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A COURSE IN CALCULUS 2

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Chapter 1
Vector and Geometry of Space

1.1 VECTORS

In this chapter we introduce vectors and coordinate system for three-dimensional space. We will see that vectors provide simple descriptions of lines and planes in space.

We first choose in space a fixed point *O* (the **origin**) and three directed lines through *O* that are perpendicular to each other, called the **coordinate axes**, labeled the *x*-axis, *y*-axis, and *z*-axis. The direction of z-axis is determined by the **right-hand-rule**: If you curl the fingers of your right hand around the z-axis in the direction of a 90^o counterclockwise rotation from the positive *x*-axis to the positive *y*-axis, then your thumb points the positive direction of the *z*-axis. The three coordinate planes divide space into 8 **octants.** There is a one-to-one correspondence between any point *P* in space and a triple (x_P, y_P, z_P) of real numbers, the **coordinates** of *P*. See Figure 1.1.1

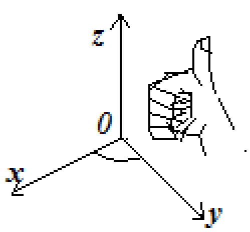


Figure 1.1.1: The coordinate axes

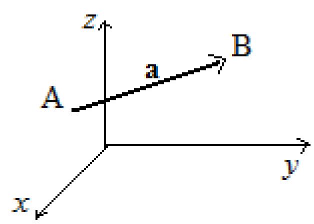


Figure 1.1.2: Vector **a** = $\overrightarrow{AB}$

A. VECTOR

The term **vector** is used to indicate a quantity that has both *magnitude* and *direction* (velocity, force,...). A vector is used to be denoted by $\overrightarrow{AB}$ or **a** ... See Figure 1.1.2.

DEFINITION 1.1.1 VECTOR AND COMPONENTS
A three-dimensional **vector** is an ordered triple $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ of real numbers. The umbers a_1, a_2, a_3 are called the **components** of **a.**

PROPERTY 1.1.1

a. Given $A(x_A, y_A, z_A)$, $B(x_B, y_B, z_B)$, then

$$\overrightarrow{AB} = \langle x_B - x_A,\ y_B - y_A,\ z_B - z_A \rangle \qquad (1.1.1)$$

b. The length of the vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2} \qquad (1.1.2)$$

DEFINITION 1.1.2 VECTOR ADDITION

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $b = \langle b_1, b_2, b_3 \rangle$, then $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1,\ a_2 + b_2,\ a_3 + b_3 \rangle$

DEFINITION 1.1.3 MULTIPLICATION A VECTOR BY A SCALAR

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, and c is a scalar (number), then $c\mathbf{a} = \langle ca_1, ca_2, ca_3 \rangle$

PROPERTY 1.1.2

If **a**, **b**, and **c** are vectors in space, k, l are scalars (numbers), then

a. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$

b. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$

c. $\mathbf{a} + \mathbf{0} = \mathbf{a}$, where **0** is the **zero vector**

d. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$

e. $k(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b}$

f. $(k + l)\mathbf{a} = k\mathbf{a} + l\mathbf{a}$

g. $(kl)\mathbf{a} = k(l\mathbf{a})$

h. $1\mathbf{a} = \mathbf{a}$

DEFINITION 1.1.4 STANDARD BASIC VECTORS

The vectors $\mathbf{i} = \langle 1,0,0 \rangle$, $\mathbf{j} = \langle 0,1,0 \rangle$, $\mathbf{k} = \langle 0,0,1 \rangle$ are called the **standard basic vectors**

PROPERTY 1.1.3

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, then $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

EXAMPLE 1.1.1

a. Given $A(-1, 2, -2)$, $B(2, -3, 0)$, find $\mathbf{a} = \overrightarrow{AB}$, and $|\mathbf{a}|$

Solution: $\mathbf{a} = \overrightarrow{AB} = \langle 3, -5, 2 \rangle$, $|\mathbf{a}| = \overrightarrow{AB} = \sqrt{3^2 + (-5)^2 + 2^2} = \sqrt{38}$.

b. A 100-kG weight hangs from two wires as shown in Figure 1.1.3. Find the tensions (forces) T_1 and T_2 in both wires and their magnitudes.

Figure 1.1.3:

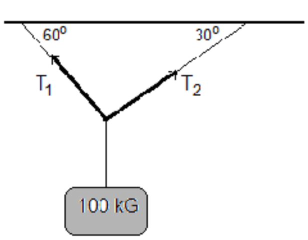
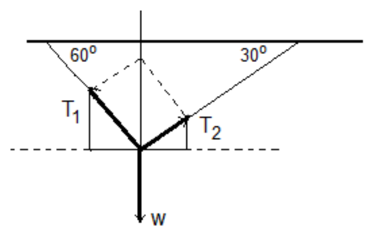


Figure 1.1.4:



Solution:

Look at *Figure* 1.1.4. Express first the forces in terms of their components, then equate the total components to zero (balance the forces).

$$\mathbf{T}_1 = (-|\mathbf{T}_1|\cos 60^0)\,\mathbf{i} + (|\mathbf{T}_1|\cos 30^0)\,\mathbf{j} = (-\tfrac{1}{2}|\mathbf{T}_1|)\,\mathbf{i} + (\tfrac{\sqrt{3}}{2}|\mathbf{T}_1|)\,\mathbf{j},$$

$$\mathbf{T}_2 = (|\mathbf{T}_2|\cos 30^0)\,\mathbf{i} + (|\mathbf{T}_2|\cos 60^0)\,\mathbf{j} = (\tfrac{\sqrt{3}}{2}|\mathbf{T}_2|)\,\mathbf{i} + (\tfrac{1}{2}|\mathbf{T}_2|)\,\mathbf{j},$$

$\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{W} = \mathbf{0} \quad \Rightarrow \quad \mathbf{T}_1 = -25\sqrt{3}\,\mathbf{i} + 75\,\mathbf{j}$ and $\mathbf{T}_2 = 25\sqrt{3}\,\mathbf{i} + 25\mathbf{j}$

B. DOT PRODUCT

DEFINITION 1.1.5 DOT PRODUCT

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the dot product of **a** and **b** is the number $\mathbf{a} \cdot \mathbf{b}$ given by

$$\boxed{\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3} \tag{1.1.3}$$

PROPERTY 1.1.4

If **a**, **b**, and **c** are vectors in space and k is a constant, then

a. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$

b. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

c. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

d. $(k\mathbf{a}) \cdot \mathbf{b} = k(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (k\mathbf{b})$

e. $\mathbf{0} \cdot \mathbf{a} = 0$

THEOREM 1.1.1

a. If θ, $0 \le \theta \le \pi$, is the angle between **a**, **b**, then

$$\boxed{\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta} \tag{1.1.4}$$

$$\boxed{\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}} \tag{1.1.5}$$

b. Vectors **a** and **b** are orthogonal if and only if

$$\mathbf{a} \cdot \mathbf{b} = 0 \qquad \text{i.e.} \qquad a_1 b_1 + a_2 b_2 + a_3 b_3 = 0 \tag{1.1.6}$$

Proof

Apply the Law of Cosines to a triangle:

$$|\mathbf{b} - \mathbf{a}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta.$$

On the other hand:

$$|\mathbf{b} - \mathbf{a}|^2 = (\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a}) = \mathbf{b} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{b} - 2\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\mathbf{a} \cdot \mathbf{b}.$$

Therefore we obtain (1.1.4).

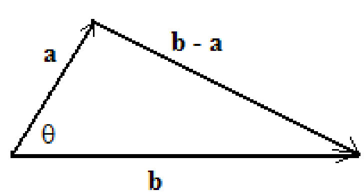


Figure 1.1.5

DEFINITION 1.1.6 DIRECTION ANGLES AND DIRECTION COSINES

a. The **direction angles** of a nonzero vector **a** are the angles α, β, and γ in the interval $[0,\pi]$ that the vector **a** makes with the positive *x*- , *y*- , and *z*- axes.
b. The cosines of these direction angles, $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are called the **direction cosines** of the vector **a**.

It follows from (1.1.5) that

$$\cos\alpha = \frac{\mathbf{a}\cdot\mathbf{i}}{|\mathbf{a}||\mathbf{i}|} = \frac{a_1}{|\mathbf{a}|}\,;\ \cos\beta = \frac{\mathbf{a}\cdot\mathbf{j}}{|\mathbf{a}||\mathbf{j}|} = \frac{a_2}{|\mathbf{a}|}\,;\ \cos\gamma = \frac{\mathbf{a}\cdot\mathbf{k}}{|\mathbf{a}||\mathbf{k}|} = \frac{a_3}{|\mathbf{a}|}\,; \qquad (1.1.7)$$

$$a_1 = |\mathbf{a}|\cos\alpha; \qquad a_2 = |\mathbf{a}|\cos\beta; \qquad a_3 = |\mathbf{a}|\cos\gamma; \qquad (1.1.8)$$

Therefore
$$\mathbf{a} = |\mathbf{a}|\langle\cos\alpha,\ \cos\beta,\ \cos\gamma\rangle \qquad (1.1.9)$$

The **unit vector** of **a** :
$$\frac{\mathbf{a}}{|\mathbf{a}|} = \langle\cos\alpha,\ \cos\beta,\ \cos\gamma\rangle \qquad (1.1.10)$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \qquad (1.1.11)$$

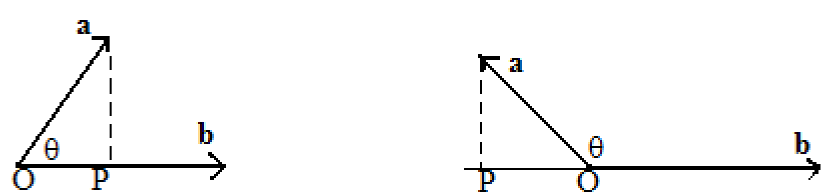
DEFINITION 1.1.7 SCALAR PROJECTION AND VECTOR PROJECTION

a. The **scalar projection** of **a** onto **b** (also called the *component of* **a** *along* **b**):

$$\boxed{\overline{\mathrm{OP}} = \mathrm{comp}_{\mathbf{b}}\mathbf{a} = |\mathbf{a}|\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}} \qquad (1.1.12)$$

b. The **vector projection** of **a** onto **b**:

$$\boxed{\overrightarrow{OP} = \mathrm{proj}_{\mathbf{b}}\,\mathbf{a} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}\right)\frac{\mathbf{b}}{|\mathbf{b}|} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b}} \qquad (1.1.13)$$



$\overline{\mathrm{OP}} > 0$, $\overrightarrow{OP} = k\mathbf{b}, k > 0$ $\qquad$ $\overline{\mathrm{OP}} < 0$, $\overrightarrow{OP} = k\mathbf{b}, k < 0$

Figure 1.1.6: Projection

EXAMPLE 1.1.2

a. Find the scalar projection of $\mathbf{a} = \langle 1,-2,-3\rangle$ onto $\mathbf{b} = \langle -2,3,-1\rangle$.

b. The force (in newtons) $\mathbf{F} = \langle 3,2,5\rangle$ moves a particle from the point $A(2, 1, 4)$ to the point $B(6, 3, 5)$, (the length unit is meter). Find the work done.

c. Find the direction cosines of $\mathbf{a} = \langle 2,3,1\rangle$

c. Find the angle between $\mathbf{a} = \langle -1, 3, -2 \rangle$ and $\mathbf{b} = \langle 1, -2, -1 \rangle$.

d. Find the unit vector of $\mathbf{a} = \langle 2, 4, -4 \rangle$.

e. Are $\mathbf{a} = \langle -1, 3, -7 \rangle$ and $\mathbf{b} = \langle 1, -2, -1 \rangle$ orthogonal?

f. Let $\mathbf{a} = \langle -1, 4, 3 \rangle$ and $\mathbf{b} = \langle 1, 1, 2 \rangle$. Find the scalar and vector projections of **b** onto **a** and of **a** onto **b**.

Answer: $\text{comp}_{\mathbf{b}}\mathbf{a} = 9/\sqrt{6}$; $\text{proj}_{\mathbf{b}}\mathbf{a} = \left\langle \frac{9}{6}, \frac{9}{6}, \frac{18}{6} \right\rangle$; $\text{comp}_{\mathbf{a}}\mathbf{b} = 9/\sqrt{26}$; $\text{proj}_{\mathbf{a}}\mathbf{b} = \left\langle \frac{-9}{26}, \frac{36}{26}, \frac{27}{26} \right\rangle$

C. CROSS PRODUCT

DEFINITION 1.1.8 CROSS PRODUCT

The **cross product** of **a** and **b** is a vector, denoted by $\mathbf{a}\times\mathbf{b}$, that satisfies

(i) $\mathbf{a}\times\mathbf{b}$ is orthogonal to both vectors **a** and **b**.

(ii) The direction of $\mathbf{a}\times\mathbf{b}$ is given by the *right-hand rule* (the fingers of the right hand curl in the direction of a rotation through an angle less than 180^{o} from **a** to **b**, then the thumb points the direction of $\mathbf{a}\times\mathbf{b}$)

(iii) $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$, where θ, $0 \le \theta \le \pi$, is the angle between **a**, **b** (1.1.14)

Note that the condition (1.1.14) means that the magnitude (length) of $\mathbf{a}\times\mathbf{b}$ is equal to **the area** of the parallelogram determined by **a** and **b**. See Figure 1.1.7

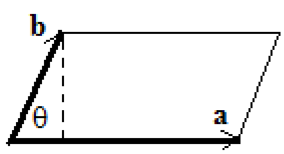


Figure 1.1.7

PROPERTIES 1.1.5 CROSS PRODUCT

a. $\mathbf{i}\times\mathbf{j} = \mathbf{k}$, $\mathbf{j}\times\mathbf{k} = \mathbf{i}$, $\mathbf{k}\times\mathbf{i} = \mathbf{j}$

b. $\mathbf{a}\times\mathbf{b} = \mathbf{0}$ if and only if a and b are parallel.

c. $\mathbf{b}\times\mathbf{a} = -(\mathbf{a}\times\mathbf{b})$

d. $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = (\mathbf{a}\times\mathbf{b}) + (\mathbf{a}\times\mathbf{c})$

e. $(k\mathbf{a})\times\mathbf{b} = k(\mathbf{a}\times\mathbf{b})$

f. $(k\mathbf{a})\times(l\mathbf{b}) = (kl)(\mathbf{a}\times\mathbf{b})$, where k and l are numbers

From these properties, it is easy to obtain the following component expression for the cross product of two vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$.

<u>THEOREM 1.1.2</u> MATRIX EXPRESSION OF THE CROSS PRODUCT

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **cross product** of **a** and **b** is determined by

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}$$

$$= (a_2b_3 - a_3b_2)\,\mathbf{i} + (a_3b_1 - a_1b_3)\,\mathbf{j} + (a_1b_2 - a_2b_1)\,\mathbf{k}$$

$$= \langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1 \rangle \tag{1.1.15}$$

Proof

$\mathbf{a}\times\mathbf{b} = \langle a_1, a_2, a_3 \rangle \times \langle b_1, b_2, b_3 \rangle = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})\times(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$. Apply properties 1.1.5.

EXAMPLE 1.1.3

a. Given $\mathbf{a} = \langle 1, 3, -2 \rangle$, $\mathbf{b} = \langle -2, 4, 1 \rangle$, find $\mathbf{a}\times\mathbf{b}$, $|\mathbf{a}\times\mathbf{b}|$.

Answer: $\langle 11, 3, 10 \rangle, \sqrt{230}$.

b. Find the area of the triangle *ABC*, *A*(2,8,12), *B*(4,5,8), *C*(1,4,10).

Answer: $\sqrt{285}/2$.

c. Find the height *AH* of the triangle *ABC*, *A*(1,6,4), *B*(2,5,8), *C*(-1,4,0).

Answer: $AH = \dfrac{|\overrightarrow{BA}\times\overrightarrow{BC}|}{|\overrightarrow{BC}|} = \sqrt{\dfrac{176}{74}} = \sqrt{\dfrac{88}{37}}$

d. Prove that

$$\boxed{\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}} \tag{1.1.16}$$

Hint: Apply (1.1.3) and (1.1.15).

D. SCALAR TRIPLE PRODUCT

DEFINITION 1.1.9 SCALAR TRIPLE PRODUCT

The **scalar triple product** of three vectors **a**, **b**, and **c,** denoted by $(\mathbf{a}, \mathbf{b}, \mathbf{c})$, is a number that is defined by the scalar product of **a** and $\mathbf{a}\times\mathbf{b}$:

$$(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) \tag{1.1.17}$$

<u>THEOREM 1.1.3</u> EXPRESSION OF THE SCALAR TRIPLE PRODUCT

a. Let $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, and $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$.

Then the scalar triple product of three vectors **a**, **b**, and **c** is determined by the determinant

$$(\mathbf{a},\mathbf{b},\mathbf{c}) = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \tag{1.1.18}$$

b. $(\mathbf{b},\mathbf{a},\mathbf{c}) = -(\mathbf{a},\mathbf{b},\mathbf{c})$, i.e., $\mathbf{b}.(\mathbf{a}\times\mathbf{c}) = -\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ (1.1.19)

c. $(\mathbf{a},\mathbf{b},\mathbf{c}) = (\mathbf{b},\mathbf{c},\mathbf{a}) = (\mathbf{c},\mathbf{a},\mathbf{b})$, i.e, $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$ (1.1.20)

Proof

The conclusion a. follows directly from (1.1.3), (1.1.15), and (1.1.16). The conclusions b. and c. follow from the determinant properties.

EXAMPLE 1.1.4

Let $\mathbf{a} = \langle 2,-1,-2\rangle$, $\mathbf{b} = \langle -1,3,1\rangle$, and $\mathbf{c} = \langle -3,2,2\rangle$. Find $(\mathbf{a},\mathbf{b},\mathbf{c})$, $(\mathbf{b},\mathbf{c},\mathbf{a})$, $(\mathbf{b},\mathbf{a},\mathbf{c})$.
Answer: -5, -5, 5. These results justify (1.1.18) and (1.1.19).

PROPERTIES 1.1.6 SCALAR TRIPLE PRODUCT

a. The volume of the parallelepiped determined by the vectors **a**, **b**, and **c** is the magnitude (absolute value) of their scalar triple product:

$$\boxed{V = |(\mathbf{a},\mathbf{b},\mathbf{c})|} \tag{1.1.21}$$

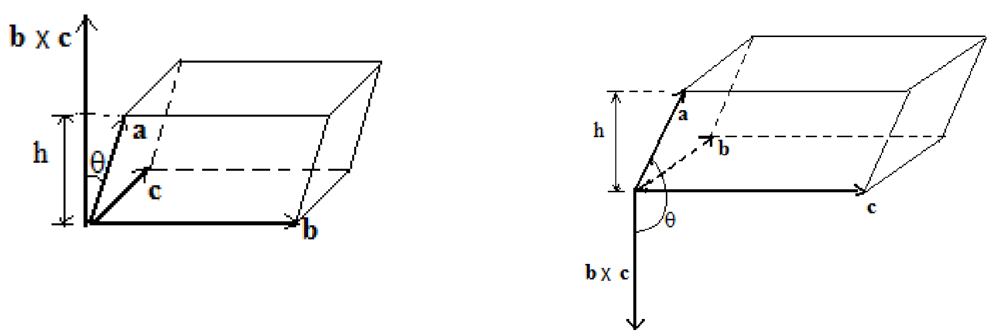
b. The vectors **a**, **b**, and **c** are **coplanar** if and only if

$$(\mathbf{a},\mathbf{b},\mathbf{c}) = 0 \tag{1.1.22}$$

Proof

a. (1.1.20) follows directly from (1.1.4) and (1.1.14):
$(\mathbf{a},\mathbf{b},\mathbf{c}) = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = |\mathbf{a}||(\mathbf{b}\times\mathbf{c})|\cos\theta$, where θ, $0 \le \theta \le \pi$, is the angle between **a** and $\mathbf{b}\times\mathbf{c}$. The length $|\mathbf{b}\times\mathbf{c}|$ is equal to the area of the parallelogram determined by **b** and **c**; the height h of the parallelepiped is the magnitude of $|\mathbf{a}|\cos\theta$. Figure 1.1.8 shows that when $0 \le \theta \le \pi/2$: $|\mathbf{a}|\cos\theta \ge 0 \Rightarrow (\mathbf{a},\mathbf{b},\mathbf{c}) \ge 0$ and when $\pi/2 < \theta \le \pi$: $|\mathbf{a}|\cos\theta < 0 \Rightarrow (\mathbf{a},\mathbf{b},\mathbf{c}) < 0$.

c. It is the consequence of a.



a) $0 \le \theta \le \pi/2$: $(\mathbf{a},\,\mathbf{b},\,\mathbf{c}) = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) \ge 0$ b) $\pi/2 < \theta \le \pi$: $(\mathbf{a},\,\mathbf{b},\,\mathbf{c}) = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) < 0$

Figure 1.1.8: The parallelepiped determined by the vectors **a**, **b**, and **c**

EXAMPLE 1.1.5

Let $\mathbf{a} = \langle 3, -3, -4 \rangle$, $\mathbf{b} = \langle 1, -3, -1 \rangle$, and $\mathbf{c} = \langle k, -2, 4 \rangle$.

a. Find the volume of the parallelepiped determined by **a**, **b**, and **c**, if $k = 5$.

b. Determine k so that **a**, **b**, and **c** are coplanar.

c. Determine whether the points $A(1, 0, 1)$, $B(2, 4, 6)$, $C(3, -1, 2)$, and $D(6, 2, 8)$ lie in the same plane.

Answer: a. 67; b. -22/9; c. Yes.

1.2 EQUATIONS OF LINES AND PLANES

A. EQUATIONS OF LINES

A line $\mathscr{L}$ is determined by a point $P_0(x_0, y_0, z_0)$ on it and its **direction vector** $\mathbf{v} = \langle a, b, c \rangle$. Let $P(x, y, z)$ be an arbitrary point on $\mathscr{L}$. Let $\mathbf{r} = \overrightarrow{OP} = \langle x, y, z \rangle$, and $\mathbf{r}_0 = \overrightarrow{OP_0} = \langle x_0, y_0, z_0 \rangle$ be the **position vectors** of P and P_0. Vector $\overrightarrow{P_0P} = \mathbf{r} - \mathbf{r}_0$ is parallel to **v**. Look at Figure 1.2.1.

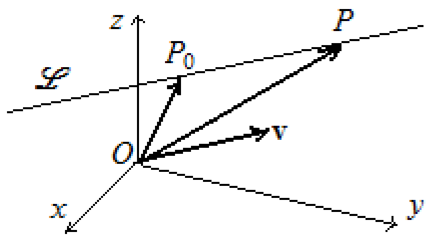


Figure 1.2.1

The **vector equations** of $\mathscr{L}$:

$$\boxed{\mathbf{r} = \mathbf{r}_0 + t\,\mathbf{v}} \tag{1.2.1}$$

The **parametric equations** of $\mathscr{L}$:

$$\boxed{x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct} \tag{1.2.2}$$

(t is the **parameter**, $t \in \mathbb{R}$)

The **symmetric equations** of $\mathscr{L}$:

$$\boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}} \tag{1.2.3}$$

If one of a, b, or $c = 0$, these equations become, for instance if $a = 0$:
(The line lies in the plane $x = x_0$ that is parallel to the xy-plane)

$$\boxed{x = x_0 \quad \frac{y - y_0}{b} = \frac{z - z_0}{c}} \tag{1.2.4}$$

If two among *a*, *b*, or *c* = 0, these equations become, for instance if *a* = 0 and *b* = 0:

$$\boxed{x = x_0 \quad y = y_0} \tag{1.2.5}$$

EXAMPLE 1.2.1

a. Find the vector equation and the parametric equations for the line that passes through the point (3, 2, -1) and is parallel to the vector $\mathbf{v} = \langle -2, -5, 1 \rangle$.
At what points does the line passes through the *xy*-plane?

Solution:

The vector equation:

$$\mathbf{r} = \langle 3-2t, 2-5t, -1+t \rangle = (3-2t)\mathbf{i} + (2-5t)\mathbf{j} + (-1+t)\mathbf{k}$$

The parametric equations:

$$x = 3-2t, \; y = 2-5t, \; z = -1+t.$$

Put $z = 0 \Rightarrow t = 1 \Rightarrow x = 1, y = -3, z = 0 \Rightarrow$

the line passes through the *xy*-plane at the point *P*(1, -3, 0).

b. Find the vector equation, the parametric equation, and the symmetric equation for the line that passes through the points *A*(4, 3, 6), *B*(2, -5, 6). Is this line parallel to the line in question a?

B. EQUATIONS OF PLANES

A plane $\mathscr{P}$ is determined by a point $P_0(x_0, y_0, z_0)$ in it and its **normal vector** $\mathbf{n} = \langle a, b, c \rangle$. Let $P(x, y, z)$ be an arbitrary point in $\mathscr{P}$. Let $\mathbf{r} = \overrightarrow{OP} = \langle x, y, z \rangle$, and $\mathbf{r}_0 = \overrightarrow{OP_0} = \langle x_0, y_0, z_0 \rangle$ be the **position vectors** of *P* and P_0. Vector $\overrightarrow{P_0P} = \mathbf{r} - \mathbf{r}_0$ is orthogonal to $\mathbf{n} = \langle a, b, c \rangle$ and so we have

The **vector equations** of $\mathscr{P}$: $\boxed{\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0 \quad \text{or} \quad \mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0}$ (1.2.6)

The **scalar equation** of $\mathscr{P}$: $\boxed{a(x-x_0) + b(y-y_0) + c(z-z_0) = 0}$ (1.2.7)

or : $\boxed{ax + by + cz + d = 0}$ (1.2.8)

where $d = -(ax_0 + by_0 + cz_0)$

EXAMPLE 1.2.2

a. Find an equation of the plane that passes through *A*(1, 3, -4) and is orthogonal to $\overrightarrow{BC}$, where *B*(-1, -3, 1), *C*(3, 4, -2).

Solution:

$\mathbf{n} = \overrightarrow{BC} = \langle 4, 7, -3 \rangle \Rightarrow 4(x+1) + 7(y+3) - 3(z-1) = 0$ or $4x + 7y - 3z + 28 = 0$

b. Find an equation of the plane $\mathscr{P}$ that passes through points $A(2, 1, -2)$, $B(-2, -1, 2)$, and $C(3, 4, 1)$. Show that $\mathscr{P}$ passes through the origin.

Solution:

Let $P(x, y, z)$ be an arbitrary point in the plane. Four points A, B, C, and P are coplanar, or three vectors $\overrightarrow{BA}, \overrightarrow{AC},$ and $\overrightarrow{AP}$ are coplanar. It follows from (1.1.21) that the equation of $\mathscr{P}$ is $9x - 8y + 5z = 0$. The coordinates of the origin $O(0, 0, 0)$ satisfy the equation of $\mathscr{P}$. It means that $\mathscr{P}$ passes through the origin O.

c. Find the point P at which the line $x = 4 - 2t,\ y = 2 - 3t,\ z = -1 + 2t$ intersects the plane $8x + 6y - z + 9 = 0$.

Answer: $P(1, -5/2, 2)$.

d. Find the formula for the distance d from a point $P_1(x_1, y_1, z_1)$ to the plane $\mathscr{P}$ $ax + by + cz + d = 0$.

Solution:

Let $P_0(x_0, y_0, z_0)$ be a point in the plane $\mathscr{P}$. The normal vector of $\mathscr{P}$ is $\mathbf{n} = \langle a, b, c \rangle$. The distance d is equal to the magnitude of the scalar projection of $\overrightarrow{P_0P_1} = \langle x_1 - x_0,\ y_1 - y_0,\ z_1 - z_0 \rangle$ onto $\mathbf{n} = \langle a, b, c \rangle$:

$$d = \left| comp_{\mathbf{n}} \overrightarrow{P_0P} \right| = \frac{|\mathbf{n} \cdot \overrightarrow{P_0P}|}{|\mathbf{n}|} = \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}}.$$

Since, $P_0 \in \mathscr{P} \Rightarrow ax_0 + by_0 + cz_0 + d = 0$, then

$$\boxed{d = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}} \tag{1.2.9}$$

1.3 CYLINDERS AND QUADRATIC SURFACES

DEFINITION 1.3.1 CYLINDERS

A **cylinder** is a surface that consists of all lines (called **rulings**) that are parallel to a given line and pass through a given plane curve.

By rotation it can be supposed that the rulings are parallel to one axis.

EXAMPLE 1.3.1

a. The surface $z = x^2$ (the equation does not involve y) is a **parabolic cylinder.** The rulings are parallel to the y-axis and pass through the parabola $z = x^2$ in the xz-plane

b. The surface $x^2 + y^2 = 4$ (the equation does not involve z) is a **circular cylinder.** The rulings are parallel to the z-axis and pass through the circle $x^2 + y^2 = 4$ in the xy-plane

c. Determine the surface $z^2 + 4x^2 = 1$

DEFINITION 1.3.2 QUADRATIC SURFACES

A **quadratic surface** is the graph of a second-degree equation

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0 .$$

By translation and rotation it can be brought into one of two standard forms

$$Ax^2 + By^2 + Cz^2 + J = 0 \quad \text{or} \quad Ax^2 + By^2 + Iz = 0$$

EXAMPLE 1.3.2

a. **Ellipsoid** $\boxed{\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1}$ (1.3.1)

b. **Elliptic paraboloid** $\boxed{\dfrac{z}{c} = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}}$ (1.3.2)

Circular paraboloid: If $a = b$

b. **Hyperbolic paraboloid** $\boxed{\dfrac{z}{c} = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}}$ (1.3.3)

d. **Hyperboloid of one sheet** $\boxed{\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1}$ (1.3.4)

e. **Hyperboloid of two sheets** $\boxed{\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = -1}$ (1.3.5)

f. **Cone** $\boxed{\dfrac{z^2}{c^2} = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}}$ (1.3.6)

1.4 CYLINDRICAL AND SPHERICAL COORDINATES

A. CYLINDRICAL COORDINATES

DEFINITION 1.4.1 CYLINDRICAL COORDINATES

A point $P(x, y, z)$ in the space can be represented by the ordered triple (r, θ, z), where (r, θ) are polar coordinates of the projection of P onto the xy-plane. The ordered triple (r, θ, z) is called the **cylindrical coordinates** of P.

