Exercise 4.4 - Solution

Are the following statements correct or not? Prove each of your claims either by means of resolution or by specifying a suitable interpretation as a counterexample.

The numbers in the transformations below stand for the respective step (c.f. Resolution slide 14/37):

1. Eliminate $\supset$ and $\equiv$.
2. Push $\neg$ inside.
3. Rename variables to make them syntactically distinct.
4. Eliminate $\exists$'s by Skolemization.
5. Move $\forall$'s to the left.
6. Distribute $\vee$ over $\wedge$ $((\alpha \wedge \beta) \vee \gamma \equiv (\alpha \vee \gamma) \wedge (\beta \vee \gamma))$
7. Simplify.

Conversions into CNF:

1. $\{\ \exists x P(x)\ ,\ \exists x Q(x)\ \} \quad \models \quad \exists x[P(x) \wedge Q(x)]$

$$
\begin{aligned}
& \{\exists x P(x), \exists x Q(x)\} \cup \{\neg \exists x[P(x) \wedge Q(x)]\} \\
\triangleq\ & \exists x P(x) \wedge \exists x Q(x) \wedge \neg \exists x[P(x) \wedge Q(x)] \\
\overset{1.}{\equiv}\ & \quad \text{— '' —} \\
\overset{2.}{\equiv}\ & \exists x P(x) \wedge \exists x Q(x) \wedge \forall x[\neg P(x) \vee \neg Q(x)] \\
\overset{3.}{\equiv}\ & \exists x P(x) \wedge \exists y Q(y) \wedge \forall z[\neg P(z) \vee \neg Q(z)] \\
\overset{4.}{\rightsquigarrow}\ & P(\boldsymbol{f}) \wedge Q(\boldsymbol{f'}) \wedge \forall z[\neg P(z) \vee \neg Q(z)] \\
\overset{5.}{\equiv}\ & \forall z[\ P(\boldsymbol{f}) \wedge Q(\boldsymbol{f'}) \wedge [\neg P(z) \vee \neg Q(z)]\ ] \\
\overset{6.}{\equiv}\ & \quad \text{— '' —} \\
\overset{7.}{\equiv}\ & \quad \text{— '' —} \\
\triangleq\ & \{\ [P(\boldsymbol{f})]\ ,\ [Q(\boldsymbol{f'})]\ ,\ [\sim P(z), \sim Q(z)]\ \}
\end{aligned}
$$

2. $\{\ \forall x P(x) \vee \forall x Q(x)\ \} \quad \models \quad \forall x[P(x) \vee Q(x)]$

$$
\begin{aligned}
& \{\forall x P(x) \vee \forall x Q(x)\} \cup \{\neg \forall x[P(x) \vee Q(x)]\} \\
\triangleq\ & [\forall x P(x) \vee \forall x Q(x)] \wedge \neg \forall x[P(x) \vee Q(x)] \\
\overset{1.}{\equiv}\ & \quad \text{— '' —} \\
\overset{2.}{\equiv}\ & [\forall x P(x) \vee \forall x Q(x)] \wedge \exists x[\neg P(x) \wedge \neg Q(x)] \\
\overset{3.}{\equiv}\ & [\forall x P(x) \vee \forall y Q(y)] \wedge \exists z[\neg P(z) \wedge \neg Q(z)] \\
\overset{4.}{\rightsquigarrow}\ & [\forall x P(x) \vee \forall y Q(y)] \wedge [\neg P(\boldsymbol{f}) \wedge \neg Q(\boldsymbol{f})] \\
\overset{5.}{\equiv}\ & \forall x \forall y[\ [P(x) \vee Q(y)] \wedge \neg P(\boldsymbol{f}) \wedge \neg Q(\boldsymbol{f})\ ] \\
\overset{6.}{\equiv}\ & \quad \text{— '' —} \\
\overset{7.}{\equiv}\ & \quad \text{— '' —} \\
\triangleq\ & \{\ [P(x), Q(y)]\ ,\ [\sim P(\boldsymbol{f})]\ ,\ [\sim Q(\boldsymbol{f})]\ \}
\end{aligned}
$$

