The Cavendish Experiment

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***Abstract*—We re-performed the Cavendish Experiment using a torsional balance apparatus. The aim of our experiment was to determine the universal gravitational constant G. Unlike Cavendish's original experiment, the experiment was performed with up-to-date instruments. We found G $2.29 \times 10^{-11}\mathrm{Nm}^2 kg^{-2}$, which is 14.61σ deviation from the accepted value of $G = 6.6743 \times 10^{-11}$. The experiment was not performed under suitable conditions and the error value was high. In addition, only about 4000 seconds of data were collected and it is suggested that different data-sets should be collected to obtain more consistent results for those who will repeat the experiment.**

I. Introduction & Theory

The experimental setup comprises small masses arranged as a dumbbell, suspended by a fiber in the form of a torsion balance. Two stationary and heavier masses are also placed on opposite sides of the smaller ones. The system is initially in equilibrium but is disturbed when the revolving fixture is rotated, and the heavier masses are positioned on the opposite sides of the smaller masses.

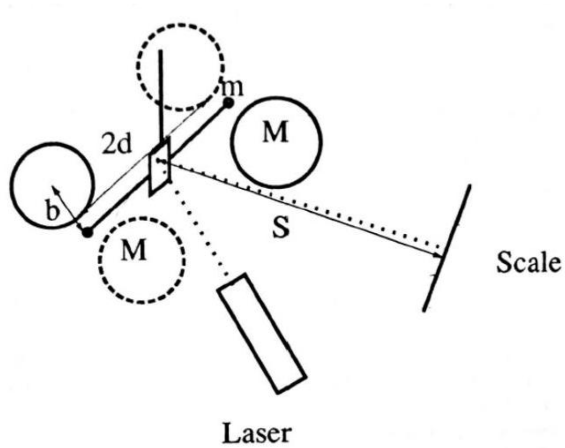

Fig. 1. Measuring the Universal Gravitational Constant with the Cavendish balance[1]

This creates a torque caused by the force of gravitation acting on the small masses.

$$\tau_g = 2Fr \tag{1}$$

Unlike in Figure 1, we have shown the radius as r instead of d.

As the arm supporting the small masses starts to rotate under the influence of gravitational attraction between the masses, a countering torque emerges due to the torsion of the wire. The thinness and length of the fiber allow even weak forces, such as gravitational forces, to produce measurable twist angles. Consequently, the twisting of the fiber generates an opposition torque in the torsion balance, which can be determined quantitatively.

$$\tau_t = \kappa\theta \tag{2}$$

It can be observed that an equilibrium angle exists at which the opposing torques acting on a system are in a state of equilibrium:

$$\tau_t = \tau_g \tag{3}$$

$$\kappa\theta_{eq} = 2Fr \tag{4}$$

The dumbbell will oscillate around the equilibrium angle θ_{eq} due to the fact that the system's potential energy is at a minimum at this point. In an ideal scenario, the oscillation would continue perpetually. The movement of the dumbbell can be expressed by a equation:

$$\ddot{\theta} + \frac{\kappa}{I}\theta = 0 \tag{5}$$

We need to find the torsion coefficient κ of the wire to obtain G.

$$\kappa = \omega_0^2 I \tag{6}$$

We can combine Eq. 6 with Eq. 4 to obtain:

$$\omega_0^2 I\theta_{eq} = 2Fr \tag{7}$$

We are required to derive an equation to represent the equilibrium angle θ_{eq}. By tracking the movement of a laser beam that is reflected off a small mirror located at the center of the dumbbell and detected by a scale positioned at a distance L, we can monitor the oscillations. Two distinct equilibrium angles θ_{eq} can be determined by positioning the heavy masses differently. The relation between the equilibrium angle and the difference between the positions of the laser beam at the two equilibrium angles is outlined by the following equation:

$$\tan 2\theta_{eq} = \frac{S}{2L} \tag{8}$$

The quantity denoted by S corresponds to the difference between the initial and final equilibrium positions of the laser beam as observed on the scale. This correlation holds true by virtue of the fact that the mirror rotates by an angle of $2\theta_{eq}$ during the transition from one equilibrium position to the other.

