Chapter-5 Numericals

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① $\vec{J} = \frac{4}{r^2}\cos\theta\,\hat{a}_r + 20e^{-2r}\sin\theta\,\hat{a}_\theta - r\sin\theta\cos\phi\,\hat{a}_\phi$ A/m^2

ⓐ Find $\vec{J}$ at $r=3$, $\theta=0°$ and $\phi=\pi$

ⓑ Total current passing through $\underbrace{r=3}_{\text{const}}$, $\underbrace{0<\theta<20°}_{\text{not const}}$, $\underbrace{0<\phi<2\pi}_{\text{not const}}$ in $\hat{a}_r$ direction

so $\hat{a}_r$ direction

If not given then check which parameter is given as constant in question

Soln ÷

$$\vec{J} = \frac{4}{r^2}\cos\theta\,\hat{a}_r + 20e^{-2r}\sin\theta\,\hat{a}_\theta - r\sin\theta\cos\phi\,\hat{a}_\phi \quad A/m^2$$

(i) $\vec{J}$ at $(r=3,\ \theta=0°,\ \phi=\pi)$

$$\vec{J} = \frac{4}{3^2}\cos 0\,\hat{a}_r + 20e^{-2\times3}\sin 0\,\hat{a}_\theta - 3\sin 0\cos 180\,\hat{a}_\phi \quad A/m^2$$

$$= \frac{4}{9}\hat{a}_r \quad A/m^2$$

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(ii) $I = \int_s \vec{J}\cdot\vec{ds}$ $\quad\left(\vec{ds} = r^2\sin\theta\,d\theta\,d\phi\,\hat{a}_r\right)$

$$= \int_s \left[\frac{4}{r^2}\cos\theta\,\hat{a}_r + 20e^{-2r}\sin\theta\,\hat{a}_\theta - r\sin\theta\cos\phi\,\hat{a}_\phi\right]\left[r^2\sin\theta\,d\theta\,d\phi\,\hat{a}_r\right]$$

$$= \int_s \frac{4}{\cancel{r^2}}\cos\theta\cdot\cancel{r^2}\sin\theta\,d\theta\,d\phi$$

calc always radian

π rad $\to 180°$ $\quad \frac{\pi}{180}$ rad $\to 1°$ $\quad 20° \to \frac{20\pi}{180}$

$$= 2\int^{\pi/9}\sin 2\theta\,d\theta\int^{2\pi}d\phi \quad = 1.47A \ \#$$

$$= 2\int_0^{\pi/9} \sin 2\theta \, d\theta \int_{\phi=0}^{2\pi} d\phi \qquad = 1.47A \ \#$$

② An aluminium conductor is 1000ft long and has a circular cross section with a diameter of 0.8 inch. If there is a dc voltage of 1.2V between the ends, find:

ⓐ the current density ⓑ the current ⓒ Power dissapated

For AL, $\sigma = 3.82\times10^{7}$ mho/m

(1 ft = 12 inch, 1 inch = 0.0254m)

Soln ÷

(i) $\vec{J} = \sigma\vec{E}$

$l = 1000ft = 1000 \times 0.3048 \ m = 304.8m$

$V = 1.2V$

$$\vec{E} = \frac{V}{l} = \frac{1.2}{304.8} = 3.937\times10^{-3} \ V/m$$

Now

$J = \sigma\vec{E}$

$= 3.82\times10^{7} \times 3.937\times10^{-3} = 1.504\times10^{5} \ A/m^2$

(ii) $I = ?$

$V = IR$

$$R = \frac{\rho l}{A} = \frac{1}{\sigma}\frac{l}{A}$$

$$I = \frac{V}{R} = \frac{1.2}{3.82\times10^{7}} \times \frac{304.8}{\pi\,(0.4\times0.0254)^2} \qquad (8 \text{ inch})$$

$$= 48.772\,A$$

(c) $P = I^2 R$

$= 58.52\,N$

A current density in certain region is given as $\vec{J} = 20\sin\theta\cos\phi\,\hat{a}_r + \frac{1}{r}\hat{a}_\phi$ A/m^2. Find (i) the average value of J_r over the surface $r=1$, $0<\theta<\frac{\pi}{2}$, $0<\phi<\frac{\pi}{2}$ (ii) $\frac{\partial\rho_v}{\partial t}$

