Chapter 4

Pairs of Random Variables

4.1 Joint Cumulative Distribution Function

4.1.1 Definitions

In an experiment that produces one random variable, events are points or intervals on a line. In an experiment that leads to two random variables X and Y, each outcome (x, y) is a point in a plane and events are points or areas in the plane. Just as the CDF of one random variable, $F_X(x)$, is the probability of the interval to the left of x, the joint CDF $F_{X,Y}(x, y)$ of two random variables is the probability of the area in the plane below and to the left of (x, y). This is the infinite region that includes the shaded area in Figure 4.1 and everything below and to the left of it.

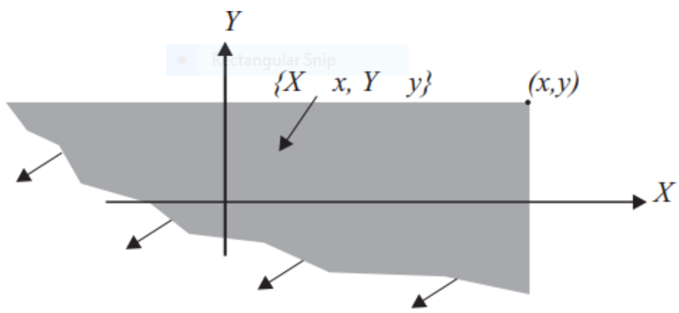


Figure 4.1: The area of the (X, Y) plane corresponding to the joint cumulative distribution function $F_{X,Y}(x, y)$

Definition 4.1 (Joint Cumulative Distribution Function) The joint **cumulative distribution function** (CDF) of random variables X and Y is

$$F_{X,Y}(x, y) = P[X < x, Y < y], \quad x, y \in \mathbb{R}. \tag{4.1}$$

4.1.2 Properties

Remark 4.1 The joint CDF has properties that are direct consequences of the definition. For example, we note that the event $\{X < x\}$ suggests that Y can have any value so long as the condition on X is met. This corresponds to the joint event $\{X < x, Y < +\infty\}$. Therefore,

$$F_X(x) = P[X < x] = P[X < x, Y < +\infty] = \lim_{y\to+\infty} F_{X,Y}(x,y) = F_{X,Y}(x,+\infty). \tag{4.2}$$

We obtain a similar result when we consider the event $\{Y < y\}$. The following theorem summarizes some basic properties of the joint CDF.

Theorem 4.1 *For any pair of random variables, X, Y,*

(a) $0 \le F_{X,Y}(x,y) \le 1$,

(b) $F_X(x) = F_{X,Y}(x,+\infty)$,

(c) $F_Y(y) = F_{X,Y}(+\infty,y)$,

(d) $F_{X,Y}(-\infty,y) = F_{X,Y}(x,-\infty) = 0$,

(e) *If $x < x_1$ and $y < y_1$, then $F_{X,Y}(x,y) \le F_{X,Y}(x_1,y_1)$,*

(f) $F_{X,Y}(+\infty,+\infty) = 1$,

(g) *If $x_1 < x_2$, $y_1 < y_2$, then*

$$P[x_1 \le X < x_2, y_1 \le Y < y_2] = F_{X,Y}(x_2,y_2) - F_{X,Y}(x_2,y_1) - F_{X,Y}(x_1,y_2) + F_{X,Y}(x_1,y_1).$$

Remark 4.2 Although its definition is simple, we rarely use the joint CDF to study probability models. It is easier to work with a probability mass function when the random variables are discrete, or a probability density function if they are continuous.

4.2 Joint Probability Mass Function

4.2.1 Definitions. Properties

Corresponding to the PMF of a single discrete random variable, we have a probability mass function of two variables.

Definition 4.2 (Joint Probability Mass Function) The joint **probability mass function** (PMF) of discrete random variables X and Y is

$$P_{X,Y}(x,y) = P[X = x, Y = y]. \tag{4.3}$$

Corresponding to S_X, the range of a single discrete random variable, we use the notation $S_{X,Y}$ to denote the set of possible values of the pair (X,Y). That is,

$$S_{X,Y} = \{(x,y) \mid P_{X,Y}(x,y) > 0\}. \tag{4.4}$$

Definition 4.3 (Joint Probability Distribution) The joint **probability distribution** of discrete ranfom variables X and Y is

$X \backslash Y$	y_1	$\dots$	y_j	$\dots$	y_n	$\sum_j$
x_1	p_{11}	$\dots$	p_{1j}	$\dots$	p_{1n}	$P[X = x_1]$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_i	p_{i1}	$\dots$	p_{ij}	$\dots$	p_{in}	$P[X = x_i]$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_m	p_{m1}	$\dots$	p_{mj}	$\dots$	p_{mn}	$P[X = x_m]$
$\sum_i$	$P[Y = y_1]$	$\dots$	$P[Y = y_j]$	$\dots$	$P[Y = y_n]$	$\sum_i \sum_j = 1$

where $S_X = \{x_1, x_2, \dots, x_m\}$, $S_Y = \{y_1, y_2, \dots, y_n\}$, $p_{ij} = P[X = x_i, Y = y_j]$, $i = 1, 2, \dots, m$; $j = 1, 2, \dots, n$.

Theorem 4.2 (a) $0 \leq p_{ij} \leq 1$ *for all* $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$.

(b) $\sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} p_{ij} = 1$.

Remark 4.3 From (4.1) and Definition 4.3, the joint cumulative distribution function of discrete random variables X and Y is

$$F_{XY}(x, y) = \sum_{x_i < x} \sum_{y_j < y} p_{ij}. \tag{4.5}$$

Example 4.1 Test two integrated circuits one after the other. On each test, the possible outcomes are a (accept) and r (reject). Assume that all circuits are acceptable with probability 0.9 and that the outcomes of successive tests are independent. Count the number of acceptable circuits X and count the number of successful tests Y before you observe the first reject. (If both tests are successful, let $Y = 2$.) Find the joint PMF of X and Y.