Using small angle approximation:

$$2\theta_{eq} \approx \frac{S}{2L} \tag{9}$$

$$\theta_{eq} = \frac{S}{4L} \tag{10}$$

$$\omega_0^2 I \frac{S}{4L} = 2Fr \tag{11}$$

where $F = G\frac{m_l m_s}{b^2}$

So we can show that:

$$\omega_0^2 I \frac{S}{4L} = G\frac{2rm_l m_s}{b^2} \tag{12}$$

where m_s is the small mass and m_l is the large mass. b is the distance between the adjacent small and large masses, $2r$ is the length of the dumbbell.

In actual practice, the presence of friction generates a damped oscillation in the system. It is assumed that the damping force caused by friction is a linear function of velocity, which can be expressed as the damping term $-a\dot{\theta}$. In this context, a positive value of a represents a damping force that opposes motion. The differential equation of Eq. 5 must be adjusted as follows[1]:

$$I\ddot{\theta} + \kappa\theta = -a\dot{\theta} \tag{13}$$

$$I\ddot{\theta} + a\dot{\theta} + \kappa\theta = 0 \tag{14}$$

$$\ddot{\theta} + 2\beta\dot{\theta} + \omega_0^2 = 0 \tag{15}$$

$\beta \equiv \frac{a}{2I}$ is the damping parameter and $\omega_0 = \sqrt{\frac{\kappa}{I}}$

we have the following solution:

$$\theta(t) = Ae^{-\beta t}\sin(\omega_d t + \delta) + C \tag{16}$$

and w_d is the damping frequency can be shown by:

$$\omega_d^2 = \omega_0^2 - \beta^2 \tag{17}$$

$$\omega_0^2 = \omega_d^2 + \beta^2 \tag{18}$$

$$\frac{\kappa}{I} = \omega_d^2 + \beta^2 \tag{19}$$

$$\kappa = I\left(\omega_d^2 + \beta^2\right) = 2mr^2\left(\omega_d^2 + \beta^2\right) \tag{20}$$

And so by using equation 12, 10 and 6:

$$G = \omega_0^2 I \frac{S}{4L} \cdot \frac{b^2}{2rm_l m_s} = \frac{b^2 \kappa\theta_{eq}}{2rm_l m_s} \tag{21}$$

Earth's mass is given by[2]:

$$M_{earth} = \frac{gR^2}{G} \tag{22}$$

So if we find G at the end of the experiment, we will be able to calculate the mass of the Earth (given the radius of the Earth and the acceleration of gravity).

II. The Experimental Setup & Method

A. Aparatus

- Movable carrying rod (for big masses)
- Torsion balance fiber
- Torsion balance rod
- Mirror
- Big masses
- Small masses
- Angular position sensor
- Laser
- Ruler
- Angular position sensor PC adapter

B. Setup

A photo of the experiment setup is shown below:

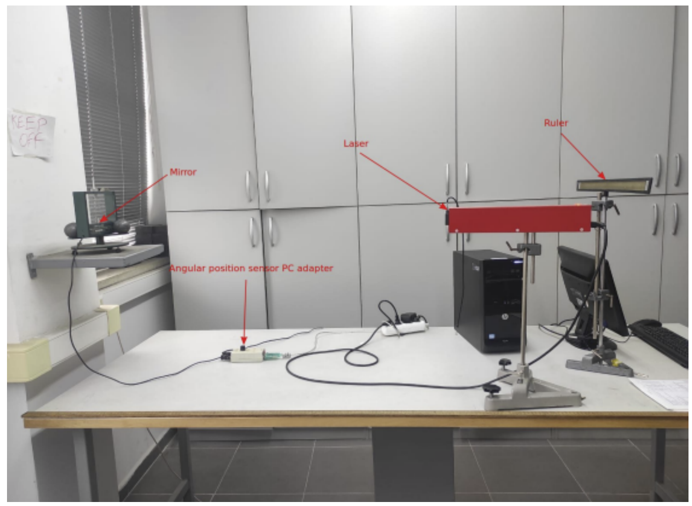

Fig. 2. The Experiment Setup

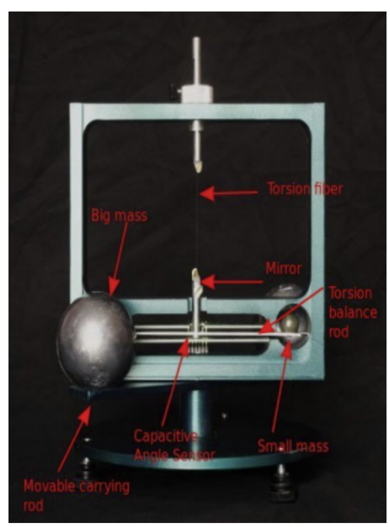

Fig. 3. Torsion balance

C. Procedure

- To begin, we power on the computer and open a data recording program.

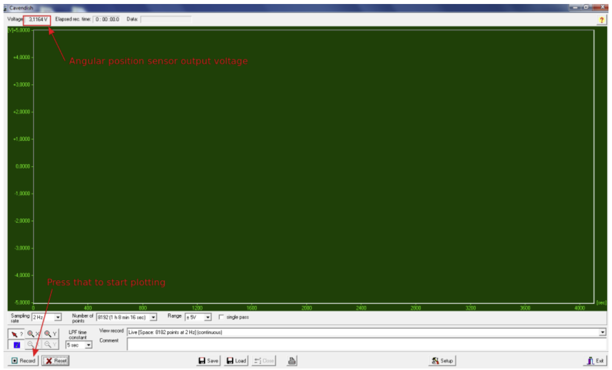

Fig. 4. Cavendish software

- Next, we measure the distance between the laser and mirror.
- Once we ensure that the laser beam is hitting the scale at the correct distance from the torsion balance, we wait for the beam to stabilize on the scale, indicating that the dumbbell is at equilibrium.
- We then move the large masses to opposite sides of the torsion balance, carefully avoiding contact with the box containing the small masses, but ensuring that the large masses are nearer the surface of the box.

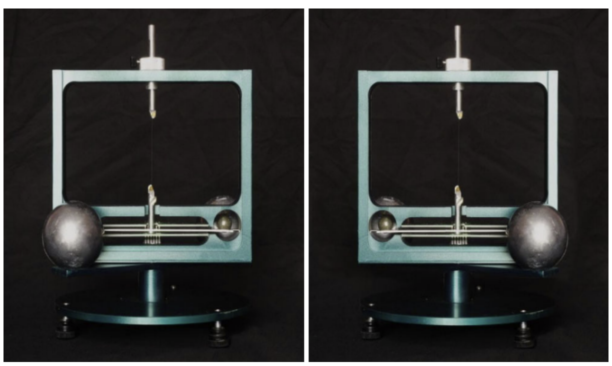

Fig. 5. Carrying rod at position A and position B

- When we observe a peak on the monitor (either upward or downward), we move the large masses to the opposite side and repeat this process until we have obtained consistent data for the peaks on the monitor, indicating resonance.
- We record data for several cycles in resonance and note the positions of the laser beam on the scale at peak angles.
- Next, we allow the system to damp on its own by stopping the movement of the large masses and waiting until it comes to rest at the equilibrium position.
- In the end, we will have data for the system's resonance and damping.

III. The Analysis & Result

A. Data

Table includes parameters we need to use and their uncertainties.

