Chapter-7

Saturday, March 1, 2025 10:49 PM

(1 → Numerical / derivation) → 8 marks

Theory type derivation numericals

Poisson's and Laplace's Equations

(Derivation like numerical v.Imp chap)

From the point of Gauss' law

$\nabla \vec{D} = \rho_v$ assuming medium can be both dielectric and conductor

or, $\nabla (\varepsilon \vec{E}) = \rho_v$

Also, $\vec{E} = -\nabla V$

So,

$\nabla (-\varepsilon \nabla V) = \rho_v$

or $\nabla \cdot \nabla V = -\dfrac{\rho_v}{\varepsilon}$

$\nabla (\nabla V)$ → vector result; grad; divergence

or $\boxed{\nabla^2 V = -\dfrac{\rho_v}{\varepsilon}}$ —— (i)

(divergence of gradient of potential)

Equation (i) is called Poisson's equation and is true for a homogeneous region in which ε is constant.

(ρ_L, ρ_S, Q) are not zero

(∇^2 → laplacian operator)

For $\rho_V = 0$

$\boxed{\nabla^2 V = 0}$ —— (ii)

which is laplace equation #

IMPORTANT EXPRESSIONS

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \qquad \text{(rectangular)}$$

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial^2 V}{\partial z^2} \qquad \text{(cylindrical)}$$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} \qquad \text{(spherical)}$$

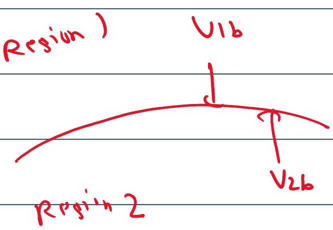
Uniqueness Theorem (Derivation V. Imp)

→ Any solution to Laplace's equation or Poisson's equation which also satisfies the boundary conditions must be the only solution that exists; it is unique. This is a statement of uniqueness theorem. The proof of uniqueness theorem for both Laplace's and Poisson's equations are shown here.

① For Laplace's equation

$$\nabla^2 V = 0$$

Let us assume V_1 and V_2 are two solutions



$$\nabla^2 V_1 = 0 \quad , \quad \nabla^2 V_2 = 0$$

or, $\nabla^2 (V_1 - V_2) = 0$ —(i)

Let the given potential values on the boundaries be V_b.

Then, $V_{1b} = V_{2b} = V_b$

or, $V_{1b} - V_{2b} = 0$ —(ii)

Consider a vector identity, which holds true for any scalar V and any vector $\vec{D}$.

$$\nabla \cdot (V\vec{D}) = V(\nabla \cdot \vec{D}) + \vec{D}\cdot(\nabla V) \quad \text{—(iii)}$$

If $V = V_1 - V_2$ $\vec{D} = \nabla(V_1 - V_2)$ then (iii) becomes

Now,

$$\nabla \cdot [(V_1-V_2)\nabla(V_1-V_2)] = (V_1-V_2)[\nabla\cdot\nabla(V_1-V_2)] + \nabla(V_1-V_2)\cdot\nabla(V_1-V_2)$$

or $$\nabla[(V_1-V_2)\nabla(V_1-V_2)] = (V_1-V_2)[\nabla^2(V_1-V_2)] + [\nabla(V_1-V_2)]^2$$

Integrating this eq^n throughout the volume that is enclosed by the boundary surfaces, we get

$$\int_{vol} \nabla[(V_1-V_2)\nabla(V_1-V_2)]\,dv = \int_{vol}(V_1-V_2)[\nabla^2(V_1-V_2)]\,dv$$

$$+ \int_{vol}[\nabla(V_1-V_2)]^2\,dv \quad \ldots\ldots (iv)$$

Using divergence theorem

$$\int_{vol} \nabla\cdot\vec{D}\,dv = \oint_s \vec{D}\cdot\vec{ds}$$

So, $$\int_{vol} \nabla[(V_1-V_2)\nabla(V_1-V_2)]\,dv = \oint_s [(V_1-V_2)\nabla(V_1-V_2)]\,\vec{ds}$$

