Algorithm and Data Structures
Lecture notes: Heapsort, Cormen Chap. 6

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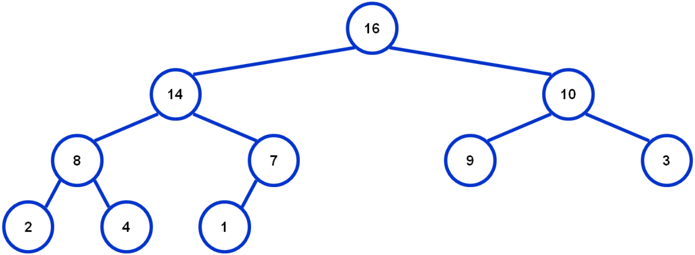
Outline

Sorting

- So far we have seen different sorting algorithms such as selection sort, and insertion, and merge sort and quicksort
  - Merge sort runs in $O(n \log n)$ both in best, average and worst-case
  - Insertion/selection sort run in $O(n^2)$, but insertion sort is fast when array is nearly sorted, runs fast in practice
- Next on the agenda : Heapsort
- Prior to describe heapsort, we introduce the heap data structure and operations on that data structure

Heap : definition

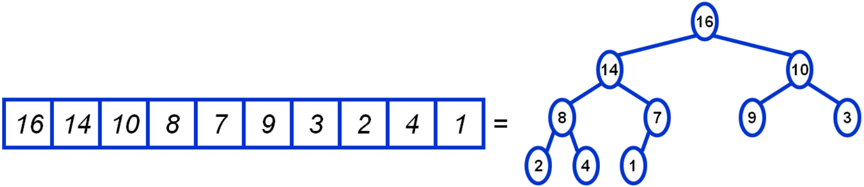
- A heap is a complete binary tree



- *Binary* because each node has at most two children
- *Complete* because each internal node, except possibly at the last level, has exactly two children

Heaps

- In practice, heaps are usually implemented as arrays



Heaps

- How to represent a complete binary tree as an array :
  - The root node is $A[1]$
  - Ordering nodes per levels starting at the root, and from left to right in a same level, then node i is $A[i]$
  - The parent of node i is $A[\lfloor i/2 \rfloor]$
  - The left child of node i is $A[2i]$
  - The right child of node i is $A[2i+1]$

16	14	10	8	7	9	3	2	4	1

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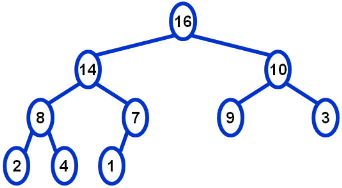


Referencing Heap Elements

function Parent(i)
  return $\lfloor i/2 \rfloor$;

function Left(i)
  return $2 \times i$;

function Right(i)
  return $2 \times i + 1$;

The Heap Property

- Heaps must satisfy the following relation :

$$A[Parent(i)] \geq A[i] \text{ for all nodes } i > 1$$

- In other words, the value of a node is at most the value of its parent

Heap Height

- The height of a node in the tree = the number of edges on the longest downward path to a leaf
- The height of a tree = the height of its root
- What is the height of an n-element heap ? Why ?
- Heap operations take at most time proportional to the height of the heap

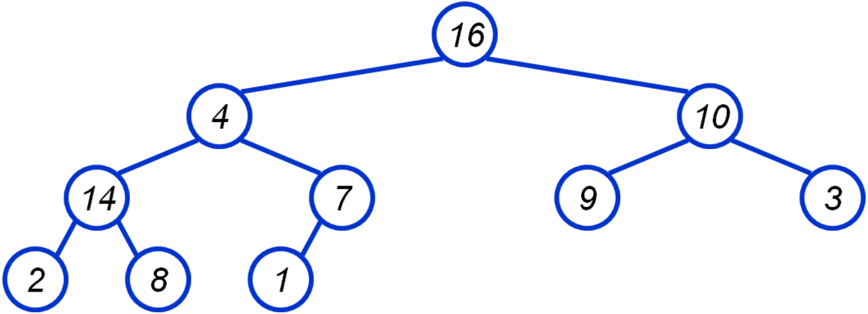
Heap Operations

There are two main heap operations : heapify and buildheap.