(along $\hat{a}_r$)

Soln:

(i) $$I = \int_S \vec{J}\cdot\vec{ds}$$ along $\hat{a}_r$ ✓

$$= \int_S \left(20\sin\theta\cos\phi\,\hat{a}_r + \frac{1}{r}\hat{a}_\phi\right)\left(r^2\sin\theta\,d\theta\,d\phi\,\hat{a}_r\right)$$

$$= \int_S 20\sin^2\theta\cos\phi\,r^2\,d\theta\,d\phi \qquad \left(\begin{matrix} r=1 \\ \text{const}\end{matrix}\right)$$

$$= 20\int_{\theta=0}^{\pi/2}\sin^2\theta\,d\theta\int_{\phi=0}^{\pi/2}\cos\phi\,d\phi$$

$$= 5\pi\ A$$

Now using

$$I = \int^{\pi/2}\int^{\pi/2} J_r\,(r^2\sin\theta\,d\theta\,d\phi) \qquad (r=1)$$

Now using

$$I = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{\pi/2} J_r \left(r^2 \sin\theta \, d\theta \, d\phi\right) \qquad (r=1)$$

$$5\pi = J_r \int_{\theta=0}^{\pi/2} \sin\theta \, d\theta \int_{\phi=0}^{\pi/2} d\phi$$

$$5\cancel{\pi} = J_r \times \frac{\cancel{\pi}}{2}$$

$$\therefore J_r = 10 \, A/m^2$$

Alternative

$$J_r = \frac{I}{\text{Area}}$$

$$\text{Area} = \int_S \vec{ds} \leftarrow \text{along } \hat{a}_r$$

$$= \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{\pi/2} \overset{(1)}{r^2} \sin\theta \, d\theta \, d\phi = \frac{\pi}{2} m^2$$

$$J_r = \frac{5\pi}{\frac{\pi}{2}} = 10 A/m^2$$

(ii) $\dfrac{\partial \rho_v}{\partial t}$

we know

$$\nabla \cdot \vec{J} = -\frac{\partial \rho_v}{\partial t}$$

So,

$$\frac{\partial \rho_v}{\partial t} = -(\nabla \cdot \vec{J})$$

$$= -\left[\frac{1}{r^2}\frac{\partial (r^2 J_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (J_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial (J_\phi)}{\partial \phi}\right]$$

$$= -\left[\frac{1}{r^2} \frac{\partial (r^2 J_r)}{\partial r} + \frac{1}{r\sin\theta} \frac{\partial (J_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta} \frac{\partial (J_\phi)}{\partial \phi} \right]$$

$$= -\left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2\, 20 \sin\theta \cos\phi \right) + 0 + 0 \right]$$

$$= -\left[\frac{1}{r^2}\, 20\sin\theta\cos\phi\, 2r \right] = -\frac{40\sin\theta\cos\phi}{r}$$

\# Given the vector current density $\vec{J} = 10\rho^2 z\,\hat{a}_\rho - 4\rho\cos^2\phi\,\hat{a}_\phi$ mA/m^2.
find the current flowing outward through the cylinder band
$\rho = 3$, $0 < \phi < 2\pi$, $2 < z < 2.8$.

constant along $\hat{a}_\rho$

Soln:

A/m^2

$$I = \int_s \vec{J} \cdot \vec{ds}$$

$$= \int_s \left[\left(10\rho^2 z\hat{a}_\rho - 4\rho\cos^2\phi\,\hat{a}_\phi \right) \times 10^{-3} \right] \left(\rho\, d\phi\, dz\, \hat{a}_\rho \right)$$

$$= \int_s 10^{-2} \rho^3 z \; d\phi\, dz$$

$$= 10^{-2} \times 3^3 \int_{\phi=0}^{2\pi} d\phi \int_{z=2}^{2.8} z\,dz$$

$$= 3.257\ \text{A}$$

The current density in a certain region is approximated by $\vec{J} = \frac{1}{r} e^{-t} \hat{a}_r$ A/m^2 in spherical co-ordinates. (a) How much

current is crossing the surface $r = 5$m and $t = 1$s (b) Find $\rho_v(r,t)$ assuming that $\rho_v \to 0$ as $t \to \infty$ (c) Find the expression for the velocity of the charge density.