$$= \oint_s [(V_{1b}-V_{2b})\nabla(V_{1b}-V_{2b})]\,\vec{ds}$$

From eq^n (ii) $V_{1b} - V_{2b} = 0$

or $$\int_{vol} \nabla[(V_1-V_2)\nabla(V_1-V_2)]\,dv = 0$$

(iv) becomes

$$0 = \int_{vol} (V_1 - V_2)\left[\nabla^2 (V_1 - V_2)\right] dv + \int_{vol} \left[\nabla (V_1 - V_2)\right]^2 dV$$

From (i)

$\nabla^2 (V_1 - V_2) = 0$

so, $0 = 0 + \int_{vol} \left[\nabla (V_1 - V_2)\right]^2 dV$

or $\int_{vol} \left[\nabla (V_1 - V_2)\right]^2 dV = 0$

For this we must have #

$\left[\nabla (V_1 - V_2)\right]^2 = 0$

or $\nabla (V_1 - V_2) = 0$ —— (v)

So, $V_1 - V_2 = $ constant

The constant can be evaluated to zero so,

$V_1 - V_2 = 0$

$\therefore V_1 = V_2$

Which proves two solutions we have assumed are identical.

(2) For Poisson's Equation

Poisson's equation is given by :-

$$\nabla^2 V = \frac{-\rho_v}{\varepsilon}$$

let us assume two solutions of Poisson's equation, V_1 and V_2

Therefore,

$$\nabla^2 V_1 = -\frac{\rho_v}{\varepsilon} \quad \text{and} \quad \nabla^2 V_2 = -\frac{\rho_v}{\varepsilon}$$

or,
$$\nabla^2 V_1 - \nabla^2 V_2 = -\frac{\rho_v}{\varepsilon} - \left(-\frac{\rho_v}{\varepsilon}\right)$$

or,
$$\boxed{\nabla^2 (V_1 - V_2) = 0 \quad \text{———} \quad (i)}$$

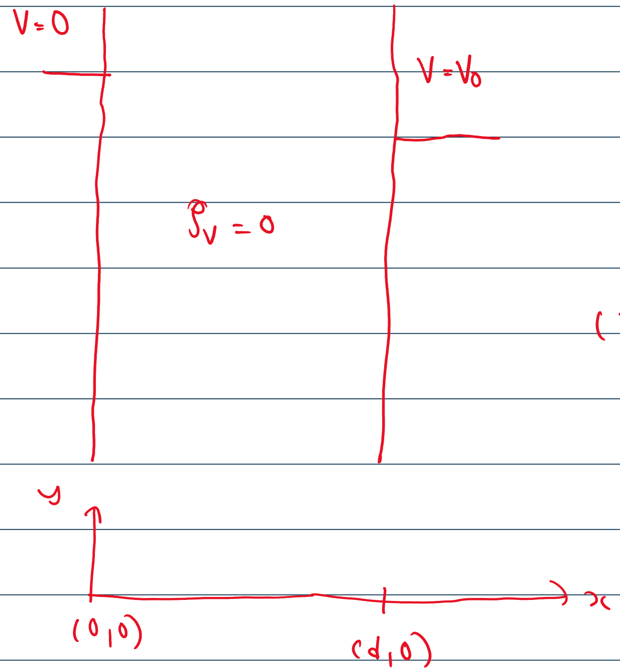
Now proceed as in the case of Laplace's equation.

Boundary Value Problems

-) Problems where the goal is to determine the electric potential V and field E within a region, given specific conditions (e.g fixed potentials or charge densities) on the boundaries of that region.

eg: If you know the voltage on the surface of a conductor, you can calculate the potential and field everywhere else in space.

One Dimensional Boundary Value Problems



(rectangular co-ordinates)

let us consider a parallel plate capacitor with potential V_0 on one place, and zero on the other. Since V is a the function of only one variable (x-coordinate in this case), it is called one dimensional boundary value problem.

