Applications of definite integrals

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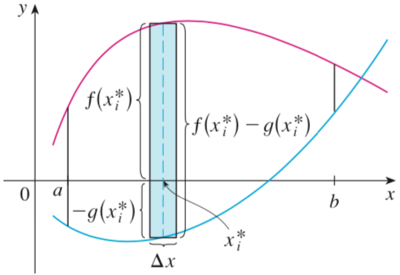
Area between curves



$$S \approx \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\Delta x_i \to \int_a^b [f(x) - g(x)]dx.$$

- The area enclosed by $y = f(x)$, $y = g(x)$ and $x = a$, $x = b$ is:

$$S = \int_a^b |f(x) - g(x)| dx.$$

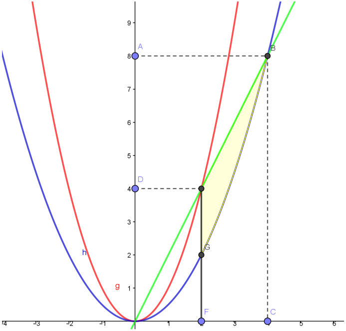
- The area enclosed by $x = f(y)$, $x = g(y)$ and $y = c$, $y = d$ is:

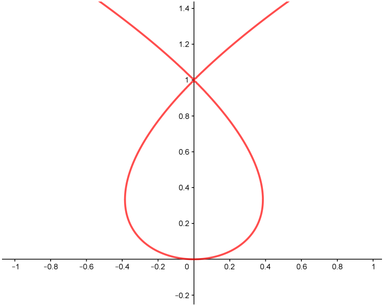
$$S = \int_c^d |f(y) - g(y)| dy.$$

Example

Find the areas enclosed by

1. the curves $y = x^2$, $y = \frac{x^2}{2}$, $y = 2x$.
2. the curve $x^2 = y(y-1)^2$.





Areas enclosed by parametric curves

Consider a region enclosed by a parametric curve which is traced out once by the parametric equations $\begin{cases} x = x(t), \\ y = y(t), \alpha \leq t \leq \beta, \end{cases}$

$$S = \int_{\alpha}^{\beta} |y(t)x'(t)| dt.$$

Example

Find the area of the region enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

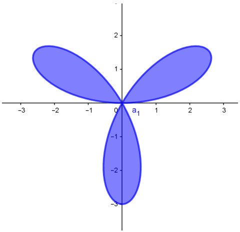
Curves given in polar coordinates

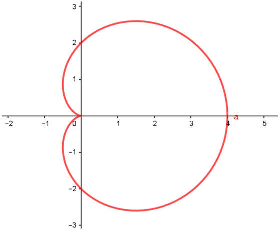
The area enclosed by the rays $\varphi = \alpha, \varphi = \beta$ and the curve $r = r(\varphi)$ is $S = \dfrac{1}{2}\displaystyle\int_{\alpha}^{\beta} r^2(\varphi)d\varphi$.

Example

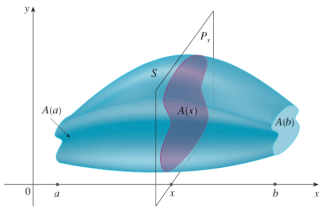
Evaluate the planar areas enclosed by the following curves

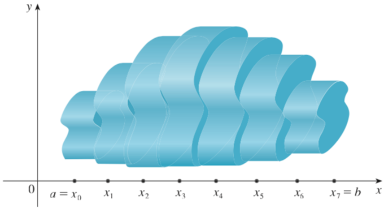
a) $r = 3\sin 3\varphi$ b) $r = 2(1 + \cos\varphi)$.





Volumes





Consider a solid S that lies between the planes $x = a$ and $x = b$. Assume that the cross-sectional area of S in the plane (P_x), through x and perpendicular to the $x-$axis is $A(x)$.
Divide S into n slabs of width Δx_i, we can approximate the volume S_i by the volume of a cylinder with base area $A(x_i^*)$ and height Δx_i. $V \approx \sum_{i=1}^{n} A(x_i^*)\Delta x_i$.

Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane (P_x), through x and perpendicular to the $x-$axis is $A(x)$, then the volume of S is

$$V = \int_a^b A(x)dx.$$

Solids of revolution

Solids of revolution are obtained by revolving a region about a line. For example, rotating the graph $y = f(x)$, $a \leq x \leq b$ about Ox

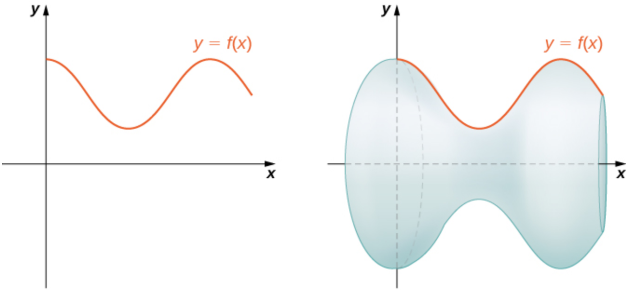


Figure: Source: https://math.libretexts.org/

Volume of a solid of revolution

The volume of the solid obtained by rotating the region bounded by $x = a$, $x = b$, $y = 0$ and $y = f(x)$ about Ox is

$$V = \pi \int_a^b f^2(x)dx.$$

The volume of the solid obtained by rotating the region bounded by $y = c$, $y = d$, $x = 0$ and $x = g(y)$ about Oy is

$$V = \pi \int_c^d g^2(y)dy.$$

Example

Find the volume of the region given by $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} \le 1$.

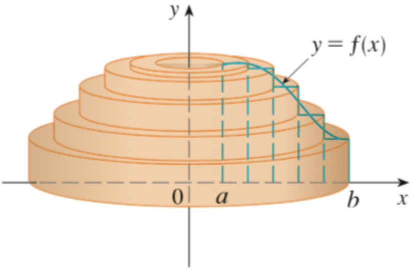
Example

Find the volum of the solids of revolution obtained by rotating

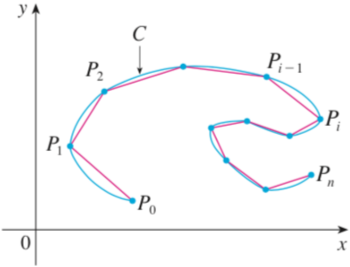
1. the region enclosed by $y = \sin x$ and the lines $y = 0, x = 0, x = \frac{\pi}{2}$ about Ox.
2. the region enclosed by $y = \sin x$ and the lines $y = 0, y = 1, x = 0$ about Oy.
3. the region enclosed by $y = \sin x$ and the lines $y = 0, x = 0, x = \frac{\pi}{2}$ about Oy.

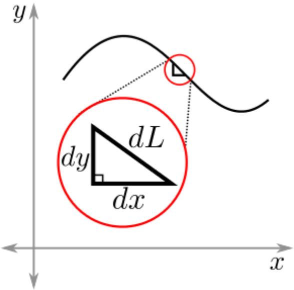
The volume of the solid obtained by rotating the region bounded by $y = f(x)$, $y = 0, x = a, x = b$ about Oy is

$$V = 2\pi \int_a^b xf(x)dx.$$



Arclength





$$L = \sum_{i=1}^{n} \widehat{P_{i-1}P_i} \approx \sum_{i=1}^{n} P_{i-1}P_i = \sum_{i=1}^{n} \sqrt{1+[f'(x_i^*)]^2}\Delta x_i.$$

The arclength of the curve

- $(\mathcal{C}) : y = f(x),\ a \leq x \leq b$

$$L = \int_a^b \sqrt{1 + f'^2(x)}dx.$$

- $(\mathcal{C}) : \begin{cases} x = x(t) \\ y = y(t) \end{cases}$, $\alpha \leq t \leq \beta$ is

$$L = \int_\alpha^\beta \sqrt{x'^2(t) + y'^2(t)}dt.$$

- $(\mathcal{C}) : r = r(\varphi),\ \alpha \leq \varphi \leq \beta$ is

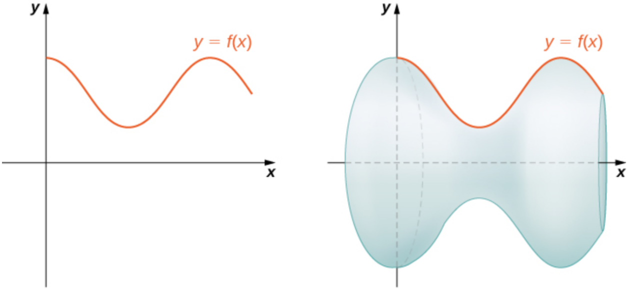
$$L = \int_\alpha^\beta \sqrt{r^2(\varphi) + r'^2(\varphi)}d\varphi.$$

Example

1. Find the arclength of the curve $y = x^{\frac{3}{2}}$, $0 \le x \le 4$.
2. Find the arclength of the curve $x = a\cos^3 t$, $y = a\sin^3 t$, $a > 0$.
3. Find the arclength of the curve $r = a(1 + \cos\varphi)$, $(a > 0)$.

