Chapter-5

Tuesday, February 18, 2025 12:23 AM

✓ 2 marks (exam) #

Feature	Gradient ($\nabla \phi$)	Divergence ($\nabla \cdot \mathbf{F}$)
Definition	Measures the rate and direction of change of a scalar field.	Measures the net outward flux (spread) of a vector field from a point.
Input	Scalar field (ϕ)	Vector field ($\mathbf{F}$)
Output	Vector field ($\nabla \phi$)	Scalar field ($\nabla \cdot \mathbf{F}$)

Current

→ electric charges in motion constitute a current. The unit of current is the ampere (A), defined as the rate of movement of charge passing a given reference point (or crossing a given reference plane) of one coulomb per second. Current is symbolized by I and therefore

$$I = \frac{dq}{dt}$$

→ Current is thus defined as the motion of positive charges, even though conduction in metals takes place through the motion of electrons.

Current Density

→ The current density is defined as the current passing through a unit surface area. It is a vector quantity and its direction is the direction of current flow and is denoted by $\vec{J}$. The unit is A/m^2.

→ The incremental current ΔI crossing an incremental surface ΔS normal to the current density is :-

$$\Delta I = J_N \Delta S$$

If the current density is not perpendicular to the surface then,

$$\Delta I = \vec{J} \cdot \vec{\Delta s}$$

Total current is:-

$$I = \int_s \vec{J} \cdot \vec{ds}$$

Consider that ΔQ of charge moves a distance Δx in time Δt. The corresponding current is:-

$$\Delta I = \frac{\Delta Q}{\Delta t} = \frac{\rho_v \Delta V}{\Delta t} = \frac{\rho_v \Delta S \Delta x}{\Delta t}$$

or, $\Delta I = \rho_v \Delta S \dfrac{\Delta x}{\Delta t}$ or, $\Delta I = \rho_v \Delta S v_x$ (v_x = x - comp of velocity)

or, $\dfrac{\Delta I}{\Delta S} = \rho_v v_x$

$\therefore$ $J_x = \rho_v v_x$ conventional current density #

$$\boxed{\text{In general } \vec{J} = \rho_v \vec{v}}$$

(VIMP)

Principal of Conservation of charge and Continuity Equation

→ It states that the charges can neither be created nor be destroyed, although equal amounts of the positive and the negative charges may be simultaneously created by separation and destroyed by recombination.

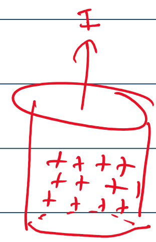
es: the positive charges on the glass rod and the negative charges on a piece of silk are simultaneously created when they are rubbed together.

Continuity of Current (continuity Equation)

→ The continuity equation is based on the principle of conservation of charge.

Consider a region bounded by a closed surface. The current through the closed surface is :-

$$I = \oint_s \vec{J} \cdot \vec{ds}$$



and this outward flow of positive charge must be balanced by a decrease of positive charge (or perhaps an increase of negative charge) with the closed surface. If the charge inside the closed surface is denoted by Q_i, the rate of decrease is $-\frac{dQ_i}{dt}$ and the principle of conservation of charge requires

$$I = \oint_s \vec{J} \cdot \vec{ds} = -\frac{dQ_i}{dt} \quad \text{—— (i)}$$

where -ve sign is interpreted as an outward flowing current.

Eqn (i) is the intergal form of continuity equation

Using the divergence theorem

$$\oint_S \vec{J} \cdot \vec{ds} = \int_{vol} (\nabla \cdot \vec{J})\, dv$$

$$Q = \int_{vol} \rho_v \, dv$$

$$\phi = \int_{vol} \rho_v \, dv$$

$\therefore$ Equation (i) becomes

$$\int_{vol} (\nabla \cdot \vec{J}) \, dv = -\frac{dQ_i}{dt} = -\frac{d}{dt}\int_{vol} \rho_v \, dv$$

In moving the time derivative of ρ_v inside the volume integral, it is necessary to use partial differentiation as ρ_v may be a function of time as well as of the space co-ordinates.

$$\int_{vol} (\nabla \cdot \vec{J}) \, dv = \int_{vol} -\frac{\partial \rho_v}{\partial t} \, dv$$

$$\boxed{\therefore \nabla \cdot \vec{J} = -\frac{\partial \rho_v}{\partial t}}$$

side note
for steady currents, charge density does not vary with time, $\frac{\partial \rho_v}{dt} = 0$ and $\nabla \cdot \vec{J} = 0$, thus steady currents are divergenceless or solenoidal #

Which is the expression of the continuity equation in differential or point form.

