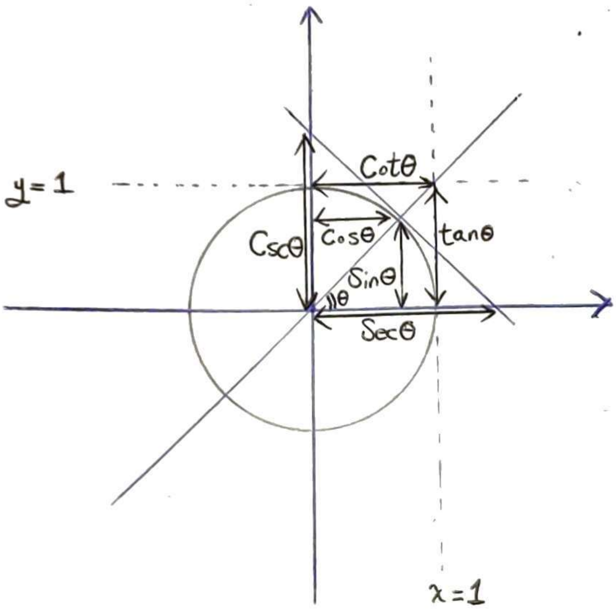
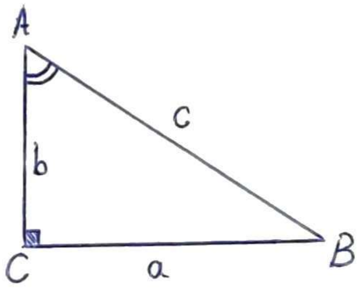
Trigonometry

① Basics





- $\sin \hat{A} = \frac{a}{c}$
- $\cos \hat{A} = \frac{b}{c}$
- $\tan \hat{A} = \frac{a}{b}$
- $\cot \hat{A} = \frac{b}{a}$
- $\sec \hat{A} = \frac{c}{b}$
- $\csc \hat{A} = \frac{c}{a}$

→ $\sec \hat{A} = \frac{1}{\cos \hat{A}}$

→ $\csc \hat{A} = \frac{1}{\sin \hat{A}}$

→ $\cot \hat{A} = \frac{1}{\tan \hat{A}}$

② Basic Relations

- $\sin^2\alpha + \cos^2\alpha = 1$
- $\sec^2\alpha = \frac{1}{\cos^2\alpha} = 1 + \tan^2\alpha$
- $\csc^2\alpha = \frac{1}{\sin^2\alpha} = 1 + \cot^2\alpha$
- $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$
- $\cot\alpha = \frac{\cos\alpha}{\sin\alpha}$

③ Mixed Angles

- $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$
- $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$
- $\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$
- $\sin 2\alpha = 2\sin\alpha\cos\alpha$
- $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$
- $\cos 2\alpha = 1 - 2\sin^2\alpha \rightarrow \sin^2\alpha = \frac{1-\cos 2\alpha}{2}$
- $\cos 2\alpha = 2\cos^2\alpha - 1 \rightarrow \cos^2\alpha = \frac{1+\cos 2\alpha}{2}$

$\Rightarrow \tan^2\alpha = \frac{1-\cos 2\alpha}{1+\cos 2\alpha}$

- $\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$

④

- $\sin^2\alpha + \cos^2\alpha = 1 - 0\sin^2 2\alpha$
- $\sin^4\alpha + \cos^4\alpha = 1 - \frac{1}{2}\sin^2 2\alpha$
- $\sin^6\alpha + \cos^6\alpha = 1 - \frac{3}{4}\sin^2 2\alpha$

⋮

⑤

- $A\sin\theta + B\cos\theta = K\sin(\theta+\alpha)$

$$\begin{cases} K = \sqrt{A^2+B^2} \\ \tan\alpha = \frac{B}{A} \end{cases}$$

⑥ Sum To Product

- $\sin p + \sin q = 2\sin\left(\frac{p+q}{2}\right)\cos\left(\frac{p-q}{2}\right)$
- $\sin p - \sin q = 2\cos\left(\frac{p+q}{2}\right)\sin\left(\frac{p-q}{2}\right)$
- $\cos p + \cos q = 2\cos\left(\frac{p+q}{2}\right)\cos\left(\frac{p-q}{2}\right)$
- $\cos p - \cos q = 2\sin\left(\frac{p+q}{2}\right)\sin\left(\frac{p-q}{2}\right)$

