Chapter 2

Random Variables and Probability Distributions

Chapter 1 defines a probability model. It begins with a physical model of an experiment. An experiment consists of a procedure and observations. The set of all possible observations, S, is the sample space of the experiment. S is the beginning of the mathematical probability model. In addition to S, the mathematical model includes a rule for assigning numbers between 0 and 1 to sets A in S. Thus for every $A \subset S$ the model gives us a probability $P[A]$, where $0 \leq P[A] \leq 1$.

In this chapter and for most of the remainder of the course, we will examine probability models that assign numbers to the outcomes in the sample space. When we observe one of these numbers, we refer to the observation as a random variable. We shall use a capital letter, say X, to denote a random variable and its corresponding small letter, x in this case, for one of its values. The set of possible values of X is the range of X. Since we often consider more than one random variable at a time, we denote the range of a random variable by the letter S with a subscript which is the name of the random variable. Thus S_X is the range of random variable X, S_Y is the range of random variable Y, and so forth. We use S_X to denote the range of X because the set of all possible values of X is analogous to S, the set of all possible outcomes of an experiment.

A probability model always begins with an experiment. Each random variable is related directly to this experiment. There are three types of relationships.

1. The random variable is the observation.

Example 2.1 The experiment is to attach a photo detector to an optical fiber and count the number of photons arriving in a one microsecond time interval. Each observation is a random variable X. The range of X is $S_X = \{0, 1, 2, \ldots\}$. In this case, S_X, the range of X, and the sample space S are identical.

2. The random variable is a function of the observation.

Example 2.2 The experiment is to test six integrated circuits and after each test observe whether the circuit is accepted (A) or rejected (R). Each observation is a sequence of six letters where each letter is either A or R. For example, $S = AARAAA$. The sample space S consists of the 64 possible sequences. A random variable related to this experiment is N, the number of accepted circuits. For outcome s, $N = 5$ circuits are accepted. The range of N is $S_N = \{0, 1, \ldots, 6\}$.

3. The random variable is a function of another random variable.

Example 2.3 In Example 2.2, the net revenue R obtained for a batch of six integrated circuits is \$5 for each circuit accepted minus \$7 for each circuit rejected. (This is because for each bad circuit that goes out of the factory, it will cost the company \$7 to deal with the customer's complaint and supply a good replacement circuit.) When N circuits are accepted, $6 - N$ circuits are rejected so that the net revenue R is related to N by the function

$$R = g(N) = 5N - 7(6 - N) = 12N - 42 \text{ dollars.}$$

Since $S_N = \{0, \ldots, 6\}$, the range of R is

$$S_R = \{-42, -30, -18, -6, 6, 18, 30\}.$$

2.1 Concept of Random Variable

2.1.1 Random Variable

Definition 2.1 (Random variable) A random variable is a function that associates a real number with each element in the sample space.

Example 2.4 Here are some random variables:

1. X, the number of students asleep in the next probability lecture.
2. Y, the number of phone calls you answer in the next hour.
3. Z, the number of minutes you wait until you next answer the phone.

Random variables X and Y are discrete random variables. The possible values of these random variables form a countable set. The underlying experiments have sample spaces that are discrete. The random variable Z can be any nonnegative real number. It is a continuous random variable. Its experiment has a continuous sample space.

2.1.2 Discrete Random Variable

Definition 2.2 (Discrete sample space) If a sample space contains a finite number of possibilities or an unending sequence with as many elements as there are whole numbers, it is called a discrete sample space.

Definition 2.3 (Discrete random variable) X is a discrete random variable if the range of X is a countable set
$$S_X = \{x_1, x_2, \dots\}.$$

Definition 2.4 (Finite random variable) X is a finite random variable if the range is a finite set
$$S_X = \{x_1, x_2, \dots, x_n\}.$$

Example 2.5 Statisticians use sampling plans to either accept or reject batches or lots of material. Suppose one of these sampling plans involves sampling independently 10 items from a lot of 100 items in which 12 are defective.

Let X be the random variable defined as the number of items found defective in the sample of 10. In this case, the random variable takes on the values $0, 1, 2, \dots, 9, 10$. X is a discrete random variable.

A random variable whose set of possible values is an entire interval of numbers is not discrete.

2.1.3 Continuous Random Variable

Until now, we have studied discrete random variables. By definition the range of a discrete random variable is a countable set of numbers. This chapter analyzes random variables that range over continuous sets of numbers. A continuous set of numbers, sometimes referred to as an interval, contains all of the real numbers between two limits. For the limits x_1 and x_2 with $x_1 < x_2$, there are four different intervals distinguished by which of the limits are contained in the interval. Thus we have definitions and notation for the four continuous sets bounded by the lower limit x_1 and upper limit x_2.

- (x_1, x_2) is the open interval defined as all numbers between x_1 and x_2 but not including either x_1 or x_2. Formally, $(x_1, x_2) = \{x | x_1 < x < x_2\}$.
- $[x_1, x_2]$ is the closed interval defined as all numbers between x_1 and x_2 including both x_1 and x_2. Formally $[x_1, x_2] = \{x | x_1 \le x \le x_2\}$.
- $[x_1, x_2)$ is the interval defined as all numbers between x_1 and x_2 including x_1 but not including x_2. Formally, $[x_1, x_2) = \{x | x_1 \le x < x_2\}$.
- $(x_1, x_2]$ is the interval defined as all numbers between x_1 and x_2 including x_2 but not including x_1. Formally, $(x_1, x_2] = \{x | x_1 < x \le x_2\}$.

Many experiments lead to random variables with a range that is a continuous interval. Examples include measuring T, the arrival time of a particle ($S_T = \{t | 0 \le t < \infty\}$); measuring V, the voltage across a resistor ($S_V = \{v | -\infty < v < \infty\}$); and measuring the phase angle A of a sinusoidal radio wave ($S_A = \{a | 0 \le a < 2\pi\}$). We will call T, V, and A continuous random variables.

Example 2.6 Suppose we have a wheel of circumference one meter and we mark a point on the perimeter at the top of the wheel. In the center of the wheel is a radial pointer that we spin. After spinning the pointer, we measure the distance, X meters, around the circumference of the wheel going clockwise from the marked point to the pointer position as shown in Figure 2.1. Clearly, $0 \leq X < 1$. Also, it is reasonable to believe that if the spin is hard enough, the pointer is just as likely to arrive at any part of the circle as at any other. For a given x, what is the probability $P[X = x] = 0$.