Using Laplace's equation

$$\nabla^2 V = 0$$

or,

$$\frac{\partial^2 V}{\partial x^2} + \cancel{\frac{\partial^2 V}{\partial y^2}}^{0} + \cancel{\frac{\partial^2 V}{\partial z^2}}^{0} = 0$$

or,

$$\frac{\partial^2 V}{\partial x^2} = 0$$

or,

$$\frac{d^2 V}{dx^2} = 0$$

Integrate twice

$$\frac{dV}{dx} = A \quad \text{(1st integration)}$$

or $V = Ax + B$ —(ii) (2nd integration)

Given,

$V = 0$ at $x = 0$

$V = V_0$ at $x = d$

From (i)

$0 = A \cdot 0 + B$

$V_0 = A \cdot d + B$

$\therefore B = 0$, $A = \frac{V_0}{d}$

From (ii)

$V = \frac{V_0}{d} x + 0$

$$\therefore \boxed{V = \frac{V_0}{d} x}$$

Using Laplace's equation we can find out the capacitance of several conductor configuration using the steps

1. Given V use $\vec{E} = -\nabla V$ to find $\vec{E}$
2. Use $\vec{D} = \varepsilon \vec{E}$ to find $\vec{D}$
3. Evaluate $\vec{D}$ at either capacitor plate, $\vec{D} = D_n \hat{a}_n$
4. Recognize that $\rho_s = D_n$
5. Find q by a surface integration over the capacitor plate and then calculate capacitance.

(4) Recognize that $\rho_s = D_N$

(5) Find Q by a surface integration over the capacitor plate and then calculate capacitance.

(Imp derivations)

(1) Find the capacitance of parallel plate capacitor

we have $V = V_0 \frac{x}{d}$

(i) $\vec{E} = -\nabla V$

$$\vec{E} = -\left(\frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z\right)$$ rectangular system

$$= -\left[\frac{\partial (V_0 \frac{x}{d})}{\partial x}\hat{a}_x + \frac{\partial (V_0 \frac{x}{d})}{\partial y}\hat{a}_y + \frac{\partial (V_0 \frac{x}{d})}{\partial z}\hat{a}_z\right]$$

$$= -\frac{V_0}{d}\hat{a}_x$$

(ii) $\vec{D} = \varepsilon\vec{E} = \varepsilon\left(-\frac{V_0}{d}\hat{a}_x\right) = -\frac{\varepsilon V_0}{d}\hat{a}_x$ (For free space $\varepsilon = \varepsilon_0$) here it is not free space it is boundary value problem so ε is used

(iii) $\vec{D}_s = \vec{D}\Big|_{x=0} = -\frac{\varepsilon V_0}{d}\hat{a}_x\Big|_{x=0} = -\frac{\varepsilon V_0}{d}\hat{a}_x$

$\hat{a}_N = \hat{a}_x \qquad \vec{D}_s = -\frac{\varepsilon V_0}{d}\hat{a}_N = D_N \hat{a}_N$

and $D_N = -\varepsilon \frac{V_0}{d}$

(iv) $\rho_s = D_N = -\varepsilon \frac{V_0}{d}$

(v) $Q = \int_S \rho_s ds = \int_S \left(-\varepsilon \frac{V_0}{d}\right) ds = -\varepsilon \frac{V_0}{d} \int ds = -\frac{\varepsilon V_0}{d} S$