7 Product To Sum

$$\sin\alpha\cos\beta = \frac{1}{2}\left(\sin(\alpha+\beta)+\sin(\alpha-\beta)\right)$$

$$\cos\alpha\cos\beta = \frac{1}{2}\left(\cos(\alpha+\beta)+\cos(\alpha-\beta)\right)$$

$$\sin\alpha\sin\beta = -\frac{1}{2}\left(\cos(\alpha+\beta)-\cos(\alpha-\beta)\right)$$

8

$$\tan p+\tan q = \frac{\sin(p+q)}{\cos p\cos q}$$

$$\cot p+\cot q = \frac{\sin(p+q)}{\sin p\sin q}$$

$$\tan p-\tan q = \frac{\sin(p-q)}{\cos p\cos q}$$

$$\cot p-\cot q = \frac{-\sin(p-q)}{\sin p\sin q}$$

9 3α (4-3-3 system)

$$\sin 3\alpha = -4\sin^{3}\alpha + 3\sin\alpha$$

$$\cos 3\alpha = 4\cos^{3}\alpha - 3\cos\alpha$$

$$\tan 3\alpha = \frac{-\tan^{3}\alpha + 3\tan\alpha}{1-3\tan^{2}\alpha}$$

10 What Is $\sin\alpha$ & $\cos\alpha$?

$$\sin\alpha = \frac{2\tan\frac{\alpha}{2}}{1+\tan^{2}\frac{\alpha}{2}}$$

$$\cos\alpha = \frac{1-\tan^{2}\frac{\alpha}{2}}{1+\tan^{2}\frac{\alpha}{2}}$$

11

$$\begin{cases} \sin^{2}\alpha = \frac{1}{2}(1-\cos 2\alpha) \\ \cos^{2}\alpha = \frac{1}{2}(1+\cos 2\alpha) \end{cases}$$

θ	$0°$	$30°$	$45°$	$60°$	$90°$
	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
Sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
Cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$-$
Cot	$-$	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0

- $y = \overbrace{\arcsin x}^{\sin^{-1}x} \rightsquigarrow \boxed{x=\sin y}$ $\quad -1\leqslant x\leqslant 1 \quad -\pi/2\leqslant y\leqslant \pi/2$
- $y = \overbrace{\arccos x}^{\cos^{-1}x} \rightsquigarrow \boxed{x=\cos y}$ $\quad -1\leqslant x\leqslant 1 \quad 0\leqslant y\leqslant \pi$
- $y = \overbrace{\arctan x}^{\tan^{-1}x} \rightsquigarrow \boxed{x=\tan y}$ $\quad \mathbb{R} \quad -\pi/2< y< \pi/2$
- $y = \overbrace{\operatorname{arccot} x}^{\cot^{-1}x} \rightsquigarrow \boxed{x=\cot y}$ $\quad \mathbb{R} \quad 0< y< \pi$
- $y = \overbrace{\operatorname{arcsec} x}^{\sec^{-1}x} \rightsquigarrow \boxed{x=\sec y}$ $\quad x\leqslant -1$ or $x\geqslant 1 \quad 0\leqslant y\leqslant \pi/2$ or $\pi/2< y\leqslant \pi$
- $y = \overbrace{\operatorname{arccsc} x}^{\csc^{-1}x} \rightsquigarrow \boxed{x=\csc y}$ $\quad x\leqslant -1$ or $x\geqslant 1 \quad -\pi/2\leqslant y<0$ or $0<y\leqslant \pi/2$

Logarithm

① Basic Definition

$$\log_b a = c \rightsquigarrow b^c = a$$

② Limitations

$$\log_v u \qquad \begin{matrix} u > 0 \\ v > 0 \\ v \neq 1 \end{matrix}$$

③ Base

⊛ Base 10 : $\log_{10} u = \log u$

⊛ Base e (Euler's number) : $\log_e u = \ln u$ (Natural Logarithm)

$e \simeq 2.7$

④ Formulas

1) $\log_a a = 1$

2) $\log_a 1 = 0$

3) $\log_K AB = \log_K A + \log_K B$

4) $\log_K \left(\frac{A}{B}\right) = \log_K A - \log_K B$

5) $\log_{B^m} A^n = \frac{n}{m} \log_B A$

6) $\log_B A = \dfrac{\log_K A}{\log_K B}$

7) $\log_b a \times \log_a b = 1$

8) $A^{\log_A K} = K$

Identities or Factorizations

① Binomial Expansions

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

② Difference of two squares

$$\boxed{(a-b)(a+b) = a^2 - b^2}$$

③

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$$

④ Sum of two cubes

$$\boxed{a^3 + b^3 = (a+b)(a^2 - ab + b^2)}$$

⑤ Difference of two cubes

$$\boxed{a^3 - b^3 = (a-b)(a^2 + ab + b^2)}$$

⑥ Euler's Factorization Method

$$(a+b+c)(a^2+b^2+c^2-ab-ac-bc) = a^3+b^3+c^3-3abc$$

⑦ Lagrange Polynomial

$$(a^2+b^2)(x^2+y^2) = (ax-by)^2 + (ay+bx)^2$$

⑧ Newton's Generalized Binomial Theoreom

$$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \dots + \binom{n}{n}a^0 b^n$$