Point Form of Ohm's law

(2 mark derive)

→ Consider a conductor as shown having uniform value of current density $\vec{J}$ and electric field intensity $\vec{E}$. If l is the length of the conductor.

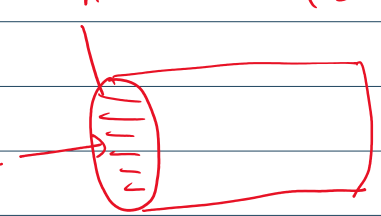


Fig : current through a conductor

The resistivity of the material is ÷

$$\rho = \frac{RA}{l}\ \Omega m \qquad \left(\rho = \frac{RA}{l} \right)$$

The conductivity of the material is ÷

$$\sigma = \frac{1}{\rho} = \frac{l}{RA}\ \frac{mho}{m} \quad \text{— (i)}$$

The potential difference between the sides of an infinitely small volume dv enclosed by the surface ds is given by ÷

$$V = EL \quad \text{— (ii)} \qquad \left(V = Ed \right)$$

Total current passing through area A is ÷

$$I = JA \quad \text{— (iii)} \qquad J = \frac{I}{A}$$

Ohm's law is ÷

$$V = IR$$

Using (ii) and (iii)

$$\rightarrow EL = JAR$$

or, $$J = \frac{EL}{AR}$$

or $$J = \frac{L}{AR} \cdot E$$

Using (i)

$$J = \sigma E \qquad \left(\sigma = \frac{L}{AR} \right)$$

In vector form $\boxed{\vec{J} = \sigma \vec{E}}$

This equation is known as point form of Ohm's law

Relaxation Time Constant (4-marks derivation) (V. imp)

→ Any charge placed inside the conductor moves towards the surface and eventually appears on the surface of the conductor as shown in the figure below.

Fig : the charges decaying at a point within the conductor and reappearing on the surface

A time constant that shows how fast the charges decay at a point within the conductor and reappear on the surface is termed as the relaxation time constant (RTC) for the conductor.

The continuity equation is :-

$$(\nabla \vec{J}) = -\frac{\partial \rho_v}{\partial t} \qquad \text{—(i)}$$

The point form of Ohm's law is :-

$$\vec{J} = \sigma \vec{E} \quad \text{—— (ii)}$$

From equations (i) and (ii)

$$\nabla \cdot (\sigma \vec{E}) = -\frac{\partial \rho_v}{\partial t}$$

free space

$$\nabla \cdot \left(\frac{\sigma}{\varepsilon} \vec{D} \right) = -\frac{\partial \rho_v}{\partial t} \qquad \left(\vec{D} = \varepsilon_0 \vec{E} \right)$$

conductor $\rightarrow$ $\vec{D} = \varepsilon \vec{E}$ $\rightarrow$ $\varepsilon = \varepsilon_r \varepsilon_0$

Assuming a homogeneous medium where σ and ε do not vary with position,

$$\nabla \cdot \vec{D} = -\frac{\varepsilon}{\sigma} \frac{\partial \rho_v}{\partial t}$$

Maxwell 1st equal is :- $\nabla \cdot \vec{D} = \rho_v$

So

$$\rho_v = -\frac{\varepsilon}{\sigma} \frac{\partial \rho_v}{\partial t}$$

or $\frac{\varepsilon}{\sigma} \frac{\partial \rho_v}{\partial t} + \rho_v = 0$

or $\frac{\partial \rho_v}{\partial t} + \frac{\sigma}{\varepsilon} \rho_v = 0 \qquad \left(\frac{dx}{dt} + kx = 0 \right)$

$\rightarrow$ soln :- $x = x_0 e^{-kt}$

For simplicity let us suppose that σ is not a function of ρ_v Then, we can obtain solution as :-

$$\rho_v = \rho_0 e^{-\left(\frac{\sigma}{\varepsilon}\right) t} \quad \text{—— (iii)}$$

Where, ρ_0 = charge density at $t = 0$

Equation (iii) shows an exponential decay of charge density at every point with a time constant of $\frac{\varepsilon}{\sigma}$

$$\rho_v = \rho_0 e^{-\frac{t}{T}} \quad \text{where} \quad T = \frac{\varepsilon}{\sigma}$$

When $t = T$, $\rho_v = \rho_0 e^{-1} = 0.37\rho_0$ ($t=2T$, $\rho_v = 0.14\rho_0$; $t=3T$, $\rho_v = 0.05\rho_0$)

i.e when time equals T, the charges within the conductor decays to 37% of the initial charges ρ_0 and simultaneously reappear on the conductor surface. This time is called the relaxation time constant for the conductor.

