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Advanced Parallel Algorithms

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8.1 Parallel Recursive

Divide and Conquer

- The principle of "divide and conquer" as follows:
  - B1: Divide the original problem into smaller problems.
  - B2: Recursive implementation with small problems.
  - B3: Combine results from small problems to obtain original problem results.
  - Small problems are independent of each other so they can be done in parallel.
- The problem is how to do steps 1 and 3 the most effectively ???

Devide and Conquer

General strategy:

Divide-and-conquer(Problem P of size n):

If P is trivial (i.e., n very small), solve P directly.
otherwise:

divide P into $q \geq 1$ *independent* subproblems $P_1,\ldots,P_q$ of smaller size $n_1,\ldots,n_q$

solve these recursively:

$S_1 \leftarrow$ Divide-and-conquer(P_1, n_1),

$\vdots$

$S_q \leftarrow$ Divide-and-conquer(P_q, n_q)

combine the subsolutions $S_1,\ldots,S_q$ to a solution S for P.

Complexity

- Considering the problem P having n-length, divided into q child problems of n/k-length (each problem has k elements, k>1), executed in parallel with p processors

$$
\text{t\_divide\_conquer}(n,p) = \begin{cases} \text{t\_run\_trivial}(n) \text{ if } n \text{ is small enough} \\ \text{t\_run\_serial}(n) \text{ if } p = 1 \\ \text{t\_divide}(n,p) + \text{t\_combine}(n,p) + q/p \\ \quad \text{t\_divide\_conquer}(k,1) \text{ if } 1 < p < q \\ \text{t\_divide}(n,p) + \text{t\_combine}(n,p) + \\ \quad \text{t\_divide\_conquer}(k,p/q) \text{ if } p > q \text{ or } p=q \end{cases}
$$

For Example (1)

- Sum of n numbers A[1..n] with p processors.
- Idea:
  - If $n = 1 \rightarrow$ return value A[1].
  - If $p = 1 \rightarrow$ run in serial mode.
  - Divide array A into 2 parts A1 and A2, each containing n/2 elements, executed in parallel:
    - Calculate recursively S1: sum of all A1's elements with p/2 processors.
    - Calculate recursively S2: sum of all A2's elements with p/2 processors.
  - Get the total S = S1+S2.

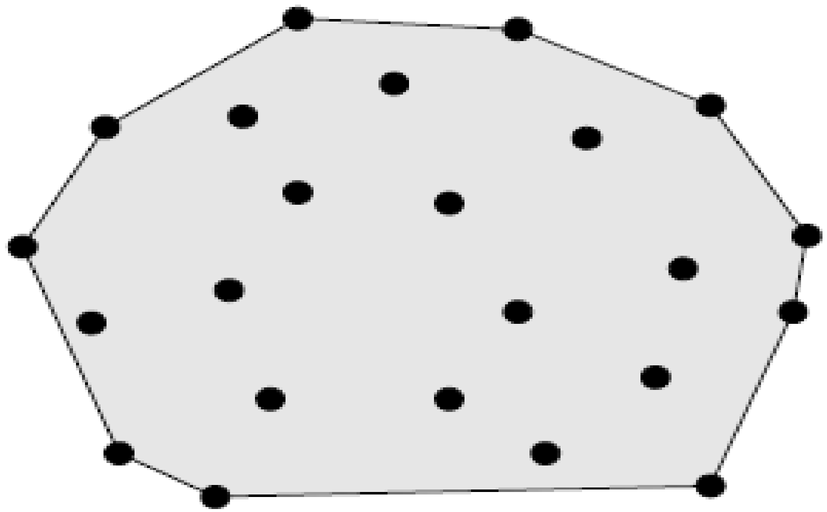
Sum of n numbers A[1..n] with p processors

```
INPUT        :    A[1..n], p bộ xử lý;
OUTPUT       :    SUM = ∑ A[i];
FUNCTION S = SUM(A,n,m,p)  // n,m la chi so dau tien va cuoi cung
BEGIN
    IF p = 1 THEN
        S = SEQUENCE_SUM(A,n,m);
    END IF.
    DO IN PARALLEL
        S1  =   SUM(A1,n,(n+m)/2,p/2);
        S2  =   SUM(A2,(n+m)/2,m,p/2);
    END DO
    S   =   S1 + S2;
END;
```

- Recursive equation: $T(n) = T(n/2) + O(1)$ (considering $p \approx n$)
- → Complexity: $O(\log n)$.
- Machine PRAM EREW.

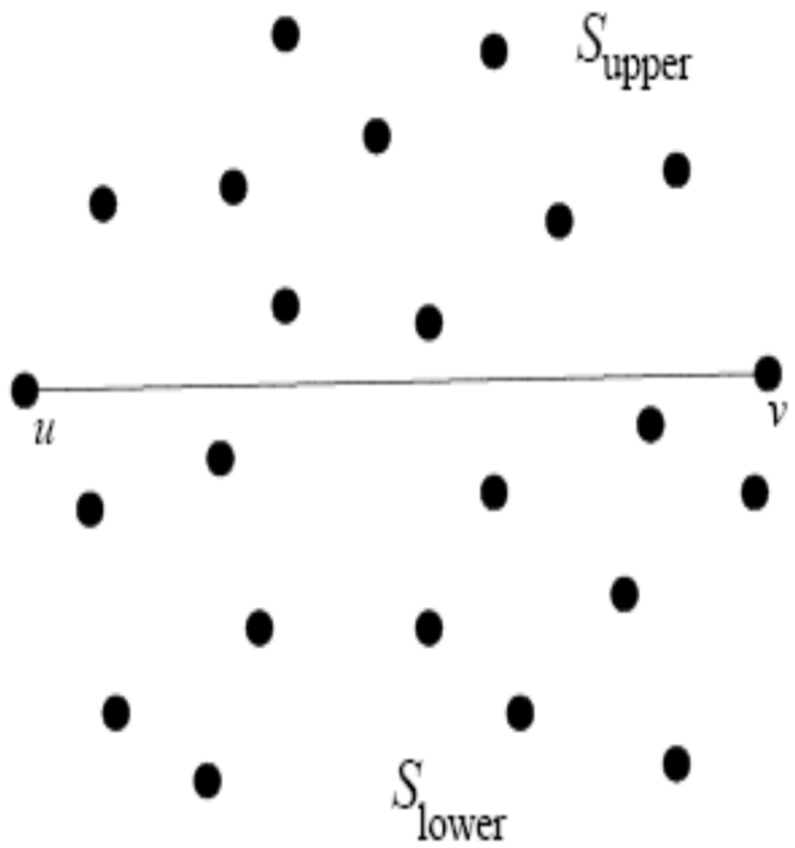
Example : convex hull

- The problem of determining the convex envelope of a set of vertices in the plane.
  - Input: n vertex (xk, yk) in the plane.
  - Output: Set of vertices that form the smallest convex polygon containing all the remaining vertices.



Parallel QuickHull

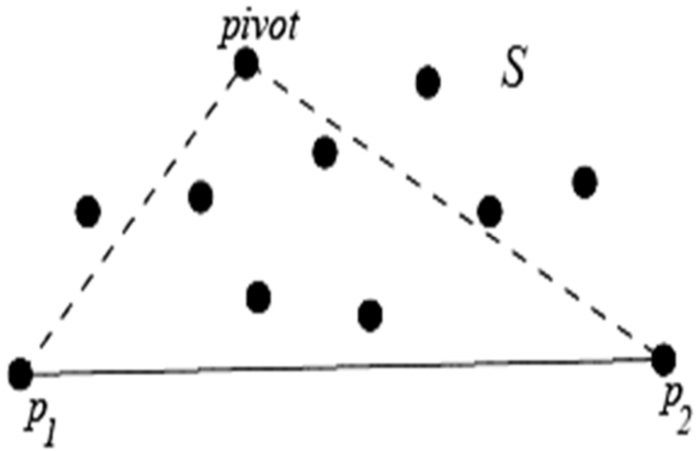
- Idea:
  - Initial: Define u,v as 2 vertices with x coordinate values being the smallest and largest → hence u, v are in convex envelope.
  - Segment (u,v) divides the initial set S into 2 upper and lower regions: S_upper and S_lower
  - Treating S_upper and S_lower in parallel.

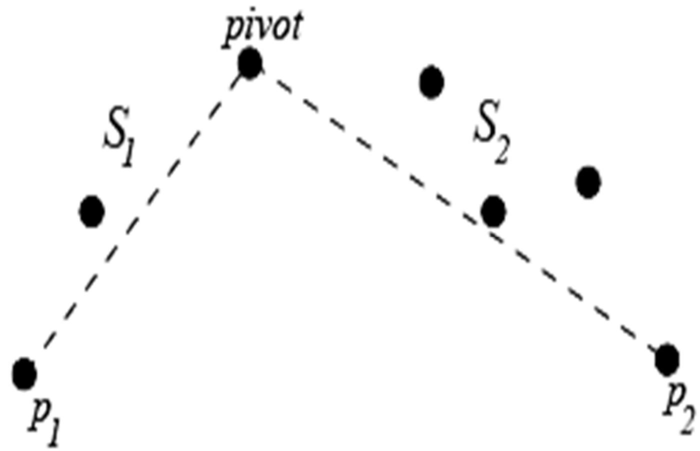


Parallel QuickHull

- Both upper hull and lower hull can be treated in the same way.
- Division step:
  - Select the pivot p as the point that has the longest distance from the (p_1 , p_2).
  - The pivot point will be on the convex envelope → points in the triangle (p, p1, p2) are eliminated.
  - The remaining points are divided into 2 parts outside the edges: (p, p_1) and $(p_2\ p)$.
  - Recursive implementation with these two parts with the steps as above.

