Discrete Structures: Homework #1

Due on 22 June 2020

Professor Jensen Section 201

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Problem 1

True or False: The statements *"If Earl studied, then Earl received an A."* and *"If Earl received an A, then Earl studied."* are logically equivalent.

Solution

False
Here, the two statements are converses of each other, which are not logically equivalent.

For instance, consider the situation where Earl got an A but did not study. In this instance, the original statement is true because Earl did not satisfy the hypothesis. However, the second statement would be false because Earl got an A without studying. Thus, the two statements are not logically equivalent.

Problem 2

True or False: The statements *"If Earl studied, then Earl received an A."* and *"If Earl did not receive an A, the Earl did not study."* are logically equivalent.

Solution

True
Here, the two statements are contrapositives of each other, which are logically equivalent.

Let p be the proposition *"Earl studied"* and q be the proposition *"Earl recieved an A."*. Note that the first statement given in the problem can be rewritten using the above statements as $p \rightarrow q$, and the second statement given in the problem as $\neg q \rightarrow \neg p$. The truth table for each statement given in the problem is the following:

p	q	$p \rightarrow q$	$\neg q \rightarrow \neg p$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Since $\neg q \rightarrow \neg p \equiv p \rightarrow q$, the statements *"If Earl studied, then Earl received an A."* and *"If Earl did not receive an A, the Earl did not study."* are logically equivalent.

Problem 3

Write the negation of the statement *"If Earl studied, then Earl received an A."*

Solution

Earl studied and did not receive an A

Let p be the proposition *"Earl studied"* and q be the proposition *"Earl recieved an A."* Note that $\neg(p \rightarrow q) \equiv p \wedge \neg q$. Thus we can write the negation of *"If Earl studied, then Earl received an A."* as *"Earl studied and did not receive an A"*.

Problem 4

Write the converse, inverse, and contrapositive of the statement *"If Popeye is a Dachshund, then Popeye has short legs."*

Solution

Converse: ***"If Popeye has short legs, then Popeye is a Dachshund"***
Inverse: ***"If Popeye is not a Dachshund, then Popeye does not have short legs"***
Contrapositive: ***"If Popeye does not have short legs, then Popeye is not a Dachshund"***

Let p be the proposition *"Popeye is a Dachshund"* and q be the proposition *"Popeye has short legs"*. Note that the converse of $p \to q$ is defined as $q \to p$, the inverse of $p \to q$ is defined as $\neg p \to \neg q$, and the contrapositive of $p \to q$ is defined as $\neg q \to \neg p$. Thus, the converse of the original statement is *"If Popeye has short legs, then Popeye is a Dachshund,"* the inverse of the original statement is *"If Popeye is not a Dachshund, then Popeye does not have short legs,"* and the contrapositive of the original statement is *"If Popeye does not have short legs, then Popeye is not a Dachshund."*

Problem 5

Use a truth table to determine whether $(p \wedge \neg q) \to r \equiv p \to (q \vee r)$.

Solution

True

p	q	r	$p \wedge \neg q$	$(p \wedge \neg q) \to r$	$q \vee r$	$p \to (q \vee r)$
T	T	T	F	T	T	T
T	T	F	F	T	T	T
T	F	T	T	T	T	T
T	F	F	T	F	F	F
F	T	T	F	T	T	T
F	T	F	F	T	T	T
F	F	T	F	T	T	T
F	F	F	F	T	F	T

In the above, since column $(p \wedge \neg q) \to r$ and $p \to (q \vee r)$ (both highlighted) have matching corresponding elements, $(p \wedge \neg q) \to r \equiv p \to (q \vee r)$. Without building the truth table, this can still be justified.
Note that $p \wedge \neg q$ can only be true when p is true and q is false, which means that the statement $(p \wedge \neg q) \to r$ can only be false when p is true, q is false, and r is false (which gives the implication $T \to F$ which is logically equivalent to F). Using similar reasoning, since $a \to b$ is only false when a is true and b is false, $p \to (q \wedge r)$ is false when p is true and $q \wedge r)$ is false, which is only true where q and r are both false. Thus, since they are both only false where p is true and both q and r are false and true otherwise, the two statements must be logically equivalent.

Problem 6

Let p, q, and r be the propositions

p: *"You get an A on the final exam"*

q: *"You do every exercise in this book"*

r: *"You get an A in this class"*

Write these statements using p, q, r and logical connectives (including negations).

a) You get an A in this class, but you do not do every exercise in this book.

b) You get an A on the final, you do every exercise in this book, and you get an A in this class.

c) To get an A in this class, it is necessary for you to get an A on the final.

d) You get an A on the final, but you don't do every exercise in this book; nevertheless, you get an A in this class.

e) Getting an A on the final and doing every exercise in this book is sufficient for getting an A in this class.

f) You will get an A in this class if and only if you either do every exercise in this book or you get an A on the final.