Solution. The sample space of the experiment is

$$S = \{aa, ar, ra, rr\}.$$

We compute

$$P[aa] = 0.81, \quad P[ar] = P[ra] = 0.09, \quad P[rr] = 0.01.$$

Each outcome specifies a pair of values X and Y. Let $g(s)$ be the function that transforms each outcome s in the sample space S into the pair of random variables (X, Y). Then

$$g(aa) = (2, 2), \quad g(ar) = (1, 1), \quad g(ra) = (1, 0), \quad g(rr) = (0, 0).$$

For each pair of values x, y, $P_{X,Y}(x, y)$ is the sum of the probabilities of the outcomes for which $X = x$ and $Y = y$. For example, $P_{X,Y}(1, 1) = P[ar]$. The joint PMF can be given as a set of labeled points in the x, y plane where each point is a possible value (probability > 0) of the pair (x, y), or as a simple list:

$$P_{X,Y}(x,y) = \begin{cases} 0.81, & x = 2, y = 2, \\ 0.09, & x = 1, y = 1, \\ 0.09, & x = 1, y = 0, \\ 0.01, & x = 0, y = 0, \\ 0, & \text{otherwise.} \end{cases}$$

A second representation of $P_{X,Y}(x, y)$ is the matrix:

$P_{X,Y}(x,y)$	$y = 0$	$y = 1$	$y = 2$
$x = 0$	0.01	0	0
$x = 1$	0.09	0.09	0
$x = 2$	0	0	0.81

(4.6)

Note that all of the probabilities add up to 1. This reflects the second axiom of probability that states $P[S] = 1$. Using the notation of random variables, we write this as

$$\sum_{x \in S_X} \sum_{y \in S_Y} P_{X,Y}(x,y) = 1.$$

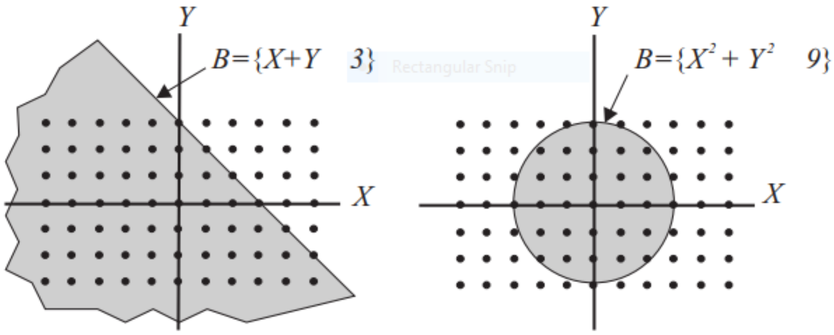


Figure 4.2: Subsets B of the (X, Y) plane. Points $(X, Y) \in S_{X,Y}$ are marked by bullets

We represent an event B as a region in the (X, Y) plane. Figure 4.2 shows two examples of events. We would like to find the probability that the pair of random variables (X, Y) is in the set B. When $(X, Y) \in B$, we say the event B occurs. Moreover, we write $P[B]$ as a shorthand for $P[(X, Y) \in B]$. The next theorem says that we can find $P[B]$ by adding the probabilities of all points (x, y) with nonzero probability that are in B.

Theorem 4.3 *For discrete random variables X and Y and any set B in the (X, Y) plane, the probability of the event $\{(X, Y) \in B\}$ is*

$$P[B] = \sum_{(x,y) \in B} P_{X,Y}(x,y). \tag{4.7}$$

Example 4.2 Continuing Example 4.1, find the probability of the event B that X, the number of acceptable circuits, equals Y, the number of tests before observing the first failure.

Solution. Mathematically, B is the event $\{X = Y\}$. The elements of B with nonzero probability are

$$B \cap S_{X,Y} = \{(0,0), (1,1), (2,2)\}.$$

Therefore,

$$P[B] = P_{X,Y}(0,0) + P_{X,Y}(1,1) + P_{X,Y}(2,2) = 0.01 + 0.09 + 0.81 = 0.91.$$

4.2.2 Marginal PMF

In an experiment that produces two random variables X and Y, it is always possible to consider one of the random variables, Y, and ignore the other one, X. In this case, we can use the methods of Chapter 2 to analyze the experiment and derive $P_Y(y)$, which contains the probability model for the random variable of interest. On the other hand, if we have already analyzed the experiment to derive the joint PMF $P_{X,Y}(x,y)$, it would be convenient to derive $P_Y(y)$ from $P_{X,Y}(x,y)$ without reexamining the details of the experiment.

Theorem 4.4 *For discrete random variables X and Y with joint PMF $P_{X,Y}(x,y)$,*

$$P_X(x) = \sum_{y \in S_Y} P_{X,Y}(x,y), \quad P_Y(y) = \sum_{x \in S_X} P_{X,Y}(x,y) \tag{4.8}$$

Remark 4.4 From Definition 4.3,

(a)

$$P[X = x_i] = \sum_{j=1}^{n} p_{ij},\ i = 1, \dots, m; \quad P[Y = y_j] = \sum_{i=1}^{m} p_{ij},\ j = 1, \dots, n.$$

(b) Marginal probability distribution of X:

X	x_1	x_2	$\dots$	x_m
$P_X(x)$	$P[X = x_1]$	$P[X = x_2]$	$\dots$	$P[X = x_m]$

(c) Marginal probability distribution of Y:

Y	y_1	y_2	$\dots$	y_n
$P_Y(y)$	$P[Y = y_1]$	$P[Y = y_2]$	$\dots$	$P[Y = y_n]$

Theorem 4.4 shows us how to obtain the probability model (PMF) of X, and the probability model of Y given a probability model (joint PMF) of X and Y. When a random variable X is part of an experiment that produces two random variables, we sometimes refer to its PMF as a marginal probability mass function. This terminology comes from the matrix representation of the joint PMF. By adding rows and columns and writing the results in the margins, we obtain the marginal PMFs of X and Y. We illustrate this by reference to the experiment in Example 4.1.

Example 4.3 In Example 4.1, we found the joint PMF of X and Y to be

$P_{X,Y}(x,y)$	$y=0$	$y=1$	$y=2$
$x=0$	0.01	0	0
$x=1$	0.09	0.09	0
$x=2$	0	0	0.81

Find the marginal PMFs for the random variables X and Y.

