Fundamentals of Optimization

Lagrangian relaxation

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Content

- Lagrange dual function
- Lagrange dual problem
- KKT condition

Lagrange dual function

- Optimization problem in the standard form

$$
\begin{array}{ll}
(P) & \text{minimize } f(x) \\
\text{s.t.} & g_i(x) \le 0,\ i = 1, 2, \ldots, m \\
& h_i(x) = 0,\ i = 1, 2, \ldots, p \\
& x \in X \subseteq R^n
\end{array}
$$

with $x \in R^n$, and assume $D = (\cap_{i=1}^{m}$ ***dom*** $g_i) \cap (\cap_{i=1}^{m}$ ***dom*** $h_i)$ is not empty.

- Denote f^* the optimal value of $f(x)$
- If f, g_i ($i = 1,2,\ldots,m$) are convex functions, h_i ($i = 1,\ldots,p$) are linear → **convex program**

Lagrange dual function

- Define Lagrangian function L: R^n x R^m x R^p → R

$$L(x, \lambda, \mu) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{i=1}^{p} \mu_i h_i(x)$$

- Lagrange dual function (or dual function)

$$q(\lambda, \mu) = inf_{x \in D} L(x, \lambda, \mu)$$

Lagrange dual problem

- For each pair (λ,μ), the Lagrange dual function provides a lower bound on the optimal value of the primal problem (original optimization problem)

$$q(\lambda,\mu) \leq f^*$$

- Question: what is the *best* lower bound → Lagrange dual problem

$$\max q(\lambda,\mu)$$
$$\lambda \geq 0$$

Lagrange Dual problem

- **Weak duality theorem** if x^* is an optimal solution to the primal problem and (λ^*,μ^*) is an optimal solution to the dual problem, then $f(x^*) \geq q(\lambda^*,\mu^*)$
- **Corollary** If there exist x^* and (λ^*,μ^*) such that $f(x^*) = q(\lambda^*,\mu^*)$, then x^* and (λ^*,μ^*) are respectively optimal solutions to the primal and dual problems

KKT Conditions

- **Theorem** (Fritz John necessary conditions) Let x* be a feasible solution of (*P*). If x^* is a local minimum of (*P*), then there exists (u,λ,μ) such that:
  - $u\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) + \sum_{i=1}^{p} \mu_i \nabla h_i(x^*) = 0$
  - $u,\lambda \geq 0,\ (u,\lambda,\mu) \neq 0$
  - $\lambda_i g_i(x^*) = 0,\ i = 1,\ldots, m$

KKT Conditions

- **Theorem** (Karush-Kuhn-Tucker (KKT) necessary conditions) Let x* be a feasible solution of (*P*) and I = {*i*: $g_i(x^*) = 0$}. Further, suppose that $\nabla h_i(x^*)$ for $i = 1,\ldots, p$ and $g_i(x^*)$ for $i \in I$ are linearly independent. If x^* is a local minimum of (*P*), then there exists (λ, μ) such that:
  - $\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) + \sum_{i=1}^{p} \mu_i \nabla h_i(x^*) = 0$
  - $\lambda \geq 0$,
  - $\lambda_i g_i(x^*) = 0,\ i = 1,\ldots, m$

KKT Conditions

- **Theorem** (KKT sufficient conditions) Let x^* be a feasible solution of (P) which is convex program (f, g_i are convex functions, h_i are linear functions). If there exists (λ, μ) such that:
  - $\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) + \sum_{i=1}^{p} \mu_i \nabla h_i(x^*) = 0$
  - $\lambda \geq 0$,
  - $\lambda_i g_i(x^*) = 0$, $i = 1, \ldots, m$

  then x^* is a global optimal solution of (P)

Examples

minimize $f(x,y) = 2x - y$

s.t. $g_1(x,y) = x^2 + y^2 - 2 \leq 0$

$g_2(x,y) = x - y - 1 \leq 0$

- $\nabla f(x,y) = [2 \quad 1]^T$, $\nabla g_1(x,y) = [2x \quad 2y]^T$, $\nabla g_2(x,y) = [1 \quad -1]^T$
- f and g_2 are linear, so they are convex
- $\nabla^2 g_1(x,y) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ which is positive definite, so g_1 is convex

Examples

minimize $f(x,y) = 2x - y$
s.t. $g_1(x,y) = x^2 + y^2 - 2 \leq 0$
$g_2(x,y) = x - y - 1 \leq 0$

- Apply the KKT necessary conditions
  - $[2 \quad -1]^T + \lambda_1[2x \quad 2y]^T + \lambda_2[1 \quad -1]^T = 0$
  - $x^2 + y^2 - 2 \leq 0$
  - $x - y - 1 \leq 0$
  - $\lambda_1(x^2 + y^2 - 2) = 0$
  - $\lambda_2(x - y - 1) = 0$
  - $\lambda_1, \lambda_2 \geq 0$

Examples

minimize $f(x,y) = 2x - y$
s.t. $g_1(x,y) = x^2 + y^2 - 2 \leq 0$
$g_2(x,y) = x - y - 1 \leq 0$

