Chapter-4

Saturday, February 15, 2025 7:20 PM

Energy and Potential

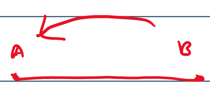
Potential Difference

→ Potential difference is defined as the work done (by an external source) in moving a unit positive charge from one point to another in an electric field.

Now,

$$\text{Potential difference } (V) = \frac{W}{Q} = \frac{-Q \int_{init}^{final} \vec{E} \cdot \vec{dl}}{Q}$$

$$\therefore V = -\int_{init}^{final} \vec{E} \cdot \vec{dl} \qquad (\text{Unit : Volt})$$



$$V_{AB} = \overset{final}{V_A} - \overset{ini}{V_B} = -\int_{B}^{A} \vec{E} \cdot \vec{dl} \qquad \left(\text{B lai A ~~kaa~~ laune ho } V_{AB} \right)$$

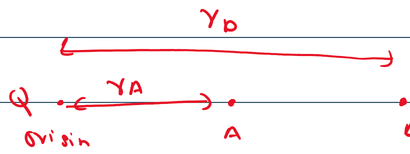
(Pot. diff between A and B

Potential at a point can be defined as the work done in bringing a unit positive charge from the zero reference (usually taken as infinity) to the point.

$$V_A = -\int_{\infty}^{A} \vec{E} \cdot \vec{dl}$$

Equipotential surface is surface composed of points having the same value of potential.

(no work is done in moving a unit charge around equipotential surface)



(r_A, θ, ϕ) (r_B, θ, ϕ)

(spherical system)

$$V_{AB} = V_A - V_B = -\int_B^A \vec{E} \cdot \vec{dl}$$

$$\vec{E} = E_r \hat{a}_r = \frac{Q}{\epsilon_0 4\pi r^2} \hat{a}_r, \quad \vec{dl} = dr\, \hat{a}_r$$

Now,

$$V_{AB} = -\int_B^A \frac{Q}{4\pi\epsilon_0 r^2} \hat{a}_r \, dr \, \hat{a}_r = -\int_B^A \frac{Q}{4\pi\epsilon_0 r^2} dr$$

$$= \frac{-Q}{4\pi\epsilon_0} \int_B^A \frac{1}{r^2} dr = \frac{-Q}{4\pi\epsilon_0} \left[\frac{r^{-2+1}}{-2+1} \right]_{r_B}^{r_A}$$

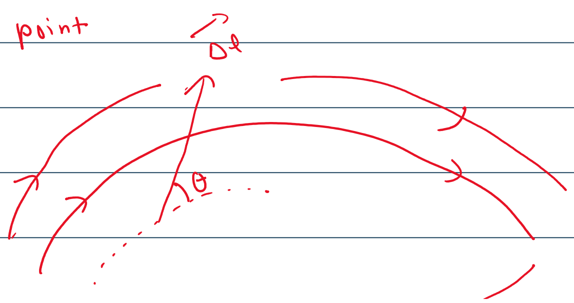
$$= \frac{Q}{4\pi\epsilon_0} \left[\frac{1}{r} \right]_{r_B}^{r_A} = \frac{Q}{4\pi\epsilon_0} \left[\frac{1}{r_A} - \frac{1}{r_B} \right]$$

$V_{AB} \rightarrow +ve$ (energy is expended)

Potential Gradient (10E) (How pd can be used to determine $\vec{E}$)

→ refers to the spatial rate of change of the electric potential in a given region of space.

Consider a region as shown in figure where both $\vec{E}$ and V varies at every point



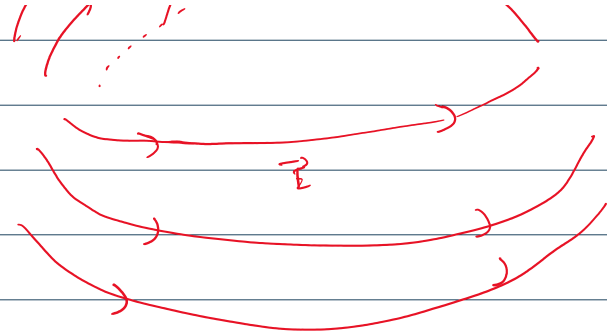


fig: Streamlines of electric field

Small work done in moving a unit charge through a small incremental distance $\vec{\Delta L}$ is given by :-