Heapify() : restore the heap property :

- Consider node i in the heap with children l and r
- Nodes l and r are each the root of a subtree, each assumed to be a heap
- Problem : Node i may violate the heap property
- Solution : let the value of node i "float down" in one of its two subtrees until the heap property is restored at node i

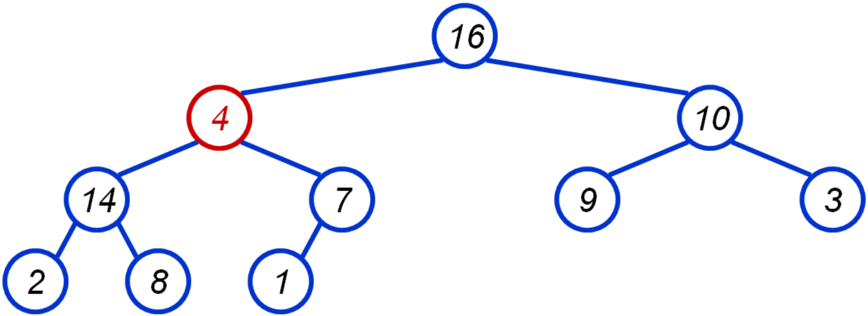
Heapify() Example



$A =$

16	4	10	14	7	9	3	2	8	1

Heapify() Example



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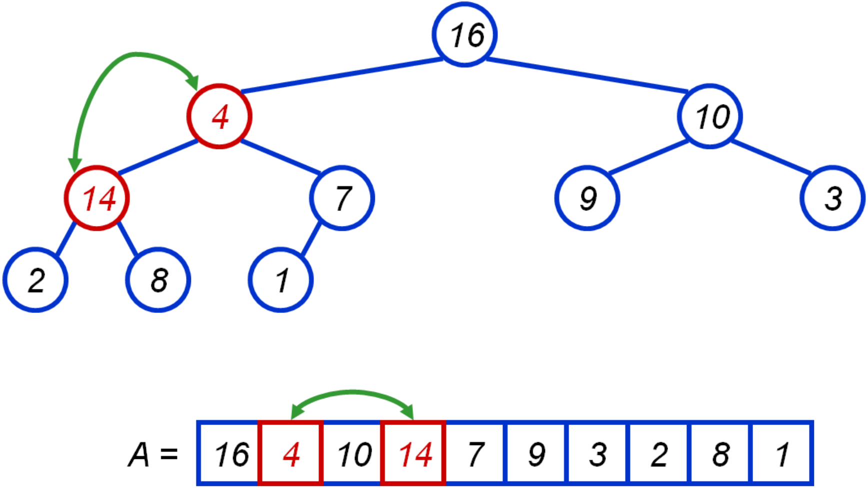
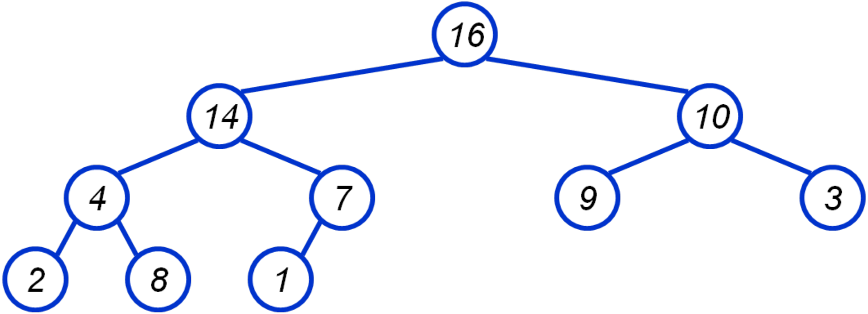
16	4	10	14	7	9	3	2	8	1