soln:

(a) $I = \oint_s \vec{J} \cdot \vec{ds}$

($r = 5$m and $t = 1$s)

Now

$$I = \oint \left(\frac{1}{r} e^{-t} \hat{a}_r \right) \cdot \left(r^2 \sin\theta \, d\theta \, d\phi \; \hat{a}_r \right)$$

$$= 5e^{-1} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \sin\theta \, d\theta \, d\phi \qquad (r = 5, \; t = 1)$$

$$= 5e^{-1} \int_{\theta=0}^{\pi} \sin\theta \, d\theta \int_{\phi=0}^{2\pi} d\phi = 23.114 \text{ A}$$

(b) $\rho_v(r,t)$ $\qquad \rho_v \to 0$ as $t \to \infty$

$$\nabla \cdot \vec{J} = -\frac{\partial \rho_v}{\partial t} \quad \text{--- (i)}$$

$$\nabla \cdot \vec{J} = \frac{1}{r^2} \frac{\partial (r^2 J_r)}{\partial r} + \frac{1}{r\sin\theta} \frac{\partial (\sin\theta \, J_\theta)}{\partial \theta} + \frac{1}{r\sin\theta} \frac{\partial J_\phi}{\partial \phi}$$

$$\vec{J} = \frac{1}{r} e^{-t} \hat{a_r} = J_r \hat{a_r} + J_\theta \hat{a_\theta} + J_\phi \hat{a_\phi}$$

$$J_r = \frac{1}{r} e^{-t}, \quad J_\theta = 0, \quad J_\phi = 0$$

So, $\nabla \cdot \vec{J} = \frac{1}{r^2} \frac{\partial (r^2 \cdot \frac{1}{r} e^{-t})}{\partial r} = \frac{1}{r^2} \cdot e^{-t} \cdot = \frac{1}{r^2} e^{-t}$

Then from (i)

$$\frac{1}{r^2} e^{-t} = -\frac{\partial \rho_v}{\partial t}$$

Integrating both side

$$\int \frac{1}{r^2} e^{-t} = -\int \frac{\partial \rho_v}{\partial t}$$

Or, $\rho_v = -\int \frac{1}{r^2} e^{-t} \, dt$ | $\int e^{-x} = -e^{-x} + C$

$\int e^{x} = e^{x} + C$

$\therefore \rho_v = \frac{1}{r^2} e^{-t} + C$ ——(ii)

Using condition that $\rho_v = 0$ as $t \to \infty$

$0 = 0 + C$ / $C = 0$

So,

$$\boxed{\rho_v = \frac{1}{r^2} e^{-t} \ C/m^3}$$

(c) $\vec{J} = a \cdot \vec{r}$

(c) $\vec{J} = \rho_v \vec{v}$

$$\boxed{\text{or } \vec{v} = \frac{\vec{J}}{\rho_v} = \frac{\frac{1}{r} e^{-t}\,\hat{a}_r}{\frac{1}{r^2} e^{-t}} = r\,\hat{a}_r \text{ m/s}}$$

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(4) If $\vec{J} = \frac{1}{r^3}\left(2\cos\theta\,\hat{a}_r + \sin\theta\,\hat{a}_\theta\right)$ A/m^2, calculate the current passing through (a) a hemispherical shell of radius 20cm, $0 < \theta < \frac{\pi}{2}$, $0 < \phi < 2\pi$

(b) a spherical shell of radius 10cm

Soln:-

$$I = \oint_S \vec{J} \cdot \vec{ds}$$

(a) $r = 20\text{cm} = 0.2\text{m} \rightarrow \text{const}$

$$I = \oint \left[\frac{1}{r^3}\left(2\cos\theta\,\hat{a}_r + \sin\theta\,\hat{a}_\theta\right)\right]\left(r^2\sin\theta\, d\theta\, d\phi\,\hat{a}_r\right)$$

$$= \oint \frac{1}{r} \times 2\cos\theta\sin\theta\, d\theta\, d\phi$$

$$= \frac{1}{0.2}\oint \sin 2\theta\, d\theta\, d\phi \quad = \frac{1}{0.2}\int_{\theta=0}^{\pi/2} \sin 2\theta\, d\theta \int_{\phi=0}^{2\pi} d\phi$$

