Poisson Statistics

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***Abstract*—We conducted an experiment to demonstrate that radioactive decay is a random process that can be described by a Poisson distribution. We used gamma-ray sources of ^{137}Cs and ^{133}Ba and recorded radiation emissions at intervals of 1 and 10 seconds using a Geiger-Müller counter. We constructed a histogram from the data and observed that both Gaussian and Poisson distributions provided a good fit to the data. In the final part of the experiment, we noted that the time interval between successive decay events followed an exponential distribution, which is another important characteristic of Poisson statistics. In the second part of the experiment, we found that the α value deviated from the expected value by 0.22σ.**

I. Introduction & Theory

The Poisson distribution, which is named after the French mathematician Simeon Denis Poisson, is a commonly used probability theory and statistical technique[1]. It's a discrete probability distribution that calculates the probability of events occurring within a fixed time or space interval. This distribution requires the independence of the time of the last event occurring and the event occurring, as well as knowledge of the constant mean rate of events. It can also be used to calculate event probabilities in specified intervals of distance, area, or volume. Ladislaus Bortkiewicz's study, which looked into the number of soldiers killed accidentally by horse kicks in the Prussian army in 1898, is credited with proving the Poisson distribution in the field of reliability engineering[2].

The Poisson distribution is a type of Binomial distribution that is utilized when there is a large number of trials and the probability of an event happening is very low. In this scenario, the product of the mean and the probability of the event is reasonable and finite.

$$\lim_{m\to\infty, p\to 0} B(n;m,p) = P(\lambda, n) \tag{1}$$

The Poisson distribution, denoted by $P(\lambda, n)$, is expressed as follows:

$$P(\lambda, n) = \frac{\lambda^n}{n!}e^{-\lambda} \tag{2}$$

It is expected that the mean of the Poisson distribution is $\lambda = mp$, since it is a special case of the binomial distribution. The standard deviation of the Poisson distribution can also be expressed as:

$$\sigma^2 = \lambda \tag{3}$$

As the Poisson distribution's mean increases, it becomes more similar to a Gaussian distribution. This can be observed when fitting both distributions to experimental data. The normalized Gaussian distribution is commonly represented using a mean and standard deviation[3]:

$$P_G(\lambda, \sigma, x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-(x-\lambda)^2}{2\sigma^2}\right) \tag{4}$$

The probability of observing n events occurring, as counted in our experiment, during a time interval t is as follows:

$$P(\alpha, t, n) = \frac{(\alpha t)^n \cdot e^{-\alpha t}}{n!} \tag{5}$$

To determine the probability of a single count occurring during a time interval of dt, we can substitute the appropriate values into Equation 5:

$$P(\alpha, dt, 1) = \frac{(\alpha dt)^{e^{-\alpha dt}}}{1!} \tag{6}$$

Next, we can find the probability of having n events within an interval t, followed by an event within dt, by simply multiplying the two probabilities in the previous equations:

$$P_P(n+1, t)dt = P(\alpha, t, n)P(\alpha, dt, 1) \tag{7}$$

$$P_P(n+1, t)dt = \frac{(\alpha t)^n e^{-\alpha t}}{n!}\frac{(\alpha dt)e^{-\alpha dt}}{1!} \tag{8}$$

If we assume that $e^{-\alpha dt}$ is equal to 1, then we get:

$$P_P(n+1, t)dt \approx \frac{(\alpha t)^n}{n!}e^{-\alpha t}\alpha^n dt \tag{9}$$

$$P_P(n+1, t) \approx \frac{(\alpha t)^n}{n!}e^{-\alpha t}\alpha^n \tag{10}$$

In our experiment, we will only consider values of n that are either 0 or 1. As n gets bigger, the $P_P(n+1, t)$ distribution will become similar to the Gaussian distribution[3].