3. $\models\ (\forall x P(x) \wedge \forall x Q(x)\,) \equiv \forall x[P(x) \wedge Q(x)]$

$$
\begin{aligned}
& \{\} \cup \{\neg[(\forall x P(x) \wedge \forall x Q(x)) \equiv \forall x[P(x) \wedge Q(x)]]\} \\
\hat{=}\;& \neg[(\forall x P(x) \wedge \forall x Q(x)) \equiv \forall x[P(x) \wedge Q(x)]] \\
\overset{1.}{\equiv}\;& \neg[[\neg(\forall x P(x) \wedge \forall x Q(x)) \vee \forall x[P(x) \wedge Q(x)]] \wedge [\neg\forall x[P(x) \wedge Q(x)] \vee (\forall x P(x) \wedge \forall x Q(x))]] \\
\overset{2.}{\equiv}\;& [\forall x P(x) \wedge \forall x Q(x) \wedge \exists x[\neg P(x) \vee \neg Q(x)]] \vee [\forall x[P(x) \wedge Q(x)] \wedge (\exists x \neg P(x) \vee \exists x \neg Q(x))] \\
\overset{3.}{\equiv}\;& [\forall u P(u) \wedge \forall v Q(v) \wedge \exists w[\neg P(w) \vee \neg Q(w)]] \vee [\forall x[P(x) \wedge Q(x)] \wedge (\exists y \neg P(y) \vee \exists z \neg Q(z))] \\
\overset{4.}{\rightsquigarrow}\;& [[\forall u P(u) \wedge \forall v Q(v) \wedge [\neg P(\boldsymbol{f}) \vee \neg Q(\boldsymbol{f})]] \vee [\forall x[P(x) \wedge Q(x)] \wedge (\neg P(\boldsymbol{f'}) \vee \neg Q(\boldsymbol{f''}))]] \\
\overset{5.}{\equiv}\;& \forall u \forall v \forall x[\,[P(u) \wedge Q(v) \wedge [\neg P(\boldsymbol{f}) \vee \neg Q(\boldsymbol{f})]] \vee [P(x) \wedge Q(x) \wedge (\neg P(\boldsymbol{f'}) \vee \neg Q(\boldsymbol{f''}))]\,] \\
\overset{6.}{\equiv}\;& \forall u \forall v \forall x[\quad [P(u) \vee P(x)] \wedge [P(u) \vee Q(x)] \wedge [P(u) \vee \neg P(\boldsymbol{f'}) \vee \neg Q(\boldsymbol{f''})] \wedge \\
& \qquad [Q(v) \vee P(x)] \wedge [Q(v) \vee Q(x)] \wedge [Q(v) \vee \neg P(\boldsymbol{f'}) \vee \neg Q(\boldsymbol{f''})] \wedge \\
& \qquad [\neg P(\boldsymbol{f}) \vee \neg Q(\boldsymbol{f}) \vee P(x)] \wedge [\neg P(\boldsymbol{f}) \vee \neg Q(\boldsymbol{f}) \vee Q(x)] \wedge [\neg P(\boldsymbol{f}) \vee \neg Q(\boldsymbol{f}) \vee \neg P(\boldsymbol{f'}) \vee \neg Q(\boldsymbol{f''})] \quad] \\
\overset{7.}{\equiv}\;& \quad \text{— '' —} \\
\hat{=}\;& \{\quad [P(u), P(x)]\,,\ [P(u), Q(x)]\,,\ [P(u), \sim P(\boldsymbol{f'}), \sim Q(\boldsymbol{f''})]\,, \\
& \qquad [Q(v), P(x)]\,,\ [Q(v), Q(x)]\,,\ [Q(v), \sim P(\boldsymbol{f'}), \sim Q(\boldsymbol{f''})]\,, \\
& \qquad [\sim P(\boldsymbol{f}), \sim Q(\boldsymbol{f}), P(x)]\,,\ [\sim P(\boldsymbol{f}), \sim Q(\boldsymbol{f}), Q(x)]\,,\ [\sim P(\boldsymbol{f}), \sim Q(\boldsymbol{f}), \sim P(\boldsymbol{f'}), \sim Q(\boldsymbol{f''})] \quad \}
\end{aligned}
$$

