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PRESENTATION TITLE

Stochastic Proceses

STOCHASTIC PROCESSES

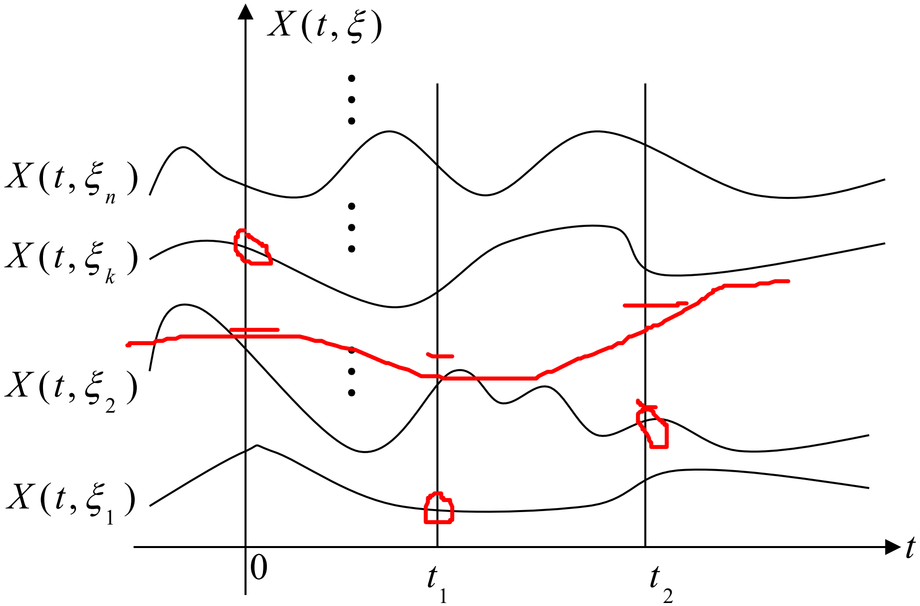
- Text

IV. Stochastic processes

- Definitions
- Stationary processes
- Spectral density function
- Ergodicity of stochastic processes
- Stochastic Data Processing Systems

IV. Stochastic processes
4.1. Definitions

- Stochastic processes
  - Let ξ denote the random outcome of an experiment. To every such outcome suppose a waveform X(t, ξ) is assigned.
  - The collection of such waveforms form a stochastic process.



IV. Stochastic processes
4.1. Definitions

- The set of $\{\xi_k\}$ and the time index *t* can be continuous or discrete (countably infinite or finite) as well.
- For fixed $\xi_i \in S$ (the set of all experimental outcomes), $X(t, \xi_i)$ is a specific time function.
- For fixed *t*, $X_1 = X(t_1, \xi_i)$ is a random variable. The ensemble of all such realizations $X(t, \xi)$ over time represents the stochastic process $X(t)$.

IV. Stochastic processes
4.1. Definitions

- Example
  - $X(t)=A\cos(\omega_0 t+\varphi)$
    - Where φ is a uniformly distributed random variable in $(0, 2\pi)$ represents a stochastic process.
- Some stochastic processes
  - Brownian motion,
  - Stock market fluctuations,
  - Various queuing systems.

IV. Stochastic processes
4.1. Definitions

- If *X*(*t*) is a stochastic process, then for fixed *t*, *X*(*t*) represents a random variable.
- Its distribution function is given by

$$F_X(x,t) = P\{X(t) \le x\}$$

- Notice that $F_x(x, t)$ depends on *t*, since for a different *t*, we obtain a different random variable.

$$f_X(x,t) \triangleq \frac{dF_X(x,t)}{dx}$$

- Derivative of $F_x(x, t)$ represents the first-order probability density function of the process *X*(*t*).

IV. Stochastic processes
4.1. Definitions

- For $t = t_1$ and $t = t_2$,
  - $X(t)$ represents two different random variables $X_1 = X(t_1)$ and $X_2 = X(t_2)$ respectively.
  - Their joint distribution is given by

$$F_X(x_1, x_2, t_1, t_2) = P\{X(t_1) \le x_1, X(t_2) \le x_2\}$$

  - Their joint density function is:

$$f_X(x_1, x_2, t_1, t_2) \triangleq \frac{\partial^2 F_X(x_1, x_2, t_1, t_2)}{\partial x_1 \ \partial x_2}$$

  - And represents the second-order density function of the process $X(t)$.

IV. Stochastic processes
4.1. Definitions

- Similarly $f_x(x_1, ..., x_n, t_1, ..., t_n)$ represents the n^{th} order density function of the process *X*(*t*).
- Complete specification of the stochastic process *X*(*t*) requires the knowledge of $f_x(x_1, ..., x_n, t_1, ..., t_n)$ for all t_i, i = 1, ..., n and for all *n*. (an almost impossible task in reality).

IV. Stochastic processes
4.1. Definitions

- Characterisctics of stochastic processes
  - Mean of a stochastic process

$$\mu(t) \triangleq E\{X(t)\} = \int_{-\infty}^{+\infty} x f_X(x,t)dx$$

    - μ(t) represents the mean value of a process *X*(*t*). In general, the mean of a process can depend on the time index *t*.
  - Autocorrelation function of a process *X*(*t*) is defined as

$$R_{XX}(t_1,t_2) \triangleq E\{X(t_1)X^*(t_2)\} = \int\int x_1 x_2^* f_X(x_1,x_2,t_1,t_2)dx_1 dx_2$$

    - It represents the interrelationship between the random variables *X*1 = *X*(*t*1) and *X*2 = *X*(*t*2) generated from the process *X*(*t*).

IV. Stochastic processes
4.1. Definitions

- Properties of autocorrelation function

$$1.\quad R_{XX}(t_1,t_2) = R^*_{XX}(t_2,t_1) = [E\{X(t_2)X^*(t_1)\}]^*$$

$$2.\quad R_{XX}(t,t) = E\{|X(t)|^2\} \geq 0.$$

(Average instantaneous power)

- 3. $R_{xx}(t_1, t_2)$ represents a nonnegative definite function, i.e., for *any* set of constants $\{a_i\}^n_{i=1}$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j^* R_{XX}(t_i,t_j) \geq 0.$$

  - This equation follows by noticing that: $E\{|Y|^2\} \geq 0$ and

$$Y = \sum_{i=1}^{n} a_i X(t_i).$$

IV. Stochastic processes
4.1. Definitions

- The **autocovariance** function of the process *X*(*t*).

