Laplace transform and applications to ODEs

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Content

Advantages of Laplace transform in ODEs

- Many practical engineering problems involve mechanical or electrical systems acted on by discontinuous or impulsive forcing terms.
  Ex: electrical circuit $LQ''(t) + RQ'(t) + \dfrac{1}{C}Q(t) = E(t) \Rightarrow$ discontinuous RHS.
- Solve linear ODEs with variable coefficients.
  Ex: $xy'' - 2y' + xy = 0$.
- More general problems.

Content

Definition

Definition

Let $f(t)\colon [0,+\infty) \to \mathbb{R}$. Laplace transform of $f(t)$ is the function:

$$F(s) = \mathcal{L}\{f\}(s) = \int_0^{+\infty} e^{-st} f(t)dt, \qquad s \in \mathbb{R}.$$

Example

1. $f(t) = e^{at} \Rightarrow \mathcal{L}\{e^{at}\}(s) = \dfrac{1}{s-a}, s > a.$
2. $f(t) = t^a, a > -1 \Rightarrow \mathcal{L}\{t^a\}(s) = \dfrac{\Gamma(a+1)}{s^{a+1}}, s > 0.$

Gamma function $\Gamma(p) = \int\limits_0^{+\infty} e^{-t} t^{p-1} dt,\ p > 0.$

Theorem (Existence of Laplace transform)

Let $f(t)$ be a function that is defined and piecewise continuous on every finite intervals $[0, T]$, $T \geq 0$. Moreover, assume that $f(t)$ is exponentially bounded, i.e., there exists $M > 0, c, t_0 \geq 0$ such that

$$|f(t)| \leq M.e^{ct}, \forall t \geq t_0.$$

Then the Laplace transform $\mathcal{L}\{f\}(s)$ exists for all $s > c$.

Sketch of proof.

Example

$f(t) = \sin t, \cos t, e^{at}, t^{a}, a > 0.$

Heaviside function $u(t-a) = \begin{cases} 1 & \text{when } t \geq a \\ 0 & \text{when } 0 \leq t < a, \end{cases} \qquad a > 0.$

From now on, we assume $f(t), g(t)$ are piecewise continuous and exponentially bounded (then their Laplace transforms exist).

Proposition

$$\mathcal{L}\{Af + Bg\}(s) = A\mathcal{L}\{f\}(s) + B\mathcal{L}\{g\}(s).$$

Fundamental Laplace transforms

$f(t)$	$F(s)$	$f(t)$	$F(s)$
1	$\frac{1}{s}, s > 0$	e^{at}	$\frac{1}{s-a}, s > a$
$\sin kt$	$\frac{k}{s^2+k^2}, s > 0$	$\cos kt$	$\frac{s}{s^2+k^2}, s > 0$
$t^a, a > -1$	$\frac{\Gamma(a+1)}{s^{a+1}}, s > 0$	$t^n, n \in \mathbb{N}^*$	$\frac{n!}{s^{n+1}}, s > 0$
$\sinh kt$	$\frac{k}{s^2-k^2}, s > \lvert k \rvert$	$\cosh kt$	$\frac{s}{s^2-k^2}, s > \lvert k \rvert$
$u(t-a),\ a \geq 0$	$\frac{e^{-as}}{s}, s > 0$		

Recall: $\Gamma(p+1) = p\Gamma(p)$, $\Gamma(n+1) = n!$, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Example

Find $\mathcal{L}\{f\}(s)$:

1. $f(t) = 3t^2 + 4\sqrt{t}$,
2. $f(t) = \sinh 3t - 2\cos 2t - u(t-2)$,
3. $f(t) = (e^{-t} + 1)^2$.

Content

Theorem

If $\mathcal{L}\{f\}(s) = \mathcal{L}\{g\}(s)$, $\forall s > c$, then $f(t) = g(t)$ at points of continuity of f and g.

Definition

If $F(s) = \mathcal{L}\{f\}(s)$, we call $f(t)$ the inverse Laplace transform of $F(s)$, and denote
$$f(t) = \mathcal{L}^{-1}\{F\}(t).$$

$$\boxed{F(s) = \mathcal{L}\{f\}(s) \Leftrightarrow f(t) = \mathcal{L}^{-1}\{F\}(t)}$$

$$f(t) \overset{\mathcal{L}}{\mapsto} F(s), \quad F(s) \overset{\mathcal{L}^{-1}}{\mapsto} f(t).$$

Proposition

$$\mathcal{L}^{-1}\{AF + BG\}(t) = A\mathcal{L}^{-1}\{F\}(t) + B\mathcal{L}^{-1}\{G\}(t)$$

Example

Find $\mathcal{L}^{-1}\{F\}(t)$:

1. $F(s) = \dfrac{3}{s^2} - \dfrac{1}{s\sqrt{s}}$,
2. $F(s) = \dfrac{e^{-2s}}{s} + \dfrac{1}{s-5}$,
3. $F(s) = \dfrac{1}{s^2 - 4s + 3}$.

Content

Shifting on $s-$domain

Theorem

$$\mathcal{L}\{e^{at}f\}(s) = F(s-a),$$
$$\mathcal{L}^{-1}\{F(s-a)\}(t) = e^{at}f(t).$$

Proof.

Example

Calculate

1. $\mathcal{L}\{e^{-2t}\sin t + e^{t}\cos 2t\}(s)$.
2. $\mathcal{L}^{-1}\{\dfrac{s+2}{s^2-6s+10}\}(t)$.

Derivatives on $t-$domain

Common assumptions: $f(t)$ is piecewise continuous and exponentially bounded, its Laplace transform $L\{f\}(s) = F(s)$.

Theorem

Assume further that $f(t)$ is differentiable, then

$$\mathcal{L}\{f'\}(s) = sF(s) - f(0).$$

Proof.

Generally,

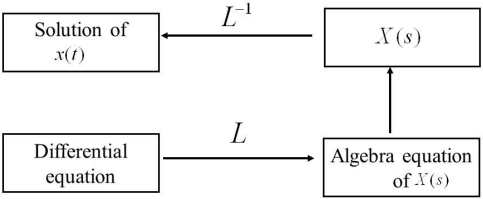
$$\mathcal{L}\{f^{(n)}\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0).$$

Example

Calculate

1. $\mathcal{L}\{\cos kt\}(s)$.
2. $\mathcal{L}\{te^{at}\}(s)$.

Application to linear ODEs with constant coefficients



Example

Solve the following ODEs:

1. $x'' - x' - 2x = 0,\ x(0) = 0, x'(0) = 2.$
2. $x'' + 4x = \cos t,\ x(0) = x'(0) = 0.$

Integrals on $t-$domain

Theorem

$$\mathcal{L}\left\{\int_0^t f(v)dv\right\}(s) = \frac{F(s)}{s}.$$

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\}(t) = \int_0^t f(v)dv.$$

Example

Calculate

1. $\mathcal{L}^{-1}\Big\{\dfrac{1}{s(s^2+4)}\Big\}(t)$.
2. $\mathcal{L}^{-1}\Big\{\dfrac{1}{s^2(s+4)}\Big\}(t)$.

Derivatives on $s-$domain

Theorem

$$\mathcal{L}\{-tf(t)\}(s) = F'(s).$$

$$\mathcal{L}^{-1}\{F'\}(t) = -tf(t).$$

Proof.

Generally,

$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n F^{(n)}(s).$$

Example

Calculate $\mathcal{L}\{t \sin kt\}(s)$.

Example (Application to linear ODEs)

Solve the following ODEs:

1. $tx'' + (3t-1)x' + 3x = 0, x(0) = 0.$