Surface area of solids of revolution

Rotating the graph $y = f(x)$, $a \leq x \leq b$ about Ox



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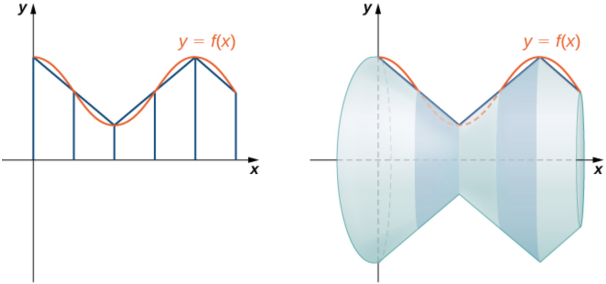

Figure: Source: https://math.libretexts.org/

The surface area of the solid of revolution obtained by

- rotating the graph $y = f(x)$, $a \leq x \leq b$ about Ox is

$$S = 2\pi \int_a^b |f(x)| \sqrt{1 + f'^2(x)} dx.$$

- rotating the curve $x = g(y)$, $c \leq y \leq d$ about Oy is

$$S = 2\pi \int_c^d |g(y)| \sqrt{1 + g'^2(y)} dy.$$

Example

Find the surface area of the solid of revolution obtained by rotating the graph $y = \sin x$, $0 \le x \le \dfrac{\pi}{2}$ about Ox.

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There are situations in which it is impossible to find the exact value of a definite integral,

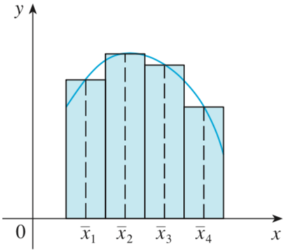
- $\displaystyle\int_0^1 e^{-x^2}dx, \int_0^1 \frac{\sin x}{x}dx.$
- the function is determined from an experiment by instrument reading or collected data, no explicit formula for $f(x)$.

Aim: Approximate value of definite integrals.

Recall: $\displaystyle\int_a^b f(x)dx = \sum_{i=1}^{n} S_i \approx \sum_{i=1}^{n} f(x_i^*)\Delta x_i$, where
$a \equiv x_0 < x_1 < \ldots < x_n \equiv b$, $x_i^* \in [x_{i-1}, x_i]$, $\Delta x_i = x_i - x_{i-1}$.
Simply choose $x_i^* \equiv$ end points.
In the following rules, we need

- formula used.
- error bound of the approximation.

Midpoint rule

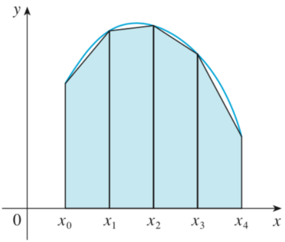


$$\int_a^b f(x)dx \approx \Delta x\,[f(\bar{x}_1) + f(\bar{x}_2) + \ldots + f(\bar{x}_n)],$$

where $\Delta x = \dfrac{b-a}{n}$, $\bar{x}_i = \dfrac{x_{i-1}+x_i}{2}$, $1 \leq i \leq n$.

If $|f''(x)| \leq M, x \in [a,b]$, then $|E_M| \leq \dfrac{M(b-a)^3}{24n^2}$.

Trapezoidal rule

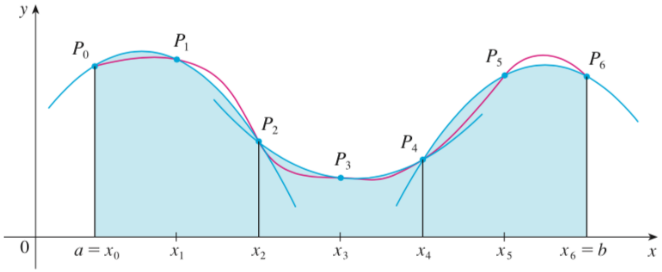


$$\int_a^b f(x)dx \approx \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + \ldots + 2f(x_{n-1}) + f(x_n)\right],$$

where $\Delta x = \dfrac{b-a}{n}$, $x_i = x_0 + i\Delta x$, $1 \le i \le n$.

If $|f''(x)| \le M, x \in [a, b]$, then $|E_T| \le \dfrac{M(b-a)^3}{12n^2}$.

Simpson's rule



In each small strip, approximate the (red) curve by a (blue) parabola.

Divide the interval into $n = 2k$ subintervals:
$a \equiv x_0, x_i = x_0 + i\Delta x, 1 \leq i \leq 2k,\ \Delta x = \dfrac{b-a}{2k}$.
Then

$$\int_a^b f(x)dx \approx \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \ldots + \right.$$
$$\left. + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right].$$

If $|f^{(4)}(x)| \leq M, x \in [a,b]$, then $|E_S| \leq \dfrac{M(b-a)^5}{180n^4}$.