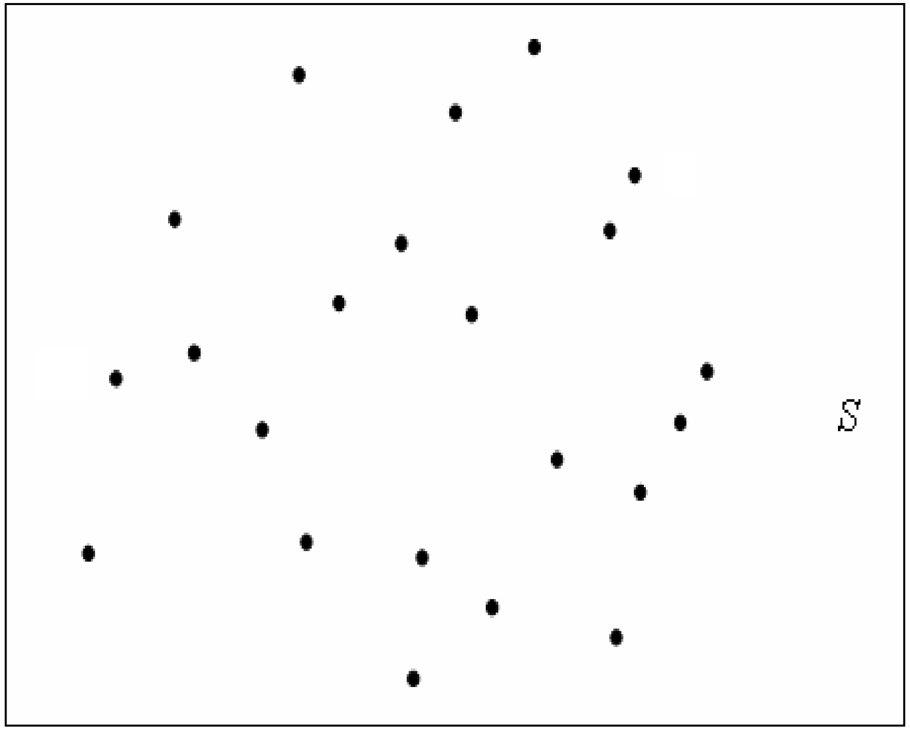
Parallel QuickHull



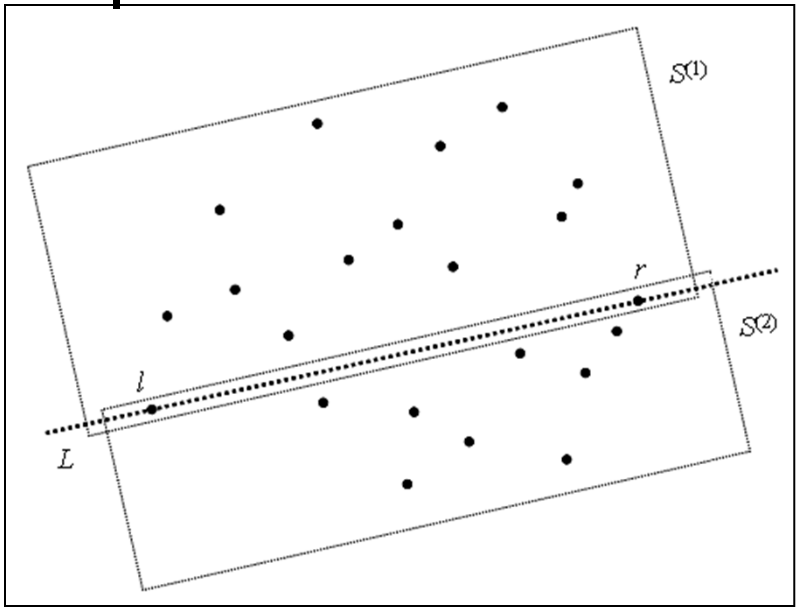


- Signs:
  - Pivot point p: max $|(p_1 - p) \times (p_2 - p)|$;
  - The vertexes are in the triangle if the total angles from that vertex are equal to 2π.
  - Angle between 2 vectors: $\cos(a,b) = (a \times b)/(|a||b|)$

Illustration steps



Uper and Lower Hull



Set of vertexes in S

Illustration steps

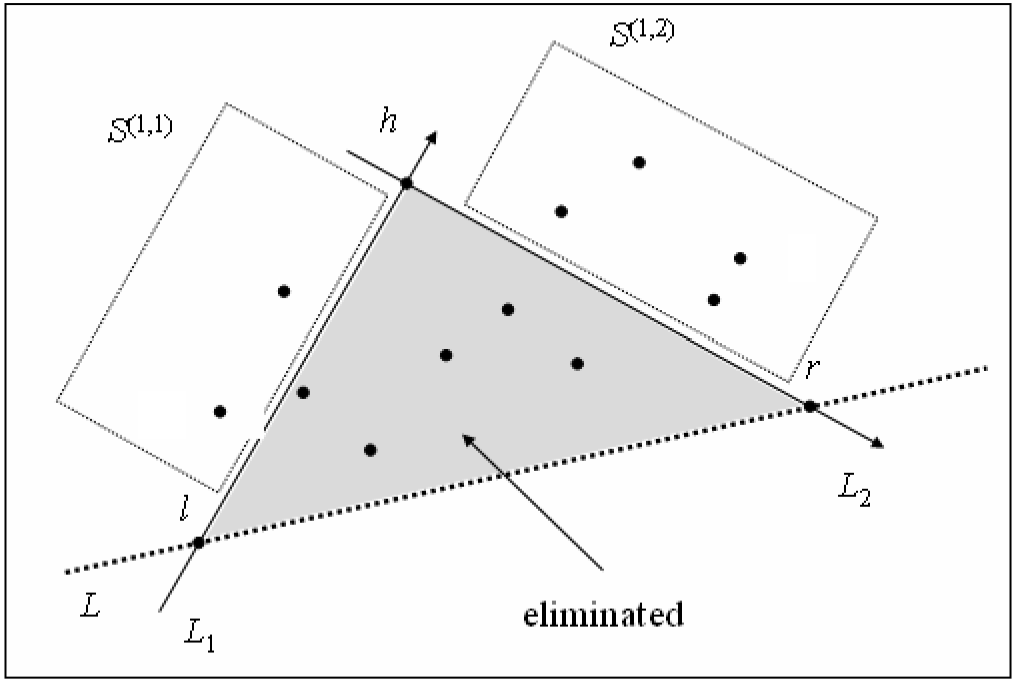
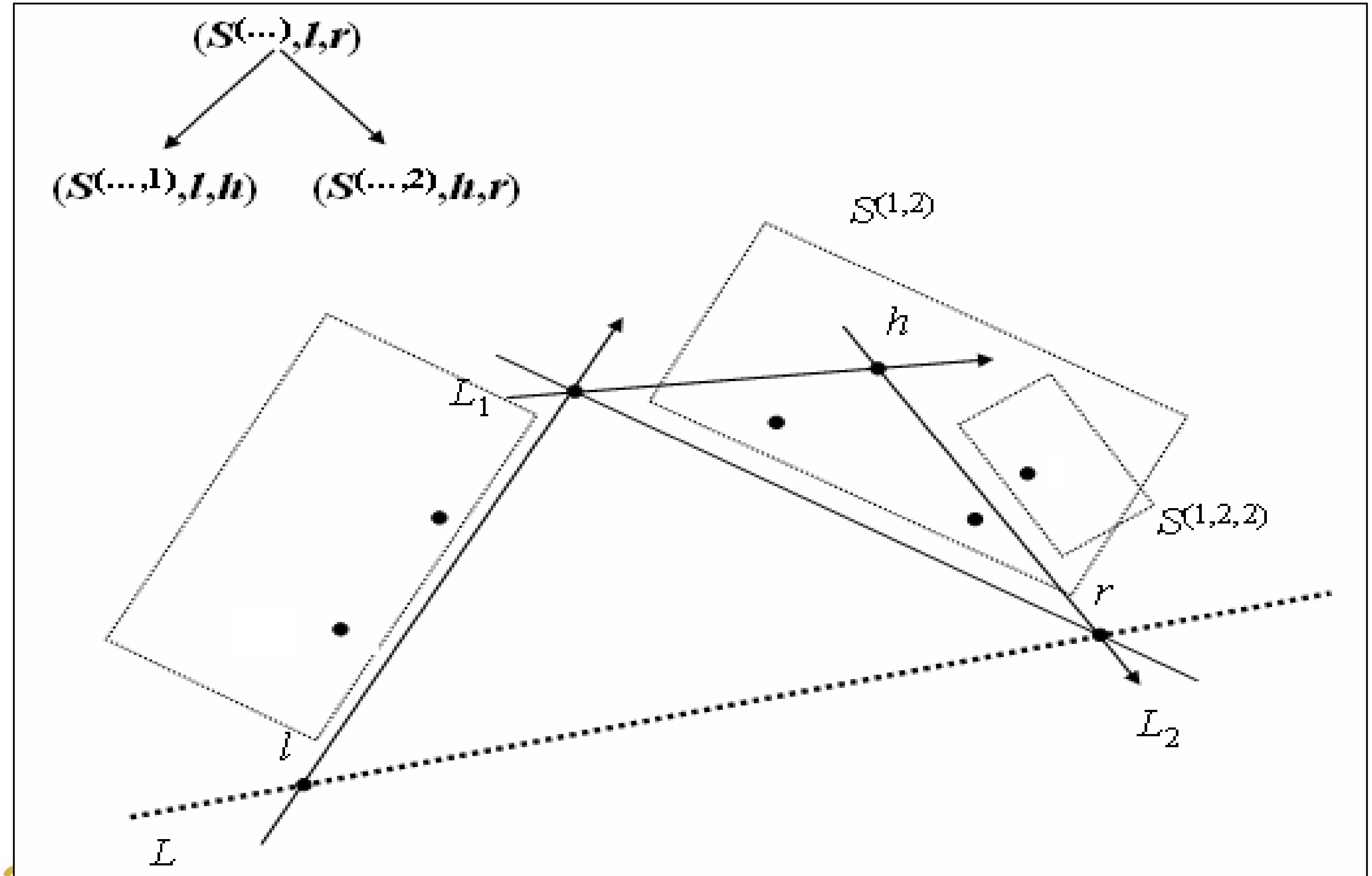


Illustration steps



Procedure QUICKHULL

```
procedure QUICKHULL(S, l, r)
begin
    if  S = {l, r} then
        return (l, r)  /* lr is an edge of H(S) */
    else
        h = FURTHEST(S, l, r)
        S(1) = p ∈ S ∋ p is on or left of line lh
        S(2) = p ∈ S ∋ p is on or left of line hr
        return QUICKHULL(S(1), l, h) ||
            (QUICKHULL(S(2), h, r) - h)
    end
end

Initial call
begin
    l0 = (x0,y0)  /* point of S with smallest abscissa */
    r0 = (x0, y0 - ε)
    result = QUICKHULL(S, l0, r0) - r0
/* The point r0 is eliminated from the final list*/
end
```

Recursive in PRAM

- If we represent the sub-problem levels of a recursive algorithm as a tree → get the k-level tree (with k = 2 → binary tree).
- The recursive idea in PRAM is to divide the set of processors into groups, each of which will correspond to a sub-tree in the tree.
- Executing in parallel with all processors, in each group of processors, doing the work corresponding to the sub-tree.