(vi) The capacitance is given as :-

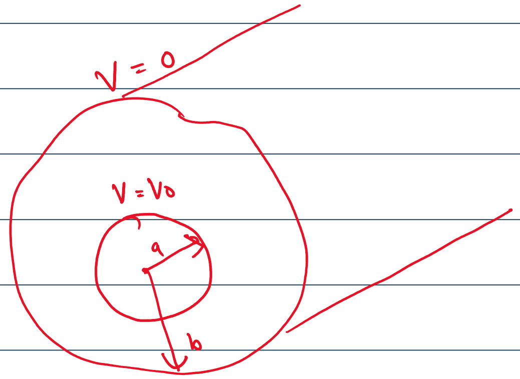
$$C = \frac{|Q|}{V_0} = \frac{\left|-\frac{\varepsilon V_0}{d} S\right|}{V_0} = \frac{\frac{\varepsilon V_0 S}{d}}{V_0} = \frac{\varepsilon S}{d}$$

$$\boxed{\therefore\ C = \frac{\varepsilon S}{d}}$$

V.Imp

(one dimensional problem in cylindrical system)

Find the capacitance of a co-axial capacitor using Laplace's equation



$\rho_{in} = a$

$\rho_{out} = b$

$V \rightarrow$ vary on ρ

Assume that V is a function of ρ only laplace equation is :-

$$\nabla^2 V = 0$$

or, $\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial \rho}\right) + \underbrace{\frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2}}_{0} + \underbrace{\frac{\partial^2 V}{\partial z^2}}_{0}$ only on ρ (laplacian cylindrical)

or, $$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial \rho}\right) = 0$$

or, $$\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial \rho}\right) = 0$$

Integrating

$$\rho\frac{\partial V}{\partial \rho} = A$$

or $$\frac{\partial V}{\partial \rho} = \frac{A}{\rho}$$

~~Inte~~ $$\partial V = \frac{A}{\rho}\partial\rho$$

Integrating

$$V = A \ln(\rho) + B \quad \text{— (i)}$$

Where A and B are constant of integration which are determined by using boundary conditions.

let $V = 0$ at $\rho = b$

$V = V_0$ at $\rho = a$, $a < b$

From (i)

$$0 = A\ln(b) + B$$

$$V_0 = A\ln(a) + B$$

$$V_0 - 0 = A\ln(a) - A\ln(b) + B - B$$

$$V_0 = A\ln\left(\frac{a}{b}\right)$$

$$A = \frac{V_0}{\ln\left(\frac{a}{b}\right)} = \frac{-V_0}{\ln\left(\frac{b}{a}\right)}$$

$$B = -A\ln(b) = \frac{V_0\ln(b)}{\ln\left(\frac{b}{a}\right)}$$

Now eq^n (i) becomes

$$V = \left[\frac{-V_0}{\ln\left(\frac{b}{a}\right)}\right]\ln(\rho) + \left[\frac{V_0\ln(b)}{\ln\left(\frac{b}{a}\right)}\right]$$

$$= \frac{V_0}{\ln\left(\frac{b}{a}\right)}\Big[\ln(b) - \ln(\rho)\Big] = \frac{V_0\ln\left(\frac{b}{\rho}\right)}{\ln\left(\frac{b}{a}\right)}$$

$$\therefore \quad V = V_0\frac{\ln\left(\frac{b}{\rho}\right)}{\ln\left(\frac{b}{a}\right)}$$

(i) $E = -\nabla V$

$$= -\left(\frac{\partial V}{\partial \rho}\hat{a}_\rho + \cancelto{0}{\frac{1}{\rho}\frac{\partial V}{\partial \phi}}\hat{a}_\phi + \cancelto{0}{\frac{\partial V}{\partial z}}\hat{a}_z\right)$$

$$= -\frac{\partial V}{\partial \rho}\hat{a}_\rho = -\frac{\partial}{\partial \rho}\left[V_0 \frac{\ln\left(\frac{b}{\rho}\right)}{\ln\left(\frac{b}{a}\right)}\right]\hat{a}_\rho$$

$$= \frac{-V_0}{\ln\left(\frac{b}{a}\right)} \frac{\partial}{\partial \rho}\left(\ln\left(\frac{b}{\rho}\right)\right)\hat{a}_\rho$$

$$= \frac{-V_0}{\ln\left(\frac{b}{a}\right)}\left[\frac{\partial \ln b}{\partial \rho} - \frac{\partial \ln \rho}{\partial \rho}\right]\hat{a}_\rho$$