Solution. To find $P_X(x)$, we note that both X and Y have range $\{0,1,2\}$. Theorem 4.4 gives

$$P_X(0)=\sum_{y=0}^{2}P_{X,Y}(0,y)=0.01,\quad P_X(1)=\sum_{y=0}^{2}P_{X,Y}(1,y)=0.18$$

$$P_X(2)=\sum_{y=0}^{2}P_{X,Y}(2,y)=0.81,\quad P_X(x)=0,\quad x\neq 0,1,2.$$

For the PMF of Y, we obtain

$$P_Y(0)=\sum_{x=0}^{2}P_{X,Y}(x,0)=0.10,\quad P_Y(1)=\sum_{x=0}^{2}P_{X,Y}(x,1)=0.09$$

$$P_Y(2)=\sum_{x=0}^{2}P_{X,Y}(x,2)=0.81,\quad P_Y(y)=0,\quad y\neq 0,1,2.$$

Referring to the matrix representation of $P_{X,Y}(x,y)$ in Example 4.1, we observe that each value of $P_X(x)$ is the result of adding all the entries in one row of the matrix. Each value of $P_Y(y)$ is a column sum. We display $P_X(x)$ and $P_Y(y)$ by rewriting the matrix in Example 4.1 and placing the row sums and column sums in the margins.

$P_{X,Y}(x,y)$	$y=0$	$y=1$	$y=2$	$P_X(x)$
$x=0$	0.01	0	0	0.01
$x=1$	0.09	0.09	0	0.18
$x=2$	0	0	0.81	0.81
$P_Y(y)$	0.10	0.09	0.81	

Note that the sum of all the entries in the bottom margin is 1 and so is the sum of all the entries in the right margin. This is simply a verification of Theorem 2.1(b), which states that the PMF of any random variable must sum to 1. The complete marginal PMF, $P_Y(y)$, appears in the bottom row of the table, and $P_X(x)$ appears in the last column of the table.

$$P_X(x)=\begin{cases}0.01, & x=0,\\ 0.18, & x=1,\\ 0.81, & x=2,\\ 0, & \text{otherwise.}\end{cases}\qquad P_Y(y)=\begin{cases}0.1, & y=0,\\ 0.09, & y=1,\\ 0.81, & y=2,\\ 0, & \text{otherwise.}\end{cases}$$

4.3 Joint Probability Density Function

4.3.1 Definitions

Definition 4.4 (Joint Probability Density Function) The joint PDF of the continuous random variables X and Y is a function $f_{X,Y}(x,y)$ with the property

$$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(u,v)dvdu. \tag{4.9}$$

Theorem 4.5

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}. \tag{4.10}$$

4.3.2 Properties

Theorem 4.6

$$P[x_1 \leq X < x_2, y_1 \leq Y < y_2] = F_{X,Y}(x_2,y_2) - F_{X,Y}(x_2,y_1) - F_{X,Y}(x_1,y_2) + F_{X,Y}(x_1,y_1). \tag{4.11}$$

Theorem 4.7 *A joint PDF $f_{X,Y}(x,y)$ has the following properties*

(a) $f_{X,Y}(x,y) \geq 0$ *for all* $(x,y) \in \mathbb{R}^2$,

(b) $\int\limits_{-\infty}^{+\infty} \int\limits_{-\infty}^{+\infty} f_{X,Y}(x,y)dxdy = 1.$

Theorem 4.8 *The probability that the continuous random variables (X,Y) are in A is*

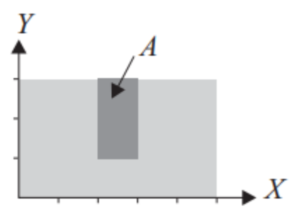
$$P[A] = \iint_A f_{X,Y}(x,y)dxdy. \tag{4.12}$$

Example 4.4 Random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} c, & 0 \leq x \leq 5,\ 0 \leq y \leq 3 \\ 0, & \text{otherwise.} \end{cases}$$

Find the constant c and $P[A] = P[2 \leq X < 3, 1 \leq Y < 3]$.

Solution.



The large rectangle in the diagram is the area of nonzero probability. Theorem 4.7 states that the integral of the joint PDF over this rectangle is 1:

$$1 = \int_0^5 \int_0^3 cdydx = 15c.$$

Therefore, $c = 1/15$. The small dark rectangle in the diagram is the event $A = \{2 \leq X < 3, 1 \leq Y < 3\}$. $P[A]$ is the integral of the PDF over this rectangle, which is

$$P[A] = \int_2^3 \int_1^3 \frac{1}{15} dv du = 2/15.$$

This probability model is an example of a pair of random variables uniformly distributed over a rectangle in the X, Y plane.

Example 4.5 Find the joint CDF $F_{X,Y}(x, y)$ when X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 2, & 0 \leq y \leq x \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Solution. The joint CDF can be found using Definition 4.4 in which we integrate the joint PDF $f_{X,Y}(x, y)$ over the area shown in Figure 4.1. To perform the integration it is extremely useful to draw a diagram that clearly shows the area with nonzero probability, and then to use the diagram to derive the limits of the integral in Definition 4.4.

4.3.3 Marginal PDF

Suppose we perform an experiment that produces a pair of random variables X and Y with joint PDF $f_{X,Y}(x, y)$. For certain purposes we may be interested only in the random variable X. We can imagine that we ignore Y and observe only X.

Theorem 4.9 *If X and Y are random variables with joint* PDF $f_{X,Y}(x, y)$,

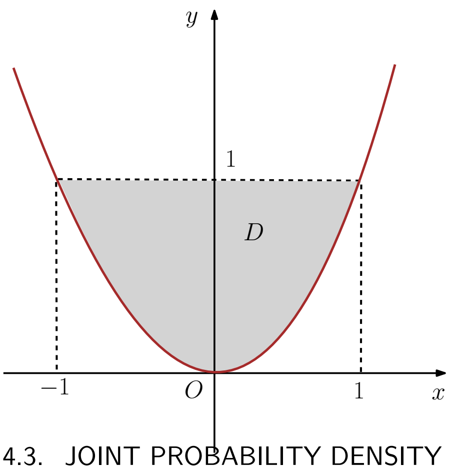
$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dy, \quad f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dx. \tag{4.13}$$

Example 4.6 The joint PDF of X and Y is

$$f_{X,Y}(x,y) = \begin{cases} \frac{5y}{4}, & -1 \leq x \leq 1,\ x^2 \leq y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find the marginal PDFs $f_X(x)$ and $f_Y(y)$.

