and $N-k$ labeled failure, is

$$P_X(x) = \frac{C_k^x C_{N-k}^{n-x}}{C_N^n}, \quad \max\{0, n-(N-k)\} \le x \le \min\{n,k\}. \tag{3.2}$$

Accordingly, the probability distribution of the hypergeometric variable X is called the **hypergeometric distribution**, and is denoted by $\mathcal{H}(N,n,k)$ (or $X \sim \mathcal{H}(N,n,k)$), since they depend on the number of successes k in the set N from which we select n items.

Theorem 3.4 *The mean and variance of the hypergeometric random variable X are*

$$\mu = E[X] = \frac{nk}{N} \quad \textit{and} \quad \sigma^2 = Var[X] = \frac{N-n}{N-1}.n.\frac{k}{N}\Big(1-\frac{k}{N}\Big).$$

3.1.3 Discrete Uniform Distribution

Definition 3.5 (Discrete uniform random variable) X is a discrete uniform random variable if the PMF of X has the form

$$P_X(x) = \begin{cases} \dfrac{1}{l-k+1}, & x = k, k+1, k+2, \dots, l \\ 0, & \text{otherwise,} \end{cases}$$

where the parameters k and l are integers such that $k < l$.

The probability distribution of the discrete uniform random variable is called the **discrete uniform distribution**, and is denoted by $\mathcal{U}(k,l)$.

To describe this discrete uniform random variable, we use the expression "X is uniformly distributed between k and l."

Example 3.6 Roll a fair die. The random variable N is the number of spots that appears on the side facing up. Therefore, N is a discrete uniform $(1,6)$ random variable and

$$P_N(n) = \begin{cases} \dfrac{1}{6}, & n = 1,2,3,4,5,6, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 3.5 *For the discrete uniform random variable X of Definition 3.5,*

$$\mu = E[X] = \frac{k+l}{2} \quad \text{and} \quad \sigma^2 = Var[X] = \frac{(l-k)(l-k+2)}{12}.$$

3.1.4 Poisson Distribution and the Poisson Process

Another discrete random variable that has numerous practical applications is the Poisson random variable. Its probability distribution provides a good model for data that represent the number of occurrences of a specified event in a given unit of time or space.

Experiments yielding numerical values of a random variable X, the number of outcomes occurring during a given time interval or in a specified region, are called **Poisson experiments**. The given time interval may be of any length, such as a minute, a day, a week, a month, or even a year.

(a) Properties of the Poisson Process

A Poisson experiment is derived from the **Poisson process** and possesses the following properties.

1. The number of outcomes occurring in one time interval or specified region of space is independent of the number that occur in any other disjoint time interval or region. In this sense we say that the Poisson process has no memory.
2. The probability that a single outcome will occur during a very short time interval or in a small region is proportional to the length of the time interval or the size of the region and does not depend on the number of outcomes occurring outside this time interval or region.
3. The probability that more than one outcome will occur in such a short time interval or fall in such a small region is negligible.

(b) Poisson Distribution $\mathcal{P}(\lambda)$

The number X of outcomes occurring during a Poisson experiment is called a **Poisson random variable**.

Definition 3.6 (Poisson random variable) X is a Poisson random variable if the PMF of X has the form

$$P_X(x) = \begin{cases} \dfrac{\lambda^x e^{-\lambda}}{x!}, & x = 0, 1, 2, \ldots \\ 0, & \text{otherwise,} \end{cases}$$

where the parameter λ is in the range $\lambda > 0$.

The symbol $e \simeq 2.71828\ldots$ is evaluated using your scientific calculator, which should have a function such as e^x.

Probability distribution of Poisson random variable is called the **Poisson distribution**, and is denoted by $\mathcal{P}(\lambda)$.

Remark 3.3 Here are some examples of experiments for which the random variable X can be modeled by the Poisson random variable:

1. The number of calls received by a switchboard during a given period of time.
2. The number of bacteria per small volume of fluid.
3. The number of customer arrivals at a checkout counter during a given minute.
4. The number of machine breakdowns during a given day.
5. The number of traffic accidents at a given intersection during a given time period.

In each example, X represents the number of events that occur in a period of time or space during which an average of λ such events can be expected to occur. The only assumptions needed when one uses the Poisson distribution to model experiments such as these are that the counts or events occur randomly and independently of one another.

Example 3.7 The number of hits at a Web site in any time interval is a Poisson random variable. A particular site has on average $\alpha = 2$ hits per second.

(a) What is the probability that there are no hits in an interval of 0.25 seconds?

(b) What is the probability that there are no more than two hits in an interval of one second?

Solution. (a) In an interval of 0.25 seconds, the number of hits H is a Poisson random variable with $\lambda = \alpha T = (2\text{hits}/s) \times (0.25s) = 0.5$ hits. The PMF of H is

$$P_H(h) = \begin{cases} \dfrac{(0.5)^h \times e^{-5}}{h!}, & h = 0, 1, 2, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

The probability of no hits is

$$P[H = 0] = P_H(0) = \frac{(0.5)^0 \times e^{-0.5}}{0!} = 0.607.$$

(b) In an interval of 1 second, $\lambda = \alpha T = (2\text{hits}/s) \times (1s) = 2$ hits. Letting J denote the number of hits in one second, the PMF of J is

$$P_J(j) = \begin{cases} \dfrac{(2)^j \times e^{-2}}{j!}, & j = 0, 1, 2, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

To find the probability of no more than two hits, we note that $\{J \leq 2\} = \{J = 0\} \cup \{J = 1\} \cup \{J = 2\}$ is the union of three mutually exclusive events. Therefore,

$$\begin{aligned} P[J \leq 2] &= P[J = 0] + P[J = 1] + P[J = 2] \\ &= P_J(0) + P_J(1) + P_J(2) \\ &= e^{-2} + \frac{2^1 \times e^{-2}}{1!} + \frac{2^2 \times e^{-2}}{2!} = 0.677. \end{aligned}$$

Theorem 3.6 *Both the mean and the variance of the Poisson distribution $\mathcal{P}(\lambda)$ are λ.*

3.1.5 Approximation of Binomial Distribution by a Poisson Distribution

Theorem 3.7 *Let X be a binomial random variable with probability distribution $\mathcal{B}(n, p)$. When $n \to \infty$, $p \to 0$, and $np \to \mu$ as $n \to \infty$ remains constant,*

$$\mathcal{B}(n, p) \to \mathcal{P}(\lambda) \quad \textit{as} \quad n \to \infty.$$

Remark 3.4 The Poisson distribution provides a simple, easy-to-compute, and accurate approximation to binomial probabilities when n is large and $\lambda = np$ is small, preferably with $np < 7$.

Example 3.8 Suppose a life insurance company insures the lives of 5000 men aged 42. If actuarial studies show the probability that any 42-year-old man will die in a given year to be 0.001, find the exact probability that the company will have to pay $X = 4$ claims during a given year.

