

Fundamentals of optimization

Linear Programming

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Content

- Linear Programs
- Geometric approach
- Simplex method

Linear programs

Standard form

$$f(x) = c_1x_1 + c_2x_2 + \dots + c_nx_n \rightarrow \max$$

$$a_{1,1}x_1 + a_{1,2}x_2 + \dots + a_{1,n}x_n \leq b_1$$

$$a_{2,1}x_1 + a_{2,2}x_2 + \dots + a_{2,n}x_n \leq b_2$$

...

$$a_{m,1}x_1 + a_{m,2}x_2 + \dots + a_{m,n}x_n \leq b_m$$

$$x_1, x_2, \dots, x_n \in R, x_1, x_2, \dots, x_n \geq 0$$

Convert general linear program forms into standard form

- $f(x) \rightarrow \min \Leftrightarrow -f(x) \rightarrow \max$
- $g(x) \geq b \Leftrightarrow -g(x) \leq -b$
- $A = B \Leftrightarrow (A \leq B) \text{ and } (A \geq B)$
- A variable $x_j \in \mathbb{R}$ can be represented by $x_j = x_j^+ - x_j^-$ where $x_j^+, x_j^- \geq 0$

Convert general linear program forms into standard form

- Example

$$f(x_1, x_2) = 3x_1 + 2x_2 \rightarrow \min$$

$$2x_1 + x_2 \leq 7$$

$$x_1 + 2x_2 = 8$$

$$x_1 - x_2 \geq 2$$

$$x_1, x_2 \in \mathbb{R}, x_2 \geq 0$$

Convert general linear program forms into standard form

- Represent: $x_1 = x_1^+ - x_1^-$

$$f(x_1^+, x_1^-, x_2) = -3x_1^+ + 3x_1^- - 2x_2 \rightarrow \max$$

$$2x_1^+ - 2x_1^- + x_2 \leq 7$$

$$x_1^+ - x_1^- + 2x_2 \leq 8$$

$$-x_1^+ + x_1^- - 2x_2 \leq -8$$

$$-x_1^+ + x_1^- + x_2 \leq 2$$

$$x_1^+, x_1^-, x_2 \in \mathbb{R}, x_1^+, x_1^-, x_2 \geq 0$$

Geometric approach

- Constraints (inequalities) form a feasible region
- Optimal points will be one of the corners of the feasible region

