

# Discrete Structures: Homework #4

Due on 13 July 2020

*Professor Jensen Section 201*

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## Problem 1

Prove or disprove the statement “The set of integers is closed under division.”

### Solution

This statement is false.

#### Proof by Counterexample

Consider  $1 \div 2$ .

Note that 1 and 2 are both integers. Since the quotient of these two integers is not also an integer, the integers cannot be closed under division.

## Problem 2

Is the statement “ $\forall x, y \in \mathbb{R}, \sqrt{x+y} = \sqrt{x} + \sqrt{y}$ ” True or False? If it is false, provide a counterexample.

### Solution

This statement is false.

#### Proof by Counterexample

Consider  $x = 9$  and  $y = 16$ . In this case, note that  $\sqrt{x+y} = \sqrt{9+16} = \sqrt{25} = 5$  and that  $\sqrt{x} + \sqrt{y} = \sqrt{9} + \sqrt{16} = 3 + 4 = 7$ .

$\therefore$  The statement  $\forall x, y \in \mathbb{R}, \sqrt{x+y} = \sqrt{x} + \sqrt{y}$  is not true.

## Problem 3

Is the statement “ $\exists x, y \in \mathbb{R}, \sqrt{x+y} = \sqrt{x} + \sqrt{y}$ ” True or False? If it is true, provide an example.

### Solution

This statement is true.

#### Proof by Example

Consider  $x = 9$  and  $y = 0$ . In this case, note that  $\sqrt{x+y} = \sqrt{9+0} = \sqrt{9} = 3$  and that  $\sqrt{x} + \sqrt{y} = \sqrt{9} + \sqrt{0} = 3 + 0 = 3$ . In fact, this is true for any  $x$ , provided that  $y = 0$ .

#### Proof by Direct Proof (because I think it's interesting)

Prove that for  $x \in \mathbb{R}$  and  $y = 0$ ,  $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$ .

**Proof:** Let  $x$  be an arbitrary real number and  $y = 0$ . Note that 0 is a solution to  $x^2 = 0$  (since  $0^2 = 0 \cdot 0 = 0$ ) and that the square root  $r$  of a number  $x$  is defined as a non-negative real number that satisfies  $r^2 = x$ . From this, we can conclude that  $\sqrt{0} = 0$ . This means that we can write  $\sqrt{x+y}$  as  $\sqrt{x+0} = \sqrt{x}$ , since  $y = 0$  and  $x+0 = x$  for all real numbers  $x$  by the Additive Identity Axiom. Continuing, note that  $\sqrt{x} = \sqrt{x} + 0$  by the Additive Identity Axiom and that since  $0 = \sqrt{0}$ , then we can write  $\sqrt{x} + 0 = \sqrt{x} + \sqrt{0}$ . Since this is in the form  $\sqrt{x} + \sqrt{y}$  where  $x$  is a real number and  $y = 0$ , the statement that for  $x \in \mathbb{R}$  and  $y = 0$ ,  $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$  holds.

This result then implies the existence of  $x, y \in \mathbb{R}$  such that  $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$ , meaning that the original statement is true.

## Problem 4

Use the definition of even and odd to prove that for all integers  $m$ , if  $m$  is even then  $3m + 5$  is odd.

### Solution

#### Proof By Direct Proof

**Proof:** Let  $m$  be an even integer. By definition of an even integer,  $m = 2k$  for some integer  $k$ . It follows, then, that  $3m + 5 = 3(2k) + 5 = 6k + 5 = 6k + 4 + 1 = (6k + 4) + 1 = 2(3k + 2) + 1$ . Because integers are closed under multiplication and addition,  $3k + 2$  must be an integer and, again because of closure under multiplication and addition,  $2(3k + 2) + 1$  is also an integer. By definition of an odd integer,  $2(3k + 2) + 1$  is an odd integer and thus that  $3m + 5$  is odd. This means then, that if  $m$  is an even integer, then  $3m + 5$  is odd.

## Problem 5

Use the definition of even and odd to prove that if  $k$  is any even integer and  $m$  is any odd integer, then  $k^2 + m^2$  is odd.

### Solution

#### Proof By Direct Proof

**Proof:** Let  $k$  be an even integer and  $m$  be an odd integer. By definition of odd and even integers,  $k = 2x$  and  $m = 2y + 1$  for some integers  $x$  and  $y$ . It follows, then, that  $k^2 + m^2 = (2x)^2 + (2y + 1)^2 = 4x^2 + 4y^2 + 4y + 1 = 2(2x^2 + 2y^2 + 2y) + 1$ . By closure under multiplication, it follows that  $2x^2 + 2y^2 + 2y$  is an integer and, by closure under multiplication and addition, that  $2(2x^2 + 2y^2 + 2y) + 1$  is an integer. By definition of an odd integer,  $2(2x^2 + 2y^2 + 2y) + 1$  must be odd. This means then, that if  $k$  is an even integer and  $m$  is an odd integer, then  $k^2 + m^2$  must be odd.