Solution. The exact probability is given by the binomial distribution as

$$P[X = 4] = \frac{5000!}{4!4996!}(0.001)^4(0.999)^{4996}$$

for which binomial tables are not available. To compute $P[X = 4]$ without the aid of a computer would be very time-consuming, but the Poisson distribution can be used to provide a good approximation to $P[X = 4]$. Computing $\lambda = np = (5000)(0.001) = 5$ and substituting into the formula for the Poisson probability distribution, we have

$$P[X = 4] \simeq \frac{5^4}{4!}e^{-5} = \frac{(625)(0.006738)}{24} = 0.175.$$

Example 3.9 In a certain industrial facility, accidents occur infrequently. It is known that the probability of an accident on any given day is 0.005 and accidents are independent of each other.

(a) What is the probability that in any given period of 400 days there will be an accident on one day?

(b) What is the probability that there are at most three days with an accident?

Solution. Let X be a binomial random variable with $n = 400$ and $p = 0.005$. Thus, $np = 2$. Using the Poisson approximation,

(a) $P[X = 1] = e^{-1}2^1 = 0.271$ and

(b) $P[X \le 3] = \sum_{x=0}^{3} \dfrac{e^{-2}2^x}{x!} = 0.857$.

3.2 Some Continuous Probability Distributions

3.2.1 Continuous Uniform Distribution $\mathcal{U}[a, b]$

One of the simplest continuous distributions in all of statistics is the **continuous uniform distribution**. This distribution is characterized by a density function that is “flat,” and thus the probability is uniform in a closed interval, say $[a, b]$.

Definition 3.7 (Uniform random variable) X is a uniform $\mathcal{U}[a, b]$ random variable if the PDF of X is

$$f_X(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x \le b, \\ 0, & \text{otherwise}, \end{cases} \tag{3.3}$$

where the two parameters are $b > a$.

Example 3.10 Suppose that a large conference room at a certain company can be reserved for no more than 4 hours. Both long and short conferences occur quite often. In fact, it can be assumed that the length X of a conference has a uniform distribution on the interval $[0, 4]$.

(a) What is the probability density function?

(b) What is the probability that any given conference lasts at least 3 hours?

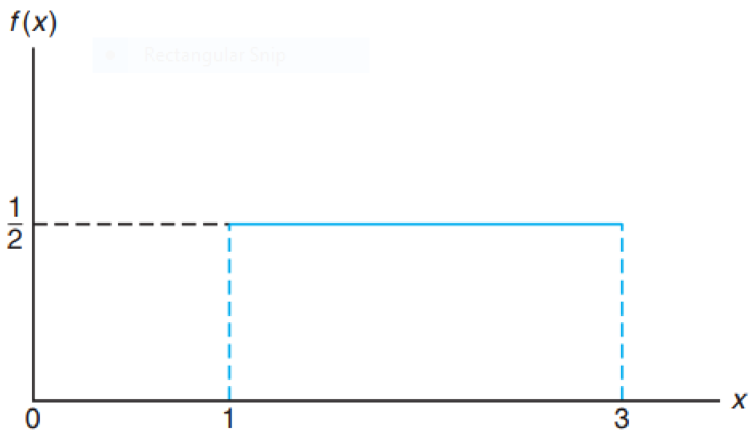

Figure 3.1: The density function for a random variable on the interval $[1, 3]$.

Solution.

(a) The appropriate density function for the uniformly distributed random variable X in this situation is

$$f(x) = \begin{cases} \dfrac{1}{4}, & 0 \le x \le 4, \\ 0, & \text{otherwise.} \end{cases}$$

(b) $P[X \ge 3] = \int_3^4 \frac{1}{4} dx = \frac{1}{4}$.

Theorem 3.8 *If X is a uniform $\mathcal{U}[a, b]$ random variable,*

(a) *The CDF of X is*

$$F_X(x) = \begin{cases} 0, & x \le a, \\ \dfrac{x-a}{b-a}, & a < x \le b, \\ 1, & x > b. \end{cases} \tag{3.4}$$

(b) *The expected value of X is* $\mu = E[X] = \dfrac{a+b}{2}$.

(c) *The variance of X is* $\sigma^2 = Var[X] = \dfrac{(b-a)^2}{12}$.

Example 3.11 The phase angle, φ, of the signal at the input to a modem is uniformly distributed between 0 and 2π radians. Find the CDF, the expected value, and the variance of φ.

Solution. From the problem statement, we identify the parameters of the uniform $\mathcal{U}[a, b]$ random variable as $a = 0$ and $b = 2\pi$. Therefore the PDF of is

$$f_\varphi(x) = \begin{cases} \dfrac{1}{2\pi}, & 0 \le x < 2\pi, \\ 0, & \text{otherwise.} \end{cases}$$

The CDF is

$$F_\varphi(x) = \begin{cases} 0, & x \le 0, \\ \dfrac{x}{2\pi}, & 0 < x \le 2\pi, \\ 1, & x > 2\pi. \end{cases}$$

The expected value is $E[\varphi] = b/2 = \pi$ radians, and the variance is $Var[\varphi] = (2\pi)^2/12 = \pi^2/3$ rad^2.

The relationship between the family of discrete uniform random variables and the family of continuous uniform random variables is fairly direct. The following theorem expresses the relationship formally.

Theorem 3.9 *Let X be a uniform $\mathcal{U}[a, b]$ random variable, where a and b are both integers. Let $K = \lceil X \rceil$. Then K is a discrete uniform $[a+1, b]$ random variable.*

3.2.2 Exponential Distribution $exp(\lambda)$

Definition 3.8 (Exponential random variable) X is an exponential $exp(\lambda)$ random variable if the PDF of X is

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & \text{otherwise}, \end{cases} \tag{3.5}$$

where the parameter $\lambda > 0$.

Example 3.12 The probability that a telephone call lasts no more than t minutes is often modeled as an exponential CDF.

$$F_T(t) = \begin{cases} 1 - e^{-t/3}, & t \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

What is the PDF of the duration in minutes of a telephone conversation? What is the probability that a conversation will last between 2 and 4 minutes?

Solution. We find the PDF of T by taking the derivative of the CDF:

$$f_T(t) = \frac{dF_T(t)}{dt} = \begin{cases} \frac{1}{3}e^{-t/3}, & t \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, observing Definition 3.8, we recognize that T is an exponential $(\lambda = 1/3)$ random variable. The probability that a call lasts between 2 and 4 minutes is

$$P[2 \leq T \leq 4] = F_T(4) - F_T(2) = e^{-2/3} - e^{-4/3} = 0.250.$$

Example 3.13 In Example 3.12, what is $E[T]$, the expected duration of a telephone call? What are the variance and standard deviation of T? What is the probability that a call duration is within ± 1 standard deviation of the expected call duration?