$$= -\frac{V_0}{\ln\left(\frac{b}{a}\right)}\left[0 - \frac{1}{\rho}\right]\hat{a}_\rho$$

$$= \frac{V_0}{\ln\left(\frac{b}{a}\right)} \frac{1}{\rho}\hat{a}_\rho$$

(ii) $\vec{D} = \varepsilon\vec{E} = \varepsilon\left[\frac{V_0}{\ln\left(\frac{b}{a}\right)} \frac{1}{\rho}\hat{a}_\rho\right] = \frac{\varepsilon V_0}{\ln\left(\frac{b}{a}\right)} \frac{1}{\rho}\hat{a}_\rho$

(i) $\vec{D_s} = D\Big|_{\rho = a} = \frac{\varepsilon V_0}{\ln\left(\frac{b}{a}\right)} \cdot \frac{1}{a}\hat{a}_\rho = \frac{\varepsilon V_0}{a} \cdot \frac{1}{\ln\left(\frac{b}{a}\right)}\hat{a}_\rho$

$\hat{a}_N = \hat{a}_\rho$

$$\vec{D_s} = \frac{\varepsilon V_0}{a} \frac{1}{\ln\left(\frac{b}{a}\right)}\hat{a}_N = D_N \hat{a}_N$$

$$\therefore D_N = \frac{\varepsilon V_0}{a} \frac{1}{\ln\left(b\right)}$$

$$\therefore \; D_N = \frac{\varepsilon V_0}{a} \frac{1}{\ln\left(\frac{b}{a}\right)}$$

(iv) $\rho_s = D_N = \frac{\varepsilon V_0}{a} \frac{1}{\ln\left(\frac{b}{a}\right)}$

(v) $$Q = \int_s \rho_s \, ds = \int_s \left[\frac{\varepsilon V_0}{a} \frac{1}{\ln\left(\frac{b}{a}\right)} \right] ds = \frac{\varepsilon V_0}{a} \frac{1}{\ln\left(\frac{b}{a}\right)} \int_s ds$$

For radius $= a$, length $= L$

$$\int_s ds = 2\pi r h = 2\pi a L$$

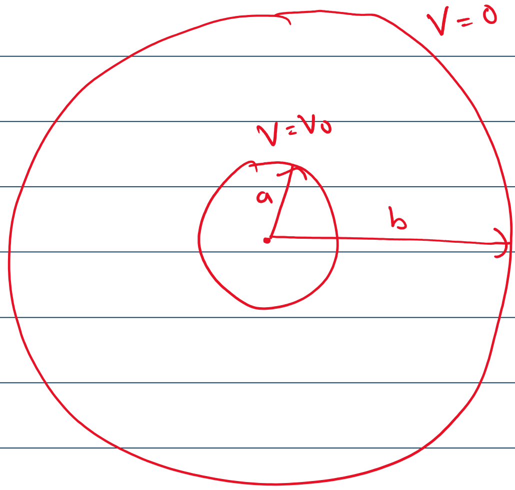
$$Q = \frac{\varepsilon V_0}{\cancel{a}} \frac{1}{\ln\left(\frac{b}{a}\right)} \cdot 2\pi \cancel{a} L = \frac{2\pi \varepsilon V_0 L}{\ln\left(\frac{b}{a}\right)}$$

The capacitance is :-

$$\boxed{C = \frac{|Q|}{V_0} = \frac{2\pi \varepsilon L}{\ln\left(\frac{b}{a}\right)}}$$

Exp

1. Find the capacitance of spherical capacitor using Laplace's equation. (one dimensional problem in spherical coordinate system)