⑨

$$(a+b+c+\dots)^2 = a^2+b^2+c^2+\dots+2(ab+ac+bc+\dots)$$

{ Derivative }

⊛ $f(x) = C \longrightarrow f'(x) = 0$

⊛ $f(x) = x^n \longrightarrow f'(x) = n x^{n-1}$, $(n \neq 0)$ $\longrightarrow$ $\boxed{f(x) = u^n \rightarrow f'(x) = n u' u^{n-1}}$

⊛ $f(x) = \sqrt{x} \longrightarrow f'(x) = \dfrac{1}{2\sqrt{x}}$ $\longrightarrow$ $\boxed{f(x) = \sqrt{u} \rightarrow f'(x) = \dfrac{u'}{2\sqrt{u}}}$

$\boxed{f(x) = \sqrt[n]{u^m} \rightarrow f'(x) = \dfrac{m u'}{n \sqrt[n]{u^{n-m}}}}$

⊛ $f(x) = e^x \longrightarrow f'(x) = e^x$ $\longrightarrow$ $\boxed{f(x) = e^u \longrightarrow f'(x) = u' e^u}$

$\boxed{f(x) = a^u \longrightarrow f'(x) = u' a^u \times \ln a}$

⊛ $f(x) = \ln x \longrightarrow f'(x) = \dfrac{1}{x}$ $\longrightarrow$ $\boxed{f(x) = \ln u \longrightarrow f'(x) = \dfrac{u'}{u}}$ $\rightsquigarrow u > 0$

$\boxed{f(x) = \log_a u \longrightarrow f'(x) = \dfrac{u'}{u} \times \dfrac{1}{\ln a}}$ $\rightsquigarrow u > 0$

⊛ $f(x) = \sin x \longrightarrow f'(x) = \cos x$ $\longrightarrow$ $\boxed{f(x) = \sin u \rightarrow f'(x) = u' \cos u}$

⊛ $f(x) = \cos x \longrightarrow f'(x) = -\sin x$ $\longrightarrow$ $\boxed{f(x) = \cos u \rightarrow f'(x) = -u' \sin u}$

⊛ $f(x) = \tan x \longrightarrow f'(x) = 1 + \tan^2 x$ $\longrightarrow$ $\boxed{f(x) = \tan u \rightarrow f'(x) = u'(1 + \tan^2 u) = u' \sec^2 u}$

$u \neq k\pi + \pi/2 \leftsquigarrow$

⊛ $f(x) = \cot x \longrightarrow f'(x) = -(1 + \cot^2 x)$ $\longrightarrow$ $\boxed{f(x) = \cot u \rightarrow f'(x) = -u'(1 + \cot^2 u) = -u' \csc^2 u}$

$u \neq k\pi \leftsquigarrow$

⊛ $f(x) = \sec x \longrightarrow f'(x) = \sec x \tan x$ $\longrightarrow$ $\boxed{f(x) = \sec u \rightarrow f'(x) = u' \sec u \tan u}$ $\rightsquigarrow u \neq k\pi + \pi/2$

⊛ $f(x) = \csc x \longrightarrow f'(x) = -\csc x \cot x$ $\longrightarrow$ $\boxed{f(x) = \csc u \rightarrow f'(x) = -u' \csc u \cot u}$ $\rightsquigarrow u \neq k\pi$

⊛ $f(x) = \arcsin x \longrightarrow f'(x) = \dfrac{1}{\sqrt{1 - x^2}}$ $\longrightarrow$ $\boxed{f(x) = \arcsin u \rightarrow f'(x) = \dfrac{u'}{\sqrt{1 - u^2}}}$ $\rightsquigarrow |u| < 1$

⊛ $f(x) = \arccos x \longrightarrow f'(x) = \dfrac{-1}{\sqrt{1 - x^2}}$ $\longrightarrow$ $\boxed{f(x) = \arccos u \rightarrow f'(x) = \dfrac{-u'}{\sqrt{1 - u^2}}}$ $\rightsquigarrow |u| < 1$