Solution. Set $D = \{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 1,\ x^2 \leq y \leq 1\}$.



We use Theorem 4.9 to find the marginal PDF $f_X(x)$. When $x < -1$ or when $x > 1$, $f_{X,Y}(x, y) = 0$, and therefore $f_X(x) = 0$. For $-1 \leq x \leq 1$,

$$f_X(x) = \int_{x^2}^1 \frac{5y}{4} dy = \frac{5(1-x^4)}{8}.$$

The complete expression for the marginal PDF of X is

$$f_X(x) = \begin{cases} \frac{5(1-x^4)}{8}, & -1 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

For the marginal PDF of Y, we note that for $y < 0$ or $y > 1$, $f_Y(y) = 0$. For $0 \le y \le 1$, we integrate over the horizontal bar marked $Y = y$. The boundaries of the bar are $x = -\sqrt{y}$ and $x = \sqrt{y}$. Therefore, for $0 \le y \le 1$,

$$f_Y(y) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{5y}{4} dx = \frac{5y}{4} x\Big|_{x=-\sqrt{y}}^{x=\sqrt{y}} = \frac{5y^{3/2}}{2}.$$

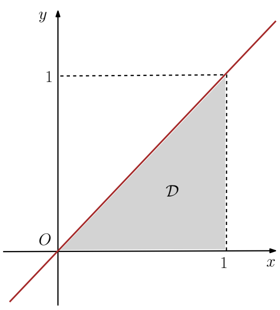
The complete marginal PDF of Y is $f_Y(y) = \begin{cases} \dfrac{5y^{3/2}}{2}, & 0 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$

Example 4.7 The joint PDF of X and Y is $f(x,y) = \begin{cases} kx, & \text{if } 0 < y < x < 1, \\ 0, & \text{otherwise.} \end{cases}$

(a) Find constant k.

(b) Find the marginal PDFs $f_X(x)$ and $f_Y(y)$.

Solution.



Set $\mathcal{D} = \{(x,y) \in \mathbb{R}^2 : 0 < x < 1;\ 0 < y < x\}$.

(a) From Theorem 4.7, $k \ge 0$ and

$$1 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)dxdy = \int_0^1 dx \int_0^x kxdy = k\int_0^1 x^2 dx = \frac{k}{3}.$$

Hence $k = 3$ and $f(x,y) = \begin{cases} 3x, & \text{if } (x,y) \in \mathcal{D}, \\ 0, & \text{otherwise.} \end{cases}$

(b) From Theorem 4.9,

$$f_X(x) = \int_{-\infty}^{+\infty} f(x,y)dy = \begin{cases} \int_0^x 3xdy, & 0 < x < 1, \\ 0, & \text{otherwise,} \end{cases} = \begin{cases} 3x^2, & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)dx = \begin{cases} \int_y^1 3xdx, & 0 < y < 1, \\ 0, & \text{otherwise,} \end{cases} = \begin{cases} \dfrac{3}{2} - \dfrac{3}{2}y^2, & 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

4.4 Expected Values

4.4.1 Expected Values. Variances

Theorem 4.10 *For random variables X and Y, the expected value of $W = g(X,Y)$ is*

$$\begin{aligned}\text{Discrete: } E[W] &= \sum_{x\in S_X}\sum_{y\in S_Y} g(x,y)P_{X,Y}(x,y),\\ \text{or } E[W] &= \sum_i\sum_j g(x_i,y_j)P[X=x_i, Y=y_j],\\ \text{Continuous: } E[W] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f_{X,Y}(x,y)dxdy.\end{aligned}$$

Theorem 4.11

$$E[g_1(X,Y)+\ldots+g_n(X,Y)] = E[g_1(X,Y)]+\ldots+E[g_n(X,Y)].$$

The following theorem describes the expected sum of two random variables, a special case of Theorem 4.11.

Theorem 4.12 *For any two random variables X and Y,*

$$E[X+Y] = E[X]+E[Y].$$

Theorem 4.13 *The variance of the sum of two random variables is*

$$Var[X+Y] = Var[X]+Var[Y]+2E[(X-\mu_X)(Y-\mu_Y)].$$

4.4.2 Covariance. Covariance Matrix. Correlation Coefficient

Definition 4.5 (Covariance) The covariance of two random variables X and Y is

$$Cov[X,Y] = E[(X-\mu_X)(Y-\mu_Y)].$$

Remark 4.5 From the properites of the expected value,

$$Cov(X,Y) = E[XY]-E[X].E[Y], \tag{4.14}$$

where $E[XY]$ is

$$E[XY] = \begin{cases}\sum\limits_i\sum\limits_j x_iy_jp_{ij}, & \text{(Discrete Random Variables)}\\ \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} xyf(x,y)dxdy, & \text{(Continuous Random Variables).}\end{cases}$$

Definition 4.6 (Covariance Matrix) The covariance matrix of two random variables X and Y is

$$\Gamma = \begin{bmatrix}Cov(X,X) & Cov(X,Y)\\ Cov(Y,X) & Cov(Y,Y)\end{bmatrix} = \begin{bmatrix}V(X) & Cov(X,Y)\\ Cov(X,Y) & V(Y)\end{bmatrix} \tag{4.15}$$

Sometimes, the notation σ_{XY} is used to denote the covariance of X and Y. The correlation of two random variables, denoted $r_{X,Y}$, is a close relative of the covariance.