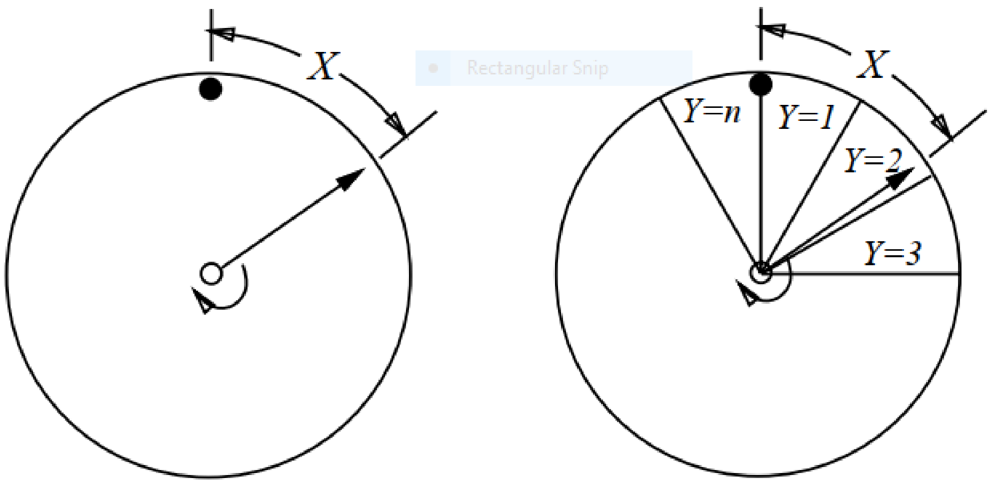

Figure 2.1: The random pointer on disk of circumference 1

This problem is surprisingly difficult. However, given that we have developed methods for discrete random variables in Subsection 2.1.2, a reasonable approach is to find a discrete approximation to X. As shown on the right side of Figure 2.1, we can mark the perimeter with n equal-length arcs numbered 1 to n and let Y denote the number of the arc in which the pointer stops. Y is a discrete random variable with range $S_Y = \{1, 2, \ldots, n\}$. Since all parts of the wheel are equally likely, all arcs have the same probability.

Definition 2.5 (Continuous sample space) If a sample space contains an infinite number of possibilities equal to the number of points on a line segment, it is called a continuous sample space.

Definition 2.6 (Continuous random variable) When a random variable can take on values on a continuous scale, it is called a continuous random variable.

Example 2.7 Let Y be the random variable defined by the waiting time, in hours, between successive speeders spotted by a radar unit. The random variable Y takes on all values y for which $y \geq 0$. Y is a continuous random variable.

2.2 Discrete Probability Distributions

2.2.1 Probability Function/ Probability Mass Function

Recall that a discrete probability model assigns a number between 0 and 1 to each outcome in a sample space. When we have a discrete random variable X, we express the probability model as a probability function (PF), probability mass function (PMF) or probability distribution (PD) $(x, P_X(x))$. The argument of a PMF (PF/PD) ranges over all real numbers.

Definition 2.7 (Probability function/Probability mass function) The set of ordered pairs $(x, P_X(x))$ is a **probability function**, or **probability mass function** (PMF) of the discrete random variable X if, for each possible outcome x,

1. $P_X(x) = P[X = x]$.
2. $P_X(x) \geq 0$.
3. $\sum_{x \in S_X} P_X(x) = 1$.

Remark 2.1 Note that $[X = x]$ is an event consisting of all outcomes s of the underlying experiment for which $X(s) = x$. On the other hand, $P_X(x)$ is a function ranging over all real numbers x. For any value of x, the function $P_X(x)$ is the probability of the event $[X = x]$.

Example 2.8 Suppose we observe three calls at a telephone switch where voice calls (V) and data calls (D) are equally likely. Let X denote the number of voice calls, Y the number of data calls, and let $R = XY$. The sample space of the experiment and the corresponding values of the random variables X, Y, and R are

	Outcomes	DDD	DDV	DVD	DVV	VDD	VDV	VVD	VVV
	P[.]	1/8	1/8	1/8	1/8	1/8	1/8	1/8	1/8
random variable	X	0	1	1	2	1	2	2	3
random variable	Y	3	2	2	1	2	1	1	0
random variable	R	0	2	2	2	2	2	2	0

What is the probability mass function of R?

Solution. We see that $R = 0$ if either outcome, DDD or VVV, occurs so that

$$P[R = 0] = P[DDD] + P[VVV] = \frac{1}{4}.$$

For the other six outcomes of the experiment, $R = 2$ so that $P[R = 2] = 6/8$. The PMF of R is

$$P_R(r) = \begin{cases} 1/4, & r = 0, \\ 3/4, & r = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.8 (Probability distribution) The **probability distribution** for a discrete random variable X is a formula, table, or graph that gives the possible values of X, and the probability associated with each value of X.

X	x_1	x_2	...
P	$P[X = x_1]$	$P[X = x_2]$	...

(2.1)

Requirements for discrete probability distribution:

1. The probability of each value of the discrete random variable is between 0 and 1, inclusive ($0 \le P[X = x_i] \le 1, i = 1, 2, \ldots$).
2. The sum of all the probabilities is 1, that is $\sum_i P[X = x_i] = 1$.

Example 2.9 A shipment of 20 similar laptop computers to a retail outlet contains 3 that are defective. If a school makes a random purchase of 2 of these computers, find the probability distribution for the number of defectives.

Solution. Let X be a random variable whose values x are the possible numbers of defective computers purchased by the school. Then X can only take the numbers 0, 1, and 2. Now

$$P[X = 0] = \frac{C_3^0 C_{17}^2}{C_{20}^2} = \frac{68}{95}; \quad P[X = 1] = \frac{51}{190}; \quad P[X = 2] = \frac{3}{190}.$$

Thus, the probability distribution of X is

X	0	1	2
$P[X = x_i]$	$\frac{68}{95}$	$\frac{51}{190}$	$\frac{3}{190}$

Theorem 2.1 *For a discrete random variable X with PMF $P_X(x)$ and range S_X. If $B \subset S_X$, the probability that X is in the set B is*

$$\boxed{P[B] = \sum_{x \in B} P_X(x)} \tag{2.2}$$

Proof. Since the events $[X = x]$ and $[X = y]$ are disjoint when $x \neq y$, B can be written as the union of disjoint events $B = \cup_{x \in B} [X = x]$. Thus,

$$P[B] = \sum_{x \in B} P[X = x] = \sum_{x \in B} P_X(x).$$

2.2.2 Cumulative Distribution Function

Definition 2.9 (Cumulative distribution function) The cumulative distribution function (CDF) $F_X(x)$ of a discrete random variable X with probability distribution $P_X(x)$ is

$$\boxed{F_X(x) = P[X < x] = \sum_{t < x} P_X(t), \quad \text{for } -\infty < x < \infty} \tag{2.3}$$

Theorem 2.2 *For any discrete random variable X with range $S_X = \{x_1, x_2, \ldots\}$ satisfying $x_1 \le x_2 \le \ldots$,*

(a) $0 \le F_X(x) \le 1$.

(b) *For all* $x_1 < x_2$, $F_X(x_1) \le F_X(x_2)$ *and* $\lim_{x \to a^-} F_X(x) = F_X(a)$ *for all* $a \in \mathbb{R}$.

(c) $F_X(-\infty) = 0$ *and* $F_X(+\infty) = 1$.

(d) *For* $x_i \in S$ *and* ε, *an arbitrarily small positive number,* $F_X(x_i + \varepsilon) - F_X(x_i) = P_X(x_i)$.