II. The Experimental Setup & Method

A. Aparatus

- Geiger-Muller counter
- Sample Holder
- ^{137}Cs and ^{133}Ba as Radiation Sources
- Lead Absorber
- Chart Recorder
- Laptop

B. Setup

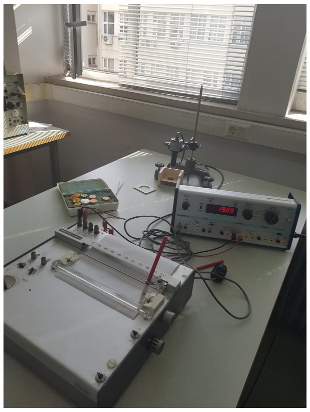

Fig. 1. The Experiment Setup with Geiger Counter and Chart Recorder

C. Method

c.1) Part 1

- To begin the experiment, we must determine the operating voltage of the Geiger tube. Afterwards, we will position a Gamma-ray source (^{137}Cs) on the sample holder below the Geiger tube unit and set the counter to 100 seconds, utilizing radioactivity and single mode. We will then document the number of counts in 100 second intervals for High Voltage ranging from 300V to 500V, with 20V intervals.
- Next, we adjusted the Geiger counter to the determined voltage of 440V as mentioned in the analysis section. Following that, we set the counter to continuous mode with 10-second intervals.
- To collect data, first adjust the position of the tray below the Geiger tube until there are approximately 100 counts in 10 seconds using the ^{137}Cs. Record the number of counts every 10 seconds for 100 intervals thereafter.
- We set 1 s interval and repeat the step.
- To obtain 100 counts in a 10-second interval, use the second Gamma-ray source, ^{133}Ba, and adjust the position of the tray. Repeat the previous two steps and collect four sets of data.

c.2) Part 2

- In the second part, investigate the time elapsed between successive counts using the chart recorder.
- Place ^{133}Ba in the sample holder and adjust the tray position.
- Start the chart recorder and run it to obtain approximately 100 peaks. Record the distance between successive pulses.

III. The Data

We have attached a table below that shows the 4 different datasets we collected for the first part. During this phase, we used a Geiger-Müller counter to measure the number of radiation emissions of ^{137}Cs and ^{133}Ba at 10-second and 1-second intervals.

TABLE I
GAMMA-RAY EMISSION COUNT

^{137}Cs-10s	^{137}Cs-1s	^{133}Ba-10s	^{133}Ba-1s
116	13	20	3
140	14	24	1
119	12	21	1
111	10	19	2
121	10	26	3
100	8	24	1
105	9	16	5
119	14	15	3
122	8	29	3
110	6	21	1
123	10	18	1
107	11	22	1
98	13	22	3
114	12	34	0
138	15	19	3
121	3	22	1
125	9	18	2
107	8	17	1
114	8	20	1
108	3	21	2
125	9	19	4
117	14	23	3
111	7	15	2
113	8	16	2
124	9	27	0
114	12	18	3
116	8	25	2
102	14	21	3
113	10	12	7
119	5	21	2
105	14	22	1
116	12	24	2
110	13	25	0
141	11	18	5
116	9	19	1
128	9	21	2
125	17	18	2
104	8	18	3
111	12	21	2
121	7	19	2
104	12	28	1
117	16	26	3
126	9	18	1
103	9	19	0
110	5	21	4
87	10	16	1
137	8	16	3
97	9	23	1
110	15	17	0
125	6	25	2
107	9	20	4
113	17	20	1
114	6	34	2
122	15	28	1
104	17	22	0
110	9	23	1
106	12	18	3
116	11	21	2
125	11	22	1
121	9	22	2
118	12	28	3
131	5	33	0
116	9	21	2
107	12	19	2
119	4	18	3
114	5	28	0
114	12	17	1
117	15	23	1
120	10	20	2
96	12	20	3
101	10	23	0
105	9	17	2
102	12	21	2

TABLE II
GAMMA-RAY EMISSION COUNT

^{137}Cs-10s	^{137}Css-1s	^{133}Ba-10s	^{133}Ba-1s
93	9	22	2
113	7	24	1
119	14	25	3
116	11	17	0
113	12	23	2
122	16	22	5
102	8	16	3
108	3	17	2
121	13	24	1
99	6	20	2
98	16	24	0
114	7	21	0
140	11	22	3
105	10	21	4
110	12	22	0
119	15	29	1
109	12	18	1
116	6	22	4
98	6	21	2
127	9	18	4
89	14	17	0
108	10	20	5
99	11	25	4
114	9	20	1
108	10	29	1
98	9	21	2
125	10	22	2

For the second part of the experiment, we collected 2 sets of data using chart recorders. We then presented these data as histograms in the analysis section.

