Discrete Structures: Homework #4

Due on 13 July 2020

Professor Jensen Section 201

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Problem 1

Prove or disprove the statement "The set of integers is closed under division."

Solution

This statement is false.

Proof by Counterexample

Consider $1 \div 2$.
Note that 1 and 2 are both integers. Since the quotient of these two integers is not also an integer, the integers cannot be closed under division.

Problem 2

Is the statement "$\forall x, y \in \mathbb{R}, \sqrt{x+y} = \sqrt{x} + \sqrt{y}$" True or False? If it is false, provide a counterexample.

Solution

This statement is false.

Proof by Counterexample

Consider $x = 9$ and $y = 16$. In this case, note that $\sqrt{x+y} = \sqrt{9+16} = \sqrt{25} = 5$ and that $\sqrt{x} + \sqrt{y} = \sqrt{9} + \sqrt{16} = 3 + 4 = 7$.
$\therefore$ The statement $\forall x, y \in \mathbb{R}, \sqrt{x+y} = \sqrt{x} + \sqrt{y}$ is not true.

Problem 3

Is the statement "$\exists x, y \in \mathbb{R}, \sqrt{x+y} = \sqrt{x} + \sqrt{y}$" True or False? If it is true, provide an example.

Solution

This statement is true.

Proof by Example

Consider $x = 9$ and $y = 0$. In this case, note that $\sqrt{x+y} = \sqrt{9+0} = \sqrt{9} = 3$ and that $\sqrt{x} + \sqrt{y} = \sqrt{9} + \sqrt{0} = 3 + 0 = 3$. In fact, this is true for any x, provided that $y = 0$.

Proof by Direct Proof (because I think it's interesting)

Prove that for $x \in \mathbb{R}$ and $y = 0$, $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$.
Proof: Let x be an arbitrary real number and $y = 0$. Note that 0 is a solution to $x^2 = 0$ (since $0^2 = 0 \cdot 0 = 0$) and that the square root r of a number x is defined as a non-negative real number that satsifies $r^2 = x$. From this, we can conclude that $\sqrt{0} = 0$. This means that we can write $\sqrt{x+y}$ as $\sqrt{x+0} = \sqrt{x}$, since $y = 0$ and $x + 0 = x$ for all real numbers x by the Additive Identity Axiom. Continuing, note that $\sqrt{x} = \sqrt{x} + 0$ by the Additive Identity Axiom and that since $0 = \sqrt{0}$, then we can write $\sqrt{x} + 0 = \sqrt{x} + \sqrt{0}$. Since this is in the form $\sqrt{x} + \sqrt{y}$ where x is a real number and $y = 0$, the statement that for $x \in \mathbb{R}$ and $y = 0$, $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$ holds.

This result then implies the existence of $x, y \in \mathbb{R}$ such that $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$, meaning that the original statement is true.

Problem 4

Use the definition of even and odd to prove that for all integers m, if m is even then $3m+5$ is odd.

Solution

Proof By Direct Proof

Proof: Let m be an even integer. By definition of an even integer, $m = 2k$ for some integer k. It follows, then, that $3m+5 = 3(2k)+5 = 6k+5 = 6k+4+1 = (6k+4)+1 = 2(3k+2)+1$. Because integers are closed under multiplication and addition, $3k+2$ must be an integer and, again because of closure of the integers under multiplication and addition, $2(3k+2)+1$ is also an integer. By definition of an odd integer, $2(3k+2)+1$ is an odd integer and thus that $3m+5$ is odd. This means then, that if m is an even integer, then $3m+5$ is odd.

Problem 5

Use the definition of even and odd to prove that if k is any even integer and m is any odd integer, then k^2+m^2 is odd.

Solution

Proof By Direct Proof

Proof: Let k be an even integer and m be an odd integer. By definition of odd and even integers, $k = 2x$ and $m = 2y+1$ for some integers x and y. It follows, then, that $k^2+m^2 = (2x)^2+(2y+1)^2 = 4x^2+4y^2+4y+1 = 2(2x^2+2y^2+2y)+1$. By closure of the integers under multiplication, it follows that $2x^2+2y^2+2y$ is an integer and, by closure of the integers under multiplication and addition, that $2(2x^2+2y^2+2y)+1$ is an integer. By definition of an odd integer, $2(2x^2+2y^2+2y)+1$ must be odd. This means then, that if k is an even integer and m is an odd integer, then k^2+m^2 must be odd.