⊛ $f(x) = \arctan x \longrightarrow f'(x) = \dfrac{1}{1 + x^2}$ $\longrightarrow$ $\boxed{f'(x) = \arctan u \rightarrow f'(x) = \dfrac{u'}{1 + u^2}}$

⊛ $f(x)=\operatorname{Arc}\operatorname{Cot}x \longrightarrow f'(x)=\frac{-1}{1+x^2} \longrightarrow \boxed{f(x)=\operatorname{Arc}\operatorname{Cot}u \longrightarrow f'(x)=\frac{-u'}{1+u^2}}$

⊛ $f(x)=\operatorname{Arc}\operatorname{Sec}x \longrightarrow f'(x)=\frac{1}{|x|\sqrt{x^2-1}} \longrightarrow \boxed{f(x)=\operatorname{Arc}\sec u \longrightarrow f'(x)=\frac{u'}{|u|\sqrt{u^2-1}}} \rightsquigarrow |u|>1$

⊛ $f(x)=\operatorname{Arc}\operatorname{Csc}x \longrightarrow f'(x)=\frac{-1}{|x|\sqrt{x^2-1}} \longrightarrow \boxed{f(x)=\operatorname{Arc}\operatorname{Csc}u \longrightarrow f'(x)=\frac{-u'}{|u|\sqrt{u^2-1}}} \rightarrow |u|>1$

⊛ $f(x)=\operatorname{Sinh}u \longrightarrow f'(x)=u'\operatorname{Cosh}u$

⊛ $f(x)=\operatorname{Cosh}u \longrightarrow f'(x)=u'\operatorname{Sinh}u$

⊛ $f(x)=\tanh u \longrightarrow f'(x)=u'\operatorname{sech}^2 u=\frac{u'}{\operatorname{Cosh}^2 u}$

⊛ $f(x)=\operatorname{Coth}u \longrightarrow f'(x)=-u'\operatorname{Csch}^2 u=\frac{-u'}{\operatorname{Sinh}^2 u}\ ,\ u\neq 0$

⊛ $f(x)=\operatorname{sech}u \longrightarrow f'(x)=-u'\operatorname{sech}u\tanh u$

⊛ $f(x)=\operatorname{Csch}u \longrightarrow f'(x)=-u'\operatorname{csch}u\operatorname{Coth}u\ ,\ u\neq 0$

~~~~~~

⊛ $f(x)=\operatorname{Arc}\operatorname{Sinh}u \longrightarrow f'(x)=\frac{u'}{\sqrt{u^2+1}}$

⊛ $f(x)=\operatorname{Arc}\operatorname{Cosh}u \longrightarrow f'(x)=\frac{u'}{\sqrt{u^2-1}}\ ,\ u>1$

⊛ $f(x)=\operatorname{Arc}\tanh u \longrightarrow f'(x)=\frac{u'}{1-u^2}\ ,\ |u|<1$

⊛ $f(x)=\operatorname{Arc}\operatorname{Coth}u \longrightarrow f'(x)=\frac{u'}{1-u^2}\ ,\ |u|>1$

⊛ $f(x)=\operatorname{Arc}\operatorname{sech}u \longrightarrow f'(x)=-\frac{u'}{u\sqrt{1-u^2}}\ ,\ 0<u<1$

⊛ $f(x)=\operatorname{Arc}\operatorname{Csch}u \longrightarrow f'(x)=-\frac{u'}{|u|\sqrt{1+u^2}}\ ,\ u\neq 0$
~~~~~~

Integral

$$\int x^m dx = \frac{x^{m+1}}{m+1} + C \quad m \neq -1$$

if $m=-1$ $\int \frac{1}{x} dx = \ln x + C$

$\int K dx = Kx + C$

$$\int u' u^m dx = \frac{u^{m+1}}{m+1} + C \quad m \neq -1$$

if $m=-1$ $\int \frac{u'}{u} dx = \ln u + C$

$$\int \sin x \, dx = -\cos x + C$$

$$\int \sin Kx \, dx = -\frac{1}{K} \cos Kx + C$$

$$\int u' \sin u = -\cos u + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \cos Kx \, dx = \frac{1}{K} \sin Kx + C$$

$$\int u' \cos u = \sin u + C$$

Reminder: $\begin{cases} \sin^2\alpha = 1/2\,(1 - \cos 2\alpha) \\ \cos^2\alpha = 1/2\,(1 + \cos 2\alpha) \end{cases}$

