NSC Math 120

2014

This document was typeset on June 23, 2014.

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1 Percentages

We review the concept of percentages.

Basic learning objectives

These are the tasks you should be able to perform with reasonable fluency **when you arrive at our next class meeting**. Important new vocabulary words are indicated *in italics*.

- Know how to compute any one of the whole, part, or percentage of a quantity given the other two.

Advanced learning objectives

In addition to mastering the basic objectives, here are the tasks you should be able to perform **after class, with practice**:

- Be able to explain the related notions of *percent change*, *percent increase*, and *percent decrease*.
- Understand the concept of a *reference amount* and its impact on percentages.
- Be able to correctly identify reference amounts in problems involving quantities that change multiple times.
- Be able to compute percentages accurately with shifting reference amounts.

Learning outcomes:

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2 Percentages

We review the concept of percentages.

The word *percent* means "for each hundred." For example, saying Joseph earns 60% as much as Elaine is saying he earns \$60 for each \$100 she earns.

To calculate a percentage, the relationship between the amount that constitutes the whole, called the *reference amount* and the amount under consideration must be found. This is done via the following formula:

$$\text{Percentage} = \frac{\text{Part}}{\text{Whole}} \times 100 \tag{1}$$

As an example,

$$\frac{15}{25} \times 100 = 60$$

means that 15 is 60% of 25.

The relationship in equation (**??**) can be used to find any of the three pieces – the percentage, the part, the whole – if the other two are known by using algebra to solve for the missing piece.

As an example, to find 70% of 220

$$\begin{aligned} 70 &= \frac{\text{Part}}{220} \times 100 \\ 70 \div 100 &= \frac{\text{Part}}{220} \\ 0.70 \times 220 &= \text{Part} \\ 154 &= \text{Part} \end{aligned}$$

and so 70% of 220 is 154.

Question 1 *The label on a 12 oz. can of* Coca-Cola *states that it contains 39 grams of carbohydrates, accounting for* 13% *of the recommended daily value. How many grams of carbohydrates are recommended daily?*

Solution

Hint:

$$13 = \frac{39}{\text{Whole}} \times 100$$

300 grams

Nice work!

Question 2 *For fiscal years 2013–2015, the Nevada legislature appropriated* 9.0% *of its* \$20.1 *billion budget to infrastructre. How many billions of dollars were appropriated for infrastructure?*

Learning outcomes:

Solution

Hint:

$$9 = \frac{\text{Part}}{\$20.1 \text{ billion}} \times 100$$

Hint: *If you use* $\$20,100,000,000$ *in your calculations, your answer will be in dollars. If you use* $\$20.1$ *in your calculations, your answer will be in billions of dollars.*

Hint: *Don't round your answer.*

1.809 billions of dollars

Nice work!

Question 3 *If my taxi fare is* $25.50 *and I pay the driver* $30, *what percent tip am I giving?*

Solution

(a) 85%

(b) 20%

(c) 17.6% ✓

(d) 15%

(e) 1.2%

Hint: *What is the* amount *of the tip? This is your* part.

Hint: *What is the amount of the basic fare? This is your* whole.

Hint:

$$\text{Percentage} = \frac{\text{Part}}{\text{Whole}} \times 100$$

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3 Simple and Compound interest

We introduce the procedures for calculating simple interest.

Basic learning objectives

These are the tasks you should be able to perform with reasonable fluency **when you arrive at our next class meeting**. Important new vocabulary words are indicated *in italics*.

- Understand the meaning of the terms *interest*, *interest rate*, *term*, and *principal*.
- Be able to perform calculations with simple interest.

Advanced learning objectives

In addition to mastering the basic objectives, here are the tasks you should be able to perform **after class, with practice**:

- Understand the use of compound interest in real-life applications.
- Be able to perform calculations with compound interest.
- Understand the concept of annul percentage yield and its relationship with simple and compound interest.

Learning outcomes:

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4 Simple Interest

We introduce the procedures for calculating simple interest.

Interest is an amount of money paid to a lender for the privilige of borrowing money. The amount borrowed is called the *principal* and the interest one pays is calculated according to an *interest rate*, which is a percentage of the principal that will be charged as interest. The *term* is the length of time the loan is in use.