Problem 6

Use the definition of even and odd to prove that the product of any two consecutive integers is even.

Solution

Proof by Direct Proof

Lemma: An arbitrary integer n is either even or odd.
Proof: By the Division Algorithm (Quotient Remainder Theorem), n can be written with unique integers q and r such that $n = 2q+r$, with $0 \leq r < 2$. Since r must be an integer, r can be only the values 0 or 1. If r is 0, then $n = 2q$, which means that n is even, by the definition of an even integer. If r is 1, then $n = 2q+1$, which means n must be an odd integer. Thus, an arbitrary integer n is either even or odd.

Proof: Let n and $n+1$ be two consecutive integers. By Lemma 1, n has to be either even or odd.

Case 1: Assume n is an even integer. By definition of an even integer, $n = 2k$ for some integer k. Additionally, $n + 1 = 2k + 1$. It follows that $n(n + 1) = (2k)(2k + 1) = 4k^2 + 2k = 2(2k^2 + k)$. By closure of integers under multiplication and addition, $2k^2 + 1$ is an integer and thus $2(2k^2 + k)$ is also an integer. By definition of an even integer, $2(2k^2 + k)$ is even, thus meaning that $n(n + 1)$ is even if n is also even.
Case 2: Assume n is an odd integer. By definition of an odd integer, $n = 2k + 1$ for some integer k. Additionally, $n+1 = 2k+1+1 = 2k+2$. It follows that $n(n+1) = (2k+1)(2k+2) = 4k^2+6k+2 = 2(2k^2+3k+1)$. By closure of integers under addition and multiplication, $2k^2 + 3k + 1$ is an integer and thus $2(2k^2 + 3k + 1)$ is also an integer. By definition of an even integer, $2(2k^2 + 3k + 1)$ is even, thus meaning that $n(n + 1)$ is even if n is odd.
Since $n(n + 1)$ is even, regardless of the parity of n, the product of two consecutive integers is always even.

Problem 7

Prove by contrapositive "For all integers a, b, and c, if $a \mid b$ and $a \nmid c$, then $a \nmid (b + c)$."

Solution

Proof By Contrapositive

Note that the contrapositive of the statement is "For all integers a, b, and c, if $a \mid (b + c)$, then $a \nmid b$ or $a \mid c$." Additionally, note that:

$$\begin{aligned}
p \to (q \vee r) &\equiv \neg p \vee q \vee r && \text{(Conditional Disjunction Equivalence)} \\
&\equiv (\neg p \vee q) \vee r && \text{(Associative Law)} \\
&\equiv \neg(\neg p \vee q) \to r && \text{(Conditional Disjunction Equivalence)} \\
&\equiv (p \wedge \neg q) \to r && \text{(De Morgan's Law)}
\end{aligned}$$

$\therefore$ It suffices to show that for all integers a, b, and c, if $a \mid (b + c)$ and $a \mid b$, then $a \mid c$.
Proof: Let a, b, and c be integers such that $b + c$ and b are both divisible by a. By definition of divisibility, $b + c = na$ and $b = ma$ for some integers n, m. By substitution, $b + c = na$ can be rewritten as $ma + c = na$. Thus, $c = na - ma = a(n - m)$. Because of closure of integers under addition, $n - m$ is an integer, and by the definition of divisibility, c is divisible by a. Thus, since for all integers a, b, and c, if $a \mid (b + c)$ and $a \mid b$, then $a \mid c$ is true, the contrapositive must be true, and thus the original statement must be true.

Direct Proof (because I think it's more intuitive than the proof by contraposition)

Note that this proof relies on a result (the Division Algorithm, also known as the Quotient-Remainder Theorem) from Chapter 4.1.
Prove that for all integers a, b, and c, if $a \mid b$ and $a \nmid c$, then $a \nmid (b + c)$.

