Laplace transform and applications to ODEs

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January 6, 2021

Content

- Shifting on $t-$domain

Heaviside function

Let $a, b \in \mathbb{R}_+$.

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2. $g(t) = \begin{cases} 0 & \text{if } 0 \le t < a \text{ or } t \ge b, \\ 1 & \text{if } a \le t < b. \end{cases}$

   $g(t) = u(t-b) - u(t-a)$

Example

Express the following functions in terms of $u(t-a)$:

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3. $h(t) = \begin{cases} 1 & \text{if } 0 \le t < \pi, \\ \cos t & \text{if } \pi \le t < 2\pi, \\ t^2 & \text{if } t \ge 2\pi. \end{cases}$

Shifting on $t-$domain

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Theorem

$$\mathcal{L}\{u(t-a)f(t)\}(s) = e^{-as}\mathcal{L}(f(t+a))(s).$$

$$\mathcal{L}^{-1}\{e^{-as}F(s)\}(t) = u(t-a)f(t-a).$$

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Example

Solve the following ODEs:

1. $y'' + 3y' + 2y = u(t-2),\ y(0) = 0, y'(0) = 1.$
2. $y'' + y = \sin t + u(t-\pi)\sin(t-\pi), y(0) = y'(0) = 0.$
3. $y'' + 4y = f(t), y(0) = y'(0) = 0$, where
$$f(t) = \begin{cases} \sin t & \text{if } 0 < t < \pi, \\ 0 & \text{if } t \geq \pi. \end{cases}$$