As an example, suppose Stephen borrows $\$1,000$ from Joanna now with the agreement that in one year he will pay her back $\$1,100$. This means that in addition to the $\$1,000$ principal, Stephen will also be paying

$$\frac{\$1,100 - \$1,000}{\$1,000} = \frac{\$100}{\$1,000} = 10\%$$

10% interest. Usually, interest is given as an *annual interest rate*. For example, suppose Stephen had borrowed $\$1,000$ from Joanna now with the agreement that in 6 months he will pay her $\$1,100$. Then the interest rate Stephen is paying is

$$10\% \div \frac{1}{2} = 20\%$$

because 6 months is only half a year, meaning if he didn't pay her back for 12 months he'd owe her $\$1,200$ instead of $\$1,100$, representing an additional $\$100$ or 10% in interest.

This type of interest, where a specific percentage of the principal is computed for a specific length of time, is called *simple interest*. The general formula for simple interest is

$$\text{Interest} = \text{Principal} \times \text{annual interest rate} \times \text{loan term} \tag{2}$$

The annual interest rate is somtimes called the *annual percentage rate* and abbreviated APR. The equation for simple interest is usually expressed in the more compact form $I = Prt$.

Example: Find the amount of simple interest charged on a loan of $\$5,000$ at 7.9% annually for 30 months.

$$I = \$5,000 \times 0.079 \times 2.5 = \$987.50$$

Note that the percentage rate is converted to a decimal ($7.9\% = 0.079$) and the loan term is expressed in years. As there are 12 months in a year, 30 months is $\frac{30}{12} = 2.5$ years.

Question 1 *How many months will it take for simple interest of* $\$500$ *to be due on a loan of* $\$10,000$ *at an annual rate of* 8%*?*

Solution

Learning outcomes:

Hint: *Solve for t in the equation*

$$\$500 = \$10,000 \times 0.08 \times t$$

Hint: *Convert your answer to months by multiplying it by 12.*

Hint: *Don't round your answer.*

7.5 months

Nice work!

In addition to paying interest, a person can also earn interest! When you save your money in the bank, the bank uses some of that money to make loans that generate interest. Because the bank is using your money, you've essentially given the bank a loan and will usually receive some interest of your own. The amount of this interest is usually far less than the amount of interest your money generates for the bank, typically only about 1%, but *c'est la vie!*

Question 2 *What percentage simple interest rate is needed to grow a* \$50 *initial investment into* \$70 *in 2 years?*

Solution

Hint: *How much interest does it take to grow* \$50 *into* \$70*?*

Hint: *Solve for r in the equation*

$$\$20 = \$50 \times r \times 2$$

Hint: *What* percentage *does your answer represent?*

20%

Nice work! If you ever find a bank offering that interest rate, call me immediately!

Question 3 *Which of the following would earn the largest dollar amount for a given principal?*

Solution

(a) An investment paying 3% annual simple interest for 4 years?
(b) An investment paying 4% annual simple interest for 3 years?
(c) An investment paying 13% annual simple interest for 1 year?
(d) An investment paying 1% annual simple interest for 10 years?
(e) An investment paying 2% annual simple interest for 7 years? ✓

Hint: *Choose a principal that's easy to work with, like* \$100.

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5 Credit

We introduce and explore the concept of consumer credit.

Basic learning objectives

These are the tasks you should be able to perform with reasonable fluency **when you arrive at our next class meeting**. Important new vocabulary words are indicated *in italics*.

- Understand the role and impact of consumer credit in today's financial environment.
- Understand the risks and benefits of credit and know about the ways in which creditworthiness is assessed.

Advanced learning objectives

In addition to mastering the basic objectives, here are the tasks you should be able to perform **after class, with practice**:

- Understand the concepts of a *finance charge*, *average daily balance*, a *billing cycle* and be able to perform calculations involving these concepts.
- Know how to answer questions about a loan by reading an *amoritization table.*

Learning outcomes:

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6 Credit

We introduce and explore the concept of consumer credit.

Consumer credit plays a critical role in the modern economy. Credit allows a person to make a purchase immediately while deferring payment for the item until later. From buying a car or home to purchasing everyday goods like groceries or gasoline, credit has come to dominate the financial landscape. It is critical that you understand the process, risks, and benefits of credit so you can make wise personal financial decisions.