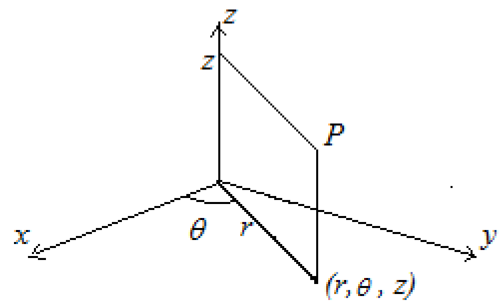


Figure 1.4.1 Cylindrical coordinates

To convert from cylindrical to rectangular coordinates we use the equations:

$$\boxed{x = r\cos\theta, \quad y = r\sin\theta, \quad z = z} \tag{1.4.1}$$

To convert from rectangular to cylindrical coordinates we use the equations:

$$\boxed{r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}, \quad z = z} \tag{1.4.2}$$

EXAMPLE 1.4.1

a. Equation $z = kr$ represents a cone $z^2 = k^2(x^2 + y^2)$.

b. Equation $r = k$ represents a cylinder $x^2 + y^2 = k^2$.

c. Equation $\theta = k$ represents a plane $y = x\tan\theta$.

d. Find the rectangular coordinates of the point with cylindrical coordinates $(3, 2\pi/3, 5)$

Answer: $(-3/2, 3\sqrt{3}/2, 5)$

B. SPHERICAL COORDINATES

DEFINITION 1.4.1 SPHERICAL COORDINATES

A point $P(x, y, z)$ in the space can be represented by the ordered triple (ρ, θ, ϕ), where θ is the same angle as in cylindrical coordinates, ϕ, $0 \le \phi \le \pi$, is the angle between the positive z-axis and $\overrightarrow{OP}$, and ρ, $\rho \ge 0$, s the distance from the origin O to P. The ordered triple (ρ, θ, ϕ) is called **spherical coordinates** of P.

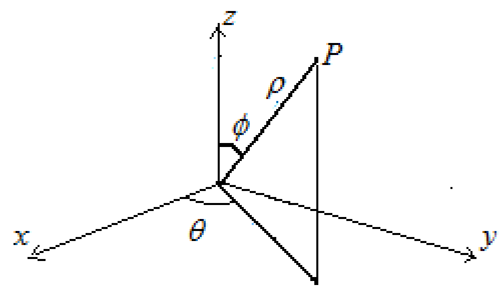


Figure 1.4.2 Spherical coordinates

To convert from spherical to rectangular coordinates we use the equations:

$$\boxed{x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi} \tag{1.4.3}$$

To convert from rectangular to spherical coordinates we use the equations:

$$\boxed{\rho^2 = x^2 + y^2 + z^2, \quad \tan\theta = \frac{y}{x}, \quad \cos\phi = \frac{z}{\rho}} \tag{1.4.4}$$

EXAMPLE 1.4.2

a. Equation $\rho = k,\ k > 0$, represents a sphere $x^2 + y^2 + z^2 = k^2$.

b. Equation $\theta = k,\ 0 \le k \le 2\pi,$ represents a plane $y = x\tan\theta$.

c. What does the equation $\phi = k,\ 0 < k < \pi,$ represent?

d. What does the equation $\rho\sin\phi = k,\ 0 < k < \pi,$ represent?

d. Find the rectangular coordinates of the point with spherical coordinates $(4,\ \pi/3,\ \pi/6)$

Answer: $\left(1,\ \sqrt{3},\ 2\sqrt{3}\right)$

EXERCISES

1.1 Find the angle between a diagonal of a cube and a diagonal of one of its faces.

1.2 Find the angle between a diagonal of a cube and one of its edges.

1.3 Prove

a. The Cauchy-Schwarz Inequality: $|\mathbf{a}\cdot\mathbf{b}| \le |\mathbf{a}||\mathbf{b}|$

b. The Triangle Inequality for vectors: $|\mathbf{a}+\mathbf{b}| \le |\mathbf{a}| + |\mathbf{b}|$

c. The Parallelogram Law: $|\mathbf{a}+\mathbf{b}|^2 + |\mathbf{a}-\mathbf{b}|^2 = 2\left(|\mathbf{a}|^2 + |\mathbf{b}|^2\right)$

1.4 Suppose that **a**, **b**, and **c** are all nonzero vectors such that $\mathbf{c} = |\mathbf{a}|\mathbf{b} + |\mathbf{b}|\mathbf{a}$. Show that **c** bisects the angle between **a** and **b**.

1.5 Given the points $A(1, 0, 0)$, $B(0, 1, 0)$, $C(0, 0, 1)$, $S(1, 1, 1)$, and $H(½, ½, ½)$. Show that *SABC* is a *regular tetrahedron* and H is its center.

1.6 A molecule of methane, CH_4, is structured with the four hydrogen atoms at the vertices of a regular tetrahedron and the carbon atom at the center. Show that the angle between the lines that joint the carbon atom to two of the hydrogen atoms is about 109.5^{o}.

1.7 Let *P* be a point not on the line *L* that passes through the points *A* and *B*. Show that the distance from *P* to *L* is

$$d = \frac{|\overrightarrow{AP}\times\overrightarrow{AB}|}{|\overrightarrow{AB}|}\ .$$

Calculate the distance, if $P(1,1,1)$, $A(0,6,8)$, $B(-1,4,7)$.

1.8 Let D be a point not on the plane P that passes through the points A, B, and C. Show that the distance from D to P is $d = \dfrac{\left|\overrightarrow{AD}\cdot\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)\right|}{\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|}$

Calculate this distance, if $D(1,2,2)$, $A(0,4,-1)$, $B(-1,2,3)$, $C(-3,2,5)$.

1.9 Given $M(0, 1, -2)$, $N(1, 3, 4)$, $A(-2, 4, -1)$, $B(-3, 1, 3)$, and $C(3, 2, 5)$.

a. Find the equation of the plane $\mathscr{P}$ that contains A, B, and C.

b. Find the equation of the line that goes through M and is perpendicular to $\mathscr{P}$.

c. Find the equation of the line $\mathscr{L}_1$ that goes through A and B.

d. Find the equation of the line $\mathscr{L}_2$ that goes through M and is parallel to $\mathscr{L}_1$.

e. Determine the distance from M to $\mathscr{P}$

f. Determine the distance between $\mathscr{L}_1$ and $\mathscr{L}_2$.

g. Find the volume of the parallelepiped determined from M, A, B, and C

h. Find the volume of the tetrahedron $MABC$

i. Find the distance between $\mathscr{L}_1$ and $\mathscr{L}_3$, the line goes through M and N.

1.10 Prove that

a. $(\mathbf{a}-\mathbf{b})\times(\mathbf{a}+\mathbf{b}) = 2(\mathbf{a}\times\mathbf{b})$

b. $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$

c. $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) + \mathbf{b}\times(\mathbf{c}\times\mathbf{a}) + \mathbf{c}\times(\mathbf{a}\times\mathbf{b}) = \mathbf{0}$

1.11 Plot the point whose cylindrical coordinates are given. Then find the rectangular coordinates of the point.

a. $(4,\ \pi/3,\ 6)$

b. $(3,\ -\pi/3,\ 5)$

1.12 Plot the point whose spherical coordinates are given. Then find the rectangular coordinates of the point.

a. $(5, -\pi/3,\ \pi/4)$

b. $(7,\ -2\pi/3,\ \pi/6)$

1.13 Plot the point whose rectangular coordinates are given. Then find the cylindrical coordinates of the point.

a. $(1,\ -1,\ 4)$

b. $(3/2,\ 3/2,\ -1)$

1.14 Plot the point whose rectangular coordinates are given. Then find the spherical coordinates of the point.

a. $(1,\ -1,\ \sqrt{2})$

b. $(-\sqrt{3},\ -3,\ 2)$

1.15 Show that the equation $(x-a_1)(x-b_1)+(y-a_2)(y-b_2)+(z-a_3)(z-b_3)=0$ represents a sphere and find its center and radius, where $(a_1,a_2,a_3)\neq(b_1,b_2,b_3)$ are fixed.

ANSWERS

1.1 $\cos\theta = 2/\sqrt{6}$ and $\cos\theta = 0$.

1.2 $\cos\theta = 1/\sqrt{3}$

1.7 $\sqrt{97/3}$

1.8 $2/\sqrt{17}$

1.9 a. $10x - 26y - 17z + 107 = 0$ b. $\dfrac{x}{10} = \dfrac{y-1}{-26} = \dfrac{z+2}{-17}$

c. $\dfrac{x+2}{1} = \dfrac{y-4}{3} = \dfrac{z+1}{-4}$ d. $\dfrac{x}{1} = \dfrac{y-1}{3} = \dfrac{z+2}{-4}$

e. $115/\sqrt{1065}$ f. $\sqrt{355/26}$

i. $\mathscr{L}_3$: $\dfrac{x}{1} = \dfrac{y-1}{2} = \dfrac{z+2}{6}$; $d = 83/\sqrt{777}$

1.15 Radius = $\frac{1}{2}|\overrightarrow{AB}|$, $A(a_1, a_2, a_3)$, $B(b_1, b_2, b_3)$

Chapter 2
Vector Functions

2.1 VECTOR FUNCTIONS AND SPACE CURVES

DEFINITION 2.1.1 VECTOR FUNCTIONS

A **vector-valued function**, or **vector function**, is a function whose domain is a set D of numbers, $D \subset \mathbb{R}$, and whose range is a set of vectors.
A vector function can be expressed by

$$\mathbf{r}(t) = \langle f(t),\ g(t),\ h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \qquad (2.1.1)$$

where $f(t)$, $g(t)$, and $h(t)$ are real-valued functions called **component functions**, and t is real variable taking values in the domain of $\mathbf{r}$.

DEFINITION 2.1.2 LIMIT OF VECTOR FUNCTIONS

If $\mathbf{r}$ is defined by (2.1.1), then

$$\lim_{t\to a}\mathbf{r}(t) = \left\langle \lim_{t\to a} f(t),\ \lim_{t\to a} g(t),\ \lim_{t\to a} h(t) \right\rangle = \lim_{t\to a} f(t)\mathbf{i} + \lim_{t\to a} g(t)\mathbf{j} + \lim_{t\to a} h(t)\mathbf{k} \qquad (2.1.2)$$

DEFINITION 2.1.3 CONTINUITY OF VECTOR FUNCTIONS

Let $\mathbf{r}$ be defined by (2.1.1).

a. $\mathbf{r}$ is called to be continuous at $t = a$ if $f(t)$, $g(t)$, and $h(t)$ are continuous at $t = a$.

b. $\mathbf{r}$ is called to be continuous on an interval I if $f(t)$, $g(t)$, and $h(t)$ are continuous on I.

We can consider the vector function $\mathbf{r}(t)$ as the position vector of the point $P\big(f(t),\ g(t),\ h(t)\big)$, i.e. $\mathbf{r} = \overrightarrow{OP}$. When the **parameter** t varies in I, the tip point P draws a **space curve** $\mathscr{C}$. Thus, any continuous vector function $\mathbf{r}(t)$ defines a continuous space curve $\mathscr{C}$ (see Figure 2.1.1) with parametric equations

$$x = f(t), \quad y = g(t), \quad z = h(t) \qquad (2.1.3)$$

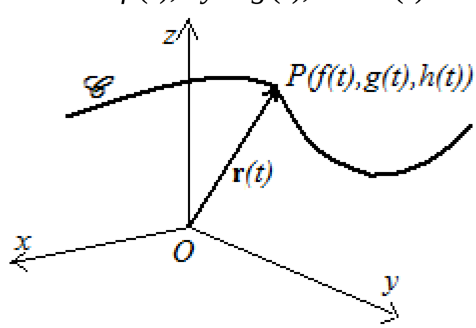


Figure 2.1.1 A space curve traced out by the tip of a position vector $\mathbf{r}(t)$

EXAMPLE 2.1.1

a. Describe the curve defined by the vector function $\mathbf{r}(t) = \langle 1+2t,\ -2-t,\ 2+3t \rangle$

Answer: The curve is a line that passes through the point (1, -2, 2) and is parallel to the vector $\mathbf{v} = \langle 2,\ -1,\ 3 \rangle$

b. Describe the curve defined by the vector function $\mathbf{r}(t) = \langle \cos t,\ \sin t,\ 2 \rangle$

Answer: The curve is a circle $x^2 + y^2 = 1$ in the plane $z = 2$.

c. Describe the curve defined by the vector function $\mathbf{r}(t) = \langle 2\cos t,\ 3\sin t,\ t \rangle$

Answer: The curve spirals upward around the cylinder $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ as t increases.

2.2 DERIVATIVES OF VECTOR FUNCTIONS

DEFINITION 2.2.1 DERIVATIVES OF VECTOR FUNCTIONS

The derivative of a vector function **r** is defined by

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} \tag{2.2.1}$$

if the limit exists. (See Figure 2.2.1)

As $h \to 0$, the vector $\overrightarrow{PQ} = \mathbf{r}(t+h) - \mathbf{r}(t)$ approaches a vector that lies on the tangent line of $\mathscr{C}$ at P. So does $\mathbf{r}'(t)$.

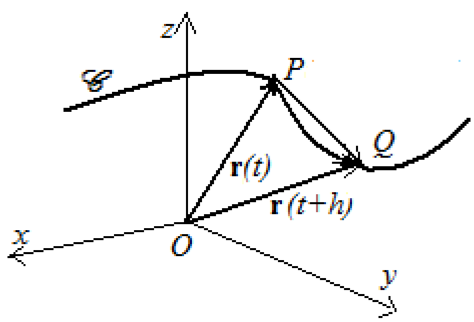


Figure 2.2.1: $\overrightarrow{PQ} = \mathbf{r}(t+h) - \mathbf{r}(t)$

DEFINITION 2.2.2 TANGENT VECTOR

If the vector function $\mathbf{r}(t)$ is differentiable, i.e. $\mathbf{r}'(t)$ exists, then

a. The vector $\mathbf{r}'(t)$ is called the **tangent vector** to the curve $\mathscr{C}$, given by $\mathbf{r}(t)$, at P.

b. The **unit tangent vector** is defined by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \tag{2.2.2}$$

THEOREM 2.2.1

If $\mathbf{r}(t) = \langle f(t),\, g(t),\, h(t)\rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, then the limit (2.2.1) exists if and only if the function $f(t),\ g(t)$, and $h(t)$ are differentiable and

$$\boxed{\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \langle f'(t),\, g'(t),\, h'(t)\rangle = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}} \qquad (2.2.3)$$

Proof : It follows directly from the definition 2.2.1 and the definition of the derivative of real functions.

EXAMPLE 2.2.1

a. Find $\mathbf{r}'(t)$ and $\mathbf{r}'(0)$ if $\mathbf{r}(t) = (3t + e^{-2t})\mathbf{i} + \sin 4t\,\mathbf{j} + \ln(t^2 + 2t + 1)\mathbf{k}$.

Answer: $\mathbf{r}'(0) = \mathbf{i} + 4\,\mathbf{j} + 2\mathbf{k}$

b. Find the unit tangent vector at $t = 1$, if $\mathbf{r}(t) = \cos 2t\,\mathbf{i} + \sin 2t\,\mathbf{j} + (1 - e^{t-1})\mathbf{k}$

Answer: $\dfrac{(-2\sin 2)\,\mathbf{i} + (2\cos 2)\,\mathbf{j} - \mathbf{k}}{\sqrt{5}}$

THEOREM 2.2.2 DIFFERENTIATION RULES

Suppose $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are differentiable vector functions, c is a scalar, and f(t) is differentiable real-valued function. Then

a. $\dfrac{d}{dt}\left[\mathbf{u}(t) + \mathbf{v}(t)\right] = \mathbf{u}'(t) + \mathbf{v}'(t)$

b. $\dfrac{d}{dt}\left[c\mathbf{u}(t)\right] = c\mathbf{u}'(t)$

c. $\dfrac{d}{dt}\left[f(t)\mathbf{u}(t)\right] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$

d. $\dfrac{d}{dt}\left[\mathbf{u}(t)\cdot\mathbf{v}(t)\right] = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$

e. $\dfrac{d}{dt}\left[\mathbf{u}(t)\times\mathbf{v}(t)\right] = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$

f. $\dfrac{d}{dt}\left[\mathbf{u}(f(t))\right] = f'(t)\mathbf{u}'(f(t))$ (*Chain Rule*)

EXAMPLE 2.2.2

a. Let $\mathbf{u}(t) = 2t\,\mathbf{i} - t^2\mathbf{j} + e^{3t}\mathbf{k},\ \mathbf{v}(t) = t^3\,\mathbf{i} - (1+t^2)\mathbf{j} + \sin t\,\mathbf{k}$. Find $\dfrac{d}{dt}\left[\mathbf{u}(t)\cdot\mathbf{v}(t)\right]$ by two ways: apply Theorem 2.2.2.d. and direct calculation (calculate $\mathbf{u}(t)\cdot\mathbf{v}(t)$, then find its derivative) .

Answer: $2t + 12t^3 + e^{3t}(3\sin t + \cos t)$

b. Show that if $|\mathbf{r}(t)| = c$ (constant), then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$.

Proof:

$\mathbf{r}(t)\cdot\mathbf{r}(t) = |\mathbf{r}(t)|^2 = c^2 =$ constant. Then Theorem 2.2.2.d. implies $2\mathbf{r}'(t)\cdot\mathbf{r}(t) = 0$, thus $\mathbf{r}'(t)\cdot\mathbf{r}(t) = 0$, i.e. $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$.

Geometrically, this result says that if a curve lies on a sphere, then the tangent vector $\mathbf{r}'(t)$ is always perpendicular to the position vector $\mathbf{r}(t)$.

2.3 INTEGRALS OF VECTOR FUNCTIONS

If $\mathbf{r}(t)$ is continuous on $[a, b]$, we can define its integral in the same way as for real-valued function.

DEFINITION 2.3.1 INTEGRAL OF VECTOR FUNCTIONS

Suppose $\mathbf{r}(t) = \langle f(t),\, g(t),\, h(t)\rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ is continuous on $[a, b]$, i.e. the component functions $f(t),\, g(t)$, and $h(t)$ are continuous on $[a, b]$. Then we can define as for scalar function: subdivide $[a, b]$ in to n subintervals by points $a = t_0 < t_1 < ... < t_n = b$. Denote the $\Delta t_i = [t_{i-1}, t_i],\ 1 \le i \le n$. Use the same notations $\Delta t_i,\ 1 \le i \le n$, for the lengths of the subintervals. Choose $t_i^* \in \Delta t_i,\ 1 \le i \le n$. Denote $\lambda = \max\{\Delta t_i, 1 \le i \le n\}$. Then

$$\int_a^b \mathbf{r}(t)dt = \lim_{\lambda\to 0}\sum_{i=1}^n \mathbf{r}(t_i^*)\Delta t_i = \lim_{\lambda\to 0}\left[\left(\sum_{i=1}^n f(t_i^*)\Delta t_i\right)\mathbf{i} + \left(\sum_{i=1}^n g(t_i^*)\Delta t_i\right)\mathbf{j} + \left(\sum_{i=1}^n h(t_i^*)\Delta t_i\right)\mathbf{k}\right]$$

$$= \left(\int_a^b f(t)dt\right)\mathbf{i} + \left(\int_a^b g(t)dt\right)\mathbf{j} + \left(\int_a^b h(t)dt\right)\mathbf{k} \qquad (2.3.1)$$

if the limit on the right hand side of (2.3.1) exists and does not depend on the division of the interval $[a, b]$ and on the choosing points $t_i^* \in \Delta t_i,\ 1 \le i \le n$.

We can extend the **Fundamental Theorem** of Calculus to continuous vector functions as follows:

$$\boxed{\int_a^b \mathbf{r}(t)dt = \mathbf{R}(t)\Big|_a^b = \mathbf{R}(b) - \mathbf{R}(a)} \qquad (2.3.2)$$

where $\mathbf{R}(t)$ is an **anti-derivative** of $\mathbf{r}(t)$, that is, $\mathbf{R}'(t) = \mathbf{r}(t)$.

EXAMPLE 2.3.1

a. Evaluate $\int_0^1 \mathbf{r}(t)dt$ if $\mathbf{r}(t) = 2t^4\,\mathbf{i} - t^{3/2}\mathbf{j} + e^{-2t}\mathbf{k}$

Answer: $\frac{2}{5}\mathbf{i} - \frac{2}{5}\mathbf{j} + \frac{1-e^{-2}}{2}\mathbf{k}$

b. Evaluate $\int_0^{\pi/2} \mathbf{r}(t)dt$ if $\mathbf{r}(t) = (t + \cos 3t)\mathbf{i} + \sin 4t\,\mathbf{j} + (t - e^t)\mathbf{k}$.

Answer: $\left(\frac{\pi^2}{8} - \frac{1}{3}\right)\mathbf{i} + \left(e^{\pi/2} - 1 - \frac{\pi^2}{8}\right)\mathbf{k}$

c. Find $\mathbf{r}(t)$ if $\mathbf{r}'(t) = (t + t^2)\mathbf{i} + 4t\,\mathbf{j} + (t - t^3)\mathbf{k}$ and $\mathbf{r}(0) = \mathbf{i} + 2\mathbf{j}$

Answer: $\left(\frac{t^2}{2} + \frac{t^3}{3} + 1\right)\mathbf{i} + 2(t^2 + 1)\mathbf{j} + \left(\frac{t^2}{2} - \frac{t^4}{4}\right)\mathbf{k}$

2.4 ARC LENGTH OF SPACE CURVES

The length of a space curve is defined in the same way as for a plane curve.

DEFINITION 2.4.1 ARC LENGTH

Suppose the continuous space curve $\mathscr{C}$ is defined by a differentiable vector function on $[a,b]$, $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$. Moreover, $f'(t)$, $g'(t)$, and $h'(t)$ are continuous on $[a,b]$ and the curve is traversed exactly once when t increases from a to b. Then the length of $\mathscr{C}$ is

$$\boxed{L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,dt = \int_a^b |\mathbf{r}'(t)|\,dt} \qquad (2.4.1)$$

PROPERTY 2.4.1

If we denote the **arc length function** of the curve $\mathscr{C}$ by the length of the part of $\mathscr{C}$ between $\mathbf{r}(a)$ and $\mathbf{r}(t)$, $a \le t \le b$, by

$$s(t) = \int_a^t \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,dt = \int_a^t |\mathbf{r}'(t)|\,dt \qquad (2.4.2)$$

then

$$\frac{ds(t)}{dt} = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} = |\mathbf{r}'(t)| \qquad (2.4.3)$$

EXAMPLE 2.4.1

a. Find the length of the circular helix $\mathbf{r}(t) = \langle a\cos t, a\sin t, t\rangle$, $a > 0$, $0 \le t \le 2\pi$.

Answer: $2\pi\sqrt{1 + a^2}$

b. Find the arc length function of the circular helix $\mathbf{r}(t) = \langle a\cos t, a\sin t, t\rangle$, $t \ge 0$, $a > 0$.

Answer: $t\sqrt{1 + a^2}$

c. Find the length of the curve $\mathbf{r}(t) = \left\langle \sin t - t\cos t,\ \cos t + t\sin t,\ t^2 \right\rangle,\ 0 \le t \le 2\pi.$

Answer: $2\pi^2\sqrt{5}$

2.5 CURVATURE

Suppose that the space curve $\mathscr{C}$, defined by a vector function $\mathbf{r}(t)$, is **smooth**, that is $\mathbf{r}$(t) is differentiable and $\mathbf{r}'(t) \neq \mathbf{0}$. The unit tangent vector $\mathbf{T}(t)$, defined in (2.2.2), indicates the direction of the curve:

$$\boxed{\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}} \tag{2.5.1}$$

$\mathbf{T}(t)$ changes its direction less or more if $\mathscr{C}$ is fairly straight or twists more sharply. The curvature of $\mathscr{C}$ at a given point is a measure of how quickly the curve changes direction at that point. Thus we can define

DEFINITION 2.5.1 CURVATURE

The **curvature** of a curve is defined by

$$\boxed{\kappa = \left|\frac{d\mathbf{T}}{ds}\right|} \tag{2.5.2}$$

where **T** is the unit tangent vector and *s* is the arc length function

The formula (2.5.1) can be expressed by

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{d\mathbf{T}/dt}{ds/dt}\right| = \frac{|d\mathbf{T}/dt|}{|ds/dt|}$$

Noting (2.4.3), we obtain

$$\boxed{\kappa = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}} \tag{2.5.3}$$

EXAMPLE 2.5.1

a. Show that the curvature of the circle of radius *a* is 1/*a*.

Solution:

We can take the circle with equation defined by

$\mathbf{r}(t) = \left\langle a\cos t,\ a\sin t,\ 0 \right\rangle,\ 0 \le t \le 2\pi.$ Then

$\mathbf{r}'(t) = \left\langle -a\sin t,\ a\cos t,\ 0 \right\rangle;\ |\mathbf{r}'(t)| = a;$ so $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \left\langle -\sin t,\ \cos t,\ 0 \right\rangle;$

$\mathbf{T}'(t) = \langle -\cos t,\, -\sin t,\, 0\rangle$; $|\mathbf{T}'(t)| = 1$. Therefore $\kappa = |\mathbf{T}'(t)| / |\mathbf{r}'(t)| = 1/a$.

b. Determine the curvature of the circular helix $\mathbf{r}(t) = \langle a\cos t,\, a\sin t,\, t\rangle$, $0 \le t \le 2\pi$.

Answer: $\dfrac{a}{a^2+1}$

The following Theorem is more convenient for calculating the curvature.

THEOREM 2.5.1

The curvature of the curve given by the vector function **r**(t) is calculated from

$$\boxed{\kappa(t) = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}} \tag{2.5.4}$$

Proof

It follows from (2.4.3) and (2.5.1) that

$$\mathbf{r}'(t) = |\mathbf{r}'(t)|\mathbf{T}(t) = s'(t)\mathbf{T}(t)$$

$$\Rightarrow \mathbf{r}''(t) = s''(t)\mathbf{T}(t) + s'(t)\mathbf{T}'(t)$$

$$\Rightarrow \mathbf{r}'(t)\times\mathbf{r}''(t) = [s'(t)\mathbf{T}(t)]\times[s''(t)\mathbf{T}(t) + s'(t)\mathbf{T}'(t)]$$

$$= [s'(t)]^2\,[\mathbf{T}(t)\times\mathbf{T}'(t)], \text{ because of } \mathbf{T}\times\mathbf{T} = \mathbf{0}$$

$$\Rightarrow |\mathbf{r}'(t)\times\mathbf{r}''(t)| = [s'(t)]^2\,|\mathbf{T}(t)\times\mathbf{T}'(t)| = [s'(t)]^2\,|\mathbf{T}(t)||\mathbf{T}'(t)| = [s'(t)]^2\,|\mathbf{T}'(t)|,$$

because $|\mathbf{T}| = 1$ and then $\mathbf{T}\perp\mathbf{T}'$, by Example 2.2.2.b.

Now, (2.5.4) follows from (2.5.3).

COROLLARY OF THEOREM 2.5.1

The curvature of a plane curve with equation $y = f(x)$ is calculated from

$$\boxed{\kappa(x) = \frac{|f''(x)|}{\left|1+[f'(x)]^2\right|^{3/2}}} \tag{2.5.5}$$

Proof

Apply (2.5.4), noting that the plane curve $y = f(x)$ can be given by a vector function $\mathbf{r}(x) = \langle x,\, f(x),\, 0\rangle$.

EXAMPLE 2.5.2

a. Show that the curvature of the circle of radius *a* is 1/*a*, applying (i) (2.5.4) , (ii) (2.5.5) .

b. Find the curvature of the curve $y = \sin x$ at the point with $x = 0$ and at $x = \pi/2$.

Answer: $\kappa(0) = 0$; $\kappa(\pi/2) = 1$

d. Find the curvature of the curve $y = x^2 + 1$ at $A(0, 1)$, $B(1, 2)$, and $C(2, 5)$.

Answer: $\kappa(0) = 2$, $\kappa(1) = 2/5^{3/2}$, and $\kappa(2) = 2/17^{3/2}$.

2.6 NORMAL AND BINORMAL VECTORS

Suppose that the space curve $\mathscr{C}$, defined by a vector function $\mathbf{r}(t)$, is **smooth**. Note that $\mathbf{T}(t)$ is the unit tangent vector of the curve $\mathscr{C}$. Then $\mathbf{T}(t)$ and $\mathbf{T}'(t)$ are orthogonal, $\mathbf{T} \perp \mathbf{T}'$, by Example 2.2.2.b.

DEFINITION 2.6.1 NORMAL AND BINORMAL VECTORS. NORMAL PLANE

a. The unit vector of $\mathbf{T}'(t)$, denoted by $\mathbf{N}(t)$,

$$\boxed{\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}} \tag{2.6.1}$$

The vector $\mathbf{N}(t)$ is called the **unit normal vector**.

b. The vector

$$\boxed{\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)} \tag{2.6.2}$$

is perpendicular to both $\mathbf{T}(t)$ and $\mathbf{N}(t)$. $\mathbf{B}(t)$ is called the **binormal vector**.