Recursive with UperHull

- Variables used:
  - Each point i corresponds to 1 variable F[i]:
    - Initial: F[i] = 1;
    - Eliminated for being in: F[i] = 0;
    - Marked as the point in the convex envelope: F[i] = 2.
  - Since each vertex set will be assigned to two bottom points, each vertex determines the current 2 bottom points through the variable: P[i] và Q[i].
- Recursive steps:
  - All processors perform in parallel to identify the T[i].
  - Update peaks P[i] and Q[i]:
    - Left vertexes (P[i],T[i]) assigned Q[i] = T[i].
    - Left vertexes (T[i],Q[i]) assigned P[i] = T[i].
  - Update again values F[i].
  - Repeat the above work until all F[i] $\neq$ 1.

8.2 Accelerated Cascading

Concepts of complexity

- In serial calculation, there is only one concept of complexity = number of steps to implement the algorithm (≈ duration): Called S(n)
- In parallel calculations there is additional concept number of operations performed on all processors: Called W(n).
- If $W_i(n)$ is the number of operations performed simultaneously at step i → we have a formula:

$$W(n) = \sum_{i=1}^{S(n)} W_i(n)$$

Example of S(n) and W(n)

- The combined problem with $n = 2^k$ values using math operation $\oplus$. Balancing tree algorithm as follows:

```
T REDUCE(sequence⟨T⟩ a, ⊕ : T × T → T)

T B[1..n]
forall i ∈ 1 : n do
    B[i] ← a_i
enddo
for h = 1  to k do
    forall i ∈ 1 : n/2^h do
        B[i] ← B[2i − 1] ⊕ B[2i]
    enddo
enddo
S ← B[1]
return S
```

Examples of S(n) and W(n)

- Define S(n) and W(n) values according to algorithm's segment codes as follows:

$$
\begin{gathered}
S_{2\text{--}4}(n) = \Theta(1) \\
S_{6\text{--}8}(n) = \Theta(1) \\
S_{5\text{--}9}(n) = kS_{6\text{--}8}(n) = \Theta(\lg n) \\
S_{10}(n) = \Theta(1) \\
S(n) = S_{2\text{--}4}(n) + S_{5\text{--}9}(n) + S_{10}(n) = \Theta(\lg n)
\end{gathered}
$$

$$
\begin{gathered}
W_{2\text{--}4}(n) = \Theta(n) \\
W_{6\text{--}8}(n,h) = \Theta(\tfrac{n}{2^h}) \\
W_{5\text{--}9}(n) = \sum_{h=1}^{k} W_{6\text{--}8}(n,h) = \Theta(n) \\
W_{10}(n) = \Theta(1) \\
W(n) = W_{2\text{--}4}(n) + W_{5\text{--}9}(n) + W_{10}(n) = \Theta(n)
\end{gathered}
$$

Accelerated Cascading Technique

- The cost of an algorithm is the number of operations that the system must perform.
- Some algorithms are called optimal if: $W(n) = \Theta(T_s(n))$. where:
  - $W(n)$: cost of parallel algorithm.
  - $T_s(n)$: the best execution time of serial algorithm.
- Accelerated Cascading technique combines non-optimal algorithm but faster execution time with optimal algorithm but slow execution time.

Example (1)

- An array L[1..n] receives integer values from 1..k with $k = O(\log_2 n)$. Determine the number of occurrences of integers appeared in array L.
- Let R[i] be the number of occurrences of value i.
- Optimal serial algorithm $T_s(n) = \Theta(n)$:

$$
\begin{array}{l}
R[1:k] \leftarrow 0 \\
\textbf{for } i = 1 \textbf{ to } n \textbf{ do} \\
\quad R[L[i]] \leftarrow R[L[i]] + 1 \\
\textbf{enddo}
\end{array}
$$

First parallel approach

- Use two-dimensional arrays: C[1..n,1..k] $C_{i,j} = \begin{cases} 1 & \text{if } L[i] = j \\ 0 & \text{otherwise} \end{cases}$

- The the number of occurrences of integer i equals the total of C[1:n,j].

$$
\begin{array}{l}
\textbf{forall } i \in 1:n, j \in 1:k \textbf{ do} \\
\quad C[i,j] \leftarrow 0 \\
\textbf{enddo} \\
\textbf{forall } i \in 1:n \textbf{ do} \\
\quad C[i, L[i]] \leftarrow 1 \\
\textbf{enddo} \\
\textbf{forall } j \in 1:k \textbf{ do} \\
\quad R[j] \leftarrow \textsc{Reduce}(C[1:n,j], +) \\
\textbf{enddo}
\end{array}
$$

Example with n = 8 & k = 4

L[1..8] = 1 2 2 3 2 1 4 2 (k = 4)

	1	2	3	4
1	0	0	0	0
2	0	0	0	0
3	0	0	0	0
4	0	0	0	0
5	0	0	0	0
6	0	0	0	0
7	0	0	0	0
8	0	0	0	0
	R1	R2	R3	R4

	1	2	3	4
1	1			
2		1		
3		1		
4			1	
5		1		
6	1			
7				1
8		1		
	R1	R2	R3	R4

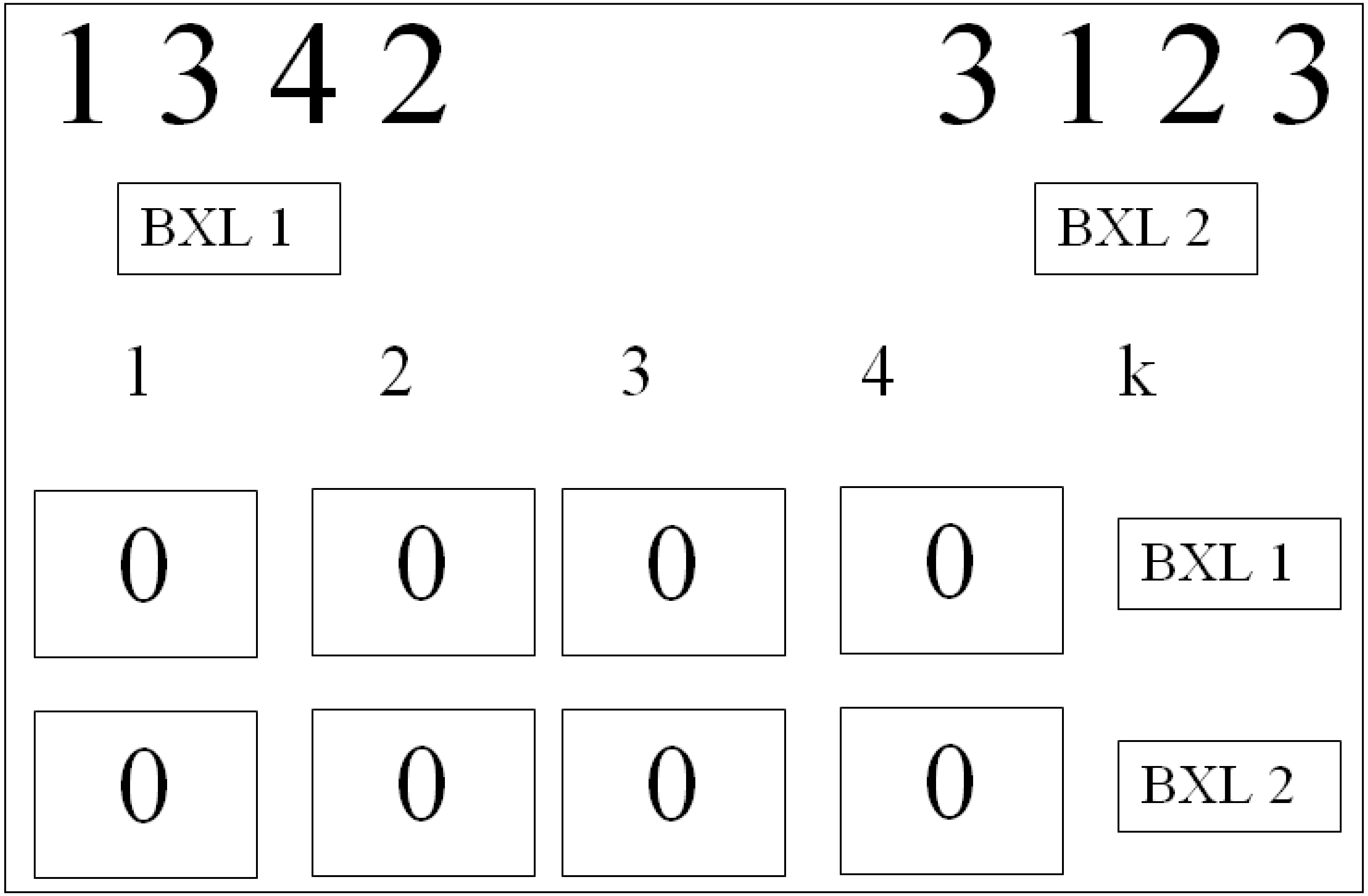
Comments …

- The number of operations performed by the first two parallel loops is $\Theta(nk)$ with $\Theta(1)$ step.
- Execution time in paragraph 3 according to binary tree model: $\Theta(\log_2 n)$.
- The real number of calculations in paragraph 3 is: $\Theta(nk)$
- This algorithm is not cost effective → not optimal.