	Value	Uncertainty	Unit
Large Mass(m_l)	1.0385	0.001	kg
Small Mass(m_s)	0.014573	0.000001	kg
Distance of Small Sphere to the Axis of Rotation(r)	0.066653	0.0000371	m
Distances between centers of large and small spheres for equilibrium positions (b)	0.0461025	0.000158	m
Moment of Inertia (I) of Torsion Balance	0.000143	8.79×10^{-7}	kg m^2

We converted relative errors to standard errors in "guide.pdf".

The laser measurements are below:

	Value	Uncertainty	Unit
Distance of Laser to Mirror(L)	1.54	0.01	m
Minimum Laser Position	0.153	0.003	m
Maximum Laser Position	0.177	0.003	m

The figure below also show the raw data obtained during experiment:

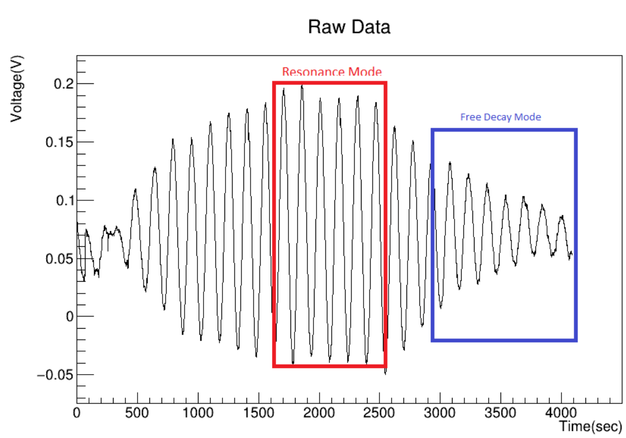

Fig. 6. Raw Data

B. Finding κ

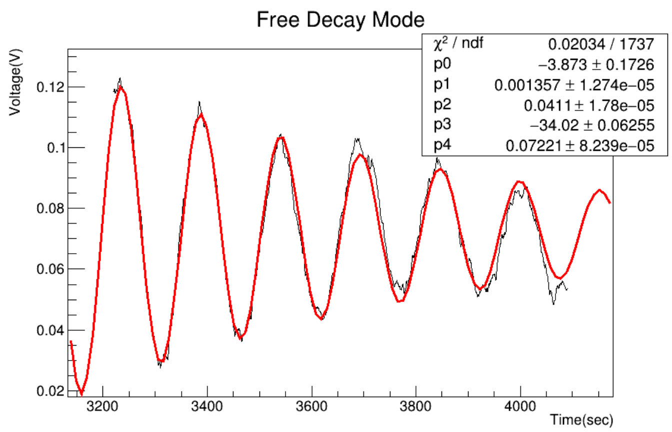

Fig. 7. Fitting for Free Decay Mode

We used the ROOT to fit the solution in Eq. 16 to data of free decay mode.

$$V(t) = 0.07221 - 3.873\exp(-0.001357t) \times \sin(0.0411t - 34.02) \tag{23}$$

Values obtained from the equation:

$$\omega_d = 0.0411 \pm 0.0000178 s^{-1} \tag{24}$$

$$\beta = 0.001357 \pm 0.00001274 s^{-1} \tag{25}$$

$$V_{\text{mean,freedecay}} = 0.07221 \pm 0.00008239 V \tag{26}$$

We can calculate torsion constant using the formula provided in the theory section.

$$\omega_0^2 = \omega_d^2 + \beta^2 \tag{27}$$

$$\omega_0^2 = 0.001691 \pm 0.000001 s^{-2} \tag{28}$$

$$\kappa = I\left(\omega_d^2 + \beta^2\right) \tag{29}$$

$$\kappa \approx 2.42 \times 10^{-7} \pm 0.000000001 kg \times m^2 \times s^{-2} \tag{30}$$

A software was used to make uncertainty calculations easier. [3]

C. Finding θ_{eq}

We used the ROOT too to fit the a sinusoidal to the resonance part of raw data.