$$\Delta V = -\vec{E}\cdot\vec{d\varphi}$$

let θ be the angle between $\vec{\Delta\varphi}$ and $\vec{E}$, then

$$\Delta V = -E\Delta L\cos\theta$$

or, $$\frac{\Delta V}{\Delta\varphi} = -E\cos\theta$$

Taking limit as $\Delta\varphi \to 0$, we have

$$\lim_{\Delta\varphi\to 0}\frac{\Delta V}{\Delta\varphi} = -E\cos\theta$$

or, $$\frac{dV}{d\varphi} = -E\cos\theta$$

or, for $\theta = 180^\circ$, $\frac{dV}{d\varphi} \to \left.\frac{dV}{d\varphi}\right|_{max}$

and $$\left.\frac{dV}{d\varphi}\right|_{max} = E \quad \text{--- (i)}$$

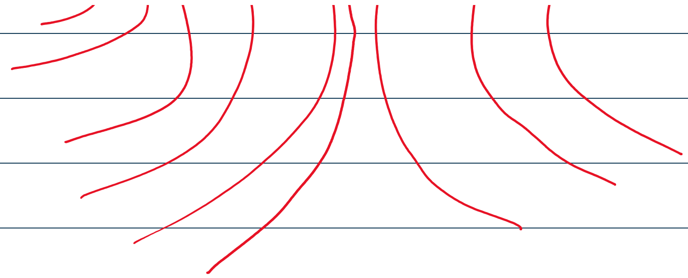


Fig: Illustration of equipotential surfaces

Equation (i) inform us the direction and magnitude of $\vec{E}$

If $\vec{\Delta l}$ is directed along equipotential surface,

$$\nabla V = -\vec{E} \cdot \vec{\Delta l} = 0$$

Since neither $\vec{E}$ nor $\vec{\Delta l}$ is zero; $\vec{E}$ must be perpendicular to this $\vec{\Delta l}$ or perpendicular to the equipotentials.

let $\hat{a}_N$ be a unit vector normal to the equipotential surface and directed towards higher potentials. Then,

$$\vec{E} = -\left.\frac{dV}{dl}\right|_{max} \hat{a}_N$$

Since $\left.\frac{dV}{dl}\right|_{max}$ occurs when $\vec{\Delta l}$ is in direction of $\hat{a}_N$, we may write

$$\left.\frac{dV}{dl}\right|_{max} = \frac{dV}{dN}$$

$$\therefore \vec{E} = -\frac{dV}{dN} \hat{a}_N \quad \text{——(ii)}$$

$$\boxed{\text{Gradient of } T = \text{grad } T = \frac{dT}{dN} \hat{a}_N} \quad \text{——(iii)}$$

From (ii) and (iii)

$\vec{E} = -\text{grad } V$ ✓ for any scalar gradient

From (ii) and (ii)

$\vec{E} = -\text{grad}\, V$ for any scallar gradient

As grad $V = \nabla V$ we can write

$$\boxed{\vec{E} = -\nabla V}$$

[Div ma $\nabla \cdot \vec{V}$] (gradient ma no .)

$\nabla \cdot \vec{D}$ (Divergence is vector) ∇V (gradient is scalar)

The Dipole (V.IMP) (Derivation)

→ An electric dipole, or simply a dipole is the name given to two point charges at equal and opposite sign sepperated by a distance which is small compared to the distance to the point p at which the value at $\vec{E}$ and V are to be evaluated.

Consider a dipole formed by two charges +Q and −Q seporated by a distance d as shown in the figure below, the +Q and −Q charges being positioned in rectangular system at $(0, 0, \frac{1}{2}d)$ and $(0, 0, -\frac{1}{2}d)$ respectively.