$= 31.415\ A$

(b) $r = 0.1\ m$ → not given so $\theta = \pi$, $\phi = 2\pi$ in spherical
upper limit

$$I = \frac{1}{0.1} \oint \sin 2\theta \, d\theta \, d\phi = \frac{2\pi}{0.1} \int_{\theta=0}^{\pi} \sin 2\theta \, d\theta = 0 \quad \#$$

\# The potential field $V = 2x^2 + 4y - 2z^2\ V$ exist in the free space surrounding a perfectly conducting surface. Point $P(4, 3, 2)$ lies on the surface (a) Give equation of the surface (b) Find the unit vector outward normal to the surface at P, assuming the origin is inside the surface.

Soln:

(a) $V\big|_{(4,3,2)} = 2 \times 4^2 + 4 \times 3 - 2 \times 2^2 = 36\ V$

eqn of surface is:

$V = 2x^2 + 4y^2 - 2z^2$

$36 = 2x^2 + 4y^2 - 2z^2$

So $2x^2 + 4y^2 - 2z^2 - 36 = 0$

(b) $\hat{a}_N = -\dfrac{\vec{E}}{|\vec{E}|}$

$\vec{E} = -\nabla V$

$$= -\left[\frac{\partial V}{\partial x} \hat{a}_x + \frac{\partial V}{\partial y} \hat{a}_y + \frac{\partial V}{\partial z} \hat{a}_z \right]$$

$$= -\left[\frac{\partial (2x^2 + 4y - 2z^2)}{\partial x} \hat{a}_x + \dots \right]$$

$$= -16\hat{a}_x - 4\hat{a}_y + 8\hat{a}_z \ \text{V/m}$$

$$|\vec{E}| = \sqrt{(-16)^2 + (-4)^2 + 8^2} = 18.33 \ \text{V/m}$$

Thus,

$$\hat{a}_N = -\frac{\vec{E}}{|\vec{E}|} = 0.8729\hat{a}_x + 0.2182\hat{a}_y - 0.4364\hat{a}_z$$

\# A current density in certain region is given as $\vec{J} = \frac{400\sin\theta}{r^2}\hat{a}_r \ \text{A/m}^2$

Find total current through surface $r = 0.8$ bounded by $0.1\pi < \theta < 0.3\pi$
$0 < \phi < 2\pi$

along $\hat{a}_r$

Now,

$$I = \int_S \vec{J}\cdot\vec{dS}$$

$$= \iint_S \frac{400\sin\theta}{\cancel{r^2}}\hat{a}_r \cdot \left(\cancel{r^2}\sin\theta\, d\theta\, d\phi\, \hat{a}_r\right)$$

$$= \int_S 400\sin^2\theta\, d\theta\, d\phi \quad (r = 0.8 \text{ put})$$

$$= 400\int_{\theta=0.1\pi}^{0.3\pi}\sin^2\theta\, d\theta \int_{\phi=0}^{2\pi} d\phi$$

$$= 561.31 \ \text{A}$$ #

\# $\vec{J} = 3r^2\cos\theta\,\hat{a}_r - r^2\sin\theta\,\hat{a}_\theta \ \text{A/m}^2$ find I $\theta = 30^\circ$, $0 < \phi < 2\pi$

4 $\vec{J} = 3r^2\cos\theta\,\hat{a}_r - r^2\sin\theta\,\hat{a}_\theta$ A/m^2 find I $\theta = 30^\circ$, $0 < \phi < 2\pi$

$0 < r < 2$

const along $\hat{a}_\theta$

$\vec{ds} = r\sin\theta\,d\phi\,dr\,\hat{a}_\theta$

now,

$$I = \oint_s (3r^2\cos\theta\,\hat{a}_r - r^2\sin\theta\,\hat{a}_\theta)(r\sin\theta\,d\phi\,dr\,\hat{a}_\theta)$$

$$= \oint_s -r^3\sin^2\theta\,d\phi\,dr \quad (\theta = 30^\circ)$$

$$= \frac{-1}{4}\oint_s r^3\,d\phi\,dr$$

$$= -\frac{1}{4}\int_0^2 r^3\,dr\int_{\phi=0}^{2\pi} d\phi$$

$$= -6.28 \text{ A}$$