Definition 4.7 (Correlation) The correlation of X and Y is $r_{X,Y} = E[XY]$, where

$$E[XY] = \begin{cases} \sum\limits_{x\in S_X} \sum\limits_{y\in S_Y} xyP_{X,Y}(x,y), & \text{(Discrete Random Variables)} \\ \int\limits_{-\infty}^{+\infty} \int\limits_{-\infty}^{+\infty} xyf(x,y)dxdy, & \text{(Continuous Random Variables).} \end{cases}$$

The following theorem contains useful relationships among three expected values: the covariance of X and Y, the correlation of X and Y, and the variance of $X+Y$.

Theorem 4.14 (a) $Cov(X,Y) = Cov(Y,X)$.

(b) $Var[X+Y] = Var[X] + Var[Y] + 2Cov[X,Y]$.

(c) $Var[X] = Cov[X,X]$, $Var[Y] = Cov[Y,Y]$.

Example 4.8 For the integrated circuits tests in Example 4.1, we found in Example 4.3 that the probability model for X and Y is given by the following matrix.

$P_{X,Y}(x,y)$	$y=0$	$y=1$	$y=2$	$P_X(x)$
$x=0$	0.01	0	0	0.01
$x=1$	0.09	0.09	0	0.18
$x=2$	0	0	0.81	0.81
$P_Y(y)$	0.10	0.09	0.81	

Find $r_{X,Y}$ and $Cov[X,Y]$.

Solution. By Definition 4.7,

$$r_{X,Y} = E[XY] = \sum_{x=0}^{2}\sum_{y=0}^{2} xyP_{X,Y}(x,y) = 1\times 1\times 0.09 + 2\times 2\times 0.81 = 3.33.$$

To use Theorem 4.14(a) to find the covariance, we find

$$E[X] = 1\times 0.18 + 2\times 0.81 = 1.80$$
$$E[Y] = 1\times 0.09 + 2\times 0.81 = 1.71.$$

Therefore, by Theorem 4.14(a), $Cov[X,Y] = 3.33 - 1.80\times 1.71 = 0.252$.

Example 4.9 The joint CDT of X and Y is

X \ Y	-1	0	1
-1	$4/15$	$1/15$	$4/15$
0	$1/15$	$2/15$	$1/15$
1	0	$2/15$	0

(a) Find $E(X), E(Y), Cov(X,Y)$.

(b) Find the CDT of X and Y.

Solution. (a) We have

$$E(X) = (-1) \times \frac{9}{15} + 0 \times \frac{4}{15} + 1 \times \frac{2}{15} = -\frac{7}{15}.$$
$$E(Y) = (-1) \times \frac{5}{15} + 0 \times \frac{5}{15} + 1 \times \frac{5}{15} = 0.$$
$$E(XY) = (-1) \times (-1) \times \frac{4}{15} + (-1) \times (1) \times \frac{4}{15} + 1 \times (-1) \times 0 + 1 \times 1 \times 0 = 0.$$

Hence $Cov(X,Y) = E(XY) - E(X) \times E(Y) = 0$.
(b) The CDT of X and Y are

X	-1	0	1
P	$9/15$	$4/15$	$5/15$

Y	-1	0	1
P	$5/15$	$5/15$	$5/15$

Associated with the definitions of covariance and correlation are special terms to describe random variables for which $r_{X,Y} = 0$ and random variables for which $Cov[X,Y] = 0$.

Definition 4.8 (Orthogonal Random Variables) Random variables X and Y are orthogonal if $r_{X,Y} = 0$.

Definition 4.9 (Uncorrelated Random Variables) Random variables X and Y are uncorrelated if $Cov[X,Y] = 0$.

Definition 4.10 (Correlation Coefficient) The correlation coefficient of two random variables X and Y is

$$\rho_{X,Y} = \frac{Cov[X,Y]}{\sqrt{Var[X]Var[Y]}} = \frac{Cov[X,Y]}{\sigma_X \sigma_Y}. \tag{4.16}$$

Remark 4.6 Note that the units of the covariance and the correlation are the product of the units of X and Y. Thus, if X has units of kilograms and Y has units of seconds, then $Cov[X,Y]$ and $r_{X,Y}$ have units of kilogram-seconds. By contrast, $\rho_{X,Y}$ is a dimensionless quantity.

Theorem 4.15

$$-1 \le \rho_{X,Y} \le 1.$$

Theorem 4.16 *If X and Y are random variables such that $Y = aX + b$*

$$\rho_{X,Y} = \begin{cases} -1, & a < 0, \\ 0, & a = 0, \\ 1, & a > 0. \end{cases}$$

4.5 Conditioning by an Event

4.5.1 Conditional Joint PMF

Definition 4.11 (Conditional Joint PMF) For discrete random variables X and Y and an event, B with $P[B] > 0$, the conditional joint PMF of X and Y given B is

$$P_{X,Y|B}(x,y) = P[X = x, Y = y|B].$$

Theorem 4.17 *For any event B, a region of the X,Y plane with $P[B] > 0$,*

$$P_{X,Y|B}(x,y) = \begin{cases} \dfrac{P_{X,Y}(x,y)}{P[B]}, & (x,y) \in B, \\ 0, & \textit{otherwise.} \end{cases}$$

4.5.2 Conditional Joint PDF

Definition 4.12 (Conditional Joint PDF) Given an event B with $P[B] > 0$, the conditional joint probability density function of X and Y is

$$f_{X,Y|B}(x,y) = \begin{cases} \dfrac{f_{X,Y}(x,y)}{P[B]}, & (x,y) \in B, \\ 0, & \text{otherwise.} \end{cases}$$

Example 4.10 X and Y are random variables with joint PDF

$$f_{X,Y|B}(x,y) = \begin{cases} \dfrac{1}{15}, & 0 \leq x \leq 5,\ 0 \leq y \leq 3, \\ 0, & \text{otherwise.} \end{cases}$$

Find the conditional PDF of X and Y given the event $B = \{X + Y \geq 4\}$.