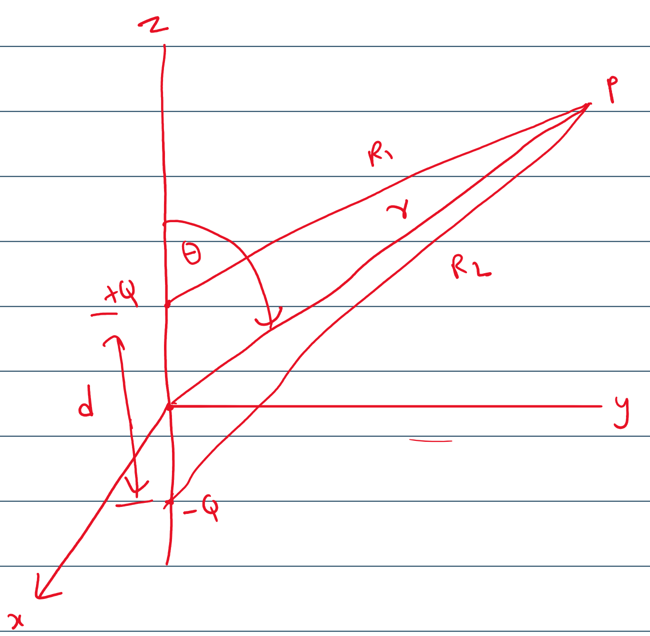


Fig: Illustration of dipole. P is the point at which field

Fig: Illustration of dipole. P is the point at which field is to be determined

let point P be described in spherical co-ordinate system as $P(r, \theta, \phi = 90^\circ)$ and the distances from Q and −Q to be R_1 and R_2.

Now, the total potential at point P is :-

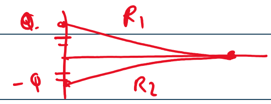
formula

$$V = \frac{+Q}{4\pi\varepsilon_0 R_1} + \frac{(-Q)}{4\pi\varepsilon_0 R_2} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$\text{or } V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{R_2 - R_1}{R_1 R_2}\right) \quad \ldots\ldots (i)$$

fig

Case-I : when point P is on z = 0 plane ($\theta = 90^\circ$) then $R_1 = R_2$



$\therefore V = 0$ so, $\vec{E} = 0$

Case-II : when point P is at large distance, R_1 will be parallel to R_2

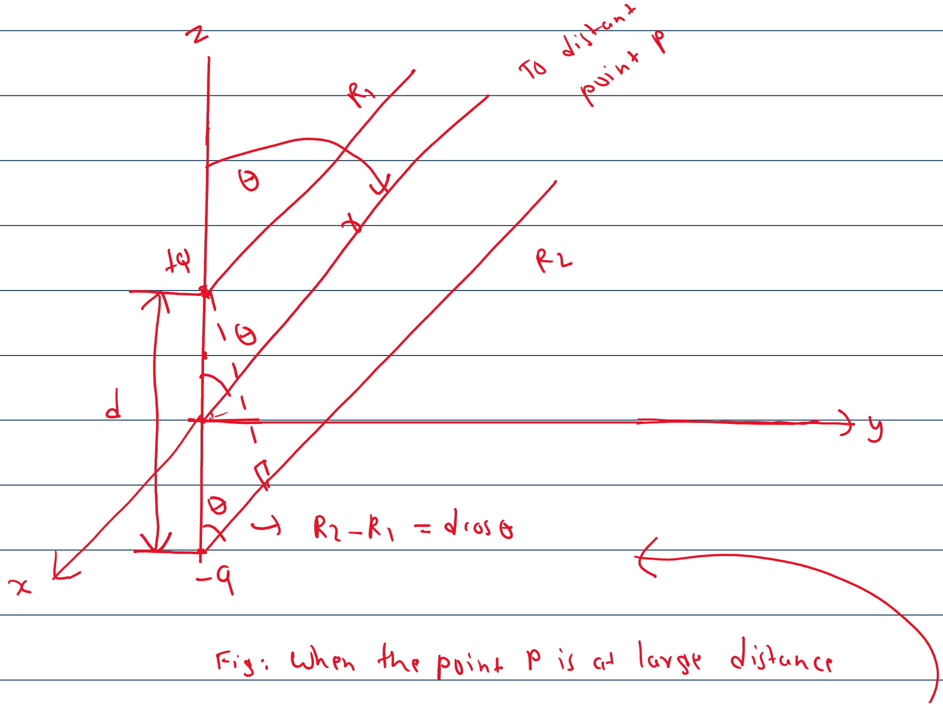


Fig: When the point P is at large distance

Fig: When the point P is at large distance

$$\cos\theta = \frac{b}{h} = \frac{R_2 - R_1}{d} \qquad \therefore\ R_2 - R_1 = d\cos\theta$$

Also, $R_1 \approx R_2$ then $R_1 R_2 = r^2$

$(R_1 = R_2 = r)$
approx

From (i)

$$V = \frac{Q}{4\pi\varepsilon_0}\,\frac{d\cos\theta}{r^2} \quad \text{—— (ii)}$$

Since $d\cos\theta = \vec{d}\cdot\hat{a}_r$

($\vec{d}\cdot\hat{a}_r = |\vec{d}|.\,|\hat{a}_r|\cos\theta = d\cos\theta$) eq^n (ii) becomes

$$V = \frac{Q}{4\pi\varepsilon_0}\,\frac{\vec{d}\cdot\hat{a}_r}{r^2}$$

($\vec{p} = Q\vec{d}$ is the dipole moment directed from $-Q$ to $+Q$
and $p = Qd$ is magnitude of dipole moment)