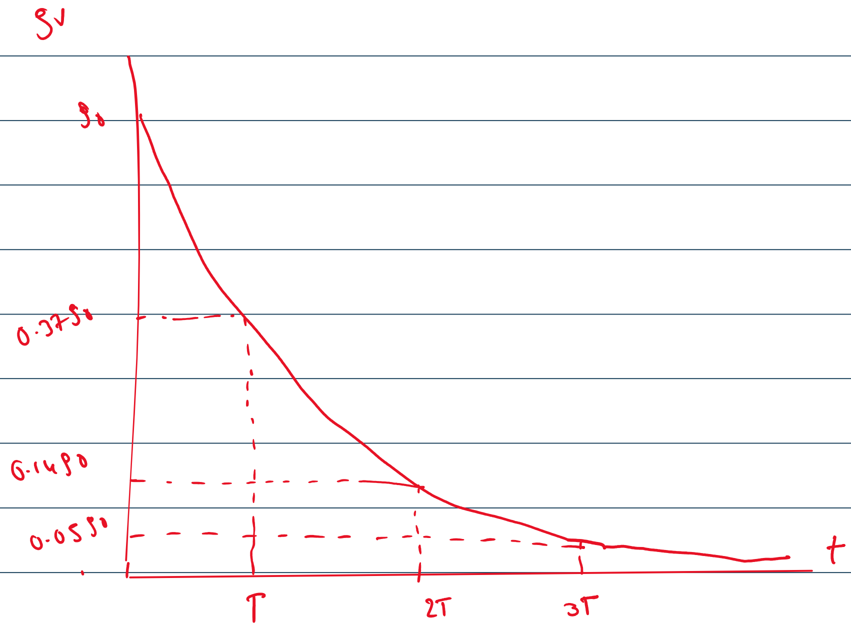


Fig: decaying of the charges within the conductors

2nd

4 Boundary Condition Between Conductor and Free Space (Chap 5, 6, 7, 8) 2mp (Derivation and num 2mp)

Properties of Perfect Conductors

1. No charge and no electric field may exist at any point within a conducting material. Charge may, however appear on the surface as a surface change density.
2. The conductor surface is equipotential surface.
3. The tangential component of an electric external field as well as electric flux density is zero.
4. The normal component of electric flux density is equal to the surface charge density on the conductor surface.

Derivation

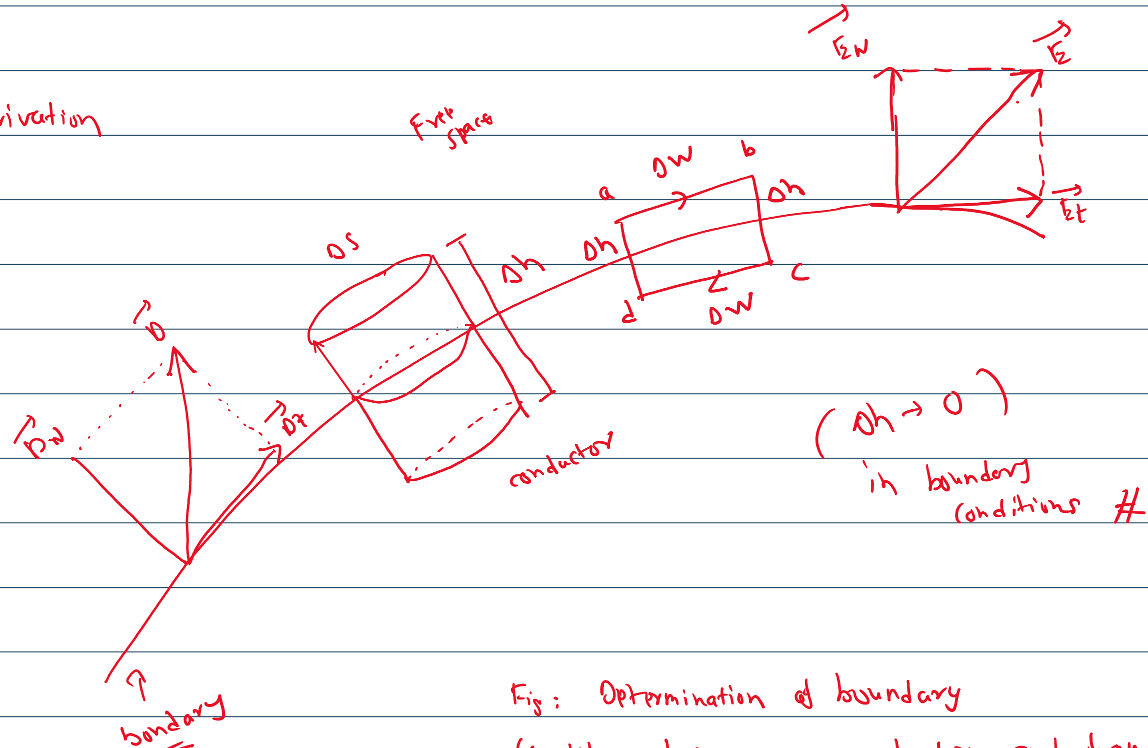


Fig: Determination of boundary conditions between a conductor and free space

Consider an arrangement which depicts a boundary between a conductor and free space showing the tangential and normal components of $\vec{D}$ and $\vec{E}$ on the free-space side of the boundary. Both fields are zero in the conductor.