Heapify() Example



Heapify() Example



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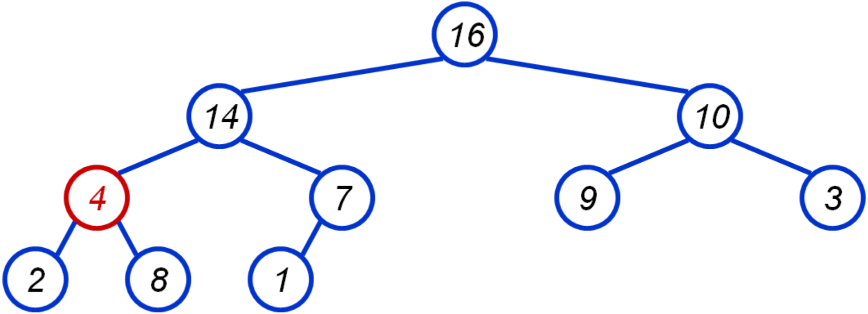
Heapify() Example



A =	16	14	10	4	7	9	3	2	8	1

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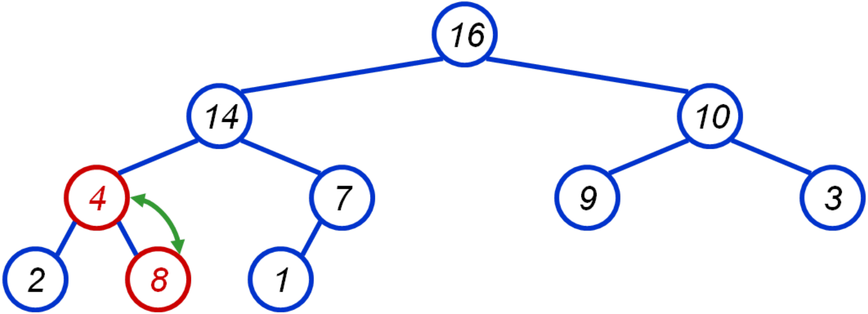
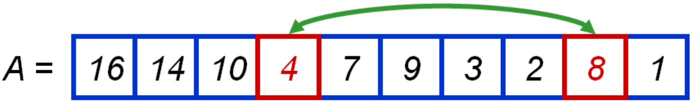
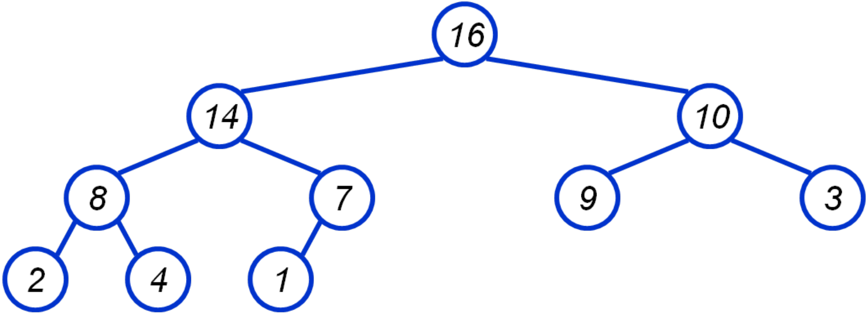


