Discrete Structures: Homework #2

Due on 30 June 2020

Professor Jensen Section 201

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Problem 1

Is the statement "$\forall$ real numbers x and y, $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$" *True* or *False*? If it is false, provide a counterexample.

Solution

False
Consider the case $x = 4$, $y = 5$. In this case, $\sqrt{x+y} = \sqrt{4+5} = \sqrt{9} = 3$. However, $\sqrt{4} + \sqrt{5} = 2 + \sqrt{5} \neq 3$.

Problem 2

Is the statement "$\exists$ real numbers x and y such that $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$" *True* or *False*? If it is true, provide an example.

Solution

True
Consider the case $x = 1$, $y = 0$. In this case, $\sqrt{x+y} = \sqrt{1+0} = \sqrt{1} = 1$ and $\sqrt{1} + \sqrt{0} = \sqrt{1} = 1$. More generally, this is true when at least one of x or y is 0 (which follows from the identity element axiom of addition for real numbers, as $\sqrt{0} = 0$ and $\sqrt{x+0} = \sqrt{x}$, meaning that $\sqrt{x+0} = \sqrt{x} + \sqrt{0}$).

Problem 3

Write a negation for each statement.

a) There exists a real number x such that $x \leq -2$
b) $\forall$ computer programs P, if P compiles without error messages, then P is correct.
c) $\forall$ integers n, $\exists$ a prime number p such that $n < p < 2n$.

Solution

a) For all real numbers x, $x > -2$
b) There exists a computer program P such that P compiles without error messages and P is not correct.
c) There exists an integer n such that for all prime numbers p, $(p \leq x) \lor (p \geq 2x)$

Part A

Let $P(x)$ be the statement $x \leq -2$.
The statement can then be written as $\exists x P(x)$. By De Morgan's Law of Quantifiers, $\neg\exists x P(x) \equiv \forall x \neg P(x)$. Additionally, note that $\neg(x \leq -2) \equiv (x > -2)$. Thus, the negation of the original statement is $\forall x(x > -2)$, which in English is "For all real numbers x, $x > -2$".

Part B

Let $E(x)$ be the statement "x compiles without error messages." and $C(x)$ be the statement "x is correct." The original statement can then be written as $\forall P(E(P) \to C(P))$. By De Morgan's Law of Quantifiers, $\neg\forall P(E(P) \to C(P)) \equiv \exists P \neg(E(P) \to C(P))$. By conditional disjunction, $\exists P \neg(E(P) \to C(P)) \equiv \exists P \neg(\neg E(P) \lor C(P))$, and by De Morgan's law of propositional logic, $\exists P \neg(\neg E(P) \lor C(P)) \equiv \exists P(E(P) \land \neg C(P))$, which in English would be "There exists a computer program P such that P compiles without error messages and P is not correct."

Part C

Let $P(x,p)$ be the statement $x < p < 2x$. The original statement can then be written as $\forall n \exists p(P(n,p))$. Note that $\neg(x < p < 2x) \equiv (p \leq x) \vee (p \geq 2x)$. By De Morgan's Law of Quantifiers,$\neg\forall n\exists p(P(n,p)) \equiv \exists n \neg \exists p(P(n,p)) \equiv \exists n \forall p(\neg P(n,p))$. Then, taking note of the negation of $P(x)$, $\exists n \forall p(\neg P(n,p)) \equiv \exists n \forall p((p \leq x) \vee (p \geq 2x))$. Written in English, the statement would be "There exists an integer n such that for all prime numbers p, $(p \leq x) \vee (p \geq 2x)$."

Problem 4

Show that $\exists x P(x) \wedge \exists x Q(x)$ and $\exists(P(x) \wedge Q(x))$ are not logically equivalent.

Solution

Let $P(x)$ be the statement $x < 1$ and $Q(x)$ be the statement $x > 1$, where $x \in \mathbb{Z}$.
The first statement is then $\exists x(x < 1) \wedge \exists x(x > 1)$. This is a true statement, as there does exist an integer x that is less than 1 (e.g. 0) and an x that is greater than 1 (e.g. 2). However, the second statement is false, as there exists no integer x that is both greater than and less than 1. $\therefore \exists x P(x) \wedge \exists x Q(x) \not\equiv \exists(P(x) \wedge Q(x))$

Problem 5

Show that $\forall x P(x) \vee \forall x Q(x)$ and $\forall x(P(x) \vee Q(x))$ are not logically equivalent.

Solution

Let $P(x)$ be the statement $x \geq 1$ and $Q(x)$ be the statement $x < 1$, where $x \in \mathbb{Z}$.
The first statement is false, since $\forall x P(x)$ is false, as not every integer is greater than or equal to 1, and $\forall x Q(x)$ is false, as not every integer is less than 1. However, the second statement is true, as every integer is greater than or equal to or less than 1. $\therefore \forall x P(x) \vee \forall x Q(x) \not\equiv \forall x(P(x) \vee Q(x))$.

Problem 6

Let $P(x,y)$ be the statement "$xy = 1$". If the domain for both variables is the set of nonzero real numbers, what are the truth values?

a) $\exists y \forall x P(x,y)$
b) $\forall x \exists y P(x,y)$

Solution

a) False
b) True