Alternatively, show validity of the $\supset$ and $\subset$ directions of $\equiv$ separately:

"$\supset$" $\models\ (\forall x P(x) \wedge \forall x Q(x)\,) \supset \forall x[P(x) \wedge Q(x)]$

$$
\begin{aligned}
& \{\} \cup \{\neg[(\forall x P(x) \wedge \forall x Q(x)) \supset \forall x[P(x) \wedge Q(x)]]\} \\
\hat{=}\;& \neg[(\forall x P(x) \wedge \forall x Q(x)) \supset \forall x[P(x) \wedge Q(x)]] \\
\overset{1.}{\equiv}\;& \neg[\neg(\forall x P(x) \wedge \forall x Q(x)) \vee \forall x[P(x) \wedge Q(x)]] \\
\overset{2.}{\equiv}\;& \forall x P(x) \wedge \forall x Q(x) \wedge \exists x[\neg P(x) \vee \neg Q(x)] \\
\overset{3.}{\equiv}\;& \forall x P(x) \wedge \forall y Q(y) \wedge \exists z[\neg P(z) \vee \neg Q(z)] \\
\overset{4.}{\rightsquigarrow}\;& \forall x P(x) \wedge \forall y Q(y) \wedge [\neg P(\boldsymbol{f}) \vee \neg Q(\boldsymbol{f})] \\
\overset{5.}{\equiv}\;& \forall x \forall y[\,P(x) \wedge Q(y) \wedge [\neg P(\boldsymbol{f}) \vee \neg Q(\boldsymbol{f})]\,] \\
\overset{6.}{\equiv}\;& \quad \text{— '' —} \\
\overset{7.}{\equiv}\;& \quad \text{— '' —} \\
\hat{=}\;& \{\,[P(x)]\,,\ [Q(y)]\,,\ [\sim P(\boldsymbol{f}), \sim Q(\boldsymbol{f})]\,\}
\end{aligned}
$$

"$\subset$" $\models\ \forall x[P(x) \wedge Q(x)] \supset (\,\forall x P(x) \wedge \forall x Q(x)\,)$

$$
\begin{aligned}
& \{\} \cup \{\neg[\forall x[P(x) \wedge Q(x)] \supset (\forall x P(x) \wedge \forall x Q(x))]\} \\
\hat{=}\;& \neg[\forall x[P(x) \wedge Q(x)] \supset (\forall x P(x) \wedge \forall x Q(x))] \\
\overset{1.}{\equiv}\;& \neg[\neg\forall x[P(x) \wedge Q(x)] \vee (\forall x P(x) \wedge \forall x Q(x))] \\
\overset{2.}{\equiv}\;& \forall x[P(x) \wedge Q(x)] \wedge (\exists x \neg P(x) \vee \exists x \neg Q(x)) \\
\overset{3.}{\equiv}\;& \forall x[P(x) \wedge Q(x)] \wedge (\exists y \neg P(y) \vee \exists z \neg Q(z)) \\
\overset{4.}{\rightsquigarrow}\;& \forall x[P(x) \wedge Q(x)] \wedge (\neg P(\boldsymbol{f}) \vee \neg Q(\boldsymbol{f'})) \\
\overset{5.}{\equiv}\;& \forall x[\,P(x) \wedge Q(x) \wedge (\neg P(\boldsymbol{f}) \vee \neg Q(\boldsymbol{f'}))\,] \\
\overset{6.}{\equiv}\;& \quad \text{— '' —} \\
\overset{7.}{\equiv}\;& \quad \text{— '' —} \\
\hat{=}\;& \{\,[P(x)]\,,\ [Q(x)]\,,\ [\sim P(\boldsymbol{f}), \sim Q(\boldsymbol{f'})]\,\}
\end{aligned}
$$