## Problem 6

Use the definition of even and odd to prove that the product of any two consecutive integers is even.

### Solution

#### Proof by Direct Proof

**Proof:** Let  $n$  and  $n + 1$  be two consecutive integers.

**Case 1:** Assume  $n$  is an even integer. By definition of an even integer,  $n = 2k$  for some integer  $k$ . Additionally,  $n + 1 = 2k + 1$ . It follows that  $n(n + 1) = (2k)(2k + 1) = 4k^2 + 2k = 2(2k^2 + k)$ . By closure under multiplication and addition,  $2k^2 + k$  is an integer and thus  $2(2k^2 + k)$  is also an integer. By definition of an even integer,  $2(2k^2 + k)$  is even, thus meaning that  $n(n + 1)$  is even if  $n$  is also even.

**Case 2:** Assume  $n$  is an odd integer. By definition of an odd integer,  $n = 2k + 1$  for some integer  $k$ . Additionally,  $n + 1 = 2k + 1 + 1 = 2k + 2$ . It follows that  $n(n + 1) = (2k + 1)(2k + 2) = 4k^2 + 6k + 2 = 2(2k^2 + 3k + 1)$ . By closure under addition and multiplication,  $2k^2 + 3k + 1$  is an integer and thus  $2(2k^2 + 3k + 1)$

is also an integer. By definition of an even integer,  $2(2k^2 + 3k + 1)$  is even, thus meaning that  $n(n + 1)$  is even if  $n$  is odd.

Since  $n(n + 1)$  is even, regardless of the parity of  $n$ , the product of two consecutive integers is always even.

## Problem 7

Prove by contrapositive “For all integers  $a$ ,  $b$ , and  $c$ , if  $a \mid b$  and  $a \nmid c$ , then  $a \nmid (b + c)$ .”

### Solution

#### Proof By Contrapositive

Note that the contrapositive of the statement is “For all integers  $a$ ,  $b$ , and  $c$ , if  $a \mid (b + c)$ , then  $a \nmid b$  or  $a \mid c$ .” Additionally, note that:

$$\begin{aligned}
 p \rightarrow (q \vee r) &\equiv \neg p \vee q \vee r && \text{(Conditional Disjunction Equivalence)} \\
 &\equiv (\neg p \vee q) \vee r && \text{(Associative Law)} \\
 &\equiv \neg(\neg p \vee q) \rightarrow r && \text{(Conditional Disjunction Equivalence)} \\
 &\equiv (p \wedge \neg q) \rightarrow r && \text{(De Morgan's Law)}
 \end{aligned}$$

$\therefore$  It suffices to show that for all integers  $a$ ,  $b$ , and  $c$ , if  $a \mid (b + c)$  and  $a \mid b$ , then  $a \mid c$ .

**Proof:** Let  $a$ ,  $b$ , and  $c$  be integers such that  $b + c$  and  $b$  are both divisible by  $a$ . By definition of divisibility,  $b + c = na$  and  $b = ma$  for some integers  $n$ ,  $m$ . By substitution,  $b + c = na$  can be rewritten as  $ma + c = na$ . Thus,  $c = na - ma = a(n - m)$ . Because of closure under addition,  $n - m$  is an integer, and by the definition of divisibility,  $c$  is divisible by  $a$ . Thus, since for all integers  $a$ ,  $b$ , and  $c$ , if  $a \mid (b + c)$  and  $a \mid b$ , then  $a \mid c$  is true, the contrapositive must be true, and thus the original statement must be true.

#### Direct Proof (because I think it's more intuitive than the proof by contraposition)

Note that this proof relies on a result (the Division Algorithm, also known as the Quotient-Remainder Theorem) from Chapter 4.1.

Prove that for all integers  $a$ ,  $b$ , and  $c$ , if  $a \mid b$  and  $a \nmid c$ , then  $a \nmid (b + c)$ .

**Lemma 1:** If  $a$  and  $b$  are integers, and  $a + b + c$  is an integer, then  $c$  must be an integer.

**Proof:** Let  $m = a + b + c$  where  $m$  is an integer. Additionally, let  $a$  and  $b$  be integers. Starting with the definition of  $m$ , we can write that  $a + b + c = m$ . By closure under subtraction, note that  $m - a - b$  is an integer. However, since  $m - a - b = c$ ,  $c$  must also be an integer.  $\therefore$  If  $a$  and  $b$  are integers, and  $a + b + c$  is an integer, then  $c$  must be an integer.