Solution. Using the PDF $f_T(t)$ in Example 3.12, we calculate the expected duration of a call:

$$E[T] = \int_{-\infty}^{+\infty} t f_T(t) dt = \int_0^{+\infty} t \frac{1}{3} e^{-t/3} dt.$$

Integration by parts yields

$$E[T] = -te^{-t/3}\Big|_0^{+\infty} + \int_0^{+\infty} e^{-t/3} dt = 3 \text{ minutes.}$$

To calculate the variance, we begin with the second moment of T:

$$E[T^2] = \int_{-\infty}^{+\infty} t^2 f_T(t) dt = \int_0^{+\infty} t^2 \frac{1}{3} e^{-t/3} dt.$$

Again integrating by parts, we have

$$E[T^2] = -t^2 e^{-t/3}\Big|_0^{+\infty} + \int_0^{+\infty} 2te^{-t/3} dt = 2\int_0^{+\infty} te^{-t/3} dt.$$

With the knowledge that $E[T] = 3$, we observe that $\int_0^{+\infty} te^{-t/3}dt = 3E[T] = 9$. Thus $E[T^2] = 6E[T] = 18$ and

$$Var[T] = E[T^2] - (E[T])^2 = 18 - 3^2 = 9.$$

The standard deviation is $\sigma_T = \sqrt{Var[T]} = 3$ minutes. The probability that the call duration is within ± 1 standard deviation of the expected value is

$$P[0 \le T \le 6] = F_T(6) - F_T(0) = 1 - e^{-2} = 0.865.$$

Theorem 3.10 *If X is an exponential $exp(\lambda)$ random variable,*

(a) $F_X(x) = \begin{cases} 1 - e^{-\lambda x}, & x \ge 0, \\ 0, & \textit{otherwise.} \end{cases}$

(b) $\mu = E[X] = 1/\lambda$.

(c) $\sigma^2 = Var[X] = 1/\lambda^2$.

The following theorem shows the relationship between the family of exponential random variables and the family of geometric random variables.

Theorem 3.11 *If X is an exponential $exp(\lambda)$ random variable, then $K = \lceil X \rceil$ is a geometric $\mathcal{G}(p)$ random variable with $p = 1 - e^{-\lambda}$.*

Example 3.14 Phone company A charges \$0.15 per minute for telephone calls. For any fraction of a minute at the end of a call, they charge for a full minute. Phone company B also charges \$0.15 per minute. However, Phone company B calculates its charge based on the exact duration of a call. If T, the duration of a call in minutes, is an exponential ($\lambda = 1/3$) random variable, what is the PDF of T? What is the expected value of T? What are the expected revenues per call $E[R_A]$ and $E[R_B]$ for companies A and B?

Solution. Because T is an exponential ($\lambda = 1/3$) random variable,

$$f_T(t) = \begin{cases} \frac{1}{3}e^{-\frac{1}{3}t}, & t \ge 0, \\ 0, & \text{otherwise.} \end{cases}$$

We have in Theorem 3.10 (and in Example 3.13),

$$E[T] = \int_{-\infty}^{+\infty} t f_T(t)dt = \frac{1}{\lambda} = 3 \quad \text{minutes per call.}$$

Therefore, for phone company B, which charges for the exact duration of a call,

$$E[R_B] = 0.15 \times E[T] = 0.45 \quad \text{dolars per call.}$$

Company A, by contrast, collects \$0.15$\lceil T \rceil$ for a call of duration T minutes. Theorem 3.11 states that $K = \lceil T \rceil$ is a geometric random variable with parameter $p = 1 - e^{-1/3}$. Therefore, the expected revenue for company A is

$$E[R_A] = 0.15 \times E[K] = \frac{0.15}{p} = \frac{0.15}{0.2834} = (0.15) \times (3.5285) = 0.5292 \quad \text{dolars per call.}$$

3.2.3 Normal Distribution $\mathcal{N}(\mu, \sigma)$

The most important continuous probability distribution in the entire field of statistics is the **normal distribution**. Its graph, called the normal curve, is the bell-shaped curve of Figure 3.2, which approximately describes many phenomena that occur in nature, industry, and research.

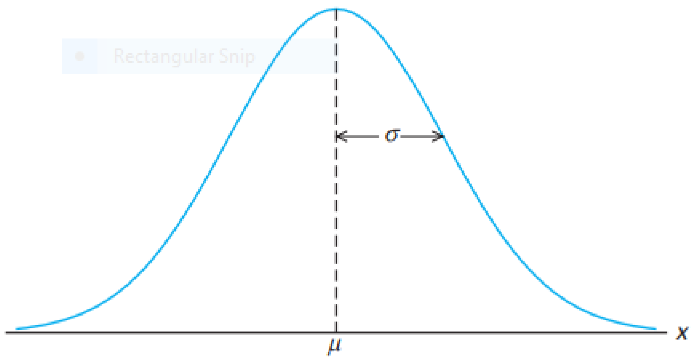

Figure 3.2: The normal curve

The normal distribution is often referred to as the **Gaussian distribution**, in honor of Karl Friedrich Gauss (1777–1855), who also derived its equation from a study of errors in repeated measurements of the same quantity.

A continuous random variable X having the bell-shaped distribution of Figure 3.2 is called a **normal random variable**. The mathematical equation for the probability distribution of the normal variable depends on the two parameters μ and σ, its mean and standard deviation, respectively. Hence, we denote the normal distribution by $\mathcal{N}(\mu, \sigma)$.

(a) Normal Distribution $\mathcal{N}(\mu, \sigma)$

Definition 3.9 The PDF of the normal random variable X, with mean μ and variance σ^2, is

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\dfrac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty, \tag{3.6}$$

where $\pi = 3.14159\ldots$ and $e = 2.71828\ldots$.

Once μ and σ are specified, the normal curve is completely determined. For example, if $\mu = 50$ and $\sigma = 5$, then the ordinates $\mathcal{N}(50, 5)$ can be computed for various values of x and the curve drawn. In Figure 3.3, we have sketched two normal curves having the same standard deviation but different means. The two curves are identical in form but are centered at different positions along the horizontal axis.

In Figure 3.4, we have sketched two normal curves with the same mean but different standard deviations. Figure 3.5 shows two normal curves having different means and different standard deviations.

