# Discrete Structures: Homework #2

Due on 30 June 2020

*Professor Jensen Section 201*

**Brian Ton**

# Problem 1

Is the statement "$\forall$ real numbers $x$ and $y$, $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$" *True* or *False*? If it is false, provide a counterexample.

### Solution

**False**
Consider the case $x = 4$, $y = 5$. In this case, $\sqrt{x+y} = \sqrt{4+5} = \sqrt{9} = 3$. However, $\sqrt{4} + \sqrt{5} = 2 + \sqrt{5} \neq 3$.

# Problem 2

Is the statement "$\exists$ real numbers $x$ and $y$ such that $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$" *True* or *False*? If it is true, provide an example.

### Solution

**True**
Consider the case $x = 1$, $y = 0$. In this case, $\sqrt{x+y} = \sqrt{1+0} = \sqrt{1} = 1$ and $\sqrt{1} + \sqrt{0} = \sqrt{1} = 1$. More generally, this is true when at least one of $x$ or $y$ is 0 (which follows from the identity element axiom of addition for real numbers, as $\sqrt{0} = 0$ and $\sqrt{x+0} = \sqrt{x}$, meaning that $\sqrt{x+0} = \sqrt{x} + \sqrt{0}$).

# Problem 3

Write a negation for each statement.

a) There exists a real number $x$ such that $x \leq -2$
b) $\forall$ computer programs $P$, if $P$ compiles without error messages, then P is correct.
c) $\forall$ integers $n$, $\exists$ a prime number $p$ such that $n < p < 2n$.

### Solution

a) For all real numbers $x$, $x > -2$
b) There exists a computer program $P$ such that $P$ compiles without error messages and P is not correct.
c) There exists an integer $n$ such that for all prime numbers $p$, $(p \leq x) \vee (p \geq 2x)$

#### Part A

Let $P(x)$ be the statement $x \leq -2$.
The statement can then be written as $\exists x P(x)$. By De Morgan's Law of Quantifiers, $\neg\exists x P(x) \equiv \forall x \neg P(x)$. Additionally, note that $\neg(x \leq -2) \equiv (x > -2)$. Thus, the negation of the original statement is $\forall x (x > -2)$, which in English is "For all real numbers $x$, $x > -2$".

#### Part B

Let $E(x)$ be the statement "$x$ compiles without error messages." and $C(x)$ be the statement "$x$ is correct." The original statement can then be written as $\forall P(E(P) \to C(P))$. By De Morgan's Law of Quantifiers, $\neg\forall P(E(P) \to C(P)) \equiv \exists P \neg(E(P) \to C(P))$. By conditional disjunction, $\exists P \neg(E(P) \to C(P)) \equiv \exists P \neg(\neg E(P) \vee C(P))$, and by De Morgan's law of propositional logic, $\exists P \neg(\neg E(P) \vee C(P)) \equiv \exists P(E(P) \wedge \neg C(P))$, which in English would be "There exists a computer program $P$ such that $P$ compiles without error messages and P is not correct."

### Part C

Let $P(x,p)$ be the statement $x < p < 2x$. The original statement can then be written as $\forall n \exists p(P(n,p))$. Note that $\neg(x < p < 2x) \equiv (p \leq x) \vee (p \geq 2x)$. By De Morgan's Law of Quantifiers,$\neg\forall n \exists p(P(n,p)) \equiv \exists n \neg \exists p(P(n,p)) \equiv \exists n \forall p(\neg P(n,p))$. Then, taking note of the negation of $P(x)$, $\exists n \forall p(\neg P(n,p)) \equiv \exists n \forall p((p \leq n) \vee (p \geq 2n))$. Written in English, the statement would be "There exists an integer $n$ such that for all prime numbers $p$, $(p \leq n) \vee (p \geq 2n)$."

## Problem 4

Show that $\exists x P(x) \wedge \exists x Q(x)$ and $\exists x(P(x) \wedge Q(x))$ are not logically equivalent.