Solution

a) $p \wedge \neg q$

b) $p \wedge q \wedge r$

c) $r \rightarrow p$

d) $p \wedge \neg q \wedge r$ (Note that the statement can be rewritten as "You get an A on the final, do not do every exercise *and* you get an A in this class", as in this case, the word *and* can replace *nevertheless* without changing the meaning)

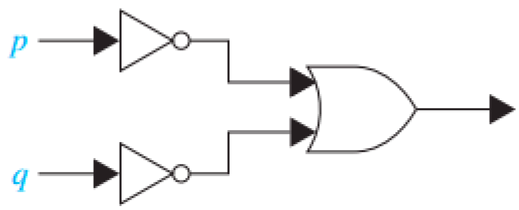
e) $(p \wedge q) \rightarrow r$

f) $r \iff (q \vee p)$

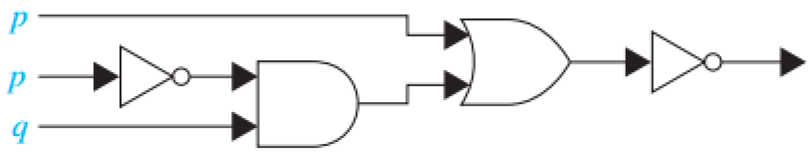
Problem 7

Find the output of each of these combinatorial circuits

a)



b)

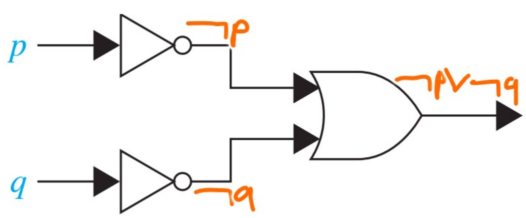


Solution

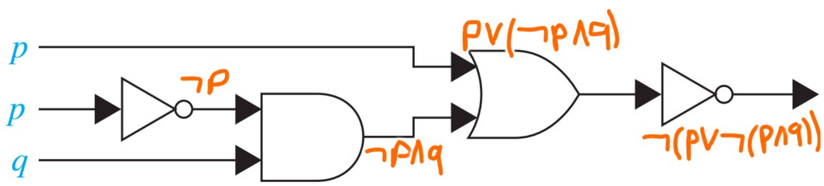
- **a**) $\neg p \vee \neg q$
- **b**) $\neg(p \vee (\neg p \wedge q))$

(Annotated Figures)

a)



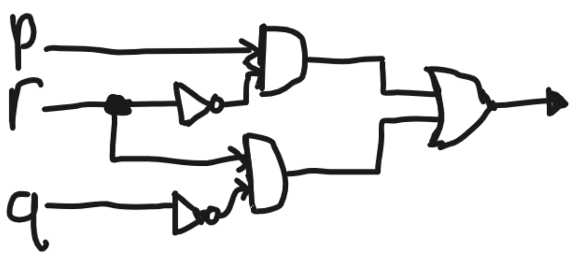
b)



Problem 8

Construct a combinatorial circuit using inverters, OR gates, and AND gates that produces the output $(p\wedge\neg r)\vee(\neg q\wedge r)$ from the input bits p, q, and r.

Solution



Problem 9

A says The two of us are both knights and B says A is a knave.

Solution

A is a Knave
B is a Knight

Assume A is a knight. As a result, B must also be a knight. However, if B is a knight as well, A cannot be a knight, as then B would be lying.
A must then be a knave. As a result, B must be a knight, as he is telling the truth in this instance. Note that when A is a knave and B is a knight, both satisfy the requirements of the problem, as A is not telling the truth (as both of them are not knights) and B is telling the truth (as A is a knave).

Problem 10

Are these system specifications consistent? Whenever the system software is being upgraded, users cannot access the file system. If users can access the file system, then they can save new files. If users cannot save new files, then the system software is not being upgraded.

Solution

System is **consistent**

Let:
u: *"The system is being upgraded"*
f: *"Users can access the file system*
s: *"Users are able to save new files*
Rewrite the sentences as:
1) $u\rightarrow\neg f$
2) $f\rightarrow s$

3) $\neg s \rightarrow \neg u$

Assume u to be true. By the third statement, we know that $\neg s$ must be false in order to keep the statement true, meaning that s must be true. By the first statement, we know that $\neg f$ must be true in order to keep the statement true, meaning that f is false. Statement 2 is then $\mathbf{F} \rightarrow \mathbf{T} \equiv \mathbf{T}$.
Since there exists a configuration of the system where each statement is true, the system must be consistent.