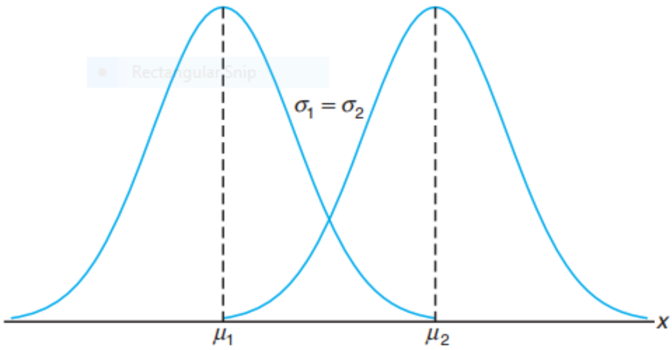

Figure 3.3: Normal curves with $\mu_1 < \mu_2$ and $\sigma_1 = \sigma_2$

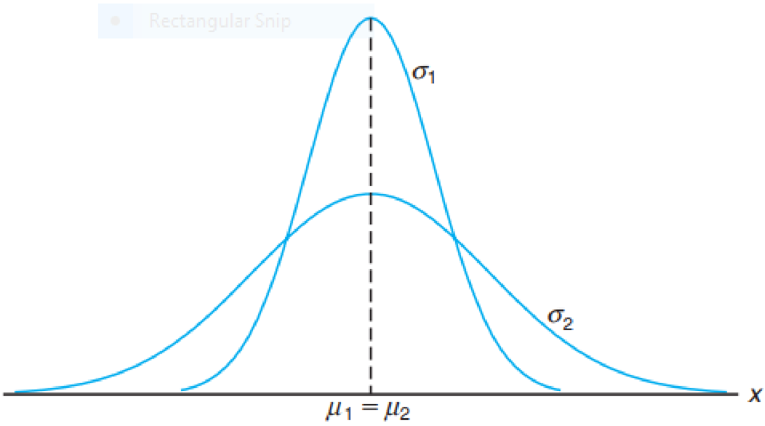

Figure 3.4: Normal curves with $\mu_1 = \mu_2$ and $\sigma_1 < \sigma_2$

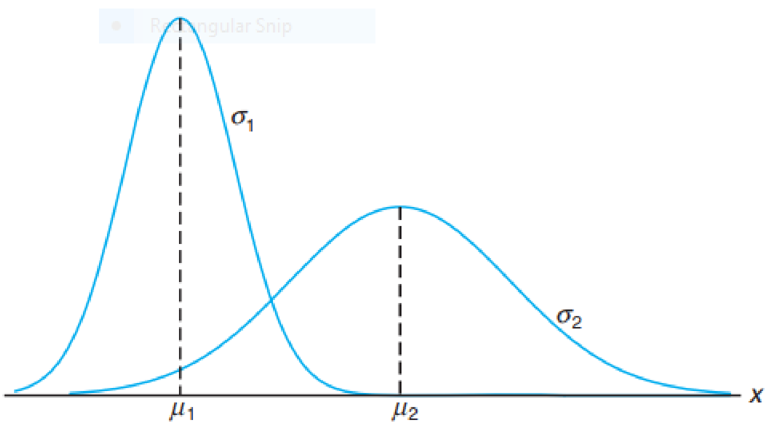

Figure 3.5: Normal curves with $\mu_1 < \mu_2$ and $\sigma_1 < \sigma_2$

Based on inspection of Figures 3.2 through 3.5 and examination of the first and second derivatives of $\mathcal{N}(\mu, \sigma)$, we list the following properties of the normal curve:

1. The mode, which is the point on the horizontal axis where the curve is a maximum, occurs at $x = \mu$.
2. The curve is symmetric about a vertical axis through the mean μ.
3. The curve has its points of inflection at $x = \mu \pm \sigma$; it is concave downward if $\mu - \sigma < X < \mu + \sigma$ and is concave upward otherwise.
4. The normal curve approaches the horizontal axis asymptotically as we proceed in either direction away from the mean.
5. The total area under the curve and above the horizontal axis is equal to 1.

Theorem 3.12 *The mean and variance of the normal random variable are μ and σ^2, respectively. Hence, the standard deviation is σ.*

Proof. To evaluate the mean, we first calculate

$$E[X-\mu] = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}(x-\mu)e^{-\dfrac{(x-\mu)^2}{2\sigma^2}}dx.$$

Setting $z = (x-\mu)/\sigma$ and $dx = \sigma dz$, we obtain

$$E[X-\mu] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} ze^{-\dfrac{z^2}{2}}dz = 0,$$

since the integrand above is an odd function of z. Hence,

$$E[X] = \mu.$$

The variance of the normal distribution is given by

$$E[(X-\mu)^2] = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}(x-\mu)^2e^{-\dfrac{(x-\mu)^2}{2\sigma^2}}dx.$$

Again setting $z = (x-\mu)/\sigma$ and $dx = \sigma dz$, we obtain

$$E[(X-\mu)^2] = \frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} z^2e^{-\dfrac{z^2}{2}}dz.$$

Integrating by parts with $u = z$ and $dv = ze^{-z^2/2}dz$ so that $du = dz$ and $v = -e^{-z^2/2}$, we find that

$$E[(X-\mu)^2] = \frac{\sigma^2}{\sqrt{2\pi}}\Big(-ze^{-z^2/2}\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-\dfrac{z^2}{2}}dz\Big) = \sigma^2(0+1) = \sigma^2.$$

Theorem 3.13 *If X is normal random variable $\mathcal{N}(\mu, \sigma)$, $Y = aX + b$ is normal random variable $\mathcal{N}(a\mu + b, a\sigma)$.*

(b) Standard Normal Distribution $\mathcal{N}(0,1)$. Area Under the Normal Curve

Put

$$Z = \frac{X-\mu}{\sigma}.$$

If X is a normal random variable with mean μ and variance σ^2 then Z is seen to be a normal random variable with mean 0 and variance 1.

Definition 3.10 (Standard normal distribution) The distribution of a normal random variable with mean 0 and variance 1 is called a standard normal distribution.

Definition 3.11 (CDF) The CDF of the standard normal random variable Z is

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{u^2}{2}} du.$$

Given a table of values of $\Phi(z)$ (see Table 3.9), we use the following theorem to find probabilities of a normal random variable with parameters μ and σ.

Theorem 3.14 *If X is a normal random variable $\mathcal{N}(\mu,\sigma)$, the CDF of X is*

$$F_X(x) = \Phi\Big(\frac{x-\mu}{\sigma}\Big).$$

The probability that X is in the interval (x_1, x_2) is

$$P[x_1 < X < x_2] = \Phi\Big(\frac{x_2-\mu}{\sigma}\Big) - \Phi\Big(\frac{x_1-\mu}{\sigma}\Big).$$

The curve of any continuous probability distribution or probability density function is constructed so that the area under the curve bounded by the two ordinates $x = x_1$ and $x = x_2$ equals the probability that the random variable X assumes a value between $x = x_1$ and $x = x_2$. Thus, for the normal curve in Figure 3.6,

$$P(x_1 < X < x_2) = \frac{1}{\sigma\sqrt{2\pi}} \int_{x_1}^{x_2} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

is represented by the area of the shaded region.

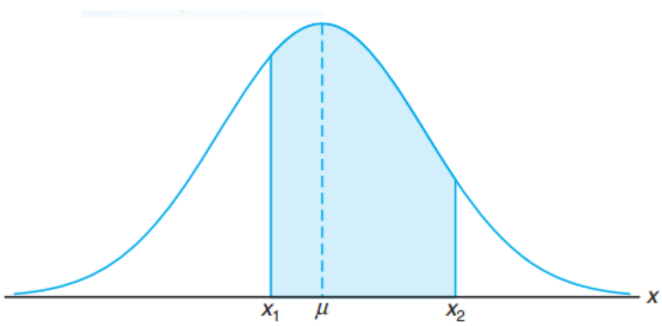

Figure 3.6: $P(x_1 < X < x_2) =$ area of the shaded region

Remark 3.5 (a) In using Theorem 3.15, we transform values of a norm random variable, X, to equivalent values of the standard normal random variable, Z. For a sample value x of the random variable X, the corresponding sample value of Z is

$$z = \frac{x-\mu}{\sigma} \quad \text{or equivalently,} \quad x = \mu + z\sigma. \tag{3.7}$$

The original and transformed distributions are illustrated in Figure 3.7. Since all the values of X falling between x_1 and x_2 have corresponding z values between z_1 and z_2, the area under the X-curve between the ordinates $x = x_1$ and $x = x_2$ in Figure 3.7 equals the area under the Z-curve between the transformed ordinates $z = z_1$ and $z = z_2$.