$$C_{XX}(t_1,t_2) = R_{XX}(t_1,t_2) - \mu_X(t_1)\mu_X^*(t_2)$$

- Examples
  - Given

$$z = \int_{-T}^{T} X(t)dt.$$

$$E[|z|^2] = \int_{-T}^{T}\int_{-T}^{T} E\{X(t_1)X^*(t_2)\}dt_1dt_2$$

$$= \int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1,t_2)dt_1dt_2$$

IV. Stochastic processes
4.1. Definitions

- Consider process X(t)

$$X(t) = a\cos(\omega_0 t + \varphi), \qquad \varphi \sim U(0, 2\pi).$$

$$\mu_X(t) = E\{X(t)\} = aE\{\cos(\omega_0 t + \varphi)\}$$

$$= a\cos\omega_0 t\, E\{\cos\varphi\} - a\sin\omega_0 t\, E\{\sin\varphi\} = 0,$$

$$\text{since} \quad E\{\cos\varphi\} = \frac{1}{2\pi}\int_0^{2\pi} \cos\varphi\, d\varphi = 0 = E\{\sin\varphi\}.$$

$$R_{XX}(t_1, t_2) = a^2 E\{\cos(\omega_0 t_1 + \varphi)\cos(\omega_0 t_2 + \varphi)\}$$

$$= \frac{a^2}{2} E\{\cos\omega_0(t_1 - t_2) + \cos(\omega_0(t_1 + t_2) + 2\varphi)\}$$

$$= \frac{a^2}{2} \cos\omega_0(t_1 - t_2).$$

IV. Stochastic processes
4.2. Stationary processes

- Strict sense and wide sense stationarity
  - Stationarity
    - Stationary processes exhibit statistical properties that are invariant to shift in the time index.
    - Thus, for example, second-order stationarity implies that the statistical properties of the pairs $\{X(t_1), X(t_2)\}$ and $\{X(t_1+c), X(t_2+c)\}$ are the same for *any* c.
    - Similarly first-order stationarity implies that the statistical properties of $X(t_i)$ and $X(t_i+c)$ are the same for any c.

IV. Stochastic processes 4.2. Stationary processes

- Strict sense stationarity
  - In strict terms, the statistical properties of a stochastic process are governed by the joint probability density function.
  - A process is n^{th}-order **Strict-Sense Stationary (S.S.S)** if

$$f_X(x_1, x_2, \cdots x_n,\ t_1, t_2 \cdots, t_n) \equiv f_X(x_1, x_2, \cdots x_n,\ t_1 + c, t_2 + c \cdots, t_n + c)$$

for *any c*, where the left side represents the joint density function of the random variables $X_1=X(t_1)$, ..., $X_n=X(t_n)$ and the right side corresponds to the joint density function of the random variables $X'_1=X(t_1+c)$, ..., $X'_n=X(t_n+c)$.

  - A process *X*(*t*) is said to be **strict-sense stationary** if equation above is true for all t_i, i=1, ..., n; n=1, 2, ... and any c.

IV. Stochastic processes
4.2. Stationary processes

- First order strict sense stationary process
  - For any c

$$f_X(x,t) \equiv f_X(x,t+c)$$

  - In particular, if c = -t, then

$$f_X(x,t) = f_X(x)$$

  - That means, the first-order density of *X*(*t*) is independent of *t*. In that case

$$E[X(t)] = \int_{-\infty}^{+\infty} x f(x) dx = \mu, \quad a \ constant. \qquad (1)$$

IV. Stochastic processes
4.2. Stationary processes

- Second order strict sense stochastic process
  - From definition, we have:

$$f_X(x_1, x_2,\ t_1, t_2) \equiv f_X(x_1, x_2,\ t_1 + c, t_2 + c)$$

  - for any c. If c is choosen so as c = $-t_2$, we get

$$f_X(x_1, x_2,\ t_1, t_2) \equiv f_X(x_1, x_2,\ t_1 - t_2)$$

  - The second order density function of a strict sense stationary process depends only on the difference of the time indices t_1-t_2=τ.

IV. Stochastic processes
4.2. Stationary processes

- The autocorrelation function is given by

$$
\begin{aligned}
R_{XX}(t_1,t_2) &\triangleq E\{X(t_1)X^*(t_2)\} \\
&= \int\int x_1 x_2^* f_X(x_1,x_2,\tau = t_1 - t_2)dx_1 dx_2 \\
&= R_{XX}(t_1 - t_2) \triangleq R_{XX}(\tau) = R_{XX}^*(-\tau), \qquad (2)
\end{aligned}
$$

- The autocorrelation function of a second order strict-sense stationary process depends only on the difference of the time indices t_2-t_1=τ

IV. Stochastic processes
4.2. Stationary processes

- Wide sense stationarity
  - A process $X(t)$ is said to be **Wide-Sense Stationary** if:

$$E\{X(t)\} = \mu \quad \text{and} \quad E\{X(t_1)X^*(t_2)\} = R_{XX}(t_1 - t_2),$$

  - Since these equations follow from (1) and (2), strict-sense stationarity always implies wide-sense stationarity.
  - In general, the converse is *not true*
    - Exception: the Gaussian process (normal process).
    - This follows, since if $X(t)$ is a Gaussian process, then by definition $X_1=X(t_1)$, ..., $X_n=X(t_n)$ are jointly Gaussian random variables for any t_1,..., t_n whose joint characteristic function is given by

$$\phi_{\underline{X}}(\omega_1, \omega_2, \cdots, \omega_n) = e^{j\sum_{k=1}^{n} \mu(t_k)\omega_k - \sum_{l,k}^{n}\sum C_{XX}(t_i, t_k)\omega_i\omega_k / 2}$$

IV. Stochastic processes
4.2. Stationary processes

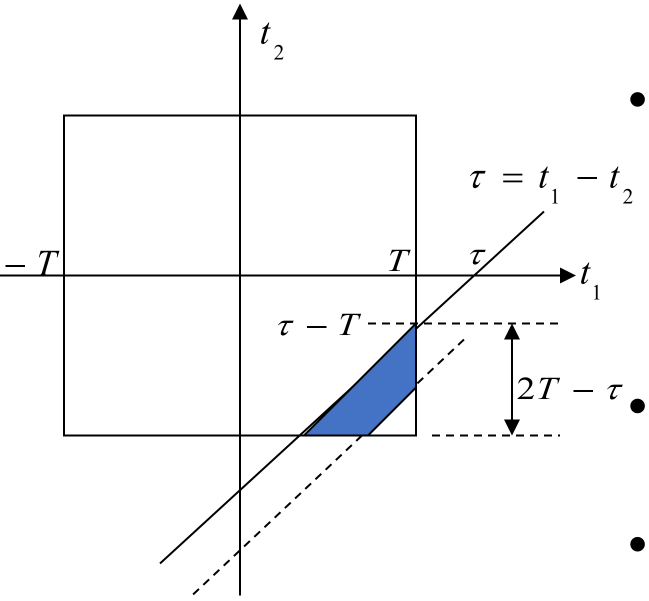
- Examples
  - The process:

$$X(t) = a\cos(\omega_0 t + \varphi), \qquad \varphi \sim U(0,2\pi).$$

    - This process is wide sense stationary but not strict sense stationary.
  - If the process X(t) has zero mean, then σ^2_z is reduced to:

$$z = \int_{-T}^{T} X(t)dt.$$

$$\sigma_z^2 = E\{|z|^2\} = \int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1 - t_2)dt_1 dt_2.$$



  - As t_1 and t_2 varies from –T to T, so $\tau=t_2-t_1$ varies from -2T to 2T.
  - $R_{xx}(\tau)$ is a constant over the shaded region in the figure on the left.