4. $\models \neg\phi$ where ϕ is $\forall x[P(x) \supset Q(x, g(x))] \wedge \exists x[P(g(x)) \wedge \neg Q(g(x), g(g(x)))]$

$$
\begin{aligned}
& \{\} \cup \{\neg\neg\phi\} \\
\stackrel{\wedge}{=}\; & \neg\neg[\forall x[P(x) \supset Q(x, g(x))] \wedge \exists x[P(g(x)) \wedge \neg Q(g(x), g(g(x)))]] \\
\stackrel{1.}{\equiv}\; & \neg\neg[\forall x[\neg P(x) \vee Q(x, g(x))] \wedge \exists x[P(g(x)) \wedge \neg Q(g(x), g(g(x)))]] \\
\stackrel{2.}{\equiv}\; & \forall x[\neg P(x) \vee Q(x, g(x))] \wedge \exists x[P(g(x)) \wedge \neg Q(g(x), g(g(x)))] \\
\stackrel{3.}{\equiv}\; & \forall x[\neg P(x) \vee Q(x, g(x))] \wedge \exists y[P(g(y)) \wedge \neg Q(g(y), g(g(y)))] \\
\stackrel{4.}{\rightsquigarrow}\; & \forall x[\neg P(x) \vee Q(x, g(x))] \wedge P(g(\boldsymbol{f})) \wedge \neg Q(g(\boldsymbol{f}), g(g(\boldsymbol{f}))) \\
\stackrel{5.}{\equiv}\; & \forall x[\, [\neg P(x) \vee Q(x, g(x))] \wedge P(g(\boldsymbol{f})) \wedge \neg Q(g(\boldsymbol{f}), g(g(\boldsymbol{f})))\,] \\
\stackrel{6.}{\equiv}\; & \quad \text{— '' —} \\
\stackrel{7.}{\equiv}\; & \quad \text{— '' —} \\
\stackrel{\wedge}{=}\; & \{\, [\sim P(x), Q(x, g(x))]\, ,\, [P(g(\boldsymbol{f}))]\, ,\, [\sim Q(g(\boldsymbol{f}), g(g(\boldsymbol{f})))]\, \}
\end{aligned}
$$

5. $\{\, \forall x \exists y[\quad (\, P(y) \supset P(f(x))\,) \wedge (\, P(f(x)) \supset Q(x, f(y))\,) \wedge (\, Q(x, f(x)) \vee P(y)\,) \quad]\, \} \quad \models \quad \forall x \exists y Q(x, y)$