IV. THE ANALYSIS & RESULT

A. Part 1

We used ROOT for our analysis, which consisted of generating a histogram of our data, identifying the best fits, and creating the plots.

To achieve reliable results, our goal is to choose a voltage value within the plateau region of the counter, where it demonstrates the highest stability. As shown below, 420V, 440V and 460V are feasible options, but we select 440V as the operating voltage.

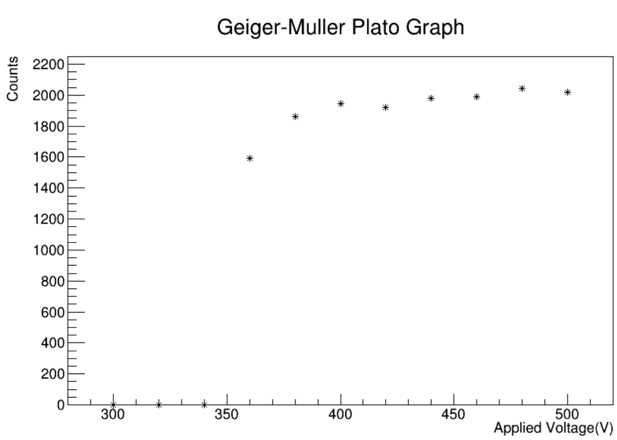

Fig. 2. Data points for Counts vs. Applied Voltage

Next, we calculate the mean and standart deviation of each data-set. The table below shows the mean and standard deviation of each distribution:

TABLE III
MEAN AND STANDARD DEVIATIONS OF GAMMA-RAYS SOURCES

$Dataset$	μ	σ	$\frac{\sqrt{\mu}}{\sigma}$
^{137}Cs - 10s	113.5	10.74	0.991959
^{137}Cs - 1s	10.22	3.254	0.982444
^{133}Ba - 10s	21.39	4.047	1.142805
^{133}Ba - 1s	1.95	1.381	1.011169

If we plot $\frac{\sqrt{\mu}}{\sigma}$ versus μ, we get the following graph:

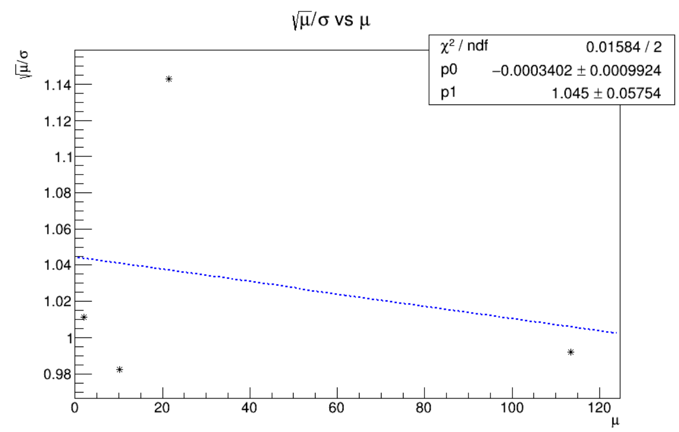

Fig. 3. $\frac{\sqrt{\mu}}{\sigma}$ versus μ

When performing a line fit on this data using the line-fit macro in ROOT, we obtained a result with a small slope and a value close to 1.00. Thus, we can assume that this value remains constant at 1.00.

We will now use ROOT to fit the Poisson and Gaussian functions to our histogram, as shown in the graphs below. This will allow us to determine the values of the chi-square test.