Geometric approach

$$f(x_1, x_2) = 3x_1 + 2x_2 \rightarrow \max$$

$$2x_1 + x_2 \leq 7$$

$$x_1 + 2x_2 \leq 8$$

$$x_1 - x_2 \leq 2$$

$$x_1, x_2 \geq 0$$

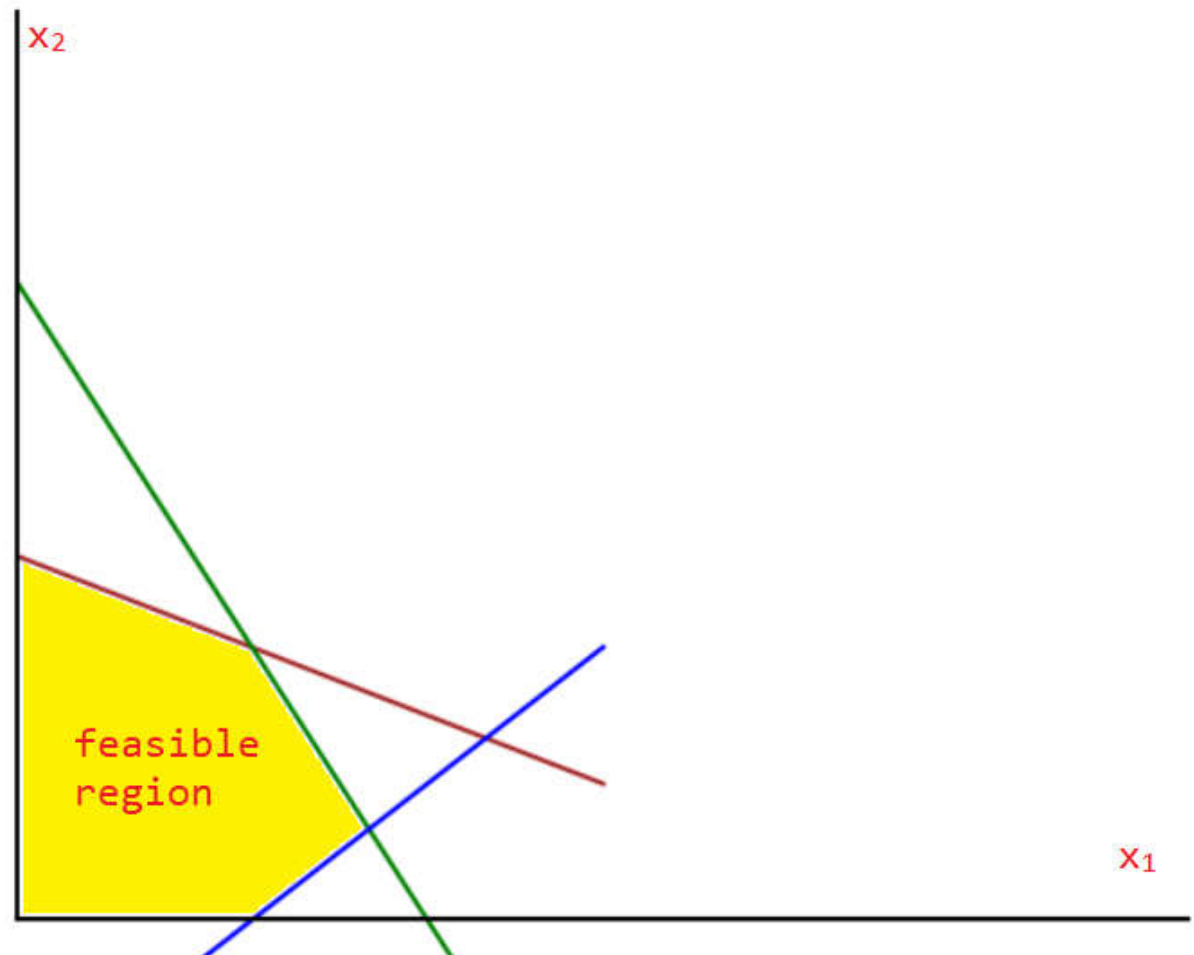
Geometric approach

$$2x_1 + x_2 \leq 7$$

$$x_1 + 2x_2 \leq 8$$

$$x_1 - x_2 \leq 2$$

$$x_1, x_2 \geq 0$$



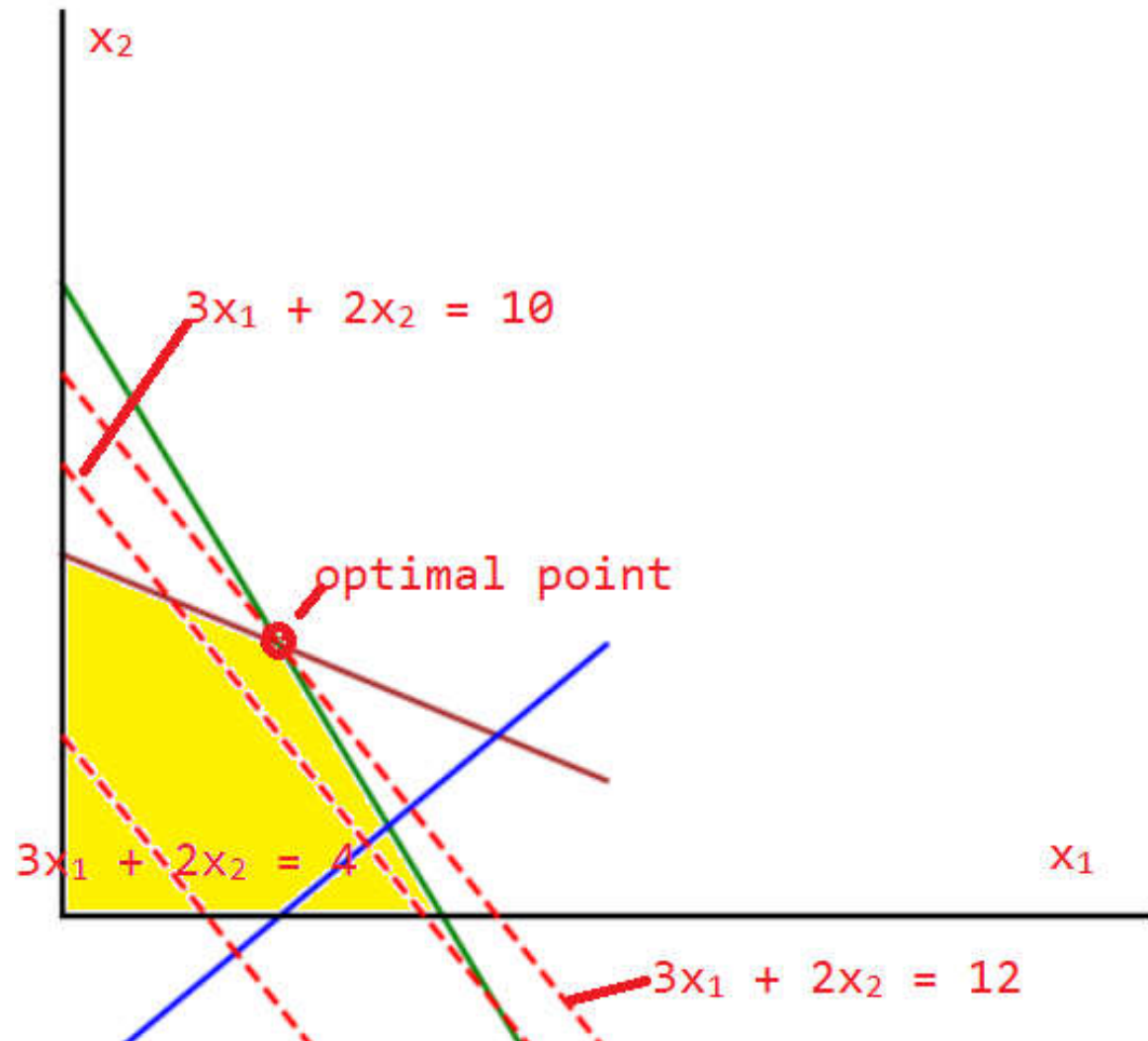
Geometric approach

$$2x_1 + x_2 \leq 7$$

$$x_1 + 2x_2 \leq 8$$

$$x_1 - x_2 \leq 2$$

$$x_1, x_2 \geq 0$$



Geometric approach

- Special cases
 - Problem does not have optimal solutions
 - Problem does not have feasible solutions

$$f(x_1, x_2) = 3x_1 + 2x_2 \rightarrow \max$$

$$-2x_1 - x_2 \leq -7$$

$$x_1 - x_2 \leq 2$$

$$x_1, x_2 \in \mathbb{R}, x_1, x_2 \geq 0$$

$$f(x_1, x_2) = 3x_1 + 2x_2 \rightarrow \max$$

$$2x_1 + x_2 \leq 7$$

$$-4x_1 - 2x_2 \leq -16$$

$$x_1, x_2 \in \mathbb{R}, x_1, x_2 \geq 0$$

Example

- A company must decide to make a plan to produce 2 products P1, P2.
 - The revenue received when selling 1 unit of P1 and P2 are respectively 5\$ and 7\$
 - The manufacturing cost when producing P1 and P2 are respectively 5\$ and 3\$
 - The storage cost in warehouses for 1 unit of P1 and P2 are respectively 2\$ and 3\$
- Compute the production plan so that
 - Total manufacturing cost is less than or equal to 200\$
 - Total storage cost is less than or equal to 150\$
 - Total revenue is maximal

Simplex method

$$f(x_1, x_2, \dots, x_n) = c_1x_1 + c_2x_2 + \dots + c_nx_n \rightarrow \max$$

$$a_{1,1}x_1 + a_{1,2}x_2 + \dots + a_{1,n}x_n \leq b_1$$

$$a_{2,1}x_1 + a_{2,2}x_2 + \dots + a_{2,n}x_n \leq b_2$$

. . .