So,
$$V = \frac{\vec{p}\cdot\hat{a}_r}{4\pi\varepsilon_0 r^2} = \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0 r^3} \qquad \left(\hat{a}_r = \frac{\vec{r}}{|\vec{r}|}\right)$$

If the dipole center is not at origin but at $\vec{r}\,'$ then
formula

$$V = \frac{\vec{p}\cdot(\vec{r} - \vec{r}\,')}{4\pi\varepsilon_0\,|\vec{r} - \vec{r}\,'|^3}$$

Then, $\vec{E} = -\nabla V$

$$= -\left(\frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial\theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}\hat{a}_\phi\right)$$

$$= -\left[\frac{\partial \left(\frac{Qd\cos\theta}{4\pi\varepsilon_0 r^2} \right)}{\partial r} \hat{a}_r + \frac{1}{r} \frac{\partial \left(\frac{Qd\cos\theta}{4\pi\varepsilon_0 r^2} \right)}{\partial \theta} \hat{a}_\theta + \frac{1}{r\sin\theta} \frac{\partial \left(\frac{Qd\cos\theta}{4\pi\varepsilon_0 r^2} \right)}{\partial \phi} \hat{a}_\phi \right]$$

$$= -\left[\frac{Qd\cos\theta}{4\pi\varepsilon_0} (-2r^{-3}) \hat{a}_r + \frac{Qd}{4\pi\varepsilon_0 r^3} (-\sin\theta) \hat{a}_\theta + 0 \right]$$

$$= -\left(-\frac{Qd\cos\theta}{2\pi\varepsilon_0 r^3} \hat{a}_r - \frac{Qd\sin\theta}{4\pi\varepsilon_0 r^3} \hat{a}_\theta \right)$$

$$\therefore \boxed{\vec{E} = \frac{Qd}{4\pi\varepsilon_0 r^3} \left(2\cos\theta\, \hat{a}_r + \sin\theta\, \hat{a}_\theta \right)}$$

(Electric field intensity due to dipole)

Energy Density in the Electrostatic Field (VImp)

→ To find the potential energy present in a system of charges, we must find the work done by an external source in positioning the charges.

Consider an empty universe at first. We don't have to do work to bring a charge Q_1 from infinity to any position as there exists no field.

Work done in positioning Q_2 in the field of $Q_1 = Q_2 V_{2,1}$ $\quad (W = QV)$

$V_{2,1}$ = the potential at the location of Q_2 due to Q_1

Work done in positioning Q_3 in the field of Q_1 and Q_2

$$= Q_3 V_{3,1} + Q_3 V_{3,2}$$

Similarly work done in positioning Q_4

$$= Q_4 V_{4,1} + Q_4 V_{4,2} + Q_4 V_{4,3}$$ and so on

The total work done in positioning all charges = potential energy of the field

$$= W_E = Q_2 V_{2,1} + Q_3 V_{3,1} + Q_3 V_{3,2} + Q_4 V_{4,1} + Q_4 V_{4,2} + Q_4 V_{4,3} + \ldots \quad (i)$$

logic #

$$\left(V = \frac{Q}{4\pi\varepsilon_0 R} \right)$$

We have,

$$Q_3 V_{3,1} = Q_3 \frac{Q_1}{4\pi\varepsilon_0 R_{13}} = Q_1 \frac{Q_3}{4\pi\varepsilon_0 R_{31}} = Q_1 V_{1,3}$$

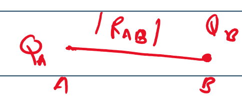
$Q_3 V_{3,1}$: potential at 3 due to moving the charge at 1

$\frac{Q_1}{4\pi\varepsilon_0 R_{13}}$: (at 3 due 1)

$\frac{Q_3}{4\pi\varepsilon_0 R_{31}}$: (at 1 due 3)

$Q_1 V_{1,3}$: (at 1 due 3)

($R_{13} = R_{31}$ scalar distance between Q_1 and Q_3)



$$V_{B,A} = \frac{Q_A}{4\pi\varepsilon_0 R_{AB}}$$

$$V_{A,B} = \frac{Q_B}{4\pi\varepsilon_0 R_{BA}}$$

from this we can conclude

Now,

$$W_E = Q_1 V_{1,2} + Q_1 V_{1,3} + Q_2 V_{2,3} + Q_1 V_{1,4} + Q_2 V_{2,4} + Q_3 V_{3,4} + \ldots \quad (ii)$$