**Proof:** Let  $a$ ,  $b$ , and  $c$  be integers such that  $b$  is divisible by  $a$  but  $c$  is not divisible by  $a$ . Note that by the definition of divisibility, since  $a \mid b$ , there exists an integer  $q_1$  such that  $b = q_1a$ . Additionally, by the Division Algorithm since  $a \nmid c$ , then there exists unique

nonzero integers  $q_2$  and  $r$  such that  $c = q_2a + r$ . Then,  $b + c = q_1a + q_2a + r = a(q_1 + q_2) + r$ . However, note that  $a(q_1 + q_2) + r$  cannot be divisible by  $a$ . If  $a(q_1 + q_2) + r$  were divisible by  $a$ , then  $a(q_1 + q_2) + r = a(q_1 + q_2 + \frac{r}{a})$  where  $\frac{r}{a}$  must be an integer by Lemma 1 since  $q_1$  and  $q_2$  are integers and  $q_1 + q_2 + \frac{r}{a}$  must be an integer by the definition of divisibility. However, this would contradict the fact that  $a \nmid c$ , since it would follow that  $r$  is divisible by  $a$  and that  $c = (q_2 + \frac{r}{a})a$  (where  $c$  is divisible by  $a$  because  $q_2$  and  $\frac{r}{a}$  are integers and due to closure under addition,  $q_2 + \frac{r}{a}$  is an integer, and therefore, by the definition of divisibility,  $c$  is divisible by  $a$ ). Thus,  $a(q_1 + q_2) + r$  cannot be divisible by  $a$ .

Since  $a \nmid (a(q_1 + q_2) + r)$  and  $a(q_1 + q_2) + r = (b + c)$ ,  $a \nmid (b + c)$ .

$\therefore$  For all integers  $a$ ,  $b$ , and  $c$ , if  $a \mid b$  and  $a \nmid c$ , then  $a \nmid (b + c)$ .

## Problem 8

Prove or disprove the statement “The square root of an irrational number is irrational.”

### Solution

#### Proof by Contradiction

Let  $x$  be an irrational number.

Assume  $\sqrt{x}$  is rational.

Since  $\sqrt{x}$  is rational, there exist integers  $p$  and  $q$  such that  $\sqrt{x} = \frac{p}{q}$ . It then follows that  $x = \frac{p^2}{q^2}$ . By closure under multiplication,  $p^2$  and  $q^2$  must be integers. However, since  $x$  can be written as the ratio of integers, it must be rational. This is a contradiction, since  $x$  was defined to be an irrational number, and thus the assumption that  $\sqrt{x}$  is rational is false.

$\therefore$  The square root of an irrational number is irrational.

## Problem 9

Is the statement “ $5\sqrt{2} - 3$  is irrational” True or False? Prove the statement if it is True. Disprove the statement if it is False. (You can use the fact that  $\sqrt{2}$  is irrational)

### Solution

#### Proof By Contradiction

**Proof:** Assume that  $5\sqrt{2} - 3$  is rational.

If  $5\sqrt{2} - 3$  is rational, then there exist integers  $p$  and  $q$  such that  $5\sqrt{2} - 3 = \frac{p}{q}$ . Adding 3 to both sides of the equation gives  $5\sqrt{2} = \frac{p}{q} + 3 = \frac{p+3q}{q}$ . Then, dividing both sides of the equation by 5 gives  $\sqrt{2} = \frac{p+3q}{5q}$ . By closure under addition and multiplication,  $p + 3q$  and  $5q$  must be integers. However, since  $\sqrt{2}$  can be written as the ratio of two integers, it must be rational. This leads to a contradiction, since  $\sqrt{2}$  is irrational. Hence, the assumption that  $5\sqrt{2} - 3$  is

rational is false.

$\therefore 5\sqrt{2} - 3$  must be irrational.

## Problem 10

Prove the statement is false. “There is a positive integer  $n$  such that  $n^2 + 6n + 5$  is prime.”

### Solution

#### Direct Proof

**Proof:** Let  $m$  be an integer of the form  $n^2 + 6n + 5$  where  $n$  is a positive integer.

Note that  $n^2 + 6n + 5 = (n + 1)(n + 5)$ . Since any  $m$  has the two factors  $(n + 1)$  and  $(n + 5)$  and since  $n + 1 \neq 1$  and  $n + 5 \neq 1$  (because  $n \geq 1$ ),  $m$  cannot be prime (since  $m$  has at least 2 positive factors, neither of which is 1).

$\therefore$  The statement “There is a positive integer  $n$  such that  $n^2 + 6n + 5$  is prime” is false.