$$a_{m,1}x_1 + a_{m,2}x_2 + \dots + a_{m,n}x_n \leq b_m$$

$$x_1, x_2, \dots, x_n \geq 0$$

Simplex method

Add slack variables $s_1, s_2, \dots, s_m \geq 0$

$$f(x_1, x_2, \dots, x_n, s_1, s_2, \dots, s_m) = c_1x_1 + c_2x_2 + \dots + c_nx_n \rightarrow \max$$

$$a_{1,1}x_1 + a_{1,2}x_2 + \dots + a_{1,n}x_n + s_1 = b_1$$

$$a_{2,1}x_1 + a_{2,2}x_2 + \dots + a_{2,n}x_n + 0 + s_2 + \dots = b_2$$

\dots

$$a_{m,1}x_1 + a_{m,2}x_2 + \dots + a_{m,n}x_n + 0 + \dots + s_m = b_m$$

$$x_1, x_2, \dots, x_n, s_1, s_2, \dots, s_m \geq 0$$

Simplex method

Consider the linear program under augmented form

$$f(x) = cx \rightarrow \max$$

$$Ax = b$$

$$x \geq 0$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$$

$$c = (c_1, c_2, \dots, c_n)$$

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \dots & \dots & \dots & \dots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{pmatrix}$$

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{pmatrix}$$

Denote $J = (1, 2, \dots, n)$, $I = (1, 2, \dots, m)$ - set of indices

Basic feasible solutions

- Denote
 - $A(j)$ the j^{th} column of matrix A
 - If $J = (j_1, j_2, \dots, j_k)$ – vector of indices (we also use J as a set of indices), then $x_J = (x_{j_1}, \dots, x_{j_k})$, $A(J) = (A(j_1), \dots, A(j_k))$ submatrix of A composed by selecting columns $A(j_1), \dots, A(j_k)$
- Suppose $\text{rank}(A) = m$
- $J_B = (j_1, j_2, \dots, j_m)$ such that columns $A(j_1), \dots, A(j_m)$ are linear independent, J_N – vector of remaining indices of J (out of J_B)
- $B = A(J_B)$ - basic matrix
- $N = A(J_N)$
- $x = (x_B, x_N)$, $x_B = x(J_B) = (x_{j_1}, \dots, x_{j_m})$ such that $Bx = b$, $x_B = B^{-1} b$, $x_N = 0$

Basic feasible solutions

Example

$$f(x) = cx \rightarrow \max$$

$$Ax = b$$

$$x \geq 0$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

$$c = (3, 2, 0, 0, 0)$$

$$A = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 \end{pmatrix}$$

$$b = \begin{pmatrix} 7 \\ 8 \\ 2 \end{pmatrix}$$

$J = (1,2,3,4,5)$ – set of variable indices, $I = (1,2,3)$ – set of constraint indices

$$J_B = (3,4,5), J_N = (1,2),$$

$$x_B = \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \\ 2 \end{pmatrix} \quad x_N = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad N = \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ 1 & -1 \end{pmatrix}$$

Simplex method

- Consider another feasible solution $x' = x + \Delta x$
 $\rightarrow Ax' = Ax \rightarrow A\Delta x = 0 \rightarrow B\Delta x_B + N\Delta x_N = 0 \rightarrow \Delta x_B = -B^{-1}N\Delta x_N$
 $\rightarrow \Delta f = c\Delta x = c_B\Delta x_B + c_N\Delta x_N = -(c_B B^{-1}N - c_N)\Delta x = -\Delta_N\Delta x_N =$
 $-\sum_{j \in J_N} \Delta_j \Delta x_j$, where $u = c_B B^{-1}$, $\Delta_N = uN - c_N$
- $\Delta x_j = x_j \geq 0, \forall j \in J_N$
- If $\Delta_j \geq 0, \forall j \in J_N$ then $\Delta f \leq 0$: x is a maximizer

Simplex method

- If there exists index $p \in J_N$ such that $\Delta_p < 0$, build another feasible solution $x' = x + \Delta x$ as follows
 - $\Delta x_p = \theta \geq 0$
 - $\Delta x_j = 0, \forall j \in J_N \setminus \{p\}$
 - $\Delta x_B = -B^{-1}N\Delta x_N = -\theta B^{-1}A(p), x'_B = x_B - \theta B^{-1}A(p),$
 - $\Delta f = -\theta \Delta_p$
- with $\theta > 0$ and small, we have $\Delta f > 0$: x is not an optimal solution because x' is better than x .
- If $B^{-1}A(p) \leq 0$, then $x'_B = x_B - \theta B^{-1}A(p) \geq 0, \forall \theta > 0$. It means that $f \rightarrow +\infty$ when $\theta \rightarrow +\infty$ (cannot find maximizer)