Problem 11

Write the negation of the statement *"The bulb in my lamp is burned out or my lamp is not plugged in."* What is the name of the logical equivalence that you applied?

Solution

"The bulb in my lamp is not burned out and my lamp is plugged in"

Let p be the proposition *"The bulb in my lamp is burned out"* and q be the proposition *"My lamp is plugged in."* The original statement can thus be written as $p \vee \neg q$. The negation of this statement can be symbolically denoted as $\neg(p \vee \neg q)$ which by De Morgan's law is equivalent to $\neg p \wedge \neg(\neg q)$ which by the double negation law is equivalent to $\neg p \wedge q$. Putting it back into words, the negation of the original statement is *"The bulb in my lamp is not burned out and my lamp is plugged in."*

Problem 12

Use De Morgan's laws to find the negation of each of the following statements

a) Kwame will take a job in industry or go to graduate school
b) Yoshiko knows Java and Calculus
c) James is young and strong.
d) Rita will move to Oregon or Washington

Solution

a) Kwame will not take a job in industry and not go to graduate school.
b) Yoshiko does not know Java or does not know Calculus.
c) James is not young or not strong.
d) Rita will not move to Oregon and will not move to Washington.

Problem 13

For each of these compound propositions use the conditional-disjunction equivalence ($p \rightarrow q \equiv \neg p \vee q$) to find an equivalent compound proposition that does not involve conditionals.

a) $\neg p \rightarrow \neg q$
b) $(p \vee q) \rightarrow \neg p$
c) $(p \rightarrow \neg q) \rightarrow (\neg p \rightarrow q)$

Solution

a) $\neg(\neg p) \vee \neg q \equiv \boxed{p \vee \neg q}$
b) $\neg(p \vee q) \vee \neg p \equiv (\neg p \wedge \neg q) \vee \neg p \equiv \boxed{\neg p}$

Problem 13 continued on next page...

c) $(\neg p\vee\neg q)\to(p\vee q)\equiv\neg(\neg p\vee\neg q)\vee(p\vee q)\equiv \boxed{(p\wedge q)\vee(p\vee q)}$

Problem 14

Show that the following statement is a tautology by using a truth table:
$[(p\to q)\wedge(q\to r)]\to(p\to r)$

Solution

p	q	r	$p\to q$	$q\to r$	$(p\to q)\wedge(q\to r)$	$p\to r$	$[(p\to q)\wedge(q\to r)]\to(p\to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Since the final (highlighted) column of the truth table contains all T values, the original statement is a tautology.

Problem 15

Show that $(p\wedge q)\to r$ and $(p\to r)\wedge(q\to r)$ are not logically equivalent.

Solution

p	q	r	$p\wedge q$	$(p\wedge q)\to r$	$p\to r$	$q\to r$	$(p\to r)\wedge(q\to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	F	T	T	T	T
T	F	F	F	T	F	T	F
F	T	T	F	T	T	T	T
F	T	F	F	T	T	F	F
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

Since not all corresponding elements between the (highlighted) columns $(p\wedge q)\to r$ and $(p\to r)\wedge(q\to r)$ are the same, $(p\wedge q)\to r\not\equiv((p\to r)\wedge(q\to r)$.

Problem 16

Use logical equivalences to verify $\neg((\neg p\wedge q)\vee(\neg p\wedge\neg q))\vee(p\wedge q)\equiv p$. Supply a reason for each step.

Problem 16 continued on next page. . .

Solution

$$\begin{aligned}
\neg((\neg p\wedge q)\vee(\neg p\wedge\neg q))\vee(p\wedge q) &\equiv(\neg(\neg p\wedge q)\wedge\neg(\neg p\wedge\neg q))\vee(p\wedge q) && \text{(De Morgan's Law)}\\
&\equiv((\neg(\neg p)\vee\neg q)\wedge(\neg(\neg p)\vee\neg(\neg q))\vee(p\wedge q) && \text{(De Morgan's Law)}\\
&\equiv((p\vee\neg q)\wedge(p\vee q))\vee(p\wedge q) && \text{(Double Negation Law)}\\
&\equiv(((p\vee\neg q)\wedge p)\vee((p\vee\neg q)\wedge q))\vee(p\wedge q) && \text{(Distributive Law)}\\
&\equiv((p\wedge(p\vee\neg q))\vee(q\wedge(p\vee\neg q)))\vee(p\wedge q) && \text{(Commutative Law)}\\
&\equiv(((p\wedge(p\vee\neg q))\vee((q\wedge p)\vee(q\wedge\neg q)))\vee(p\wedge q) && \text{(Distributive Law)}\\
&\equiv(((p\wedge(p\vee\neg q))\vee((q\wedge p)\vee\mathbf{F})))\vee(p\wedge q) && \text{(Negation Law)}\\
&\equiv(((p\wedge(p\vee\neg q))\vee(q\wedge p))\vee(p\wedge q) && \text{(Identity Law)}\\
&\equiv(p\vee(q\wedge p))\vee(p\wedge q) && \text{(Absorption Law)}\\
&\equiv(p\vee(p\wedge q))\vee(p\wedge q) && \text{(Commutative Law)}\\
&\equiv p\vee(p\wedge q) && \text{(Absorption Law)}\\
&\equiv p && \text{(Absorption Law)}
\end{aligned}$$