$$
\begin{aligned}
& \{\forall x \exists y[(P(y) \supset P(f(x))) \wedge (P(f(x)) \supset Q(x, f(y))) \wedge (Q(x, f(x)) \vee P(y))]\} \cup \{\neg \forall x \exists y Q(x, y)\} \\
\stackrel{\wedge}{=}\; & \forall x \exists y[(P(y) \supset P(f(x))) \wedge (P(f(x)) \supset Q(x, f(y))) \wedge (Q(x, f(x)) \vee P(y))] \wedge \neg \forall x \exists y Q(x, y) \\
\stackrel{1.}{\equiv}\; & \forall x \exists y[(\neg P(y) \vee P(f(x))) \wedge (\neg P(f(x)) \vee Q(x, f(y))) \wedge (Q(x, f(x)) \vee P(y))] \wedge \neg \forall x \exists y Q(x, y) \\
\stackrel{2.}{\equiv}\; & \forall x \exists y[(\neg P(y) \vee P(f(x))) \wedge (\neg P(f(x)) \vee Q(x, f(y))) \wedge (Q(x, f(x)) \vee P(y))] \wedge \exists x \forall y \neg Q(x, y) \\
\stackrel{3.}{\equiv}\; & \forall x \exists y[(\neg P(y) \vee P(f(x))) \wedge (\neg P(f(x)) \vee Q(x, f(y))) \wedge (Q(x, f(x)) \vee P(y))] \wedge \exists v \forall z \neg Q(v, z) \\
\stackrel{4.}{\rightsquigarrow}\; & \forall x[(\neg P(\boldsymbol{f'}(x)) \vee P(f(x))) \wedge (\neg P(f(x)) \vee Q(x, f(\boldsymbol{f'}(x)))) \wedge (Q(x, f(x)) \vee P(\boldsymbol{f'}(x)))] \wedge \forall z \neg Q(\boldsymbol{f''}, z) \\
\stackrel{5.}{\equiv}\; & \forall x \forall z[(\neg P(\boldsymbol{f'}(x)) \vee P(f(x))) \wedge (\neg P(f(x)) \vee Q(x, f(\boldsymbol{f'}(x)))) \wedge (Q(x, f(x)) \vee P(\boldsymbol{f'}(x))) \wedge \neg Q(\boldsymbol{f''}, z)] \\
\stackrel{6.}{\equiv}\; & \quad \text{— '' —} \\
\stackrel{7.}{\equiv}\; & \quad \text{— '' —} \\
\stackrel{\wedge}{=}\; & \{[\sim P(\boldsymbol{f'}(x)), P(f(x))]\, ,\, [\sim P(f(x)), Q(x, f(\boldsymbol{f'}(x)))]\, ,\, [Q(x, f(x)), P(\boldsymbol{f'}(x))]\, ,\, [\sim Q(\boldsymbol{f''}, z)]\}
\end{aligned}
$$

6. $\{\, \forall x \exists y P(x, y)\, ,\, \neg \exists z P(z, a)\, \} \models \exists y \forall x P(x, y)$

$$
\begin{aligned}
& \{\forall x \exists y P(x, y), \neg \exists z P(z, a)\} \cup \{\neg \exists y \forall x P(x, y)\} \\
\stackrel{\wedge}{=}\; & \forall x \exists y P(x, y) \wedge \neg \exists z P(z, a) \wedge \neg \exists y \forall x P(x, y) \\
\stackrel{1.}{\equiv}\; & \quad \text{— '' —} \\
\stackrel{2.}{\equiv}\; & \forall x \exists y P(x, y) \wedge \forall z \neg P(z, a) \wedge \forall y \exists x \neg P(x, y) \\
\stackrel{3.}{\equiv}\; & \forall x \exists y P(x, y) \wedge \forall z \neg P(z, a) \wedge \forall v \exists w \neg P(w, v) \\
\stackrel{4.}{\rightsquigarrow}\; & \forall x P(x, \boldsymbol{f}(x)) \wedge \forall z \neg P(z, a) \wedge \forall v \neg P(\boldsymbol{f'}(v), v) \\
\stackrel{5.}{\equiv}\; & \forall x \forall z \forall v[\, P(x, \boldsymbol{f}(x)) \wedge \neg P(z, a) \wedge \neg P(\boldsymbol{f'}(v), v)\,] \\
\stackrel{6.}{\equiv}\; & \quad \text{— '' —} \\
\stackrel{7.}{\equiv}\; & \quad \text{— '' —} \\
\stackrel{\wedge}{=}\; & \{\, [P(x, \boldsymbol{f}(x))]\, ,\, [\sim P(z, a)]\, ,\, [\sim P(\boldsymbol{f'}(v), v)]\, \}
\end{aligned}
$$

Note that these transformations into CNF are only necessary for proving the statements with resolution. If a statement is not true, this is shown by specifying an interpretation as counterexample. A CNF transformation is not required in this case, but only shown here for the sake of completeness.