IV. Stochastic processes
4.3. Power spectrum

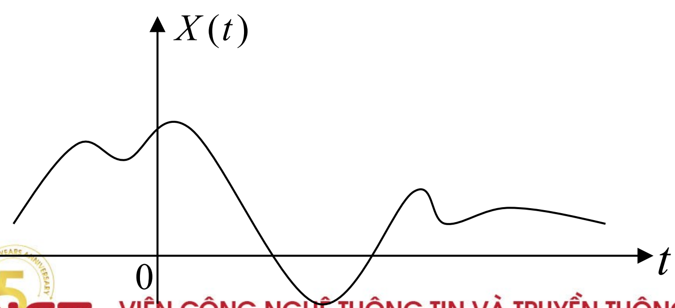
- Power spectrum
  - For a deterministic signal *x*(t)
    - The spectrum is well defined: If X(ω) represents its Fourier transform,

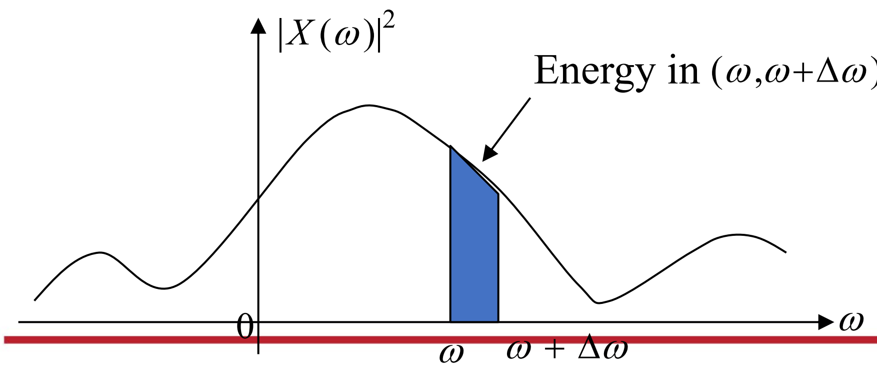
$$X(\omega)=\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt,$$

    - then $|X(\omega)|^2$ represents its energy spectrum. This follows from Parseval's theorem since the signal energy is given by

$$\int_{-\infty}^{+\infty} x^2(t)dt=\frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(\omega)|^2\, d\omega=E.$$

    - Thus $|X(\omega)|^2\Delta\omega$ represents the signal energy in the band $(\omega, \omega+\Delta\omega)$





IV. Stochastic processes
4.3. Power spectrum

- For stochastic processes,
  - A direct application of Fourier transform generates a sequence of random variables for every ω
  - For a stochastic process, $E\{|X(t)|^2\}$ represents the ensemble average power (instantaneous energy) at the instant t.
  - Partial Fourier transform of a process $X(t)$ based on $(-T, \ T)$ is given by

$$X_T(\omega) = \int_{-T}^{T} X(t) e^{-j\omega t} dt$$

  - The power distribution associated with that realization based on $(-T, T)$ is represented by

$$\frac{|X_T(\omega)|^2}{2T} = \frac{1}{2T} \left| \int_{-T}^{T} X(t) e^{-j\omega t} dt \right|^2$$

IV. Stochastic processes
4.3. Power spectrum

- The average power distribution based on (– T, T) is ensemble average of power distribution for ω

$$P_T(\omega) = E\left\{\frac{|X_T(\omega)|^2}{2T}\right\} = \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} E\{X(t_1)X^*(t_2)\}e^{-j\omega(t_1-t_2)}dt_1dt_2$$

$$= \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1,t_2)e^{-j\omega(t_1-t_2)}dt_1dt_2$$

- We have this represents the power distribution of $X(t)$ based on (– T, T).
- If $X(t)$ is assumed to be w.s.s, then

$$R_{XX}(t_1,t_2) = R_{XX}(t_1-t_2)$$

- and we have

$$P_T(\omega) = \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1-t_2)e^{-j\omega(t_1-t_2)}dt_1dt_2.$$

IV. Stochastic processes
4.3. Power spectrum

- Let $\tau = t_2 - t_1$, we obtain

$$P_T(\omega) = \frac{1}{2T}\int_{-2T}^{2T} R_{XX}(\tau)e^{-j\omega\tau}(2T - |\tau|)d\tau$$

$$= \int_{-2T}^{2T} R_{XX}(\tau)e^{-j\omega\tau}(1 - \tfrac{|\tau|}{2T})d\tau \ \geq \ 0$$

- This is the power distribution of the w.s.s. process $X(t)$ based on $(-T, T)$.
- Leting $T \to \infty$, we obtain

$$S_{XX}(\omega) = \lim_{T\to\infty} P_T(\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau)e^{-j\omega\tau}d\tau \geq 0$$

- $S_{xx}(\omega)$ is the *power spectral density* of the w.s.s process $X(t)$.

IV. Stochastic processes
4.3. Power spectrum

- Khinchin-Wiener theorem
  - The autocorrelation function and the power spectrum of a w.s.s process form a Fourier transform pair.

$$R_{XX}(\omega) \xleftrightarrow{\text{F}\cdot\text{T}} S_{XX}(\omega) \geq 0.$$

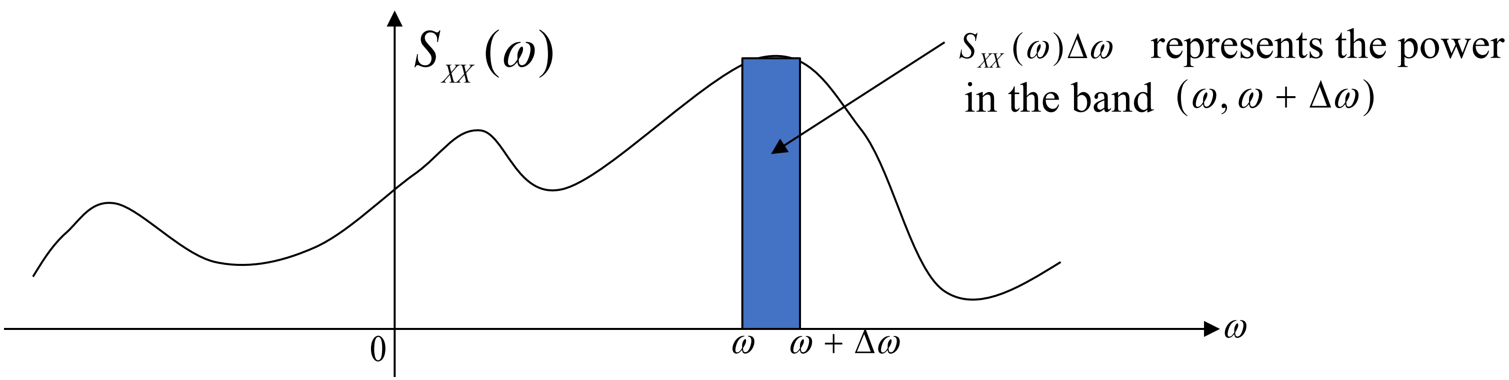
$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega)e^{j\omega\tau}d\omega$$

- For τ = 0, $S_{XX}(\omega) = \lim_{T\to\infty} P_T(\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau)e^{-j\omega\tau}d\tau \geq 0$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega)d\omega = R_{XX}(0) = E\{|X(t)|^2\} = P, \quad \textit{the total power.}$$

IV. Stochastic processes
4.3. Power spectrum

- The area under $S_{xx}(\omega)$ represents the total power of the process $X(t)$, and hence $S_{xx}(\omega)$ truly represents the power spectrum.