(e) $F_X(x) = F_X(x_i)$ *for all* x *such that* $x_i < x \le x_{i+1}$.

Theorem 2.3 *For all* $b \ge a$,

$$F_X(b) - F_X(a) = P[a \le X < b].$$

Proof. To prove this theorem, express the event $Eab = \{a < X \le b\}$ as a part of a union of disjoint events. Start with the event $Eb = \{X < b\}$. Note that Eb can be written as the union

$$Eb = \{X < b\} = \{X < a\} \cup \{a \le X < b\} = Ea \cup Eab.$$

Note also that Ea and Eab are disjoint so that $P[Eb] = P[Ea] + P[Eab]$. Since $P[Eb] = F_X(b)$ and $P[Ea] = F_X(a)$, we can write $F_X(b) = F_X(a) + P[a \le X < b]$. Therefore

$$P[a \le X < b] = F_X(b) - F_X(a).$$

Example 2.10 In Example 2.8, we found that random variable R has PMF

$$P_R(r) = \begin{cases} 1/4, & r = 0, \\ 3/4, & r = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Find and sketch the CDF of random variable R.

Solution. From the PMF $P_R(r)$, random variable R has CDF

$$F_R(r) = P[R < r] = \begin{cases} 0, & r \le 0, \\ 1/4, & 0 < r \le 2, \\ 1, & r > 2. \end{cases}$$

Consider any finite random variable X with possible values (nonzero probability) between $x_{\min}$ and $x_{\max}$. For this random variable, the numerical specification of the CDF begins with

$$F_X(x) = 0, \quad x \le x_{\min},$$

and ends with

$$F_X(x) = 1, \quad x > x_{\max}.$$

Example 2.11 If a car agency sells 50% of its inventory of a certain foreign car equipped with side airbags,

(a) find a formula for the probability distribution $P_X(x)$ of the number of cars with side airbags among the next 4 cars sold by the agency, X.

(b) find the cumulative distribution function of the random variable X; using $F_X(x)$, verify that $P_X(2) = 3/8$.

Solution.

(a) Since the probability of selling an automobile with side airbags is 0.5, the $2^4 = 16$ points in the sample space are equally likely to occur. Therefore, the denominator for all probabilities, and also for our function, is 16. To obtain the number of ways of selling 3 cars with side airbags, we need to consider the number of ways of partitioning 4 outcomes into two cells, with 3 cars with side airbags assigned to one cell and the model without side airbags assigned to the other. This can be done in $C_4^3 = 4$ ways. In general, the event of selling x models with side airbags and $4 - x$ models without side airbags can occur in C_4^x ways, where x can be 0, 1, 2, 3, or 4. Thus, the probability distribution $P_X(x) = P[X = x]$ is

$$P_X(x) = \frac{1}{16}C_4^x, \quad \text{for } x = 0, 1, 2, 3, 4.$$

(b) Direct calculations of the probability distribution $P_X(0) = 1/16$. $P_X(1) = 1/4$, $P_X(2) = 3/8$, $P_X(3) = 1/4$, and $P_X(4) = 1/16$. Therefore

$$F_X(x) = \begin{cases} 0, & \text{for } x \leq 0, \\ \frac{1}{16}, & \text{for } 0 < x \leq 1, \\ \frac{5}{16}, & \text{for } 1 < x \leq 2, \\ \frac{11}{16}, & \text{for } 2 < x \leq 3, \\ \frac{15}{16}, & \text{for } 3 < x \leq 4, \\ 1, & \text{for } x > 4. \end{cases}$$

Now

$$P_X(2) = F_X(3) - F_X(2) = \frac{11}{16} - \frac{5}{16} = \frac{3}{8}.$$

2.3 Continuous Probability Distributions

2.3.1 Cumulative Distribution Function

Example 2.6 shows that when X is a continuous random variable, $P[X = x] = 0$ for $x \in S_X$. This implies that when X is continuous, it is impossible to define a probability mass function $P_X(x)$. On the other hand, we will see that the cumulative distribution function, $F_X(x)$ in Definition 2.9, is a very useful probability model for a continuous random variable. We repeat the definition here.

Definition 2.10 The **cumulative distribution function** (CDF) of random variable X is

$$\boxed{F_X(x) = P[X < x], \quad x \in \mathbb{R}} \tag{2.4}$$

Remark 2.2 The key properties of the CDF, described in Theorem 2.2 and Theorem 2.3, apply to all random variables. Graphs of all cumulative distribution functions start at zero on the left and end at one on the right. All are nondecreasing, and, most importantly, the probability that the random variable is in an interval is the difference in the CDF evaluated at the ends of the interval.

Theorem 2.4 *For any random variable X,*

(a) $F_X(-\infty) = 0$.

(b) $F_X(+\infty) = 1$.

(c) $P[a \le X < b] = F_X(b) - F_X(a)$.

Remark 2.3 Although these properties apply to any CDF, there is one important difference between the CDF of a discrete random variable and the CDF of a continuous random variable. Recall that for a discrete random variable X, $F_X(x)$ has zero slope everywhere except at values of x with nonzero probability. At these points, the function has a discontinuity in the form of a jump of magnitude $P_X(x)$. By contrast, the defining property of a continuous random variable X is that $F_X(x)$ is a continuous function of X.

Definition 2.11 (Continuous random variable) X is a continuous random variable if the CDF $F_X(x)$ is a continuous function.

Example 2.12 In the wheel-spinning experiment of Example 2.6, find the CDF of X.

Solution.

$$F_X(x) = \begin{cases} 0, & x \le 0, \\ x, & 0 < x \le 1, \\ 1, & x > 1. \end{cases}$$

2.3.2 Probability Density Function

Definition 2.12 (Probability density function) The probability density function (PDF) of a continuous random variable X is

$$\boxed{f_X(x) = \frac{dF_X(x)}{dx}} \tag{2.5}$$

Example 2.13 Figure 2.2 depicts the PDF of a random variable X that describes the voltage at the receiver in a modem. What are probable values of X?

Solution. Note that there are two places where the PDF has high values and that it is low elsewhere. The PDF indicates that the random variable is likely to be near -5V (corresponding to the symbol 0 transmitted) and near $+5$ V (corresponding to a 1 transmitted). Values far from ± 5 V (due to strong distortion) are possible but much less likely.

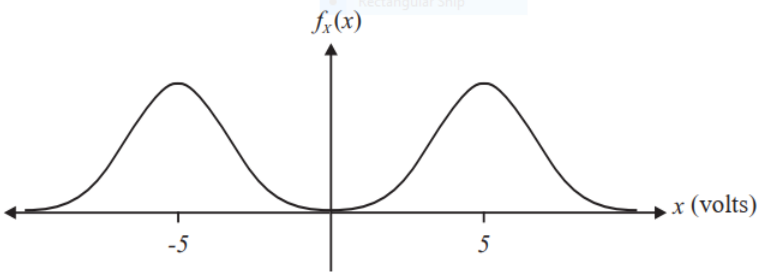

Figure 2.2: The PDF of the modem receiver voltage X

Remark 2.4 Another reason why the PDF is the most useful probability model is that it plays a key role in calculating the expected value of a random variable, the subject of the next section. Important properties of the PDF follow directly from Definition 2.12 and the properties of the CDF.

