Discrete Structures: Homework #5

Due on 17 July 2020

Professor Jensen Section 201

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Problem 1

Let the universal set be the set $\{a, b, c, d, e, f, g\}$ and let $A = \{a, d, f\}$ and $B = \{d, g\}$.

a) Find $A \cup B$.
b) Find $A \cap B$.
c) Find $A - B$.
d) Find $B - A$.
e) Find $\overline{A}$. Note that $\overline{A} = U - A$
f) Find $A \times B$.
g) Find $\mathcal{P}(B)$.

Solution

a) $A \cup B = \{a, d, f, g\}$
b) $A \cap B = \{d\}$.
c) $A - B = \{a, f\}$.
d) $B - A = \{g\}$.
e) $\overline{A} = \{b, c, e, g\}$
f) $A \times B = \{(a, d), (d, d), (f, d), (a, g), (d, g), (f, g)\}$.
g) $\mathcal{P}(B) = \{\emptyset, \{d\}, \{g\}, \{d, g\}\}$.

Problem 2

Consider the set $S = \{1, 2, 3, 4, 5, 6\}$. Answer each question *Yes* or *No*.

a) Is $\{\{1, 4, 5\}, \{2, 3\}, \{2, 6\}\}$ a partition of S?
b) Is $\{\{1, 2, 5\}, \{3\}, \{4, 6\}\}$ a partition of S?
c) Is $\{\{1, 4\}, \{2, 3\}, \{6\}\}$ a partition of S?

Solution

a) *No*
b) *Yes*
c) *No*

Explanation

Part A
Since not all elements within the proposed partition are disjoint (i.e. the sets $\{2, 3\}$ and $\{2, 6\}$), it cannot be a partition of S.
Part B
Since all elements within the proposed partition are disjoint and the union of them is equal to S, it is a partition of S.
Part C
Since 5 is not in any of the elements within the proposed partition, the union of the elements is not equal to S.

Problem 3

Suppose A is a set with 8 elements. What is $|\mathcal{P}(A)|$?

Solution

$|\mathcal{P}(A)| = 2^8 = 256$

Explanation

Note that $|\mathcal{P}(A)| = 2^{|A|}$.
This can be proven using induction.
Base Case: Show $|\mathcal{P}(\emptyset)| = 2^{|\emptyset|}$.
$|\emptyset| = 0$
$|\mathcal{P}(\emptyset)| = |\{\emptyset\}| = 1$
$2^{|\emptyset|} = 2^0 = 1 = |\mathcal{P}(\emptyset)|$.
Induction Hypothesis: Assume for some fixed $k \geq 0, \forall$ sets S, if $|S| = k$, then $|\mathcal{P}(s)| = 2^k$.
Induction Step: Let T be a set with $|T| = k + 1$. Let $S = T - \{a\}$ for an arbitrary element a in T. Thus, $|S| = k$. Let B denote the set that contains all the elements in $\mathcal{P}(S)$ with a adjoined to them. Here, note that $|\mathcal{P}(S)| = |B|$, since the construction of B preserves the equality of cardinality. Then, note that $\mathcal{P}(T) = \mathcal{P}(S) \cup B$. Hence $|\mathcal{P}(T)| = |\mathcal{P}(S)| + |B|$. Since $|\mathcal{P}(S)| = |B|$, $|\mathcal{P}(T)| = 2 \cdot |\mathcal{P}(S)|$. By the inductive step, since $|\mathcal{P}(S)| = 2^k$, $|\mathcal{P}(T)| = 2 \cdot 2^k = 2^{k+1} = 2^{|T|}$.
$\therefore$ By induction, $|\mathcal{P}(A)| = 2^{|A|}$ for some arbitrary set A.

Problem 4

Define sets A and B as follows:
$A = \{m \in \mathbb{Z} \mid m = 3a \text{ for some integer a}\}$
$B = \{n \in \mathbb{Z} \mid n = 3b - 3 \text{ for some integer b}\}$
Show $A = B$.

Solution

Part 1

Show: $A \subseteq B$.
Let $x \in A$. By definition of A, $x = 3a$ for some $a \in \mathbb{Z}$. Let $b = a + 1$. Note that by closure under addition, b must be an integer. By substitution, $3b - 3 = 3(a + 1) - 3 = 3a + 3 - 3 = 3a = x$. Hence, $x = 3b - 3$ for some $b \in \mathbb{Z}$ (namely $b = a + 1$). In other words, $x \in B$ and $A \subseteq B$.

Part 2

Show: $B \subseteq A$.
Let $x \in B$. By definition of B, $x = 3b - 3$ for some $b \in \mathbb{Z}$. Let $a = b - 1$. Note that by closure under addition, a must be an integer. By substitution, $3a = 3(b - 1) = 3b - 3 = x$. Thus, $x = 3a$ for some $a \in \mathbb{Z}$ (namely $a = b - 1$). In other words, $x \in A$ and $B \subseteq A$.

$\therefore$ Since $A \subseteq B$ and $B \subseteq A$, $A = B$.

Problem 5

Prove the statement using the subset method if it is true and find a counterexample if it is false. Assume all sets are subsets of a universal set U.
a) For all sets A, B, and C, $A - (B - C) = (A - B) - C$.
b) For all sets A and B, $A \cap (A \cup B) = A$.

Solution

a) False
b) True

Part A

Let $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 3\}$, and $C = \{3\}$.
Here, $A - (B - C) = A - (\{2\}) = \{1, 3, 4, 5\}$.
Additionally, $(A - B) - C = (\{1, 4, 5\}) - C = \{1, 4, 5\}$.
Note that $A - (B - C) = \{1, 3, 4, 5\} \neq (A - B) - C = \{1, 4, 5\}$. Hence, the statement is false.

Part B

Part 1
Show: $A \cap (A \cup B) \subseteq A$.
Let $x \in A \cap (A \cup B)$. By definition of intersection, $x \in A$ and $x \in (A \cup B)$. Therefore, by simplification, $x \in A$ and thus $A \cap (A \cup B) \subseteq A$.

Part 2
Show: $A \subseteq A \cap (A \cup B)$.
Let $x \in A$. By definition of union, if an element is in $A \cup B$, then the element must be in A or B. Hence, since $x \in A$, $x \in (A \cup B)$. Furthermore, by definition of an intersection, if an element is in both A and B, then it must also be in $A \cap B$. Since $x \in A$ and $x \in (A \cup B)$, $x \in A \cap (A \cup B)$ and thus $A \subseteq A \cap (A \cup B)$.

$\therefore$ Since $A \cap (A \cup B) \subseteq A$ and $A \subseteq A \cap (A \cup B)$, $A \cap (A \cup B) = A$.

Problem 6

State De Morgan's Laws for sets.

Solution

1. The complement of the union of two sets is the intersection of the complements of the two sets.
2. The complement of the intersection of two sets is the union of the complements of the two sets.