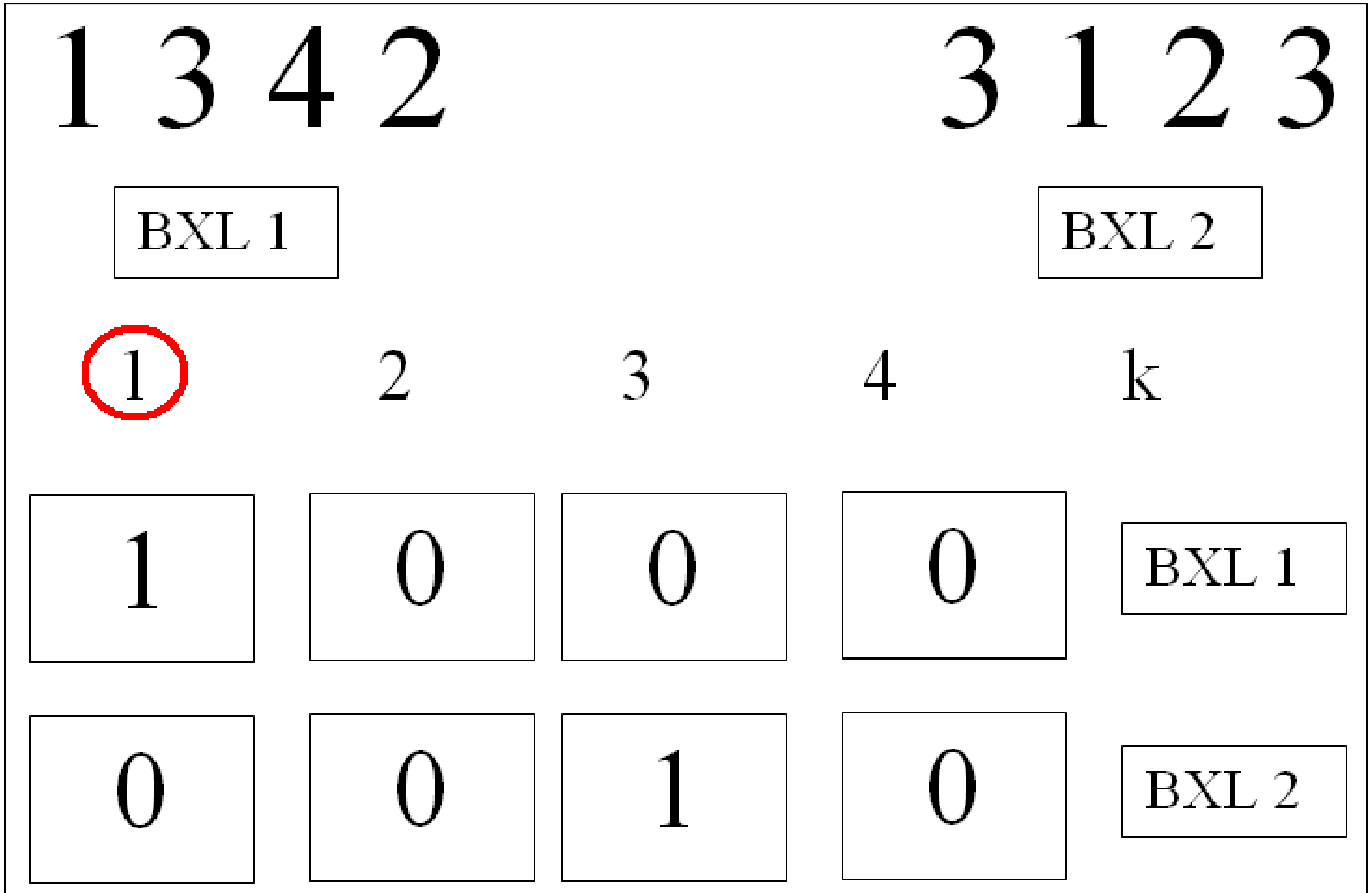
Using Accelerated Cascading

- With m=n/k, setup array Ĉ[1..m.1.k] corresponding to array L. That is, dividing L into m sub-array of k elements.
- Using m processor in order to scan from 1:k, i.e. each processor performs 1 optimal serial algorithm to determine the number of occurrentce of j $\in$ 1:k.
    - The number of steps executed is $\Theta(k) = \Theta(\log_2 n)$,
    - The cost of execution is $\Theta(mk) = \Theta(n)$.
- R[j] determined by summing each column Ĉ[1:m,j]:
    - Using balancing tree algorithm: cost $\Theta(m)$
    - Total cost is $\Theta(mk) = \Theta(n)$. → cost optimization