DEFINITION 2.6.2 NORMAL PLANE AND OSCULATING PLANE

a. The plane determined by $\mathbf{N}(t)$ and $\mathbf{B}(t)$ at the point $P(t) \in \mathscr{C}$, is called the **normal plane** of $\mathscr{C}$ at $P(t)$.

b. The plane determined by $\mathbf{N}(t)$ and $\mathbf{T}(t)$ at the point $P(t) \in \mathscr{C}$, is called the **osculating plane** of $\mathscr{C}$ at $P(t)$.

c. The circle, that lies in the osculating plane of $\mathscr{C}$ at $P(t)$, has the same unit tangent vector $\mathbf{T}(t)$, lies on the concave side of $\mathscr{C}$, and has radius $\rho = 1/\kappa(t)$, it means that has the same curvature $\kappa(t)$ as $\mathscr{C}$ at $P(t)$, is called the **osculating circle** of $\mathscr{C}$ at $P(t)$. (See Figure 2.6.1)

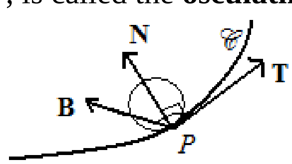


Figure 2.6.1

The normal plane is orthogonal to the tangent vector $\mathbf{T}(t)$, while the osculating plane is the one that comes closest to containing the part of $\mathscr{C}$ near the point $P(t)$.

EXAMPLE 2.6.1

Find the radius and graph the osculating circle of the curve $y = 2\sin x$ at $P(\pi/2, 2)$.

Solution: $\kappa(x) = \dfrac{|-2\sin x|}{|1+4\cos^2 x|^{3/2}} \Rightarrow \kappa(\pi/2) = 2$ and $\rho = 1/2$. See Figure 2.6.2

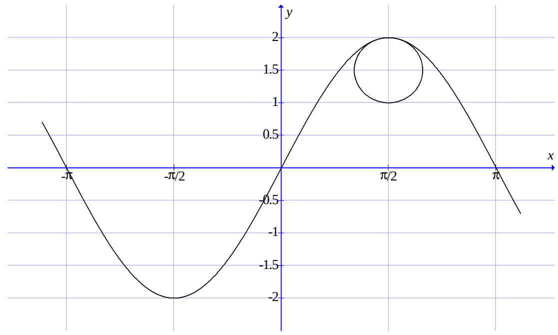


Figure 2.6.2

2.7 MOTION IN SPACE: VELOCITY AND ACCELERATION

Suppose a particle moves on a smooth space curve $\mathscr{C}$, defined by a vector function $\mathbf{r}(t)$. The **velocity vector v**(t) is defined from

$$\mathbf{v}(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} = \mathbf{r}'(t) \tag{2.7.1}$$

The **speed** is, see (2.4.3),

$$|\mathbf{v}(t)| = |\mathbf{r}'(t)| = \frac{ds}{dt} \tag{2.7.2}$$

The **acceleration vector a**(t) is

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) \tag{2.7.3}$$

EXAMPLE 2.7.1

a. Find the velocity and acceleration vectors and speed of a particle with position vector

$\mathbf{r}(t) = \langle 2t,\ e^{3t},\ 3t^2 \rangle$

Answer: $\mathbf{v}(t) = \langle 2,\ 3e^{3t},\ 6t \rangle$, $\mathbf{a}(t) = \langle 0,\ 9e^{3t},\ 6 \rangle$, $|\mathbf{v}(t)| = \sqrt{4+9e^{6t}+36t^2}$.

b. A moving particle starts at an initial position $\mathbf{r}(0) = \langle 2, -3,\ 4 \rangle$ with initial velocity $\mathbf{v}(0) = \langle 1,\ 5,\ -4 \rangle$. Find its velocity and position at time t if $\mathbf{a}(t) = \langle 2,\ t^2,\ e^{2t} \rangle$. Find its speed at t = 1.

Answer:

$$\mathbf{v}(t) = \left\langle 2t+1,\ \frac{t^3}{3}+5,\ \frac{e^{2t}}{2}-4 \right\rangle,\ \mathbf{r}(t) = \left\langle t^2+t+2,\ \frac{t^4}{12}-3,\ \frac{e^{2t}}{4}+4 \right\rangle,\ |\mathbf{v}(1)| = \sqrt{9+\frac{256}{9}+\frac{(e-8)^2}{4}}$$

c. An object with mass m and that moves in an elliptical path in the plane xOy with constant angular speed ω has position vector $\mathbf{r}(t) = a\cos\omega t\,\mathbf{i} + b\sin\omega t\,\mathbf{j}$. Find the force acting on the object and show that it is directed toward the origin. (Such a force is called a *centripetal* force).

EXERCISES

1. Show that the curve with parametric equations $x = t\cos t,\ y = t\sin t,\ z = t$ lies on the cone $x^2 + y^2 = z^2$

2. Show that the curve with parametric equation $x = t^2 - 1, y = 1 + 2t,\ z = 1 - 3t^2$ passes through the points $A(0, 3, -2)$, $B(8, 7, -26)$, and $C(0, -1, -2)$ but not $D(3, -3, -2)$.

3. Try to sketch the curve of intersection of the circular cylinder $x^2 + y^2 = 1$ and parabolic cylinder $z = x^2$. Then find the parametric equations for this curve.

4. Suppose that **u** and **v** are vector functions that possess limits as $t \to t_0$. Prove

   a. $\lim\limits_{t\to t_0} \mathbf{u}(t)\cdot\mathbf{v}(t) = \left[\lim\limits_{t\to t_0}\mathbf{u}(t)\right]\cdot\left[\lim\limits_{t\to t_0}\mathbf{v}(t)\right]$

   b. $\lim\limits_{t\to t_0} \mathbf{u}(t)\times\mathbf{v}(t) = \left[\lim\limits_{t\to t_0}\mathbf{u}(t)\right]\times\left[\lim\limits_{t\to t_0}\mathbf{v}(t)\right]$

5. Find the unit tangent vector, the normal vector, and the binormal vector at the point with the given value of *t*.

   a. $\mathbf{r}(t) = \left\langle t^2,\ \tfrac{2}{3}t^3,\ t\right\rangle,\ t = 1.$ b. $\mathbf{r}(t) = \left\langle e^{2t}\sin t,\ e^{2t}\cos t,\ e^{2t}\right\rangle,\ t = 0$

6. The curves $\mathbf{r}_1(t) = \langle \sin \pi t,\ \cos t - 1,\ 2t\rangle$ and $\mathbf{r}_2(t) = \left\langle t^2,\ t,\ t + 2t^2\right\rangle$ intersect at the origin. Find the angle of intersection.

7. Find the derivative of $\mathbf{u}(t)\cdot\mathbf{v}(t)$ and $\mathbf{u}(t)\times\mathbf{v}(t)$ if

   $\mathbf{u}(t) = \left\langle t + e^{-2t},\ e^{-t},\ e^{2t}\right\rangle,\ \mathbf{v}(t) = \left\langle \sin t,\ \cos 2t,\ t^2\right\rangle$

8. Prove that

   a. $\dfrac{d}{dt}\left[\mathbf{r}(t)\times\mathbf{r}'(t)\right] = \left[\mathbf{r}(t)\times\mathbf{r}''(t)\right]$

   b. $\dfrac{d}{dt}\left(\mathbf{r}(t)\cdot\left[\mathbf{r}'(t)\times\mathbf{r}''(t)\right]\right) = \mathbf{r}(t)\cdot\left[\mathbf{r}'(t)\times\mathbf{r}'''(t)\right]$

   c. $\dfrac{d}{dt}\left|\mathbf{r}(t)\right| = \dfrac{1}{\left|\mathbf{r}(t)\right|}\left[\mathbf{r}(t)\cdot\mathbf{r}'(t)\right]$

9. Find the curvature and the equation of the osculating circle of the ellipse $9x^2 + 4y^2 = 36$ at the points (2, 0) and (0, 3).

10. Find the velocity vector, acceleration vector, and speed of a moving particle with the given position function.
    a. $\mathbf{r}(t) = \left\langle \sqrt{2}\, t,\, e^{t},\, e^{-t} \right\rangle$. b. $\mathbf{r}(t) = e^{t} \left\langle \cos t,\, \sin t,\, t \right\rangle$

11. A force 20 N acts directly upward from the *xy*-plane on an object with mass 4 kg. The object starts at the origin with initial velocity $\mathbf{v}(0) = \langle 1, -2, 2 \rangle$. Find its position function and its speed at time *t*.

12. Find the curvature of the trajectory in Exercise11.

13. A projectile is fired from the ground with angle of elevation α, initial position at the origin, and initial speed v_0. Suppose that air resistance is negligible and the only external force is due to gravity, find the position vector $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$ of the projectile. What value of α maximizes the horizontal distance traveled and what is the maximum height in this case? Determine these values if $v_0 = 98\, m/s$.

Answers

3. $x = \cos t,\ y = \sin t,\ z = \cos^2 t$.

5. a. $\left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right\rangle$; $\left\langle -\frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \right\rangle$; $\left\langle -\frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right\rangle$.

   b. $\left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle$; $\left\langle \frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}, 0 \right\rangle$; $\left\langle \frac{2}{3\sqrt{5}}, \frac{4}{3\sqrt{5}}, \frac{-5}{3\sqrt{5}} \right\rangle$

9. $\frac{2}{9},\ (x+2.5)^2 + y^2 = \frac{81}{4}$; $\frac{3}{4},\ x^2 + (y - \frac{5}{3})^2 = \frac{16}{9}$.

11. $\mathbf{r}(t) = \left\langle t, -2t, \frac{5}{2}t^2 + 2t \right\rangle$; $|\mathbf{v}(t)| = \sqrt{25t^2 + 20t + 9}$

Chapter 3
DOUBLE INTEGRALS

3.1 DEFINITIONS AND PROPERTIES

We extend the definite integral of one variable functions to double integral of two variable functions. Double integrals give us to compute areas, masses,..., and center of plane regions.

DEFINITION 3.1.1 DOUBLE INTEGRAL

Let $f(x,y)$ be defined on the domain $D, D \subset G = [a,b]\times[c,d] \subset \mathbb{R}^2$. Divide the rectangle G into sub-domains (sub-rectangles) by lines

$$a = x_0 < x_1 < x_2 < ... < x_m = b \text{ and } c = y_0 < y_1 < y_2 < ... < y_n = d$$

Denote $\Delta x_i = x_i - x_{i-1}, 1 \le i \le m,\ \Delta y_j = y_j - y_{j-1}, 1 \le j \le n,$ and

$$S_{m,n} = \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta x_i \Delta y_j \tag{3.1.1}$$

where $x_i^* \in [x_{i-1}, x_i], 1 \le i \le m, \quad y_j^* \in [y_{j-1}, y_j], 1 \le j \le n,$

and with convention $f(x_i^*, y_j^*) = 0$, if $(x_i^*, y_j^*) \notin D$

Denote $\lambda_{m,n} = \max\{\Delta x_i, \Delta y_j : 1 \le i \le m, 1 \le j \le n\}$

If there exists a unique and finite limit

$$\lim_{\lambda_{m,n}\to 0} S_{m,n} = \lim_{\lambda_{m,n}\to 0} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta x_i \Delta y_j = L \tag{3.1.2}$$

then the limit is called **double integral** of $f(x,y)$ on the domain D and denoted by

$$\iint_D f(x,y)dA \text{ or } \iint_D f(x,y)dxdy$$

Therefore we can write

$$\iint_D f(x,y)dA = \iint_D f(x,y)dxdy = \lim_{\lambda_{m,n}\to 0} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta x_i \Delta y_j \tag{3.1.3}$$

<u>**THEOREM 3.1.1**</u> EXISTENCE CONDITION FOR DOUBLE INTEGRAL

If $f(x,y)$ is continuous on the domain $D, D \subset G = [a,b]\times[c,d] \subset \mathbb{R}^2$, then the double integral defined in (3.1.3) exists.

PROPERTY 3.1.1 DOUBLE INTEGRALS

Suppose double integrals of $f(x,y)$, $g(x,y)$ exist on the domains D, D_1, D_2, where D_1, D_2 do not overlap except perhaps on their boundary, and $D_1 \cup D_2 = D$ .k is a constant.

a. $\iint_D k\, f(x,y)dxdy = k\iint_D f(x,y)dxdy$, ($k$ is a constant). (3.1.4)

b. $\iint\limits_D [f(x,y)+g(x,y)]dxdy = \iint\limits_D f(x,y)dxdy + \iint\limits_D g(x,y)dxdy$ (3.1.5)

c. $f(x,y) \ge g(x,y) \Rightarrow \iint\limits_D f(x,y)dxdy \ge \iint\limits_D g(x,y)dxdy$ (3.1.6)

d. $\iint\limits_D f(x,y)dxdy = \iint\limits_{D_1} f(x,y)dxdy + \iint\limits_{D_2} f(x,y)dxdy$ (3.1.7)

e. $\iint\limits_D dxdy = \text{Area}(D)$ (3.1.8)

f. $m \le f(x,y) \le M$ on $D \Rightarrow m.\text{Area}(D) \le \iint\limits_D f(x,y)dxdy \le M.\text{Area}(D)$ (3.1.9)

PROPERTY 3.1.2 VOLUMES AND AREA - DOUBLE INTEGRALS

a. Let *S* be a solid that lies above domain *D* in the *xy*-plane and under a surface with equation $z = f(x,y), f(x,y) \ge 0$ on D. Then the volume *V* of *S* can be expressed by double integral

$$V = \iint\limits_D f(x,y)dxdy \qquad (3.1.10)$$

b. Let *S* be a solid that is determined by two surfaces: upper surface with equation $z = f_2(x,y), (x,y) \in D$ and lower surface with equation $z = f_1(x,y), (x,y) \in D.$ Then the volume *V* of *S* can be expressed by double integral

$$V = \iint\limits_D [f_2(x,y) - f_1(x,y)]dxdy \qquad (3.1.11)$$

c. In (3.1.10), if $f(x,y) = 1$ on D then we obtain the area *A* of D:

$$A = \iint\limits_D dxdy \qquad (3.1.12)$$

From the definition and Property 3.1.1.f of double integral we obtain the following properties.

PROPERTY 3.1.3 MIDPOINT RULE FOR DOUBLE INTEGRALS

If the domain *D* is closed and $f(x,y)$ is continuous on the domain *D*, then there exist a point $(x_0, y_0) \in D$ so that

$$\iint\limits_D f(x,y)dxdy = f(x_0,y_0).\text{Area}(D) \qquad (3.1.13)$$

PROPERTY 3.1.4 APPROXIMATE CALCULATION

The double integral can be approximately calculated by

$$\iint\limits_D f(x,y)dxdy \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta x_i \Delta y_j \qquad (3.1.14)$$

3.2 ITERATED INTEGRALS

Suppose that $f(x, y)$ is continuous on the domain *D*. Double integral can be evaluated by calculating two single integrals, as shown below.

CASE 1 If *D* is a rectangle, $D = [a,b]\times[c,d] \subset \mathbb{R}^2$, then

$$\iint_D f(x,y)dxdy = \int_a^b\left[\int_c^d f(x,y)dy\right]dx = \int_c^d\left[\int_a^b f(x,y)dx\right]dy \qquad (3.2.1)$$

EXAMPLE 3.2.1

a. Evaluate $I = \iint_D x^2(x+y)dxdy$ where $D = [0,1]\times[0,2]$ in two ways:

$$\int_0^1\left[\int_0^2 x^2(x+y)dy\right]dx \quad \text{and} \quad \int_0^2\left[\int_0^1 x^2(x+y)dx\right]dy$$

Answer: 7/6.

b. Find the volume of the solid *S* bounded by three coordinates planes, the elliptic paraboloid $z = 32-(x^2+4y^2)$, and the planes $x = 1$, $y = 1$.

Answer: $V = \int_0^1\left[\int_0^1 (32-x^2-4y^2)dy\right]dx = \dfrac{91}{3}$

CASE 2 If $D = \{(x,y): \varphi_1(x) \le y \le \varphi_2(x), a \le x \le b\} \subset \mathbb{R}^2$, then

$$\iint_D f(x,y)dxdy = \int_a^b\left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy\right]dx \qquad (3.2.2)$$

EXAMPLE 3.2.2

a. Find the mass of a plane lamina *D* that is bounded by two parabolas $y = 3x^2$, $y = 8+x^2$, and has mass density function $\rho(x,y) = 1+y$.

Solution:

Two parabolas $y = 3x^2$ and $y = 8+x^2$ intersect at $M_1(-2, 12)$ and $M_2(2, 12)$. That is $D = \{(x,y): 3x^2 \le y \le 8+x^2, -2 \le x \le 2\}$. Therefore

$$M = \iint_D \rho(x,y)dxdy = \int_{-2}^{2}\left[\int_{3x^2}^{8+x^2}(1+y)dy\right]dx = \frac{704}{5}$$

b. Find the volume of the tetrahedron *T* bounded by the coordinate planes and the plane $x+3y+z=3$.

Solution:

The base D of T in the xy-plane is $D=\left\{(x,y): 0\le y\le \frac{3-x}{3}, 0\le x\le 3\right\}$.

The equation of the upper surface is $z=3-x-3y$. Therefore $V=\iint_D (3-x-3y)dxdy = 3/2$

CASE 3 If $D=\{(x,y):\psi_1(y)\le x\le \psi_2(y), c\le y\le d\}\subset \mathbb{R}^2$, then

$$\iint_D f(x,y)dxdy=\int_c^d\left[\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx\right]dy \tag{3.2.3}$$

EXAMPLE 3.2.3

Evaluate $I=\iint_D x^2y\,dxdy$, where D is bounded by $xy=1,\ y=1, y=2$, and $x=0$.

Solution:

$D=\left\{(x,y):0\le x\le \frac{1}{y}, 1\le y\le 2\right\}$. $I=\iint_D x^2y\,dxdy=\frac{1}{6}$

Note: If the domain D is complicated, we divide D into sub-domains that belong to one of the cases 1, 2, 3, and apply (3.1.7) of Property 3.1.1.d.

3.3 CHANGE OF VARIABLES IN DOUBLE INTEGRALS

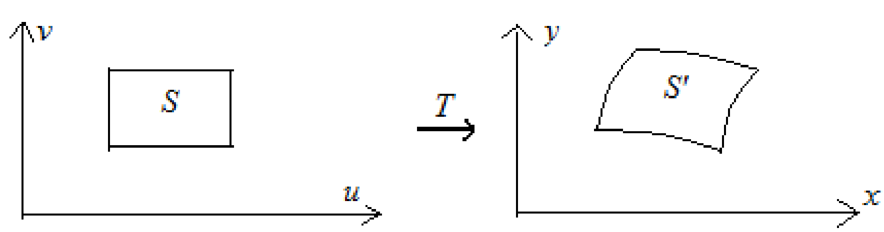
In Chapter 7 (Calculus 1) we have a formula for **change variable**: if f is continuous on an interval $[a,b]$, $x'(t)$ is continuous on $[\alpha,\beta]$, the range of $x(t)$ belongs to $[a,b]$, and $x(\alpha)=a, x(\beta)=b$, then

$$\int_a^b f(x)dx=\int_\alpha^\beta f\left(x(t)\right)x'(t)dt \tag{3.3.1}$$

How a change of variables (x,y) to (u,v) affects a double integral?

Suppose a 1-1 transformation T from D_{uv} in uv-plane to D_{xy} in xy-plane is defined by differentiable functions $x=x(u,v),\ y=y(u,v)$, or in vector function: $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+0\mathbf{k}$. A small rectangle S of sides $\Delta u,\Delta v$ in uv-plane becomes a small parallelogram S' in xy-plane with secant vectors:

$$\Delta\mathbf{a}\approx\Delta u\frac{\partial\mathbf{r}}{\partial u}=\Delta u\,\mathbf{r}_u=\Delta u\left(\frac{\partial x}{\partial u}\mathbf{i}+\frac{\partial y}{\partial u}\mathbf{j}+0\mathbf{k}\right),\ \Delta\mathbf{b}\approx\Delta v\frac{\partial\mathbf{r}}{\partial v}=\Delta v\,\mathbf{r}_v=\Delta v\left(\frac{\partial x}{\partial v}\mathbf{i}+\frac{\partial y}{\partial v}\mathbf{j}+0\mathbf{k}\right)$$



The area of S: $A(S) = \Delta u\,\Delta v$. The area of S': $A(S') \approx |\Delta\mathbf{a}\times\Delta\mathbf{b}| = |\mathbf{r}_u \times \mathbf{r}_v|\,\Delta u\,\Delta v$, where

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial x/\partial u & \partial y/\partial u & 0 \\ \partial x/\partial v & \partial y/\partial v & 0 \end{vmatrix} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix}\mathbf{k} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}\mathbf{k}$$

The **Jacobian** of the transformation T is defined by

$$\boxed{J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}} \qquad (3.3.2)$$

Therefore

$$\boxed{A(S') = |J|\,\Delta u\,\Delta v = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\Delta u\,\Delta v = \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right|\Delta u\,\Delta v} \qquad (3.3.3)$$

PROPERTY 3.5.1 CHANGE OF VARIABLE IN A DOUBLE INTEGRAL

Suppose a 1-1 transformation T from D_{uv} in the uv-plane to D_{xy} in the xy-plane is defined by differentiable functions $x = x(u,v),\ y = y(u,v)$. Suppose $f(x,y)$ is continuous on D_{xy}. Then

$$\boxed{\begin{aligned} \iint_{D_{xy}} f(x,y)\,dx\,dy &= \iint_{D_{uv}} f\left(x(u,v), y(u,v)\right)|J|\,du\,dv \\ &= \iint_{D_{uv}} f\left(x(u,v), y(u,v)\right)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv \end{aligned}} \qquad (3.3.4)$$

EXAMPLE 3.3.1

a. Evaluate the plane area limited by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1,\ a > 0,\ b > 0.$

Solution:

Change variables: $\dfrac{x}{a} = u,\ \dfrac{y}{b} = v \Rightarrow J = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab;\ D \leftrightarrow D_{uv} = u^2 + v^2 \le 1$

$\Rightarrow A(D) = \iint_D dxdy = ab\iint_{D_{uv}} dudv = \pi ab$

b. Evaluate $\iint_D e^{(x+y)/(x-y)}\,dxdy$, where D is the trapezium with vertices (0, -1), (1, 0), (0, -2) and (2, 0). (Hint: Use the change of variable $u = x + y,\ v = x - y$)

Solution:

Use the change of variable $u = x+y,\ v = x-y$. Then $D \leftrightarrow D_{uv}$, where D_{uv} is the trapezium with vertices (-2, 2), (2, 2), (1, 1) and (-1, 1).

$$u = x+y,\ v = x-y \Rightarrow x = \frac{u+v}{2};\ y = \frac{u-v}{2} \Rightarrow |J| = \frac{1}{2}$$

$$\Rightarrow \iint_D e^{(x+y)/(x-y)}\, dxdy = \iint_{D_{uv}} e^{u/v}\, |J|\, dudv = \frac{1}{2}\int_1^2 dv \int_{-v}^{v} e^{u/v}\, |J|\, dudv = \frac{3}{4}\left(e - e^{-1}\right)$$

c. Evaluate $\iint_D y\, dxdy$, where D is bounded by the x-axis, and the parabolas $y^2 = 4-4x$ and $y^2 = 4+4x$ such that $y \geq 0$. (Hint: Use the change of variable $x = u^2 - v^2,\ y = 2uv$)

Solution:

$D \leftrightarrow D_{uv} = [0, 1]\times[0, 1]$.

$$x = u^2 - v^2,\ y = 2uv \Rightarrow J = 4\left(u^2+v^2\right) \Rightarrow \iint_D y\, dxdy = \iint_{D_{uv}} 2uv\, |J|\, dudv = 2$$

3.4 DOUBLE INTEGRALS IN POLAR COORDINATES

Remind that $$x = r\cos\theta,\ y = r\sin\theta$$

$$\Rightarrow J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r \qquad (3.4.1)$$

Therefore, as given in Section 3.3,

$$\boxed{\iint_{D_{xy}} f(x,y)\, dx\, dy = \iint_{D_{r\theta}} f\left(r\cos\theta, r\sin\theta\right) r dr d\theta} \qquad (3.4.2)$$

<u>CASE 1</u> If the domain D is a polar rectangle, $D = \left\{(r,\theta) : a \leq r \leq b, \alpha \leq \theta \leq \beta\right\}$, then

$$\iint_D f(x,y)dxdy = \int_\alpha^\beta \left[\int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\right] d\theta = \int_a^b \left[\int_\alpha^\beta f(r\cos\theta, r\sin\theta) d\theta\right] r\, dr \qquad (3.4.3)$$

<u>CASE 2</u> If $D = \left\{(r,\theta) : r_1(\theta) \leq r \leq r_2(\theta), \alpha \leq \theta \leq \beta\right\}$, then

$$\iint_D f(x,y)dxdy = \int_\alpha^\beta \left[\int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\right] d\theta \qquad (3.4.4)$$

EXAMPLE 3.4.1

a. Evaluate $I = \iint_D (x^2 + 2y^2)dxdy$, where D is the domain in the first quadrant and bounded by two circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 16$.

Solution:

$$I = \iint_D (x^2+2y^2)dxdy = \int_0^{\pi/2}\left[\int_1^4 r^2(1+\sin^2\theta)\,r\,dr\right]d\theta$$
$$= \int_0^{\pi/2}\left[\int_1^4 \frac{3-\cos 2\theta}{2} r^3\,dr\right]d\theta = \frac{765}{16}\pi .$$

b. Evaluate $I = \iint_D \left(2+\sqrt{x^2+y^2}\right)dxdy$, where D is the domain bounded by the circle $x^2+y^2=2y$.

Solution:

The polar equation of the circle $x^2+y^2=2y$ is $r=2\sin\theta,\ 0\le\theta\le\pi$.

$$I = \iint_D \left(2+\sqrt{x^2+y^2}\right)dxdy = \int_0^{\pi}\left[\int_0^{2\sin\theta}(2+r)\,r\,dr\right]d\theta = \int_0^{\pi}\left[r^2+\frac{r^3}{3}\right]\Bigg|_0^{2\sin\theta}d\theta = 2\left(\pi+\frac{16}{9}\right)$$

3.5 APPLICATION OF DOUBLE INTEGRALS

3.5.1. AREA AND VOLUME

See (3.1.10)-(3.1.12), Property 3.1.2.

EXAMPLE 3.5.1

a. Find the volume of the solid inside both the cylinder $x^2+y^2=4$ and the ellipsoid $4x^2+4y^2+z^2=64$

Solution :

$$D=\{(r,\theta): r\le 2,\ 0\le\theta\le 2\pi\}$$

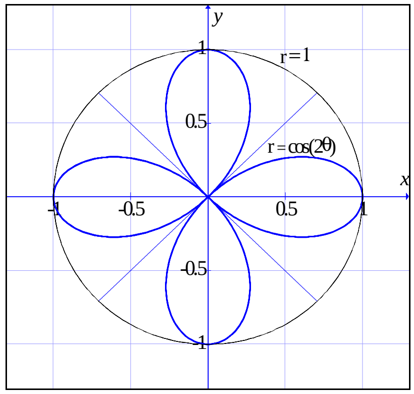
$$V = \iint_D 2\sqrt{64-4(x^2+y^2)}\,dx\,dy = \int_0^{2\pi}\left[\int_0^2 4\sqrt{16-r^2}\ \ rdr\right]d\theta = \frac{64\pi}{3}\left(8-3\sqrt{3}\right).$$

b. Find the volume of the solid bounded by two paraboloids $z=3-x^2-y^2$ and $z=x^2+y^2-1$.

Solution: $$V = \iint_D [(3-x^2-y^2)-(x^2+y^2-1)]\,dx\,dy = \int_0^{2\pi}\left[\int_0^{\sqrt{2}}(4-2r^2)\,r\,dr\right]d\theta = 4\pi .$$

c. Find the area of one loop of the four-leaved rose $r=\cos 2\theta$

Solution:



$$D=\{(r,\theta):0\le r\le \cos 2\theta, -\pi/4\le\theta\le\pi/4\}.$$

$$A=\iint_D dx\,dy=\int_{-\pi/4}^{\pi/4}\left[\int_0^{\cos 2\theta} r\,dr\right]d\theta=\pi/8$$

d. Find the volume of the solid between two paraboloids $z=3(x^2+y^2)$, $z=x^2+y^2$, and inside the cylinder $2y=x^2+y^2$.

Solution :

$$D=\{(r,\theta):0\le r\le 2\sin\theta, 0\le\theta\le\pi\}$$

$$V=\iint_D[(3(x^2+y^2)-(x^2+y^2)]\,dx\,dy=\int_0^{\pi}\left[\int_0^{2\sin\theta}2r^2.r\,dr\right]d\theta=3\pi.$$

e. Find the volume of the solid above the cone $z=\sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=1$.

Answer:

$$D=\left\{(r,\theta):r=\sqrt{2}/2,\ 0\le\theta\le 2\pi\right\};\ V=\frac{2\pi}{3}\left(1-\frac{1}{\sqrt{2}}\right)$$

3.5.2. DENSITY AND MASS

Suppose a thin plate or a lamina occupies a region D on the xy-plane and its mass **density** at a point (x, y) in D is given by $\rho(x,y)$. Suppose $\rho(x,y)$ is continuous in D. This means that

$$\rho(x,y)=\lim_{\varepsilon\to 0}\frac{\Delta m}{\Delta A} \tag{3.5.1}$$

where ΔA is the area of an ε–neighborhood G_ε of (x, y), and Δm is the mass of G_ε.

The total mass of the lamina is

$$\boxed{m=\iint_D \rho(x,y)\,dx\,dy} \tag{3.5.2}$$

EXAMPLE 3.5.2

a. Find the mass of the lamina that occupies the region *D* bounded by $x = y^2, y = x-2,$ and has the density $\rho(x,y) = x$

Solution

$$m = \iint_D x\,dx\,dy = \int_{-1}^{2}\left[\int_{y^2}^{y+2} x\,dx\right]dy = \frac{1}{2}\int_{-1}^{2}\left(4+4y+y^2-y^4\right)dy = \frac{36}{5}$$

b. Find the mass of the lamina that occupies the part of the disk $x^2+y^2=1$ in the first quadrant if the density at (x,y) is proportional to its distance from the *x*-axis.