Theorem 2.5 *For a continuous random variable X with* PDF $f_X(x)$,

(a) $f_X(x) \geq 0$ *for all x,*

(b) $\int\limits_{-\infty}^{+\infty} f_X(x)dx = 1$,

(c) $F_X(x) = \int\limits_{-\infty}^{x} f_X(u)du$.

Proof. The first statement is true because $F_X(x)$ is a nondecreasing function of x and therefore its derivative, $f_X(x)$, is nonnegative. The second statement follows from the second one and Theorem 2.4(b). The third fact follows directly from the definition of $f_X(x)$ and the fact that $F_X(-\infty) = 0$.

Theorem 2.6

$$\boxed{P[a \leq X < b] = \int_a^b f_X(x)dx} \tag{2.6}$$

Proof. From Theorem 2.5(b) and Theorem 2.4,

$$P[a \leq X < b] = P[X < b] - P[X < a] = F_X(b) - F_X(a) = \int_a^b f_X(x)dx.$$

Remark 2.5 (a) $P[X = a] = 0$ for continuous random variables. This implies that

$$P[X \geq a] = P[X > a] \quad \text{and} \quad P[X \leq a] = P[X < a].$$

This is not true in general for discrete random variables.

(b) The probability that X will fall into a particular interval say, from a to b is equal to the area under the curve between the two points a and b. This is the shaded area in Figure 2.3.

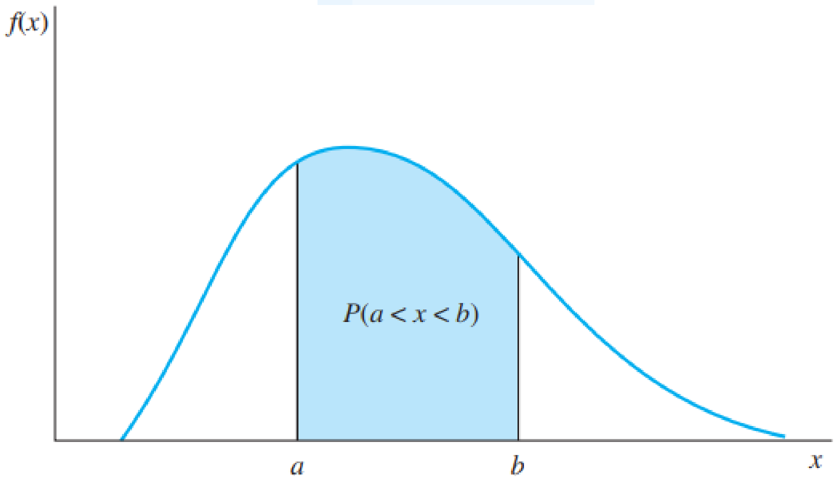

Figure 2.3: The probability distribution $f_X(x)$; $P[a < X < b]$ is equal to the shaded area under the curve

Example 2.14 For the experiment in Examples 2.6 and 2.12, find the PDF of X and the probability of the event $\{1/4 \leq X < 3/4\}$.

Solution. Taking the derivative of the CDF in Equation (2.5), $f_X(x) = 0$, when $x < 0$ or $x \geq 1$. For x between 0 and 1 we have $f_X(x) = dF_X(x)/dx = 1$. Thus the PDF of X is

$$f_X(x) = \begin{cases} 1, & 0 \leq x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

The fact that the PDF is constant over the range of possible values of X reflects the fact that the pointer has no favorite stopping places on the circumference of the circle. To find the probability that X is between $1/4$ and $3/4$, we can use either Theorem 2.4 or Theorem 2.6. Thus

$$P[1/4 \leq X < 3/4] = F_X(3/4) - F_X(1/4) = 1/2,$$

and equivalently,

$$P[1/4 \leq X < 3/4] = \int_{1/4}^{3/4} f_X(x)dx = \int_{1/4}^{3/4} dx = 1/2.$$

Example 2.15 Consider an experiment that consists of spinning the pointer in Example 2.6 three times and observing Y meters, the maximum value of X in the three spins. The CDF of Y is

$$F_Y(y) = \begin{cases} 0, & y \leq 0, \\ y^3, & 0 < y \leq 1, \\ 1, & y > 1. \end{cases}$$

Find the PDF of Y and the probability that Y is between $1/4$ and $3/4$.

Solution. Applying Definition 2.12,

$$f_Y(y) = \begin{cases} 3y^2, & 0 < y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 2.4 or Theorem 2.6 can be used to calculate the probability of observing Y between $1/4$ and $3/4$:

$$P[1/4 < Y < 3/4] = F_Y(3/4) - F_Y(1/4) = (3/4)^3 - (1/4)^3 = 13/32,$$

and equivalently,

$$P[1/4 < Y < 3/4] = \int_{1/4}^{3/4} f_Y(y)dy = \int_{1/4}^{3/4} 3y^2 dy = 13/32.$$

2.4 Mathematical Expectation

2.4.1 Expected/Mean of a Random Variable

The average value of a collection of numerical observations is a statistic of the collection, a single number that describes the entire collection. Statisticians work with several kinds of averages. The ones that are used the most are the mean, the median, and the mode.

Example 2.16 For one quiz, 10 students have the following grades (on a scale of 0 to 10):

$$9, 5, 10, 8, 4, 7, 5, 5, 8, 7$$

Find the mean, the median, and the mode.

Solution. The sum of the ten grades is 68. The mean value is $68/10 = 6,8$. The median is 7 since there are four scores below 7 and four scores above 7. The mode is 5 since that score occurs more often than any other. It occurs three times.

Example 2.16 and the preceding comments on averages apply to observations collected by an experimenter. We use probability models with random variables to characterize experiments with numerical outcomes. A parameter of a probability model corresponds to a statistic of a collection of outcomes. Each parameter is a number that can be computed from the PMF or CDF of a random variable. The most important of these is the expected value of a random variable, corresponding to the mean value of a collection of observations. We will work with expectations throughout the course.

Definition 2.13 (Mode) A mode of random variable X is a number x_{mod} satisfying

$$\boxed{P_X(x_{mod}) \ge P_X(x) \quad \text{for all} \quad x} \tag{2.7}$$

Definition 2.14 (Median) A median, x_{med}, of random variable X is a number that satisfies

$$\boxed{P[X < x_{med}] = P[X \ge x_{med}]} \tag{2.8}$$

Example 2.17 The probability density function of the continuous random variable X is

$$f_X(x) = \begin{cases} \frac{3}{4}x(2-x), & 0 \le x \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

What is x_{mod}? What is x_{med}?

Solution. Applying Theorem 2.5(c),

$$F_X(x) = \begin{cases} 0, & x \le 0, \\ \dfrac{3}{4}\left(x^2 - \dfrac{x^3}{3}\right), & 0 < x \le 2, \\ 1, & x > 2. \end{cases}$$

So x_{med} is a solution of the equation $F_X(x) = \dfrac{1}{2}$, or $x^3 - 3x^2 + 2 = 0$ with $0 < x \le 2$. Hence $x_{med} = 1$.