Lemma 1: If a and b are integers, and $a + b + c$ is an integer, then c must be an integer.
Proof: Let $m = a + b + c$ where m is an integer. Additionally, let a and b be integers. Starting with the definition of m, we can write that $a + b + c = m$. By closure of integers

under subtraction, note that $m - a - b$ is an integer. However, since $m - a - b = c$, c must also be an integer.
$\therefore$ If a and b are integers, and $a + b + c$ is an integer, then c must be an integer.

Proof: Let a, b, and c be integers such that b is divisible by a but c is not divisible by a. Note that by the definition of divisibility, since $a \mid b$, there exists an integer q_1 such that $b = q_1 a$. Additionally, by the Division Algorithm since $a \nmid c$, then there exists unique nonzero integers q_2 and r such that $c = q_2 a + r$. Then, $b + c = q_1 a + q_2 a + r = a(q_1 + q_2) + r$. However, note that $a(q_1 + q_2) + r$ cannot be divisible by a. If $a(q_1 + q_2) + r$ were divisible by a, then $a(q_1 + q_2) + r = a(q_1 + q_2 + \frac{r}{a})$ where $\frac{r}{a}$ must be an integer by Lemma 1 since q_1 and q_2 are integers and $q_1 + q_2 + \frac{r}{a}$ must be an integer by the definition of divisibility. However, this would contradict the fact that $a \nmid c$, since it would follow that r is divisible by a and that $c = (q_2 + \frac{r}{a})a$ (where c is divisible by a because q_2 and $\frac{r}{a}$ are integers and due to closure of the integers under addition, $q_2 + \frac{r}{a}$ is an integer, and therefore, by the definition of divisiblility, c is divisible by a). Thus, $a(q_1 + q_2) + r$ cannot be divisible by a.
Since $a \nmid (a(q_1 + q_2) + r)$ and $a(q_1 + q_2) + r = (b + c)$, $a \nmid (b + c)$.
$\therefore$ For all integers a, b, and c, if $a \mid b$ and $a \nmid c$, then $a \nmid (b + c)$.

Problem 8

Prove or disprove the statement "The square root of an irrational number is irrational."

Solution

Proof by Contradiction

Let x be an irrational number.
Assume $\sqrt{x}$ is rational.
Since $\sqrt{x}$ is rational, there exist integers p and q such that $\sqrt{x} = \frac{p}{q}$. It then follows that $x = \frac{p^2}{q^2}$. By closure of integers under multiplication, p^2 and q^2 must be integers. However, since x can be written as the ratio of integers, it must be rational. This is a contradiction, since x was defined to be an irrational number, and thus the assumption that $\sqrt{x}$ is rational is false.
$\therefore$ The square root of an irrational number is irrational.

Problem 9

Is the statement "$5\sqrt{2} - 3$ is irrational" True or False? Prove the statement if it is True. Disprove the statement if it is False. (You can use the fact that $\sqrt{2}$ is irrational)

Solution

Proof By Contradiction

Proof: Assume that $5\sqrt{2}-3$ is rational.
If $5\sqrt{2}-3$ is rational, then there exist integers p and q such that $5\sqrt{2}-3=\frac{p}{q}$. Adding 3 to both sides of the equation gives $5\sqrt{2}=\frac{p}{q}=\frac{p+3q}{q}$. Then, dividing both sides of the equation by 5 gives $\sqrt{2}=\frac{p+3q}{5q}$. By closure of integers under addition and multiplication, $p+3q$ and $5q$ must be integers. However, since $\sqrt{2}$ can be written as the ratio of two integers, it must be rational. This leads to a contradiction, since $\sqrt{2}$ is irrational. Hence, the assumption that $5\sqrt{2}-3$ is rational is false.
$\therefore$ $5\sqrt{2}-3$ must be irrational.

Problem 10

Prove the statement is false. "There is a positive integer n such that n^2+6n+5 is prime."

Solution

Direct Proof

Proof: Let m be an integer of the form n^2+6n+5 where n is a positive integer.
Note that $n^2+6n+5=(n+1)(n+5)$. Since any m has the two factors $(n+1)$ and $(n+5)$ and since $n+1\neq 1$ and $n+5\neq 1$ (because $n\geq 1$), it cannot be prime (since it has at least 2 positive factors, neither of which is 1, meaning that it does not have only the factors 1 and itself).
$\therefore$ The statement "There is a positive integer n such that n^2+6n+5 is prime" is false.