A is function of r only

laplace equation is :-

$$\nabla^2 V = 0$$

or, $$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \overbrace{\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right)}^{0} + \overbrace{\frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}}^{0} \right] = 0$$

or, $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = 0$$

or, $$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = 0$$

or, Since V is not a function of θ or ϕ but only r, the partial derivative may be replaced by ordinary derivative

$$\frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) = 0$$

Integrating

$$r^2 \frac{dV}{dr} = A$$

or, $dV = \frac{A}{r^2} dr$

or, $V = A\left(\frac{r^{-2+1}}{-2+1}\right) + B$

or $V = -\frac{A}{r} + B$ —— (i)

let $V = 0$ at $r = b$

$V = V_0$ at $r = a$, $a < b$

From (i)

$$0 = -\frac{A}{b} + B$$

$$V_0 = -\frac{A}{a} + B$$

$$V_0 = -\frac{A}{a} + \frac{A}{b} + 0$$

$$V_0 = \frac{A}{b} - \frac{A}{a}$$

$$A:= V_0 = A\left(\frac{1}{b} - \frac{1}{a}\right)$$

$\therefore A = \frac{V_0}{\left(\frac{1}{b} - \frac{1}{a}\right)}$ $\quad B = \frac{A}{b} = \frac{V_0}{b\left(\frac{1}{b} - \frac{1}{a}\right)}$

Now eqn (i) becomes

$$V = -\frac{V_0}{r\left(\frac{1}{b}-\frac{1}{a}\right)} + \frac{V_0}{b\left(\frac{1}{b}-\frac{1}{a}\right)} = \frac{V_0}{\left(\frac{1}{b}-\frac{1}{a}\right)}\left(\frac{1}{b}-\frac{1}{r}\right)$$

$$\therefore \quad V = V_0 \frac{\left(\frac{1}{b}-\frac{1}{r}\right)}{\left(\frac{1}{b}-\frac{1}{a}\right)}$$

(ii) $\vec{E} = \nabla V$

$$= -\left(\frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{a}_\phi\right)$$

$$= -\frac{\partial V}{\partial r}\hat{a}_r$$

$$= -\frac{\partial}{\partial r}\left[V_0 \frac{\left(\frac{1}{b}-\frac{1}{r}\right)}{\left(\frac{1}{b}-\frac{1}{a}\right)}\right]\hat{a}_r$$

$$= \frac{-V_0}{\left(\frac{1}{b}-\frac{1}{a}\right)}\frac{\partial}{\partial r}\left(\frac{1}{b}-\frac{1}{r}\right)\hat{a}_r$$

$$= \frac{-V_0}{\left(\frac{1}{b}-\frac{1}{a}\right)}\left(0+\frac{1}{r^2}\right)\hat{a}_r$$

$$= \frac{-V_0}{r^2\left(\frac{1}{b}-\frac{1}{a}\right)} \hat{a}_r = \frac{V_0}{r^2\left(\frac{1}{a}-\frac{1}{b}\right)} \hat{a}_r$$

(ii) $$\vec{D} = \varepsilon \vec{E} = \frac{\varepsilon V_0}{r^2\left(\frac{1}{a}-\frac{1}{b}\right)} \hat{a}_r$$

(iii) $$\vec{D}_s = \vec{D}\Big|_{r=a} = \frac{\varepsilon V_0}{r^2\left(\frac{1}{a}-\frac{1}{b}\right)} \hat{a}_r \Bigg|_{r=a} = \frac{\varepsilon V_0}{a^2\left(\frac{1}{a}-\frac{1}{b}\right)} \hat{a}_r$$

$$\hat{a}_N = \hat{a}_r$$

$$\vec{D}_s = \frac{\varepsilon V_0}{a^2\left(\frac{1}{a}-\frac{1}{b}\right)} \hat{a}_N = D_N \hat{a}_N$$

$$D_N = \frac{\varepsilon V_0}{a^2\left(\frac{1}{a}-\frac{1}{b}\right)}$$

(iv) $$\rho_s = D_N = \frac{\varepsilon V_0}{a^2\left(\frac{1}{a}-\frac{1}{b}\right)}$$