($\rightarrow$ +ve $\leftarrow$ −ve) ($\uparrow$ +ve $\downarrow$ −ve)

(Direction)

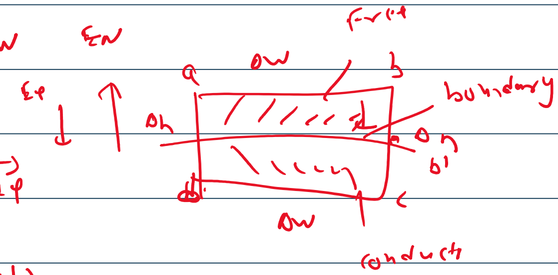
For calculating tangential field

$$\int_{abcda} \vec{E}\cdot\vec{dl} = 0$$

or $$\int_a^b \vec{E}\cdot\vec{dl} + \int_b^c \text{''} + \int_c^d \text{''} + \int_d^a \vec{E}\cdot\vec{dl} = 0$$

Now,

$$\overset{(\longrightarrow)}{\int_a^b \vec{E}\cdot\vec{dl}} = E_t\,\Delta w\cos 0 = E_t\,\Delta w$$



$$\int_b^c \vec{E}\cdot\vec{dl} = \int_{fs\,(b}^{b'} \vec{E}\cdot\vec{dl} + \int_{con\,(b')}^{c} \vec{E}\cdot\vec{dl}$$

$$= E_N \cdot \frac{1}{2}\Delta h\cos(180^\circ) \;(\downarrow) + 0 \quad (\text{within conductor})$$

$$= -\underbrace{E_N}_{\text{at } b}\frac{1}{2}\Delta h$$

$$\int_c^d \vec{E}\cdot\vec{dl} = 0 \quad (\text{within conductor} = 0)$$

$$\int_d^a \vec{E}\cdot\vec{dl} = \int_{d\,(con)}^{d'} \vec{E}\cdot\vec{dl} + \int_{d'}^{a} \vec{E}\cdot\vec{dl}$$

$$= 0 + E_{N\,at\,a}\,\frac{1}{2}\Delta h\cos 0^\circ \;(\uparrow)$$

$$= E_{N\,at\,a}\,\frac{1}{2}\Delta h$$

Now,

Now,

$$E_t \Delta w - \frac{1}{2}\Delta h\, E_{N,\text{at } b} + 0 + \frac{1}{2}\Delta h\, E_{N,\text{at } a} = 0$$

let ~~Δw~~ $\Delta h \to 0$ and keep Δw small and finite

or, $E_t \Delta w - 0 + 0 = 0$

or, $E_t \Delta w = 0$

Since $\Delta w \neq 0$

$$\boxed{E_t = 0}$$

For determining normal components

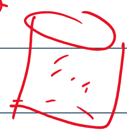
$$\oint_S \vec{D}\cdot\vec{ds} = \Delta Q$$

or, $$\int_{top} \vec{D}\cdot\vec{ds} + \int_{bottom} \vec{D}\cdot\vec{ds} + \int_{side} \vec{D}\cdot\vec{ds} = \Delta Q$$

or, $D_N \Delta S + 0 + 0 = \Delta Q$

The second term is zero because no electric field exists within a conductor and the third term is zero because the height of the cylinder Δh is infinitely small.

cylindrical surface



or, $D_N \Delta S = \rho_S \Delta S$ ($= \Delta Q$)

so, $D_N = \rho_S$

Hence, the desired boundary conditions for the conductor-to-free space boundary in electrostatics are :-

Hence, the desired boundary conditions for the conductor-to-free space boundary in electrostatics are ÷

$$\boxed{D_t = \varepsilon_0 E_t = 0 \,, \quad D_N = \varepsilon_0 E_N = \rho_s}$$

Similarly if we calculate boundary conditions between the conductor and dielectric, the results are ÷

$$\boxed{D_t = \varepsilon_0 \varepsilon_r E_t = 0 \,, \quad D_N = \varepsilon_0 \varepsilon_r E_N = \rho_s}$$