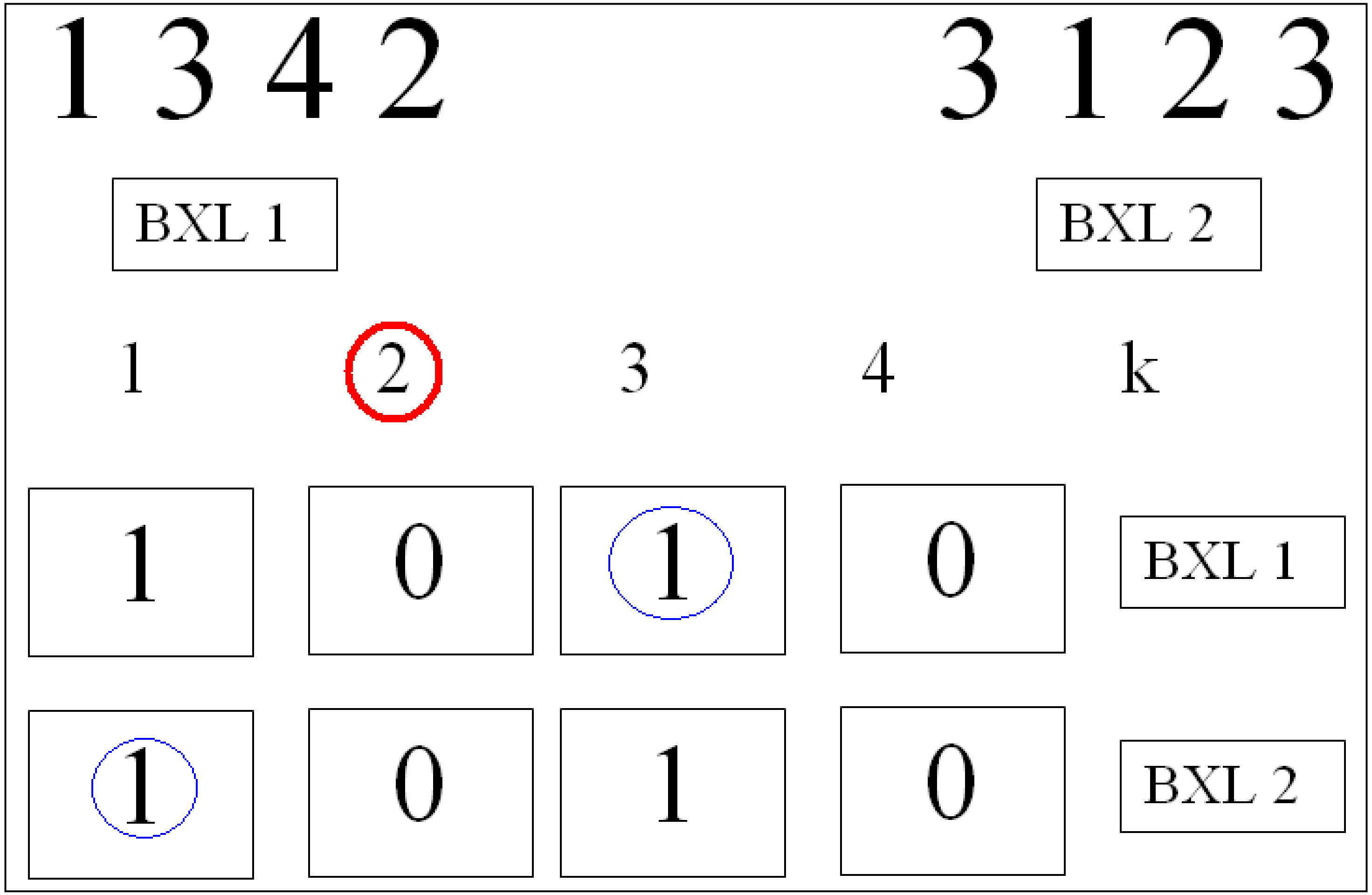
Using Accelerated Cascading



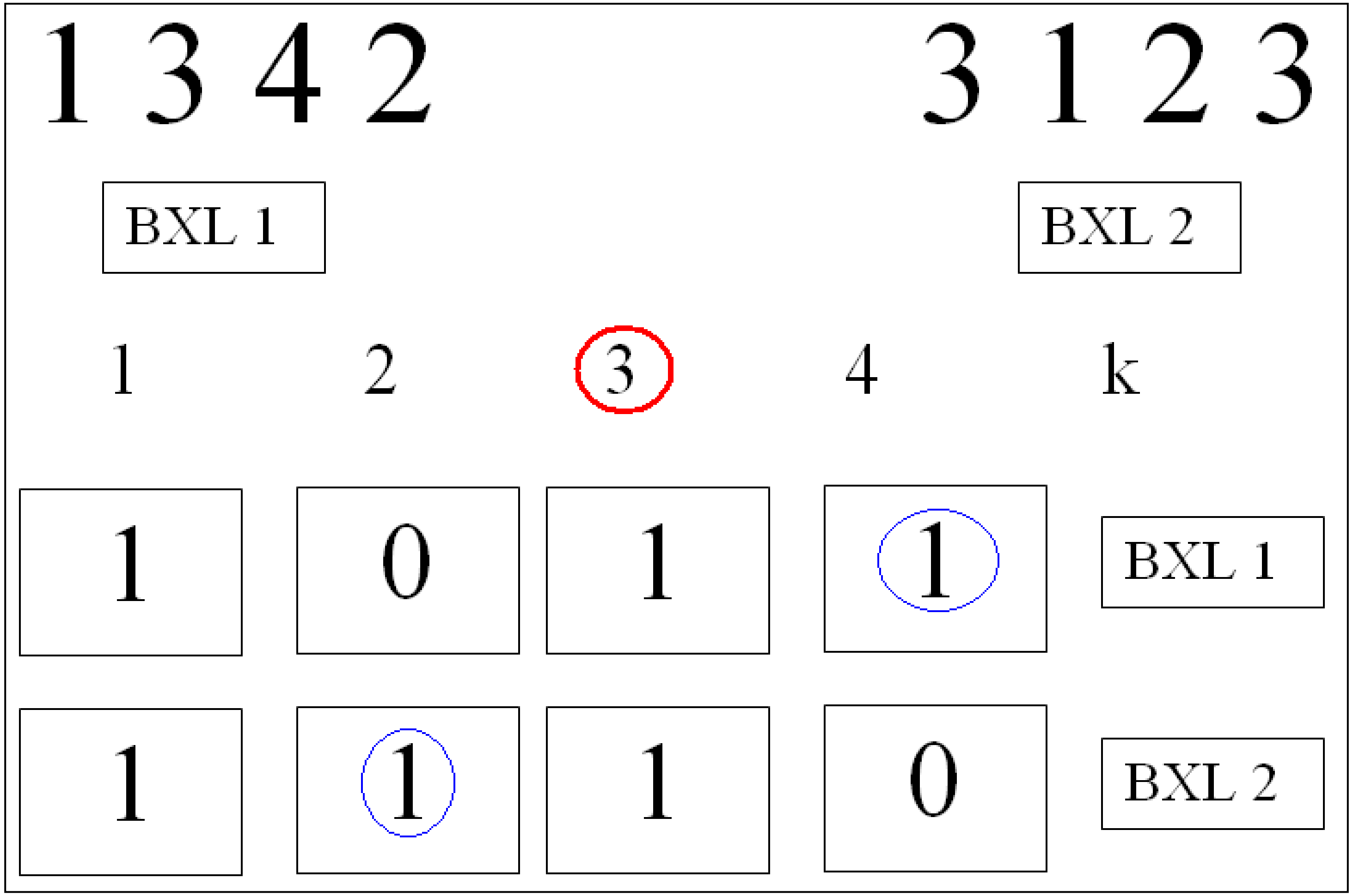
Using Accelerated Cascading



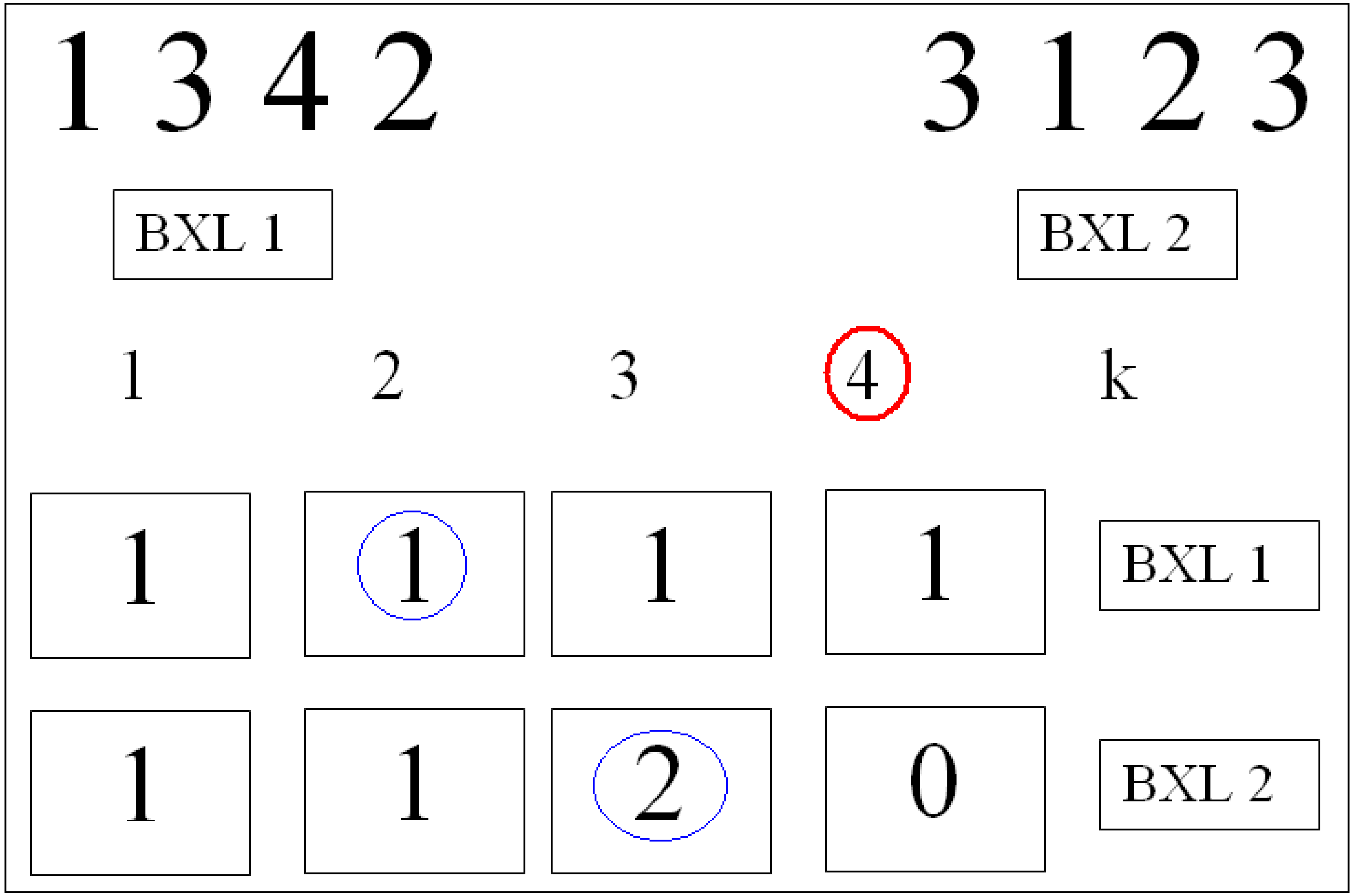
Using Accelerated Cascading



Using Accelerated Cascading



Using Accelerated Cascading



Optimal algorithm

Input: Sequence $L[1..n]$ with values in the range $1:k$

Output: Sequence $R[1..k]$ the occurrence counts for the values in L

integer $\hat{C}[1..m, 1..k]$
forall $i \in 1:m, j \in 1:k$ **do**
    $\hat{C}[i, j] \leftarrow 0$
enddo
forall $i \in 1:m$ **do**
    for $j = 1$ **to** k **do**
        $\hat{C}[i, L[(i-1)k + j]] \leftarrow \hat{C}[i, L[(i-1)k + j]] + 1$
    enddo
enddo
forall $j \in 1:k$ **do**
    $R[j] \leftarrow \text{REDUCE}(\hat{C}[1:m, j], +)$
enddo

Example (2): find Max

- Determine $X_i = \max \{ X_1, X_2, ..X_n \}: X_i \geq X_j \ \forall\ j \in 1{:}n$.
- Algorithm with PRAM EREW: $O(\log_2 n)$ step with the cost of O(n), using O(n) processors and balanced tree model.
- Consider the following algorithm with PRAM CRCW:

Input: *An array A of p distinct elements*

Output: *Boolean Array of M, M(i) =1 if and only if A(i) is the maximum element.*

begin

1. **for** $1 \leq i,j \leq p$ **pardo**
   if $A(i) \geq A(j)$ **then** *Set* $B(i,j):=1$
   else *Set* $B(i,j):=0$
2. **for** $1 \leq i \leq p$ **pardo**
   Set $M(i):=B(i,1) \cap ... \cap B(i,p)$

end.

Find Max with PRAM CRCW

- The algorithm above has 2 parts:
  - Part 1 can be done in parallel using n^2 processor with O(1) steps → Costs $O(n^2)$.
  - Part 2 can be done with PRAM CRCW: determine the value M[i] needs O(1) step, cost O(n) → total cost: O(n).
  - If PRAM CRCW is selected, the problem is quickly resolved with O(1) steps with $O(n^2)$ processors, hence the total cost is $O(n^2)$.

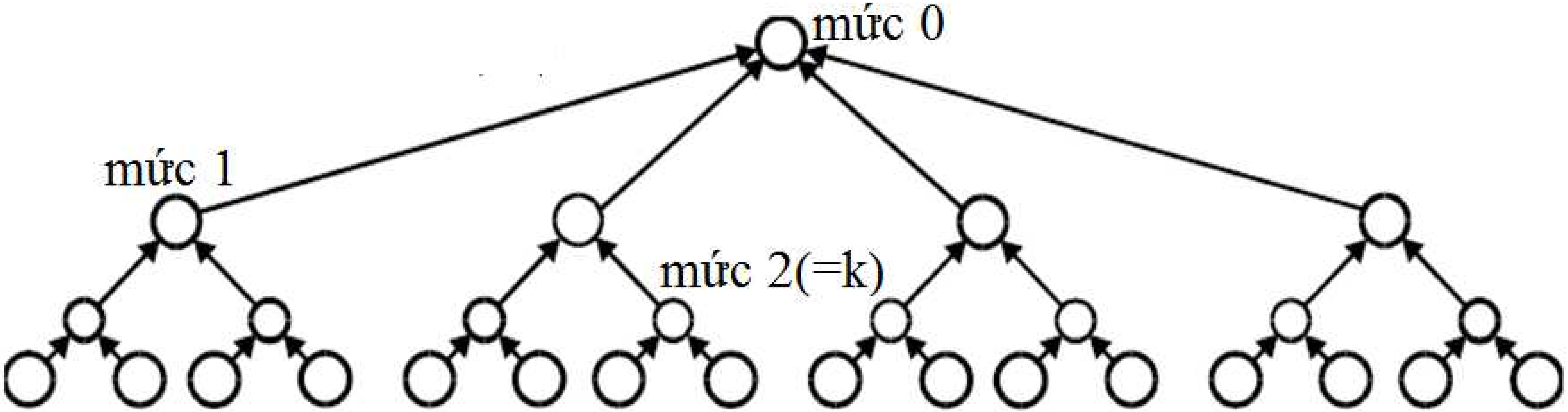
Find Max: Accelerated Cascading

- Let (W(n),T(n)) be the number of operation and time duration of the algorithm.
- Find-Max problem:
  - (1) Balance-Tree with EREW: (O(n), O(log2n)): optimal but slow
  - (2) Use CRCW with n2 processors: (O(n2), O(1)) : not optimal but fast.
- (3) Build a new algorithms on the DLDT tree with $(O(n.\log_2\log_2 n),O(\log_2\log_2 n))$;
- Apply (1) and (3) with Accelerated Cascading technique → new algorithm : $(O(n), O(\log_2\log_2 n))$.

