Differentiation of functions of single variable

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Agenda

Outline

Recall

Definition

A proposition is a declarative sentence that is either true or false.

Given two propositions P and Q.

- Negation operator $\bar{P}$.
- Implication operator $P \Rightarrow Q$. The implication proposition is false when P is true, Q is false.
  Ex: $P =$"$\{x_n\}$ is a convergent sequence";
  $Q =$"$\{x_n\}$ is a bounded sequence".
- Biconditional operator $P \Leftrightarrow Q$.
  Ex: $P =$ "$\lim_{n\to\infty} x_n = 0$";
  $Q =$: $\lim_{n\to\infty}(x_n + 1) = 1$".

Types of mathematical proof - Logical reasoning

- Direct proof $P \Rightarrow Q$.
  Ex. Prove that if a, b are consecutive integers, then $a+b$ is an odd number.
  Transitivity philosophy: $(P \Rightarrow Q, Q \Rightarrow R) \Rightarrow (P \Rightarrow R)$.
- Proof by contradiction $(P \Rightarrow Q) \Leftrightarrow (\bar{Q} \Rightarrow \bar{P})$.
  Ex. Prove that if n^2 is an odd number, then n is an odd number.
- Proof by induction.
  We want to show the property $T(n)$ for all $n \in \mathbb{N}^*$.
  1. If $k = 1$, $T(1)$ is true.
  2. If $T(k)$ is true, then $T(k+1)$ is true.

  Then $T(n)$ holds for all $n \in \mathbb{N}$.

  Ex.Show that for all $n \in \mathbb{N}$: $1 + 2 + \ldots + n = \dfrac{n(n+1)}{2}$.

Absolute values and properties

Definition

$$|a| = \begin{cases} a & \text{when } a \geq 0, \\ -a & \text{when } a < 0. \end{cases}$$

Proposition (Properties)

1. $|x| < a \Leftrightarrow -a < x < a.$
2. $|x| > b > 0 \Leftrightarrow (x > b) \textit{ or } (x < -b).$

Outline

Definition

Definition

Let $X \subset \mathbb{R}$. A function $f: X \to \mathbb{R}$ is a rule that assigns to each element $x \in X$ a unique value $y = f(x) \in \mathbb{R}$

$$f: X \to \mathbb{R}, x \mapsto y = f(x).$$

X is called the domain of f, and $f(X) = \{f(x), x \in X\}$ is called the range of f.

$\Gamma(f) = \{(x, f(x)) \mid x \in X\}$ is the graph of f.

Examples

Notation:

$$\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}, \mathbb{R}^* = \{x \in \mathbb{R} \mid x \neq 0\}.$$

1. $f(x) = a^x$, $0 < a \neq 1$, with the domain $\mathbb{R}$ and the range $\mathbb{R}_+^*$.
2. $f(x) = \log_a x$, $0 < a \neq 1$, the domain is $\mathbb{R}_+^*$.
3. $f(x) = x^\alpha$ has domains that are dependent on $\alpha \in \mathbb{R}$.

3. $f(x) = x^\alpha$ has domains that are dependent on $\alpha \in \mathbb{R}$.
   - When $\alpha \in \mathbb{N}$, the domain is $\mathbb{R}$.
   - When $\alpha \in \mathbb{Z}_-$, the domain is $\mathbb{R}^*$.
   - When $\alpha = \frac{1}{p}$, $p \in \mathbb{N}^*$, the domain is $\mathbb{R}_+$ when p is even, and is $\mathbb{R}$ when p is odd.
   - When $\alpha \in \mathbb{R}, \alpha > 0$, the domain is $\mathbb{R}_+$. When $\alpha < 0$ the domain is $\mathbb{R}_+^*$.

Examples

Find the domain of the following functions

1. $y = \sqrt{\log x}$.
2. $y = \dfrac{\log_2(3^x - 9)}{\sqrt[5]{3 - x}}$.

Bounded functions

Definition

A function $f : X \to \mathbb{R}$ is said to be bounded if there exists a constant M such that

$$\forall x \in X \Rightarrow -M \leq f(x) \leq M.$$

Example

1. $y = \sin x$, $x \in \mathbb{R}$, $y = \cos \frac{1}{x}$, $x \in \mathbb{R}^*$, is bounded by 1.
2. $y = \tan x$, $y = \cot x$ are unbounded on their domains.