Taking the derivative of the PDF $f_X(x)$, $g(x) := f'_X(x) = \begin{cases} 0, & x \le 0, \\ \dfrac{3}{2}(1-x), & 0 < x < 2, \\ 0, & \text{otherwise.} \end{cases}$

We can see the function $g(x)$ reaches maximum at $x = 1$, so $x_{mod} = 1$.

Definition 2.15 (Expected value/Mean value) Let X be a random variable with probability distribution $P_X(x)$, or $f_X(x)$. The mean value, or expected value, of X is

$$\boxed{\mu_X = E[X] = \sum_{x \in S_X} x P_X(x) \quad \text{if } X \text{ is discrete}} \tag{2.9}$$

and

$$\boxed{\mu_X = E[X] = \int_{-\infty}^{+\infty} x f_X(x) dx \quad \text{if } X \text{ is continuous}} \tag{2.10}$$

Remark 2.6 Expectation is a synonym for expected value. Sometimes the term mean value is also used as a synonym for expected value. We prefer to use mean value to refer to a statistic of a set of experimental outcomes (the sum divided by the number of outcomes) to distinguish it from expected value, which is a parameter of a probability model. If you recall your studies of mechanics, the form of Definition 2.15 may look familiar. Think of point masses on a line with a mass of $P_X(x)$ kilograms at a distance of x meters from the origin. In this model, μ_X in Definition 2.15 is the center of mass. This is why $P_X(x)$ is called probability mass function.

Example 2.18 Random variable R in Example 2.8 has PMF

$$P_R(r) = \begin{cases} 1/4, & r = 0, \\ 3/4, & r = 2, \\ 0. & \text{otherwise.} \end{cases}$$

What is $E[R]$?

Solution.

$$E[R] = \mu_R = (0)P_R(0) + (2)P_R(2) = (0)\left(\frac{1}{4}\right) + (2)\left(\frac{3}{4}\right) = \frac{3}{2}.$$

Example 2.19 In Example 2.14, we found that the stopping point X of the spinning wheel experiment was a uniform random variable with PDF

$$f_X(x) = \begin{cases} 1, & 0 \le x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find the expected stopping point $E[X]$ of the pointer.

Solution.

$$E[X] = \int_{-\infty}^{+\infty} x f_X(x) dx = \int_0^1 x dx = 1/2 \text{ meter}.$$

Example 2.20 In Example 2.15, find the expected value of the maximum stopping point Y of the three spins:

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y) dy = \int_0^1 y(3y^2) dy = 3/4 \text{ meter}.$$

2.4.2 Functions of a Random Variable

In many practical situations, we observe sample values of a random variable and use these sample values to compute other quantities. One example that occurs frequently is an experiment in which the procedure is to measure the power level of the received signal in a cellular telephone. An observation is x, the power level in units of milliwatts. Frequently engineers convert the measurements to decibels by calculating $y = 10\log_{10} x$ dBm (decibels with respect to one milliwatt). If x is a sample value of a random variable X, Definition 2.1 implies that y is a sample value of a random variable Y. Because we obtain Y from another random variable, we refer to Y as a derived random variable.

Definition 2.16 (Mathematical function) Each sample value y of a derived random variable Y is a mathematical function $g(x)$ of a sample value x of another random variable X. We adopt the notation $Y = g(X)$ to describe the relationship of the two random variables.

Example 2.21 The random variable X is the number of pages in a facsimile transmission. Based on experience, you have a probability model $P_X(x)$ for the number of pages in each fax you send. The phone company offers you a new charging plan for faxes: \$0.10 for the first page, \$0.09 for the second page, etc., down to \$0.06 for the fifth page. For all faxes between 6 and 10 pages, the phone company will charge \$0.50 per fax. (It will not accept faxes longer than ten pages.) Find a function $Y = g(X)$ for the charge in cents for sending one fax.

Solution. The following function corresponds to the new charging plan.

$$Y = g(X) = \begin{cases} 10.5X - 0.5X^2, & 1 \le X \le 5, \\ 50, & 6 \le X \le 10. \end{cases}$$

You would like a probability model $P_Y(y)$ for your phone bill under the new charging plan. You can analyze this model to decide whether to accept the new plan.

Theorem 2.7 *For a discrete random variable X, the* PMF *of $Y = g(X)$ is*

$$\boxed{P_Y(y) = \sum_{x:g(x)=y} P_X(x)} \tag{2.11}$$

If we view $X = x$ as the outcome of an experiment, then Theorem 2.7 says that $P[Y = y]$ equals the sum of the probabilities of all the outcomes $X = x$ for which $Y = y$.

We encounter many situations in which we need to know only the expected value of a derived random variable rather than the entire probability model. Fortunately, to obtain this average, it is not necessary to compute the PMF or CDF of the new random variable. Instead, we can use the following property of expected values.

Theorem 2.8 *Let X be a random variable with probability distribution $P_X(x)$, or $f_X(x)$. The expected value of the random variable $Y = g(X)$ is*

$$\boxed{\mu_Y = E[g(X)] = \sum_{x \in S_X} g(x) P_X(x) \quad \textit{if X is discrete}} \tag{2.12}$$

and

$$\boxed{\mu_Y = E[g(X)] = \int_{-\infty}^{+\infty} g(x) f_X(x) dx \quad \textit{if X is continuous}} \tag{2.13}$$

Example 2.22 In Example 2.21, suppose all your faxes contain $1, 2, 3$, or 4 pages with equal probability. Find the PMF and expected value of Y, the charge for a fax.

Solution. From the problem statement, the number of pages X has PMF

$$P_X(x) = \begin{cases} 1/4, & x = 1, 2, 3, 4, \\ 0, & \text{otherwise.} \end{cases}$$

The charge for the fax, Y, has range $S_Y = \{10, 19, 27, 34\}$ corresponding to $S_X = \{1, 2, 3, 4\}$. Here each value of Y results in a unique value of X. Hence,

$$P_Y(y) = \begin{cases} 1/4, & x = 10, 19, 27, 34, \\ 0, & \text{otherwise.} \end{cases}$$

The expected fax bill is $E[Y] = \dfrac{1}{4}(10 + 19 + 27 + 34) = 22.5$ cents.

Example 2.23 In Example 2.22,

$$P_X(x) = \begin{cases} 1/4, & x = 1, 2, 3, 4, \\ 0, & \text{otherwise} \end{cases}$$

and

$$Y = g(X) = \begin{cases} 10.5X - 0.5X^2, & 1 \le X \le 5, \\ 50, & 6 \le X \le 10. \end{cases}$$

What is $E[Y]$?