Tree DLDT

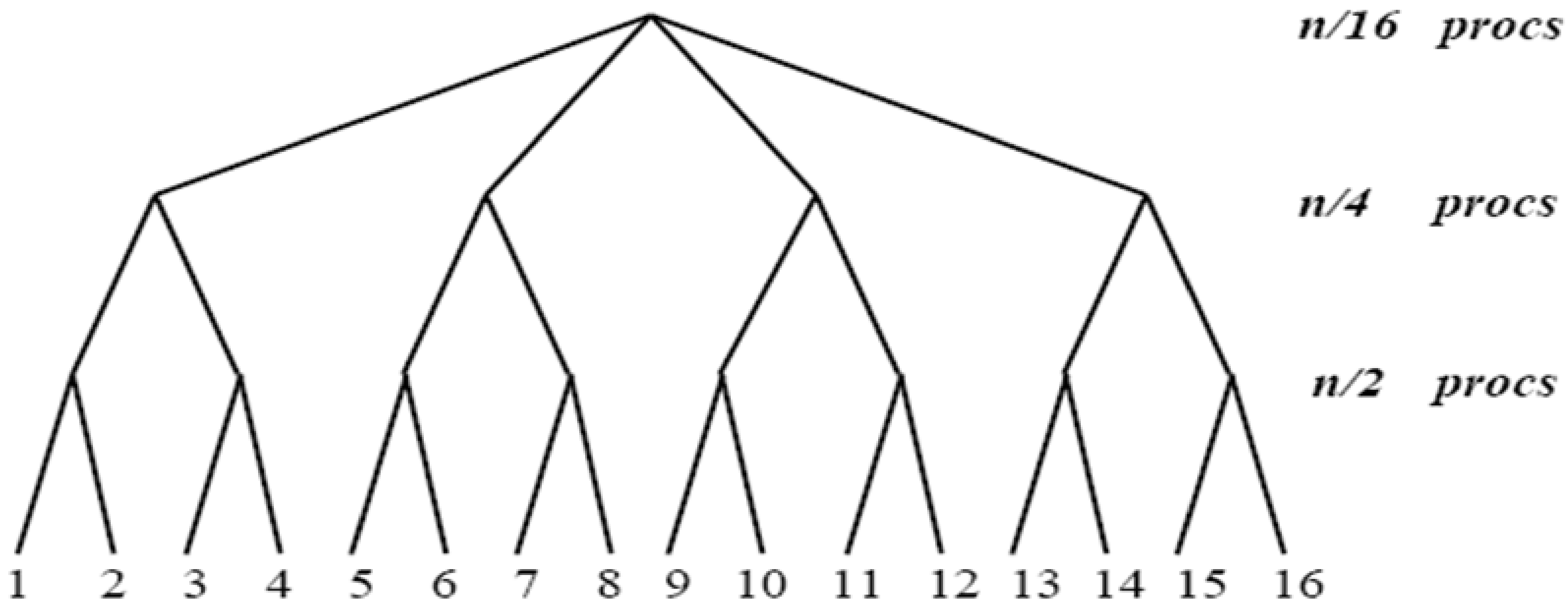
- DLDT = Doubly Logarithmic Depth Tree.
- This is recursive tree.
  - DLDT(n) is a tree with n leaves. ($n = 2^{2^k}$).
  - With k = 0 → n = 2 → tree has 1 root and 2 leaves.
  - With k > 0, tree is recursively constructed as follows:
    - Root has $\sqrt{n} = 2^{2^{k-1}}$ sub-trees.
    - Each sub-tree has $\sqrt{n}$ leaves : DLDT($\sqrt{n}$).
- Comment: The number of leaves in the tree with the root node at level i equals the number of leaves in the tree with the root node at level i+1.

Tree DLDT



- Degree of node u is the number of child nodes of u.
- Thanks to $n = 2^{2^k}$ then the root has degree of $2^{2^{k-1}} = \sqrt{n}$
- Node at level i has degree of $2^{2^{k-i-1}}$ với $0 \leq i < k$.
- Node at level k-1 has 2 child nodes
- Node at level k has 2 leaves

Tree DLDT



- The depth of the tree is : $k+1 = \log_2 \log_2 n + 1$.
- Let n be the number of the leave of the DLDT tree

Find-Max: DLDT Tree

- Comments:
  - At level 0 (root) we have n processors used to determine the Maximum value of obtained results returned from the child node ($\sqrt{n}$) at level 1. The algorithm can be applied using PRAM CRCW with (O(n),O(1)).
  - At level 1, dividing n processors into $m = \sqrt{n}$ groups. Each group corresponds to 1 node; we determine the Maximum value from $\sqrt{m}$ child node at level 2. The algorithm can be applied using PRAM CRCW to each node with (O(m),O(1));
  - At level i, each node corresponds to 1 group $2^{2^{k-i}}$ processors. Each node is the father of $2^{2^{k-i-1}}$ child nodes. The algorithm can be applied using PRAM CRCW to each node with $(O(2^{2^{k-i}}),O(1))$.

Find-Max: DLDT Tree

- Algorithm's idea:
  - Done with k repeating steps (from level k-1 = loglogn -1 to level 0 (root node)).
  - Executing the algorithm from the bottom to up.
  - At the i_th iteration step:
    - Perform in parallel with n processors divided into: $2^{2^k-2^{k-i}}$ groups, each group contains $2^{2^{k-i}}$ processors (because each child node is assigned to 1 processor =>processor group's number = child node group's number).
    - Using algorithms with CRCW for each node, the largest value of child nodes is stored at parent nodes.

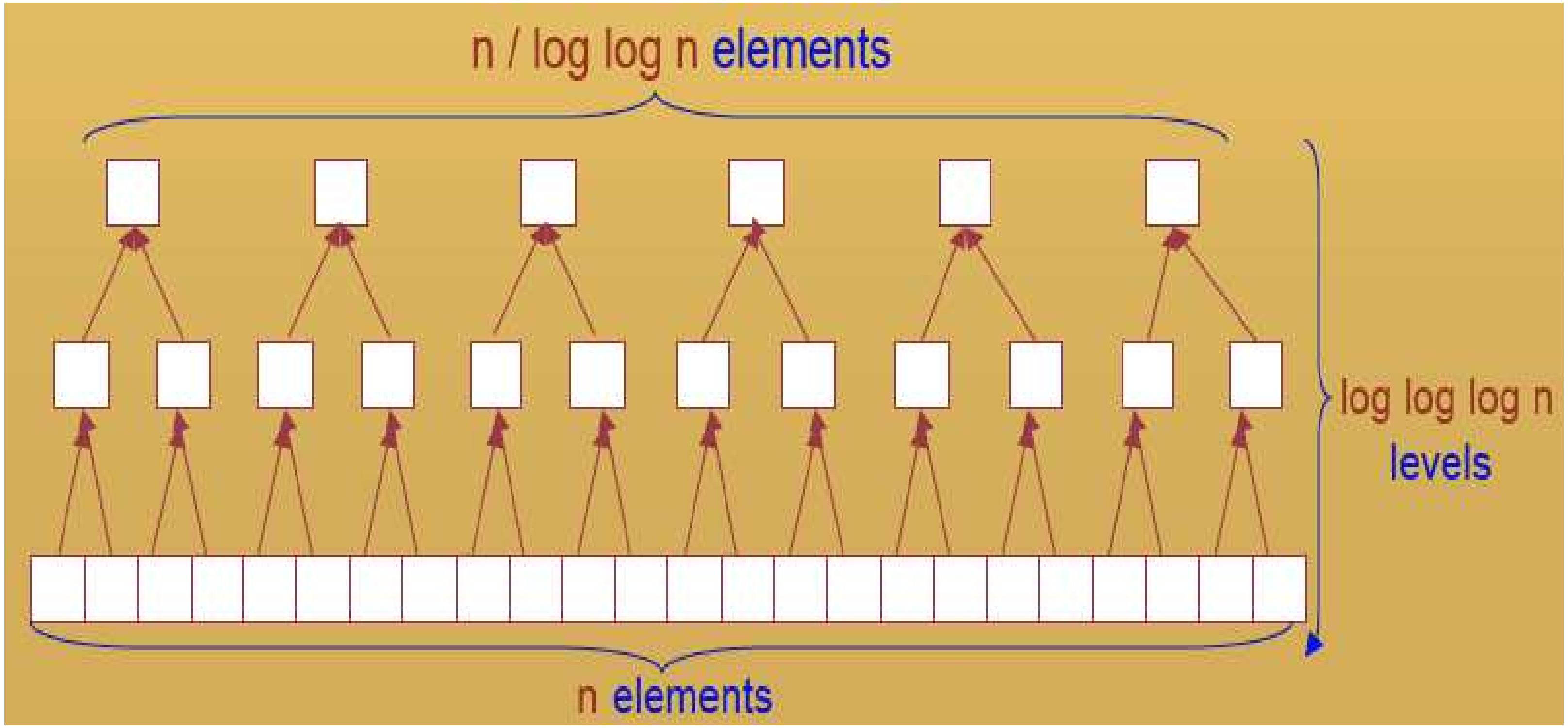
Find-Max: DLDT tree

- Performance evaluation:
  - Time duration: $O(k)=O(\log_2\log_2 n)$ repeating steps.
  - Cost evaluation:
    - Each node in i_th iteration step performs $O(2^{2^{k-i}})$ operations or $O(n^{2^{-1}})$
    - At i_th step, there is $2^{2^{k}-2^{k-i}} = n^{1-2^{-i}}$ nodes → total cost at each i_th iteration step: $W_i(n) = O(n^{1-2^{-1}} \text{x } n^{2^{-1}}) = O(n)$.
    - The cost of the entire algorithm is:

$W(n) = k*W_i(n)= O(n.k) = O(n.\log_2\log_2 n)$.