Monotone functions

Definition

Given a function $f\colon X \to \mathbb{R}$, an interval $I \subset X$.

f is called increasing on I if

$$f(x_1) \le f(x_2) \text{ whenever } x_1 < x_2,\ \forall x_1, x_2 \in I.$$

f is called decreasing on I if

$$f(x_1) \ge f(x_2) \text{ whenever } x_1 < x_2,\ \forall x_1, x_2 \in I.$$

f is strictly increasing/decreasing if the strict inequality occurs.

1. $y = \sin x$, $x \in \left[0, \frac{\pi}{2}\right]$ is an increasing function.
2. $y = \sin x$, $x \in \left[\frac{\pi}{2}, \pi\right]$, $y = \cot x, x \in (0, \pi)$ are decreasing functions.

Even functions. Odd functions

Definition

A function $f : X \to \mathbb{R}$ is called an even function if

- X is symmetric about the origin O.
- $f(-x) = f(x)$ for all $x \in X$.

A function $f : X \to \mathbb{R}$ is called an odd function if

- X is symmetric about O.
- $f(-x) = -f(x)$ for all $x \in X$.

Graphs of even functions are symmetric about Oy.
Graphs of odd functions are point symmetric with center O.

Example

- $\sin x, \tan x, \cot x$ are odd functions. $\cos x$ is an even function.
- $y = x^{2n}$ are even functions, $y = x^{2n+1}$ are odd functions, $n \in \mathbb{N}^*$.
- $f(x) = \sqrt[3]{(1-x)^2} + \sqrt[3]{(1+x)^2}$.
- $f(x) = \sin x + \cos x$.

Periodic functions

Definition

A function $f : X \to \mathbb{R}$ is said to be periodic if there exists a $p > 0$ such that

- for all $x \in X$, $x + p \in X$,
- $f(x) = f(x + p)$ for all $x \in X$.

The smallest possible value of p is called the period T of f.

In literature, such p is also called "period", then T is called fundamental/primitive/basic period.

Graph of a periodic function

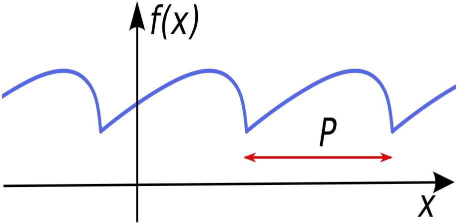


Figure: Graph of a periodic function. Source: wikipedia.

Examples

1. $y = \sin x$, $y = \cos x$ are periodic functions with period 2π.
2. $y = \tan x$, $y = \cot x$ are periodic functions with period π.
3. $y = \sin 2x + \cos 3x$ is a periodic function whose period is the least common multiple of π and $\frac{2\pi}{3}$, $T = 2\pi$.

Composite functions

Definition

Let X, Y be subsets of $\mathbb{R}$ and $f\colon X \to Y$, $g\colon Y \to \mathbb{R}$ be two functions. Then the rule assigning x to $g[f(x)]$ is the so-called the composition of g and f

$$g \circ f\colon X \to \mathbb{R},$$
$$x \mapsto z = g[f(x)].$$

Inverse functions

Definition

Let $f\colon X \to Y$ be a bijection. Then to each element $y \in Y$, there exists a unique element $x \in X$ such that $y = f(x)$. Therefore, $y \mapsto x$ determines a function

$$\begin{aligned} g\colon Y &\to X, \\ y &\mapsto x, y = f(x). \end{aligned}$$

g is called the inverse function of f.

Example (Inverse trigonometric functions)

1. $y = \sin x\colon \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$ is a bijection, its inverse is the function $\arcsin x\colon [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
2. $y = \cos x\colon [0, \pi] \to [-1, 1]$ is a bijection, its inverse is the function $\arccos x\colon [-1, 1] \to [0, \pi]$.
3. $y = \tan x\colon \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ is a bijection, its inverse is the function $\arctan x\colon \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
4. $y = \cot x\colon (0, \pi) \to \mathbb{R}$ is a bijection, its inverse is the function $\operatorname{arccot} x\colon \mathbb{R} \to (0, \pi)$.

$y = \arcsin x$ and $y = \arctan x$ are increasing functions.
$y = \arccos x$ and $y = \operatorname{arccot} x$ are decreasing functions.