Answer: *k*/3, where *k* is the proportional coefficient.

c. Find the mass of a triangular lamina with vertices (0, 0), (1, 0), and (0, 2) if the density is $\rho(x,y) = 1+3x+y$

Answer: $m = 8/3$

3.5.3. MOMENTS AND CENTER OF MASS

Suppose a thin plate or a lamina occupies a region *D* on *xy*-plane. Its mass **density** at a point (*x*, *y*) in *D* is given by $\rho(x,y)$.

The **moment about the *x*-axis** of the lamina is determined by

$$\boxed{M_x = \iint_D y.\rho(x,y)\,dx\,dy} \tag{3.5.3}$$

The **moment about the *y*-axis** of the lamina is determined by

$$\boxed{M_y = \iint_D x.\rho(x,y)\,dx\,dy} \tag{3.5.4}$$

The **center of mass** $(\overline{x}, \overline{y})$ of the lamina is determined by

$$\boxed{\overline{x} = \frac{M_y}{m} = \frac{\iint_D x\,\rho(x,y)\,dx\,dy}{\iint_D \rho(x,y)\,dx\,dy}, \quad \overline{y} = \frac{M_x}{m} = \frac{\iint_D y.\rho(x,y)\,dx\,dy}{\iint_D \rho(x,y)\,dx\,dy}} \tag{3.5.5}$$

Note:

The moment M_x (M_y) measures the ***tendency*** of the lamina **to rotate** about the *x*-axis (about the *y*-axis). The center of mass is the point where a single particle of mass m would have the same moments as the mass lamina.

EXAMPLE 3.5.3

Find the center of mass of the laminas given in Example 3.5.2

a. D is bounded by $x = y^2, y = x-2,$ and has the density $\rho(x,y) = x;\ m = 36/5$:

$$M_x = \iint_D y.x\,dx\,dy = \int_{-1}^{2}\left[y\int_{y^2}^{y+2} x\,dx\right]dy = \frac{45}{8};\ M_y = \iint_D x.x\,dx\,dy = \frac{423}{28};\ (\bar{x},\bar{y}) = \left(\frac{2115}{1008},\frac{225}{288}\right)$$

d. The lamina occupies the part of the disk $x^2 + y^2 = 1$ in the first quadrant and the density at (x,y) is proportional to its distance from the x-axis. $m = k/3$.

Answer: $M_x = \iint_D y\rho(x,y)\,dx\,dy = \dfrac{k\pi}{16};\ M_y = \iint_D x\rho(x,y)\,dx\,dy = \dfrac{k}{8};\ (\bar{x},\bar{y}) = \left(\dfrac{3}{8},\dfrac{3\pi}{16}\right)$

c. The lamina is a triangle with vertices (0, 0), (1, 0), and (0, 2) and the density is $\rho(x,y) = 1 + 3x + y$. $m = 8/3$

Answer: $(\bar{x},\bar{y}) = \left(\dfrac{3}{8},\dfrac{11}{16}\right)$

3.5.4. MOMENT OF INERTIA

The **moment of inertia about the *x*-axis** of the lamina is determined by

$$\boxed{I_x = \iint_D y^2\rho(x,y)\,dx\,dy} \tag{3.5.6}$$

The **moment of inertia about the *y*-axis** of the lamina is determined by

$$\boxed{I_y = \iint_D x^2\rho(x,y)\,dx\,dy} \tag{3.5.7}$$

The **moment of inertia about the origin** of the lamina is determined by

$$\boxed{I_0 = \iint_D \left(x^2+y^2\right)\rho(x,y)\,dx\,dy = I_x + I_y} \tag{3.5.8}$$

EXAMPLE 3.5.4

Find the moments of inertia I_x, I_y, and I_0 of a disk D with density $\rho(x,y) = \sqrt{x^2+y^2}$, center the origin, and radius a.

Answer:

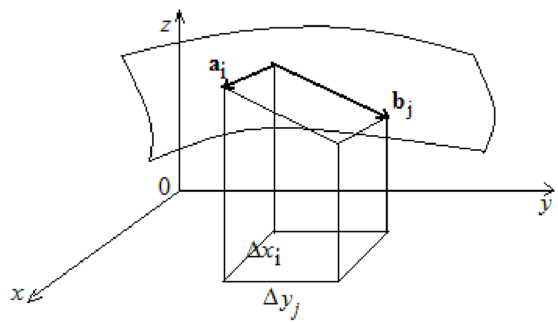
By the symmetry, $I_x = I_y$, then $I_0 = \dfrac{2\pi a^5}{5};\ I_x = I_y = \dfrac{I_0}{2} = \dfrac{\pi a^5}{5}$.

3.5.5. SURFACE AREA

Let S be a surface $z = f(x,y),\ (x,y) \in D \subset \mathbb{R}^2$, where $f(x,y)$ has continuous partial derivatives.

How can we define and determine the area of S? As usually, we divide S into patches S_{ij} by the planes perpendicular to the x-axis and the y-axis. See Figure below. The area $A(S_{ij})$ of S_{ij} is approximated by the area $A(T_{ij})$ of a parallelogram T_{ij} on the tangent plane at some point $P_{ij}\left(x_i, y_j, f(x_i, y_j)\right) \in S_{ij}$ with two sides determined by tangent vectors $\mathbf{a}_i$ and $\mathbf{b}_j$ in the direction of the x-axis and the y-axis, respectively:

$$\mathbf{a}_i = \left\langle \Delta x_i,\, 0,\, f_x(x_i, y_j)\Delta x_i \right\rangle,\ \ \mathbf{b}_j = \left\langle 0,\, \Delta y_j,\, f_y(x_i, y_j)\Delta y_j \right\rangle$$



Clearly,

$$\mathbf{a}_i \times \mathbf{b}_j = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Delta x_i & 0 & f_x(x_i, y_j)\Delta x_i \\ 0 & \Delta y_j & f_y(x_i, y_j)\Delta y_j \end{vmatrix} = \Delta x_i \Delta y_j \left\langle -f_x(x_i, y_j),\, -f_y(x_i, y_j),\, 1 \right\rangle$$

Therefore

$$A(S_{i,j}) \approx A(T_{i,j}) = \left|\mathbf{a}_i \times \mathbf{b}_j\right| = \sqrt{1 + \left[f_x(x_i, y_j)\right]^2 + \left[f_y(x_i, y_j)\right]^2}\ \Delta x_i\, \Delta y_j$$

and the **surface area**, $A(S)$, of S is determined from

$$\boxed{A(S) = \iint_D \sqrt{1 + \left[f_x(x, y)\right]^2 + \left[f_y(x, y)\right]^2}\ dx\, dy} \qquad (3.5.9)$$

EXAMPLE 3.5.5

a. Evaluate the surface area of the plane $z = ax + by + c$ that is limited by the cylinder $\dfrac{x^2}{p^2} + \dfrac{y^2}{q^2} = 1,\ p > 0,\ q > 0.$

Answer: $A(S) = \iint_D \sqrt{1 + a^2 + b^2}\ dx\, dy = \pi pq\sqrt{1 + a^2 + b^2}$

b. Show that the surface area of the sphere of radius a is $A(S) = 4\pi a^2$.

c. Evaluate the surface area of the paraboloid $z = a(x^2 + y^2),\ 0 \le z \le h,\ a > 0,\ h > 0.$

Answer: $\dfrac{\pi}{6a^2}\left[(1 + 4ah)^{3/2} - 1\right]$

EXERCISES

1. Evaluate the double integral of the function $f(x, y)$ on the domain D. Find the average value of f on the domain D.
   a. $f = x^2 + 2y^2$, $D = \{(x, y) \,:\, 0 \le x \le 1,\ 1 \le y \le 2\}$
   b. $f = x - 3y^2$, D is the triangle ABC, $A(1, 3)$, $B(3, 5)$, $C(5, 0)$.
   c. $f = x^2 - y$, D is the quadrilateral $ABEF$, $A(0, 3)$, $B(2, 5)$, $E(4, 3)$ and $F(3, 1)$.
   d. $f = x^2 - y$, D is bounded by x- and y-axis and the line going through $A(1, 3)$ and $B(4, 1)$.

2. Find the volume of the solid S bounded by $z = 8 - \dfrac{x^2}{2} - y^2$, the planes $x = 2$ and $y = 2$. and the three coordinate planes.

3. Find the volume of the solid S in the first octant bounded by the cylinder $z = 9 - y^2$ and the plane $x = 2$.

4. Find the volume of the solid S lying under the elliptic paraboloid $x^2/4 + y^2/9 + z = 1$ and above the square $[-1, 1] \times [-2, 2]$.

5. Find the volume of the solid S lying under the paraboloid $2z = x^2 + y^2$ and above the region in the xy-plane bounded by $y = 2x$ and $y = x^2$.

6. Evaluate the double integral of the function $f(x, y)$ on the domain D.
   a. $f = 2x - y$, $D = \{(x, y) : x^2 + y^2 \le 4\}$.
   b. $f = x\cos y$, D is bounded by $y = 0$, $y = x^2$, and $x = 1$.
   Answer: a. 0; b. $(1 - \cos 1)/2$;

7. Find the volume of the given solid.
   a. Under the paraboloid $z = x^2 + y^2$ and above the region in the xy-plane bounded by $y = x^2, x = y^2$.
   b. Bounded by the cylinder $x^2 + z^2 = 9$, the plane $x + 2y = 2$, and the three coordinate planes.

8. Find the center of mass of the lamina that occupies the region D and has the given density function ρ.
   a. $D = \{(x, y) \,:\, -1 \le x \le 1,\ 0 \le y \le 1\}$; $\rho(x, y) = x^2$
   b. $D = \{(x, y) \,:\, x^2 + y^2 \le 1,\ x \ge 0,\ y \ge 0\}$; $\rho(x, y) = ky$, k is a positive constant.

9. Find the area of the region D in the xy-plane, where $D = \{(x, y) \,:\, 1 \le x^2 + y^2 \le 2y\}$.

10. Find the surface area of the part of the surface $z = 3x^2 + 4y$ that lies above the triangle ABC in the xy-plane, $O(0, 0)$, $A(1, 2)$, $B(1, 0)$.

11. Find the surface area of the part of the plane $z = 3x + 4y + 2$ that lies above the rectangle $D = [0, 5] \times [1, 4]$ in the xy-plane.

12. Evaluate $\iint_D (3x+4y)\,dxdy$, where D is bounded by $y = x,\ y = x-2,\ y = -2x,\ y = 3-2x$ using the transformation $x = \frac{1}{3}(u+v),\ y = \frac{1}{3}(v-2u)$.

13. Evaluate the surface area of the ellipsoid $\dfrac{x^2+y^2}{a^2}+\dfrac{z^2}{b^2}=1,\ a>0,\ b>0,\ a\neq b.$

ANSWERS

2. 24; 3. 36; 4. 166/27;

5. 108/35; 6. a. 0; b. $(1-\cos 1)/2$;

7. a. 6/35; b. $\dfrac{9}{2}\arcsin(2/3)+\dfrac{1}{6}(11\sqrt{5}-27)$;

8. a. (0, 1/2); b. $(3/8,\ 3\pi/16)$;

9. $\dfrac{\pi}{3}+\dfrac{\sqrt{3}}{2}$; 10. $53^{3/2}/54$;

11. $15\sqrt{26}$; 12. 11/3

13. Case 1: $b>a \Rightarrow A(S)=4\pi b^2\left[\dfrac{a^2}{b^2}+\dfrac{1}{\alpha}\left(\dfrac{\pi}{2}-\arctan\dfrac{1}{\alpha}\right)\right]$; where $\alpha=\sqrt{\frac{b^2}{a^2}-1}>0$

Case 2: $b<a \Rightarrow A(S)=4\pi b^2\left[\dfrac{a^2}{b^2}+\dfrac{1}{2\alpha}\ln\left|\dfrac{1+\alpha}{1-\alpha}\right|\right]$, where $\alpha=\sqrt{1-\frac{b^2}{a^2}},\ 0<\alpha<1$

Chapter 4
TRIPLE INTEGRALS

4.1 DEFINITIONS AND PROPERTIES

We extend the double integral of two-variable functions to the triple integral of three-variable functions. Triple integrals give us to compute volumes, masses,..., and center of solids.

DEFINITION 4.1.1 TRIPLE INTEGRAL

Let $f(x,y,z)$ be defined on the domain $D, D \subset G = [a,b]\times[c,d]\times[s,t] \subset \mathbb{R}^3$. Divide the rectangular box G into sub-boxes by planes

$$a = x_0 < x_1 < x_2 < ... < x_m = b,\ c = y_0 < y_1 < y_2 < ... < y_n = d \text{ and } s = z_0 < z_1 < z_2 < ... < z_p = t$$

Denote

$$\Delta x_i = x_i - x_{i-1}, 1 \le i \le m,\ \ \Delta y_j = y_j - y_{j-1}, 1 \le j \le n,\ \Delta z_k = z_k - z_{k-1}, 1 \le k \le p,$$

$$S_{m,n,p} = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} f(x_i^*, y_j^*, z_k^*)\Delta x_i \Delta y_j \Delta z_{kj} \tag{4.1.1}$$

where

$$x_i^* \in [x_{i-1},\ x_i],\ 1 \le i \le m,\ \ y_j^* \in [y_{j-1},\ y_j],\ 1 \le j \le n,\ z_k^* \in [z_{k-1},\ x_k],\ 1 \le k \le p,$$

$$f(x_i^*, y_j^*) = 0, \text{ if } (x_i^*, y_j^*) \notin D, \text{ in convention.}$$

Denote

$$\lambda_{m,n,p} = \max\{\Delta x_i, \Delta y_j, \Delta z_k : 1 \le i \le m, 1 \le j \le n, 1 \le k \le p\}$$

If there exists an unique and finite limit

$$\lim_{\lambda_{m,n,p}\to 0} S_{m,n,p} = \lim_{\lambda_{m,n}\to 0} \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} f(x_i^*, y_j^*, z_k^*)\Delta x_i \Delta y_j \Delta z_k = L \tag{4.1.2}$$

then the limit is called **triple integral** of $f(x,y,z)$ on the domain D and denoted by

$$\iiint_D f(x,y,z)\,dV \ \text{ or } \ \iiint_D f(x,y,z)\,dx\,dy\,dz$$

Therefore we can write

$$\boxed{\iiint_D f(x,y,z)\,dV = \iiint_D f(x,y,z)\,dx\,dy\,dz = \lim_{\lambda_{m,n}\to 0} \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} f(x_i^*, y_j^*, z_k^*)\Delta x_i \Delta y_j \Delta z_k} \tag{4.1.3}$$

THEOREM 4.1.1 EXISTENCE CONDITION FOR DOUBLE INTEGRAL

If $f(x,y,z)$ is continuous on the domain $D, D \subset G = [a,b]\times[c,d]\times[s,t] \subset \mathbb{R}^3$, then the triple integral defined in (4.1.3) exists.

PROPERTY 4.1.1 TRIPLE INTEGRALS

Suppose that triple integrals of $f(x,y,z)$, $g(x,y,z)$ exist on the domains D, D_1, D_2, where D_1, D_2 do not overlap except perhaps on their boundary, and $D_1 \cup D_2 = D$, and k is a constant.

a. $\iiint\limits_D k.f(x,y,z)dxdydz = k\iiint\limits_D f(x,y,z)dxdydz$, ($k$ is a constant). (4.1.4)

b. $\iiint\limits_D [f(x,y,z)+g(x,y,z)]dxdydz = \iiint\limits_D f(x,y,z)dxdydz + \iiint\limits_D g(x,y,z)dxdydz$ (4.1.5)

c. $f(x,y,z) \ge g(x,y,z)$ on $D \Rightarrow \iiint\limits_D f(x,y,z)dxdydz \ge \iiint\limits_D g(x,y,z)dxdydz$ (4.1.6)

d. $\iiint\limits_D f(x,y,z)dxdydz = \iiint\limits_{D_1} f(x,y,z)dxdydz + \iiint\limits_{D_2} f(x,y,z)dxdydz$ (4.1.7)

e. $\iiint\limits_D dxdydz = \mathrm{V}(D)$ (4.1.8)

f. $m \le f(x,y,z) \le M$ on $D \Rightarrow m.\mathrm{V}(D) \le \iiint\limits_D f(x,y,z)dxdydz \le M.\mathrm{V}(D)$ (4.1.9)

PROPERTY 4.1.2 MIDPOINT RULE FOR TRIPLE INTEGRALS

If the domain *D* is closed and $f(x,y,z)$ is continuous on the domain *D*, then there exist a point $(x_0, y_0, z_0) \in D$ so that

$$\iiint\limits_D f(x,y,z)dxdydz = f(x_0,y_0,z_0).V(D) \qquad (4.1.10)$$

PROPERTY 4.1.3 APPROXIMATE CALCULATION

The triple integral can be approximately calculated by

$$\iiint\limits_D f(x,y,z)\,dx\,dy\,dz \approx \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} f(x_i^*, y_j^*, z_k^*)\Delta x_i \Delta y_j \Delta z_k \qquad (4.1.11)$$

4.2 ITERATED INTEGRALS

Suppose that $f(x,y,z)$ is continuous on the domain *D*. The triple integral (4.1.3) can be evaluated by calculating three single integrals, as shown below.

<u>**CASE 1**</u> If *D* is a rectangular box, $D = [a,b]\times[c,d]\times[s,t] \subset \mathbb{R}^3$, then

$$\iiint\limits_D f(x,y,z)dxdydz = \int_a^b \left[\int_c^d \left(\int_s^t f(x,y,z)dz\right)dy\right]dx = \int_a^b\int_c^d\int_s^t f(x,y,z)dz\,dy\,dx \qquad (4.2.1)$$

CASE 2 If $D=\{(x,y,z): z_1(x,y)\le z\le z_2(x,y),(x,y)\in G\subset \mathbb{R}^2\}\subset \mathbb{R}^3$, then

$$\iiint_D f(x,y,z)dxdydz=\iint_G\left[\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dz\right]dxdy \tag{4.2.2}$$

a. If $G=\{(x,y): y_1(x)\le y\le y_2(x),\ a\le x\le b\}\subset \mathbb{R}^2$ then (4.2.2) can be written in detail as

$$\iiint_D f(x,y,z)dxdydz=\int_a^b\int_{y_1(x)}^{y_2(x)}\int_{z_1(x,y)}^{z_1(x,y)} f(x,y,z)dz\,dy\,dx \tag{4.2.3}$$

b. If $G=\{(x,y): x_1(y)\le x\le x_2(y), c\le y\le d\}\subset \mathbb{R}^2$ then (4.2.2) can be written in detail as

$$\iiint_D f(x,y,z)dxdydz=\int_c^d\int_{x_1(y)}^{x_2(y)}\int_{z_1(x,y)}^{z_1(x,y)} f(x,y,z)dz\,dx\,dy \tag{4.2.4}$$

EXAMPLE 4.2.1

a. Evaluate $I=\iiint_D z(x+y)dxdydz$ where $D=[0,1]\times[0,2]\times[0,3]$

Answer: $I=\iiint_D z(x+y)dxdydz=\int_0^1\int_0^2\int_0^3(x+y)zdzdydx=\dfrac{27}{2}$

b. Evaluate $I=\iiint_D xy\,dxdydz$ where $D=\{(x,y,z): x\ge 0, y\ge 0, 0\le z\le 1-x-y\}$

Answer: $I=\iiint_D xydxdydz=\int_0^1\int_0^{1-x}\int_0^{1-x-y} xy\,dzdydx=\dfrac{1}{120}$

c. Evaluate $I=\iiint_D (x+y)\,dxdydz$, where

$D=\{(x,y,z): x+y-4\le z\le 5-(x+y), 0\le x\le 1, 0\le y\le 2)\}$.

Answer: $\iiint_D (x+y)dxdydz=\iint_G\left[\int_{x+y-4}^{5-(x+y)}(x+y)dz\right]dxdy=\dfrac{49}{3}$, $G=\{(x,y): 0\le x\le 1, 0\le y\le 2\}$

4.3 CHANGE OF VARIABLES IN TRIPLE INTEGRALS

In Chapter 3 we have seen how a change of variables (x,y) to (u,v) affects a double integral. Suppose a 1-1 transformation T from D_{uvw} in uvw-space to D_{xyz} in xyz-space is defined by differentiable functions

$$T:\quad x=x(u,v,w),\ y=y(u,v,w),\ z=z(u,v,w) \tag{4.3.1}$$

Similarly as in Section 3.3 of Chapter 3, The **Jacobian** of the transformation *T* is defined by

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \tag{4.3.2}$$

Therefore

$$\boxed{\iiint_{D_{xyz}} f(x,y,z)\,dx\,dydz = \iiint_{D_{uvw}} f\left(x(u,v,w), y(u,v,w),\ z(u,v,w)\right) |\,J\,|\,du\,dvdw} \tag{4.3.3}$$

4.4 TRIPLE INTEGRALS IN CYLINDRICAL AND SPHERICAL COORDINATES

Look at Section 1.4 in Chapter 1 for definitions of **cylindrical coordinates** and **spherical coordinates**. See Figures 5.1.1 – 5.1.2.

To convert from rectangular to cylindrical coordinates and inversely, we use

$$r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}, \qquad z = z \tag{4.4.1}$$

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z \tag{4.4.2}$$

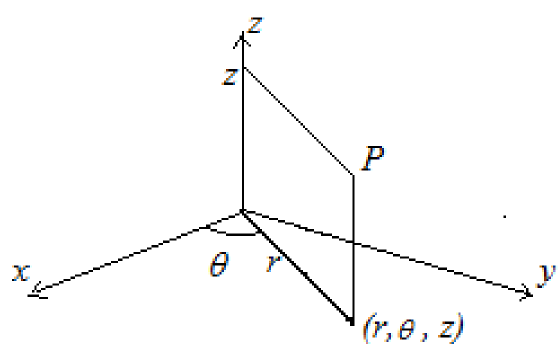


Figure 5.1.1 Cylindrical coordinates

Note that the **Jacobian**, determined from (4.3.2) is

$$J = \frac{\partial(x,y,z)}{\partial(z,r,\theta)} = r \tag{4.4.3}$$

Therefore

$$\boxed{dxdydz = r\,drd\theta dz} \tag{4.4.4}$$

To convert from rectangular to spherical coordinates and inversely, we use

$$\rho = \sqrt{x^2 + y^2 + z^2}, \quad \tan\theta = \frac{y}{x}, \quad \cos\phi = \frac{z}{\sqrt{x^2 + y^2 + z^2}} \tag{4.4.5}$$

$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi \tag{4.4.6}$$

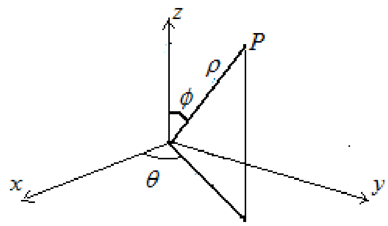


Figure 5.1.2 Spherical coordinates

Note that the Jacobian, determined from (4.3.2) is

$$J = \frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} = \rho^2 \sin\phi \tag{4.4.7}$$

Therefore

$$\boxed{dx\,dy\,dz = \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi} \tag{4.4.8}$$

EXAMPLE 4.4.1

a. Evaluate $I = \iiint\limits_D \sqrt{x^2 + y^2}\,dxdydz$, where $D = \left\{(x, y, z) : x^2 + y^2 \le 1,\ 1 - x^2 - y^2 \le z \le 5\right\}$

Answer: $I = \int\limits_0^{2\pi}\int\limits_0^1\int\limits_{1-r^2}^5 r^2 dzdrd\theta = \dfrac{46\pi}{15}$

b. Evaluate $I = \iiint\limits_D \left(x^2 + y^2\right)dxdydz$, where $D = \left\{(x, y, z) : \sqrt{x^2 + y^2} \le z \le 3\right\}$.

Answer: $I = \int\limits_0^{2\pi}\int\limits_0^3\int\limits_r^3 r^3 dz\,dr\,d\theta = \dfrac{243\pi}{10}$

c. Evaluate $I = \iiint\limits_D e^{-(x^2+y^2+z^2)^{3/2}}\,dxdydz$, where D is the ball of center (0, 0, 0) and radius a.

Answer: $\dfrac{4\pi}{3}\left(1 - e^{-a^3}\right)$

4.5 APPLICATION OF TRIPLE INTEGRALS

4.5.1. VOLUME

$$\boxed{V(D) = \iiint\limits_D dx\,dy\,dz} \tag{4.5.1}$$

EXAMPLE 4.5.1

a. Find the volume of the solid: $D=\{(x,y,z):1\le x^2+y^2\le 4, 0\le z\le 1+x^2+y^2\}$

Answer: $V=\int_0^{2\pi}\int_1^{2}\int_0^{1+r^2} r\,dzdrd\theta=\dfrac{21\pi}{2}$

b. Find the volume of the solid: $D=\{(x,y,z):x^2+y^2+z^2\le 1,\ x^2+y^2+(z-1)^2\le 1\}$

Answer: $V=\int_0^{2\pi}\int_0^{\frac{\sqrt{3}}{2}}\int_{1-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r\,dzdrd\theta=\dfrac{5\pi}{12}$

c. Use spherical coordinates to find the volume of the solid that lies above the cone $z=\sqrt{x^2+y^2}$ and below the sphere $z=x^2+y^2+z^2$

Answer: $V=\int_0^{\pi/4}\int_0^{2\pi}\int_0^{\cos\phi}\rho^2\sin\phi\,d\rho d\theta d\phi=\dfrac{\pi}{8}$

4.5.2. DENSITY AND MASS

Suppose a solid occupies a region D on the space and its mass **density** at a point (x, y, z) in D is given by $\rho(x,y,z)$. Suppose $\rho(x,y,z)$ is continuous in D. It means that

$$\rho(x,y,z)=\lim_{\varepsilon\to 0}\frac{\Delta m}{\Delta V} \tag{4.5.2}$$

where ΔV is the volume of an ε–neighborhood G_ε of (x, y, z), and Δm is the mass of G_ε.
The total mass of the solid is

$$\boxed{m=\iiint_D \rho(x,y,z)\,dx\,dy\,dz} \tag{4.5.3}$$

EXAMPLE 4.5.2

a. Find the mass of the solid D = $\{(x,y,z):0\le y\le 2, 0\le x\le\sqrt{4-y^2}, 0\le z\le y\}$ if the density $\rho(x,y,z)=x$.

Answer: 2

b. Find the mass of the cube D = $[0,a]\times[0,a]\times[0,a]$ if the density is $\rho=x^2+y^2+z^2$.
Answer: a^5

4.5.3. MOMENTS, CENTER OF MASS, AND MOMENTS OF INERTIA

Suppose a solid occupies a region D on the space. Its mass **density** at a point (x, y, z) in D is given by $\rho(x,y,z)$.

The **moment about the *yz*-plane, *xz*-plane, and *xy*-plane,** of the solid is determined by

$$\boxed{M_{yz} = \iiint\limits_D x.\rho(x,y,z)\,dx\,dy\,dz} \tag{4.5.4}$$

$$\boxed{M_{xz} = \iiint\limits_D y.\rho(x,y,z)\,dx\,dy\,dz} \tag{4.5.5}$$

$$\boxed{M_{xy} = \iiint\limits_D z.\rho(x,y,z)\,dx\,dy\,dz} \tag{4.5.6}$$

The **center of mass** $(\bar{x},\bar{y},\bar{z})$ of the lamina is determined by

$$\boxed{\bar{x} = \frac{M_{yz}}{m},\quad \bar{y} = \frac{M_{xz}}{m},\quad \bar{z} = \frac{M_{xy}}{m}} \tag{4.5.7}$$

If the density is constant, the center of mass is called the **centroid** of *D*.

The **moment of inertia about the *x*-axis, *y*-axis,** and ***z*-axis** are determined by

$$\boxed{I_x = \iiint\limits_D \left(y^2+z^2\right)\rho(x,y,z)\,dx\,dy\,dz} \tag{4.5.8}$$

$$\boxed{I_y = \iiint\limits_D \left(x^2+z^2\right)\rho(x,y,z)\,dx\,dy\,dz} \tag{4.5.9}$$

$$\boxed{I_z = \iiint\limits_D \left(x^2+y^2\right)\rho(x,y,z)\,dx\,dy\,dz} \tag{4.5.10}$$

EXAMPLE 4.5.3

Find the center of mass of the cube given in Example 4.5.2.b

Answer: $(7a/12,\ 7a/12,\ 7a/12)$

EXERCISES

4.1 Find the volume of the solid S bounded by four cylinders
$y = 3x^2$, $y = 8 + x^2$, $z = y^2$, and $z = (1+y)^2$.

4.2 Find the volume of the solid, in the first octant, bounded by the coordinate planes and between the planes $x + y + z = 12$, and $3x + 4y + 5z = 60$.

4.3 Given $\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\, dz\, dy\, dx$

a. Rewrite the integral in other orders.

b. Evaluate the integral, if $f(x,y,z) = 1 + 2x + z$

4.4 Find the moments of inertia of the cube $D = [0,a]\times[0,a]\times[0,a]$ if the density is k (constant).

4.5 Find the mass and the moments of inertia of the solid bounded by the cylinder $x^2 + y^2 = a^2$ and planes $z = 0,\ z = h$, (a >0, h > 0), if the mass density at a point M is proportional to the distance from the xy-plane to M.

4.6 Find the mass and the moments of inertia of a ball of center 0 and radius a if the mass density at a point M is proportional to the distance from the origin to M.

4.7 Find the volume of a solid bounded by the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

4.8 Find the volume of the solid that lies above the cone $\phi = \pi/3$ and below the sphere $\rho = 4\cos\phi$. Determine its centroid.

4.9 Find the mass and center of mass of the solid S bounded by the paraboloid $z = 4(x^2 + y^2)$ and the plane $z = a, a > 0,$ if S has the constant density K.

4.10 a. Find the centroid of a solid homogeneous hemisphere of radius a, and

b. Find the moment of inertia of the solid about a diameter of its base.