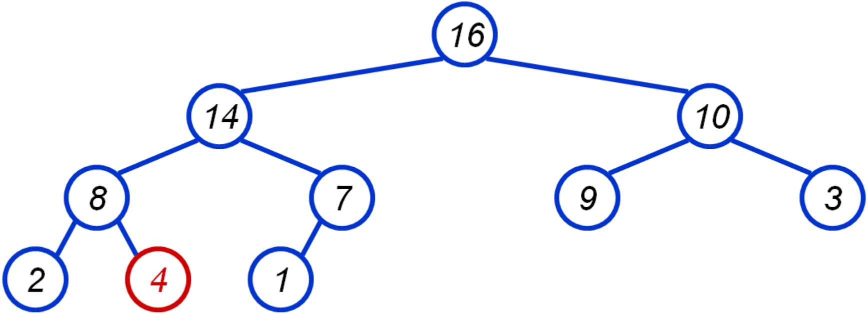


Heapify() Example



A =									
16	14	10	8	7	9	3	2	4	1

Heapify() Example



$A =$

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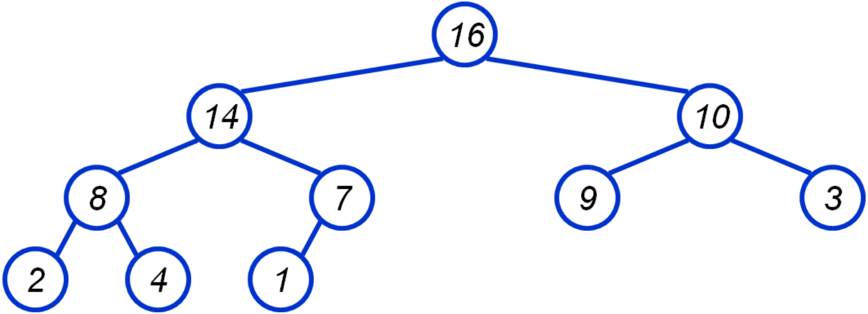
Heapify() Example



$A =$

16	14	10	8	7	9	3	2	4	1

Algorithm for heapify

An array $A[]$, where heap_size(A) returns the dimension of A

```
Heapify(A, i)
  l = Left(i) ; r = Right(i) ;
  if (l ≤ heap_size(A) & A[l] > A[i])
    largest = l ;
  else
    largest = i ;
  if (r ≤ heap_size(A) & A[r] > A[largest])
    largest = r ;
  if (largest != i)
    Swap(A, i, largest) ;
    Heapify(A, largest) ;
```

Example : heapify

This array $A = [23, 11, 14, 9, 13, 10, 1, 5, 7, 12]$ is not a heap as Parent(5) $= \lfloor \frac{i}{2} \rfloor = 2$, A[2] $=$ 11 in the array, which violates the max-heap property as A[5] $=$ 13 is greater than A[2].

Call heapify(A,2) which swap A[2] with $Right(2) = 2i + 1 = A[5]$

$$A = [23, 13, 14, 9, 11, 10, 1, 5, 7, 12]$$

Left(5) $=$ 10 and $A[10] > A[5]$ which violates the heap property $A[Parent(i)] \geq A[i]$. Thus heapify continues, swap A[5] with $Left(5) = 2i = A[10]$

$$A = [23, 13, 14, 9, 12, 10, 1, 5, 7, 11]$$

Analyzing Heapify()

- Number of basic operations performed before calling itself ?
- How many times can Heapify() recursively call itself ?
- What is the worst-case running time of Heapify() on a heap of size n ?

Analyzing Heapify()

- The work done in Heapify() is in $O(1)$
- If the heap at i has n elements, how many elements can the subtrees at l or r have? Answer : at most 2n/3 (worst case : bottom row 1/2 full)
- So time taken by Heapify() is given by the recurrence

$$T(n) \leq T(2n/3) + \Theta(1)$$

- Master Theorem applies to solve this recurrence, which corresponds to the case 2 of the restricted Master Theorem.