$$\int \sin^3 x = -\cos x + \frac{1}{3} \cos^3 x + C$$

$$\int \cos^3 x = \sin x - \frac{1}{3} \sin^3 x + C$$

$$\int \tan x = -\ln|\cos x| + C = \ln|\sec x| + C$$

$$\int \tan Kx = -\frac{1}{K} \ln|\cos Kx| + C$$

$$\int \sec^2 x \, dx = \int \frac{1}{\cos^2 x} dx = \int (1 + \tan^2 x) dx = \tan x + C$$

$$\int (1 + \tan^2 Kx) \, dx = \frac{1}{K} \tan Kx + C$$

$$\int u'(1 + \tan^2 u) = \tan u + C$$

$$\int \cot x \, dx = \ln|\sin x| + C = -\ln|\csc x| + C$$

$$\int \cot Kx \, dx = \frac{1}{K} \ln|\sin Kx| + C$$

$$\int \csc^2 x \, dx = \int \frac{1}{\sin^2 x} dx = \int (1 + \cot^2 x) dx = -\cot x + C$$

$$\int (1 + \cot^2 Kx) \, dx = -\frac{1}{K} \cot Kx + C$$

$$\int u'(1 + \cot^2 u) = -\cot u + C$$

$$\int \sec x \, dx = \int \frac{1}{\cos x} dx = \ln|\sec x + \tan x| + C = \ln\left|\tan\left(\frac{x}{2} + \frac{\pi}{2}\right)\right| + C$$

$$\int \sec^2 x \, dx = \int \frac{1}{\cos^2 x} dx = \int (1 + \tan^2 x) \, dx = \tan x + C$$

$$\int \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + C$$

$$\int \sec x \tan x = \sec x + C$$

$$\int \csc x \, dx = \int \frac{1}{\sin x} dx = \ln|\csc x - \cot x| + C = \ln\left|\tan\left(\frac{x}{2}\right)\right| + C$$

$$\int \csc^2 x \, dx = \int \frac{1}{\sin^2 x} dx = \int (1 + \cot^2 x) \, dx = -\cot x + C$$

$$\int \csc^3 x \, dx = -1/2 \cot x \csc x + 1/2 \ln|\csc x - \cot x| + C$$

$$\int \csc x \cot x \, dx = -\csc x + C$$

$$\int e^x dx = e^x + C$$

$$\int e^{ax+b} = \frac{1}{a} e^{ax+b}$$

$$\int a^x dx = \frac{a^x}{\ln a} + C \quad (a>0\ ,\ a \neq 1)$$

$$\int \ln x\, dx = x \ln x - x$$

$$\int x \operatorname{Sin} cx\, dx = \frac{\operatorname{Sin} cx}{c^2} - \frac{x \operatorname{Cos} cx}{c} + C$$

$$\int u\, dv = uv - \int v\, du$$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a} \operatorname{Arctan}\left(\frac{x}{a}\right) + C \quad a \neq 0$$

$$\int \frac{u'}{a^2+u^2} = \frac{1}{a} \operatorname{Arctan}\left(\frac{u}{a}\right) + C$$

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + C$$

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \ln\left|\frac{x+a}{x-a}\right| + C$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \operatorname{ArcSin}\left(\frac{x}{a}\right) + C \quad a \neq 0$$

$$\int \frac{u'}{\sqrt{a^2-u^2}} = \operatorname{ArcSin}\left(\frac{u}{a}\right) + C$$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x+\sqrt{x^2-a^2}\right| + C$$

$$\int \frac{u'}{\sqrt{u^2-a^2}} = \ln\left|u+\sqrt{u^2-a^2}\right| + C$$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left|x+\sqrt{x^2+a^2}\right| + C$$

$$\int \frac{u'}{\sqrt{u^2+a^2}} = \ln\left|u+\sqrt{u^2+a^2}\right|$$

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a} \operatorname{Arc Sec}\left(\frac{x}{a}\right) + C$$

$$\int \operatorname{Sinh} x\, dx = \operatorname{Cosh} x + C$$

$$\int \operatorname{Cosh} x\, dx = \operatorname{Sinh} x + C$$

$$\int \tanh x\, dx = \ln(\operatorname{Cosh} x) + C$$

$$\int \operatorname{Coth} x\, dx = \ln|\operatorname{Sinh} x| + C$$

$$\int \operatorname{Sech}^2 x\, dx = \tanh x + C$$

$$\int \operatorname{Csch}^2 x\, dx = -\operatorname{Coth} x + C$$

$$\int \operatorname{Sech} x \tanh x\, dx = -\operatorname{Sech} x + C$$

$$\int \operatorname{Csch} x \operatorname{Coth} x\, dx = -\operatorname{Csch} x + C$$