# Derivatives

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# Content

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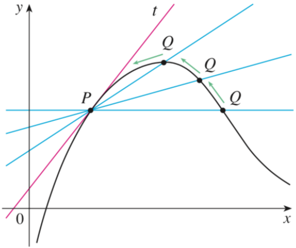
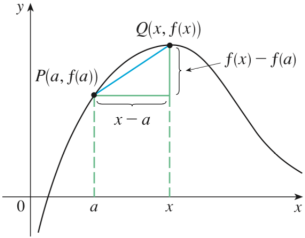
# Geometrical illustration



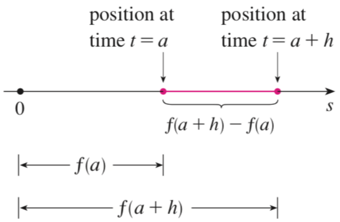

The slope of the secant $PQ$

$$k_{PQ} = \frac{f(x) - f(a)}{x - a}$$

The slope of the tangent line to the graph of $f(x)$ at $P$

$$k = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

# Physical illustration




The average velocity over the time interval $[a, a+h]$ is

$$v_m = \frac{f(a+h) - f(a)}{h}$$

The instantaneous velocity at $t = a$ is

$$v = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

# The derivative of a function

## Definition

Let $f(x)$ be defined on $(a, b)$, $x_0 \in (a, b)$. $f(x)$ is said to have a derivative at $x_0$ if there exists

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} =: f'(x_0).$$

Another notation: set $x - x_0 = \Delta x$, we can write

$$\frac{df}{dx}(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$$

$\Delta x$: the increment in $x$, $\Delta f$: the corresponding increment in $f$.

# One sided derivative

## Definition

- Let $f(x)$ be defined on $[x_0, x_0+\varepsilon)$, $\varepsilon > 0$. The righthand derivative at $x_0$ is

$$f'_{+}(x_0) = \lim_{x \to x_0^{+}} \frac{f(x) - f(x_0)}{x - x_0}.$$

- Let $f(x)$ be defined on $(x_0 - \varepsilon, x_0]$, $\varepsilon > 0$. The lefthand derivative at $x_0$ is

$$f'_{-}(x_0) = \lim_{x \to x_0^{-}} \frac{f(x) - f(x_0)}{x - x_0}.$$

Note that $\exists\, f'(x_0) \Leftrightarrow \exists\, f'_{+}(x_0) = f'_{-}(x_0)$.

## Definition

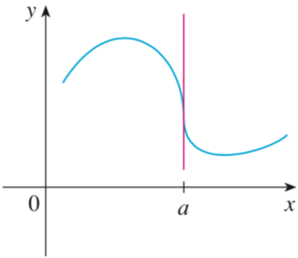
$f(x)$ is said to be differentiable at $x = x_0$ if there exists $f'(x_0)$.
$f(x)$ has derivative on the open interval $(a, b)$ if $f(x)$ has a derivative **at all points** $x \in (a, b)$.
$f(x)$ has derivative on the closed interval $[a, b]$ if $f(x)$ has derivative on the open interval $(a, b)$, lefthand derivative at $b$, and righthand derivative at $a$.

The tangent line to the graph of $f(x)$ at $x_0$ is

$$y = f(x_0) + f'(x_0)(x - x_0).$$

## Remark

If $\lim\limits_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \infty$, the tangent line to the graph of $f(x)$ at $(x_0, f(x_0))$ is perpendicular to $Ox$.



## Example

1. Compute $f'(0)$, where $f(x) = \begin{cases} \frac{1-\cos 2x}{\ln(1+x)} & \text{khi } x \neq 0, \\ 0 & \text{khi } x = 0. \end{cases}$
2. Compute $h'(1)$, where $h(x) = \begin{cases} x^2 - 5x + 4 & \text{if } x \geq 1, \\ 2^x - 2 & \text{if } x < 1. \end{cases}$
3. $f(x) = a^x$, $0 < a \neq 1$.

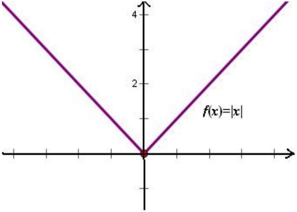
# Connection between differentiability and continuity

- If $f$ is differentiable at $x$, then $f$ is continuous at $x$.

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x) - f(x_0)}{\Delta x}$$

$$\Rightarrow f(x_0 + \Delta x) - f(x_0) = f'(x_0)\Delta x + o(\Delta x) \xrightarrow{\Delta x \to 0} 0.$$

- The converse is wrong. For instance, $f(x) = |x|$, the function is continuous at $x = 0$ but is not differentiable at $x = 0$.



# Differentiation rules

**Theorem**

*Let $f(x), g(x)$ be defined and differentiable on $(a, b)$. Then $f(x) \pm g(x)$, $f(x).g(x)$, $\frac{f(x)}{g(x)}$ are also differentiable on $(a, b)$ and*

- $\left[f(x) \pm g(x)\right]' = f'(x) \pm g'(x)$.
- $\left[f(x).g(x)\right]' = f'(x)g(x) + f(x)g'(x)$.
- $\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$, *at* $g(x) \neq 0$.
- *The chain rule:* $(f \circ g)'(x) = f'[g(x)].g'(x)$.

**Example**

$(\sin u(x))' = u'(x) \cos u(x)$.