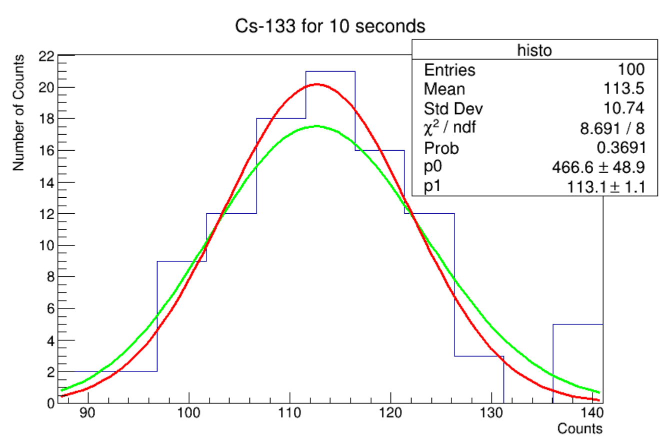

Fig. 4. ^{137}Cs-10s plots and fits

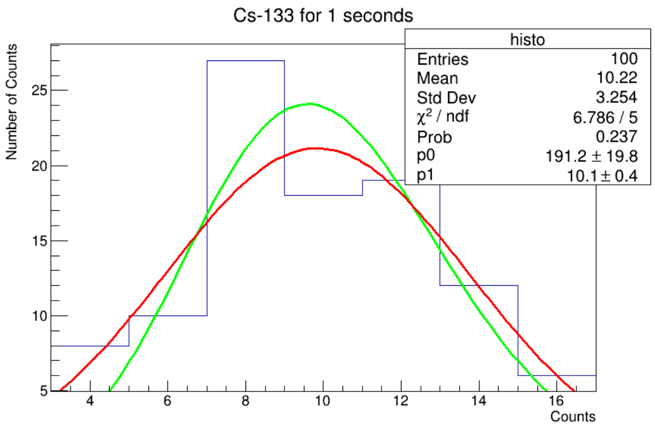

Fig. 5. ^{137}Cs-1s plots and fits

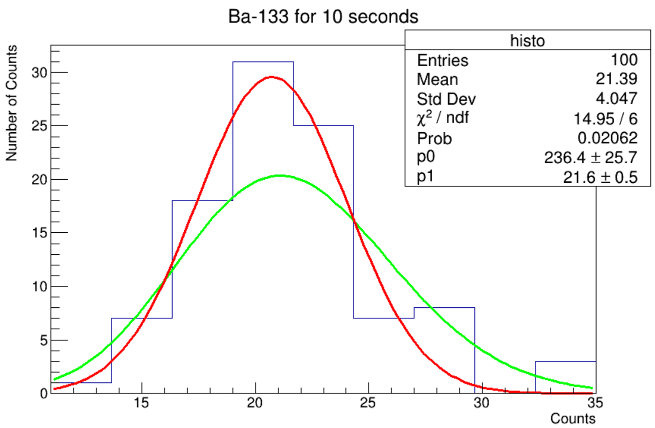

Fig. 6. ^{133}Ba-10s plots and fits

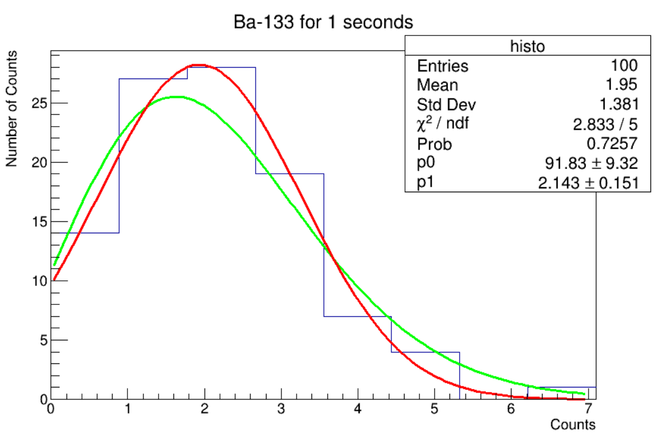

Fig. 7. ^{133}Ba-1s plots and fits

In the legend section of these graphs, p0 and p1 denote the values for the Poisson distribution. The parameter values for the Gaussian distribution were calculated also using ROOT. Although we do not have knowledge of the parameter values, it is evident from the histograms that the Gaussian and Poisson distributions possess a similar appearance.