$\therefore\neg((\neg p\wedge q)\vee(\neg p\wedge\neg q))\vee(p\wedge q)\equiv p$

Problem 17

Determine whether each of these compound propositions is satisfiable. If the compound proposition is satisfiable, provide an assignment of truth values to the variables that makes the proposition true.

a) $(p\vee\neg q)\wedge(\neg p\vee q)\wedge(\neg p\vee\neg q)$
b) $(p\to q)\wedge(p\to\neg q)\wedge(\neg p\to q)\wedge(\neg p\to\neg q)$
c) $(p \iff q)\wedge(\neg p \iff q)$

Solution

a) The compound proposition is satisfiable when p and q are both false.
b) Unsatisfiable
c) Unsatisfiable

Part A

The Truth Table For $(p\vee\neg q)\wedge(\neg p\vee q)\wedge(\neg p\vee\neg q)$ is the following:

p	q	$\neg p$	$\neg q$	$p\vee\neg q$	$\neg p\vee q$	$\neg p\vee\neg q$	$(p\vee\neg q)\wedge(\neg p\vee q)\wedge(\neg p\vee\neg q)$
T	T	F	F	T	T	F	F
T	F	F	T	T	F	T	F
F	T	T	F	F	T	T	F
F	F	T	T	T	T	T	T

Since the last (highlighted) column has a true value in the row where both p and q are false, the proposition is satisfied with those values.

Part B

Problem 17 continued on next page. . .

The Truth Table For $(p \to q) \land (p \to \neg q) \land (\neg p \to q) \land (\neg p \to \neg q)$ is the following:

p	q	$\neg p$	$\neg q$	$p \to q$	$p \to \neg q$	$\neg p \to q$	$\neg p \to \neg q$	$(p \to q) \land (p \to \neg q) \land (\neg p \to q) \land (\neg p \to \neg q)$
T	T	F	F	T	F	T	T	F
T	F	F	T	F	T	T	T	F
F	T	T	F	T	T	T	F	F
F	F	T	T	T	T	F	T	F

Since the table is false for all values in the last (highlighted) column, $(p \to q) \land (p \to \neg q) \land (\neg p \to q) \land (\neg p \to \neg q)$ is unsatisfiable.
I also want to show a "slick" way to do this (as opposed to the method shown in class):

$$
\begin{aligned}
(p \to q) \land (p \to \neg q) \land (\neg p \to q) \land (\neg p \to \neg q) &\equiv (p \to q) \land (p \to \neg q) \land (\neg p \to q) \land (q \to p) && (\neg p \to \neg q \equiv q \to p \text{ By Contrapositive}) \\
&\equiv (p \to q) \land (q \to p) \land (p \to \neg q) \land (\neg p \to q) && (\text{Commutative Property}) \\
&\equiv (p \to q) \land (q \to p) \land (p \to \neg q) \land (\neg q \to p) && (\neg p \to q \equiv \neg q \to p \text{ By Contrapositive}) \\
&\equiv (p \iff q) \land (p \to \neg q) \land (\neg q \to p) && (\text{Page 29, Table 8}) \\
&\equiv (p \iff q) \land (p \iff \neg q) && (\text{Page 29, Table 8})
\end{aligned}
$$

The last statement is a contradiction, as for the statement $a \iff b$ to be true, then a and b must have the same value. However, for the last statement to be true, p must have the same value as both q and $\neg q$ which cannot be true , and thus the proposition is unsatisfiable.

Part C

p	q	$\neg p$	$p \iff q$	$\neg p \iff q$	$(p \iff q) \land (\neg p \iff q)$
T	T	F	T	F	F
T	F	F	F	T	F
F	T	T	F	T	F
F	F	T	T	F	F

Since the table is false for all values in the last (highlighted) column, $(p \iff q) \land (\neg p \iff q)$ is unsatisfiable. This can also be justified using the reasoning found in Part B, where $(p \iff q) \land (p \iff \neg q)$ means that p must have the same value as both q and $\neg q$ which cannot be true , and thus the proposition is unsatisfiable.