Improper integrals

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Content

Last time: definite integrals $\int_a^b f(x)dx$ where $-\infty < a, b < +\infty$, $f(x)$ is bounded on $[a, b]$.
Extend the notion of integrals

- Infinite intervals (improper integrals of type 1)
- Functions with infinite discontinuity (improper integrals of type 2)

Content

Definition

Let $f(x)$ be defined on $[a, +\infty)$ and **integrable** on the interval $[a, A]$, $a \leq A < \infty$.

The improper integral of $f(x)$ on $[a, +\infty)$:

$$\int_a^{+\infty} f(x)dx = \lim_{A \to +\infty} \int_a^A f(x)dx.$$

If the limit exists (as a finite number), we say the integral converges. Otherwise, we say it diverges.

Example

Evaluate the improper integrals

$$a) \int_{-1}^{+\infty} \frac{dx}{1+x^2} \qquad b) \int_{0}^{+\infty} \sin x dx \qquad c) \int_{1}^{+\infty} \frac{dx}{x^\alpha}.$$

Remark

$$\int\limits_{a}^{+\infty} \frac{dx}{x^\alpha}, \ (a>0), \text{ converges} \Leftrightarrow \alpha > 1.$$

Definition

1. If $\int_{A'}^{a} f(x)dx$ exists for all $A' \leq a$, then:

$$\int_{-\infty}^{a} f(x)dx = \lim_{A' \to -\infty} \int_{A'}^{a} f(x)dx$$

if the limit exists as a finite number.

2. If both $\int_{-\infty}^{a} f(x)dx$ and $\int\limits_{a}^{+\infty} f(x)dx$ are convergent, then we define

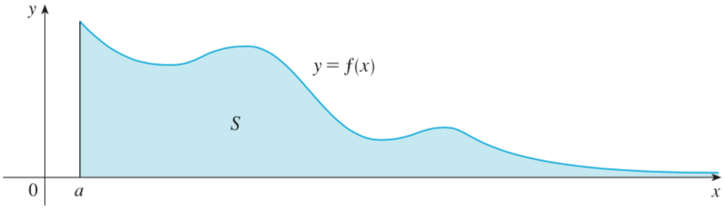
$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int\limits_{a}^{+\infty} f(x)dx.$$

Example

Evaluate the integrals

$$a) \int_{-\infty}^{+\infty} \frac{dx}{(x^2+1)^2} \qquad b) \int_{-\infty}^{0} e^{2x}(x+2)dx.$$

Geometric interpretation



$$\int_a^{+\infty} f(x)dx < \infty \Leftrightarrow S < \infty.$$

Comparison theorems

Assume that $f(x)$ and $g(x)$ are nonnegative continuous functions on $[a, +\infty)$.

Theorem (Comparison test 1)

If $\mathbf{f}(\mathbf{x}) \leq \mathbf{g}(\mathbf{x})$ *for* $x \geq a$*, then*

- *If* $\displaystyle\int_a^{+\infty} g(x)dx$ *converges, then* $\displaystyle\int_a^{+\infty} f(x)dx$ *also converges.*
- *If* $\displaystyle\int_a^{+\infty} f(x)dx$ *diverges, then* $\displaystyle\int_a^{+\infty} g(x)dx$ *also diverges.*

Theorem (Comparison test 2)

Assume that $\lim_{x\to+\infty} \frac{f(x)}{g(x)} = k \in (0,\infty)$*, then both* $\int_a^{+\infty} f(x)dx$ *and* $\int_a^{+\infty} g(x)dx$ *either converge or diverge.*

We also write, $f(x) \sim kg(x)$ as $x \to +\infty$.

Proposition

i) If $\displaystyle\lim_{x\to+\infty}\frac{f(x)}{g(x)} = 0$ *and* $\displaystyle\int_a^{+\infty} g(x)dx$ *converges, then* $\displaystyle\int_a^{+\infty} f(x)dx$ *converges.*

ii) If $\displaystyle\lim_{x\to+\infty}\frac{f(x)}{g(x)} = \infty$ *and* $\displaystyle\int_a^{+\infty} g(x)dx$ *diverges, then* $\displaystyle\int_a^{+\infty} f(x)dx$ *diverges.*

Example

Test for convergence

$$a)\ \int_{\pi}^{\infty} \frac{\sin^2 x}{x^2 + \ln x}dx \qquad b)\ \int_{1}^{+\infty} \frac{\ln(1+3x)}{\sqrt{x+1}}dx$$

$$c)\ \int_{2}^{+\infty} \frac{(1+x)\sqrt{x}}{x^3 - 2x + 1}dx \qquad d)\ \int_{1}^{+\infty} \left(e^{-\frac{4}{x^2}} - 1\right)dx$$

Absolute and conditional convergence

Definition

If $\displaystyle\int_a^{+\infty} |f(x)|dx$ **converges**, then we say $\displaystyle\int_a^{+\infty} f(x)dx$ converges absolutely.