Simplex method

- Denote $B^{-1}A(p) = (x_{j_1,p}, x_{j_2,p}, \dots, x_{j_m,p})^\top$ and

$$\theta_i = \begin{cases} \frac{x_i}{x_{i,p}} & \text{if } x_{i,p} > 0 \\ +\infty, & \text{otherwise} \end{cases} \quad \forall i \in J_B$$

→ select $\theta = \theta_q = \min\{\theta_i \mid i \in J_B\}$ (we have $\theta < +\infty$, otherwise the objective function is unbounded)

Simplex method

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 $J_B = (j_1, j_2, \dots, j_m)$  s.t.  $B = A(J_B)$  is a basic
 $J_N$  – vector of remaining indices of  $J$  (out of  $J_B$ )
While stop condition not reach{
     $u = c_B B^{-1}$ ,  $\Delta_N = uN - c_N$ ,  $x_N = 0$ ,  $x_B = B^{-1}b$ 
    if  $\Delta_N \geq 0$  then {
        print('found optimal solution!'), break
    } else {
         $p = \arg\min\{\Delta_j \mid j \in J_N\}$ 
         $Y = B^{-1}A(p)$ 
        if  $Y \leq 0$  then {
            print('objective is unbounded'), Return null
        } else {
             $q = \arg\min\{x_i/Y_i \mid i \in J_B \text{ s.t. } Y_i > 0\}$ 
            remove  $q$  from  $J_B$  and add  $p$  to  $J_B$ 
            remove  $p$  from  $J_N$  and add  $q$  to  $J_N$ 
             $B = A(J_B)$ ,  $N = A(J_N)$ 
        }
    }
}
Return  $(x_B, x_N)$  where  $x_B = B^{-1}b$ ,  $x_N = 0$ 

```

Simplex method

Example

$$\begin{aligned}f(x_1, x_2) &= 3x_1 + 2x_2 \rightarrow \max \\2x_1 + x_2 &\leq 7 \\x_1 + 2x_2 &\leq 8 \\x_1 - x_2 &\leq 2 \\x_1, x_2 &\geq 0\end{aligned}$$



Add 3 slack variables x_3, x_4, x_5

$$\begin{aligned}f(x_1, x_2) &= 3x_1 + 2x_2 \rightarrow \max \\2x_1 + x_2 + x_3 &= 7 \\x_1 + 2x_2 + x_4 &= 8 \\x_1 - x_2 + x_5 &= 2 \\x_1, x_2, x_3, x_4, x_5 &\geq 0\end{aligned}$$

Simplex method

Init: $J_B = (3,4,5)$, $J_N = (1,2)$

$$x_B = \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \\ 2 \end{pmatrix} \quad x_N = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad N = \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ 1 & -1 \end{pmatrix}$$

$$f(x) = 0$$

Simplex method

Step 1: $\Delta_N = (-3 \quad -2) \rightarrow$ select $p = 1$

$Y = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \rightarrow$ select $q = 5, J_B = (1,3,4), J_N = (2,5)$

$$x_B = \begin{pmatrix} x_1 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \quad x_N = \begin{pmatrix} x_2 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$f(x) = 6$$

$$B = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad N = \begin{pmatrix} 1 & 0 \\ 2 & 0 \\ -1 & 1 \end{pmatrix}$$

$$B^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & -1 \end{pmatrix}$$

Simplex method