ANSWERS

4.1 $260 + \dfrac{4}{15}$; 4.2 312; 4.3 2/15; 4.4 $I_x = I_y = I_z = \frac{2}{3}ka^5$

4.5 $m = \frac{1}{2}k\pi a^2h^2,\ I_x = I_y = \frac{1}{4}k\pi a^2h^2(h^2 + \frac{a^2}{2}),\ I_z = \frac{1}{4}k\pi a^4h^2$.

4.6 $m = k\pi a^4,\ I_x = I_y = I_z = \frac{4}{9}k\pi a^6$. 4.7 $\dfrac{4}{3}\pi abc$; 4.8 10π; (0, 0, 2.1);

4.9 $K\pi a^2/8$; $(0,\ 0,\ 2a/3)$; 4.10 a. $\left(0, 0, \dfrac{3}{8}a\right)$; b. $4K\pi a^5/15$;

Chapter 5

LINE INTEGRALS

5.1 LINE INTEGRALS

We extend the single integral over a curve $\mathscr{C}$. Such integrals are called *line integrals*.

DEFINITION 5.1.1 LINE INTEGRALS ON PLANE

Let $f(x,y)$ be defined on an open domain D that includes the curve $\mathscr{C}$. Divide $\mathscr{C}$ into sub-arcs that are labeled together with their lengths by $\Delta s_i, 1 \le i \le n,$ and choose any points $P_i^*(x_i^*, y_i^*) \in \Delta s_i$ $,1 \le j \le n,$ see Figure 5.1.1, and form the sum

$$S_n = \sum_{i=1}^{n} f(x_i^*, y_j^*)\Delta s_i \tag{5.1.1}$$

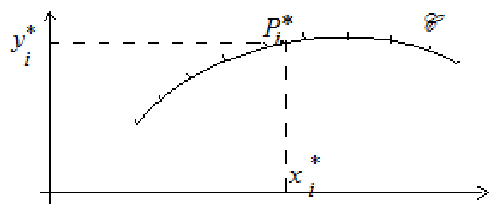


Figure 5.1.1

Denote $\lambda_n = \max\{\Delta s_i : 1 \le i \le n\}$. If there exists an unique and finite limit L

$$\lim_{\lambda_n \to 0} S_n = \lim_{\lambda_n \to 0} \sum_{i=1}^{n} f(x_i^*, y_j^*)\Delta s_i = L \tag{5.1.2}$$

then the limit is called **line integral** of $f(x,y)$ on the curve $\mathscr{C}$ and denoted by

$$\int_{\mathscr{C}} f(x,y)\,ds \tag{5.1.3}$$

Therefore

$$\boxed{\int_{\mathscr{C}} f(x,y)\,ds = \lim_{\lambda_n \to 0} S_n = \lim_{\lambda_n \to 0} \sum_{i=1}^{n} f(x_i^*, y_j^*)\Delta s_i} \tag{5.1.4}$$

<u>THEOREM 5.1.1</u> EXISTENCE CONDITION FOR LINE INTEGRALS

a. If the equation of the curve $\mathscr{C}$ is $y = \psi(x),\ a \le x \le b,$ where $\psi'(x)$ is continuous on the interval $[a, b]$, (we say that the curve is **smooth**) and $f(x,y)$ is continuous on an open domain D that contains $\mathscr{C}$. Then the line integral defined by (5.1.4) exists and it is evaluated from

$$\boxed{\int_{\mathscr{C}} f(x,y)\,ds = \int_a^b f\left(x,\psi(x)\right)\sqrt{1+\left[\psi'(x)\right]^2}\,dx} \qquad (5.1.5)$$

b. If the parametric equations of the curve $\mathscr{C}$ is $x = x(t),\ y = y(t),\ \alpha \le t \le \beta,$ where $x'(t)$ and $y'(t)$ are continuous on $[\alpha,\ \beta]$, (we say that the curve is **smooth**), and $f(x,y)$ is continuous on the domain D that contains $\mathscr{C}$. Then the line integral defined by (5.1.4) exists and it is evaluated from

$$\boxed{\int_{\mathscr{C}} f(x,y)\,ds = \int_\alpha^\beta f\left(x(t),y(t)\right)\sqrt{\left[x'(t)\right]^2+\left[y'(t)\right]^2}\,dt} \qquad (5.1.6)$$

Note: We do not care about the direction of the curve. But we suppose always that $a < b$ in (5.1.5) and $\alpha < \beta$ in (5.1.6).

PROPERTY 5.1.1 LINE INTEGRALS

a. If $\mathscr{C}$ is a **piecewise-smooth** curve, that is, $\mathscr{C}$ is a union of a finite number of smooth curves $\mathscr{C}_i$, $1 \le i \le m$. Then

$$\int_{\mathscr{C}} f(x,y)\,ds = \int_{\mathscr{C}_1} f(x,y)\,ds + \int_{\mathscr{C}_2} f(x,y)\,ds + \cdots + \int_{\mathscr{C}_m} f(x,y)\,ds \qquad (5.1.7)$$

b. $\int_{\mathscr{C}} k.f(x,y)\,ds = k.\int_{\mathscr{C}} f(x,y)\,ds$, ($k$ is a constant). (5.1.8)

c. $$\int_{\mathscr{C}} \left[f(x,y)+g(xy)\right]ds = \int_{\mathscr{C}} f(x,y)\,ds + \int_{\mathscr{C}} g(x,y)\,ds \qquad (5.1.9)$$

d. $$f(x,y) \ge g(x,y) \text{ on } D \supset \mathscr{C} \Rightarrow \int_{\mathscr{C}} f(x,y)\,ds \ge \int_{\mathscr{C}} g(x,y)\,ds \qquad (5.1.10)$$

e. $$m \le f(x,y) \le M \text{ on } D \supset \mathscr{C} \Rightarrow m.L(\mathscr{C}) \le \int_{\mathscr{C}} f(x,y)ds \le M.L(\mathscr{C}) \qquad (5.1.11)$$

$L(\mathscr{C})$ = length of $\mathscr{C}$.

f. If $\rho(x,y)$ is the **linear mass density** of a thin wire shaped like the curve $\mathscr{C}$. Then the mass of the wire is determined from

$$m = \int_{\mathscr{C}} \rho(x,y)\,ds \qquad (5.1.12)$$

The center of mass of the wire is located at the point $(\overline{x},\overline{y})$, where

$$\overline{x} = \frac{1}{m}\int_{\mathscr{C}} x\rho(x,y)\,ds, \qquad \overline{y} = \frac{1}{m}\int_{\mathscr{C}} y\rho(x,y)\,ds \qquad (5.1.13)$$

g. The length of $\mathscr{C}$ is determined from

$$L(\mathscr{C}) = \int_{\mathscr{C}} ds \qquad (5.1.14)$$

EXAMPLE 5.1.1

a. Find the length, mass, and mass center of a thin wire shaped like the parabola $y = x^2, 0 \le x \le 1$, with linear mass density $\rho(x, y) = x$.

Solution:

i. $L(\mathscr{C}) = \int\limits_{\mathscr{C}} ds = \int\limits_0^1 \sqrt{1+4x^2}\,dx = \dfrac{2\sqrt{5}+\ln\left(2+\sqrt{5}\right)}{4}$, by integration by parts. (5.1.15)

ii. $m = \int\limits_{\mathscr{C}} x\,ds = \int\limits_0^1 x\sqrt{1+4x^2}\,dx = \dfrac{5\sqrt{5}-1}{12}$

iii. $\overline{x} = \dfrac{1}{m}\int\limits_{\mathscr{C}} x\rho(x,y)\,ds = \dfrac{1}{m}\int\limits_0^1 x^2\sqrt{1+4x^2}\,dx = \dfrac{1}{m}J = \dfrac{3\left[18\sqrt{5}-\ln\left(2+\sqrt{5}\right)\right]}{16\left(5\sqrt{5}-1\right)}$

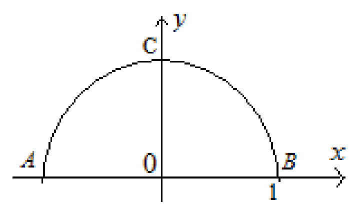
$\left\{ J = \int\limits_0^1 x^2\sqrt{1+4x^2}\,dx = \dfrac{18\sqrt{5}-\ln\left(2+\sqrt{5}\right)}{64}, \text{ by integration by parts and (5.1.15)} \right\}$

iv. $\overline{y} = \dfrac{1}{m}\int\limits_{\mathscr{C}} y\rho(x,y)\,ds = \dfrac{1}{m}\int\limits_0^1 x^2.x\sqrt{1+4x^2}\,dx = \dfrac{1}{m}K = \dfrac{25\sqrt{5}+1}{10\left(5\sqrt{5}-1\right)}$

$\left\{ K = \int\limits_0^1 x^2.x\sqrt{1+4x^2}\,dx = \dfrac{25\sqrt{5}+1}{120}, \text{ by integration by parts} \right\}$

b. Evaluate $I = \int\limits_{\mathscr{C}} (1+x+x^2y)\,ds$, $\mathscr{C}$: the circumference of the upper half of the unit disk.

Solution:



$\mathscr{C} = \mathscr{C}_1 + \mathscr{C}_2$, $\mathscr{C}_1 = AOB$, $\mathscr{C}_2 = ACB$,

$I_1 = \int\limits_{\mathscr{C}_1} (1+x+x^2y)\,ds = 2$, $I_2 = \int\limits_{\mathscr{C}_2} (1+x+x^2y)\,ds = \pi + \frac{2}{3}$, $I = \pi + \frac{8}{3}$

DEFINITION 5.1.2 LINE INTEGRALS IN SPACE

Let $f(x, y, z)$ be defined on an open domain D that contains a curve $\mathscr{C}$ in space. Divide $\mathscr{C}$ into sub-arcs that are labeled together with their lengths by Δs_i, $1 \le i \le n$, and choose any points $M_i^*(x_i^*, y_i^*, z_i^*) \in \Delta s_i$ $, 1 \le j \le n$, and form the sum

$$S_n = \sum_{i=1}^{n} f(x_i^*, y_j^*, z_i^*)\Delta s_i \tag{5.1.16}$$

Denote $\lambda_n = \max\{\Delta s_i : 1 \le i \le n\}$. If there exists an unique and finite limit

$$\lim_{\lambda_n \to 0} S_n = \lim_{\lambda_n \to 0} \sum_{i=1}^{n} f(x_i^*, y_j^*, z_j^*) \Delta s_i \tag{5.1.17}$$

then the limit is called **line integral** of $f(x, y, z)$ on the curve $\mathscr{C}$ and denoted by

$$\int_{\mathscr{C}} f(x, y, z)\, ds \tag{5.1.18}$$

Therefore we can write

$$\boxed{\int_{\mathscr{C}} f(x, y, z)\, ds = \lim_{\lambda_n \to 0} S_n = \lim_{\lambda_n \to 0} \sum_{i=1}^{n} f(x_i^*, y_j^*, z_j^*) \Delta s_i} \tag{5.1.19}$$

<u>THEOREM 5.1.1</u> EXISTENCE CONDITION FOR LINE INTEGRALS

If the parametric equations of the curve $\mathscr{C}$ is $x = x(t),\ y = y(t),\ z = z(t),\ \alpha \le t \le \beta$, where $x'(t)$, $y'(t)$ and $z'(t)$ are continuous on $[\alpha, \beta]$, (we say that the curve is **smooth**), and $f(x, y, z)$ is continuous on an open domain D that contains $\mathscr{C}$. Then the line integral defined by (5.1.19) exists and it is evaluated from

$$\boxed{\int_{\mathscr{C}} f(x, y, z)\, ds = \int_{\alpha}^{\beta} f\left(x(t), y(t), z(t)\right) \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt} \tag{5.1.20}$$

EXAMPLE 5.1.2

a. Find the mass of a thin wire shaped like the circular helix given by the equations $x = a\cos t,\ y = a\sin t,\ z = t,\ 0 \le t \le 2\pi$, with linear mass density $\rho(x, y, z) = x^2 z$.

Solution:

$$m = \int_{\mathscr{C}} \rho(x, y, z)\, ds = \int_{\mathscr{C}} x^2 z\, ds = \int_{0}^{2\pi} a^2 t \cos^2 t \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + 1}\, dt = \pi^2 a^2 \sqrt{1 + a^2}$$

b. Evaluate $\int_{\mathscr{C}} (xy + z)\, ds$, where $\mathscr{C}$ is the triangle OAB, $A(1,1,0)$, $B(1,1,1)$.

Solution:

$$I = \int_{\mathscr{C}} (xy + z)\, ds = \int_{OA} (xy + z)\, ds + \int_{AB} (xy + z)\, ds + \int_{OB} (xy + z)\, ds = \frac{9 + 2\sqrt{2} + 5\sqrt{3}}{6}$$

5.2 LINE INTEGRALS OF VECTOR FIELDS

Firstly, let us explain the notions and terminologies of regions in $\mathbb{R}^2$

Simple region type I, see Figure 1. **Simple region type II**, see Figure 2.

Simple region: both of type I and type II, see Figure 5.4.1 and 5.4.2, Section 5.4

Union of finite simple regions type I and type II, see Figure 3, $D = D_1 \cup D_2 \cup D_3$.

Simply-connected region, see Figure 1, 2, and 3.
Not simply-connected region, see Figure 4.
Connected region, see Figure 1, 2, 3, and 4.

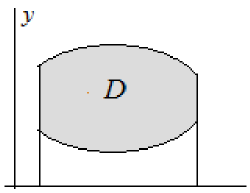


Figure 1

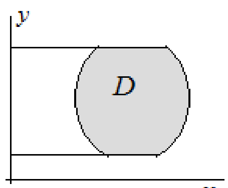


Figure 2

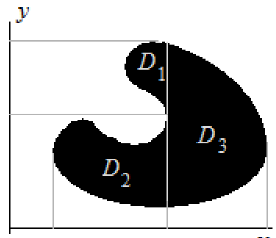


Figure 3

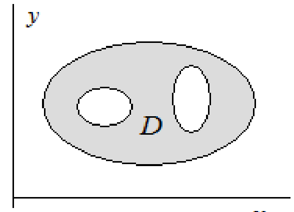


Figure 4

DEFINITION 5.2.1 VECTOR FIELD ON $\mathbb{R}^2$

Let $D \subset \mathbb{R}^2$. A vector field on D is a function $\mathbf{F}$ that assign to each point $(x, y) \in D$ a two-dimensional vector

$$\mathbf{F}(x, y) = \langle P(x, y),\, Q(x, y)\rangle = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j} \tag{5.2.1}$$

DEFINITION 5.2.2 VECTOR FIELD ON $\mathbb{R}^3$

Let $D \subset \mathbb{R}^3$. A vector field on D is a function $\mathbf{F}$ that assign to each point $(x, y, z) \in D$ a three-dimensional vector

$$\mathbf{F}(x, y, z) = \langle P(x, y, z),\, Q(x, y, z), R(x, y, z)\rangle = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\mathbf{k} \tag{5.2.2}$$

The *air velocity vectors* that indicate the wind speed and direction at points on D; the *force field* that associates a force vector with each point in a region D (e.g. the gravitational force field); the *magnetic field*... are examples for vector fields.

DEFINITION 5.2.3 THE WORK DONE BY A FORCE FIELD

Let $D \subset \mathbb{R}^3$. Let $\mathbf{F}$ be a force field, as given in (5.2.2), defined on an open $D \subset \mathbb{R}^3$ that contains a curve $\mathscr{C}$ given by a vector function

$$\mathbf{r}(t) = \langle x(t),\ y(t),\ z(t)\rangle, \alpha \le t \le \beta .$$

The end points of $\mathscr{C}$ are $M_\alpha = (x(\alpha),\ y(\alpha),\ z(\alpha))$ and $M_\beta = (x(\beta),\ y(\beta),\ z(\beta))$, according to $t = \alpha$ and $t = \beta$, respectively. How can we determine the work W done by the force field $\mathbf{F}$ in moving a particle along the curve $\mathscr{C}$ from M_α to M_β?

Divide $\mathscr{C}$ into small sub-arcs that are labeled together with their lengths by Δs_i, $1 \le i \le n$, and choose any points $M_i^*(x_i^*, y_i^*, z_i^*) \in \Delta s_i$, corresponding to the parameter value t_i^*, $1 \le j \le n$. The work done by the force $\mathbf{F}$ in moving a particle along the sub-arc Δs_i, is approximately $\mathbf{F}(x_i^*, y_j^*, z_i^*).\mathbf{T}(t_i^*)\Delta s_i$, where $\mathbf{T}(t_i^*)$ is the unit tangent vector at M_i^*, $1 \le i \le n$. Form the sum

$$\sum_{i=1}^{n} \mathbf{F}(x_i^*, y_j^*, z_i^*).\mathbf{T}(t_i^*)\Delta s_i \tag{5.2.3}$$

Denote $\lambda_n = \max\{\Delta s_i : 1 \le i \le n\}$. If there exists an unique and finite limit

$$\lim_{\lambda \to 0} \sum_{i=1}^{n} \mathbf{F}(x_i^*, y_j^*, z_i^*).\mathbf{T}(t_i^*)\Delta s_i = I \qquad (5.2.4)$$

then the work W is defined as the limit I.

DEFINITION 5.2.4 LINE INTEGRALS OF VECTOR FIELD ON $\mathbb{R}^3$

Let $D \subset \mathbb{R}^3$ and let **F** be a vector field defined on $D \subset \mathbb{R}^3$ that contains a curve $\mathscr{C}$ given by a vector function $\mathbf{r}(t), \alpha \le t \le \beta$. Use the same method as for the above work problem, we arrive to the limit (5.2.4) and define this limit as in (5.2.5) and call it as the **line integral of F along $\mathscr{C}$** from A to B and denoted by

$$\int_{\mathscr{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathscr{C}} \mathbf{F} \cdot \mathbf{T}\, ds = \lim_{\lambda \to 0} \sum_{i=1}^{n} \mathbf{F}(x_i^*, y_j^*, z_i^*).\mathbf{T}(t_i^*)\Delta s_i \qquad (5.2.5)$$

THEOREM 5.2.1 EXISTENCE CONDITION FOR LINE INTEGRALS

If the curve $\mathscr{C}$ is **smooth** and **F** is continuous on an open domain D that contains $\mathscr{C}$, then the line integral defined by (5.2.5) exists.

PROPERTY 5.2.1 EXPRESSION FOR LINE INTEGRALS OF VECTOR FIELD ON $\mathbb{R}^3$

a. $$\int_{\mathscr{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathscr{C}} \langle P(x,y,z), Q(x,y,z), R(x,y,z)dz \rangle \cdot \langle dx, dy, dz \rangle$$

$$= \int_{\mathscr{C}} P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz \qquad (5.2.6)$$

b. If the curve $\mathscr{C}$ from A to B is given by a vector function

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle,\ \alpha \le t \le \beta,$$

then

$$\int_{\mathscr{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\alpha}^{\beta} \left\langle P\big(x(t), y(t), z(t)\big),\ Q\big(x(t), y(t), z(t)\big), R\big(x(t), y(t), z(t)\big) \right\rangle \cdot \langle x'(t), y'(t), z'(t) \rangle\, dt$$

$$= \int_{\alpha}^{\beta} \Big[P\big(x(t), y(t), z(t)\big)x'(t) + Q\big(x(t), y(t), z(t)\big)y'(t) + R\big(x(t), y(t), z(t)\big)z'(t) \Big] dt \qquad (5.2.7)$$

c. Denote the curve $\mathscr{C}$ from A to B by $\mathscr{C}_{AB}$ and the curve $\mathscr{C}$ from B to A by $\mathscr{C}_{BA}$. Then

$$\int_{\mathscr{C}_{BA}} \mathbf{F} \cdot d\mathbf{r} = -\int_{\mathscr{C}_{AB}} \mathbf{F} \cdot d\mathbf{r} \qquad (5.2.8)$$

EXAMPLE 5.2.1

a. Evaluate $\int_{\mathscr{C}} \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = e^{x-1}\mathbf{i} + xy\mathbf{j}$ and $\mathscr{C}$ is given by $\mathbf{r} = t^2\mathbf{i} + t^3\mathbf{j},\ 0 \le t \le 1.$

Solution:

$$\int_{\mathscr{C}} \mathbf{F}\cdot d\mathbf{r} = \int_{\mathscr{C}} e^{x-1}dx + xydy = \frac{11}{8} - \frac{1}{e}$$

b. Evaluate $\int_{\mathscr{C}} \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = ye^{x^3-1}\mathbf{i} + (x+y)\mathbf{j}$ and $\mathscr{C}$ is given by $y = x^2, -1 \le x \le 2.$

Solution:

$$\int_{\mathscr{C}} \mathbf{F}\cdot d\mathbf{r} = \int_{\mathscr{C}} ye^{x^3-1}dx + (x+y)dy = \frac{e^7 - e^{-2}}{3} + \frac{27}{2}$$

c. Find the work done by the force field $\mathbf{F}(x,y,z) = x^2\,\mathbf{i} + xy\,\mathbf{j} + yz\,\mathbf{k}$ in moving a particle along the elliptic helix given by the equations $x = a\cos t,\ y = b\sin t,\ z = 2t,\ t \in \mathbb{R}$, $a, b > 0$,

i. From $A(a,0,0)$ to $B(-a,0,2\pi)$.

ii. From $B(-a,0,2\pi)$ to $A(a,0,0)$.

Solution:

i. $A(a,0,0)$ and $B(-a,0,2\pi)$ are corresponding to $t = 0$ *and* $t = \pi$, respectively.

$$\int_{\mathscr{C}_{AB}} \mathbf{F}\cdot d\mathbf{r} = \frac{2}{3}a(b^2 - a^2) + 4b\pi$$

ii. $$\int_{\mathscr{C}_{BA}} \mathbf{F}\cdot d\mathbf{r} = -\left[\frac{2}{3}a(b^2 - a^2) + 4b\pi\right]$$

d. The force exerted by an electric charge Q at the origin on a charge q at a point (x,y,z) with position vector $\mathbf{r} = \langle x,y,z\rangle$ is $\mathbf{F} = \varepsilon Qq\,\mathbf{r}/|\mathbf{r}|^3$, where ε is a constant. Find the work done by the force as the particle moves from $A(1,\ 0,\ 0)$ to $B(0,\ 1,\ \pi/2)$.

i. Along a straight line.

ii. Along the helix $\mathscr{C}: x = \cos t,\ y = \sin t,\ z = t,\ 0 \le t \le \frac{\pi}{2}$.

Solution:

i. $\mathscr{C}: x = 1 - t,\ y = t,\ z = \frac{\pi}{2}t,\ 0 \le t \le 1;$

$$W = \int_{\mathscr{C}} \mathbf{F}\cdot d\mathbf{r} = \varepsilon Qq \int_{\mathscr{C}} \frac{xdx + ydy + zdz}{\left(x^2 + y^2 + z^2\right)^{3/2}} = \varepsilon Qq\left(1 - \frac{1}{\sqrt{1 + \frac{\pi^2}{4}}}\right)$$

ii. $\mathscr{C}: x = \cos t,\ y = \sin t,\ z = t,\ 0 \le t \le \frac{\pi}{2}$

$$\int_{\mathscr{C}} \mathbf{F}\cdot d\mathbf{r} = \varepsilon Qq\left(1 - \frac{1}{\sqrt{1 + \frac{\pi^2}{4}}}\right)$$

5.3 THE FUNDAMENTAL THEOREM FOR LINE INTEGRALS

The Fundamental Theorem (see Chapter 7, Calculus 1) shows that:
If $f'(x)$ is continuous on [a, b], then

$$\int_a^b f'(x)dx = f(x)\Big|_a^b = f(b) - f(a) \tag{5.3.1}$$

We have the same result for line integral

DEFINITION 5.3.1 GRADIENT VECTOR FIELD

a. The gradient vector of $f = f(x, y)$ is defined by

$$\boxed{\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}} \tag{5.3.2}$$

b. The gradient vector of $f = f(x, y, z)$ is defined by

$$\boxed{\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}} \tag{5.3.3}$$

<u>THEOREM 5.3.1</u> THE FUNDAMENTAL THEOREM FOR LINE INTEGRALS

Let $\mathscr{C}$ be a **smooth** curve, in plane (or in space), given by $\mathbf{r}(t), \alpha \le t \le \beta$. Let f be a differentiable function of two (or three) variables whose gradient vector ∇f is continuous on an open domain D that contains $\mathscr{C}$. Then

$$\boxed{\int_{\mathscr{C}} \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(\beta)) - f(\mathbf{r}(\alpha))} \tag{5.3.4}$$

Proof:

$$\int_{\mathscr{C}} \nabla f \cdot d\mathbf{r} = \int_\alpha^\beta \left[\frac{\partial f}{\partial x}x'(t) + \frac{\partial f}{\partial y}y'(t) + \frac{\partial f}{\partial y}z'(t)\right]dt$$

$$= \int_\alpha^\beta \frac{df(\mathbf{r}(t))}{dt}dt = f(\mathbf{r}(\beta)) - f(\mathbf{r}(\alpha)).$$

DEFINITION 5.3.2 CONSERVATIVE VECTOR FIELD

A vector field **F** is called a **conservative vector field** if there is a scalar function f so that **F** is the gradient of f

$$\mathbf{F} = \nabla f \tag{5.3.5}$$

EXAMPLE 5.3.1

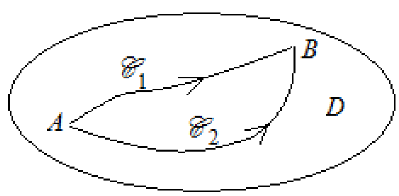
a. The vector field $\mathbf{F} = K\mathbf{r}/|\mathbf{r}|^3$ in Example 5.2.1.c is conservative because (5.3.3) is satisfied with $f(x, y, z) = -K/\sqrt{x^2 + y^2 + z^2}$.

b. Show that the vector field $\mathbf{F} = \mathbf{r}\, e^{-|\mathbf{r}|^2}$ is conservative: determine f such that $\mathbf{F} = \nabla f$?

◙®◙ $f(x, y, z) = -\dfrac{1}{2} e^{-(x^2+y^2+z^2)} \Rightarrow \mathbf{F} = \nabla f = \langle x, y, z\rangle e^{-(x^2+y^2+z^2)} = \mathbf{r}\, e^{-|\mathbf{r}^2|}$ ◙®◙

DEFINITION 5.3.3 INDEPENDENT OF PATH

If $\mathbf{F}$ is a continuous vector field on a domain D, we say that the line integral $\int_{\mathscr{C}} \mathbf{F}\cdot d\mathbf{r}$ is **independent of path** if $\int_{\mathscr{C}_1} \mathbf{F}\cdot d\mathbf{r} = \int_{\mathscr{C}_2} \mathbf{F}\cdot d\mathbf{r}$ for any two curves $\mathscr{C}_1$ and $\mathscr{C}_2$ in D that have the same initial and terminal points.



THEOREM 5.3.2

$\int_{\mathscr{C}} \mathbf{F}\cdot d\mathbf{r}$ is **independent of path** in D if and only if

$$\oint_{\mathscr{C}^*} \mathbf{F}\cdot d\mathbf{r} = 0$$

for any closed path $\mathscr{C}^*$ in D.

Proof: Clearly.

THEOREM 5.3.3

Let $\mathbf{F} = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$ be a vector field in an ***open connected*** domain D.

Then $\int_{\mathscr{C}} \mathbf{F}\cdot d\mathbf{r}$ is **independent of path** in D *if and only if* $\mathbf{F}$ is **conservative** .

Proof:

"If" : It follows from (5.3.4), Theorem 5.3.1.

"Only if": Take the case of $\mathbb{R}^2$. Suppose the line integral of $\mathbf{F} = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$ is independent of path.

Denote for fixed point $A(a, b)$.

$$f(x,y)=\int_{\mathscr{C}}\mathbf{F}\cdot d\mathbf{r}=\int_{\mathscr{C}_{AB}}\mathbf{F}\cdot d\mathbf{r}+\int_{\mathscr{C}_{BC}}\mathbf{F}\cdot d\mathbf{r}$$

Consider firstly the curve $\mathscr{C}$ as ABC in Figure 5.3.1, where $A(a,b)$, $B(x_1,y)$, $C(x,y)$. Because $\int_{\mathscr{C}_{AB}}\mathbf{F}\cdot d\mathbf{r}$ does not depend on *x*, then

$$\frac{\partial f(x,y)}{\partial x}=\frac{\partial}{\partial x}\int_{\mathscr{C}}\mathbf{F}\cdot d\mathbf{r}=\frac{\partial}{\partial x}\int_{\mathscr{C}_{BC}}\mathbf{F}\cdot d\mathbf{r}$$

$$=\frac{\partial}{\partial x}\int_{\mathscr{C}_{BC}}Pdx+Qdy=\frac{\partial}{\partial x}\left(\int_{x_1}^{x}P(x,y)dx\right)=P(x,y).$$

Consider secondly the curve $\mathscr{C}$ as ABC in Figure 5.3.2, where $A(a,b)$, $B(x,y_1)$, $C(x,y)$. Because $\int_{\mathscr{C}_{AB}}\mathbf{F}\cdot d\mathbf{r}$ does not depend on *y*, then

$$\frac{\partial f(x,y)}{\partial y}=\frac{\partial}{\partial y}\int_{\mathscr{C}}\mathbf{F}\cdot d\mathbf{r}=\frac{\partial}{\partial y}\int_{\mathscr{C}_{BC}}\mathbf{F}\cdot d\mathbf{r}$$

$$=\frac{\partial}{\partial y}\int_{\mathscr{C}_{BC}}Pdx+Qdy=\frac{\partial}{\partial x}\left(\int_{y_1}^{y}Q(x,y)dy\right)=Q(x,y).$$

Therefore $\mathbf{F}=\nabla f$.