Solution. We calculate $P[B]$ by integrating $f_{X,Y}(x,y)$ over the region B.

$$P[B] = \int_0^3 \int_{4-y}^5 \frac{1}{15} dxdy = \frac{1}{15} \int_0^3 (1+y)dy = \frac{1}{2}.$$

Definition 4.12 leads to the conditional joint PDF

$$f_{X,Y|B}(x,y) = \begin{cases} \dfrac{2}{15}, & 0 \leq x \leq 5,\ 0 \leq y \leq 3,\ x+y \geq 4 \\ 0, & \text{otherwise.} \end{cases}$$

4.5.3 Conditional Expected Value

Theorem 4.18 (Conditional Expected Value) *For random variables X and Y and an event B of nonzero probability, the conditional expected value of $W = g(X,Y)$ given B is*

$$\text{Discrete: } E[W|B] = \sum_{x \in S_X} \sum_{y \in S_Y} g(x,y) P_{X,Y|B}(x,y),$$

$$\text{Continuous: } E[W|B] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(x,y) f_{X,Y|B}(x,y) dxdy.$$

4.5.4 Conditional Variance

Definition 4.13 (Conditional Variance) The conditional variance of the random variable $W = g(X, Y)$ is
$$Var[W|B] = E([(W - E[W|B])^2|B].$$

Theorem 4.19
$$Var[W|B] = E[W^2|B] - (E[W|B])^2.$$

Example 4.11 Continuing Example 4.10, find the conditional expected value of $W = XY$ given the event $B = \{X + Y \geq 4\}$.

Solution. From Theorem 4.18,
$$\begin{aligned} E[XY|B] &= \int_0^3 \int_{4-y}^5 \frac{2}{15} xy dx dy = \frac{1}{15} \int_0^3 x^2 \Big|_{4-y}^5 y dy = \frac{1}{15} \int_0^3 (9y + 8y^2 - y^3) dy \\ &= \frac{123}{20}. \end{aligned}$$

4.6 Conditioning by a Random Variable

In Section 4.5, we use the partial knowledge that the outcome of an experiment $(x, y) \in B$ in order to derive a new probability model for the experiment. Now we turn our attention to the special case in which the partial knowledge consists of the value of one of the random variables: either $B = \{X = x\}$ or $B = \{Y = y\}$.

4.6.1 Conditional PMF

Definition 4.14 (Conditional PMF) For any event $Y = y$ such that $P_Y(y) > 0$, the conditional PMF of X given $Y = y$ is
$$P_{X|Y}(x|y) = P[X = x|Y = y].$$

The following theorem contains the relationship between the joint PMF of X and Y and the two conditional PMFs, $P_{X|Y}(x|y)$ and $P_{Y|X}(y|x)$.

Theorem 4.20 *For random variables X and Y with joint PMF $P_{X,Y}(x, y)$, and x and y such that $P_X(x) > 0$ and $P_Y(y) > 0$,*
$$P_{X,Y}(x, y) = P_{X|Y}(x|y) P_Y(y) = P_{Y|X}(y|x) P_X(x).$$

4.6.2 Conditional Expected Value of a Function

Theorem 4.21 (Conditional Expected Value of a Function) *X and Y are discrete random variables. For any $y \in S_Y$, the conditional expected value of $g(X, Y)$ given $Y = y$ is*
$$E[g(X, Y)|Y = y] = \sum x \in S_X g(x, y) P_{X|Y}(x|y).$$

The conditional expected value of X given $Y = y$ is a special case of Theorem 4.21:

$$E[X|Y = y] = \sum x \in S_X x P_{X|Y}(x|y). \tag{4.17}$$

4.6.3 Conditional PDF

Now we consider the case in which X and Y are continuous random variables. We observe $\{Y = y\}$ and define the PDF of X given $\{Y = y\}$. We cannot use $B = \{Y = y\}$ in Definition 4.12 because $P[Y = y] = 0$. Instead, we define a conditional probability density function, denoted as $f X|Y(x|y)$.

Definition 4.15 (Conditional PDF) For y such that $f_Y(y) > 0$, the conditional PDF of X given $\{Y = y\}$ is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}.$$

Definition 4.15 implies

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}. \tag{4.18}$$

Example 4.12 Returning to Example 4.5, random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 2, & 0 \le y \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

For $0 \le x \le 1$, find the conditional PDF $f_{Y|X}(y|x)$. For $0 \le y \le 1$, find the conditional PDF $f_{X|Y}(x|y)$.

Solution. For $0 \le x \le 1$, Theorem 4.9 implies

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dy = \int_0^x 2dy = 2x.$$

The conditional PDF of Y given X is

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \begin{cases} 1/x, & 0 \le y \le x, \\ 0, & \text{otherwise.} \end{cases}$$

Given $X = x$, we see that Y is the uniform $(0, x)$ random variable. For $0 \le y \le 1$, Theorem 4.9 implies

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dx = \int_y^1 2dx = 2(1-y).$$

Furthermore, Equation (4.18) implies

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} 1/(1-y), & y \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Conditioned on $Y = y$, we see that X is the uniform $(y, 1)$ random variable.

Example 4.13 The joint probability density function of random variables X and Y is

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{1}{x}, & \text{if } 0 < y < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the PDFs $f_X(x)$, $f_Y(y)$.

(b) Find the conditional PDFs $f_{X|Y}(x|y)$, $f_{Y|X}(y|x)$.

Solution. (a) The PDFs of X and Y are:

$$f_X(x) = \int_{-\infty}^{+\infty} f(x,y)dy = \begin{cases} \int_0^x \dfrac{1}{x} dy, & 0 < x < 1, \\ 0, & \text{otherwise,} \end{cases} = \begin{cases} 1, & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)dx = \begin{cases} \int_y^1 \dfrac{1}{x} dx, & 0 < y < 1, \\ 0, & \text{otherwise,} \end{cases} = \begin{cases} -\ln y, & 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(b) The conditional PDFs are

$$f_X(x|y) = \frac{f(x,y)}{f_Y(y)} = \begin{cases} -\dfrac{1}{x \ln y}, & 0 < y < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$f_Y(y|x) = \frac{f(x,y)}{f_X(x)} = \begin{cases} \dfrac{1}{x}, & 0 < y < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

We can include both expressions for conditional PDFs in the following formulas.