Step 2: $\Delta_N = (-5 \quad 3) \rightarrow \text{select } p = 2$

$Y = \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix} \rightarrow \text{select } q = 3, J_B = (1, 2, 4), J_N = (3, 5)$

$$x_B = \begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix} \quad x_N = \begin{pmatrix} x_3 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$f(x) = 11$$

$$B = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & -1 & 0 \end{pmatrix} \quad N = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$B^{-1} = \begin{pmatrix} 1/3 & 0 & 1/3 \\ 1/3 & 0 & -2/3 \\ -1 & 1 & 1 \end{pmatrix}$$

Simplex method

Step 3: $\Delta_N = (5/3 \quad -1/3) \rightarrow$ select $p = 5$

$Y = \begin{pmatrix} 1/3 \\ -2/3 \\ 1 \end{pmatrix} \rightarrow$ select $q = 4$, $J_B = (1, 2, 5)$, $J_N = (3, 4)$

$$x_B = \begin{pmatrix} x_1 \\ x_2 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} \quad x_N = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$f(x) = 12$$

$$B = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & -1 & 1 \end{pmatrix} \quad N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$B^{-1} = \begin{pmatrix} 2/3 & -1/3 & 0 \\ -1/3 & 2/3 & 0 \\ -1 & 1 & 1 \end{pmatrix}$$

Simplex method

Step 3: $\Delta_N = (4/3 \quad 1/3) > 0 \rightarrow \text{STOP, found optimal solution}$

$$x_B = \begin{bmatrix} x_1 \\ x_2 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} \quad x_N = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$f(x) = 12$$

Simplex method

Tabular form

c_B	B	x_B	c_1	...	c_p	...	c_n	θ
			A_1	...	A_p	...	A_n	
c_{j_1}	A_{j_1}	x_{j_1}	$x_{j_1,1}$		$x_{j_1,p}$		$x_{j_1,n}$	θ_{j_1}
...
c_q	A_{i_0}	x_q	$x_{q,1}$		$x_{q,p}$		$x_{q,n}$	θ_q
...
c_{j_m}	A_{j_m}	x_{j_m}	$x_{j_m,1}$		$x_{j_m,p}$		$x_{j_m,n}$	θ_{j_m}
Δ			Δ_1	...	Δ_p	...	Δ_n	

Simplex method

Tabular form

- Update schema

- Update $x_{i,j}$: $x'_{i,j} = x_{i,j} - (x_{q,j} * x_{i,p}) / x_{q,p}$, $\forall i \in J_B \setminus \{q\}$, $\forall j \in J \setminus \{p\}$
- $\Delta'_j = \Delta_j - (x_{q,j} * \Delta_p) / x_{q,p}$, $\forall j \in J \setminus \{p\}$
- $J_B = J_B \setminus \{q\} \cup \{p\}$
- On row q : $x'_{q,j} = x_{q,j} / x_{q,p}$, $\forall j \in J$
- On column p : $x'_{q,p} = 1$, $x'_{i,p} = 0$, $\forall i \in J_B \setminus \{q\}$
- $x'_q = x_q / x_{q,p}$
- $\Delta_p = 0$

Simplex method

Tabular form

Example

$$\begin{aligned}f(x_1, x_2, x_3, x_4, x_5) &= 3x_1 + 2x_2 \rightarrow \max \\2x_1 + x_2 + x_3 &= 7 \\x_1 + 2x_2 + x_4 &= 8 \\x_1 - x_2 + x_5 &= 2 \\x_1, x_2, x_3, x_4, x_5 &\geq 0\end{aligned}$$

Simplex method

Tabular form

c_B	B	x_B	3	2	0	0	0	θ
			A_1	A_2	A_3	A_4	A_5	
0	A_3	7	2	1	1	0	0	
0	A_4	8	1	2	0	1	0	