- The nonnegative-definiteness property of the auto-correlation function translates into the "nonnegative" property for its Fourier transform (power spectrum).

$$R_{XX}(\tau) \text{ nonnegative-definite} \quad \Leftrightarrow \quad S_{XX}(\omega) \geq 0.$$

IV. Stochastic processes 4.3. Power spectrum

- If $X(t)$ is a real w.s.s process, then

$$R_{XX}(\tau) = R_{XX}(-\tau)$$

- So that

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

$$= \int_{-\infty}^{+\infty} R_{XX}(\tau) \cos \omega\tau d\tau$$

$$= 2\int_{0}^{\infty} R_{XX}(\tau) \cos \omega\tau d\tau = S_{XX}(-\omega) \geq 0$$

- The power spectrum is an even function, (in addition to being real and nonnegative).

IV. Stochastic processes
4.4. Ergodicity

- Time averages
  - Given wide-sense stationary process X(t).
  - Time averages
    - Mean

$$\mathrm{n} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathrm{x}(t)dt$$

    - Autocorrelation

$$\mathrm{R}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathrm{x}(t+\tau)\mathrm{x}(t)dt$$

    - These limits are random variables.
    - Problems:

$$\mathrm{n} \overset{?}{=} E\{\mathrm{x}(t)\} \qquad \mathrm{R}(\tau) \overset{?}{=} E\{\mathrm{x}(t+\tau)\mathrm{x}(t)\}$$

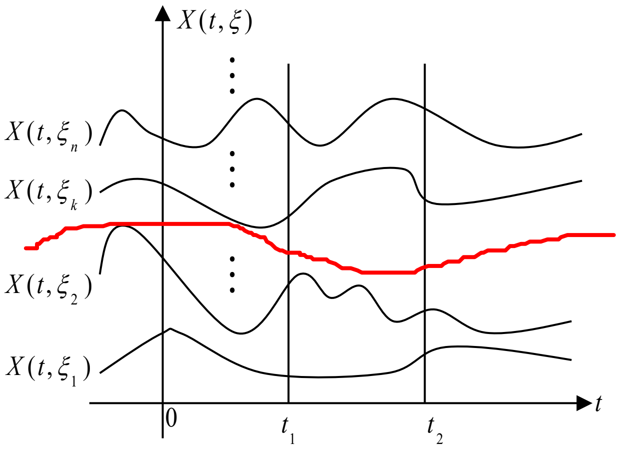
IV. Stochastic processes
4.4. Ergodicity

- Ergodicity
  - X(t) is ergodic if in the most general form if all its statistics can be determined from a single function $X(t, \zeta)$ of the process.
  - X(t) is ergodic if time averages equal ensemble averages (expected values)

IV. Stochastic processes
4.4. Ergodicity

- Ergodicity of the mean
  - Time average of a given process X(t)

$$n_T = \frac{1}{2T}\int_{-T}^{T} x(t)\,dt$$



  - n_T is a random variable.
  - Since E{X(t)} is a constant, we have

$$E\{n_T\} = E\{X(t)\} = \eta$$

  - The variance of n_T is given by:

$$\sigma_{n_T}^{\ 2} = \frac{1}{T}\int_{0}^{2T}\left(1 - \frac{\tau}{2T}\right)[R(\tau)-\eta^2]\,d\tau$$

  - $R(\tau)$ is the autocorrelation of X(t).
  - If this variance tends to zero with $T\to\infty$, then n_T tends to its expected value.

IV. Stochastic processes
4.4. Ergodicity

- Ergodic theorem for E{X(t)}

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\mathrm{x}(t)dt = E\{\mathrm{x}(t)\} = \eta$$

$$iff$$

$$\lim_{T\to\infty}\frac{1}{T}\int_{0}^{2T}\left(1 - \frac{\tau}{2\mathrm{T}}\right)[R(\tau)-\eta^2]d\tau = 0$$

IV. Stochastic processes
4.4. Ergodicity

- Ergodicity of autocorrelation
  - We form the average:

$$R_T(\lambda) = \frac{1}{2T}\int_{-T}^{T} x(t+\lambda)x(t)dt$$

  - We have

$$E\{R_T(\lambda)\} = \frac{1}{2T}\int_{-T}^{T} E\{x(t+\lambda)x(t)\}dt = R(\lambda)$$

  - For a given λ, $R_T(\lambda)$ is the time average of the process $\Phi(t)=x(t+\lambda)x(t)$
  - The mean of the process $\Phi(t)$ is given by

$$E\{\Phi(t)\}=E\{x(t+\lambda)x(t)\} = R(\lambda)$$

IV. Stochastic processes
4.4. Ergodicity

- Its autocorrelation

$R_{\Phi\Phi}(\tau)=E\{x(t+\lambda+\tau)x(t+\tau)x(t+\lambda)x(t)\}$

- Hence with $w(t) = \Phi(t)$, we have
- Ergodicity theorem for autocorrelation
  - For a given λ

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\mathrm{x}(t+\lambda)\mathrm{x}(t)dt = E\{\mathrm{x}(t+\lambda)\mathrm{x}(t)\} = R(\lambda)$$

iff

$$\lim_{T\to\infty}\frac{1}{T}\int_{0}^{2T}\left(1-\frac{\tau}{2\mathrm{T}}\right)[R_{\Phi\Phi}(\tau)-R^2(\lambda)]d\tau = 0$$

IV. Stochastic processes
4.4. Ergodicity

- Ergodicity of the distribution function
  - We determine first order distribution F(x) = E{X(t)≤x} of a given process X(t) by a suitable time average.
  - Consider the process

$$\mathrm{y}(t)=\begin{cases}1 & \text{if } \mathrm{x}(t)\le x\\ 0 & \text{if } \mathrm{x}(t)>x\end{cases}$$

  - Its mean is given by
  - Its autocorrelation

$$E\{\mathrm{y}(t)\}=1.P\{\mathrm{x}(t)\le x\}=F(x)$$

  - Where F(x,x;τ) is the second order distribution of x(t)

$$E\{\mathrm{y}(t+\tau)\mathrm{y}(t)\}=1.P\{\mathrm{x}(t+\tau)\le x,\mathrm{x}(t)\le x\}=F(x,x;\tau)$$