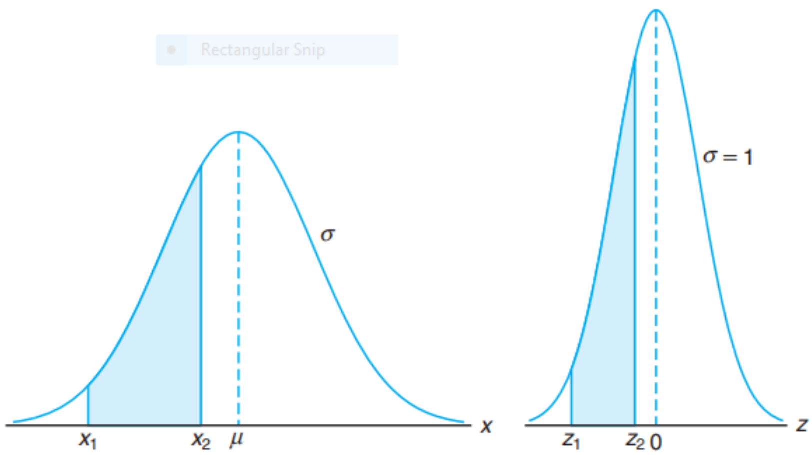

Figure 3.7: The original and transformed normal distributions

(b) The probability distribution for Z, shown in Figure 3.8, is called the standardized normal distribution because its mean is 0 and its standard deviation is 1. Values of Z on the left side of the curve are negative, while values on the right side are positive. The area under the standard normal curve to the left of a specified value of Z say, z_0 is the probability $P[Z \le z_0]$. This cumulative area is recorded in Table 3.9 and is shown as the shaded area in Figure 3.8.

Example 3.15 Suppose your score on a test is $x = 46$, a sample value of the Gaussian (61, 10) random variable. Express your test score as a sample value of the standard normal random variable, Z.

Solution. Equation (3.7) indicates that $z = (46-61)/10 = -1.5$. Therefore your score is 1.5 standard deviations less than the expected value.

Remark 3.6 To find probabilities of norm random variables, we use the values of (z) presented in Table 3.9. Note that this table contains entries only for $z \ge 0$. For negative values of z, we apply the following property of $\Phi(z)$.

Theorem 3.15

$$\Phi(-z) = 1 - \Phi(z).$$

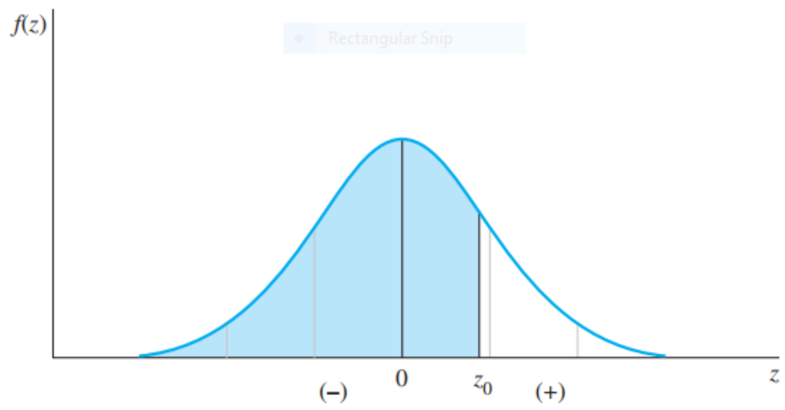

Figure 3.8: Standardized normal distribution

Remark 3.7 Figure 3.9 displays the symmetry properties of $\Phi(z)$. Both graphs contain the standard normal PDF. In Figure 3.9(a), the shaded area under the PDF is $\Phi(z)$. Since the area under the PDF equals 1, the unshaded area the PDF is $1 - \Phi(z)$. In Figure 3.9(b), the shaded area on the right is $1 - \Phi(z)$ and the shaded area on the left is $\Phi(-z)$. This graph demonstrates that $\Phi(-z) = 1 - \Phi(z)$.

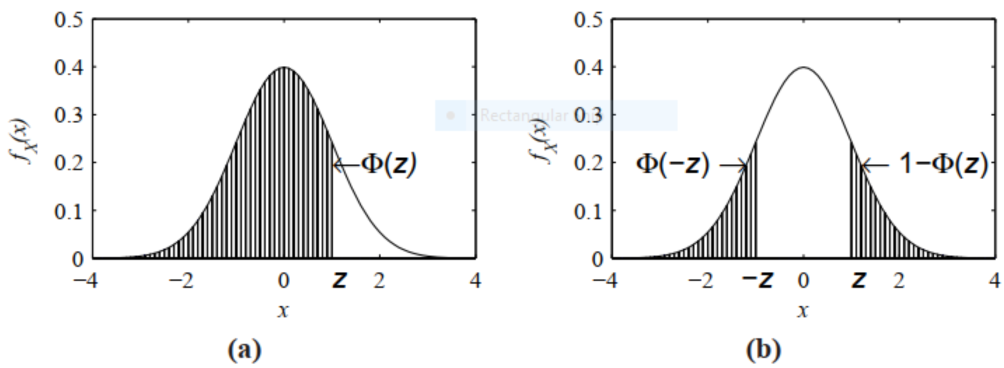

Figure 3.9: Standardized normal distribution

Example 3.16 Find the probability that a normally distributed random variable will fall within these ranges:

(a) One standard deviation of its mean.

(b) Two standard deviations of its mean.

Solution. (a) Since the standard normal random variable z measures the distance from the mean in units of standard deviations, you need to find

$$P[-1 \leq Z \leq 1] = \Phi(1) - \Phi(-1) = \Phi(1) - [1 - \Phi(1)] = 0.84134 - 0.15866 = 0.68268.$$

Remember that you calculate the area between two z-values by subtracting the tabled entries for the two values.

(b) As in part (a), $P[-2 \leq Z \leq 2] = 0.97725 - 0.02275 = 0.9545$.

Example 3.17 If X is the norm random variable $\mathcal{N}(61, 10)$, what is $P[X \leq 46]$?

Solution. Applying Theorem 3.14, Theorem 3.15 and the result of Example 3.15, we have

$$P[X \leq 46] = \Phi(-1.5) = 1 - \Phi(1.5) = 1 - 0.93319 = 0.06681.$$

Example 3.18 If X is a Gaussian random variable with $\mu = 61$ and $\sigma = 10$, what is $P[51 < X \leq 71]$?

Solution. Applying Equation (3.7), we find that the event $\{51 < X \leq 71\}$ corresponds to $\{-1 < Z \leq 1\}$. The probability of this event is

$$\Phi(1) - \Phi(-1) = \Phi(1) - [1 - \Phi(1)] = 2\Phi(1) - 1 = 0.68268.$$

Example 3.19 Studies show that gasoline use for compact cars sold in the United States is normally distributed, with a mean of 25.5 miles per gallon (mpg) and a standard deviation of 4.5 mpg.