Theorem 4.22

$$f_{X,Y}(x,y) = f_{Y|X}(y|x) f_X(x) = f_{X|Y}(x|y) f_Y(y).$$

4.6.4 Conditional Expected Value of a Function

Definition 4.16 (Conditional Expected Value of a Function) For continuous random variables X and Y and any y such that $f_Y(y) > 0$, the conditional expected value of $g(X,Y)$ given $Y = y$ is

$$E[g(X,Y)|Y = y] = \int_{-\infty}^{+\infty} g(x,y) f_{X|Y}(x|y) dx.$$

The conditional expected value of X given $Y = y$ is a special case of Definition 4.16:

$$E[X|Y = y] = \int_{-\infty}^{+\infty} x f_{X|Y}(x|y) dx. \tag{4.19}$$

4.6.5 Conditional Expected Value

Definition 4.17 (Conditional Expected Value) The conditional expected value $E[X|Y]$ is a function of random variable Y such that if $Y = y$ then $E[X|Y] = E[X|Y = y]$.

Example 4.14 For random variables X and Y in Example 4.5, we found in Example 4.12 that the conditional PDF of X given Y is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} 1/(1-y), & y \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find the conditional expected values $E[X|Y = y]$ and $E[X|Y]$.

Solution. Given the conditional PDF $f_{X|Y}(x|y)$, we perform the integration

$$\begin{aligned} E[X|Y = y] &= \int_{-\infty}^{+\infty} x f_{X|Y}(x|y) dx \\ &= \int_y^1 \frac{1}{1-y} x dx = \frac{x^2}{2(1-y)}\Big|_{x=y}^{x=1} = \frac{1+y}{2}. \end{aligned}$$

Since $E[X|Y = y] = (1 + y)/2$, $E[X|Y] = (1 + Y)/2$.

4.7 Independent Random Variables

Chapter 1 presents the concept of independent events. We have that events A and B are independent if and only if the probability of the intersection is the product of the individual probabilities, $P[AB] = P[A]P[B]$.

Applying the idea of independence to random variables, we say that X and Y are independent random variables if and only if the events $\{X = x\}$ and $\{Y = y\}$ are independent for all $x \in S_X$ and all $y \in S_Y$. In terms of probability mass functions and probability density functions we have the following definition.

4.7.1 Definitions

Definition 4.18 (Independent Random Variables) Random variables X and Y are independent if and only if

$$\begin{aligned} &\text{Discrete:} \quad P_{X,Y}(x,y) = P_X(x)P_Y(y), \\ &\text{Continuous:} \quad f_{X,Y}(x,y) = f_X(x)f_Y(y). \end{aligned}$$

Because Definition 4.18 is an equality of functions, it must be true for all values of x and y. Theorem 4.20 implies that if X and Y are independent discrete random variables, then

$$P_{X|Y}(x|y) = P_X(x), \quad P_{Y|X}(y|x) = P_Y(y). \tag{4.20}$$

Theorem 4.22 implies that if X and Y are independent continuous random variables, then

$$f_{X|Y}(x|y) = f_X(x), \quad f_{Y|X}(y|x) = f_Y(y). \tag{4.21}$$

Example 4.15

$$f_{X,Y}(x,y) = \begin{cases} 4xy, & 0 \le x \le 1,\ 0 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Are X and Y independent?

Solution. The marginal PDFs of X and Y are

$$f_X(x) = \begin{cases} 2x, & 0 \le x \le 1, \\ 0, & \text{otherwise,} \end{cases} \qquad f_Y(y) = \begin{cases} 2y, & 0 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

It is easily verified that $f_{X,Y}(x,y) = f_X(x)f_Y(y)$ for all pairs (x,y) and so we conclude that X and Y are independent.

4.7.2 Properties

Theorem 4.23 *For independent random variables X and Y,*

(a) $E[g(X)h(Y)] = E[g(X)]E[h(Y)]$,

(b) $r_{X,Y} = E[XY] = E[X]E[Y]$,

(c) $Cov[X,Y] = \rho_{X,Y} = 0$,

(d) $Var[X+Y] = Var[X] + Var[Y]$,

(e) $E[X|Y=y] = E[X]$ *for all* $y \in S_Y$,

(f) $E[Y|X=x] = E[Y]$ *for all* $x \in S_X$.

These results all follow directly from the joint PMF for independent random variables. We observe that Theorem 4.4.23(c) states that independent random variables are uncorrelated. We will have many occasions to refer to this property. It is important to know that while $Cov[X,Y] = 0$ is a necessary property for independence, it is not sufficient. There are many pairs of uncorrelated random variables that are not independe.

Example 4.16 Random variables X and Y have a joint PMF given by the following matrix

$P_{X,Y}(x,y)$	$y=-1$	$y=0$	$y=1$
$x=-1$	0	0.25	0
$x=1$	0.25	0.25	0.25

(4.22)

Are X and Y independent? Are X and Y uncorrelated?

Solution. For the marginal PMFs, we have $P_X(-1) = 0.25$ and $P_Y(-1) = 0.25$. Thus

$$P_X(-1)P_Y(-1) = 0.0625 \neq P_{X,Y}(-1,-1) = 0,$$

and we conclude that X and Y are not independent.

To find $Cov[X, Y]$, we calculate

$$E[X] = 0.5, \quad E[Y] = 0, \quad E[XY] = 0.$$

Therefore, Theorem 4.14(a) implies

$$Cov[X, Y] = E[XY] - E[X]E[Y] = \rho_{X,Y} = 0,$$

and by definition X and Y are uncorrelated.

Theorem 4.24 If $X_1, X_2, \ldots, X_n$ are independent random variables having normal distributions with means $\mu_1, \mu_2, \ldots, \mu_n$ and variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$, respectively, then the random variable

$$Y = a_1X_1 + a_2X_2 + \cdots + a_nX_n$$

has a normal distribution with mean

$$\mu_Y = a_1\mu_1 + a_2\mu_2 + \cdots + a_n\mu_n$$

and variance

$$\sigma_Y^2 = a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + \cdots + a_n^2\sigma_n^2.$$

Problems – Chapter 4

Problem 4.1 Random variables X and Y have the joint CDF

$$F_{X,Y}(x,y) = \begin{cases} (1-e^{-x})(1-e^{-y}), & x \geq 0,\ y \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) What is $P[X \leq 2, Y \leq 3]$?