Solution. Applying Theorem 2.8 we have

$$\begin{aligned} E[Y] &= \sum_{x=1}^{4} P_X(x)g(x) \\ &= \frac{1}{4}[(10.5)(1) - (0.5)(1)^2] + \frac{1}{4}[(10.5)(2) - (0.5)(2)^2] \\ &\quad + \frac{1}{4}[(10.5)(3) - (0.5)(3)^2] + \frac{1}{4}[(10.5)(4) - (0.5)(4)^2] \\ &= \frac{1}{4}[10 + 19 + 27 + 34] = 22.5 \text{ cents.} \end{aligned}$$

Theorem 2.9 *For any random variable X,*

(i) $E[X - \mu_X] = 0.$

(ii) $E[aX + b] = aE[X] + b.$

Corollary 2.1 (i) *Setting $a = 0$, we see that $E[b] = b$.*

(ii) *Setting $b = 0$, we see that $E[aX] = aE[X]$.*

Example 2.24 Recall that in Examples 2.8 and 2.18, we found that R has PMF

$$P_R(r) = \begin{cases} 1/4, & r = 0, \\ 3/4, & r = 2, \\ 0 & \text{otherwise,} \end{cases}$$

and expected value $E[R] = 3/2$. What is the expected value of $V = g(R) = 4R + 7$?

Solution. From Theorem 2.9(ii),

$$E[V] = E[g(R)] = 4E[R] + 7 = 4 \times \frac{3}{2} + 7 = 13.$$

We can verify this result by applying Theorem 2.8. Using the PMF $P_R(r)$ given in Example 2.8, we can write

$$E[V] = g(0)P_R(0) + g(2)P_R(2) = 7 \times \frac{1}{4} + 15 \times \frac{3}{4} = 13.$$

Theorem 2.10 *The expected value of the sum or difference of two or more functions of a random variable X is the sum or difference of the expected values of the functions. That is,*

$$\boxed{E[g(X) \pm h(X)] = E[g(X)] \pm E[h(X)]} \tag{2.14}$$

Example 2.25 Let X be a random variable with probability distribution as follows:

X	0	1	2	3
$P_X(x)$	1/3	1/2	0	1/6

Find the expected value of $Y = (X-1)^2$.

Solution. Applying Theorem 2.10 to the function $Y = (X-1)^2$, we can write

$$E[(X-1)^2] = E[X^2] - 2E[X] + E[1].$$

From Corollary 2.1, $E[1] = 1$, and by direct computation,

$$E[X] = (0)\left(\frac{1}{3}\right) + (1)\left(\frac{1}{2}\right) + (2)(0) + (3)\left(\frac{1}{6}\right) = 1 \quad \text{and}$$

$$E[X^2] = (0)\left(\frac{1}{3}\right) + (1)\left(\frac{1}{2}\right) + (4)(0) + (9)\left(\frac{1}{6}\right) = 2.$$

Hence,

$$E[(X-1)^2] = 2 - (2)(1) + 1 = 1.$$

2.5 Variance and Standard Deviation

In Subsection 2.4.1, we describe an average as a typical value of a random variable. It is one number that summarizes an entire probability model. After finding an average, someone who wants to look further into the probability model might ask, "How typical is the average?" or, "What are the chances of observing an event far from the average?" In the example of the midterm exam, after you find out your score is 7 points above average, you are likely to ask, "How good is that? Is it near the top of the class or somewhere near the middle?" A measure of dispersion is an answer to these questions wrapped up in a single number. If this measure is small, observations are likely to be near the average. A high measure of dispersion suggests that it is not unusual to observe events that are far from the average.

The most important measures of dispersion are the standard deviation and its close relative, the variance. The variance of random variable X describes the difference between X and its expected value.

2.5.1 Definitions

Definition 2.17 (Variance) The variance of random variable X is

$$\boxed{Var[X] = E\left[(X-\mu_X)^2\right]} \tag{2.15}$$

Definition 2.18 (Standard deviation) The standard deviation of random variable X is

$$\boxed{\sigma_X = Var[X]} \tag{2.16}$$

Remark 2.7 It is useful to take the square root of $Var[X]$ because σ_X has the same units (for example, exam points) as X. The units of the variance are squares of the units of the random variable (exam points squared). Thus σ_X can be compared directly with the expected value. Informallywe think of outcomes within $\pm\sigma_X$ of μ_X as being in the center of the distribution. Thus if the standard deviation of exam scores is 12 points, the student with a score of +7 with respect to the mean can think of herself in the middle of the class. If the standard deviation is 3 points, she is likely to be near the top. Informally, we think of sample values within σ_X of the expected value, $x \in [\mu_X - \sigma_X, \mu_X + \sigma_X]$, as "typical" values of X and other values as "unusual."

Because $(X - \mu_X)^2$ is a function of X, $Var[X]$ can be computed according to Theorem 2.8.

$$\boxed{\sigma_X^2 = Var[X] = \sum_{x \in S_X} (x - \mu_X)^2 P_X(x) \quad \text{if } X \text{ is discrete}} \tag{2.17}$$

and

$$\boxed{\sigma_X^2 = Var[X] = \int_{-\infty}^{+\infty} (x - \mu_X)^2 f_X(x) dx \quad \text{if } X \text{ is continuous}} \tag{2.18}$$

2.5.2 Properties and Examples

Theorem 2.11 $Var[X] = E[X^2] - \mu_X^2$ *where*

$$\boxed{E[X^2] = \sum_{x \in S_X} x^2 P_X(x) \quad \textit{if } X \textit{ is discrete}} \tag{2.19}$$

and

$$\boxed{E[X^2] = \int_{-\infty}^{+\infty} x^2 f_X(x) dx \quad \textit{if } X \textit{ is continuous}} \tag{2.20}$$

Remark 2.8 Our interpretation of expected values of discrete random variables carries over to continuous random variables. $E[X]$ represents a typical value of X, and the variance describes the dispersion of outcomes relative to the expected value. Furthermore, if we view the PDF $f_X(x)$ as the density of a mass distributed on a line, then $E[X]$ is the center of mass.

Definition 2.19 (Moment) For random variable X:

(a) The nth **moment** is $E[X^n]$.

(b) The nth **central moment** is $E[(X - \mu_X)^n]$.

Example 2.26 In Example 2.8, we found that random variable R has PMF

$$P_R(r) = \begin{cases} 1/4, & r = 0, \\ 3/4, & r = 2, \\ 0 & \text{otherwise.} \end{cases}$$

In Example 2.18, we calculated $E[R] = \mu_R = 3/2$. What is the variance of R?

Solution. In order of increasing simplicity, we present three ways to compute $Var[R]$.