TABLE IV
VALUES OF χ^2 TESTS

$Dataset$	$Poisson\chi^2$	$Poisson\chi_v{}^2$	$Gaussian\chi^2$	$Gaussian\chi_v{}^2$
$137Cs - 10s$	8.691	1.086375	6.98848	0.998354286
$137Cs - 1s$	6.786	1.3572	4.06876	1.01719
$133Ba - 10s$	14.95	2.491666667	8.69768	1.739536
$133Ba - 1s$	2.833	0.5666	2.01982	0.504955

According to the theory, as the mean increases, a Poisson distribution tends to approach a Gaussian distribution. This property is supported by both the χ^2 values and histogram plots.

B. Part 2

In the second part of the analysis, we explore the importance of datasets with n=0 and n=1 when compared to the Poisson distribution. The results we obtain should match the equations:

$$P_P(0+1,t) = \frac{(\alpha t)^0 e^{-\alpha t}\alpha}{(0)!} \tag{11}$$

$$P_P(1+1,t) = \frac{(\alpha t)^1 e^{-\alpha t}\alpha}{1!} \tag{12}$$

Below are histograms and exponential fits of these data-sets:

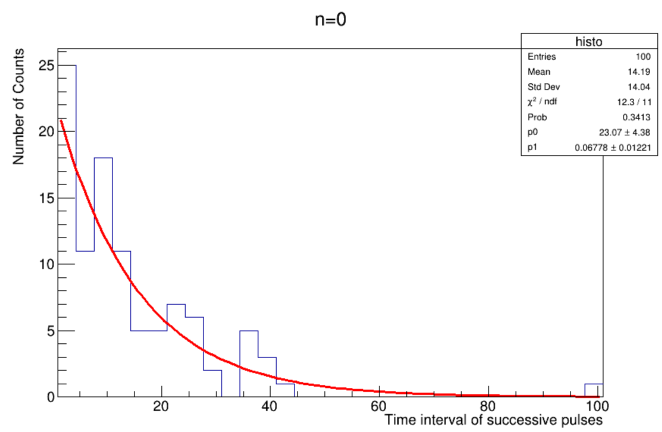

Fig. 8. Histogram and exp fitting for n=0

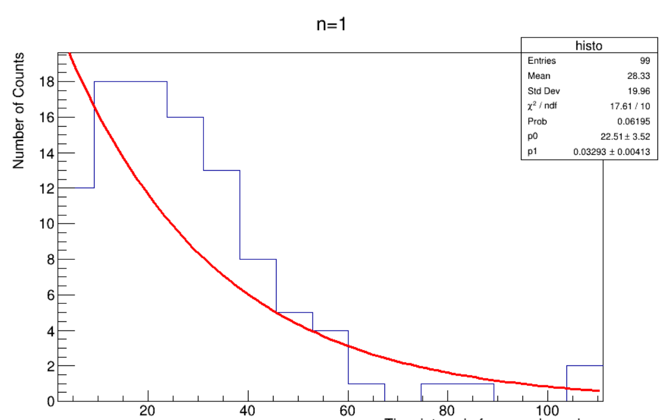

Fig. 9. Histogram and exp fitting for n=1

The chi-square test values are as shown in the graph. In addition, α values are as follows:

$$\alpha_{n=0} = 0.06778 \pm 0.01221 \tag{13}$$

$$\alpha_{n=1} = 0.03293 \pm 0.00413 \tag{14}$$

For n=0, we find the expected alpha value as follows:

$$\alpha_{expected(n=0)} = \frac{numberofpeaks}{Totaltime} = \frac{100}{1419} = 0.07047 \tag{15}$$

And the value we found is the following number of sigma away from the expected value:

$$\frac{\alpha_{expected} - \alpha_{n=0}}{\sigma_\alpha} = \frac{0.07047 - 0.06778}{0.01221} = 0.22\sigma away \tag{16}$$

V. The Conclusion

During the first part of the experiment, we analyzed the histograms we created from the collected data and fit them with Gaussian and Poisson distributions. We observed a strong correlation between the Gaussian and Poisson distributions, as well as with the histogram. Additionally, the chi-square tests of these distributions also showed a correlation. However, only in the 10-second intervals of the ^{133}Ba isotope were the chi-square values relatively higher compared to the others. In such cases of reduced randomness, collecting more than 100 data points may result in more consistent outcomes. This is because collecting less data reduces randomness. Unluckily, this data-set may have a meaningless correlation.