$$T(n) \in \Theta(\log n)$$

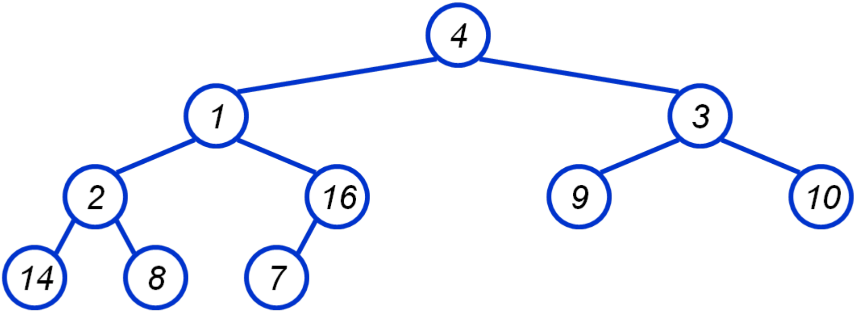
Heap Operations : BuildHeap()

- We can build a heap in a bottom-up manner by running Heapify() on successive subarrays
  - Note : for array of length n, all elements in range $A[\lfloor n/2 \rfloor + 1..n]$ are heaps (Why ?)
  - Walk backwards through the array from n/2 to 1, calling Heapify() on each node.
- given an unsorted array A, make A a heap

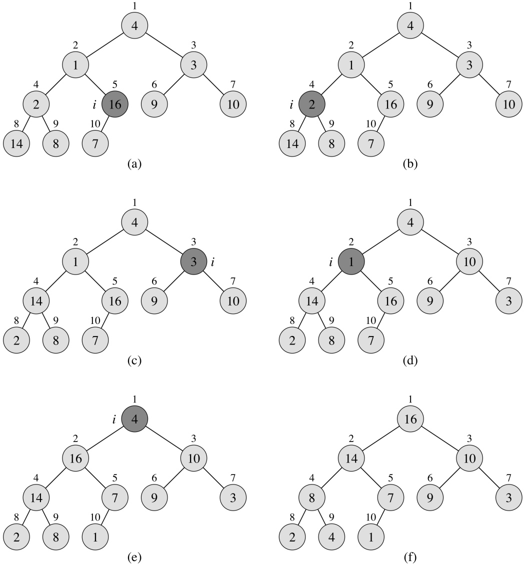
```
BuildHeap(A)
  heap_size(A) = length(A);
  for (i = ⌊length(A)/2⌋ downto 1)
    Heapify(A, i);
```

BuildHeap() Example

Work through example $A = [4, 1, 3, 2, 16, 9, 10, 14, 8, 7]$



A	4	1	3	2	16	9	10	14	8	7



BuildHeap : a second example

Run the algorithm $BuildHeap(A)$ on the array $A = [5, 3, 17, 10, 84, 19, 6, 22, 9]$

Find $i = \lfloor \frac{length(A)}{2} \rfloor = \lfloor \frac{9}{2} \rfloor = 4$, the entry in A where BuildHeap starts

BuildHeap starts at $A[4]$, from which heapify is run. Left(4) $= 8$, A[8] $= 10$; Right(4) $= 9$, A[9] $= 9$, so nothing to change

Next $A[3] = 17$, Left(3) $= 6$, A[6] $= 19$; Right(3) $= 7$, A[7] $= 6$. $A[6] > A[3]$, so heapify is needed on $A[3]$, yielding
$A[5, 3, 17, 22, 84, 19, 6, 10, 9]$
$A[5, 3, 19, 22, 84, 17, 6, 10, 9]$

Next $A[2] = 3$, and so on

$A[5, 84, 19, 22, 3, 17, 6, 10, 9]$
$A[84, 5, 19, 22, 3, 17, 6, 10, 9]$
$A[84, 22, 19, 5, 3, 17, 6, 10, 9]$
$A[84, 22, 19, 10, 3, 17, 6, 5, 9]$

Analyzing BuildHeap()

- Each call to Heapify() takes $O(\log n)$ time
- There are $O(n)$ such calls (specifically, $\lfloor n/2 \rfloor$)
- Thus the running time is $O(n \log n)$
  - Is this a correct asymptotic upper bound ?
  - Is this an asymptotically tight bound ?
- A tighter bound is $O(n)$