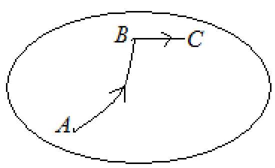


Figure 5.3.1

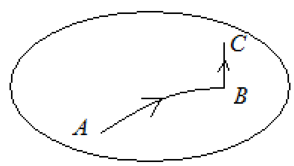


Figure 5.3.2

5.4 GREEN'S THEOREM

Consider a domain *D* in the *xy*-plane that is bounded by a simple closed curve $\mathscr{C}$. The **positive direction** of the curve is chosen so that *D* is always on the left when a point traverses $\mathscr{C}$ in this direction. Denote by $\mathscr{C}^+$ the positively oriented.

THEOREM 5.4.1 GREEN'S THEOREM

Let $\mathbf{F}=P(x,y)\,\mathbf{i}+Q(x,y)\,\mathbf{j}$, where $P(x,y)$ and $Q(x,y)$ have continuous partial derivatives on an *open region G that contains D*, where *D* is a closed region bounded by a *piecewise-smooth* and *simple closed* curve $\mathscr{C}$. Then

$$\boxed{\oint_{\mathscr{C}^+}\mathbf{F}\cdot d\mathbf{r}=\oint_{\mathscr{C}^+}P(x,y)\,dx+Q(x,y)\,dy=\iint_D\left(\frac{\partial Q(x,y)}{\partial x}-\frac{\partial P(x,y)}{\partial y}\right)dxdy}\qquad(5.4.1)$$

Proof :

(For the special case when *D* is a **simple region**, i.e. *D* is both of type I and type II, see Figures 5.4.1 and 5.4.2 below.)

$\iint_D \frac{\partial Q(x,y)}{\partial x} dxdy = \int_c^d \left(\int_{x_3(y)}^{x_4(y)} \frac{\partial Q(x,y)}{\partial x} dx \right) dy$, see Figure 5.4.2, $\mathscr{C}_3 : x = x_3(y), \mathscr{C}_4 : x = x_4(y)$,

$$= \int_c^d \left(Q[x_4(y), y] - Q[x_3(y), y] \right) dy = \int_c^d Q[x_4(y), y] dy + \int_d^c Q[x_3(y), y] dy = \oint_{\mathscr{C}^+} Q(x,y) dy$$

Then

$$\iint_D \frac{\partial Q(x,y)}{\partial x} dxdy = \oint_{\mathscr{C}^+} Q(x,y) dy \tag{5.4.2}$$

$\iint_D -\frac{\partial P(x,y)}{\partial y} dxdy = -\int_a^b \left(\int_{y_1(x)}^{y_2(x)} \frac{\partial P(x,y)}{\partial y} dy \right) dx$, see Figure 5.4.1, $\mathscr{C}_1 : y = y_1(x), \mathscr{C}_2 : y = y_2(x)$,

$$= -\int_a^b \left(P[x, y_2(x)] - P[x, y_1(x)] \right) dx$$

$$= \int_b^a P[x, y_2(x)] dx + \int_a^b P[x, y_1(x)] dx = \oint_{\mathscr{C}^+} P(x,y) dx$$

Then
$$\iint_D -\frac{\partial P(x,y)}{\partial y} dxdy = \oint_{\mathscr{C}^+} P(x,y) dx \tag{5.4.3}$$

Adding (5.4.2) and (5.4.3), we obtain (5.4.1).

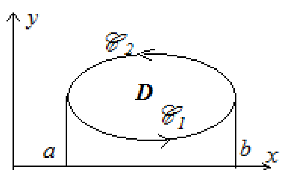


Figure 5.4.1

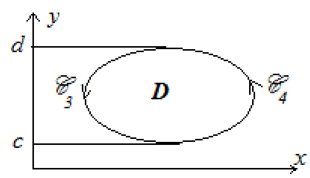


Figure 5.4.2

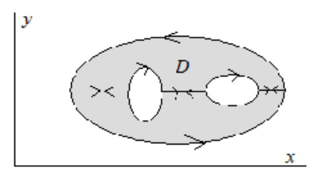


Figure 5.4.3

NOTE: The Green's Theorem can be extended to the case where D is not simply-connected region as given in Figure 5.4.3. In this case, the closed curve $\mathscr{C}$ is not simple.

COROLLARY 5.4.1 OF GREEN'S THEOREM

a. The area of a domain D in G is determined by (5.4.1) if $\frac{\partial Q(x,y)}{\partial x} - \frac{\partial P(x,y)}{\partial y} = 1$.

For example

$$\boxed{A(D) = \oint_{\mathscr{C}^+} xdy = -\oint_{\mathscr{C}^+} ydx = \frac{1}{2} \oint_{\mathscr{C}^+} xdy - ydx} \tag{5.4.4}$$

b. Let $\mathbf{F} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$, where $P(x,y)$ and $Q(x,y)$ have continuous partial derivatives on an *open simply-connected region* G and that

$$\frac{\partial Q(x,y)}{\partial x} = \frac{\partial P(x,y)}{\partial y} \quad \text{throughout } G \tag{5.4.5}$$

Then

(i) For any *simple closed path* $\mathscr{C}$ in G that bounds a region D

$$\oint_{\mathscr{C}} P(x,y)\,dx + Q(x,y)\,dy = 0 \qquad (5.4.6)$$

(ii) $\int_{\mathscr{C}} \mathbf{F}\cdot d\mathbf{r} = \int_{\mathscr{C}} P(x,y)\,dx + Q(x,y)\,dy$ is independent of path in G. (5.4.7)

(iii) $\mathbf{F}$ is a *conservative vector field*, it means that there is a scalar function f so that

$$\mathbf{F} = \nabla f \qquad (5.4.8)$$

EXAMPLE 5.4.1

a. Evaluate $\oint_{\mathscr{C}^+} (x^2 + \sin x)\,dx + xy\,dy$ where $\mathscr{C}$ is $OABO$, $A(1,0)$, $B(0,1)$.

Solution:

$$\oint_{\mathscr{C}^+} (x^2 + \sin x)\,dx + xy\,dy = \iint_D ydxdy = \frac{1}{6}$$

b. Evaluate $\oint_{\mathscr{C}^+} (e^x - 2yx^2)\,dx + (2xy^2 - 3y)\,dy$ where $\mathscr{C}$ is the circle $x^2 + y^2 = 2$.

Solution:

$$\oint_{\mathscr{C}^+} (e^x - 2yx^2)\,dx + (2xy^2 - 3y)\,dy = \iint_D 2(x^2 + y^2)dxdy = 4\pi .$$

c. Using the line integral to show that the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is πab.

d. Evaluate $\int_{\mathscr{C}} \left[xy + (y^2/2) + e^x \sin(e^x)\right]dx + \left[(x^2/2) + ye^{-y^2} + xy\right]dy$

where $\mathscr{C} = \overset{\frown}{OA}: y = \sqrt{x},\ 0 \le x \le 1.$

Answer: $\frac{3}{2} - \frac{1}{2e} + \cos(1) - \cos(e)$

e. Evaluate $\oint_{\mathscr{C}^+} (\sin y - 2x^2)\,dx + (2y + x\cos y)\,dy$ where $\mathscr{C}$ is the curve $x^2 + xy + y^2 = 1$.

Answer: 0.

We define now two operations that can be performed on vector field $\mathbf{F}$ and that have important applications in vector calculus to fluid flow and electricity and magnetism: one, $\operatorname{curl}\mathbf{F}$, produces a vector field and the other, $\operatorname{div}\mathbf{F}$, produces a scalar field; both concern with differentiation.

5.5 CURL

DEFINITION 5.5.1 VECTOR DIFFERENTIAL OPERATOR "DEL" ∇

The **vector differential operator "del"** ∇ denoted formally by

$$\boxed{\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}} \quad (5.5.1)$$

is defined as when it operates on a scalar function $f = f(x, y, z)$ it gives the **gradient** of *f*:

$$\boxed{\nabla f = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} = f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}} \quad (5.5.2)$$

DEFINITION 5.5.2 CURL OF VECTOR FIELDS

Let $\mathbf{F} = \mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$ be a vector field. Define

$$\boxed{\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F}} \quad (5.5.3)$$

$$\boxed{\begin{aligned}&\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F}\\ &\qquad = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}\end{aligned}} \quad (5.5.4)$$

EXAMPLE 5.5.1

a. $\mathbf{F} = x^2 z\,\mathbf{i} + 2yxz^2\,\mathbf{j} - 5y^2 z\,\mathbf{k}$; $\operatorname{curl}\mathbf{F}$?

$$\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 z & 2xyz^2 & -5y^2 z \end{vmatrix}$$

$$= \left(-10yz - 4xyz\right)\mathbf{i} + \left(x^2 - 0\right)\mathbf{j} + \left(2yz^2 - 0\right)\mathbf{k} = -2yz(5+2x)\mathbf{i} + x^2\mathbf{j} + 2yz^2\mathbf{k}$$

b. $\mathbf{F} = (x + y^3 z)\mathbf{i} + (x^2 + yz)\,\mathbf{j} + (x^2 - 2y^3)\mathbf{k}$; $\operatorname{curl}\mathbf{F}$?

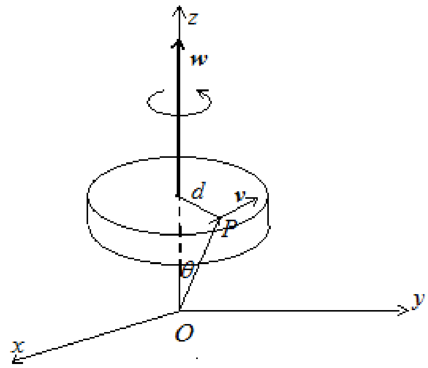
Answer: $\operatorname{curl}\mathbf{F} = \left\langle -(6y^2 + y),\ (y^3 - 2x),\ (2x - 3y^2 z)\right\rangle$

c. $\mathbf{F} = 2xy^3 z\,\mathbf{i} + 3x^2 y^2 z\,\mathbf{j} + x^2 y^3\,\mathbf{k}$; $\operatorname{curl}\mathbf{F}$?

Answer: $\operatorname{curl}\mathbf{F} = \mathbf{0}$.

EXAMPLE 5.5.2 CONNECTION BETWEEN THE CURL VECTOR AND ROTATIONS

Let B be a rigid body rotating about the z-axis.



The rotation can be described by the vector $\mathbf{w} = \omega\mathbf{k}$, where ω is the **angular speed** of B. Let $\mathbf{r} = \langle x, y, z \rangle$ be the **position vector** of P. Let $\mathbf{v}$ be the **tangential speed** of P.

Let θ be the angle between the z-axis and vector $\mathbf{r} = \langle x, y, z \rangle = \overrightarrow{OP}$.

$$d = OP\sin\theta = |\mathbf{r}|\sin\theta, \quad |\mathbf{v}| = \omega d = \omega|\mathbf{r}|\sin\theta = |\mathbf{w}||\mathbf{r}|\sin\theta, \quad \mathbf{v} \perp Oz \text{ and } OP.$$

Therefore

$\mathbf{v} = \mathbf{w} \times \mathbf{r}$

$$\mathbf{v} = \mathbf{w}\times\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = -\omega y\,\mathbf{i} + \omega x\,\mathbf{j} \Rightarrow \operatorname{curl}\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} = 2\omega\mathbf{k} = 2\mathbf{w}$$

It shows that $\operatorname{curl}\mathbf{v}$ and $\mathbf{w}$ have the same direction and $|\operatorname{curl}\mathbf{v}| = 2\omega = 2|\mathbf{w}|$.

It means that **the higher the angular speed the greater the curl of the tangential speed**.

DEFINITION 5.5.3

If $\operatorname{curl}\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is called **irrotational**.

THEOREM 5.5.1

If $f = f(x, y, z)$ has *continuous second-order partial derivatives*, then

$$\operatorname{curl}(\nabla f) = \mathbf{0} \tag{5.5.5}$$

Proof

Since $\nabla f = \dfrac{\partial f}{\partial x}\mathbf{i} + \dfrac{\partial f}{\partial y}\mathbf{j} + \dfrac{\partial f}{\partial z}\mathbf{k}$, then by (5.5.4),

$$\operatorname{curl}(\nabla f) = \nabla\times(\nabla f) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}$$

$$= (f_{zy} - f_{yz})\,\mathbf{i} + (f_{xz} - f_{zx})\,\mathbf{j} + (f_{yx} - f_{yx})\,\mathbf{k} = \mathbf{0},$$

because $f_{zy} = f_{yz}$, $f_{xz} = f_{zx}$, $f_{yx} = f_{yx}$, see Theorem in Section 8.4, Calculus 1.

COROLLARY 5.5.1 OF THEOREM 5.5.1

If $\mathbf{F} = P(x,y,z)\,\mathbf{i} + Q(x,y,z)\,\mathbf{j} + R(x,y,z)\,\mathbf{k}$ is a *conservative vector field*, where P, Q, and R have *continuous partial derivatives*, then $\operatorname{curl}\mathbf{F} = \mathbf{0}$, (**F** is irrotational).

Proof

Since **F** is a *conservative vector field*, $\mathbf{F} = \nabla f$, by (5.3.5). Then (5.5.5) implies that $\operatorname{curl}\mathbf{F} = \mathbf{0}$.

The converse statement of the Corollary 5.5.1 is also true if D is simply-connected. We can combine these statements in the following theorem.

<u>**THEOREM 5.5.2**</u>

Let $\mathbf{F} = P(x,y,z)\,\mathbf{i} + Q(x,y,z)\,\mathbf{j} + R(x,y,z)\,\mathbf{k}$ be defined on a ***simply-connected*** domain *D*. Let P, Q, and R have ***continuous partial derivatives*** on *D*. Then

$$\mathbf{F} \text{ is conservative} \iff \operatorname{curl}\mathbf{F} = \mathbf{0} \tag{5.5.6}$$

EXAMPLE 5.5.3

a. Show that $\mathbf{F} = e^{2y}z^3\,\mathbf{i} + (2xe^{2y}z^3 + 2yz)\,\mathbf{j} + (3xe^{2y}z^2 + y^2 - 1/z^2)\,\mathbf{k}$ is conservative. Determine f such that $\mathbf{F} = \nabla f$.

Solution

It is conservative, because

$$\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^{2y}z^3 & 2xe^{2y}z^3 + 2yz & 3xe^{2y}z^2 + y^2 - 1/z^2 \end{vmatrix} = 0\,\mathbf{i} + 0\,\mathbf{j} + 0\,\mathbf{k} = \mathbf{0}.$$

Then there exists a scalar function *f* such that $\mathbf{F} = \nabla f$. Thus

$$f_x = e^{2y}z^3, \qquad f_y = 2xe^{2y}z^3 + 2yz, \qquad f_z = 3xe^{2y}z^2 + y^2 - 1/z^2$$

$$f(x,y,z) = \int f_x(x,y,z)\,dx = \int e^{2y}z^3 dx = xe^{2y}z^3 + g(y,z) = xe^{2y}z^3 + g(y,z)$$

$$\Rightarrow f_y(x,y,z) = 2xe^{2y}z^3 + g_y(y,z) = 2xe^{2y}z^3 + 2yz \quad \Rightarrow g_y(y,z) = 2yz$$

$$\Rightarrow g(y,z) = \int g_y(y,z)\,dy = \int 2yz\,dy = y^2z + h(z)$$

$$\Rightarrow f(x,y,z) = xe^{2y}z^3 + g(y,z) = xe^{2y}z^3 + y^2z + h(z)$$

$$\Rightarrow f_z(x,y,z) = 3xe^{2y}z^2 + y^2 + h'(z) = 3xe^{2y}z^2 + y^2 - \frac{1}{z^2}$$

$$\Rightarrow h'(z) = -\frac{1}{z^2} \;\Rightarrow h(z) = \frac{1}{z} + C$$

Therefore

$$f(x,y,z) = xe^{2y}z^3 + y^2z + h(z) = xe^{2y}z^3 + y^2z + \frac{1}{z} + C.$$ You can verify that $\mathbf{F} = \nabla f$.

b. Verify wherther the vector fields **F** given in Example 5.5.1.a and b are conservative.

c. Show that the vector field **F** given in Example 5.5.1.c is conservative. Determine *f* such that $\mathbf{F} = \nabla f$?

d. Show that the vector field **F** below is conservative. Determine f so that $\mathbf{F} = \nabla f$.

$$\mathbf{F} = \left\langle -2xye^{-x^2y};\; -x^2e^{-x^2y} + \frac{2y}{y^2+z^4+1};\; \frac{4z^3}{y^2+z^4+1} + 2ze^{z^2} \right\rangle$$

5.6 DIVERGENCE

DEFINITION 5.6.1 DIVERGENCE

Let $\mathbf{F} = P(x,y,z)\,\mathbf{i} + Q(x,y,z)\,\mathbf{j} + R(x,y,z)\,\mathbf{k}$, and $\partial P/\partial x,\ \partial Q/\partial y,\ \partial R/\partial z$ exist, then the **divergence** of **F** is a **scalar function** defined by

$$\boxed{\operatorname{div}\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}} \tag{5.6.1}$$

Using the symbol of the gradient operator "del" we can write symbolically the divergence as the **dot product**

$$\boxed{\operatorname{div}\mathbf{F} \;=\; \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \;=\; \nabla \,.\, \mathbf{F}} \tag{5.6.2}$$

EXAMPLE 5.6.1

a. $\mathbf{F} = (2x^2e^y + y^2z)\,\mathbf{i} + (xy^3 - xz)\,\mathbf{j} + (2x + 4yz^4)\,\mathbf{k}$

Then $\operatorname{div}\mathbf{F} = 4xe^y + 3xy^2 + 16yz^3$

b. $\mathbf{F} = (xe^{-x} + x^2z)\,\mathbf{i} + (e^xy^3 - yz)\,\mathbf{j} + (zx^2 + ye^{-2z})\,\mathbf{k}$

Then $\operatorname{div}\mathbf{F} = e^{-x}(1-x) + 2xz + 3e^xy^2 - z + x^2 - 2ye^{-2z}$

THEOREM 5.6.1

Let $\mathbf{F} = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$, and P, Q, R have continuous second-order partial derivatives, then

$$\boxed{\operatorname{div}(\operatorname{curl}\mathbf{F}) = 0} \tag{5.6.3}$$

Proof

Apply (5.5.3) and (5.6.1).

EXAMPLE 5.6.2

a. $\mathbf{F} = \left\langle xy - 10yz,\ yz + x^2e^z,\ zx - 2yx^2 \right\rangle$. Evaluate $\operatorname{div}\mathbf{F}$.

Answer: $\operatorname{div}\mathbf{F} = x + y + z$

b. $\mathbf{G} = \left\langle -10yz - 4xyz,\ x^2,\ 2yz^2 \right\rangle$. Evaluate $\operatorname{div}\mathbf{G}$.

c. $\mathbf{G} = \left\langle -6y^2 - y,\ y^3 - 2x,\ 2x - 3y^2z \right\rangle$. Evaluate $\operatorname{div}\mathbf{F}$.

The reason for the name **divergence** is from the context of fluid (or gas) flow. If **F** is the velocity of a fluid, then $\operatorname{div}\mathbf{F}$ represents the net rate of change of the mass of the fluid flowing from the point (x,y,z) per unit volume, i.e. $\operatorname{div}\mathbf{F}$ measures the tendency of the fluid to diverge from the point (x,y,z). If $\operatorname{div}\mathbf{F} = 0$, then **F** is said to be **incompressible**.

5.7 VECTOR FORM OF GREEN'S THEOREM

Consider a vector field on the plane $\mathbf{F} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$. Then

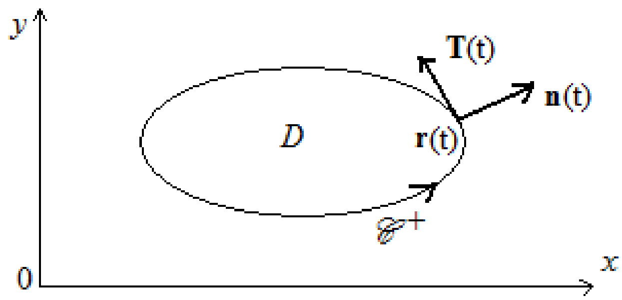
$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\mathbf{k}; \quad \operatorname{curl}\mathbf{F}\cdot\mathbf{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \tag{5.7.1}$$

Then the formula (5.4.1) of Green's Theorem can be rewritten in the vector form

$$\boxed{\oint_{\mathscr{C}^+} \mathbf{F}\cdot d\mathbf{r} = \oint_{\mathscr{C}^+} (\mathbf{F}\cdot\mathbf{T})\,ds = \oint_{\mathscr{C}^+} P(x,y)\,dx + Q(x,y)\,dy = \iint_D (\operatorname{curl}\mathbf{F})\cdot\mathbf{k}\,dxdy} \tag{5.7.2}$$

Formula (5.7.2) expresses the integral of the **tangential component** of **F** along the closed curve $\mathscr{C}^+$ as the double integral of the **vertical component** of $\operatorname{curl}\mathbf{F}$ over the domain D enclosed by $\mathscr{C}^+$.

What about the integral of the **normal component** of **F** along the closed curve $\mathscr{C}^+$?



If the vector equation of $\mathscr{C}^+$ is

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} \qquad \alpha \le t \le b\,,$$

then the unit tangent vector **T** is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{|\mathbf{r}'(t)|}\left(x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j}\right),$$

and the unit outward normal vector **n** is (note that **n** is perpendicular to **T** and to **k**)

$$\mathbf{n}(t) = \mathbf{T}\times\mathbf{k} = \frac{1}{|\mathbf{r}'(t)|}\langle x'(t),\ y'(t),\ 0\rangle \times \langle 0,\ 0,\ 1\rangle$$

$$= \frac{1}{|\mathbf{r}'(t)|}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x'(t) & y'(t) & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{y'(t)\,\mathbf{i} - x'(t)\,\mathbf{j}}{|\mathbf{r}'(t)|}$$

It is easy to provide the following calculations

$$\oint_{\mathscr{C}^+} \mathbf{F}\cdot\mathbf{n}\,ds = \int_{\alpha}^{\beta} [\mathbf{F}(t)\cdot\mathbf{n}(t)]\,|\,\mathbf{r}'(t)\,|\,dt$$

$$= \int_{\alpha}^{\beta} \frac{1}{|\mathbf{r}'(t)|}\left[P\left(x(t), y(t)\right) y'(t) - Q\left(x(t), y(t)\right) x'(t)\right] |\,\mathbf{r}'(t)\,|\,dt$$

$$= \int_{\alpha}^{\beta} \left[P\left(x(t), y(t)\right) y'(t) - Q\left(x(t), y(t)\right) x'(t)\right] dt$$

$$= \oint_{\mathscr{C}^+} -Q\,dx + P\,dy = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dxdy, \text{ by Green's formula}\,.$$

Therefore

$$\boxed{\oint_{\mathscr{C}^+} \mathbf{F}\cdot\mathbf{n}\,ds = \iint_D \operatorname{div}\mathbf{F}(x, y)\,dxdy} \qquad (5.7.3)$$

Formula (5.7.3) expresses the integral of the **normal component** of **F** along the closed curve $\mathscr{C}^+$ in the *xy*-plane as the double integral of $\operatorname{div}\mathbf{F}$ over the domain *D* enclosed by $\mathscr{C}^+$.

EXERCISES

5.1 Evaluate $\int_{\mathscr{C}} xy^4 ds,$ where $\mathscr{C}$ is the right half of the circle $x^2 + y^2 = 16$.

5.2 Find the mass and the center of mass of a thin wire of shape of the upper half of the unit circle $x^2 + y^2 = 1.$ The density is $\rho = 1 + x^2 y$.

5.3 Show that the line integral $\int_{\mathscr{C}} \mathbf{F} \cdot d\mathbf{r}$ is independent of path and evaluate the integral.

a. $\mathbf{F} = \left\langle 2x\sin y,\ x^2\cos y - 3y^2 \right\rangle$, $\mathscr{C}$ is any path from $(-1,0)$ to $(5,1)$.

b. $\mathbf{F} = \left\langle 2x\cos y - y\cos x,\ -x^2\sin y - \sin x \right\rangle$, $\mathscr{C}$ is any path from $(0,0)$ to $(1,1)$.

5.4 Find the work done by the force field in moving an object.

a. $\mathbf{F} = \left\langle x^2y^3,\ x^3y^2 \right\rangle$, from $A(0,0)$ to $B(2,1)$, by any path.

b. $\mathbf{F} = \left\langle y,\ xy,\ xyz \right\rangle$, from $A(0,\ 0,\ 0)$ to $B(1,\ 2,\ 3)$ on a straight line .

5.5 Evaluate the line integral by two methods: directly and using Green's Theorem.

a. $\oint_{\mathscr{C}^+} xy^2 dx + x^3 dy$, $\mathscr{C}$ is the rectangle with vertices $(0,0), (2,0), (2,3),$ and $(0,3)$

b. $\oint_{\mathscr{C}^+} e^y dx + 2xe^y dy$, $\mathscr{C}$ is the circumference of the square with vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$

c. $\oint_{\mathscr{C}^+} (y + e^{\sqrt{x}})dx + (2x + \cos^2 y)dy$, $\mathscr{C}$ is the boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$

5.6 Let $\mathbf{F} = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j} = \dfrac{-y\,\mathbf{i} + x\,\mathbf{j}}{x^2 + y^2}$.

a. Show that $\dfrac{\partial Q(x,y)}{\partial x} = \dfrac{\partial P(x,y)}{\partial y}$ in $D = \sqcup \setminus \{(0,0)\}$

b. Show that $\int_{\mathscr{C}_1} \mathbf{F} \cdot d\mathbf{r}$ and $\int_{\mathscr{C}_2} \mathbf{F} \cdot d\mathbf{r}$, where $\mathscr{C}_1$ and $\mathscr{C}_2$ are the lower half and upper half of the circle $x^2 + y^2 = 1$ from $(-1,0)$ to $\left(1,0\right)$, are not the same. Does this contradict (5.4.7)?

5.7 Find the curl and the divergence of the vector

a. $\mathbf{F} = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k};$

b. $\mathbf{F} = e^x \sin y\,\mathbf{i} + e^x \cos y\,\mathbf{j} + z\,\mathbf{k};$

c. $\mathbf{F} = \dfrac{x}{x^2 + y^2 + z^2}\mathbf{i} + \dfrac{y}{x^2 + y^2 + z^2}\mathbf{j} + \dfrac{z}{x^2 + y^2 + z^2}\mathbf{k};$

d. $\mathbf{F} = xe^{yz}\,\mathbf{i} + ye^{xz}\,\mathbf{j} + ze^{xy}\,\mathbf{k}$;

5.8 Determine whether or not the vector field is conservative. If it is conservative, find the function f such that $\mathbf{F} = \nabla f$.

a. $\mathbf{F} = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$;

b. $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$;

c. $\mathbf{F} = 2xy\,\mathbf{i} + (x^2 + 2yz)\,\mathbf{j} + y^2\,\mathbf{k}$;

d. $\mathbf{F} = e^x\,\mathbf{i} + e^z\,\mathbf{j} + e^y\,\mathbf{k}$;

e. $\mathbf{F} = yze^{xz}\,\mathbf{i} + e^{xz}\,\mathbf{j} + xye^{xz}\,\mathbf{k}$;

5.9 Evaluate the integrals $\oint_{\mathscr{C}^+} \mathbf{F}\cdot\mathbf{T}\,ds$ and $\oint_{\mathscr{C}^+} \mathbf{F}\cdot\mathbf{n}\,ds$

a. $\mathbf{F} = xy\,\mathbf{i} + y^2\,\mathbf{j}$, $\mathscr{C}^+$: $x^2 + y^2 - 2y = 0$.

b. $\mathbf{F} = -ye^{x^2+y^2}\,\mathbf{i} + xe^{x^2+y^2}\,\mathbf{j}$, $\mathscr{C}^+$: $x^2 + y^2 = 1$. *Answer*: e , 0.

c. $\mathbf{F} = 4xy\,\mathbf{i} + (x^2 + 2y^2)\,\mathbf{j}$, $\mathscr{C}^+$: $x^2 + y^2 - 2x = 0, y \ge 0$, and $y = 0,\ 0 \le x \le 2$

5.10 Prove that

a. $\operatorname{curl}(\mathbf{F} + \mathbf{G}) = \operatorname{curl}\mathbf{F} + \operatorname{curl}\mathbf{G}$;

b. $\operatorname{curl}(f\,\mathbf{F}) = f\operatorname{curl}\mathbf{F} + (\nabla f)\times\mathbf{F}$.

ANSWERS

5.1 $\frac{2}{5}\times 4^6$;

5.2 $m = \pi + \frac{8}{3}$; $(\bar{x}, \bar{y}) = \left(0, \frac{3(5\pi + 16)}{8(3\pi + 8)}\right)$;

5.3 a. $25\sin 1 - 1.$; b. $\cos 1 - \sin 1$.

5.4 a. 8/3; b. 41/6;

5.5 a. 6; b. e – 1; c. 1/3

5.6 b. $-\pi$ and π. No (why?)