(a) What percentage of compacts get 30 mpg or more?

(b) In times of scarce energy resources, a competitive advantage is given to an automobile manufacturer who can produce a car that has substantially better fuel economy than the competitors' cars. If a manufacturer wishes to develop a compact car that outperforms 95% of the current compacts in fuel economy, what must the gasoline use rate for the new car be?

Solution. To solve this problem, you must first find the z-value corresponding to $x = 30$. Substituting into the formula for z, you get $z = 1$. The proportion of compacts that get 30 mpg or more is equal to

$$P[X \geq 30] = 1 - P[Z < 1] = 1 - 0.84134 = 0.15866.$$

The percentage exceeding 30 mpg is $100(0.15866) \simeq 15.87\%$.

(b) The gasoline use rate X has a normal distribution with a mean of 25.5 mpg and a standard deviation of 4.5 mpg. You need to find a particular value say, x_0 such that $P[X < x_0] = 0.95$. This is the 95th percentile of the distribution of gasoline use rate x. Since the only information you have about normal probabilities is in terms of the standard normal random variable z, start by standardizing the value of x_0: $z_0 = \dfrac{x_0 - 25.5}{4.5}$. Since the value of z_0 corresponds to x_0, it must also have area 0.95 to its left. If you look in the interior of Table 3 in Appendix I, you will find that the area 0.9500 is exactly halfway between the areas for $z = 1.64$ and $z = 1.65$. Thus, z_0 must be exactly halfway between 1.64 and 1.65, or $z_0 = \dfrac{x_0 - 25.5}{4.5} = 1.645$ Solving for x_0, you obtain $x_0 = \mu + z_0\sigma = 25.5 + (1.645)(4.5) = 32.9$.

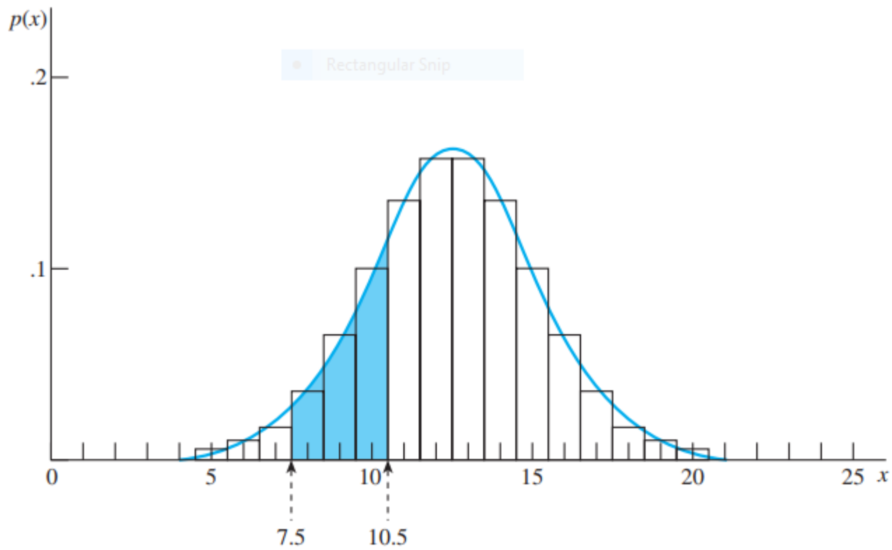

Figure 3.10: The binomial probability distribution for $n = 25$ and $p = 0.5$ and the approximating normal distribution with $\mu = 12.5$ and $\sigma = 2.5$

(c) The Normal Approximation to the Binomial Probability Distribution

Probabilities associated with binomial experiments are readily obtainable from the formula $\mathcal{B}(n, p)$ of the binomial distribution when n is small. In the previous section, we illustrated how the Poisson distribution can be used to approximate binomial probabilities when n is quite large and p is very close to 0 or 1. Both the binomial and the Poisson distributions are discrete. We now state a theorem that allows us to use areas under the normal curve to approximate binomial properties when n is sufficiently large.

Theorem 3.16 *If X is a binomial random variable with mean $\mu = np$ and variance $\sigma^2 = npq$, then the limiting form of the distribution of*

$$Z = \frac{X - np}{\sqrt{npq}},$$

as $n \to \infty$, is the standard normal distribution $\mathcal{N}(0, 1)$.

It turns out that the normal distribution with $\mu = np$ and $\sigma^2 = npq$ not only provides a very accurate approximation to the binomial distribution when n is large and p is not extremely close to 0 or 1 but also provides a fairly good approximation even when n is small and p is reasonably close to $1/2$.

To illustrate the normal approximation to the binomial distribution, we first draw the histogram for $\mathcal{B}(15, 0.4)$ and then superimpose the particular normal curve having the same mean and variance as the binomial variable X. Hence, we draw a normal curve with $\mu = np = (15)(0.4) = 6$ and $\sigma^2 = npq = (15)(0.4)(0.6) = 3.6$. The histogram of $\mathcal{B}(15, 0.4)$ and the corresponding superimposed normal curve, which is completely determined by its mean and variance, are illustrated in Figure 3.11.

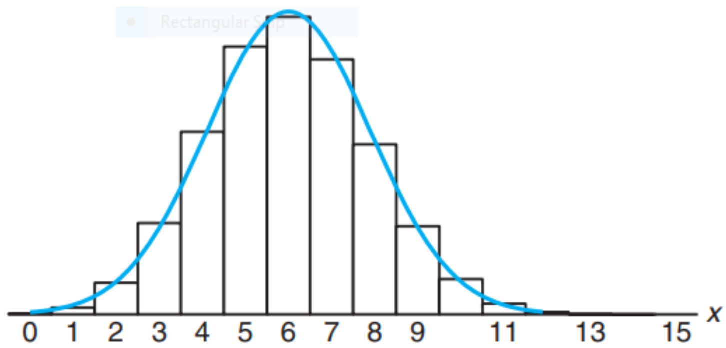

Figure 3.11: Normal approximation of $\mathcal{B}(15, 0.4)$

In our illustration of the normal approximation to the binomial, it became apparent that if we seek the area under the normal curve to the left of, say, x, it is more accurate to use $x + 0.5$. This is a correction to accommodate the fact that a discrete distribution is being approximated by a continuous distribution. The correction $+0.5$ is called a continuity correction. The foregoing discussion leads to the following formal normal approximation to the binomial.

Definition 3.12 (Normal Approximation to the Binomial Distribution) Let X be a binomial random variable with n trials and probability p of success. The probability distribution of X is approximated using a normal curve with

$$\mu = np \quad \text{and} \quad \sigma = \sqrt{npq},$$

and

$$P[x_1 < X < x_2] = \sum_{k=x_1}^{x_2} C_n^k (p)^k (1-p)^{n-k} \simeq \Phi\Big(\frac{x_2 + 0.5 - \mu}{\sigma}\Big) - \Phi\Big(\frac{x_1 - 0.5 - \mu}{\sigma}\Big) \qquad (3.8)$$

and the approximation will be good if np and $n(1-p)$ are greater than or equal to 5.