Analyzing BuildHeap() : Tight

Heap-properties of an n-element heap

- Height $= \lfloor \log n \rfloor$
- At most $\lceil \frac{n}{2^{h+1}} \rceil$ nodes of any height h
- The time for Heapify on a node of height h is $O(h)$

$$
\begin{aligned}
\sum_{h=0}^{\lfloor \log n \rfloor} \lceil \frac{n}{2^{h+1}} \rceil O(h) &= O(n \sum_{h=0}^{\lfloor \log n \rfloor} \frac{h}{2^h}) \\
&= O(n \sum_{h=0}^{\infty} \frac{h}{2^h}) \\
&= O(n)
\end{aligned}
$$

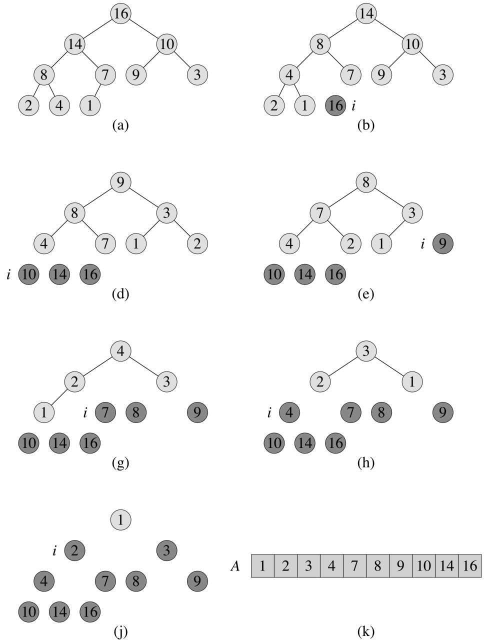
Analyzing BuildHeap() : Tight

$$
\begin{aligned}
\sum_{h=0}^{\infty} \frac{h}{2^h} &= \sum_{h=0}^{\infty} h(\frac{1}{2})^h \\
&= \sum_{h=0}^{\infty} hx^h \text{ where } x = \frac{1}{2} \\
&= \frac{1/2}{(1-\frac{1}{2})^2} \text{ the closed form of } \sum_{h=0}^{\infty} h(\frac{1}{2})^h \\
&= 2
\end{aligned}
$$

Heapsort

- Given BuildHeap(), a sorting algorithm is easily constructed :
  - Maximum element is at A[1]
  - Swap A[1] with element at A[n], A[n] now contains correct value
  - Decrement heap_size[A]
  - Restore heap property at A[1] by calling Heapify()
  - Repeat, always swapping A[1] for A[heap_size(A)]

```
Heapsort(A)
  BuildHeap(A) ;
  for (i = length(A) downto 2)
    Swap(A[1], A[i]) ;
    heap_size(A) = heap_size(A) - 1 ;
    Heapify(A, 1) ;
```


(a)
(b)
(c)
(d)
(e)
(f)
(g)
(h)
(i)
(j)
A 1 2 3 4 7 8 9 10 14 16
(k)

Analyzing Heapsort

- The call to BuildHeap() takes $O(n)$ time
- Each of the n - 1 calls to Heapify() takes $O(\log n)$ time
- Thus the total time taken by HeapSort()

$$
\begin{aligned}
&= O(n) + (n-1)O(\log n) \\
&= O(n) + O(n \log n) \\
&= O(n \log n)
\end{aligned}
$$

Note, like merge sort, the running time of heapsort is independent of the initial state of the array to be sorted. So best case and average case of heapsort are in $O(n \log n)$

Priority Queues

- Heapsort is a nice algorithm, but in practice Quicksort is faster
- But the heap data structure is useful for implementing priority queues :
  - A data structure like queue or stack, but where a value or key is associated to each element, representing the priority of the corresponding element. The element with the highest priority is served first
  - Supports the operations Insert(), Maximum(), and ExtractMax()

Priority Queue Operations

- **function** Insert(S, x) inserts the element x into set S
- **function** Maximum(S) returns the element of S with the maximum key
- **function** ExtractMax(S) removes and returns the element of S with the maximum key
- Think how to implement these operations using a heap ?