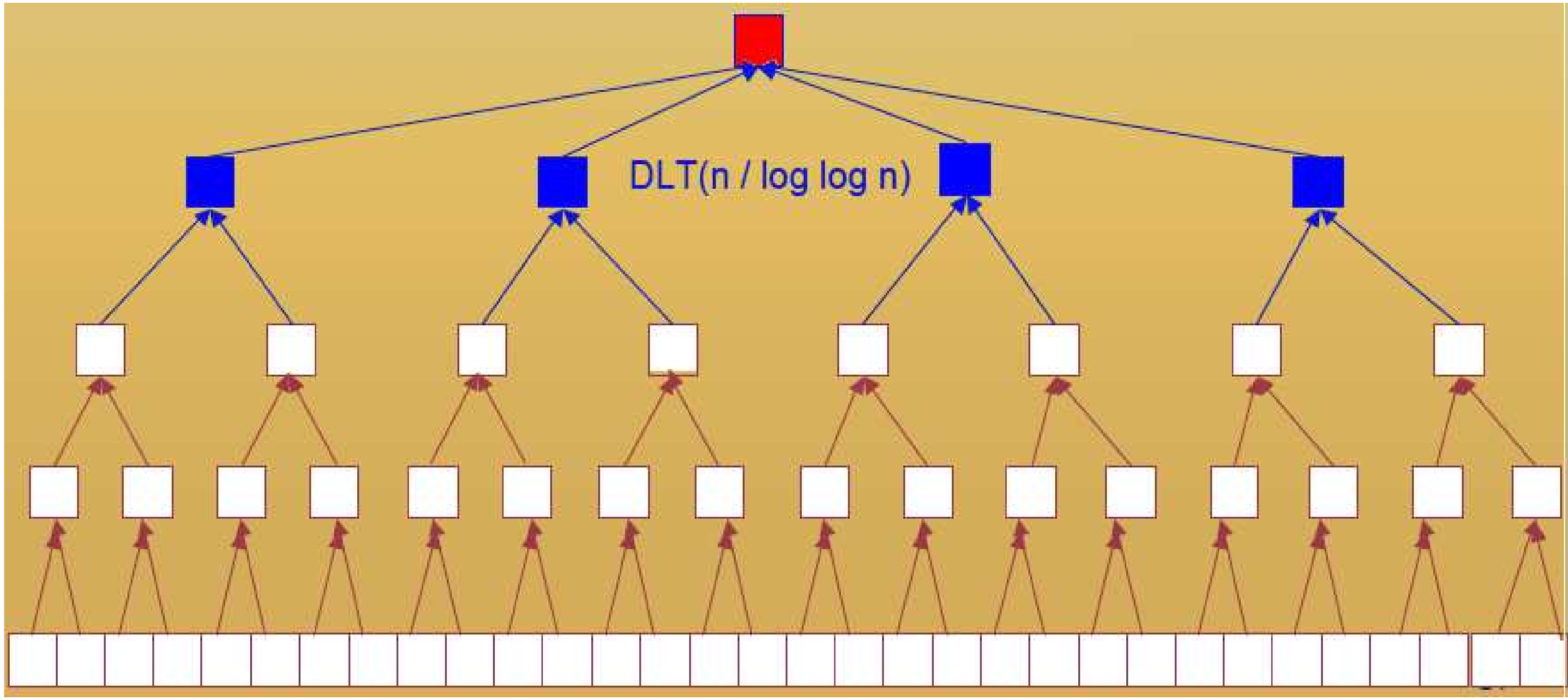
Using Accelerated Cascading

- Step 1: Using balancing tree technique with [logloglogn] level from the bottom.



Using Accelerated Cascading

- Step 2. Performing the algorithm on the DLDT tree with the number of nodes $m = n/\log_2\log_2 n$.



Using Accelerated Cascading

- Step 1. Using balanced tree:
  - After each step of the balancing tree, the number of nodes decreases by 1/2 (done with $\log_2\log_2\log_2 n$ serial steps =>T(n) = $\log_2\log_2\log_2 n$).
  - With k= $\log_2\log_2\log_2 n$, after this step the remaining number of nodes equals $m = n/2^k = n/\log_2\log_2 n$.
  - Cost of step 1: W(n) = O(n).
- Step 2. Using DLDT algorithm to the m other nodes:
  - $T(n) = O(\log_2\log_2 m) = O(\log_2\log_2 n)$.
  - $W(n) = O(m \text{ x } \log_2\log_2 m) = O(m \text{ x } \log_2\log_2 n) = O(n)$.
- Conclusion: New algorithm (O(n), $O(\log_2\log_2 n)$).

8.3 Pipeline

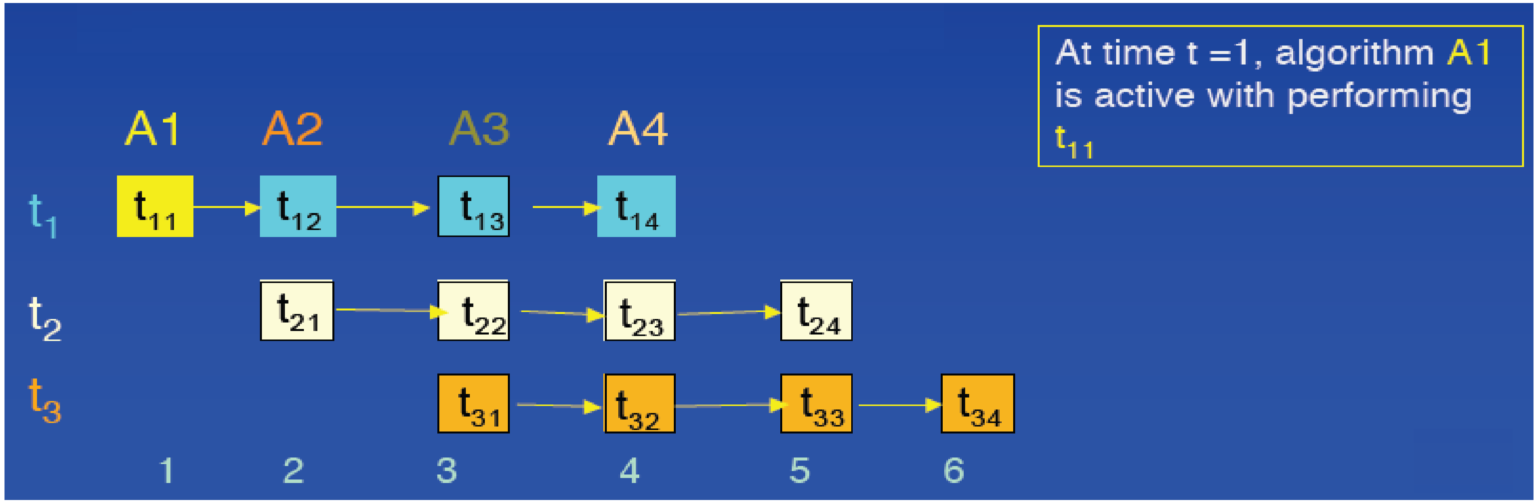
Pipeline Technique

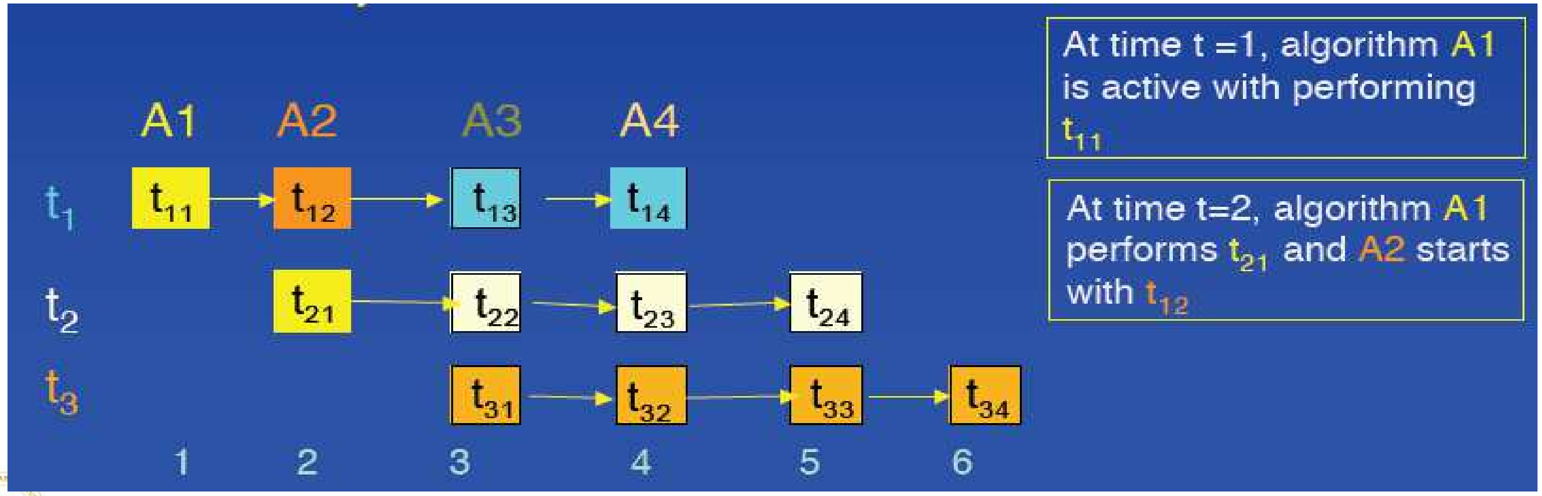
- Widely used to accelerate the executing time of many problems, including:
  - Each main problem can be divided into several child problems.
  - These sub-problems can depend on each other in an order of execution.
  - At each moment, the processors perform an algorithm for sub-problems of each main problem in parallel (make sure the execution order is constant).

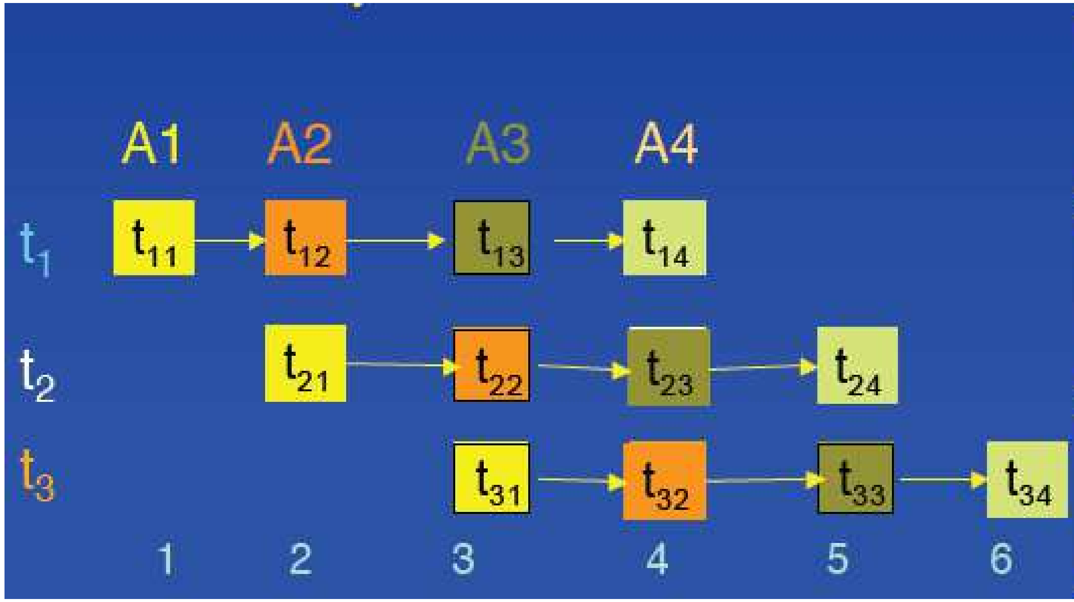
Pipeline Mechanism

- Considering n problems: $t_1, t_2, \ldots, t_n$ need to do.
- Each t_i can be divided into a set of sub-problems: $\{t_{i,1}, t_{i,2}, \ldots, t_{i,m}\}$ so that $t_{i,k}$ must be terminated before starting $t_{i,k+1}$.
- Assuming that at each step j = 1..m, algorithm step A_j will be done with sub-problems: $t_{1,j}, t_{2,j}, \ldots, t_{n,j}$.