5.7 a. 0; 3π; b. $2\pi e$; 0; c. $-\pi$; $16/3$;

Chapter 6

SURFACE INTEGRALS

In this chapter two kinds of surface integrals are taken into consideration: surface integrals of scalar functions and surface integrals of vector fields

6.1 SURFACE INTEGRALS OF SCALAR FUNCTIONS

PROBLEM 6.1.1 SURFACE MASS

Suppose the differentiable function $z = z(x, y),\ (x, y) \in D \subset \mathbb{R}^2$, define a smooth surface $\mathscr{S}$ in space. Suppose the mass density of the surface is determined by $\rho = \rho(x, y, z),\ (x, y, z) \in \mathscr{S}$. How can we define and determine the mass of $\mathscr{S}$? As usually, we divide *S* into patches S_{ij} by the planes perpendicular to the *x*-axis and the *y*-axis. See Figure 6.1.1. The area $A(S_{ij})$ of S_{ij} is approximated by the area $A(T_{ij})$ of a parallelogram T_{ij} on the tangent plane at some point $P_{ij}\left(x_i, y_j, z(x_i, y_j)\right) \in S_{ij}$ with two sides determined by tangent vectors $\mathbf{a}_i$ and $\mathbf{b}_j$ in the direction of the *x*-axis and the *y*-axis, respectively:

$$\mathbf{a}_i = \left\langle \Delta x_i,\ 0,\ z_x(x_i, y_j)\Delta x_i \right\rangle,\ \ \mathbf{b}_j = \left\langle 0,\ \Delta y_j,\ z_y(x_i, y_j)\Delta y_j \right\rangle$$

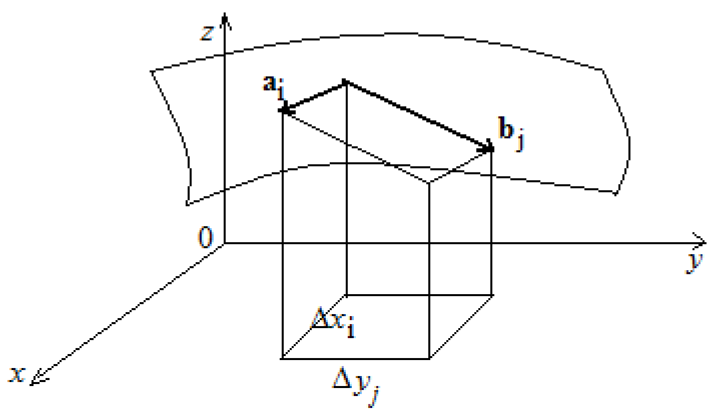


Figure 6.1.1

Clearly,

$$\mathbf{a}_i \times \mathbf{b}_j = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Delta x_i & 0 & z_x(x_i, y_j)\Delta x_i \\ 0 & \Delta y_j & z_y(x_i, y_j)\Delta y_j \end{vmatrix} = \Delta x_i \Delta y_j \left\langle -z_x(x_i, y_j), -z_y(x_i, y_j), 1 \right\rangle \qquad (6.1.1)$$

Therefore

$$A(S_{i,j}) \approx A(T_{i,j}) = \left|\mathbf{a}_i \times \mathbf{b}_j\right| = \sqrt{1 + \left[z_x(x_i, y_j)\right]^2 + \left[z_y(x_i, y_j)\right]^2}\ \Delta x_i \,\Delta y_j\ , \qquad (6.1.2)$$

the mass of S_{ij} is approximated by

$$m(S_{i,j}) \approx \rho\left(x_i, y_j, z(x_i, y_j)\right) \cdot A(S_{i,j})$$

$$\approx \rho\left(x_i, y_j, z(x_i, y_j)\right) \cdot \sqrt{1 + \left[z_x(x_i, y_j)\right]^2 + \left[z_y(x_i, y_j)\right]^2}\ \Delta x_i\, \Delta y_i$$

and the mass of $\mathscr{S}$ is approximated by

$$\sum_{i,j} \left[\rho\left(x_i, y_j, z(x_i, y_j)\right) \cdot \sqrt{1 + \left[z_x(x_i, y_j)\right]^2 + \left[z_y(x_i, y_j)\right]^2}\ \Delta x_i\, \Delta y_i \right] \tag{6.1.3}$$

Denote

$$\lambda = \max(\Delta x_i, \Delta y_j) \tag{6.1.4}$$

The mass of $\mathscr{S}$ is defined by the limit

$$\lim_{\lambda \to 0} \sum_{i,j} \left[\rho\left(x_i, y_j, z(x_i, y_j)\right) \cdot \sqrt{1 + \left[z_x(x_i, y_j)\right]^2 + \left[z_y(x_i, y_j)\right]^2}\ \Delta x_i\, \Delta y_i \right] \tag{6.1.5}$$

if the limit exists and is independent of the way of dividing $\mathscr{S}$ and choosing the points P_{ij}.

DEFINITION 6.1.1 SURFACE INTEGRALS

Suppose the differentiable function $z = z(x, y),\ (x, y) \in D \subset \mathbb{R}^2$, define a smooth surface $\mathscr{S}$ in space. Suppose $f = f(x, y, z)$ is defined on $\mathscr{S}$. By the same way provided in Problem 6.1.1 with $\rho(x, y, z)$ replaced by $f = f(x, y, z)$ we arrive to the sum and limit

$$\sum_{i,j} \left[f\left(x_i, y_j, z(x_i, y_j)\right) \cdot \sqrt{1 + \left[z_x(x_i, y_j)\right]^2 + \left[z_y(x_i, y_j)\right]^2}\ \Delta x_i\, \Delta y_i \right] \tag{6.1.6}$$

$$\lim_{\lambda \to 0} \sum_{i,j} \left[f\left(x_i, y_j, z(x_i, y_j)\right) \cdot \sqrt{1 + \left[z_x(x_i, y_j)\right]^2 + \left[z_y(x_i, y_j)\right]^2}\ \Delta x_i\, \Delta y_i \right] \tag{6.1.7}$$

If the limit (6.1.5) exists and is independent of the way of dividing $\mathscr{S}$ and choosing the points P_{ij}, then we denote the limit by

$$\iint_{\mathscr{S}} f(x, y, z)\, dS \tag{6.1.8}$$

and it is called the **surface integral of** $f = f(x, y, z)$ **over the surface** $\mathscr{S}$.

Therefore

$$\iint_{\mathscr{S}} f(x, y, z) dS = \lim_{\lambda \to 0} \sum_{i,j} \left[f\left(x_i, y_j, z(x_i, y_j)\right) \cdot \sqrt{1 + \left[z_x(x_i, y_j)\right]^2 + \left[z_y(x_i, y_j)\right]^2}\ \Delta x_i\, \Delta y_i \right] \tag{6.1.9}$$

It follows from (6.1.7) that **the surface integral** can be expressed by the double integral

$$\boxed{\iint_{\mathscr{S}} f(x,y,z)dS = \iint_{D}\left[f\left(x,y,z(x,y)\right)\cdot\sqrt{1+\left[z_x(x,y)\right]^2+\left[z_y(x,y)\right]^2}\right] dx\,dy} \qquad (6.1.10)$$

The definition and (6.1.4) result to the expression for **the mass** of a surface $\mathscr{S}$ with mass density $\rho(x,y,z)$:

$$\boxed{m(\mathscr{S}) = \iint_{\mathscr{S}} \rho(x,y,z)\,dS = \iint_{D}\left[\rho\left(x,y,z(x,y)\right)\cdot\sqrt{1+\left[z_x(x,y)\right]^2+\left[z_y(x,y)\right]^2}\right] dx\,dy} \qquad (6.1.11)$$

From the existence condition for double integrals we obtain the following Theorem:

THEOREM 6.1.1 EXISTENCE CONDITION FOR SURFACE INTEGRALS

Let $z = z(x,y),\ (x,y)\in D\subset \mathbb{R}^2$ be the equation of the surface $\mathscr{S}$. Suppose z has continuous partial derivative on D. Suppose $f = f(x,y,z)$ is defined on $\mathscr{S}$ and $f = f\left(x,y,z(x,y)\right)$ is continuous on D. Then the surface integral (6.1.8) exists.

PROPERTY 6.1.1

1. $\iint_{\mathscr{S}} dS = A(\mathscr{S})$

2. $\iint_{\mathscr{S}} k.f(x,y,z)\,dS = k.\iint_{\mathscr{S}} f(x,y,z)\,dS$, $k =$constant.

3. $\iint_{\mathscr{S}} \left[f_1(x,y,z) + f_2(x,y,z)\right] dS = \iint_{\mathscr{S}} f_1(x,y,z)\,dS + \iint_{\mathscr{S}} f_2(x,y,z)\,dS.$

4. If $\mathscr{S}$ is a union, $\mathscr{S} = \mathscr{S}_1 \cup \mathscr{S}_2$, where $\mathscr{S}_1$ and $\mathscr{S}_2$ intersect maybe only along their boundaries then

$$\iint_{\mathscr{S}} f(x,y,z)\,dS = \iint_{\mathscr{S}_1} f(x,y,z)\,dS + \iint_{\mathscr{S}_2} f(x,y,z)\,dS.$$

5. If $\mathscr{S}$ is a piecewise-smooth surface, that is $\mathscr{S} = \mathscr{S}_1 \cup \ldots \cup \mathscr{S}_k$ where $\mathscr{S}_1,\ldots,\mathscr{S}_k$ are smooth surfaces and intersect only along their boundaries then

$$\iint_{\mathscr{S}} f(x,y,z)\,dS = \iint_{\mathscr{S}_1} f(x,y,z)\,dS + \cdots + \iint_{\mathscr{S}_k} f(x,y,z)\,dS.$$

6. Suppose the mass density of the surface $\mathscr{S}$ is determined by $\rho = \rho(x,y,z),\ (x,y,z)\in\mathscr{S}$. Then the mass m and the **center of mass** $(\overline{x},\overline{y},\overline{z})$ of the surface $\mathscr{S}$ are determined from

$$m = \iint_{\mathscr{S}} \rho(x,y,z)\,dS$$

$$\boxed{\overline{x} = \frac{1}{m}\iint_{\mathscr{S}} x.\rho(x,y,z)\,dS,\quad \overline{y} = \frac{1}{m}\iint_{\mathscr{S}} y.\rho(x,y,z)\,dS,\quad \overline{z} = \frac{1}{m}\iint_{\mathscr{S}} z.\rho(x,y,z)\,dS} \qquad (6.1.12)$$

EXAMPLE 6.1.1

Evaluate the surface integral.

a. $I = \iint_{\mathscr{S}} xy\, dS$, where $\mathscr{S}$ is the surface $z = 2\sqrt{6}\,x + 3y^2$, $0 \le x \le 2, 0 \le y \le 2$

Solution:

$$I = \iint_{\mathscr{S}} xy\, dS = \int_0^2 x dx \int_0^2 y\sqrt{1 + 24 + 36y^2}\, dy = \frac{1036}{27}$$

b. $I = \iint_{\mathscr{S}} (x^2 + y^2)\, dS$, where $\mathscr{S}$ is the surface $z = x^2 + y^2$, $x^2 + y^2 \le 4$

Solution:

$$I = \iint_D (x^2 + y^2)\sqrt{1 + 4x^2 + 4y^2}\, dx\, dy = \frac{\left(1 + 391\sqrt{17}\right)\pi}{60}$$

c. $I = \iint_{\mathscr{S}} x^2 yz\, dS$, where $\mathscr{S}$ is the surface $z = 1 + 2x + 3y$, $0 \le x \le 3, 0 \le y \le 2$

Solution:

$$I = \iint_{\mathscr{S}} x^2 yz\, dS = \int_0^3 dx \int_0^2 x^2 y(1 + 2x + 3y)\sqrt{1 + 2^2 + 3^2}\, dy = 171\sqrt{14}$$

d. $\iint_{\mathscr{S}} x\, dS$, where $\mathscr{S}$ is the surface $y = x^2 + 4z$, $0 \le x \le 2, 0 \le z \le 2$

Solution:

$$\iint_{\mathscr{S}} x\, dS = \int_0^2 dx \int_0^2 x\sqrt{1 + (2x)^2 + 4^2}\, dz = \frac{1}{6}\left(33\sqrt{33} - 17\sqrt{17}\right)$$

e. Find the center of mass of a thin hemisphere of radius *a* and constant mass density *k*.

Solution:

The surface can be determined by: $z = \sqrt{a^2 - (x^2 + y^2)}$, $(x, y) \in D : x^2 + y^2 \le a^2$

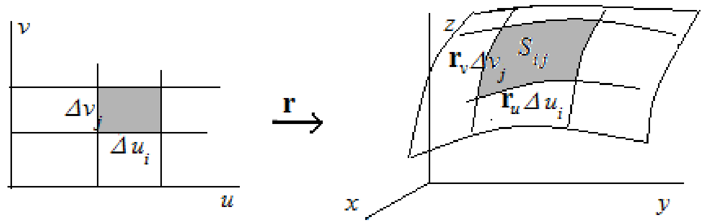
$$m = \iint_{\mathscr{S}} k(x, y, z)\, dS = k\iint_{\mathscr{S}} dS = k\iint_D \sqrt{\frac{a^2}{a^2 - \left(x^2 + y^2\right)}}\, dxdy = 2\pi k a^2$$

$$\iint_{\mathscr{S}} y\rho(x, y, z)\, dS = 0\,,\ \iint_{\mathscr{S}} x\rho(x, y, z)\, dS = 0\,,\ \iint_{\mathscr{S}} z\rho(x, y, z)\, dS = \pi k a^3,\ \left(\overline{x}, \overline{y}, \overline{z}\right) = \left(0,\ 0,\ a/2\right)$$

PARAMETRIC SURFACE

Suppose $\mathscr{S}$ is determined by a ***vector equation of two parameters*** *u* and *v*:

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle,\ (u, v) \in D \subset \mathbb{R}^2 \qquad (6.1.13)$$



The area $A(S_{ij})$ of S_{ij} is approximated by the area $A(T_{ij})$ of a parallelogram T_{ij} determined by two vectors $\Delta u_i\,\mathbf{r}_u$ and $\Delta v_j\,\mathbf{r}_v$:

$$A(S_{i,j}) \approx A(T_{i,j}) = |\Delta u_i \mathbf{r}_u \times \Delta v_j \mathbf{r}_v| = |\mathbf{r}_u \times \mathbf{r}_v|\,\Delta u_i\,\Delta v_j \qquad (6.1.14)$$

where

$$\mathbf{r}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle, \quad \mathbf{r}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle \qquad (6.1.15)$$

Therefore

$$\boxed{\iint_{\mathscr{S}} f(x,y,z)\,dS = \iint_D \left[f\big(x(u,v), y(u,v), z(u,v)\big) \cdot |\mathbf{r}_u \times \mathbf{r}_v| \right] du\,dv} \qquad (6.1.16)$$

EXAMPLE 6.1.2

a. Show that (6.1.10) is a particular form of (6.1.16)

Solution:

The equation $z = z(x,y),\ (x,y) \in D \subset \mathbb{R}^2$, of the surface $\mathscr{S}$ may be regarded as a parametric equation with parameters x, y: $\mathbf{r} = \mathbf{r}(x,y) = \langle x,\ y,\ z(x,y) \rangle,\ (x,y) \in D \subset \mathbb{R}^2$. Then

$$\mathbf{r}_x = \langle 1, 0, z_x \rangle,\ \mathbf{r}_y = \langle 0, 1, z_y \rangle,\ \mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & z_x \\ 0 & 1 & z_y \end{vmatrix} = \langle -z_x, -z_y, 1 \rangle,\ |\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{1 + z_x^2 + z_y^2}\ .$$

b. Evaluate

$I = \iint_{\mathscr{S}} (x + yz)\,dS$, where $\mathscr{S}$ is determined by the parametric equations

$$x = uv,\ y = u + v,\ z = u - v,\ u^2 + v^2 \le 1,\ u \ge 0, v \ge 0.$$

Solution:

$$\mathbf{r}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle = \langle v, 1, 1 \rangle,\ \mathbf{r}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle = \langle u, 1, -1 \rangle,\ |\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{2\left[(u^2 + v^2) + 2\right]}$$

$$I = \iint_D (uv + u^2 - v^2)\sqrt{2\left[(u^2 + v^2) + 2\right]}\ dudv = \frac{16 - 3\sqrt{6}}{30},\ \text{ where } D: u^2 + v^2 \le 1,\ u \ge 0, v \ge 0.$$

c. Show that the surface area of a sphere of radius a is $4\pi a^2$, using surface integral in two ways:

i. Equation of the sphere: $x^2 + y^2 + z^2 = a^2$.

ii. The Parametric Equation of the sphere:
$\mathbf{r} = \mathbf{r}(\phi, \theta) = \langle a \sin\phi \cos\theta,\ a \sin\phi \sin\theta,\ a \cos\phi \rangle; \phi \in [0, \pi], \theta \in [0, 2\pi],\ |\mathbf{r}_\phi \times \mathbf{r}_\theta| = a^2 \sin\phi$

6.2 SURFACE INTEGRALS OF VECTOR FIELDS

DEFINITION 6.2.1 ORIENTED SURFACES

Consider an orientable (two-sided) surface $\mathscr{S}$ that has a tangent plane at any point except at boundary points. There are two unit normal vectors at each points: $\mathbf{n}_1$ and $\mathbf{n}_2 = -\mathbf{n}_1$. If it is possible to choose a unit normal vector $\mathbf{n}$ at every such point (x, y, z) so that it varies continuously over $\mathscr{S}$, then $\mathscr{S}$ is called an **oriented surface**. One choice of **n** provides $\mathscr{S}$ with an **orientation**.

For the closed surface (the boundary of a solid region) the **positive orientation** is defined as the normal vectors point **outward from the solid**.

Note that the Mobius surface has only one side!

Mobius surface

DEFINITION 6.2.2 SURFACE INTEGRALS OF VECTOR FIELDS

Let $\mathbf{F} = \langle P(x, y, z),\, Q(x, y, z),\, R(x, y, z)\rangle$ be defined on an **oriented surface** S with unit normal vector **n**, then **surface integral of F over** $\mathscr{S}$ is defined as ***the surface integral of the normal component of*** **F**

$$\iint_{\mathscr{S}} \mathbf{F}\cdot d\mathbf{S} = \iint_{\mathscr{S}} \mathbf{F}\cdot\mathbf{n}\, dS \qquad (6.2.1)$$

This integral is called the **flux** of **F** across the **oriented surface** $\mathscr{S}$.

PROPERTY 6.2.1

1. $\iint_{\mathscr{S}^-} \mathbf{F}\cdot d\mathbf{S} = -\iint_{\mathscr{S}^+} \mathbf{F}\, d\mathbf{S}$, where $\mathscr{S}^+$ and $\mathscr{S}^-$ denote two different sides of $\mathscr{S}$.
2. $\iint_{\mathscr{S}} k\,\mathbf{F}\cdot d\mathbf{S} = k\iint_{\mathscr{S}} \mathbf{F}\, d\mathbf{S}$, $k =$constant.
3. $\iint_{\mathscr{S}} (\mathbf{F}_1 + \mathbf{F}_2)\cdot d\mathbf{S} = \iint_{\mathscr{S}} \mathbf{F}_1\cdot d\mathbf{S} + \iint_{\mathscr{S}} \mathbf{F}_2\cdot d\mathbf{S}$
4. If $\mathscr{S}$ is a union of two surfaces, $\mathscr{S} = \mathscr{S}_1 \cup \mathscr{S}_2$, where $\mathscr{S}_1$ and $\mathscr{S}_2$ intersect maybe only along their boundaries then

   $\iint_{\mathscr{S}} \mathbf{F}\cdot d\mathbf{S} = \iint_{\mathscr{S}_1} \mathbf{F}\cdot d\mathbf{S} + \iint_{\mathscr{S}_2} \mathbf{F}\cdot d\mathbf{S}.$

GRADIENT, TANGENT PLANE, AND NORMAL LINE

Suppose that the smooth surface $\mathscr{S}$ in space is defined by the differentiable function $z = z(x, y)$, or, $g(x, y, z) = z - z(x, y) = 0,\ (x, y) \in D \subset \mathbb{R}^2$.

Suppose $\mathscr{S}$ is a smooth surface with equation $g(x, y, z) = k$. Let $P(x_0, y_0, z_0)$ be a point on $\mathscr{S}$. Let $\mathscr{C}$ be a curve on $\mathscr{S}$ and passes through the point *P*. The curve $\mathscr{C}$ is described by a continuous vector function $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$. Let t_0 be the parameter value corresponding to *P*: $P = \left(x(t_0), y(t_0), z(t_0)\right) = (x_0, y_0, z_0)$. Since $\mathscr{C}$ is on $\mathscr{S}$, any point $\left(x(t), y(t), z(t)\right)$ must satisfy the equation of $\mathscr{S}$:

$$g(x(t), y(t), z(t)) = k$$

Then

$$\frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dt} + \frac{\partial g}{\partial z}\frac{dz}{dt} = 0 \ \Rightarrow\ \nabla g \cdot \mathbf{r}'(t) = 0$$

$$\Rightarrow \text{ at } P = \left(x(t_0), y(t_0), z(t_0)\right) = (x_0, y_0, z_0) : \nabla g(x_0, y_0, z_0) \perp \mathbf{r}'(t_0)$$

But $\mathbf{r}'(t_0)$ is the tangent vector to $\mathscr{C}$ at *P*, then the gradient vector $\nabla g(x_0, y_0, z_0)$ is perpendicular to any curve $\mathscr{C}$ on $\mathscr{S}$ passing through *P*. Thus **the gradient vector** $\nabla g(x_0, y_0, z_0)$ **is perpendicular to the tangent plane to the surface** $\mathscr{S}$ at *P*.

Therefore the equation of the tangent plane at the point $P(x_0, y_0, z_0)$ is

$$\boxed{g_x(x_0, y_0, z_0)(x - x_0) + g_y(x_0, y_0, z_0)(y - x_0) + g_z(x_0, y_0, z_0)(z - x_0) = 0} \tag{6.2.2}$$

and the equation of the normal line to S at $P(x_0, y_0, z_0)$ is

$$\boxed{\frac{(x - x_0)}{g_x(x_0, y_0, z_0)} = \frac{(y - x_0)}{g_y(x_0, y_0, z_0)} = \frac{(z - x_0)}{g_z(x_0, y_0, z_0)}} \tag{6.2.3}$$

EXPRESSION 6.2.1 OF SURFACE INTEGRALS

a. Suppose that $\mathscr{S}$ is determined by $z = z(x, y)$ or $g(x, y, z) = z - z(x, y) = 0,\ (x, y) \in D \subset \mathbb{R}^2$, and is **upward** oriented. It means that the *z*-component of the unit normal vector **n** is positive. (see Figure 6.2.1 below). Let $\mathbf{F} = \langle P(x, y, z),\ Q(x, y, z),\ R(x, y, z)\rangle$ be defined on $\mathscr{S}$. Then

$$\mathbf{n} = \frac{\nabla g}{|\nabla g|} = \frac{\left\langle -z_x(x, y), -z_y(x, y), 1\right\rangle}{\sqrt{1 + \left(z_x(x, y)\right)^2 + \left(z_y(x, y)\right)^2}}$$

$$dS = \sqrt{1 + \left[z_x(x, y)\right]^2 + \left[z_y(x, y)\right]^2}\ dx\,dy \tag{6.2.4}$$

Therefore (6.2.1) has the expression as the double integral below

$$\boxed{\iint_{\mathscr{S}} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_D \left[-P\left(x, y, z(x, y)\right) z_x(x, y) - Q\left(x, y, z(x, y)\right) z_y + R\left(x, y, z(x, y)\right)\right] dx\,dy} \tag{6.2.5}$$

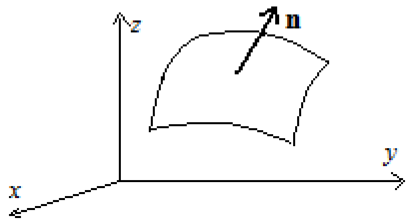


Figure 6.2.1: The upward vector **n**

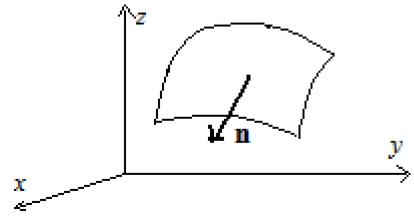


Figure 6.2.2: The downward vector **n**

b. Suppose $\mathscr{S}$ is oriented by the **downward** unit normal vector **n**, (see Figure 6.2.2), then

$$\mathbf{n} = -\frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|} = \frac{\langle z_x(x,y),\ z_y(x,y), -1\rangle}{\sqrt{1+\left(z_x(x,y)\right)^2+\left(z_y(x,y)\right)^2}},$$

$$dS = \sqrt{1+\left[z_x(x,y)\right]^2+\left[z_y(x,y)\right]^2}\,dx\,dy \ \text{, see (6.2.4)}$$

$$\boxed{\iint_{\mathscr{S}} \mathbf{F}\cdot\mathbf{n}\,dS = \iint_D \left[P\left(x,y,z(x,y)\right)z_x(x,y) + Q\left(x,y,z(x,y)\right)z_y - R\left(x,y,z(x,y)\right)\right]dx\,dy} \quad (6.2.6)$$

You can write similarly the expressions for the surface integrals if $\mathscr{S}$ is given by an equation of the form

$$x = x(y,z),\ (y,z)\in D\subset\mathbb{R}^2 \quad \text{or} \quad y = y(x,z),\ (x,z)\in D\subset\mathbb{R}^2$$

EXPRESSION 6.2.2 OF SURFACE INTEGRALS

Let $\mathscr{S}$ be the oriented smooth surface in space.

Let $\mathbf{F} = \langle P(x,y,z),\ Q(x,y,z),\ R(x,y,z)\rangle$ be the vector field defined on $\mathscr{S}$.

Let the unit normal vector of $\mathscr{S}$ be expressed by the *direction cosines*, (see 1.1, Chapter 1)

$$\mathbf{n} = \langle\cos\alpha, \cos\beta, \cos\gamma,\rangle \quad (6.2.7)$$

Then

$$\mathbf{F}\cdot\mathbf{n} = \langle P(x,y,z),\ Q(x,y,z),\ R(x,y,z)\rangle\cdot\langle\cos\alpha,\cos\beta,\cos\gamma\rangle$$
$$= P(x,y,z)\cos\alpha + Q(x,y,z)\cos\beta + R(x,y,z)\cos\gamma \quad (6.2.8)$$

Therefore (6.2.1) can be expressed as below

$$\boxed{\iint_{\mathscr{S}} \mathbf{F}\cdot d\mathbf{S} = \iint_{\mathscr{S}} \mathbf{F}\cdot\mathbf{n}\,dS = \iint_{\mathscr{S}} \left[P(x,y,z)\cos\alpha + Q(x,y,z)\cos\beta + R(x,y,z)\cos\gamma\right]dS} \quad (6.2.9)$$

Remark

To evaluate the integral in the right hand side of (6.2.9), we need to divide the surface $\mathscr{S}$ into sub-surfaces that can be expressed by one of the following equations:

$$z = z(x,y),\ (x,y)\in D_{xy}\subset\mathbb{R}^2 \ \text{or} \ x = x(y,z),\ (y,z)\in D_{yz}\subset\mathbb{R}^2 \ \text{or} \ y = y(x,z),\ (x,z)\in D_{xz}\subset\mathbb{R}^2$$

- If $\mathscr{S}$ has the equation $x = x(y,z),\ (y,z) \in D_{yz}$ then

$$\iint_{\mathscr{S}} P(x,y,z)\cos\alpha\, dS = \begin{cases} \displaystyle\iint_{D_{yz}} P(x(y,z),y,z)\,dydz & \text{if } 0 \le \alpha < \dfrac{\pi}{2} \\ 0 & \text{if } \alpha = \dfrac{\pi}{2} \\ -\displaystyle\iint_{D_{yz}} P(x(y,z),y,z)\,dydz & \text{if } \dfrac{\pi}{2} < \alpha \le \pi \end{cases} \tag{6.2.10}$$

- If $\mathscr{S}$ has the equation $y = y(x,z),\ (x,z) \in D_{xz}$ then

$$\iint_{\mathscr{S}} Q(x,y,z)\cos\beta\, dS = \begin{cases} \displaystyle\iint_{D_{xz}} Q(x,y(x,z),z)\,dxdz & \text{if } 0 \le \beta < \dfrac{\pi}{2} \\ 0 & \text{if } \beta = \dfrac{\pi}{2} \\ -\displaystyle\iint_{D_{xz}} Q(x,y(x,z),z)\,dxdz & \text{if } \dfrac{\pi}{2} < \beta \le \pi \end{cases} \tag{6.2.11}$$

- If $\mathscr{S}$ has the equation $z = z(x,y),\ (x,y) \in D_{xy}$ then

$$\iint_{\mathscr{S}} R(x,y,z)\cos\gamma\, dS = \begin{cases} \displaystyle\iint_{D_{xy}} R(x,y,z(x,y))\,dxdy & \text{if } 0 \le \gamma < \dfrac{\pi}{2} \\ 0 & \text{if } \gamma = \dfrac{\pi}{2} \\ -\displaystyle\iint_{D_{xy}} R(x,y,z(x,y))\,dxdy & \text{if } \dfrac{\pi}{2} < \gamma \le \pi \end{cases} \tag{6.2.12}$$

EXAMPLE 6.2.1

a. Evaluate $I = \iint_{\mathscr{S}} \mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F} = \langle x,\ y,\ z\rangle$

and $\mathscr{S}$ is the upper half-sphere, upward oriented, $z = 16 - x^2 - y^2,\ \ x^2 + y^2 \le 16$

Solution

Apply (6.2.5), $I = \iint_D \left[-x(-2x) - y(-2y) + \left(16 - x^2 - y^2\right)\right] dx\,dy = \iint_D \left(16 + x^2 + y^2\right) dx\,dy = 384\pi$

b. Evaluate $I = \iint_{\mathscr{S}} \mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F} = \langle x/z,\ y/z,\ z-2\rangle$

and $\mathscr{S}$ is the upper surface, upward oriented, of $z = 4 - x^2 - y^2,\ \ x^2 + y^2 \le 2$

Solution

Apply (6.2.5),

$$I = \iint_D \left[-\left(\frac{x}{4-x^2-y^2} \right)(-2x) - \left(\frac{y}{4-x^2-y^2} \right)(-2y) + \left(4-x^2-y^2-2\right) \right] dx\,dy = (8\ln 2 - 2)\pi$$

c. Find the flux of the vector field $\mathbf{F} = \langle x,\ z,\ y \rangle$ outward through the sphere $x^2+y^2+z^2=a^2$.