Priority queue : extracting the max element

```
ExtractMax(A)
   max = A[1]
   A[1] = A[A.heap-size]
   A.heap-size = A.heap-size -1
   Heapify(A,1)
   return max
```

Since Heapify runs in $\log n$, extracting the largest element of a priority queue based on a heap takes $\log n$

Priority queue : inserting an element

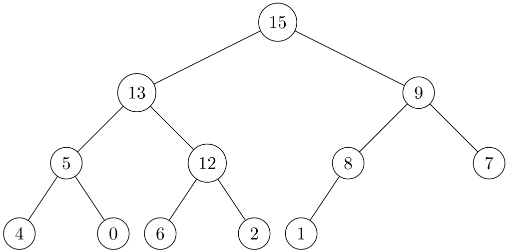
```
Insert(A, key)
   A.heap-size = A.heap-size + 1
   A[A.heap-size] = key
   i = A.heap-size
   while i > 1 and A[Parent(i)] < A[i]
      swap(A[i], A[Parent(i)])
      i = Parent(i)
```

The number of iterations execute by the while loop is bound above by $\log n$, therefore inserting an element of a priority queue based on a heap takes $\log n$

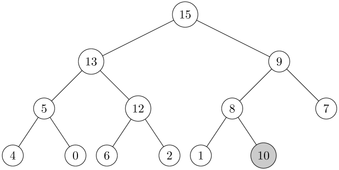
Example : Inserting an element

$Insert(A, 10)$ on the heap $A = [15, 13, 9, 5, 12, 8, 7, 4, 0, 6, 2, 1]$

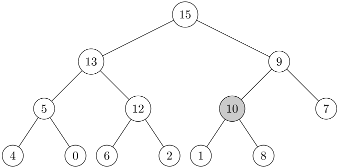
- Original heap



- Add the key 10 to the next position in the heap (corresponding to a new entry extending the array A).



- Since the parent key is smaller than 10, the nodes are swapped



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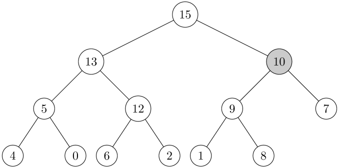

New exercises

1. Convert the array $A = [10, 26, 52, 76, 13, 8, 3, 33, 60, 42]$ into a maximum heap
2. Is this array [23, 17, 14, 6, 13, 10, 1, 5, 7, 12] a heap ? If not make it a heap.
3. Run the algorithm $BuildHeap(A)$ on the array $A = [12, 28, 36, 1, 37, 13, 4, 25, 3]$. Show each step of your work using the array representation of the modified heap
4. Heapsort $A[12, 28, 4, 37]$. Important, show each step of your work using the array representation
5. Heapsort $A = [2567563212968244]$ (very long)

New exercises continue

6. What are the minimum and maximum numbers of elements in a heap of height h?
7. Where in a heap might the smallest element reside?
8. Is an array that is in reverse sorted order a heap?
9. Using the example Insert(A,10) in your class notes, show the steps in the execution of Insert(A,3) on the priority queue $A = [15, 13, 9, 5, 12, 8, 7, 4, 0, 6, 2, 1]$ implemented using a heap
10. Similarly to the previous question, show the steps in the execution of ExtractMax(A) on the priority queue $A = [15, 13, 9, 5, 12, 8, 7, 4, 0, 6, 2, 1]$ implemented using a heap

New exercises continue

11. A d-ary heap is like a binary heap, but instead of 2 children, nodes have d children.
    11.1 Explain how would you represent a 3-ary heap in an array, i.e. give the formulas for $Parent(i)$, $Left(i)$ and $Right(i)$
    11.2 What is the height of a 3-ary heap of n elements ?
    11.3 Sketch the idea of a heapify routine for a 3-ary heap
    11.4 Give an implementation of ExtractMax() for a priority queue based on a 3-ary heap
12. Show how to implement a regular FIFO queue using a "min"-priority queue
13. Show how to implement a stack using a "max"-priority queue