Pipeline Mechanism



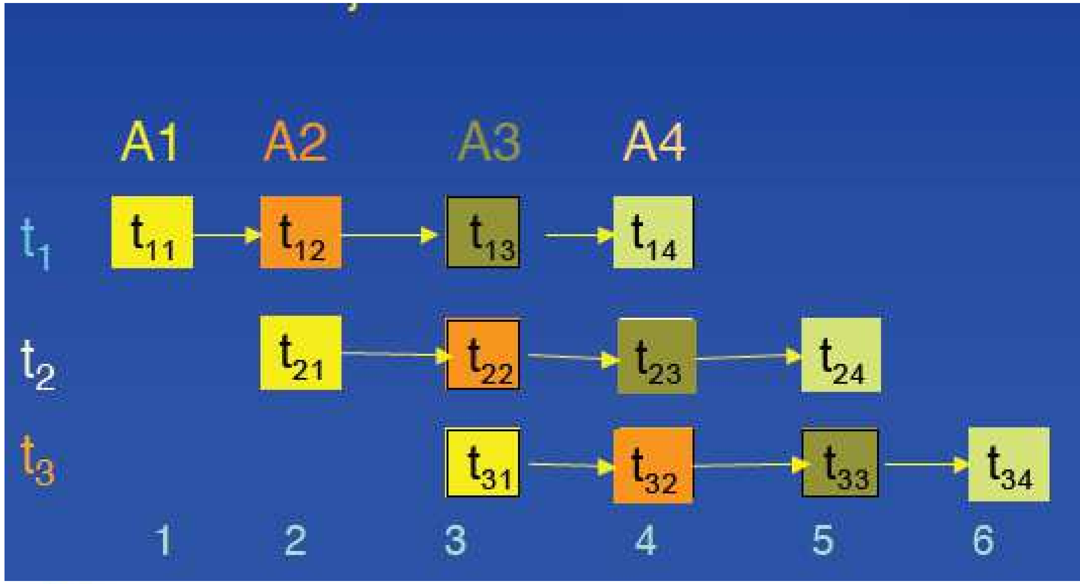




At time t=1, algorithm A1 is active with performing t_{11}

At time t=2, algorithm A1 performs t_{21} and A2 starts with t_{12}

Last subtask completed after execution of m + (n -1) subtasks



Suppose that there are m subtasks per task, n tasks and time O(t) per subtask.

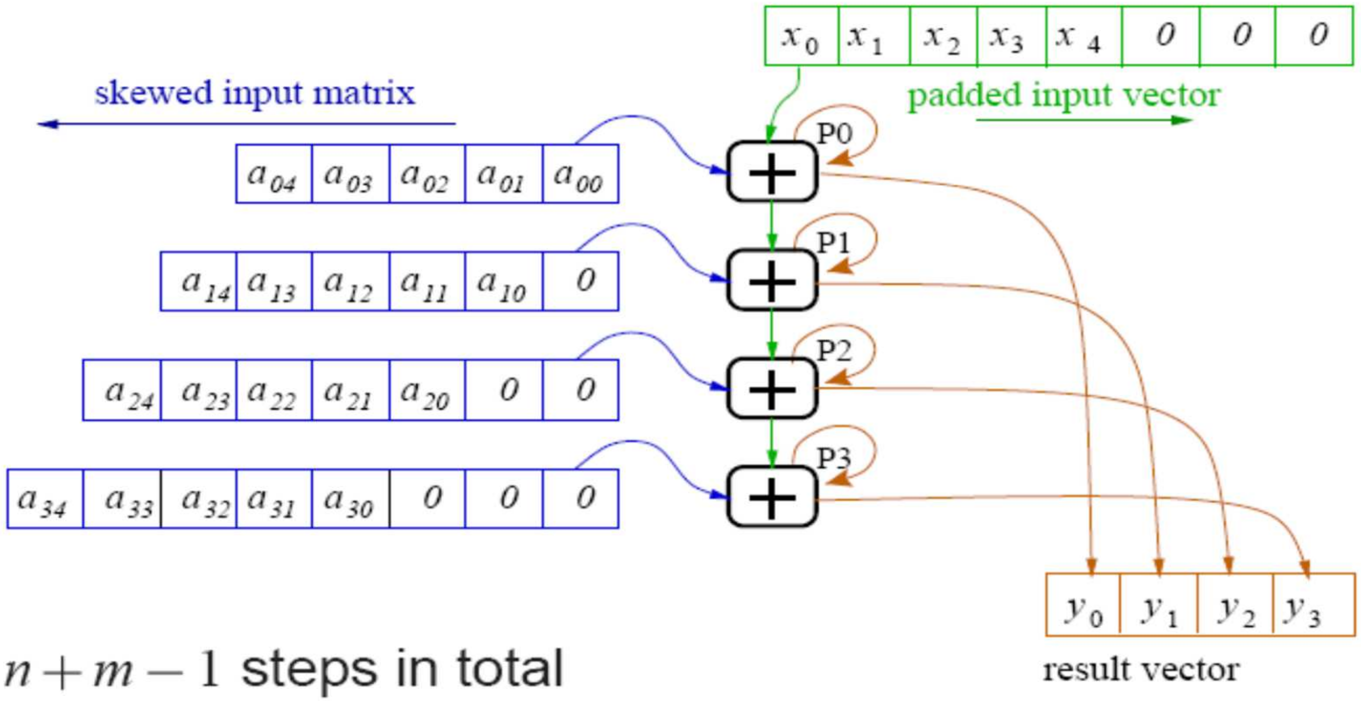
without pipelining:
$T = O(m.n.t)$

with pipelining:
$T = O((n+m) t)$

Example: matrix by vector

multiply matrix $A \in \mathbb{R}^{n,m}$ by vector $x \in \mathbb{R}^{m}$ $\rightarrow$ vector $y \in \mathbb{R}^{n}$

$$y_i = \sum_{j=0}^{m-1} a_{ij} x_j \qquad i = 0, \ldots, n-1$$



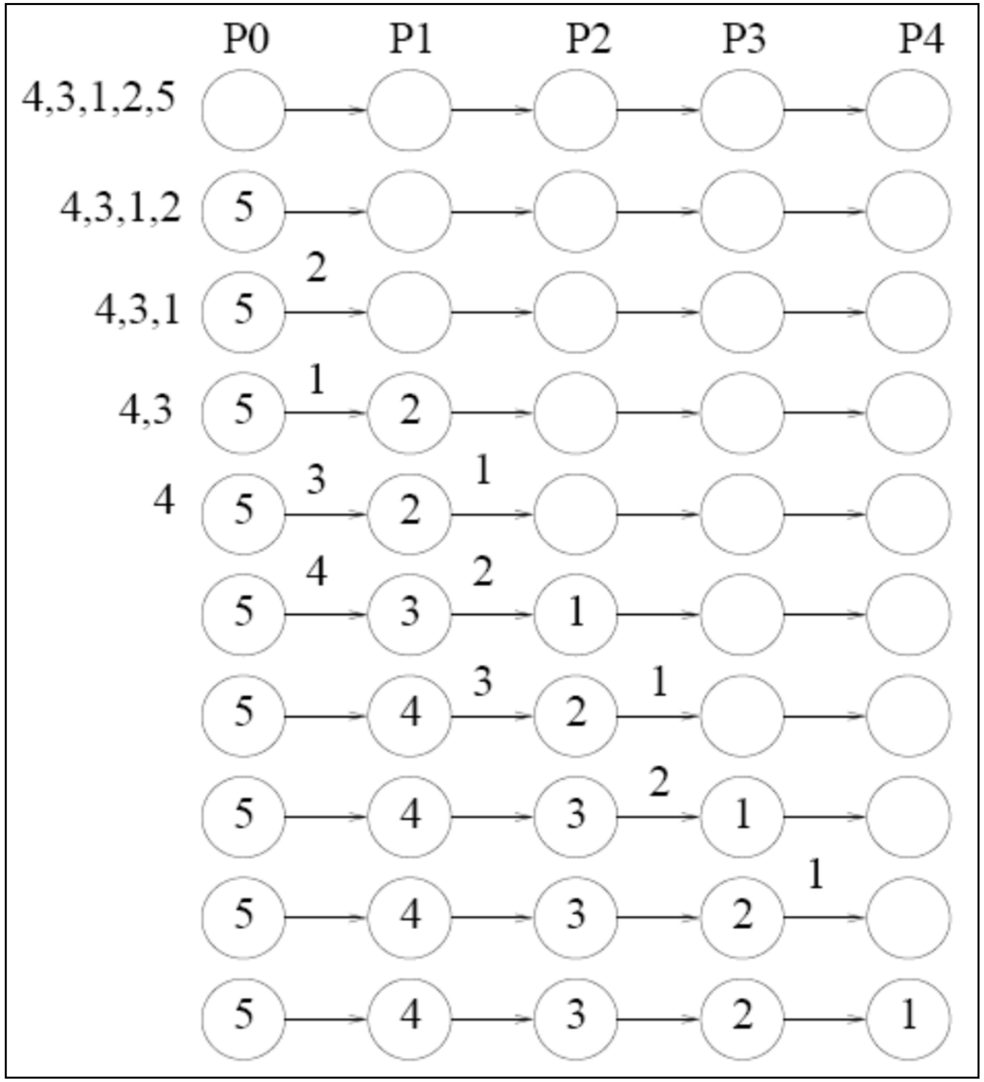
$n + m - 1$ steps in total

Example: multiply the matrix by the vector

- Illustrating with: matrix 4x4 and vector 4x1.
- Step1:
  - P_0 received (X_0 and A_{00});
  - Calculate the product of X_0 and A_{00}, then saved to Y_0 .
  - Push X_0 down to P_1 and move X_1 to the top of the range.
- Step 2:
  - P_0 received (X_1 and A_{01});P_1 received (X_0 and A_{10});
  - Calculate the corresponding products and save to Y_0 and Y_1 .
  - Push X_0, X_1 down, move X_2 to the top of the range, ...repeat until Step 8.

Parallel Insertion Sort

- Algorithm idea:
  - Values that will be sorted go into the range of processors one by one.
  - At each processor:
    - Read the value just received.
    - Compare with current value.
    - Move a value to the next processor.



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Thank you for your attentions!