1. From Definition 2.17, define

$$W = (R - \mu_R)^2 = (R - 3/2)^2.$$

The PMF of W is

$$P_W(w) = \begin{cases} 1/4, & w = (0 - 3/2)^2 = 9/4, \\ 3/4, & w = (2 - 3/2)^2 = 1/4, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$Var[R] = E[W] = (1/4)(9/4) + (3/4)(1/4) = 3/4.$$

2. Recall that Theorem 2.8 produces the same result without requiring the derivation of $P_W(w)$.

$$\begin{aligned} Var[R] &= E[(R - \mu_R)^2] \\ &= (0 - 3/2)^2 P_R(0) + (2 - 3/2)^2 P_R(2) = 3/4. \end{aligned}$$

3. To apply Theorem 2.11, we find that

$$E[R^2] = 0^2 P_R(0) + 2^2 P_R(2) = 3.$$

Thus Theorem 2.11 yields

$$Var[R] = E[R^2] - \mu_R^2 = 3 - (3/2)^2 = 3/4.$$

Note that $(X - \mu_X)^2 \geq 0$. Therefore, its expected value is also nonnegative. That is, for any random variable X

$$\boxed{Var[X] \geq 0} \tag{2.21}$$

The following theorem is related to Theorem 2.9(ii).

Theorem 2.12 $Var[aX + b] = a^2 Var[X]$.

Example 2.27 Find the variance and standard deviation of the pointer position in Example 2.6.

Solution. To compute $Var[X]$, we use Theorem 2.11, $Var[X] = E[X^2] - \mu_X^2$. We calculate $E[X^2]$ directly from Theorem 2.11,

$$E[X^2] = \int_{-\infty}^{+\infty} x^2 f_X(x)dx = \int_0^1 x^2 dx = 1/3.$$

In Example 2.19, we have $E[X] = 1/2$. Thus $Var[X] = 1/3 - (1/2)^2 = 1/12$, and the standard deviation is $\sigma[X] = \sqrt{Var[X]} = 1/\sqrt{12} = 0.289$ meters.

Example 2.28 Find the variance and standard deviation of Y, the maximum pointer position after three spins, in Example 2.15.

Solution. We proceed as in Example 2.27. We have $f_Y(y)$ from Example 2.15 and $E[Y] = 3/4$ from Example 2.20:

$$E[Y^2] = \int_{-\infty}^{+\infty} y^2 f_Y(y)dy = \int_0^1 y^2(3y^2)dy = 3/5.$$

Thus the variance is

$$Var[Y] = 3/5 - (3/4)^2 = 3/80 \text{ m}^2,$$

and the standard deviation is $\sigma[Y] = 0.194$ meters.

Theorem 2.13 *Let X be a random variable with probability distribution $P_X(x)$, or $f_X(x)$. The variance of the random variable $Y = g(X)$ is*

$$\boxed{\sigma_Y^2 = E[g(X) - \mu_{g(X)}]^2 = \sum_{x \in S_X} [g(x) - \mu_{g(X)}]^2 P_X(x) \quad \textit{if } X \textit{ is discrete}} \tag{2.22}$$

and

$$\boxed{\sigma_Y^2 = E[g(X) - \mu_{g(X)}]^2 = \int_{-\infty}^{+\infty} [g(x) - \mu_{g(X)}]^2 f_X(x)dx \quad \textit{if } X \textit{ is continuous}} \tag{2.23}$$

Example 2.29 Calculate the variance of $g(X) = 2X + 3$, where X is a random variable with probability distribution

X	0	1	2	3
$P_X(x)$	1/4	1/8	1/2	1/8

Solution. First, we find the mean of the random variable $2X + 3$. According to Theorem 2.8,

$$\mu_{2X+3} = E[2X+3] = \sum_{x=0}^{3} (2x+3)P_X(x) = 6.$$

Now, using Theorem 2.13, we have

$$\sigma_{2X+3}^2 = E[(2X+3-6)^2] = E[(4X^2 - 12X + 9)] = \sum_{x=0}^{3} (4x^2 - 12x + 9)P_X(x) = 4.$$

Example 2.30 Let X be a random variable with density function

$$f_X(x) = \begin{cases} \dfrac{x^2}{3}, & -1 < x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

Find the expected value and the variance of $g(X) = 4X + 3$.

Solution. By Theorem 2.8 we have

$$E[4X+3] = \int_{-1}^{2} \frac{(4x+3)x^2}{3} dx = \frac{1}{3} \int_{-1}^{2} (4x^3 + 3x^2) dx = 8.$$

Now, using Theorem 2.13,

$$\begin{aligned} \sigma^2_{4X+3} &= E[(4X+3) - 8^2] = E[(4X-5)^2] \\ &= \int_{-1}^{2} (4x-5)^2 \frac{x^2}{3} dx = \frac{51}{5}. \end{aligned}$$

Chapter Summary

Discrete Random Variables

1. The random variable X transforms outcomes of an experiment to real numbers. Note that X is the name of the random variable. A possible observation is x, which is a number. S_X is the range of X, the set of all possible observations x.

2. The PMF $P_X(x)$ is a function that contains the probability model of the random variable X. The PMF gives the probability of observing any x. $P_X(.)$ contains our information about the randomness of X.

3. The expected value $E[X] = \mu_X$ and the variance $Var[X]$ are numbers that describe the entire probability model. Mathematically, each is a property of the PMF $P_X(.)$. The expected value is a typical value of the random variable. The variance describes the dispersion of sample values about the expected value.

4. A function of a random variable $Y = g(X)$ transforms the random variable X into a different random variable Y. For each observation $X = x$, $g(.)$ is a rule that tells you how to calculate $y = g(x)$, a sample value of Y.

   Although $P_X(.)$ and $g(.)$ are both mathematical functions, they serve different purposes here. $P_X(.)$ describes the randomness in an experiment. On the other hand, $g(.)$ is a rule for obtaining a new random variable from a random variable you have observed.

Continuous Random Variables

1. A random variable X is continuous if the range S_X consists of one or more intervals. Each possible value of X has probability zero.

2. The PDF $f_X(x)$ is a probability model for a continuous random variable X. The PDF $f_X(x)$ is proportional to the probability that X is close to x.

3. The expected value $E[X]$ of a continuous random variable has the same interpretation as the expected value of a discrete random variable. $E[X]$ is a typical value of X.

4. A function of a random variable transforms a random variable X into a new random variable $Y = g(X)$. If X is continuous, we find the probability model of Y by deriving the CDF, $F_Y(y)$, from $F_X(x)$ and $g(x)$.

Problems – Chapter 2

Discrete Random Variables

Problem 2.1 The random variable N has PMF

$$P_N(n) = \begin{cases} c(1/2)^n, & n = 0, 1, 2, \\ 0, & \text{otherwise.} \end{cases}$$

(a) What is the value of the constant c?

(b) What is $P[N \leq 1]$?

Problem 2.2 The random variable V has PMF

$$P_V(v) = \begin{cases} cv^2, & v = 1, 2, 3, 4, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the value of the constant c.

(b) Find $P[V \in \ u^2 | u = 1, 2, 3, \dots]$.

(c) Find the probability that V is an even number.

(d) Find $P[V > 2]$.

Problem 2.3 Suppose when a baseball player gets a hit, a single is twice as likely as a double which is twice as likely as a triple which is twice as likely as a home run. Also, the player's batting average, i.e., the probability the player gets a hit, is 0.300. Let B denote the number of bases touched safely during an at-bat. For example, $B = 0$ when the player makes an out, $B = 1$ on a single, and so on. What is the PMF of B?