Exercises

1. Insertion sort and merge sort are stable algorithms while heapsort and quicksort are not. Can you explain why this is so ? Solution : Because for insertion and merge sort, two objects with equal keys appear in the same order in sorted output as they appear in the input array to be sorted, which is not necessary the case for heapsort and quicksort.
2. Run $Heapify(A, 3)$ on the array $A = [27, 17, 3, 16, 13, 10, 1, 5, 7, 12, 4, 8, 9, 0]$
   Solution : $A[27, 17, 3, 16, 13, 10, 1, 5, 7, 12, 4, 8, 9, 0]$
   $A[27, 17, 10, 16, 13, 3, 1, 5, 7, 12, 4, 8, 9, 0]$
   $A[27, 17, 10, 16, 13, 9, 1, 5, 7, 12, 4, 8, 3, 0]$

Exercises

4. *Heapify*(A, i) in the class notes is a recursive algorithm. Write an equivalent iterative algorithm.

```
MAX-HEAPIFY(A, i)
  while true
    l = LEFT(i)
    r = RIGHT(i)
    if l <= A.heap-size and A[l] > A[i]
      largest = l
    else largest = i
    if r <= A.heap-size and A[r] > A[largest]
      largest = r
    if largest == i
      return
    exchange A[i] with A[largest]
    i = largest
```

Exercises

6. Run $Heapsort(A)$ on the the array $A = [3, 15, 2, 29, 6, 14, 25, 7, 5]$
   $A[5, 13, 2, 25, 7, 17, 20, 8, 4]$
   $A[5, 13, 20, 25, 7, 17, 2, 8, 4]$
   $A[5, 25, 20, 13, 7, 17, 2, 8, 4]$
   $A[25, 5, 20, 13, 7, 17, 2, 8, 4]$
   $A[25, 13, 20, 5, 7, 17, 2, 8, 4]$
   $A[25, 13, 20, 8, 7, 17, 2, 5, 4]$
   $A[4, 13, 20, 8, 7, 17, 2, 5, 25]$
   $A[20, 13, 4, 8, 7, 17, 2, 5, 25]$
   $A[20, 13, 17, 8, 7, 4, 2, 5, 25]$
   $A[5, 13, 17, 8, 7, 4, 2, 20, 25]$
   $A[17, 13, 5, 8, 7, 4, 2, 20, 25]$
   $A[2, 13, 5, 8, 7, 4, 17, 20, 25]$
   $A[13, 2, 5, 8, 7, 4, 17, 20, 25]$
   $A[13, 8, 5, 2, 7, 4, 17, 20, 25]$
   $A[4, 8, 5, 2, 7, 13, 17, 20, 25]$
   $A[8, 4, 5, 2, 7, 13, 17, 20, 25]$
   $A[8, 7, 5, 2, 4, 13, 17, 20, 25]$
   $A[4, 7, 5, 2, 8, 13, 17, 20, 25]$
   $A[7, 4, 5, 2, 8, 13, 17, 20, 25]$
   $A[2, 4, 5, 7, 8, 13, 17, 20, 25]$
   $A[5, 4, 2, 7, 8, 13, 17, 20, 25]$
   $A[2, 4, 5, 7, 8, 13, 17, 20, 25]$
   $A[4, 2, 5, 7, 8, 13, 17, 20, 25]$
   $A[2, 4, 5, 7, 8, 13, 17, 20, 25]$

Exercises

What is the running time of *Heapsort* on an array A of length n that is sorted in decreasing order? Solution : It is the same as if the array was already sorted in increasing order. The algorithm will need to convert it to a heap that will take $O(n)$. Afterwards, however, there are $n - 1$ calls to $heapify()$, each call performs $\log n$ operations.