IV. Stochastic processes
4.4. Ergodicity

- We form the time average

$$y_T = \frac{1}{2T}\int_{-T}^{T} y(t)\,dt$$

- We have

$$E\{y_T\} = E\{y(t)\} = F(x)$$

- The variance of y_T is given by

$$\frac{1}{T}\int_{0}^{2T}\left(1 - \frac{\tau}{2T}\right)[R(\tau)-\eta^2]\,d\tau = 0$$

- Where $R(\tau)$ and η are replaced by $F(x, x;\tau)$ and $F(x)$

IV. Stochastic processes
4.4. Ergodicity

- Ergodic theorem for distribution function
  - For a given x,

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\mathrm{y}(t)dt = F(x)$$

$$iff$$

$$\lim_{T\to\infty}\frac{1}{T}\int_{0}^{2T}\left(1-\frac{\tau}{2\mathrm{T}}\right)[F(x,x;\tau)-F^{2}(x)]d\tau = 0$$

Stochastic Data Processing System Models

Stochastic Data Processing System Models

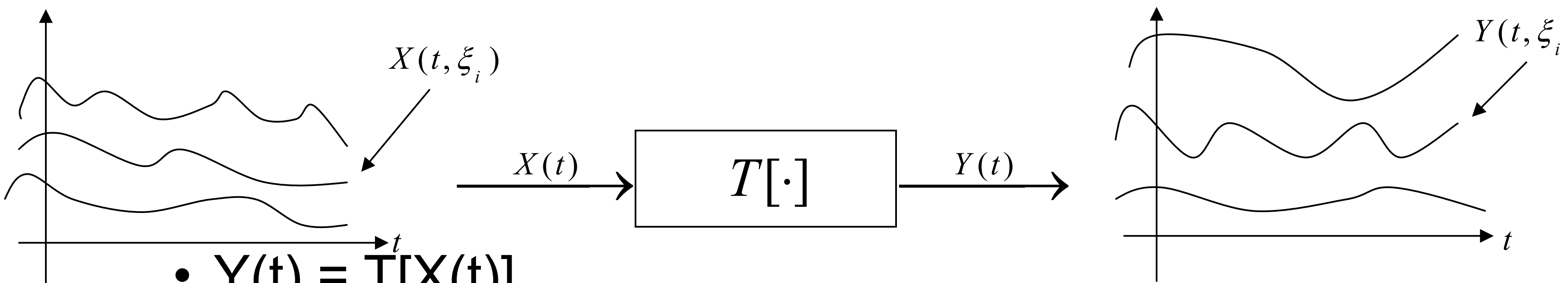
- Classification of Linear Systems

LTI systems

- Systems without memory
- Systems with memory
- Time Invariant systems
- Autocorrelation of output processes
- Power Spectrum and Khinchin-Wiener Theorem

LTI systems with stochastic input

- Deterministic system transformation



  - Y(t) = T[X(t)].
  - Problems formulation: goal: to study the output process statistics in term of the input process statistics and the system function
    - Is the output process stochastic ?
    - How are statistics of the output process ?
    - What is the relation between input and output processes ?

LTI systems with stochastic input

- Scope: Response of the deterministics systems under an actions of stochastic processes

Deterministic systems

Memoryless systems

$$Y(t) = g[X(t)]$$

Systems with memory

Time varying systems

$$Y(t) = \int_{-\infty}^{+\infty} h(t-\tau)X(\tau)d\tau$$
$$= \int_{-\infty}^{+\infty} h(\tau)X(t-\tau)d\tau.$$

Time Invariant systems

Linear systems

$$Y(t) = L[X(t)]$$

Linear time-invariant systems

$$X(t) \longrightarrow \boxed{h(t)} \longrightarrow Y(t)$$

4.1 Systems with stochastic inputs

- Memoryless systems
  - Response of memoryless systems
    - $Y(t) = g[X(t)]$
    - First order distribution and density: $F_Y(y,; t)$ và $f_Y(y; t)$ in relation with $F_X(x; t)$, $f_X(x; t)$:

$$F_Y(y) = P(Y(\xi) \le y) = P\big(g(X(\xi)) \le y\big) = P\big(X(\xi) \le g^{-1}(-\infty, y]\big).$$

    - Expectation:

$$E\{Y(t)\} = \int_{-\infty}^{\infty} g(x) f_X(x;t)\,dx$$

4.1 Systems with stochastic inputs

- Second order ststistics:
  - Second order density $f_Y(y_1, y_2; t_1, t_2$ of Y(t) can be determined in term of $f_X(x_1, x_2; t_1, t_2)$
  - Autocorrelation: $E\{Y(t_1)Y(t_2)\}$:

$$E\{Y(t_1)Y(t_2)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1)g(x_2)f_X(x_1, x_2; t_1, t_2)dx_1dx_2$$

- n order statistics: n order density function of Y(t)

$$f_Y(y_1, y_2, \ldots y_n; t_1, t_2, \ldots, t_n)$$

$(t_1) = g(X(t_1)), \ldots, Y(t_n) = g(X(t_n))$.

4.1 Systems with stochastic inputs

Memoryless Systems:

The output *Y*(*t*) in this case depends only on the present value of the input *X*(*t*). i.e.,

$$Y(t) = g\{X(t)\}$$

Strict-sense stationary input → Memoryless system → Strict-sense stationary output.

Wide-sense stationary input → Memoryless system → Need *not* be stationary in any sense.

X(*t*) stationary Gaussian with $R_{XX}(\tau)$ → Memoryless system → *Y*(*t*) stationary,but *not* Gaussian with $R_{XY}(\tau) = \eta R_{XX}(\tau)$.

4.1 Hệ thống TTBB dưới tác dụng của quá trình ngẫu nhiên

- Khảo sát tính dừng của tín hiệu đầu ra.
  - Khi tín hiệu đầu vào là tín hiệu dừng theo nghĩa hẹp, tín hiệu ra cũng dừng theo nghĩa hẹp;
  - Nếu đầu vào X(t) là dừng theo bậc N, đáp ứng Y(t) cũng dừng theo bậc N;
  - Nếu X(t) là dừng trong khoảng thì Y(t) cũng dừng trong khoảng đó;
  - Nếu X(t) là dừng theo nghĩa rộng, Y(t) có thể không dừng theo mọi nghĩa.
  - Ví dụ:
    - Bộ thu nhận theo luật bình phương: $Y(t) = X^2(t)$

4.1 Systems with stochastic inputs

- COnsider the memoryless system

$$g(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$$

4.1 Systems with stochastic inputs

Linear Systems: $L[\cdot]$ represents a linear system if

$$L\{a_1X(t_1)+a_2X(t_2)\}=a_1L\{X(t_1)\}+a_2L\{X(t_2)\}.$$

Let

$$Y(t)=L\{X(t)\}$$

represent the output of a linear system.