Remark

The graphs of f and g are symmetric about the line $y = x$.

$$(x, f(x)) \in \Gamma(f) \Leftrightarrow (g(y), y) \in \Gamma(f).$$

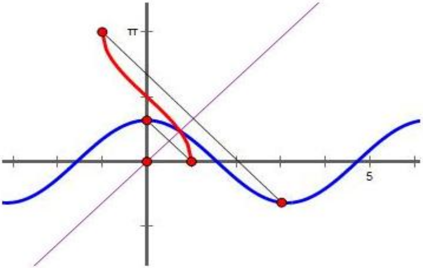


Figure: Graphs of $y = \cos x$ and $y = \arccos x$. Source: Wikipedia

Outline

Elementary functions

Chapter 1: Functions

Nguyen Thu Huong

Essential functions: exponential, logarithmic, power, trigonometric and inverse trigonometric functions.

$$
\begin{aligned}
& a^x, x^\alpha, \log_a x, \\
& \sin x, \cos x, \tan x, \cot x, \\
& \arcsin x, \arccos x, \arctan x, \operatorname{arccot} x.
\end{aligned}
$$

Hyperbolic functions

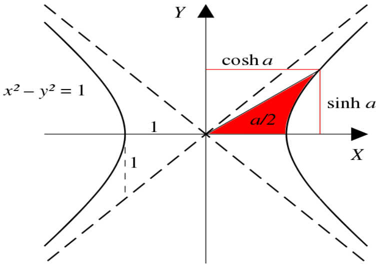
Definition

Hyperbolic sine: $\sinh x = \dfrac{e^x - e^{-x}}{2}$.

Hyperbolic cosine: $\cosh x = \dfrac{e^x + e^{-x}}{2}$.

Hyperbolic tangent: $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

Hyperbolic cotangent: $\coth x = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}, x \neq 0$.



Source: Wikipedia

Elementary functions are functions of one variable built from essential functions using the four elementary operations (sum, subtraction, product, division), composition of mappings and inverse functions.

Example

1. $f(x) = \sqrt[3]{\tan x} + x\sqrt{x}e^x$
2. $g(x) = \dfrac{\sqrt{\cos x}}{\tan \sqrt{x}}$

Outline

Definition

A number sequence is a function $f : \mathbb{N}^* \to \mathbb{R}$, $n \mapsto f(n) =: a_n$.

We denote $\{a_n\}_{n\geq 1}$.

Definition

The sequence $\{a_n\}$ is said to converge to $L < \infty$ when $n \to \infty$ if and only if for all $\varepsilon > 0$, there exists $N_0(\varepsilon)$ such that $|a_n - L| < \varepsilon$ for all $n \geq N_0$.
We write $\lim\limits_{n\to\infty} a_n = L < \infty$.

Properties

1. If the limit $\lim_{n\to\infty} a_n$ exists, then it is unique.
2. If the sequence $\{a_n\}$ is convergent, then it is bounded, i.e. there exists $M > 0$ such that $|a_n| \leq M$, for all n.
3. If $a_n \geq A$ for all n, then $\lim_{n\to\infty} a_n \geq A$.
4. If $\lim_{n\to\infty} a_n \neq 0$, then $a_n \neq 0$ for n large enough.

Limits law

Proposition

Assume that $\lim_{n\to\infty} a_n = L_1$, $\lim_{n\to\infty} b_n = L_2$, *then we have*

- $\lim_{n\to\infty} (a_n \pm b_n) = L_1 + \pm L_2$.
- $\lim_{n\to\infty} a_n . b_n = L_1 . L_2$.
- $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{L_1}{L_2}$, $(L_2 \neq 0, b_n \neq 0)$.

We can also define infinite limits, i.e. $L = \infty$.

Squeeze theorem

Theorem

Let $\{a_n\}, \{b_n\}, \{c_n\}$ be sequences that $a_n \leq b_n \leq c_n$ for $n \geq N_0$ and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$. Then $\lim\limits_{n\to\infty} b_n = L$.

Convergence criterion

Theorem (Cauchy's criterion)

The sequence $\{a_n\}$ is convergent if and only if it is a Cauchy sequence, i.e.

$$\forall \varepsilon > 0, \exists N_0(\varepsilon) \text{ such that } |a_m - a_n| < \varepsilon, \forall m, n \geq N_0.$$

Definition

A sequence $\{a_n\}$ is said to be **increasing** if $a_n \leq a_{n+1}, \forall n \in \mathbb{N}$.
A sequence $\{a_n\}$ is said to be **decreasing** if $a_n \geq a_{n+1}, \forall n \in \mathbb{N}$.

Theorem (Monotone convergence theorem)

If a sequence is increasing and bounded from above, then it is convergent.
If a sequence is decreasing and bounded from below, then it is convergent.

Test for convergence of the following sequences

1. $a_n = \dfrac{\sin n}{\sqrt{n^2+1}}$.
2. $a_n = 1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \dots \dfrac{1}{n^2}$.
3. $a_n = \left(1 + \dfrac{1}{n}\right)^n$.
   This sequence is increasing and converges to $e = 2,7182\dots$