- Apply the KKT necessary conditions (rewrite)
  - $2 + 2\lambda_1 x + \lambda_2 = 0$ (1)
  - $-1 + 2\lambda_1 y - \lambda_2 = 0$ (2)
  - $x^2 + y^2 - 2 \leq 0$ (3)
  - $x - y - 1 \leq 0$ (4)
  - $\lambda_1(x^2 + y^2 - 2) = 0$ (5)
  - $\lambda_2(x - y - 1) = 0$ (6)
  - $\lambda_1, \lambda_2 \geq 0$ (7)

Examples

minimize $f(x,y) = 2x - y$
s.t. $g_1(x,y) = x^2 + y^2 - 2 \leq 0$
$g_2(x,y) = x - y - 1 \leq 0$

- There are 4 cases
  - (I) $\lambda_1 = \lambda_2 = 0$
  - (II) $\lambda_1 = 0, \lambda_2 > 0$
  - (III) $\lambda_1 > 0, \lambda_2 = 0$
  - (IV) $\lambda_1 > 0, \lambda_2 > 0$

Examples

minimize $f(x,y) = 2x - y$
s.t. $g_1(x,y) = x^2 + y^2 - 2 \leq 0$
$g_2(x,y) = x - y - 1 \leq 0$

- Case (I) $\lambda_1 = \lambda_2 = 0$, from (1) and (2) → NOT POSSIBLE

Examples

minimize $f(x,y) = 2x - y$

s.t. $g_1(x,y) = x^2 + y^2 - 2 \le 0$

$g_2(x,y) = x - y - 1 \le 0$

- Case (II) $\lambda_1 = 0, \lambda_2 > 0$, from (1) and (2) we have
  - $2 + \lambda_2 = 0$
  - $-1 - \lambda_2 = 0$ (NOT POSSIBLE as $\lambda_2 > 0$)

Examples

minimize $f(x,y) = 2x - y$
s.t. $g_1(x,y) = x^2 + y^2 - 2 \leq 0$
$g_2(x,y) = x - y - 1 \leq 0$

- Case (III) $\lambda_1 > 0, \lambda_2 = 0$, we have
  - $2 + 2\lambda_1 x = 0$ (8)
  - $-1 + 2\lambda_1 y = 0$ (9)
  - $x^2 + y^2 - 2 = 0$ (10)
  - $x - y - 1 \leq 0$ (11)
- Solve this equation, we get $x = -1$ and $y = 1$, $\lambda_1 = 1$, $\lambda_2 = 0$. This solution satisfies (1) - (7) → $x = -1$, $y = 1$ is a global optimal solution to the problem (based on the KKT sufficient conditions)

Examples

minimize $f(x,y) = 2x - y$

s.t. $g_1(x,y) = x^2 + y^2 - 2 \le 0$

$g_2(x,y) = x - y - 1 \le 0$

- Case (IV) $\lambda_1 > 0, \lambda_2 > 0$, we have
  - $2 + 2\lambda_1 x + \lambda_2 = 0$ (12)
  - $-1 + 2\lambda_1 y - \lambda_2 = 0$ (13)
  - $x^2 + y^2 - 2 = 0$ (14)
  - $x - y - 1 = 0$ (15)
  - $\lambda_1, \lambda_2 \ge 0$ (16)

Examples

minimize $f(x,y) = 2x - y$
s.t. $g_1(x,y) = x^2 + y^2 - 2 \le 0$
$g_2(x,y) = x - y - 1 \le 0$

- Case (IV) $\lambda_1 > 0, \lambda_2 > 0$, we have
  - From (15), we have x = y + 1, replace in (12) and (13), we have
    - $2 + 2\lambda_1(y + 1) + \lambda_2 = 0$ (17)
    - $-1 + 2\lambda_1 y - \lambda_2 = 0$ (18)
  - Subtract (17) and (18), we have $3 + 2\lambda_1 + 2\lambda_2 = 0$ (NOT POSSIBLE as $\lambda_1 > 0, \lambda_2 > 0$)

Lagrange relaxation for Integer Programming

$$
\begin{array}{lll}
Z_{IP} = & \text{minimize } f(x) = & c^{\mathrm{T}}x \\
& \text{s.t.} & Ax \geq b \\
& & Dx \geq d \\
& & x \text{ integer}
\end{array}
$$

- Let $X = \{x \text{ integer} \mid Dx \geq d\}$
- Assume optimizing over X can be solved easily, but adding constraint $Ax \geq b$ makes the problem too difficult

Lagrange relaxation for Integer Programming

$$
\begin{array}{lll}
Z(\lambda) = & \min_x & c^T x + \lambda^T (b - Ax) \\
& \text{s.t.} & Dx \geq d \\
& & x \text{ integer}
\end{array}
$$

- For a fixed λ, $Z(\lambda)$ is assumed to be computed easily
- Important: compute the best lower bound

$$Z_D = \max_{\lambda \geq 0} Z(\lambda)$$

Lagrange relaxation for Integer Programming

- Subgradient method for computing Z_D

```
Choose starting point λ^(0) (e.g., λ^(0) = 0); k = 0
while (STOP condition not reach) {
    x^(k) is the solution of Z(λ^(0))
    Compute subgradient s^(k) = b - Ax^(k) of function Z at λ^(k)
    if s^(k) = 0 then BREAK
    λ^(k+1) = max{0, λ^(k) + α^(k)s^(k)}  /* α^(k) denote the step size */
    k = k + 1
}
```