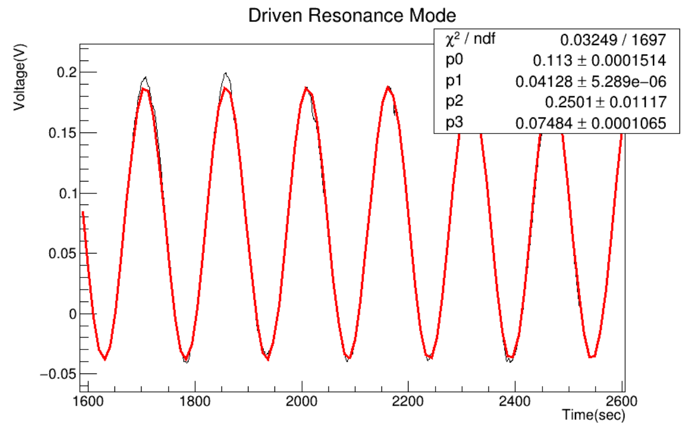

Fig. 8. Fitting for resonance mode of data

$$V(t) = 0.07484 + 0.113\sin(0.04128t + 0.2501) \tag{31}$$

V value obtained from the this equation is:

$$V_{\text{mean,resonance}} = 0.07484 \pm 0.0001065 V \tag{32}$$

Let's find ΔV_{eq}:

$$\Delta V_{eq} = |V_{\text{mean,freedecay}} - V_{\text{mean,resonance}}| = 0.00263 \pm 0.0001346 V \tag{33}$$

Now we will find ΔV_{res} :

$$\Delta V_{\text{res}} = 2|k| \tag{34}$$

k is the amplitude of the sinusoidal that fitted to resonance data.

$$\Delta V_{\text{res}} = 0.226 \pm 0.0003 \tag{35}$$

The difference in resonance state angle can be calculated by measuring the deflection of a laser beam:

$$\tan\left(\Delta\theta_{res}\right) = \frac{S}{2L} \tag{36}$$

where $S = 2.4$ cm± 0.4 cm and $L = 154$ cm$\pm$ 1 cm Angle difference using small angle approximation:

$$\tan\left(\Delta\theta_{\text{res}}\right) \approx \Delta\theta_{\text{res}} = 0.008 \pm 0.001 rad \tag{37}$$

We need to find C conversion constant:

$$\Delta\theta_{res} = C \cdot \Delta V_{\text{res}} \tag{38}$$

$$C = 0.035 \pm 0.0044 rad \times V^{-1} \tag{39}$$

From V_{eq} in Eq. we can find θ_{eq} using C:

$$\theta_{eq} = C \cdot V_{\text{eq}} \tag{40}$$

$$\theta_{eq} = 0.00009 \pm 0.00001 rad \tag{41}$$

Because of $\kappa\theta_{eq} = F2r$ and $F = G\frac{m_s m_l}{b^2}$, we can show that:

$$G = \frac{\kappa\theta_{eq} b^2}{2rm_l m_s} \tag{42}$$

and when we make the calculations:

$$G = 2.29 \times 10^{-11} \pm 0.3 \times 10^{-11} \mathrm{Nm}^2 kg^{-2} \quad (43)$$

IV. The Conclusion

If we compare the value we found with the real value of $G = 6.6743 \times 10^{-11}$ [2] :

$$\Delta\sigma_{G_{exp}} = \frac{|G - G_{exp}|}{\sigma_{G_{exp}}} = 14.61\sigma_{G_{exp}} \quad (44)$$

This value is relatively high. Our goal was to find a value less than 3σ. One reason for this is that the θ_{eq} value was smaller than expected compared to other experiments. If we had calculated a larger value for ΔV_{res}, we would have found a G value that is closer to reality. The small ΔV may be due to insufficient damping time. Additionally, if we increase the range of the free decay data, our fitting function may give a better value. The experiment was carried out in a crowded environment, which may have greatly affected the results due to the presence of objects that could be affected by the environment. Some values, such as mass and inertia, were given to us beforehand. The accuracy of these values and their uncertainties may have caused the experiment to be inaccurate. If we had also calculated the density of the Earth, we would have found a value quite far from the true value. We avoided this calculation, assuming that the experiment was not very successful.