Problem 2.4 In a package of M&Ms, Y, the number of yellow M&Ms, is uniformly distributed between 5 and 15.

(a) What is the PMF of Y?

(b) What is $P[Y < 10]$?

(c) What is $P[Y > 12]$?

(d) What is $P[8 \leq Y \leq 12]$?

Problem 2.5 When a conventional paging system transmits a message, the probability that the message will be received by the pager it is sent to is p. To be confident that a message is received at least once, a system transmits the message n times.

(a) Assuming all transmissions are independent, what is the PMF of K, the number of times the pager receives the same message?

(b) Assume $p = 0.8$. What is the minimum value of n that produces a probability of 0.95 of receiving the message at least once?

Problem 2.6 When a two-way paging system transmits a message, the probability that the message will be received by the pager it is sent to is p. When the pager receives the message, it transmits an acknowledgment signal (ACK) to the paging system. If the paging system does not receive the ACK, it sends the message again.

(a) What is the PMF of N, the number of times the system sends the same message?

(b) The paging company wants to limit the number of times it has to send the same message. It has a goal of $P[N \leq 3] \geq 0.95$. What is the minimum value of p necessary to achieve the goal?

Problem 2.7 The random variable X has CDF

$$F_X(x) = \begin{cases} 0, & x \leq -1, \\ 0.2, & -1 < x \leq 0, \\ 0.7, & 0 < x \leq 1, \\ 1, & x > 1. \end{cases}$$

(a) Draw a graph of the CDF.

(b) Write $P_X(x)$, the PMF of X. Be sure to write the value of $P_X(x)$ for all x from $-\infty$ to ∞.

Problem 2.8 The random variable X has CDF

$$F_X(x) = \begin{cases} 0, & x \leq -3, \\ 0.4, & -3 < x \leq 5, \\ 0.8, & 5 < x \leq 7, \\ 1, & x \geq 7. \end{cases}$$

(a) Draw a graph of the CDF.

(b) Write $P_X(x)$, the PMF of X.

Problem 2.9 Let X have the uniform PMF

$$P_X(x) = \begin{cases} 0.01, & x = 1, 2, \ldots, 100, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find a mode x_{mod} of X. If the mode is not unique, find the set X_{mod} of all modes of X.

(b) Find a median x_{med} of X. If the median is not unique, find the set X_{med} of all numbers x that are medians of X.

Problem 2.10 Find the expected value of the random variable X in Problem 2.7.

Problem 2.11 Find the expected value of the random variable X in Problem 2.8.

Problem 2.12 In an experiment to monitor two calls, the PMF of N, the number of voice calls, is

$$P_N(n) = \begin{cases} 0.2, & n = 0, \\ 0.7, & n = 1, \\ 0.1, & n = 2, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find $E[N]$, the expected number of voice calls.

(b) Find $E[N^2]$, the second moment of N.

(c) Find $Var[N]$, the variance of N.

(d) Find σ_N, the standard deviation of N.

Problem 2.13 Find the variance of the random variable X in Problem 2.7.

Problem 2.14 Show that the variance of $Y = aX + b$ is $Var[Y] = a^2 Var[X]$.

Problem 2.15 Given a random variable X with mean μ_X and variance σ_X^2, find the mean and variance of the standardized random variable

$$Y = \frac{(X - \mu_X)}{\sigma_X}.$$

Continuous Random Variables

Problem 2.16 The cumulative distribution function of random variable X is

$$F_X(x) = \begin{cases} 0, & x \leq -1, \\ (x+1)/2, & -1 < x \leq 1, \\ 1, & x > 1. \end{cases}$$

(a) What is $P[X > 1/2]$?

(b) What is $P[-1/2 < X \leq 3/4]$?

(c) What is $P[|X| \leq 1/2]$?

(d) What is the value of a such that $P[X \leq a] = 0.8$?

Problem 2.17 The cumulative distribution function of the continuous random variable V is

$$F_V(v) = \begin{cases} 0, & v \leq -5, \\ c(v+5)^2, & -5 < v \leq 7, \\ 1, & v > 7. \end{cases}$$

(a) What is c?

(b) What is $P[V > 4]$?

(c) $P[-3 < V \leq 0]$?

(d) What is the value of a such that $P[V > a] = 2/3$?

Problem 2.18 The random variable X has probability density function

$$f_X(x) = \begin{cases} cx, & 0 \leq x \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

Use the PDF to find

(a) the constant c,

(b) $P[0 \leq X \leq 1]$,

(c) $P[-1/2 \leq X \leq 1/2]$,

(d) the CDF $F_X(x)$.

Problem 2.19 The cumulative distribution function of random variable X is

$$F_X(x) = \begin{cases} 0, & x \leq -1, \\ (x+1)/2, & -1 < x \leq 1, \\ 1, & x > 1. \end{cases}$$

Find the PDF $f_X(x)$ of X.

Problem 2.20 Continuous random variable X has PDF

$$f_X(x) = \begin{cases} 1/4, & -1 \leq x \leq 3, \\ 0, & \text{otherwise.} \end{cases}$$

Define the random variable Y by $Y = h(X) = X^2$.

(a) Find $E[X]$ and $Var[X]$.

(b) Find $h(E[X])$ and $E[h(X)]$.

(c) Find $E[Y]$ and $Var[Y]$.

Problem 2.21 Random variable X has CDF

$$F_X(x) = \begin{cases} 0, & x \leq 0, \\ x/2, & 0 < x \leq 2, \\ 1, & x > 2. \end{cases}$$

(a) What is $E[X]$?

(b) What is $Var[X]$?

Problem 2.22 The cumulative distribution function of the continuous random variable X is

$$F(x) = \begin{cases} 0, & x \le 0 \\ \dfrac{1}{2} - k\cos x, & 0 < x \le \pi \\ 1, & x > \pi. \end{cases}$$

(a) What is k?

(b) What is $P[0 < X < \dfrac{\pi}{2}]$?

(c) What is $E[X]$?

Problem 2.23 The cumulative distribution function of the continuous random variable X is

$$F(x) = \begin{cases} 0, & x \le -a \\ A + B\arcsin\dfrac{x}{a}, & x \in (-a, a) \\ 1, & x \ge a. \end{cases}$$

(a) What are A and B?

(b) What is the PDF $f_X(x)$?

Problem 2.24 The cumulative distribution function of the continuous random variable X is $F(x) = a + b\arctan x$, $(-\infty < x < +\infty)$

(a) What are a and b?

(b) What is the PDF $f_X(x)$?

(c) What is $P[-1 < X < 1]$?

Problem 2.25 The cumulative distribution function of the continuous random variable X is $F(x) = 1/2 + 1/\pi \arctan x/2$. What is the value of x_1 such that $P(X > x_1) = 1/4$?

Problem 2.26 The continuous random variable X has PDF $f(x) = ae^{-|x|}$, $(-\infty < x < +\infty)$. Define the random variable Y by $Y = X^2$.

(a) What is a?

(b) What is the CDF $F_Y(x)$?

(c) What is $E[X]$? What is $Var[X]$?