Now Adding (i) and (ii)

$$2W_E = Q_1 (V_{1,2} + V_{1,3} + V_{1,4} + \ldots)$$
$$+ Q_2 (V_{2,1} + V_{2,3} + V_{2,4} + \ldots)$$
$$+ Q_3 (V_{3,1} + V_{3,2} + V_{3,4} + \ldots)$$
$$+ \ldots \quad (iii)$$

Here $V_{1,2} + V_{1,3} + V_{1,4} + \ldots = V_1$ which is the potential at the location of Q_1 due to presence of $Q_2, Q_3, Q_4 \ldots$

$V_{2,1} + V_{2,3} + V_{2,4} + \ldots = V_2$ " " " at $Q_1, Q_3, Q_4 \ldots$

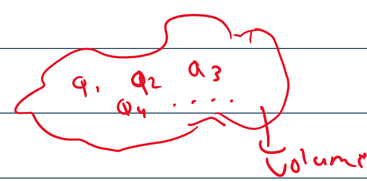
Eqn (iii) becomes

$$2W_E = Q_1V_1 + Q_2V_2 + Q_3V_3 + \cdots$$

or $W_E = \frac{1}{2}\left(Q_1V_1 + Q_2V_2 + Q_3V_3 + \cdots\right)$

$$\boxed{\therefore\ W_E = \frac{1}{2}\sum_{m=1}^{n=N} Q_m V_m}$$

For continuous charge distribution



$$Q = \int_{vol} \rho_v \, dv$$

$$W_E = \frac{1}{2}\int_{vol} (\underbrace{\rho_v dv}_{\text{volume}})\ \overset{\text{potential}}{V}$$

$$= \frac{1}{2}\int_{vol} (\rho_v V)\, dv$$

From maxwell's 1st equation

$$\nabla \cdot \vec{D} = \rho_v$$

$$W_E = \frac{1}{2}\int (\nabla\cdot\vec{D}) V \, dv \qquad \text{—— (iv)}$$

Consider a vector identity which is true for any scalar V and any vector $\vec{D}$.

div → vector   gradient → scalar

$$\nabla(V\vec{D}) = V(\nabla\cdot\vec{D}) + \vec{D}(\nabla V)$$

→ need to memorize this

need to memorize this identity

or, $V(\nabla \cdot \vec{D}) = \nabla(V\vec{D}) - \vec{D}(\nabla V)$ — #

Now (iv) becomes

$$W_E = \frac{1}{2}\int_{vol}\left(\nabla(V\vec{D}) - \vec{D}(\nabla V)\right) dv$$

or, $W_E = \frac{1}{2}\int_{vol} \nabla(V\vec{D})\, dv - \frac{1}{2}\int_{vol} \vec{D}(\nabla V)\, dv$

Using divergence theorem → $\oint_S \vec{D} \cdot \vec{ds} = \int_{vol} (\nabla \cdot \vec{D})\, dv$

$$\int_{vol} \nabla \cdot (V\vec{D})\, dv = \oint_S (V\vec{D}) \cdot \vec{ds}$$

single term

Now,

$$W_E = \frac{1}{2}\oint_S (V\vec{D}) \cdot \vec{ds} - \frac{1}{2}\int_{vol} \vec{D}(\nabla V)\, dv$$

V is inversely proportional to r, $\vec{D}$ is inversely proportional to r^2 and $ds \propto r^2$. For $r \to \infty$, the 1st integral becomes zero (we are considering large universe of charges so)

so, $W_E = -\frac{1}{2}\int_{vol} \vec{D}(\nabla V)\, dv$

we know, $\vec{E} = -\nabla V$ and $\vec{D} = \varepsilon_0 \vec{E}$

$$\boxed{W_E = \frac{1}{2}\int_{vol} \vec{D}\cdot\vec{E}\, dv}$$

now, $d W_E \rightarrow$ remove integration

$$w_E = \frac{dW_E}{dv} = \frac{1}{2}\vec{D}\cdot\vec{E} = \frac{1}{2}\varepsilon_0 E^2 \quad (\vec{E}\cdot\vec{E} = E^2)$$

$\therefore$ $\boxed{w_E = \frac{1}{2}\vec{D}\cdot\vec{E} = \frac{1}{2}\varepsilon_0 E^2}$ is the required expression for energy density in the electrostatic field