Remark 3.8 (a) Since the normal distribution is continuous, the area under the curve at any single point is equal to 0. Keep in mind that this result applies only to continuous random variables. Because the binomial random variable X is a discrete random variable, the probability that X takes some specific value say, $X = 11$ will not necessarily equal 0.

(b) Figures 3.10 and 3.12 show the binomial probability histograms for $n = 25$ with $p = 0.5$ and $p = 0.1$, respectively. The distribution in Figure 3.10 is exactly symmetric.

(c) If you superimpose a normal curve with the same mean, $\mu = np$, and the same standard deviation, $\sigma = \sqrt{npq}$, over the top of the bars, it "fits" quite well; that is, the areas under the curve are almost the same as the areas under the bars. However, when the probability of success, p, gets small and the distribution is skewed, as in Figure 3.12, the symmetric normal curve no longer fits very well. If you try to use the normal curve areas to approximate the area under the bars, your approximation will not be very good.

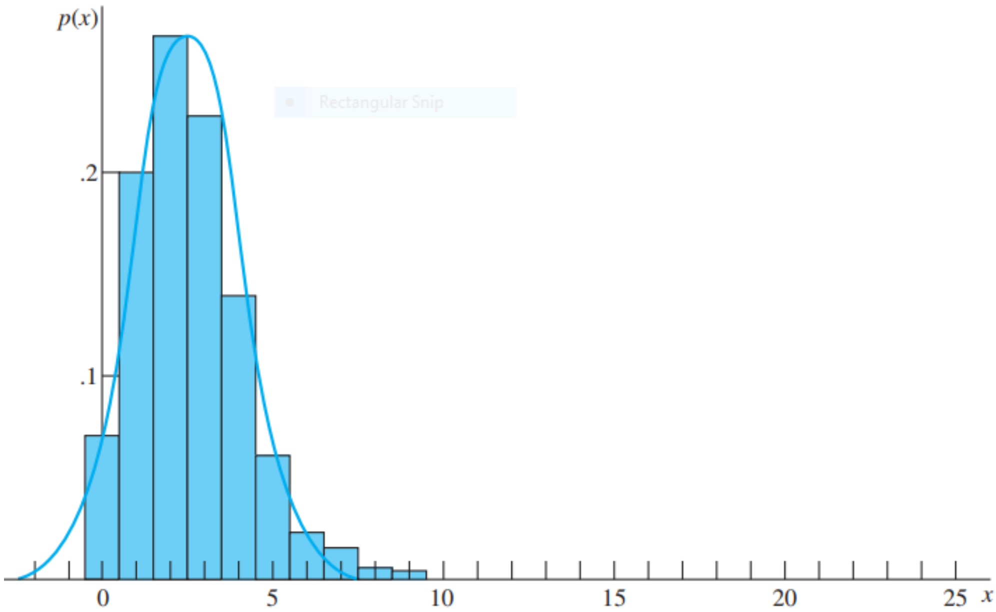

Figure 3.12: The binomial probability distribution and the approximating normal distribution for $n = 25$ and $p = 0.1$

Example 3.20 Use the normal curve to approximate the probability that $X = 8, 9$, or 10 for a binomial random variable with $n = 25$ and $p = 0.5$. Compare this approximation to the exact binomial probability.

Solution. You can find the exact binomial probability for this example because there are cumulative binomial tables for $n = 25$,

$$P[X=8]+P[X=9]+P[X=10] = \left(C_{25}^{8}+C_{25}^{9}+C_{25}^{10}\right)(0.5)^{25} \simeq 0.190535.$$

To use the normal approximation, first find the appropriate mean and standard deviation for the normal curve: $\mu = np = 12.5$, $\sigma = \sqrt{npq} = 2.5$. It follows from (3.8) that

$$P[8 \le X \le 10] = \Phi(-0.8) - \Phi(-2) = 0.18911.$$

You can compare the approximation, 0.18911, to the actual probability, 0.190535. They are quite close!

Remark 3.9 The normal approximation to the binomial probabilities will be adequate if both $np > 5$ and $n(1-p) > 5$.

Example 3.21 The reliability of an electrical fuse is the probability that a fuse, chosen at random from production, will function under its designed conditions. A random sample of 1000 fuses was tested and $X = 27$ defectives were observed. Calculate the approximate probability of observing 27 or more defectives, assuming that the fuse reliability is 0.98.

Solution. The probability of observing a defective when a single fuse is tested is $p = 0.02$, given that the fuse reliability is 0.98. Then $\mu = np = 20$, $\sigma = \sqrt{npq} = 4.43$.

The probability of 27 or more defective fuses, given $n = 1000$, is

$$P[X \geq 27] = P[X = 27] + P[X = 28] + \cdots + P[X = 1000].$$

It is appropriate to use the normal approximation to the binomial probability because $np = 20$ and $nq = 980$ are both greater than 5. So

$$\begin{aligned} P[27 \leq X \leq 1000] &= \Phi\Big(\frac{1000 + 0.5 - 20}{4.43}\Big) - \Phi\Big(\frac{27 - 0.5 - 20}{4.43}\Big) \\ &= 1 - \Phi(1.47) = 1 - 0.92922 = 0.07078. \end{aligned}$$

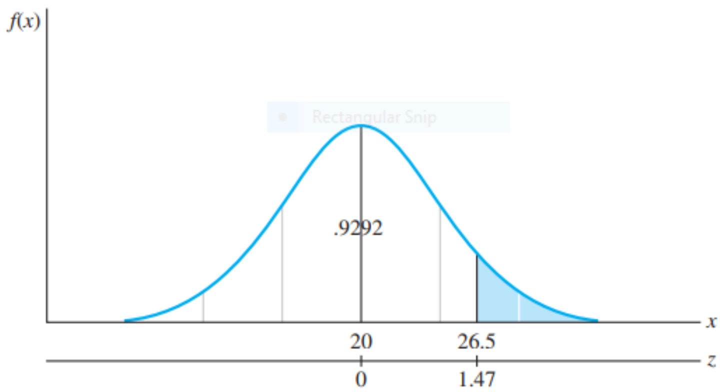

Figure 3.13: Normal approximation to the binomial for Example 3.21

$$\text{Table the values of standard normal CDF } \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-t^2}{2}} dt \tag{3.9}$$