### Solution

Let $P(x)$ be the statement $x < 1$ and $Q(x)$ be the statement $x > 1$, where $x \in \mathbb{Z}$.
The first statement is then $\exists x(x < 1) \wedge \exists x(x > 1)$. This is a true statement, as there does exist an integer x that is less than 1 (e.g. 0) and an x that is greater than 1 (e.g. 2). However, the second statement is false, as there exists no integer x that is both greater than and less than 1. $\therefore \exists x P(x) \wedge \exists x Q(x) \not\equiv \exists(P(x) \wedge Q(x))$

## Problem 5

Show that $\forall x P(x) \vee \forall x Q(x)$ and $\forall x(P(x) \vee Q(x))$ are not logically equivalent.

### Solution

Let $P(x)$ be the statement $x \geq 1$ and $Q(x)$ be the statement $x < 1$, where $x \in \mathbb{Z}$.
The first statement is false, since $\forall x P(x)$ is false, as not every integer is greater than or equal to 1, and $\forall x Q(x)$ is false, as not every integer is less than 1. However, the second statement is true, as every integer is greater than or equal to or less than 1. $\therefore \forall x P(x) \vee \forall x Q(x) \not\equiv \forall x(P(x) \vee Q(x))$.

## Problem 6

Let $P(x,y)$ be the statement "$xy = 1$". If the domain for both variables is the set of nonzero real numbers, what are the truth values?

a) $\exists y \forall x P(x,y)$
b) $\forall x \exists y P(x,y)$

### Solution

a) False
b) True

### Part A

Note that the statement is equivalent to saying that there exists a nonzero real $y$ that is the multiplicative inverse of all nonzero real $x$.
By Theorem 4 in Appendix A1.2 (also proven in the section below, as its proof is not given in the book), the multiplicative inverse of a nonzero real number $x$ is unique (i.e. there is only 1 multiplicative inverse for each nonzero real x). Thus, the only way for the original statement to be true is if all nonzero real $x$ share the same singular multiplicative inverse $y$ (a universal generalization of the fact that $y$, according

to the statement, is a multiplicative inverse of an arbitrary nonzero real $x$, and by Theorem 4, the only multiplicative inverse of an arbitrary $x$). However, since 3 has the (unique) multiplicative inverse of $\frac{1}{3}$ and 2 has the (unique) multiplicative inverse $\frac{1}{2}$, it cannot be the case that there exists a $y$ that is the multiplicative inverse of all $x$, and thus the statement is false.

***Remark 1*** *In the above proof, Theorem 4 is needed to cover the case in which both $\frac{1}{x}$ and a constant $y$ where $\frac{1}{x} \neq y$ are multiplicative inverses of an arbitrary $x$, as then it would be the case that $y$ is a shared multiplicative inverse of all $x$ and thus the statement would be true.*

***Remark 2*** *After going to office hours on Tuesday, I realized that the proof can actually be done easier if done in "reverse." Instead of considering the statement as "there exists y that is the multiplicative inverse of all x," due to the nature of multiplicative inverses, one can actually reverse the statement, considering the statement as "there exists a y that has every real number x as its multiplicative inverse." The result then, trivially follows from Theorem 4, as it says that all multiplicative inverses must be unique.*

**Theorem 4 (Appendix 1.2)** *The multiplicative inverse of a nonzero real number is unique.*

*Proof.* Let $x$ be a nonzero real number.
Let $a$ and $b$ both be the multiplicative inverse of x.
By the inverse law of multiplication:
$x \cdot a = a \cdot x = 1$
$x \cdot b = b \cdot x = 1$
We can then write:

$$\begin{aligned}
a &= a \cdot 1 && \text{(Multiplicative Identity Law)} \\
&= a \cdot (x \cdot b) && (x \cdot b = b \cdot x = 1) \\
&= (a \cdot x) \cdot b && \text{(Associative Law for Multiplication)} \\
&= 1 \cdot b && (x \cdot a = a \cdot x = 1) \\
&= b && \text{(Multiplicative Identity Law)}
\end{aligned}$$

Since $a = b$, $a$ and $b$ must be the same multiplicative inverse, and thus the multiplicative inverse of a nonzero real number $x$ is unique. □

**Part B**

Note that the statement is equivalent to saying that for every nonzero real $x$, there exists a nonzero real $y$ that is its multiplicative inverse. This follows trivially from the inverse law of multiplication of the field axioms.