Series with sign changing terms

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Example (Series with sign changing terms)

- $1 - 1 + 1 - 1 + \ldots + 1 - 1 + \ldots$
- $\sin 1 + \sin 2 + \sin 3 + \sin 4 + \ldots$
- $1 - \frac{1}{3!} + \frac{1}{5!} + \ldots + \frac{(-1)^k}{(2k+1)!} + \ldots$

Absolute convergence

Definition

$\sum_{n=1}^{\infty} a_n$ is absolutely convergent $\Leftrightarrow \sum_{n=1}^{\infty} |a_n|$ is convergent.

Proposition

If $\sum_{n=1}^{\infty} a_n$ converges absolutely then $\sum_{n=1}^{\infty} a_n$ also converges.

Proof

Suppose that the series $\sum_{n=1}^{\infty} a_n$ converges absolutely, i.e. $\sum_{n=1}^{\infty} |a_n|$ converges.
We have $0 \leq a_n + |a_n| \leq 2|a_n|$ for all n, hence, by the comparison test the series $\sum_{n=1}^{\infty} (a_n + |a_n|)$ converges.
Since $a_n = (a_n + |a_n|) - |a_n|$, we conclude that $\sum_{n=1}^{\infty} a_n$ also converges.

In case $\sum_{n=1}^{\infty} a_n$ does not converge absolutely, $\sum_{n=1}^{\infty} a_n$ might converge or diverge, which must be verified by another test.

Definition

$\sum_{n=1}^{\infty} a_n$ is conditionally convergent $\Leftrightarrow \sum_{n=1}^{\infty} |a_n|$ is divergent and $\sum_{n=1}^{\infty} a_n$ is convergent.

$\sum_{n=1}^{\infty} a_n$ is absolutely convergent $\Leftrightarrow \sum_{n=1}^{\infty} |a_n|$ is convergent and (hence) $\sum_{n=1}^{\infty} a_n$ is convergent.

Example

Test for convergence

$$\text{a)}\ \sum_{n=1}^{\infty}(-1)^n\frac{n}{2^n} \qquad\qquad \text{b)}\ \sum_{n=1}^{\infty}\frac{\sin n^2}{\sqrt{n^3}}$$

Solution:

a) We have

$$D=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(-1)^{n+1}.(n+1)}{2^{n+1}}\frac{2^n}{(-1)^n.n}\right|$$
$$=\lim_{n\to\infty}\frac{n+1}{2n}=\frac{1}{2}<1,$$

so, the series converges absolutely.

b) We have

$$0 \le \left| \frac{\sin n^2}{\sqrt{n^3}} \right| \le \frac{1}{\sqrt{n^3}}, \quad \forall n \in \mathbb{N}^*.$$

$\sum \frac{1}{n^{\frac{3}{2}}}$ converges, hence by the comparison tets, the series $\sum_{n=1}^{\infty} \frac{\sin n^2}{\sqrt{n^3}}$ converges absolutely.

Example

Conditionally convergent series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

Remark

We can also apply the D'Alembert and Cauchy tests to series with sign changing terms

$$D = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|, \; C = \lim_{n\to\infty} \sqrt[n]{|a_n|}.$$

If $D < 1$, ($C < 1$) then the series converges absolutely (hence converges).
If $D > 1$, ($C > 1$) then the series diverges.

Indeed,

$$D = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1 \text{ or } C = \lim_{n\to\infty} \sqrt[n]{|a_n|} > 1 \Rightarrow \lim_{n\to\infty} a_n \neq 0.$$

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Alternating series

Definition

An alternating series is a series of the form

$$-a_1 + a_2 - a_3 + \ldots + a_{2n} - a_{2n+1} + \ldots = \sum_{n=1}^{\infty} (-1)^n a_n$$

or

$$a_1 - a_2 + a_3 - \ldots + a_{2n-1} - a_{2n} + \ldots = \sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

where $a_n > 0$.

Example

Which one of the following is an alternating series?

a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{\cos n\pi}{n^2+\pi^2}$ b) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{\sin n}{n}$ c) $\displaystyle\sum_{n=2}^{\infty}(-1)^{n^2}\frac{\ln n}{n\sqrt{n}}$

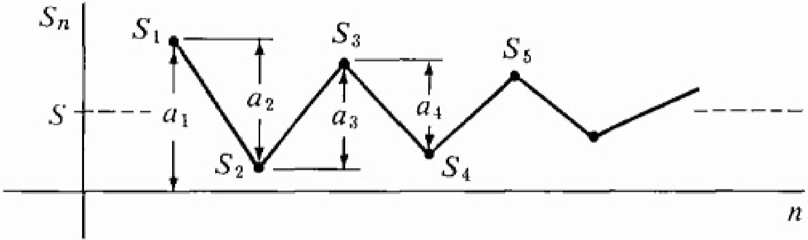
a) positive series $(-1)^n.\cos n\pi = 1$.
b) is not an alternating series as $\sin n$ changes sign arbitrarily.
c) is an alternating series as $\frac{\ln n}{n\sqrt{n}} > 0$, $(-1)^{n^2} = (-1)^n$.