(b) What is the marginal CDF, $F_X(x)$?

(c) What is the marginal CDF, $F_Y(y)$?

Problem 4.2 Random variables X and Y have CDF $F_X(x)$ and $F_Y(y)$. Is $F(x,y) = F_X(x)F_Y(y)$ a valid CDF? Explain your answer.

Problem 4.3 Random variables X and Y have the joint PMF

$$P_{X,Y}(x,y) = \begin{cases} cxy, & x = 1,2,4,\ y = 1,3, \\ 0, & \text{otherwise.} \end{cases}$$

(a) What is the value of the constant c?

(a) What is $P[Y < X]$?

(b) What is $P[Y > X]$?

(c) What is $P[Y = X]$?

(d) What is $P[Y = 3]$?

Problem 4.4 Random variables X and Y have the joint PMF

$$P_{X,Y}(x,y) = \begin{cases} c|x+y|, & x = -2,0,2,\ y = -1,0,1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) What is the value of the constant c?

(b) What is $P[Y < X]$?

(c) What is $P[Y > X]$?

(d) What is $P[Y = X]$?

(e) What is $P[X < 1]$?

Problem 4.5 Given the random variables X and Y in Problem 4.3, find

(a) The marginal PMFs $P_X(x)$ and $P_Y(y)$,

(b) The expected values $E[X]$ and $E[Y]$,

(c) The standard deviations σ_X and σ_Y.

Problem 4.6 Given the random variables X and Y in Problem 4.4, find

(a) The marginal PMFs $P_X(x)$ and $P_Y(y)$,

(b) The expected values $E[X]$ and $E[Y]$,

(c) The standard deviations σ_X and σ_Y.

Problem 4.7 Random variables X and Y have the joint PDT

X \ Y	1	2	3
1	0.12	0.15	0.03
2	0.28	0.35	0.07

(a) Are X and Y independent?.

(b) Find the marginal PDTs of X and Y.

(c) Find the PDT of Z, where $Z = XY$.

(d) Find $E(Z)$. Proof $E(Z) = E(X).E(Y)$.

Problem 4.8 Random variables X and Y have the joint PDT

X \ Y	-1	0	1
-1	$4/15$	$1/15$	$4/15$
0	$1/15$	$2/15$	$1/15$
1	0	$2/15$	0

(a) Find $E(X)$, $E(Y)$, and $Cov(X,Y)$.

(b) Are X and Y independent?

(c) Find the marginal PDTs of X and Y.

Problem 4.9 Random variables X and Y have the joint PDT

X \ Y	1	2	3
1	0.17	0.13	0.25
2	0.10	0.30	0.05

(a) Find the marginal PDTs of X and Y.

(b) Find the covariance matrix of X and Y.

(c) Find the correlation coefficient of two random variables X and Y.

(d) Are X and Y independent?

Problem 4.10 Random variables X and Y have the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} c, & x+y \leq 1,\ x \geq 0,\ y \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) What is the value of the constant c?

(b) What is $P[X \leq Y]$?

(c) What is $P[X+Y \leq 1/2]$?

Problem 4.11 Random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} cxy^2, & 0 \leq x \leq 1,\ 0 \leq y \leq 1, \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the constant c.

(b) Find $P[X > Y]$ and $P[Y < X^2]$.

(c) Find $P[\min(X,Y) \leq 1/2]$.

(d) Find $P[\max(X,Y) \leq 3/4]$.

Problem 4.12 X and Y are random variables with the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 2, & x+y \leq 1,\ x \geq 0,\ y \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) What is the marginal PDF $f_X(x)$?

(b) What is the marginal PDF $f_Y(y)$?

Problem 4.13 Random variables X and Y have the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} cy, & 0 \leq y \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Draw the region of nonzero probability.

(b) What is the value of the constant c?

(c) What is $F_X(x)$?

(d) What is $F_Y(y)$?

(e) What is $P[Y \leq X/2]$?

Problem 4.14 Given random variables X and Y in Problem 4.4 and the function $W = X + 2Y$, find

(a) The probability mass function $P_W(w)$,

(b) The expected value $E[W]$,

(c) $P[W > 0]$.

Problem 4.15 Random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} x+y, & 0 \le x \le 1,\ 0 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let $W = \max(X,Y)$.

(a) What is S_W, the range of W?

(b) Find $F_W(w)$ and $f_W(w)$.

Problem 4.16 For the random variables X and Y in Problem 4.3, find

(a) The expected value of $W = Y/X$,

(b) The correlation, $E[XY]$,

(c) The covariance, $Cov[X,Y]$,

(d) The correlation coefficient, $\rho_{X,Y}$,

(e) The variance of $X+Y$, $Var[X+Y]$.

Problem 4.17 Random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} (x+y)/3, & 0 \le x \le 1,\ 0 \le y \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

(a) What are $E[X]$ and $Var[X]$?

(b) What are $E[Y]$ and $Var[Y]$?

(c) What is $Cov[X,Y]$?

(d) What is $E[X+Y]$?

(e) What is $Var[X+Y]$?

Problem 4.18 Random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 5x^2/2, & -1 \le x \le 1,\ 0 \le y \le x^2, \\ 0, & \text{otherwise.} \end{cases}$$

Let $A = \{Y \le 1/4\}$.

(a) What is the conditional PDF $f_{X,Y|A}(x,y)$?

(b) What is $f_{Y|A}(y)$?

(c) What is $E[Y|A]$?

(d) What is $f_{X|A}(x)$?

(e) What is $E[X|A]$?

Problem 4.19 X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} (4x+2y)/3, & 0 \le x \le 1,\ 0 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) For which values of y is $f_{X|Y}(x|y)$ defined? What is $f_{X|Y}(x|y)$?

(b) For which values of x is $f_{Y|X}(y|x)$ defined? What is $f_{Y|X}(y|x)$?

Problem 4.20 The joint PDF of two random variables X and Y is

$$f_{X,Y}(x,y) = \begin{cases} kx, & \text{if } 0 < y < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the constant k.

(b) Find the PDFs of X and Y.

(c) Are X and Y independent?