x	0	1	2	3	4	5	6	7	8	9
0,0	0,50000	50399	50798	51197	51595	51994	52392	52790	53188	53586
0,1	53983	54380	54776	55172	55567	55962	56356	56749	57142	57535
0,2	57926	58317	58706	59095	59483	59871	60257	60642	61026	61409
0,3	61791	62172	62556	62930	63307	63683	64058	64431	64803	65173
0,4	65542	65910	66276	66640	67003	67364	67724	68082	68439	68739
0,5	69146	69447	69847	70194	70544	70884	71226	71566	71904	72240
0,6	72575	72907	73237	73565	73891	74215	74537	74857	75175	75490
0,7	75804	76115	76424	76730	77035	77337	77637	77935	78230	78524
0,8	78814	79103	79389	79673	79955	80234	80511	80785	81057	81327
0,9	81594	81859	82121	82381	82639	82894	83147	83398	83646	83891
1,0	84134	84375	84614	84850	85083	85314	85543	85769	85993	86214
1,1	86433	86650	86864	87076	87286	87493	87698	87900	88100	88298
1,2	88493	88686	88877	89065	89251	89435	89617	89796	89973	90147
1,3	90320	90490	90658	90824	90988	91149	91309	91466	91621	91774
1,4	91924	92073	92220	92364	92507	92647	92786	92922	93056	93189
1,5	93319	93448	93574	93699	93822	93943	94062	94179	94295	94408
1,6	94520	94630	94738	94845	94950	95053	95154	95254	95352	95449
1,7	95543	95637	95728	95818	95907	95994	96080	96164	96246	96327
1,8	96407	96485	96562	96638	96712	96784	96856	96926	96995	97062
1,9	97128	97193	97257	97320	97381	97441	97500	97558	97615	97670
2,0	97725	97778	97831	97882	97932	97982	98030	98077	98124	98169
2,1	98214	98257	98300	98341	98382	98422	99461	98500	98537	98574
2,2	98610	98645	98679	98713	98745	98778	98809	98840	98870	98899
2,3	98928	98956	98983	99010	99036	99061	99086	99111	99134	99158
2,4	99180	99202	99224	99245	99266	99285	99305	99324	99343	99361
2,5	99379	99396	99413	99430	99446	99261	99477	99492	99506	99520
2,6	99534	99547	99560	99573	99585	99598	99609	99621	99632	99643
2,7	99653	99664	99674	99683	99693	99702	99711	99720	99728	99763
2,8	99744	99752	99760	99767	99774	99781	99788	99795	99801	99807
2,9	99813	99819	99825	99831	99836	99841	99846	99851	99856	99861
3,0	0,99865	3,1	99903	3,2	99931	3,3	99952	3,4	99966	
3,5	99977	3,6	99984	3,7	99989	3,8	99993	3,9	99995	
4,0	999968									
4,5	999997									
5,0	99999997									

Problems – Chapter 3

Problem 3.1 (see Problem 2.4) In a package of M&Ms, Y, the number of yellow M&Ms, is uniformly distributed between 5 and 15.

(a) What is the PMF of Y?

(b) What is $P[Y < 10]$?

(c) What is $P[Y > 12]$?

(d) What is $P[8 \leq Y \leq 12]$?

Problem 3.2 (see Problem 2.5) When a conventional paging system transmits a message, the probability that the message will be received by the pager it is sent to is p. To be confident that a message is received at least once, a system transmits the message n times.

(a) Assuming all transmissions are independent, what is the PMF of K, the number of times the pager receives the same message?

(b) Assume $p = 0.8$. What is the minimum value of n that produces a probability of 0.95 of receiving the message at least once?

Problem 3.3 (see Problem 2.6) When a two-way paging system transmits a message, the probability that the message will be received by the pager it is sent to is p. When the pager receives the message, it transmits an acknowledgment signal (ACK) to the paging system. If the paging system does not receive the ACK, it sends the message again.

(a) What is the PMF of N, the number of times the system sends the same message?

(b) The paging company wants to limit the number of times it has to send the same message. It has a goal of $P[N \leq 3] \geq 0.95$. What is the minimum value of p necessary to achieve the goal?

Problem 3.4 Four microchips are to be placed in a computer. Two of the four chips are randomly selected for inspection before assembly of the computer. Let X denote the number of defective chips found among the two chips inspected. Find the probability mass and distribution function of X if

(a) Two of the microchips were defective.

(b) One of the microchips was defective.

(c) None of the microchips was defective.

Problem 3.5 (Final exam 20191) A four engine plane can fly if at least two engines work.

(a) If the engines operate independently and each malfunctions with probability q, what is the probability that the plane will fly safely?

(b) A two engine plane can fly if at least one engine works and if an engine malfunctions with probability q, what is the probability that plane will fly safely?

(c) Which plane is the safest?

Problem 3.6 (Final exam 20191) A rat maze consists of a straight corridor, at the end of which is a branch; at the branching point the rat must either turn right or left. Assume 10 rats are placed in the maze, one at a time.

(a) If each is choosing one of the two branches at random, what is the distribution of the number that turn right?

(b) What is the probability at least 9 will turn the same way?

Problem 3.7 A student who is trying to write a paper for a course has a choice of two topics, A and B. If topic A is chosen, the student will order 2 books through interlibrary loan, while if topic B is chosen, the student will order 4 books. The student feels that a good paper necessitates receiving and using at least half the books ordered for either topic chosen.

(a) If the probability that a book ordered through interlibrary loan actually arrives on time is 0.9 and books arrive independently of one another, which 2 topics should the student choose to maximize the probability of writing a good paper?

(b) What if, the arrival probability is only 0.5 instead of 0.9?

Problem 3.8 The number of phone calls at a post office in any time interval is a Poisson random variable. A particular post office has on average 2 calls per minute.

(a) What is the probability that there are 5 calls in an interval of 2 minutes?

(b) What is the probability that there are no calls in an interval of 30 seconds?

(c) What is the probability that there are no less than one call in an interval of 10 seconds?

Problem 3.9 (see Problem 1.42) An airline sells 200 tickets for a certain flight on an airplane that has only 198 seats because, on the average, 1 percent of purchasers of airline tickets do not appear for the departure of their flight. Determine the probability that everyone who appears for the departure of this flight will have a seat.

Problem 3.10 (see Problem 1.43) A midterm test has 4 multiple choice questions with four choices with one correct answer each. If you just randomly guess on each of the 4 questions, what is the probability that you get exactly 2 questions correct? Assume that you answer all and you will get (+5) points for 1 question correct, (-2) points for 1 question wrong. Let X is number of points that you get. Find the probability mass function of X.

Problem 3.11 Y is an exponential random variable with the PDF $f_X(x)$ is

$$f(x) = \begin{cases} 5e^{-5x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

(a) What is $E[X]$?

(b) What is $P[0,4 < X < 1]$?

Problem 3.12 X is a Gaussian random variable with $E[X] = 0$ and $\sigma_X = 0,4$.

(a) What is $P[X > 3]$?

(b) What is the value of c such that $P[3 - c < X < 3 + c] = 0,9$?

Problem 3.13 Let X be an exponential random variable with parameter and define $Y = [X]$, the largest integer in X, (ie. $[x] = 0$ for $0 \leq x < 1$, $[x] = 1$ for $1 \leq x < 2$ etc.)

(a) Find the probability function for Y.

(b) Find $E(Y)$.

(c) Find the distribution function of Y.

(d) Let Y represent the number of periods that a machine is in use before failure. What is the probability that the machine is still working at the end of 10th period given that it does not fail before 6th period?

Problem 3.14 Starting at 5:00 am, every half hour there is a flight from San Francisco airport to Los Angeles International Airport. Suppose that none of these planes sold out and that they alwayshave room for passengers. A person who wants to fly LA arrives at the airport at a random time between 8:45–9:45 am. Find the probability that she waits at most 10 minutes and at least 15 minutes.

Problem 3.15 X is a Gaussian random variable with $E[X] = 0$ and $P[|X| \leq 10] = 0.1$. What is the standard deviation σ_X?