The α value obtained for $n = 0$ in the second part of the experiment was consistent with the expected value, with only a difference of 0.22σ. This is a good result. However, the alpha value for $n = 1$ was lower than expected. This was likely due to a miscalculation in the data used to draw the histogram. Overall, taking into account the successful result for $n = 0$, we can conclude that this part of the experiment was a success.

In summary, the data sets indicate that the Poisson and Gaussian distributions produced comparable results, which were analyzed through both graphical and numerical methods. The results demonstrate that Radioactive Decay is a random process that can be explained using Poisson statistics.

References

[1] S D Poisson. “Mmoire sur la probabilit des rsultats moyens des observations”. In: *Histoire de l’cole normale suprieure* 2 (1837), pp. 1–110.

[2] L Bortkiewicz. *Das Gesetz der kleinen Zahlen*. Leipzig: Teubner, 1898.

[3] E. Gülmez. *Advanced Physics Experiments*. 1st. Boğaziçi University Publications, 1999.

VI. Appendix

A. Codes

The code I use to fit the exponential function to the histogram in Part 2:

```
{
  TCanvas* c1 = new TCanvas();
  c1->SetWindowSize(900, 600);

  TH1F *histo = new
      TH1F("histo","Histogram",15,2,111.001 );

 std::ifstream file("n1values.txt");

 float datum;
 while (file>>datum) histo->Fill(datum);
 histo->Draw();
 histo->SetTitle("n=1");
 histo->GetXaxis()->SetTitle("Time interval
      of successive pulses");
 histo->GetYaxis()->SetTitle("Number of
      Counts");
 gStyle->SetOptFit(1111);
TF1 *expo_fit = new
    TF1("expo_fit","[0]*exp(-[1]*x)",0,111);
  expo_fit->SetParameters(100,1);
  expo_fit->SetLineColor(kRed);
  expo_fit->SetLineWidth(3);
  histo->Fit(expo_fit,"R");

}
```

Code for line-fit:

```
{
 TCanvas* c1 = new TCanvas();
  c1->SetWindowSize(900, 600);

 TGraph *mygraph = new TGraph("mean-std.txt");
 mygraph->Draw("A*");

 gStyle->SetOptFit(1);
 mygraph->SetTitle("#sqrt{#mu}/#sigma vs
      #mu");
 mygraph->GetXaxis()->SetTitle("#mu");
 mygraph->GetYaxis()->SetTitle("#sqrt{#mu}/#sigma");

 TF1 *fnew = new TF1("fnew","[0]*x+[1]",0,6);
 fnew->SetParameters(0,1); // arbitrary
      starting parameters
 fnew->SetLineColor(kBlue); // draw in blue
      color
 fnew->SetLineStyle(2); // draw dotted line
 mygraph->Fit(fnew);

}
```

Code for histograms-Gaussian and Poisson fit :

```
{
  TCanvas* c1 = new TCanvas();
  c1->SetWindowSize(900, 600);

  TH1F *histo = new
      TH1F("histo","Histogram",11,87,141.001 );

 std::ifstream file("Cs-440V-10s.txt");
```

```
  float datum;
  while (file>>datum) histo->Fill(datum);
  histo->Draw();
  histo->SetTitle("Cs-133 for 10 seconds");
  histo->GetXaxis()->SetTitle("Counts");
  histo->GetYaxis()->SetTitle("Number of
      Counts");

 TF1* poisson_fit = new TF1("poisson_fit",
     "[0]*TMath::Poisson(x,[1])", 87, 142);

  poisson_fit->SetParameters(500, 100);
  // Cosmetics
  poisson_fit->SetLineColor(kGreen);
  poisson_fit->SetLineWidth(3);
  histo->Fit(poisson_fit, "R+");


  TF1* gauss_fit = new TF1("gauss_fit",
      "[0]*TMath::Gaus(x,[1],[2])", 87, 142);
  gauss_fit->SetParameters(500, 100, 11);
  // Cosmetics
  gauss_fit->SetLineColor(kRed);
  gauss_fit->SetLineWidth(3);
  histo->Fit(gauss_fit, "R+");
  gStyle->SetOptFit(1111);
}
```