Time-Invariant System: $L[\cdot]$ represents a time-invariant system if

$$Y(t)=L\{X(t)\}\Rightarrow L\{X(t-t_0)\}=Y(t-t_0)$$

i.e., shift in the input results in the same shift in the output also.

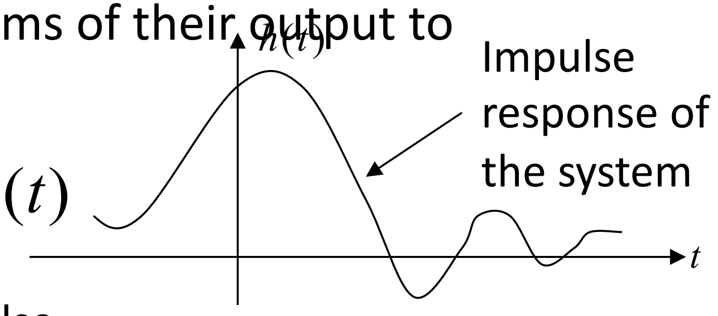
If $L[\cdot]$ satisfies both equations, then it corresponds to a linear time-invariant (LTI) system.

LTI systems can be uniquely represented in terms of their output to a delta function

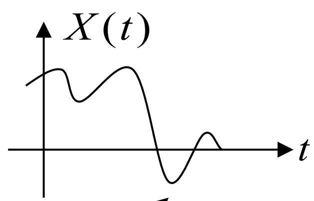
$$\delta(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow h(t)$$

Impulse

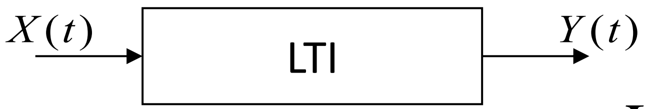
Impulse response

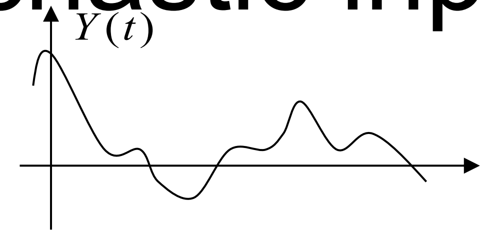


4.1 Systems with stochastic inputs



arbitrary input





$$Y(t) = \int_{-\infty}^{+\infty} h(t-\tau)X(\tau)d\tau$$

$$= \int_{-\infty}^{+\infty} h(\tau)X(t-\tau)d\tau$$

Eq. follows by expressing $X(t)$ as

$$X(t) = \int_{-\infty}^{+\infty} X(\tau)\delta(t-\tau)d\tau$$

and we have

$$Y(t) = L\{X(t)\}.$$

$$Y(t) = L\{X(t)\} = L\{\int_{-\infty}^{+\infty} X(\tau)\delta(t-\tau)d\tau\}$$

$$= \int_{-\infty}^{+\infty} L\{X(\tau)\delta(t-\tau)d\tau\}$$

By Linearity

$$= \int_{-\infty}^{+\infty} X(\tau)L\{\delta(t-\tau)\}d\tau$$

By Time-invariance

$$= \int_{-\infty}^{+\infty} X(\tau)h(t-\tau)d\tau = \int_{-\infty}^{+\infty} h(\tau)X(t-\tau)d\tau.$$

4.1 Systems with stochastic inputs

- Output statistics
  - Mean of the output

$$\mu_Y(t) = E\{Y(t)\} = \int_{-\infty}^{+\infty} E\{X(\tau)h(t-\tau)d\tau\}$$

$$= \int_{-\infty}^{+\infty} \mu_X(\tau)h(t-\tau)d\tau = \mu_X(t) * h(t).$$

  - Cross-correlation of input-output

$$R_{XY}(t_1,t_2) = E\{X(t_1)Y^*(t_2)\}$$

$$= E\{X(t_1)\int_{-\infty}^{+\infty} X^*(t_2-\alpha)h^*(\alpha)d\alpha\}$$

$$= \int_{-\infty}^{+\infty} E\{X(t_1)X^*(t_2-\alpha)\}h^*(\alpha)d\alpha$$

$$= \int_{-\infty}^{+\infty} R_{XX}(t_1,t_2-\alpha)h^*(\alpha)d\alpha$$

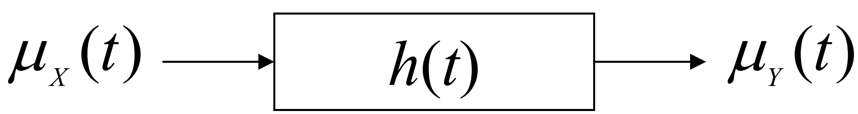
$$= R_{XX}(t_1,t_2) * h^*(t_2).$$

4.1 Systems with stochastic inputs

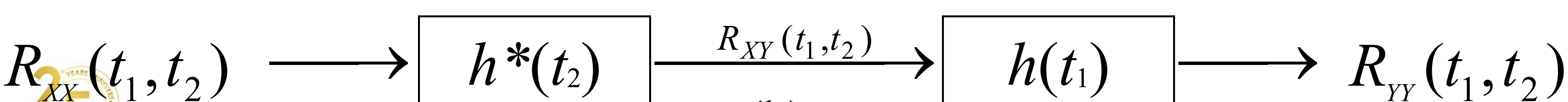
- Autocorrelation of output

$$
\begin{aligned}
R_{YY}(t_1,t_2) &= E\{Y(t_1)Y^*(t_2)\} \\
&= E\{\int_{-\infty}^{+\infty} X(t_1-\beta)h(\beta)d\beta\, Y^*(t_2)\} = \int_{-\infty}^{+\infty} E\{X(t_1-\beta)Y^*(t_2)\}h(\beta)d\beta \\
&= \int_{-\infty}^{+\infty} R_{XY}(t_1-\beta,t_2)h(\beta)d\beta = R_{XY}(t_1,t_2) * h(t_1)
\end{aligned}
$$

$$R_{YY}(t_1,t_2) = R_{XX}(t_1,t_2) * h^*(t_2) * h(t_1).$$



(a)



(b)

4.1 Systems with stochastic inputs

- If X(t) - wide sense stationary $\mu_X(t) = \mu_X$

$$\mu_Y(t) = \mu_X \int_{-\infty}^{+\infty} h(\tau)d\tau = \mu_X c, \quad a\ constant.$$

$$R_{XX}(t_1, t_2) = R_{XX}(t_1 - t_2)$$

- X(t), Y(t) are jointly w.s.s

$$R_{XY}(t_1, t_2) = \int_{-\infty}^{+\infty} R_{XX}(t_1 - t_2 + \alpha)h^*(\alpha)d\alpha$$

$$= R_{XX}(\tau) * h^*(-\tau) = R_{XY}(\tau), \quad \tau = t_1 - t_2.$$

- Y(t) is w.s.s.

$$R_{YY}(t_1, t_2) = \int_{-\infty}^{+\infty} R_{XY}(t_1 - \beta - t_2)h(\beta)d\beta, \quad \tau = t_1 - t_2$$

$$= R_{XY}(\tau) * h(\tau) = R_{YY}(\tau).$$

$$R_{YY}(\tau) = R_{XX}(\tau) * h^*(-\tau) * h(\tau).$$

4.1 Systems with stochastic inputs

The output process is also wide-sense stationary.