# Differentiation formulas

$$C' = 0$$

$$(e^x)' = e^x$$

$$(\ln x)' = \frac{1}{x}$$

$$(\sin x)' = \cos x$$

$$(\tan x)' = \frac{1}{\cos^2 x}$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$

$$(\arctan x)' = \frac{1}{1+x^2}$$

$$(x^\alpha)' = \alpha x^{\alpha-1}, \alpha \in \mathbb{R}$$

$$(a^x)' = a^x \ln a, 0 < a \neq 1$$

$$(\log_a x)' = \frac{1}{x \ln a}, 0 < a \neq 1$$

$$(\cos x)' = -\sin x$$

$$(\cot x)' = -\frac{1}{\sin^2 x}$$

$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$$

$$(\text{arccot}\, x)' = -\frac{1}{1+x^2}$$

## Example

1. CMR $\arcsin x + \arccos x = \frac{\pi}{2}$, $\forall x \in [-1, 1]$.
2. CMR $2 \arctan x + \arcsin \frac{2x}{1+x^2} = \pi$, $\forall x \geq 1$.

# Derivative of the inverse function

**Theorem**

*Assume that* $f(x)\colon (a,b) \to (c,d)$ *has an inverse* $f^{-1}(x)\colon (c,d) \to (a,b)$. *If* $f(x)$ *is differentiable at* $x_0 \in (a,b)$, $f'(x_0) \neq 0$ *and* $f^{-1}(x)$ *is continuous at* $y_0 = f(x_0)$, *then the inverse function* $f^{-1}(x)$ *is also differentiable at* $y_0$. *Moreover,*

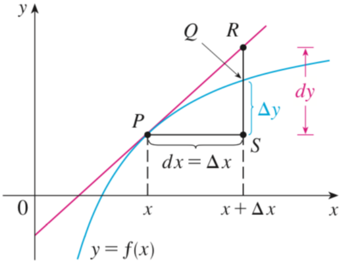
$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$

**Example**

Let $f(x) = \ln(1+x^2) + 2x + 2$ be a function whose inverse function is $g(x)$. Compute $g'(2)$.

# Content

# Geometrical interpretation



# Linear approximation. Differentials

**Definition**

Let $f(x)\colon (a,b) \to \mathbb{R}$, $x_0 \in (a,b)$. If we can write

$$f(x_0 + \Delta x) - f(x_0) = A\Delta x + \alpha(\Delta x),$$

where $A \in \mathbb{R}$ and $\boldsymbol{\alpha(\Delta x) = o(\Delta x)}$ as $\Delta x \to 0$, we say that $f(x)$ is differentiable at $x_0$ and the differential of $f(x)$ at $x_0$ is

$$df(x_0) = A\Delta x.$$

$f(x)$ has derivative at $x_0 \Leftrightarrow f(x)$ is differentiable at $x_0$ and $A = f'(x_0)$.

Choose $f(x) \equiv x$, we have $df(x) = dx = \Delta x$.
Hence, $df(x_0) = f'(x_0)dx$. $df$ is a dependent variable, which depends on $x$ and $dx$.

# Applications of linear approximation

$$\boxed{f(x_0 + \Delta x) \approx f(x_0) + f'(x_0).\Delta x}$$

## Example

Approximate the following values $\sin 62^\circ$, $\sqrt[3]{\frac{2+0,06}{2-0,06}}$.

# Invariance property of the first order differential

Given a differentiable function $y = f(x)$, we have

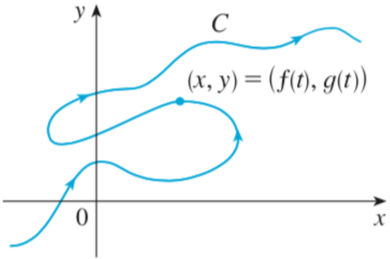
$$dy = f'(x)dx.$$

Assume that $x$ is a dependent variable, namely $x = g(t)$
$\Rightarrow y = f(g(t))$.
We write in terms of $t$

$$dy = (f \circ g)'(t)dt = f'(g(t))g'(t)dt = f'(x)dx.$$

The differential $dy = f'(x)dx$ is invariant, whether $x$ is an independent or a dependent variable.

# Parametric curves




It is impossible to describe this curve by an equation $y = y(x)$. Assume that $f(t), g(t)$ are functions of the third variable, the parameter $t$. For each $t$, we determine a point $M(f(t), g(t))$.When $t$ varies, $M$ also varies and traces out a parametric curve $C$.

## Tangents to parametric curves

Assume that $f'(t) \neq 0 \, \forall \, t \in (a, b)$, then $x$ has an inverse: $t = f^{-1}(x)$, we can rewrite $y = g(f^{-1}(x))$.
It is obvious that $y = y(x)$ is differentiable and if $x'(t)$

$$y'(x) = \frac{dy}{dx} = \frac{\frac{dg}{dt}}{\frac{df}{dt}} = \frac{g'(t)}{f'(t)}.$$

If $\dfrac{dx}{dt} = 0$, $\left(\dfrac{dy}{dt} \neq 0\right)$, the tangent is horizontal.
If $\dfrac{dy}{dt} = 0$, $\left(\dfrac{dx}{dt} \neq 0\right)$, the tangent is vertical.

## Example

Compute the derivative $y'(x)$ of a function given by

a) $\begin{cases} x = e^{t^2}, \\ y = te^{t^2}. \end{cases}$ b) $\begin{cases} x = t - e^{t}, \\ y = 2t + e^{-t^2}. \end{cases}$