(v) $$Q = \int_s \rho_s \, ds = \frac{\varepsilon V_0}{a^2\left(\frac{1}{a}-\frac{1}{b}\right)} \int_s ds$$

(v) $Q = \int_S \rho_s \, ds = \dfrac{\varepsilon V_0}{a^2\left(\frac{1}{a}-\frac{1}{b}\right)} \int_S ds$

For radius a, $\int_S ds = 4\pi r^2 = 4\pi a^2$

$$Q = \frac{\varepsilon V_0}{a^2\left(\frac{1}{a}-\frac{1}{b}\right)} \times 4\pi a^2 = \frac{4\pi\varepsilon V_0}{\left(\frac{1}{a}-\frac{1}{b}\right)}.$$

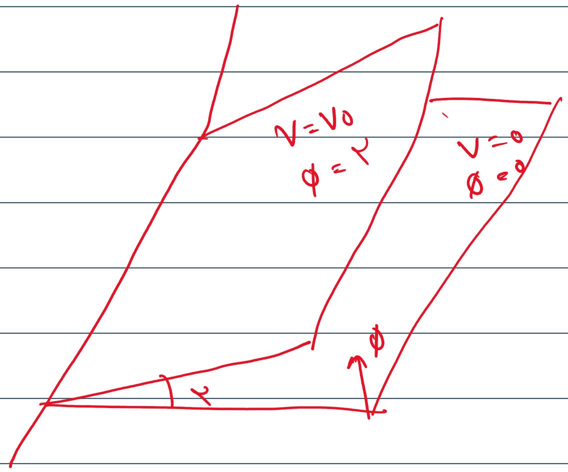
The capacitance is given as ÷

$$C = \frac{|Q|}{V_0} = \frac{4\pi\varepsilon}{\left(\frac{1}{a}-\frac{1}{b}\right)}$$

$$\therefore \boxed{C = \frac{4\pi\varepsilon}{\left(\frac{1}{a}-\frac{1}{b}\right)}}$$

Extra

Q Find the electric field intensity due to two infinite radial planes as shown in figure with an interior angle α. An infinitesimal insulating gap exists at $\rho = 0$. (cylindrical)



V is function of ϕ only

$\nabla^2 V = 0$

or, $\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \frac{1}{\rho^2}\left(\frac{\partial^2 V}{\partial\phi^2}\right) + \frac{\partial^2 V}{\partial z^2} = 0$

or, $\frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2} = 0$

or, $\frac{\partial^2 V}{\partial\phi^2} = 0$

or, $\frac{d^2 V}{d\phi^2} = 0$

or Integrating

$\frac{dV}{d\phi} = A$

Integrating

$$V = A\phi + B \quad \text{— (i)}$$

Given, $V = 0$ at $\phi = 0$

$V = V_0$ at $\phi = \alpha$

$$0 = 0 + B$$

$$V_0 = A\alpha + B$$

$$B = 0, \quad A = \frac{V_0}{\alpha}$$

Now (i) becomes

$$V = \frac{V_0}{\alpha}\phi$$

Now,

$$E = -\nabla V$$

$$= -\left(\overbrace{\frac{\partial V}{\partial \rho}}^{0}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\hat{a}_\phi + \overbrace{\frac{\partial V}{\partial z}}^{0}\hat{a}_z\right)$$

$$= -\frac{1}{\rho}\frac{\partial V}{\partial \phi}\hat{a}_\phi$$

$$= -\frac{1}{\rho}\frac{\partial}{\partial \phi}\left(\frac{V_0}{\alpha}\phi\right)\hat{a}_\phi$$

$$= -\frac{V_0}{\rho\alpha}\frac{\partial \phi}{\partial \phi}\hat{a}_\phi$$

$$= -\frac{V_0}{\rho\alpha}\hat{a}_\phi$$

$$= -\frac{V_0}{\rho 2} \hat{a}_\phi$$

$$\boxed{\therefore \vec{E} = -\frac{V_0}{\rho 2} \hat{a}_\phi}$$