Note: $\displaystyle\int_a^{+\infty} f(x)dx$ converges absolutely$\Rightarrow$ converges.

Definition

If $\displaystyle\int_a^{+\infty} |f(x)|dx$ **diverges** and $\displaystyle\int_a^{+\infty} f(x)dx$ **converges**, then we say $\displaystyle\int_a^{+\infty} f(x)dx$ converges conditionally.

Sign-changing functions

Example

Test for convergence

$$a)\int_2^{+\infty}\frac{x\cos 2x}{x^3+3}dx,\quad b)\int_{-\infty}^{+\infty}\frac{e^{-x^2}\cos x}{1+x^2}dx.$$

Example

Conditionally convergent integrals

$$a)\int_1^{+\infty}\frac{\sin x}{x}dx \qquad b)\int_1^{\infty}\cos x^2 dx.$$

Content

Definition

Let $f(x)$ be continuous on $[a, b)$ and $\lim_{x \to b^-} f(x) = \infty$. The improper integral of $f(x)$ on $[a, b)$:

$$\int_a^b f(x)dx = \lim_{A \to b^-} \int_a^A f(x)dx.$$

If the limit exists (as a finite number), we say the integral is convergent. Otherwise, we say it is divergent .

$x = b$: singular point of the integral.

Example

Evaluate the integrals

$$a)\int_0^2 \frac{dx}{\sqrt{4-x^2}} \qquad\qquad b)\int_0^1 \frac{dx}{(1-x)^\alpha}$$

Remark

$$\int_a^b \frac{dx}{(b-x)^\alpha} \text{ converges} \Leftrightarrow \alpha < 1.$$
$$\int_a^b \frac{dx}{(x-a)^\alpha} \text{ converges} \Leftrightarrow \alpha < 1.$$

Definition

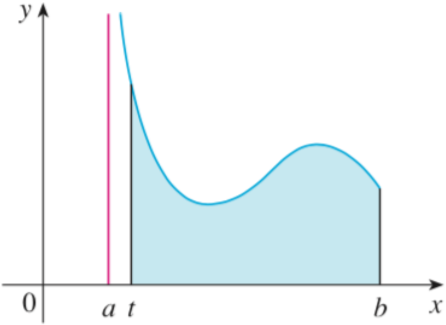
Let $f(x)$ be continuous on $(a, b]$ and $\lim\limits_{x\to a^+} f(x) = \infty$. The improper integral of $f(x)$ on $(a, b]$:

$$\int_a^b f(x)dx = \lim_{A\to a^+} \int_A^b f(x)dx.$$

Let $f(x)$ be continuous on $[a, b] \setminus \{c\}$ and c is an infinite discontinuity. If both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then the improper integral of $f(x)$ on $(a, b]$:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Geometric interpretation



$$\int_a^b f(x)dx < \infty \Leftrightarrow S < \infty.$$

Nonnegative functions

Assume $f(x)$ and $g(x)$ are nonnegative continuous functions on $[a, b)$ and $\int_a^b f(x)dx$, $\int_a^b g(x)dx$ have **unique singular point** $x = b$.

Theorem

If $0 \leq f(x) \leq g(x)$ for $x \in [a, b)$, then

- *If $\int_a^b g(x)dx$ converges, then $\int_a^b f(x)dx$ also converges.*
- *If $\int_a^b f(x)dx$ diverges, then $\int_a^b g(x)dx$ also diverges.*

Theorem

If $\lim\limits_{x\to b^-} \dfrac{f(x)}{g(x)} = k$, $0 < k < \infty$, *then both* $\int_a^b f(x)dx$ *and* $\int_a^b g(x)dx$ *are either convergent or divergent.*

Proposition

- *If* $\lim\limits_{x\to b^-} \dfrac{f(x)}{g(x)} = 0$ *and* $\int_a^b g(x)dx$ *converges* $\int_a^b f(x)dx$ *converges.*
- *If* $\lim\limits_{x\to b^-} \dfrac{f(x)}{g(x)} = \infty$ *and* $\int_a^b g(x)dx$ *diverges* $\int_a^b f(x)dx$ *diverges.*

Example

Test for convergence

$$a)\ \int_0^1 \frac{\ln(1+\sqrt{x})}{e^{\tan x}-1}dx \qquad b)\ \int_0^1 \frac{dx}{\sqrt[3]{x^2(1-x)}}$$

$$c)\ \int_0^{\infty} \frac{dx}{\sqrt{x+2x^3}}$$

Absolute and conditional convergence

Definition

If $\int_a^b |f(x)|dx$ **converges**, then we say $\int_a^b f(x)dx$ converges absolutely.

Note: If $\int_a^b f(x)dx$ converges absolutely, then it also converges.

Definition

If $\int_a^b |f(x)|dx$ **diverges** and $\int_a^b f(x)dx$ **converges** then we say $\int_a^b f(x)dx$ converges conditionally.

Example

Test for convergence $\displaystyle\int_0^\infty \frac{\sin 2x}{x\sqrt{x}}dx$.