Discrete Structures: Final Exam

Due on 23 July 2020

Professor Jensen Section 201

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Problem 1

Write the contrapositive of "If $a \nmid b$ and $a \mid c$, then $a \nmid (b+c)$."

Solution

If $a \mid (b+c)$ then $a \mid b$ or $a \nmid c$.

Explanation

Let p be the statement $a \nmid b$, q be the statement $a \mid c$, and r be the statement $a \nmid (b+c)$.
Symbolically, the original statement can thus be written as $(p \wedge q) \to r$. The contrapositive would then be $\neg r \to \neg(p \wedge q)$ which by De Morgan's law would be $\neg r \to (\neg p \vee \neg q)$, which in English would be "If $a \mid (b+c)$ then $a \mid b$ or $a \nmid c$."

Problem 2

Consider the function f from set X to set Y. Write the negation of the following definition for f to be one-to-one.
$\forall x_1, x_2 \in X$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

Solution

$\exists x_1, x_2 \in X$, such that $f(x_1) = f(x_2)$ and $x_1 \neq x_2$.

Explanation

Writing the original statement in full symbolic form, it becomes $\forall x_1, x_2 \in X, (f(x_1) = f(x_2) \to x_1 = x_2)$.
Note that we use logical equivalences to find the negation.

$$
\begin{aligned}
\neg\forall x_1, x_2 \in X, (f(x_1) = f(x_2) \to x_1 = x_2) &\equiv \exists x_1, x_2 \in X, \neg(f(x_1) = f(x_2) \to x_1 = x_2) && \text{(De Morgan's Law)} \\
&\equiv \exists x_1, x_2 \in X, (f(x_1) = f(x_2) \wedge \neg(x_1 = x_2)) && (\neg(p \to q) \equiv p \wedge \neg q) \\
&\equiv \exists x_1, x_2 \in X, (f(x_1) = f(x_2) \wedge (x_1 \neq x_2))
\end{aligned}
$$

Putting this back to match the original statement, $\exists x_1, x_2 \in X$, such that $f(x_1) = f(x_2)$ and $x_1 \neq x_2$.

Problem 3

$\sqrt{3}$ is irrational. Use proof by contradiction to prove the statement "$4\sqrt{3} - 7$ is irrational."

Solution

Proof By Contradiction Assume $4\sqrt{3} - 7$ is rational.
By definition of a rational number, $4\sqrt{3} - 7 = \frac{p}{q}$ for some integers p and q. Hence, we can write $4\sqrt{3} = \frac{p}{q} + 7$. This would mean that $4\sqrt{3} = \frac{p+7q}{q}$. Then, $\sqrt{3} = \frac{p+7q}{4q}$. Note that by closure of the integers under addition and multiplication, $p + 7q$ and $4q$ is an integer. However, this would mean that $\sqrt{3}$ can be written as the ratio of two integers, which would mean that it is rational by the definition of a rational number. This is a contradiction, by the given information. Hence, the initial assumption that $4\sqrt{3} - 7$ is rational is false, meaning that $4\sqrt{3} - 7$ is irrational.

Problem 4

A sequence $m_1, m_2, m_3, ...$ is defined by letting $m_1 = 1$ and $m_k = 2m_{k-1} + 1$, for all integers $k \geq 2$. Use induction to show that $m_n = 2^n - 1$, for all integers $n \geq 1$.

Solution

Proof By Induction
Base Case: Show that $m_1 = 2^n - 1$ for $n \geq 1$. By definition, $m_1 = 1$. Note that for $n = 1$, $2^1 - 1 = 2 - 1 = 1 = m_1$. This completes the basis step.
Inductive Hypothesis: Assume that $m_k = 2^k - 1$ for some fixed integer $k \geq 1$.
Inductive Step: By definition of the sequence:

$$\begin{aligned} m_{k+1} &= 2m_k + 1 \\ &= 2(2^k - 1) + 1 && \text{(Inductive Hypothesis)} \\ &= 2^{k+1} - 2 + 1 \\ &= 2^{k+1} - 1 \end{aligned}$$

Hence, by induction, $m_n = 2^n - 1$.