This gives rise to the following representation

$X(t)$ wide-sense stationary process → LTI system $h(t)$ → $Y(t)$ wide-sense stationary process.

(a)

$X(t)$ strict-sense stationary process → LTI system $h(t)$ → $Y(t)$ strict-sense stationary process (see Text for proof)

(b)

$X(t)$ Gaussian process (also stationary) → Linear system → $Y(t)$ Gaussian process (also stationary)

(c)

4.1 Systems with stochastic inputs

- Theorem:
  - For linear systems:

$$E\{L[X(t)]\} = L[E\{X(t)\}]$$

4.1 Systems with stochastic inputs

- White noise
  - $W(t)$ is said to be a white noise process if

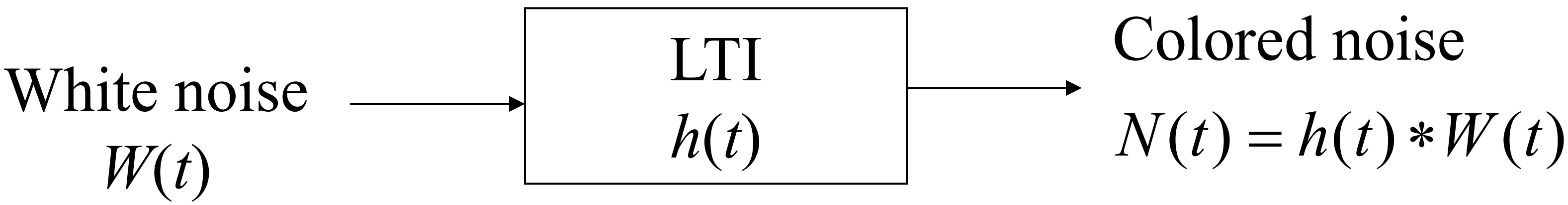
$$R_{WW}(t_1, t_2) = q(t_1)\delta(t_1 - t_2),$$

  - $E[W(t_1)\ W^*(t_2)] = 0$ unless $t_1 = t_2$
  - $W(t)$ is said to be wide-sense stationary (w.s.s) white noise if
    - $E[W(t)]$ = constant, and

$$R_{WW}(t_1, t_2) = q\delta(t_1 - t_2) = q\delta(\tau).$$

  - If $W(t)$ is also a Gaussian process (white Gaussian process), then all of its samples are independent random variables

4.1 Systems with stochastic inputs



- For w.s.s. white noise input $W(t)$, we have

$$E[N(t)] = \mu_W \int_{-\infty}^{+\infty} h(\tau)d\tau, \quad a\ constant$$

$$R_{nn}(\tau) = q\delta(\tau) * h^*(-\tau) * h(\tau) = qh^*(-\tau) * h(\tau) = q\rho(\tau)$$

  - where

$$\rho(\tau) = h(\tau) * h^*(-\tau) = \int_{-\infty}^{+\infty} h(\alpha)h^*(\alpha+\tau)d\alpha.$$

- Thus the output of a white noise process through an LTI system represents a (colored) noise process.
  - . White noise need not be Gaussian.
    - “White” and “Gaussian” are two different concepts.

4.2 Discrete Time Stochastic Processes

- Definition
  - A discrete time stochastic process $X_n = X(nT)$ is a sequence of random variables.
  - The mean, autocorrelation and auto-covariance functions of a discrete-time process are:

$$\mu_n = E\{X(nT)\}$$

$$R(n_1, n_2) = E\{X(n_1T)X^*(n_2T)\}$$

$$C(n_1, n_2) = R(n_1, n_2) - \mu_{n_1}\mu_{n_2}^*$$

4.2 Discrete Time Stochastic Processes

- Tính dừng
  - Strict sense stationarity and wide-sense stationarity definitions apply here also.
  - X(nT) is wide sense stationary if :

$$E\{X(nT)\} = \mu, \quad a\ constant$$
$$E[X\{(k+n)T\}X^*\{(k)T\}] = R(n) = r_n = r_{-n}^*$$

    - $R(n_1, n_2) = R(n_1 - n_2) = R^*(n_2 - n_1)$
  - The positive-definite property of the autocorrelation sequence
    - $$T_n = \begin{pmatrix} r_0 & r_1 & r_2 & \cdots & r_n \\ r_1^* & r_0 & r_1 & \cdots & r_{n-1} \\ & & \vdots & & \\ r_n^* & r_{n-1}^* & \cdots & r_1^* & r_0 \end{pmatrix} = T_n^*$$, n = 0, 1, 2, …, ∞

4.2 Discrete Time Stochastic Processes

- Output of the LTI systems
  - X(nT) - W.S.S, LTI system h(nT), Y(nT) - response.
  - Cross-correlation of input-output are:

$$R_{XY}(n) = R_{XX}(n) * h^*(-n)$$

$$R_{YY}(n) = R_{XY}(n) * h(n)$$

$$R_{YY}(n) = R_{XX}(n) * h^*(-n) * h(n).$$

  - Thus wide-sense stationarity from input to output is preserved for discrete-time systems also

4.3. Auto Regressive Moving Average (ARMA) Processes

- Consider an input – output representation of LTIs

$$X(n) = -\sum_{k=1}^{p} a_k X(n-k) + \sum_{k=0}^{q} b_k W(n-k),$$

  - where $X(n)$ may be considered as the output of a system $\{h(n)\}$ driven by the input $W(n)$
  - The transfer function

$$W(n) \longrightarrow \boxed{h(n)} \longrightarrow X(n)$$

$$X(z)\sum_{k=0}^{p} a_k z^{-k} = W(z)\sum_{k=0}^{q} b_k z^{-k}, \qquad a_0 \equiv 1$$

$$H(z) = \sum_{k=0}^{\infty} h(k) z^{-k} = \frac{X(z)}{W(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_q z^{-q}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_p z^{-p}} = \frac{B(z)}{A(z)}$$

$$X(n) = \sum_{k=0}^{\infty} h(n-k) W(k).$$

4.3. Auto Regressive Moving Average (ARMA) Processes

- Transfer function has p poles and q zeros

- The output undergoes regression over *p* of its previous values and at the same time a moving average based on W(n), W(n-1) ..., W(n-q) of the input over $(q + 1)$ values is added to it (ARMA(p, q))
- The input $\{W(n)\}$ represents a sequence of uncorrelated random variables of zero mean μ = 0 and constant variance σ^2_w so that $R_{WW}(n) = \sigma_W^2 \delta(n).$
  - If {W(n)})}is normally distributed then the output {X(n also represents a strict-sense stationary normal process.
  - If q = 0, AR(p) – all-pole process.
  - if p = 0, represent MA(q) process