Solution

For the upper half of the sphere, $\mathscr{S}_1 : z = \sqrt{a^2-x^2-y^2}\,, (x,y) \in D : x^2+y^2 \le a^2$, apply (6.2.5):

$$\iint_{\mathscr{S}_1} \mathbf{F}\cdot d\mathbf{S} = \iint_D \left[-x\left(\frac{-x}{\sqrt{a^2-x^2-y^2}} \right) - \left(\sqrt{a^2-x^2-y^2}\right)\left(\frac{-y}{\sqrt{a^2-x^2-y^2}} \right) + y \right] dx\,dy$$

$$= \iint_D \left(\frac{x^2}{\sqrt{a^2-x^2-y^2}} + 2y \right) dx\,dy$$

For the lower half of the sphere, $\mathscr{S}_2 : z = -\sqrt{a^2-x^2-y^2}\,, x^2+y^2 \le a^2$, apply (6.2.6):

$$\iint_{\mathscr{S}_2} \mathbf{F}\cdot d\mathbf{S} = \iint_D \left[x\left(\frac{x}{\sqrt{a^2-x^2-y^2}} \right) + \left(-\sqrt{a^2-x^2-y^2}\right)\left(\frac{y}{\sqrt{a^2-x^2-y^2}} \right) - y \right] dx\,dy$$

$$= \iint_D \left(\frac{x^2}{\sqrt{a^2-x^2-y^2}} - 2y \right) dx\,dy \qquad \Rightarrow \iint_{\mathscr{S}} \mathbf{F}\cdot d\mathbf{S} = \iint_{\mathscr{S}_1} \mathbf{F}\cdot d\mathbf{S} + \iint_{\mathscr{S}_2} \mathbf{F}\cdot d\mathbf{S} = \frac{4\pi a^3}{3}$$

d. A fluid with density k flows with velocity $\mathbf{v} = \langle y,\ 1,\ z \rangle$. Find the rate of flow upward through the paraboloid $\mathscr{S}:\ z = 9-(x^2+y^2)/4,\ x^2+y^2 \le 36$.

Solution

The rate of flow upward through the paraboloid $\mathscr{S}$ is, applying (6.2.5),

$$R = k\iint_{\mathscr{S}} \mathbf{v}\cdot d\mathbf{S} = k\iint_D \left[-y\left(\frac{-x}{2}\right) - 1\left(\frac{-y}{2}\right) + \left(9 - \frac{x^2+y^2}{4}\right) \right] dxdy\ = 162k\,\pi$$

e. Find the flux of the vector field $\mathbf{F} = \langle x,\ 2y,\ 3z \rangle$ outward through the cube $[-1,2]^3$.

Solution:

Apply (6.2.10) – (6.2.12) to six faces of the cube:

For the face $\mathscr{S}_1:\ z=2,\ (x,y)\in D_1 = [-1,2]^2,\ \alpha=\beta=\pi/2,\ \gamma=0,$

$$\iint_{\mathscr{S}_1} \mathbf{F}\cdot d\mathbf{S} = \iint_{\mathscr{S}_1} 3z\,dxdy = \iint_{D_1} 3\times 2\,dxdy = 6\times 9 = 54$$

For the face $\mathscr{S}_2:\ z=-1,\ (x,y)\in D_1 = [-1,2]^2,\ \alpha=\beta=\pi/2,\ \gamma=\pi,$

$$\iint_{\mathscr{S}_2} \mathbf{F}\cdot d\mathbf{S} = -\iint_{\mathscr{S}_2} 3z\,dxdy = -\iint_{D_1} 3\times(-1)\,dxdy = 3\times 9 = 27$$

$$\ldots\ldots \quad \Rightarrow \iint_{\mathscr{S}} \mathbf{F}\cdot d\mathbf{S} = \sum_{i=1}^{6} \iint_{\mathscr{S}_i} \mathbf{F}\cdot d\mathbf{S} = 162$$

f. Find the flux of the vector field $\mathbf{F} = \langle x+2y,\ 2y+3z,\ 3z+x\rangle$ outward through the triangular pyramid $OABC,\ O(0,0,0),\ A(1,0,0),\ B(0,1,0),\ C(0,0,1)$.

Answer: $\iint_{\mathscr{S}} \mathbf{F}\cdot d\mathbf{S} = \sum_{i=1}^{4} \iint_{\mathscr{S}_i} \mathbf{F}\cdot d\mathbf{S} = -\frac{1}{6} - \frac{1}{3} - \frac{1}{2} + 2 = \boxed{1}$

EXAMPLE 6.2.2 THE ELECTRIC FIELD

Suppose an electric charge Q is located at the origin. According to Coulomb's Law, the electric force $\mathbf{F}$ exerted by the charge Q on a charge q located at $M(x,y,z)$ is

$$\mathbf{F}(\mathbf{r}) = \frac{\varepsilon q Q}{|\mathbf{r}|^3}\mathbf{r}, \text{ where } \mathbf{r} = \langle x,y,z\rangle \tag{6.2.13}$$

where ε is a constant.

The force exerted by the charge Q (located at the origin) on a unit chart, q = 1, located at any point $M(x,y,z),\ (x,y,z)\in \mathbb{R}^3$, is called the **electric field** of Q:

$$\mathbf{E}(\mathbf{r}) = \frac{\varepsilon Q}{|\mathbf{r}|^3}\mathbf{r}, \text{ where } \mathbf{r} = \langle x,y,z\rangle \tag{6.2.14}$$

Find the **electric flux** of $\mathbf{E}$ given in (6.2.14) outward through the sphere $\mathscr{S}$: $x^2+y^2+z^2=a^2$.

Solution

Method 1

On the sphere $\mathscr{S}$: $\mathbf{E}(\mathbf{r}) = \frac{\varepsilon Q}{a^3}\mathbf{r} = \frac{\varepsilon Q}{a^3}\langle x,y,z\rangle$, where $x^2+y^2+z^2=a^2$.

By the same way in calculating the flux in Example 6.2.1 b:

- For the upper half of the sphere, $\mathscr{S}_1: z=\sqrt{a^2-x^2-y^2}\,,(x,y)\in D: x^2+y^2\le a^2$, apply (6.2.5),

$$\iint_{\mathscr{S}_1} \mathbf{E}\cdot d\mathbf{S} = \frac{\varepsilon Q}{a^3}\iint_D \left[-x\left(\frac{-x}{\sqrt{a^2-x^2-y^2}}\right) - y\left(\frac{-y}{\sqrt{a^2-x^2-y^2}}\right) + \sqrt{a^2-x^2-y^2}\right] dx\,dy$$
$$= 2\pi\varepsilon Q$$

- For the lower half of the sphere, $\mathscr{S}_2: z=-\sqrt{a^2-x^2-y^2}\,, x^2+y^2\le a^2$, apply (6.2.6),

$$\iint_{\mathscr{S}_2} \mathbf{E}\cdot d\mathbf{S} = \frac{\varepsilon Q}{a^3}\iint_D \left[x\left(\frac{x}{\sqrt{a^2-x^2-y^2}}\right) + y\left(\frac{y}{\sqrt{a^2-x^2-y^2}}\right) - \left(-\sqrt{a^2-x^2-y^2}\right)\right] dx\,dy$$

$$= 2\pi\varepsilon Q\,.$$

Therefore the **electric flux** of **E** given in (6.2.11) through the sphere $\mathscr{S}$: $x^2+y^2+z^2=a^2$ is

$$\boxed{\iint_{\mathscr{S}} \mathbf{E}\cdot d\mathbf{S} = 4\pi\varepsilon Q} \tag{6.2.15}$$

Method 2

Note that, **E** have the same direction as the unit outward normal vector **n** of the sphere $\mathscr{S}$ $x^2+y^2+z^2=a^2$, see (6.2.14).

Therefore

$$\mathbf{E}(\mathbf{r})\cdot\mathbf{n} = \frac{\varepsilon Q}{|\mathbf{r}|^3}\mathbf{r}\cdot\mathbf{n} = \frac{\varepsilon Q}{|\mathbf{r}|^3}|\mathbf{r}| = \frac{\varepsilon Q}{|\mathbf{r}|^2} = \frac{\varepsilon Q}{a^2} \text{ on the sphere},$$

and

$$\iint_{\mathscr{S}} \mathbf{E}\cdot d\mathbf{S} = \iint_{\mathscr{S}} \mathbf{E}\cdot\mathbf{n}\,dS = \iint_{\mathscr{S}} \frac{\varepsilon Q}{a^2}\,dS$$

$$= \frac{\varepsilon Q}{a^2}\,\text{Area}(\mathscr{S}) = \frac{\varepsilon Q}{a^2}4\pi a^2 = 4\pi\varepsilon Q\,.$$

From (6.2.15), Q can be expressed by the electric flux as below

$$\boxed{Q = \frac{1}{4\pi\varepsilon}\iint_{\mathscr{S}} \mathbf{E}\cdot d\mathbf{S} = \varepsilon_0\iint_{\mathscr{S}} \mathbf{E}\cdot d\mathbf{S}} \tag{6.2.16}$$

where

$$\varepsilon_0 = \frac{1}{4\pi\varepsilon} \approx 8.8546\times 10^{-2}\ \mathrm{C}^2/\mathrm{N\,m}^2 \tag{6.2.17}$$

Formula (6.2.16) gives a result of **Gauss's Law**, one of the important laws of **electrostatics**. The Law says that ***the net charge*** *enclosed by any* ***closed surface*** $\mathscr{S}$ is expressed by (6.2.16).

6.3 STOKES' THEOREM

Looking back formula (5.7.2) in Chapter 5, for the vector field **F** in the plane

$$\int_{\mathscr{C}^+} \mathbf{F}\cdot d\mathbf{r} = \int_{\mathscr{C}^+} (\mathbf{F}\cdot\mathbf{T})\,ds = \int_{\mathscr{C}^+} P(x,y)\,dx + Q(x,y)\,dy = \iint_D (\text{curl}\,\mathbf{F})\cdot\mathbf{k}\,dxdy \tag{6.3.1}$$

Because **k** is the unit normal vector of the upper side of "surface" *D* on the *xy*-plane, the above formula can be rewrite in surface integral sense as below

$$\int_{\mathscr{C}^+} \mathbf{F}\cdot d\mathbf{r} = \int_{\mathscr{C}^+} (\mathbf{F}\cdot\mathbf{T})\,ds = \int_{\mathscr{C}^+} P(x,y)\,dx + Q(x,y)\,dy$$

$$= \iint_D (\operatorname{curl}\mathbf{F})\cdot\mathbf{k}\,dxdy = \iint_D (\operatorname{curl}\mathbf{F})\cdot\mathbf{n}\,dS \qquad (6.3.2)$$

This formula says that **the line integral around the boundary $\mathscr{C}^+$ of *D* of the tangential component of F is equal to the surface integral of the normal component of the curl of F across the *D***.

We have also the same conclusion for the vector field **F** in the space as given in the Theorem by Stokes that we accept.

THEOREM 6.3.1 STOKES' THEOREM

Let $\mathscr{S}$ be an oriented piecewise-smooth surface that is bounded by a simple, **closed**, piecwise-smooth curve $\mathscr{C}^+$, where the positive direction of $\mathscr{C}$ coincides with the direction of **n** in *the right hand rule* sense**.** Let **F** be a vector field with components having continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains $\mathscr{S}$. Then

$$\int_{\mathscr{C}^+} \mathbf{F}\cdot d\mathbf{r} = \int_{\mathscr{C}^+} (\mathbf{F}\cdot\mathbf{T})\,ds$$

$$= \int_{\mathscr{C}^+} P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$$

$$= \iint_{\mathscr{S}} \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_{\mathscr{S}} \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS \qquad (6.3.3)$$

COROLLARY 6.3.1

If $\operatorname{curl}\mathbf{F} = \mathbf{0}$, then **F** is conservative and $\int_{\mathscr{C}} \mathbf{F}\cdot d\mathbf{r} = 0$ for any closed curve $\mathscr{C}$.

Proof

If $\operatorname{curl}\mathbf{F} = \mathbf{0}$, then by (6.3.3), $\int_{\mathscr{C}^+} \mathbf{F}\cdot d\mathbf{r} = 0$ for any closed curve $\mathscr{C}$. Therefore, by Corollary 5.4.1, **F** is conservative.

Remark

This Corollary is the converse statement of Corollary 5.5.1, from which follows Theorem 5.5.2

EXAMPLE 6.3.1

a. Evaluate $\int_{\mathscr{C}^+} \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F} = \langle -y^2, x, yz\rangle$ and $\mathscr{C}^+$ is the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 1$. The direction of $\mathscr{C}^+$ coincides with the positive direction of the *z*-axis.

Solution

Method 1: Direct calculation of the line integral.

The parametric equations of $\mathscr{C}^+$: $x=\cos t,\ y=\sin t,\ z=1-\cos t-\sin t,\ 0\le t\le 2\pi$.

$$\int_{\mathscr{C}^+}\mathbf{F}\cdot d\mathbf{r}=\int_0^{2\pi}\left(-y^2(t)x'(t)+x(t)y'(t)+y(t)z(t)z'(t)\right)dt$$

$$=\int_0^{2\pi}\left[\sin^3 t+\cos^2 t+\sin t(1-\cos t-\sin t)(1-\cos t-\sin t)'\right]dt=2\pi\ .$$

Method 2: Apply Stokes'Theorem'

$$\mathbf{F}=\left\langle -y^2,\ x,\ xz\right\rangle \Rightarrow \operatorname{curl}\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ -y^2 & x & yz\end{vmatrix}=\left\langle z,\ 0,\ (1+2y)\right\rangle$$

$$\int_{\mathscr{C}^+}\mathbf{F}\cdot d\mathbf{r}=\iint_{\mathscr{S}}\operatorname{curl}\mathbf{F}\cdot d\mathbf{S}=\iint_{\mathscr{S}}\operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS\ \text{, apply (6.2.5) where } z=1-x-y,$$

$$=\iint_D\left[-(1-x-y)(-1)+0(-1)+(1+2y)\right]dxdy$$

$$=\iint_D(2-x+y)\,dxdy=\int_0^{2\pi}d\theta\int_0^1(2-r\cos\theta+r\sin\theta)\,rdr=2\pi\ .$$

b. Evaluate $\iint_{\mathscr{S}}\operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS$ where $\mathbf{F}=\left\langle -yz,\ xz,\ xy\right\rangle$ and $\mathscr{S}$ is the upper part, upward oriented, of the sphere $x^2+y^2+z^2=25$ that lies inside the cylinder $x^2+y^2=9$.

Solution

Method 1:

Direct calculation of the surface integral:

$$\mathbf{F}=\left\langle -yz,\ xz,\ xy\right\rangle \Rightarrow \operatorname{curl}\mathbf{F}=\left\langle 0,\ -2y,\ 2z\right\rangle$$

$$\iint_{\mathscr{S}}\operatorname{curl}\mathbf{F}\cdot d\mathbf{S}=\iint_{\mathscr{S}}\operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS=\iint_D\left[0-(-2y)\frac{-y}{\sqrt{25-x^2-y^2}}+2\sqrt{25-x^2-y^2}\right]dxdy=72\pi$$

Method 2: Apply Stokes'Theorem'

The parametric equations of the boundary $\mathscr{C}^+$ of $\mathscr{S}$: $x=3\cos t,\ y=3\sin t,\ z=4,\ 0\le t\le 2\pi$

$$\iint_{\mathscr{S}}\operatorname{curl}\mathbf{F}\cdot d\mathbf{S}=\int_{\mathscr{C}^+}\mathbf{F}\cdot d\mathbf{r}=\int_0^{2\pi}[-y(t)z(t)x'(t)+x(t)z(t)y'(t)+x(t)y(t)z'(t)]\,dt=72\pi$$

c. Use Stockes' Theorem to evaluate $\int_{\mathscr{C}^+}\mathbf{F}\cdot d\mathbf{r}$ where

$\mathbf{F}=\langle x+y^2,\ y+z^2,\ z+x^2\rangle$ and $\mathscr{C}$ is the triangle with vertices (1, 0, 0), (0, 1, 0), (0, 0, 1), oriented counterclockwise as viewed from above.

Solution:

$$\operatorname{curl}\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ x+y^2 & y+z^2 & z+x^2\end{vmatrix}=\langle -2z,\ -2x,\ -2y\rangle$$

The equation of the plane containing the triangle:

$$x+y+z=1 \text{ or } z=1-x-y,\ (x,y)\in D=\{(x,y):0\le y\le 1-x, 0\le x\le 1\}.$$

Applying (6.2.4) and (6.3.3), we obtain

$$\int_{\mathscr{C}^+}\mathbf{F}\cdot d\mathbf{r}=\iint_{\mathscr{S}}\operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS=\iint_D[-2(1-x-y)-2x-2y]\,dxdy=\iint_D -2\,dxdy=-1$$

6.4 THE DIVERGENCE THEOREM

Formula (5.7.3) in Chapter 5 expresses the integral of the **normal component** of **F** along the closed curve $\mathscr{C}^+$ in the *xy*-plane as the double integral of $\operatorname{div}\mathbf{F}$ over the domain *D* enclosed by the curve

$$\int_{\mathscr{C}^+}\mathbf{F}\cdot\mathbf{n}\,ds=\iint_D \operatorname{div}\mathbf{F}(x,y)\,dxdy \qquad (6.4.1)$$

Note that $\mathscr{C}^+$ is the positively oriented boundary of the domain *D* in the *xy*-plane. The following theorem gives the similar result for the vector field **F** in space, the domain *D* in space, and the boundary surface $\mathscr{S}$ of *D* with positive orientation, i.e. with outward orientation.

THEOREM 6.4.1 THE DIVERGENCE THEOREM

Suppose that *D* is a simple solid region (or *D* is a finite union of simple solid regions), and $\mathscr{S}$ is its boundary surface positively oriented, and $\mathbf{F}=\langle P(x,y,z),\ Q(x,y,z),\ R(x,y,z)\rangle$ is a vector field where $P(x,y,z)$, $Q(x,y,z)$, and $R(x,y,z)$ have continuous partial derivatives on an open region that contains *D*. Then

$$\iint_{\mathscr{S}}\mathbf{F}\cdot d\mathbf{S}=\iint_{\mathscr{S}}\mathbf{F}\cdot\mathbf{n}\,dS=\iiint_D \operatorname{div}\mathbf{F}\,dxdydz=\iiint_D\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dxdydz \qquad (6.4.2)$$

We accept the Theorem.

It follows from the Midpoint Rule For Triple Integrals, see Property 4.1.2 of Chapter 4, and the Divergence Theorem that, there exist a point $M_0=(x_0,y_0,z_0)$ in *D* so that the flux of F across outward the surface $\mathscr{S}$ can be express by

$$\iint_{\mathscr{S}}\mathbf{F}\cdot d\mathbf{S}=\iint_{\mathscr{S}}\mathbf{F}\cdot\mathbf{n}\,dS=\iiint_D \operatorname{div}\mathbf{F}\,dxdydz=\operatorname{div}\mathbf{F}(M_0)\times \mathrm{V}(D) \qquad (6.4.3)$$

where V(*D*) denotes the volume of *D*.

If D is a ball of a given center $M=(x,y,z)$ and of a radius a then from (6.4.3) and the continuity of $\operatorname{div}\mathbf{F}$, by the hypotheses of the Divergence Theorem,

$$\boxed{\operatorname{div}\mathbf{F}(M)=\lim_{a\to 0}\frac{1}{\mathrm{V}(D)}\iint_{\mathscr{S}}\mathbf{F}\cdot d\mathbf{S}} \tag{6.4.4}$$

This equation says that $\operatorname{div}\mathbf{F}$ at a given point $M=(x,y,z)$, $\operatorname{div}\mathbf{F}(M)$, is ***the net rate of outward flux per unit volume*** at M. If $\operatorname{div}\mathbf{F}(M)>0$, M is called a **source.** If $\operatorname{div}\mathbf{F}(M)<0$, M is called a **sink**.

EXAMPLE 6.4.1

a. Find the flux of the vector field $\mathbf{F}=\langle x,\ z,\ y\rangle$ outward through the sphere $x^2+y^2+z^2=a^2$.

Solution

We have done the direct calculation in Example 6.2.1b and obtained $\iint_{\mathscr{S}}\mathbf{F}\cdot d\mathbf{S}=\dfrac{4\pi a^3}{3}$.

Let us calculate it applying (6.4.2). Since $\mathbf{F}=\langle x,\ z,\ y\rangle$ then $\operatorname{div}\mathbf{F}=1$ and obtain easily that result:

$$\iint_S\mathbf{F}\cdot\mathbf{n}\,dS=\iiint_D\operatorname{div}\mathbf{F}\,dxdydz=\iiint_D dxdydz=\text{Volume}(D)=\frac{4\pi a^3}{3}$$

b Find the flux of the vector field $\mathbf{F}=\langle x,\ 2y,\ 3z\rangle$ outward through the cube $[-1,2]^3$.

Solution

The direct calculation in Example 6.2.1d gives $\iint_{\mathscr{S}}\mathbf{F}\cdot d\mathbf{S}=162$.

Let us calculate it applying (6.4.2). Since $\mathbf{F}=\langle x,\ 2y,\ 3z\rangle$ then $\operatorname{div}\mathbf{F}=6$ and we obtain easily the same result:

$$\iint_S\mathbf{F}\cdot\mathbf{n}\,dS=\iiint_D\operatorname{div}\mathbf{F}\,dxdydz=\iiint_D 6\,dxdydz=6\times\text{Volume}(D)=6\times 3^3=162$$

c. Find the flux of the vector field $\mathbf{H}=\langle x+2y,\ 2y+3z,\ 3z+x\rangle$ outward through the triangular pyramid $OABC,\ O(0,0,0),\ A(1,0,0),\ B(0,1,0),\ C(0,0,1)$.

Solution

Example 6.2.1.f gives $\iint_{\mathscr{S}}\mathbf{F}\cdot d\mathbf{S}=1$. We can obtain easily the result applying (6.4.2): $\mathbf{F}=\langle x+2y,\ 2y+3z,\ 3z+x\rangle$, then $\operatorname{div}\mathbf{F}=6$, and then

$$\iint_{\mathscr{S}}\mathbf{F}\cdot\mathbf{n}\,dS=\iiint_D\operatorname{div}\mathbf{F}\,dxdydz=\iiint_D 6\,dxdydz=6\times\text{Volume}(D)=6\left(\frac{1}{3}\times\frac{1}{2}\right)=1$$

EXERCISES

6.1 Find the area of

a. The ellipse cut from the plane $z = cx$ by the cylinder $x^2 + y^2 = 1$.

b. The surface $x^2 - 2\ln x + \sqrt{15}\, y - z = 0$ above the square $D: 1 \le x \le 2,\ 0 \le y \le 1,$ in the xy-plane.

c. The part of the paraboloid $z = x^2 + y^2$ which lies under the plane $z = 6$.

6.2 Evaluate the surface integral.

a. $I = \iint_{\mathscr{S}} (x^2 z + y^2 z)\, dS,$ where $\mathscr{S}$ is the surface (hemisphere) $x^2 + y^2 + z^2 = 4,\ z \ge 0$

b. $I = \iint_{\mathscr{S}} xyz\, dS,$ where $\mathscr{S}$ is the surface with parametric equations

$$x = uv,\ y = u - v,\ z = u + v,\ (u,v) \in D: u^2 + v^2 \le 1, u \ge v \ge 0.$$

c. $I = \iint_{\mathscr{S}} y^2\, dS,$ where $\mathscr{S}: x^2 + y^2 + z^2 = a^2$.

d. $I = \iint_{\mathscr{S}} z\, dS,$ where $\mathscr{S}$ is the closed surface that contains the cylinder $x^2 + y^2 = a^2$ and the planes $z = 0$ and $z = y + a,\ a > 0.$

e. $I = \iint_{\mathscr{S}} yz\, dS,$ where $\mathscr{S}$ is the surface with parametric equations

$$x = uv,\ y = u + v,\ z = u - v,\ u^2 + v^2 \le 1.$$

6.3 Use Stockes' Theorem to evaluate

a. $\iint_{\mathscr{S}} \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, dS$, where $\mathbf{F} = \left\langle x^2 e^{yz},\ y^2 e^{xz},\ z^2 e^{xy} \right\rangle$ and $\mathscr{S}$ is the hemisphere with equation $x^2 + y^2 + z^2 = 4,\ z \ge 0$, oriented upward.

b. The work done by the force field $\mathbf{F} = \left\langle x^x + z^2, y^y + x^2, z^z + y^2 \right\rangle$ when a particle moves under its influence around the close edge of the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies in the first octant, in a counterclockwise direction as viewed from above.

6.4 Use the Divergence Theorem to evaluate $\iint_{\mathscr{S}} \mathbf{F} \cdot d\mathbf{S}$

a. $\mathbf{F} = \left\langle z^2 x,\ \frac{1}{3} y^3 + \sin z,\ x^2 z + y^2 \right\rangle$

$\mathscr{S}$ is the upward oriented top half of the sphere $x^2 + y^2 + z^2 = 1$.

b. $\mathbf{F} = \left\langle z^2 y^4,\ 4x^2 y^3 + \sin z,\ 2x^2 z + y^2 \right\rangle$ and $\mathscr{S}$ is the outward oriented surface of the cube $[-1, 1]^3$.

c. $\mathbf{F} = \langle -z^2x,\ 7z^2y,\ 3z^3 + y^2 \rangle$ and $\mathscr{S}$ is the outward oriented surface of the sphere $x^2 + y^2 + z^2 = a^2$.

d. $\mathbf{F} = \langle -xy,\ 3y^2,\ 3zy \rangle$ and $\mathscr{S}$ is the outward oriented surface of the tetrahedron with vertices $(0,\ 0,\ 0), (1,\ 0,\ 0), (0,\ 1,\ 0),$ and $(0,\ 0,\ 1)$.

e. $\mathbf{F} = \langle 6xy^2,\ 3x^2e^{2z},\ 2z^3 \rangle$ and $\mathscr{S}$ is the outward oriented surface of the solid bounded by the cylinder $x^2 + y^2 = 4$ and the planes $x = -1,$ and $x = 2$.

f. $\mathbf{F} = \langle x^5,\ \frac{10}{3}x^2y^3,\ 5zy^4 \rangle$ and $\mathscr{S}$ is the outward oriented surface of the solid bounded by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

6.5 Use the Divergence Theorem to evaluate

a. $\iint\limits_{\mathscr{S}} (2x^2 + 3y^2 + 4z^2)\, dS$, $\mathscr{S}$ is the sphere $x^2 + y^2 + z^2 = a^2$.

b. $\iint\limits_{\mathscr{S}} \dfrac{-x^2 + 2y^2 + 4z^2}{\sqrt{\dfrac{x^2}{16} + \dfrac{y^2}{81} + \dfrac{z^2}{625}}}\, dS$, $\mathscr{S}$ is the ellipsoid $\dfrac{x^2}{4} + \dfrac{y^2}{9} + \dfrac{z^2}{25} = 1$.

c. $\iint\limits_{\mathscr{S}} \mathbf{F} \cdot \mathbf{n}\, dS$, $\mathbf{F} = \langle 2x^3 + y^2,\ e^{2z},\ 6y^2z \rangle$,

$\mathscr{S}$ is the outward oriented surface of the solid bounded by $x^2 + y^2 = 4,\ z = -1,\ z = 2.$

6.6 Verify that the Stokes' Theorem is true for the given vector field $\mathbf{F}$ and surface $\mathscr{S}$.

a. $\mathbf{F} = \langle 3y,\ 4z,\ -6x \rangle$, $\mathscr{S}$ is the part of $z = x^2 + y^2 - 9, x^2 + y^2 \le 9,$ oriented downward.

b. $\mathbf{F} = \langle y,\ z,\ x \rangle$, $\mathscr{S}$ is the part of $x + y + z = -1, x \le 0, y \le 0,\ z \le 0,$ oriented downward.

6.7 Verify that the Divergence Theorem is true for the given vector field $\mathbf{F}$ on the domain D.

a. $\mathbf{F} = \langle 3xy,\ 4y,\ -6xz \rangle$, D is the cube $[0,1]^3$.

b. $\mathbf{F} = \langle x^2,\ xy,\ 3xz \rangle$, D is the region bounded by $z = x^2 + y^2$ and $z = 9$.

c. $\mathbf{F} = \langle xy,\ zy,\ xz \rangle$, D is the region bounded by $x^2 + y^2 = 4, z = -1$ and $z = 4$.

d. $\mathbf{F} = \langle ax,\ ay,\ az \rangle, a > 0,$ and D is the ball $x^2 + y^2 + z^2 \le a^2$.

6.8 Use Stockes' Theorem to evaluate $\iint\limits_{\mathscr{S}} \text{curl}\, \mathbf{F} \cdot \mathbf{n}\, dS$

a. $\mathbf{F} = \langle e^x yz,\ y^2z,\ 2z \rangle$, $\mathscr{S}$ is the part of the hemisphere $x^2 + y^2 + z^2 = 9,\ x \ge 0,$ that lies inside the cylinder $y^2 + z^2 = 4$, oriented in the direction of the positive x-axis.

b. $\mathbf{F} = \langle xyz,\ xy,\ x^2yz \rangle$, $\mathscr{S}$ consists of the top and the four sides but not the bottom of the cube $[0, 1]^3$, oriented outward.

ANSWERS

6.1 a. $\pi\sqrt{c^2+1}$; b. $3+2\ln 2$; c. $62\pi/3$

6.2 a. $I=16\pi$; b. $I=\dfrac{23\sqrt{6}}{280}-\dfrac{16}{105}=\dfrac{69\sqrt{6}-128}{840}$; c. $\dfrac{4}{3}\pi a^4$; d. $\left(\frac{3}{2}+\sqrt{2}\right)\pi a^3$

6.3 a. 0; b. 16;

6.4 a. $13\pi/20$ (*Hint*: Add the disk oriented downward)

b. 16. c. $4\pi a^5$; d. 1/3; e. 144π; f. $5\pi/12$

6.5 a. $12\pi a^4$; b. 4560π; c. 144π

6.8 a. -4π; b. $1/2$;

REFERENCES

[1] James Stewart, CALCULUS, Fourth edition, Brooks/Cole, 1999.

[2] George B. Thomas, Jr., CALCULUS, Pearson Education, Inc., 2006.

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