0	A_5	2	1	-1	0	0	1	
Δ			-3	-2	0	0	0	

Simplex method

Tabular form

c_B	B	x_B	3	2	0	0	0	θ
			A_1	A_2	A_3	A_4	A_5	
0	A_3	7	2	1	1	0	0	
0	A_4	8	1	2	0	1	0	
0	A_5	2	1	-1	0	0	1	
Δ			-3	-2	0	0	0	



Select $p = 1$

Simplex method

Tabular form

c_B	B	x_B	3	2	0	0	0	θ
			A_1	A_2	A_3	A_4	A_5	
0	A_3	7	2	1	1	0	0	7/2
0	A_4	8	1	2	0	1	0	8
0	A_5	2	1	-1	0	0	1	2
Δ			-3	-2	0	0	0	



Select $p = 1$

Simplex method

Tabular form

c_B	B	x_B	3	2	0	0	0	θ
			A_1	A_2	A_3	A_4	A_5	
0	A_3	7	2	1	1	0	0	7/2
0	A_4	8	1	2	0	1	0	8
0	A_5	2	1	-1	0	0	1	2
Δ			-3	-2	0	0	0	

Simplex method

Tabular form

c_B	B	x_B	3	2	0	0	0	θ
			A_1	A_2	A_3	A_4	A_5	
0	A_3	3	2	3	1	0	-2	
0	A_4	6	1	3	0	1	-1	
0	A_5	2	1	-1	0	0	1	
Δ			-3	-5	0	0	3	

Update Δ , x_B , $x_{i,j}$ except row A_5 , and except columns 1

Simplex method

Tabular form

c_B	B	x_B	3	2	0	0	0	θ
			A_1	A_2	A_3	A_4	A_5	
0	A_3	3	0	3	1	0	-2	
0	A_4	6	0	3	0	1	-1	
3	A_1	2	1	-1	0	0	1	
Δ			0	-5	0	0	3	

Replace A_5 by A_1 in the B , Update row corresponding to A_1 , and column 1

Simplex method

Tabular form

c_B	B	x_B	3	2	0	0	0	θ
			A_1	A_2	A_3	A_4	A_5	
0	A_3	3	0	3	1	0	-2	
0	A_4	6	0	3	0	1	-1	
3	A_1	2	1	-1	0	0	1	
Δ			0	-5	0	0	3	

Select $p = 2$

Simplex method

Tabular form

c_B	B	x_B	3	2	0	0	0	θ
			A_1	A_2	A_3	A_4	A_5	
0	A_3	3	0	3	1	0	-2	1
0	A_4	6	0	3	0	1	-1	2
3	A_1	2	1	-1	0	0	1	$+\infty$
Δ			0	-5	0	0	3	

Select $p = 2$

Select $q = 1$

Simplex method

Tabular form

c_B	B	x_B	3	2	0	0	0	θ
			A_1	A_2	A_3	A_4	A_5	
0	A_3	3	0	3	1	0	-2	
0	A_4	3	0	3	-1	1	1	
3	A_1	3	1	-1	1/3	0	1/3	
Δ			0	-5	5/3	0	-1/3	

Update Δ , x_B , $x_{i,j}$ except row A_3 , and except columns 2

Simplex method

Tabular form

c_B	B	x_B	3	2	0	0	0	θ
			A_1	A_2	A_3	A_4	A_5	
2	A_2	1	0	1	1/3	0	-2/3	
0	A_4	3	0	0	-1	1	1	
3	A_1	3	1	0	1/3	0	1/3	
Δ			0	0	5/3	0	-1/3	

Replace A_3 by A_2 in B , update x_B , $x_{i,j}$ on row A_3 , and columns 2

Simplex method

Tabular form

c_B	B	x_B	3	2	0	0	0	θ
			A_1	A_2	A_3	A_4	A_5	
2	A_2	1	0	1	1/3	0	-2/3	$+\infty$
0	A_4	3	0	0	-1	1	1	3
3	A_1	3	1	0	1/3	0	1/3	$+\infty$
Δ			0	0	5/3	0	-1/3	

Select $p = 5$

Simplex method

Tabular form

c_B	B	x_B	3	2	0	0	0	θ
			A_1	A_2	A_3	A_4	A_5	
2	A_2	1	0	1	1/3	0	-2/3	$+\infty$
0	A_4	3	0	0	-1	1	1	3
3	A_1	3	1	0	1/3	0	1/3	$+\infty$
Δ			0	0	5/3	0	-1/3	

Select $p = 5$

Select $q = 4$

Simplex method

Tabular form

c_B	B	x_B	3	2	0	0	0	θ
			A_1	A_2	A_3	A_4	A_5	
2	A_2	3	0	1	-1	2/3	-2/3	
0	A_4	3	0	0	-1	1	1	
3	A_1	2	1	0	2/3	-1/3	1/3	
Δ			0	0	4/3	1/3	-1/3	

Simplex method

Tabular form

c_B	B	x_B	3	2	0	0	0	θ
			A_1	A_2	A_3	A_4	A_5	
2	A_2	3	0	1	-1/3	2/3	0	
0	A_5	3	0	0	-1	1	1	
3	A_1	2	1	0	2/3	-1/3	0	
Δ			0	0	4/3	1/3	0	

STOP, found optimal solution!!!!