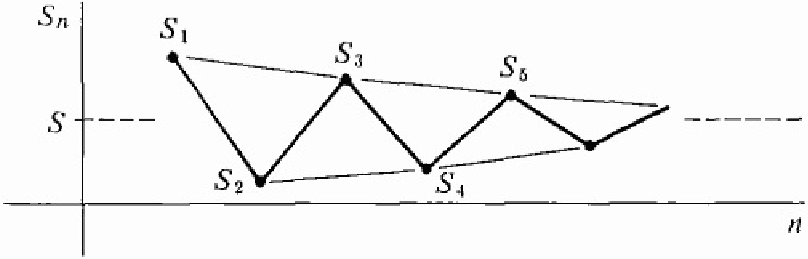
Alternating series test

Theorem (Leibniz test)

If the positive sequence $a_1, a_2, \ldots, a_n, \ldots$ decreasingly tends to 0 *as* $n \to \infty$ *then the alternating series* $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ *converges and the sum* $S \leq a_1$.

Proof.





- $\{S_{2m}\}$ is increasing and bounded from above,
  $\exists \lim\limits_{m\to\infty} S_{2m} = S$.
- $\{S_{2m+1}\}$ is decreasing and bounded from below,
  $\exists \lim\limits_{m\to\infty} S_{2m+1} = S'$.
- $S_{2m+1} = a_{2m+1} + S_{2m}$, as $\lim\limits_{n\to\infty} a_n = 0$, passing to the limit $S' = S$.

Example

Test for convergence:

$$\text{a)}\ \sum_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt{n}} \quad \text{b)}\ \sum_{n=3}^{\infty} (-1)^n \frac{\ln n}{n} \quad \text{c)}\ \sum_{n=1}^{\infty} \frac{(-1)^{n^2}.n}{\sqrt{2n^2+1}}$$

a) $a_n = \dfrac{1}{n\sqrt{n}} > 0$, alternating series. It is obvious that $\{\frac{1}{n\sqrt{n}}\}$ is a decreasing sequence which tends to 0. By Leibniz test, the given series converges.

b) $a_n = \dfrac{\ln n}{n} > 0$, alternating series.

$\lim\limits_{n\to\infty} \dfrac{\ln n}{n} = 0$ because $\lim\limits_{x\to\infty} \dfrac{\ln x}{x} \overset{L'}{=} \lim\limits_{x\to\infty} \dfrac{1}{x} = 0$.

Consider the function $f(x) = \dfrac{\ln x}{x}$, we have $f'(x) = \dfrac{1-\ln x}{x^2} < 0$ for all $x \geq 3$.Hence, $f(x)$ is a decreasing function on $[3, +\infty)$, in particular, $f(n+1) < f(n)$, or $\boxed{a_{n+1} < a_n}$ for all $n \geq 3$.

By the Leibniz test, the series converges.

c) We have $\left|\dfrac{(-1)^{n^2}.n}{\sqrt{2n^2+1}}\right| = \dfrac{n}{\sqrt{2n^2+1}} \to \dfrac{1}{\sqrt{2}} \neq 0.$

The general term does not converge to 0, hence the series is divergent. The first condition in Leibniz test is not fulfilled, then we have to use other criteria for testing for convergence.

Remark

Leibniz test is a sufficient condition for an alternating series to converge.

DON'T FORGET

a) The series is absolutely convergent as $\sum \frac{1}{n\sqrt{n}}$ converges.

Content

- A sum of **finite** terms is associative and commutative.
  $(a+b)+c = a+(b+c) = a+c+b$
- In a series, we cannot commute or regroup terms

Example

Consider $S = \sum_{n=2}^{\infty} \frac{(-1)^n}{n} = \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \ldots$

$S = \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{6} - \frac{1}{7}\right) + \ldots > 0.$

$S = \left(\frac{1}{2} - \frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{7} - \frac{1}{9}\right) + \ldots < 0.$

- Associativity and commutativity hold for **absolutely convergent series**.

Properties of absolutely convergent series

Proposition

1. *The terms of an absolutely convergent series can be rearranged in any order or regrouped, and all resulting series will converge to the same sum.*
2. *The terms of a conditionally convergent series can be suitably rearranged or regrouped such that the resulting series may diverge or converge to any desired sum.*

Product of two series

Definition

Given two series $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$. Procduct of these two series is the series

$$\Big(\sum_{n=1}^{\infty} a_n\Big)\Big(\sum_{n=1}^{\infty} b_n\Big) = \sum_{n=1}^{\infty} c_n, \text{ where } c_n = \sum_{k=1}^{n} a_k b_{n+1-k}.$$

Proposition

The sum, difference and product of two absolutely convergent series is absolutely convergent.

THANK YOU FOR YOUR ATTENTION!