4.3. Auto Regressive Moving Average (ARMA) Processes

- Autoregressive process AR(1)
  - An AR(1) process has the form:

$$X(n) = aX(n-1) + W(n)$$

  - The corresponding system transfer :

$$H(z) = \frac{1}{1-az^{-1}} = \sum_{n=0}^{\infty} a^n z^{-n}$$

    - if |a| < 1 - stable system.
    - System impulse response: $h(n) = a^n, \qquad |a| < 1$
    - Output autocorrelation:

$$R_{XX}(n) = \sigma_W^2 \delta(n) * \{a^{-n}\} * \{a^n\} = \sigma_W^2 \sum_{k=0}^{\infty} a^{|n|+k} a^k = \sigma_W^2 \frac{a^{|n|}}{1-a^2}$$

4.3. Auto Regressive Moving Average (ARMA) Processes

- Tự tương quan chuẩn hóa của đầu ra:

$$\rho_X(n) = \frac{R_{XX}(n)}{R_{XX}(0)} = a^{|n|}, \quad |n| \geq 0.$$

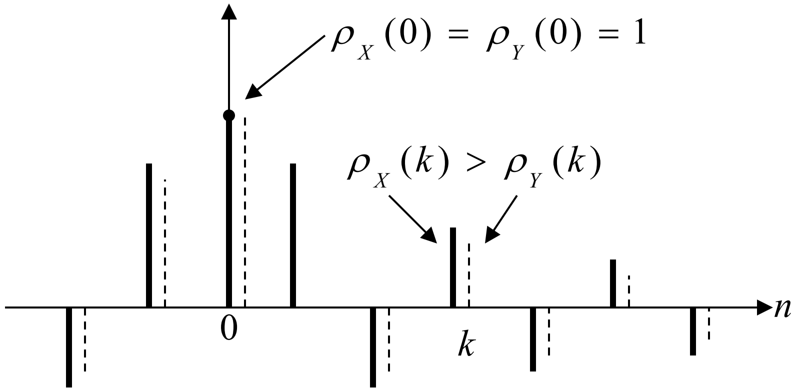
- Trường hợp quá trình X(t) kết hợp với nhiễu V(t):
  - V(t) là chuỗi ngẫu nhiên không tương quan với giá trị trung bình 0 và tự tương quan σ_V^2.
  - Y(t) = X(t) + V(t)

$$R_{YY}(n) = R_{XX}(n) + R_{VV}(n) = R_{XX}(n) + \sigma_V^2 \delta(n) = \sigma_W^2 \frac{a^{|n|}}{1-a^2} + \sigma_V^2 \delta(n)$$

$$\rho_Y(n) = \frac{R_{YY}(n)}{R_{YY}(0)} = \begin{cases} 1 & n = 0 \\ c\, a^{|n|} & n = \pm 1, \pm 2, \cdots \end{cases} \qquad c = \frac{\sigma_W^2}{\sigma_W^2 + \sigma_V^2 (1-a^2)} < 1.$$

4.3. Auto Regressive Moving Average (ARMA) Processes

- the effect of superimposing an error sequence on an AR(1) model AR(1)



4.3. Auto Regressive Moving Average (ARMA) Processes

- Autoregressive AR(2)

$$X(n) = a_1 X(n-1) + a_2 X(n-2) + W(n)$$

  - Transfer function

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} = \frac{1}{1 - a_1 z^{-1} - a_2 z^{-2}} = \frac{b_1}{1 - \lambda_1 z^{-1}} + \frac{b_2}{1 - \lambda_2 z^{-1}}$$

$$h(0) = 1,\ h(1) = a_1,\ h(n) = a_1 h(n-1) + a_2 h(n-2),\quad n \geq 2$$

    - with λ_1, λ_2 - poles of the system, the impulse response:
    - Và các hệ thức: $h(n) = b_1 \lambda_1^n + b_2 \lambda_2^n, \quad n \geq 0$

$$b_1 + b_2 = 1, \quad b_1\lambda_1 + b_2\lambda_2 = a_1. \quad \lambda_1 + \lambda_2 = a_1, \quad \lambda_1\lambda_2 = -a_2,$$

    - H(z) stable follows $|\lambda_1| < 1,\ |\lambda_2| < 1.$

4.3. Auto Regressive Moving Average (ARMA) Processes

- Autocorrelation of output process
  - Autocorrelation:

$$R_{XX}(n) = E\{X(n+m)X^*(m)\}$$
$$= E\{[a_1X(n+m-1) + a_2X(n+m-2)]X^*(m)\}$$
$$+ E\{W(n+m)X^*(m)\}$$
$$= a_1R_{XX}(n-1) + a_2R_{XX}(n-2)$$

  - Correlation coefficient:

$$\rho_X(n) = \frac{R_{XX}(n)}{R_{XX}(0)} = a_1\rho_X(n-1) + a_2\rho_X(n-2).$$

4.3. Auto Regressive Moving Average (ARMA) Processes

- ARIMA processes (Autoregressive Integrated moving average)
  - For modeling time series
  - Consider ARMA processes: W – white noise

$$X(n) = -\sum_{k=1}^{p} a_k X(n-k) + \sum_{k=0}^{q} b_k W(n-k),$$

$$(1 + \sum_{i=1}^{p} D^i)\, X(n) = (b_0 + \sum_{k=1}^{q} D^k) W(n)$$

  - D^i – delay operator, or lag operator
  - An ARIMA(*p'*,*d*,*q*) process is a particular case of an ARMA(*p*,*q*) process having the autoregressive polynomial with *d* unit roots and *p' = p-d* .

4.3. Auto Regressive Moving Average (ARMA) Processes

- Autocorrelation

$$R_{XX}(n) = R_{WW}(n) * h^*(-n) * h(n) = \sigma_W^2 h^*(-n) * h(n)$$

$$= \sigma_W^2 \sum_{k=0}^{\infty} h^*(n+k) * h(k)$$

$$= \sigma_W^2 \left(\frac{|b_1|^2 (\lambda_1^*)^n}{1-|\lambda_1|^2} + \frac{b_1^* b_2 (\lambda_1^*)^n}{1-\lambda_1^* \lambda_2} + \frac{b_1 b_2^* (\lambda_2^*)^n}{1-\lambda_1 \lambda_2^*} + \frac{|b_2|^2 (\lambda_2^*)^n}{1-|\lambda_2|^2} \right)$$

$$\rho_X(n) = \frac{R_{XX}(n)}{R_{XX}(0)} = c_1 \lambda_1^{*n} + c_2 \lambda_2^{*n}$$

4.3. Auto Regressive Moving Average (ARMA) Processes

- Moving average process

$$X(n) = \sum_{k=0}^{q} b_k W(n-k),$$

  - q zeros; without poles;
  - Non-regressive systems;
  - Impulse response and transfer function:

$$H(z) = \sum_{k=0}^{\infty} h(k) z^{-k} = b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_q z^{-q}$$

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Thank you for your attentions!