AI Globalübung

ÜB5

@Ex 5.3)

Claim: $P(A \wedge B_1) + \ldots + P(A \wedge B_k) = P(A \wedge [B_1 \vee \ldots \vee B_k])$

Proof: by induction on k

IB: $(k=1)\checkmark \quad | \quad (k=0)\checkmark$

IS: $P(A \wedge B_1) + \ldots + P(A \wedge B_{k+1}) \overset{\text{ind.}}{=} P(A \wedge [B_1 \vee \ldots \vee B_k]) + P(A \wedge B_{k+1})$

$\overset{\text{axiom 3}}{=} P([A \wedge [B_1 \vee \ldots \vee B_k]] \vee [A \wedge B_{k+1}])$

$+ P(A \wedge [B_1 \vee \ldots \vee B_k]] \wedge [A \wedge B_{k+1}])$

$= P(A \wedge [B_1 \vee \ldots \vee B_{k+1}]) + \underbrace{P(A \wedge [[B_1 \vee \ldots \vee B_k] \wedge B_{k+1}])}_{\overset{*}{=} 0}$

axiom 3:

$P(X \vee Y) = P(X) + P(Y) - P(X \wedge Y)$

$\Leftrightarrow P(X) + P(Y) = P(X \vee Y) + P(X \wedge Y)$

Proof of (*): $P([B_1 \vee \ldots \vee B_k] \wedge B_l) = 0$ for $l > k$

Proof: by induction on k

IB: $(k=1)\checkmark \quad | \quad (k=0) \quad P(FALSE \wedge B_l) = P(FALSE) = 0$

IS: $P([B_1 \vee \ldots \vee B_{k+1}] \wedge B_l)$ (with $l > k+1$)

$= P([[B_1 \vee \ldots \vee B_k] \wedge B_l] \vee [B_{k+1} \wedge B_l])$

$\underset{\text{Hint 1}}{\overset{\text{induction}}{=}} P(B_{k+1} \wedge B_l) = 0 \quad \square$

$\Rightarrow P(A|B_1) \cdot P(B_1) + \ldots + P(A|B_k) \cdot P(B_k) = P(A \wedge B_1) + \ldots + P(A \wedge B_k)$

$\overset{\text{Claim}}{=} P(A \wedge [B_1 \vee \ldots \vee B_m]) \underset{\text{Hint 2}}{=} P(A)$

Ex 5.4) a) $P(Q_i|W) = 0.95 \quad i = 1,2,3$

$P(Q_1|\neg W) = 0.3$

$P(Q_2|\neg W) = 0.5$

$P(Q_3|\neg W) = 0.1$

$P(W) = \frac{4}{5} = 0.8$

b) $P(W|Q_1) \overset{\text{Bayes}}{=} \frac{P(Q_1|W) \cdot P(W)}{P(Q_1)} \overset{\text{Ex 5.3}}{=} \frac{P(Q_1|W) \cdot P(W)}{P(Q_1|W) \cdot P(W) + P(Q_1|\neg W) \cdot \underbrace{P(\neg W)}_{= 1 - P(W)}}$

$= \frac{0.95 \cdot 0.8}{0.95 \cdot 0.8 + 0.3 \cdot (1 - 0.8)} = \frac{0.76}{0.82} \approx 0.927$

$P(Q_1, Q_2|W) = P(Q_1|W) \cdot P(Q_2|W) = 0.95 \cdot 0.95 = 0.9025$

$P(Q_1, Q_2|\neg W) = P(Q_1|\neg W) \cdot P(Q_2|\neg W) = 0.3 \cdot 0.5 = 0.15$

$P(Q_3|Q_1, Q_2, W) = P(Q_3|W) = 0.95$

conditional independence of the Q_i given W (*)

(*) A and B are cond. indep. given C $\Leftrightarrow P(A|B,C) = P(A|C)$

$\Rightarrow P(A,B|C) = \frac{P(A,B,C)}{P(C)} = \frac{P(A,B,C)}{P(B,C)} \cdot \frac{P(B,C)}{P(C)}$

$= P(A|B,C) \cdot P(B|C) \overset{!}{=} P(A|C) \cdot P(B|C)$

$\Rightarrow P(A|B,C) = \frac{P(A,B,C)}{P(B,C)} = \frac{P(A,B,C)}{P(C)} \cdot \frac{P(C)}{P(B,C)} = \frac{P(A,B|C)}{P(B|C)}$

$\overset{!}{=} \frac{P(A|C) \cdot P(B|C)}{P(B|C)} = P(A|C)$

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Ex 5.3) — ÜB5

c) Bayesian Update:

$$P(W \mid Q_1, Q_2, \neg Q_3) = \alpha \cdot P(W) \cdot P(Q_1 \mid W) \cdot P(Q_2 \mid W) \cdot \overbrace{P(\neg Q_3 \mid W)}^{1 - P(Q_3 \mid W)}$$

$$= \alpha \cdot 0{,}8 \cdot 0{,}95 \cdot 0{,}95 \cdot (1 - 0{,}95) = 0{,}0361 \cdot \alpha \approx 0{,}0572 = 57{,}2\,\%$$

$$P(\neg W \mid Q_1, Q_2, \neg Q_3) = \alpha \cdot P(\neg W) \cdot P(Q_1 \mid \neg W) \cdot P(Q_2 \mid \neg W) \cdot \overbrace{P(\neg Q_3 \mid \neg W)}^{1 - P(Q_3 \mid \neg W)}$$

$$= \alpha \cdot (1 - 0{,}8) \cdot 0{,}3 \cdot 0{,}5 \cdot (1 - 0{,}1) = 0{,}0270\, \alpha \approx 0{,}428 = 42{,}8\,\%$$

$$\begin{array}{r} 0{,}0361 \cdot \alpha \\ 0{,}0270 \cdot \alpha \\ \hline 1 = 0{,}0631\, \alpha \end{array}$$

$$\Rightarrow \alpha = \frac{1}{0{,}0631} \approx 15{,}84786\ldots$$

d) $P(W \mid Q_1, \neg Q_2, \neg Q_3) = \beta \cdot 0{,}8 \cdot 0{,}95 \cdot (1 - 0{,}95)(1 - 0{,}95) = 0{,}0019\, \beta \approx 0{,}066 = 6{,}6\,\%$

$P(\neg W \mid Q_1, \neg Q_2, \neg Q_3) = \beta \cdot (1 - 0{,}8) \cdot 0{,}3 \cdot (1 - 0{,}5)(1 - 0{,}1) = 0{,}0270 \cdot \beta \approx 0{,}934 = 93{,}4\,\%$

$1 = 0{,}0289 \cdot \beta$

$\Rightarrow \beta = \frac{1}{0{,}0289}$

e) Conditional Independence!

$\not\Rightarrow P(Q_1, Q_2) = P(Q_1) \cdot P(Q_2)$ (i.e. general/non-cond. independence)