One suggestion for those who want to repeat the experiment is to repeat the data collection process several times and find the average values of V, κ, and θ. This may bring the result closer to the true value. It may also be more useful to choose larger uncertainty values when calculating errors, preferring the safer option.

References

[1] E. Gülmez. *Advanced Physics Experiments*. 1st. Boğaziçi University Publications, 1999.

[2] OpenStax. *Newton's Law of Universal Gravitation*. https://phys.libretexts.org/@go/page/4048. Access date: 27 Mart 2023. 2016.

[3] *Uncertainty Calculator*. https://uncertaintycalculator.com/. Access date: 26 March 2023.

V. Appendix

A. Codes

Draw raw data:

```
{
  // Defining a Canvas
  TCanvas* c1 = new TCanvas();
  c1->SetWindowSize(900, 600);

  // Defining the graph object. Read from a
      txt file.
  TGraphErrors* mygraph = new
      TGraphErrors("cavendish.txt");


  // Setting the title of the graph
  mygraph->SetTitle("Raw Data");

  // Drawing the graph
  mygraph->Draw("");

  // Naming the axes
  mygraph->GetXaxis()->SetTitle("Time(sec)");
  mygraph->GetYaxis()->SetTitle("Voltage(V)");
}
```

The code I use to fit to free decay mode data:

```
{
  // Defining a Canvas
  TCanvas* c1 = new TCanvas();
  c1->SetWindowSize(900, 600);

  // Read from a txt file.
  TGraphErrors* mygraph = new
      TGraphErrors("damp.txt");


  // Setting the title of the graph
  mygraph->SetTitle("Free Decay Mode");

  // Drawing the graph
  mygraph->Draw("");

  // Naming the axes
  mygraph->GetXaxis()->SetTitle("Time(sec)");
  mygraph->GetYaxis()->SetTitle("Voltage(V)");

   // Defining and fitting an exponential
       function in a given range
  TF1* expo_fit = new TF1("expo_fit",
      "[0]*exp(-[1]*x)*sin([2]*x +[3]) + [4]",
      3000, 5000);
   // I tried to different paramater values to
       fitting
  expo_fit->SetParameters(0,0.001,
      0.03,2,0.1);
  expo_fit->SetLineColor(kRed);
  expo_fit->SetLineWidth(3);
  gStyle->SetOptFit(1);
  // Fitting and plotting
  mygraph->Fit(expo_fit);
}
```

The Code sinusoidal fitting to resonance mode:

```
{
  // Defining a Canvas
  TCanvas* c1 = new TCanvas();
  c1->SetWindowSize(900, 600);

  // Read from a txt file.
  TGraphErrors* mygraph = new
      TGraphErrors("reso.txt");


  // Setting the title of the graph
  mygraph->SetTitle("Driven Resonance Mode");

  // Drawing the graph
  mygraph->Draw("");

  // Naming the axes
  mygraph->GetXaxis()->SetTitle("Time(sec)");
  mygraph->GetYaxis()->SetTitle("Voltage(V)");
```

```
// Defining and fitting an exponential
    function in a given range
TF1* expo_fit = new TF1("expo_fit",
    "[0]*sin([1]*x +[2]) + [3]", 1500, 3000);
// I tried to different paramater values to
    fitting
expo_fit->SetParameters(1, 0.04,1,0.1);
expo_fit->SetLineColor(kRed);
expo_fit->SetLineWidth(3);
gStyle->SetOptFit(1);
// Fitting and plotting
mygraph->Fit(expo_fit);
```