A person's ability to gain credit, referred to as credit worthiness, is dependent on such things as the person's history of credit use, projected ability to repay borrowed money, and the amount of debt the person currently carries. Data about a person's creditworthiness is collected by a credit reporting agency. Three national agencies collect and compile information about consumers and individually calculate a numeric score between 300 and 850, with higher scores indicate higher creditworthiness:

Equifax `httP://www.equifax.com`

Experian `http://www.experian.com`

Trans Union `http://www.transunion.com`

Each company collects slightly different data and so calculates slightly different scores, though each score is referred to as a FICO score.[1] When you apply for a type of credit, the lender requests your credit score from one or more of the national agencies and sets the terms of credit based on what the scores say about your creditworthiness. The difference between a high FICO score and a low FICO score can be a significant reduction in interest rate, resulting in the potential savings of thousands of dollars in interest over the life of a long-term loan.

Question 1 *Use the calculator at* `http://www.myfico.com/myfico/creditcentral/loanrates.aspx` *to answer the following question. For a 60-month new auto loan of $10,000, how much interest does a person pay over the life of the loan with each FICO score ranges shown?*

FICO Score	*Interest paid*
720–850	*$*
690–719	*$*
660–689	*$*
620–659	*$*
590–619	*$*
500–589	*$*

The federal government regulates credit scores via the **F**air and **A**ccurate **C**redit **T**ransactions Act, or FACT Act. Basic information about the FACT act may be

Learning outcomes:

[1] The FICO score is named after an early collector of credit history, the **F**air **I**saac **Co**operation.

found at `http://www.fdic.gov/consumers/consumer/alerts/facta.html`. Additionally, the federal government tracks statistics involving the use of consumer credit that can be accessed through the U.S. Census Bureau. You will find the answers to the following questions on the Census Bureau's statistical abstract for Banking, Finance, & Insurance at `http://www.census.gov/compendia/statab/cats/banking_finance_insurance/payment_systems_consumer_credit_mortgage_debt.html`.

Question 2 *In 2012, approximately how many credit cards were held by American consumers?*

Solution

(a) 110 million cards

(b) 1.1 billion cards ✓

(c) 330 million cards

(d) 33 million cards

(e) 65 million cards

Question 3 *In 2012, approximately how much money was spent using credit cards by American consumers?*

Solution

(a) \$2.4 billion

(b) \$24 billion

(c) \$240 billion

(d) \$2.4 trillion ✓

(e) \$24 trillion

Question 4 *In 2012, approximately how much outstanding credit card debt did American consumers carry?*

Solution

(a) \$87 million

(b) \$870 million

(c) \$8.7 billion

(d) \$87 billion

(e) \$870 billion ✓

Question 5 *In 2010, approximately what percent of credit card accounts were delinquent? (A delinquent account is past 30 days due and still accruing interest charges.)*

Solution

(a) 6.1%
(b) 8%
(c) 4.9% ✓
(d) 2.8%
(e) 7.5%

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7 Insurance

We explore the mathematical concepts related to risk and insurance.

Basic learning objectives

These are the tasks you should be able to perform with reasonable fluency **when you arrive at our next class meeting**. Important new vocabulary words are indicated *in italics*.

- Understand the basic role that risk assessment plays in insurance calculations.
- Perform computations involving simplified risk and insurance settings.

Advanced learning objectives

In addition to mastering the basic objectives, here are the tasks you should be able to perform **after class, with practice**:

- Understand how probability distributions allow for the assessment of risk.
- Perform computations involving risk using the normal distribution.

Learning outcomes:

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8 Insurance

We explore the mathematical concepts related to risk and insurance.

Understanding insurance involves understanding risk. Understanding risk involves understanding probability. An event with low probability will generally be cheaper to insure against than an equally damaging event with higher probability.

Insurance companies employ people skilled in a branch of applied mathematics known as actuarial science. These actuaries perform complex calculations to help companies set prices for their policies, known as premiums, that will allow the company to satisfy the legitimate claims against those policies, cover business expenses, and make a profit.

We will examine simplified situations involving insurance to get a sense of the concepts and computations involved.

In the mid-17$^{\text{th}}$ century, a London merchant named John Graunt collected and analyzed mortality rolls (lists of burials), originally kept to track the Plague, and published a short work entitled *Natural and Political Observations Made upon the Bills of Mortality*. The most important idea in Graunt's work was the analysis of the distribution of lifespans for a group of people born at the same time. His London Life Table showed the percentage of people who would likely live to a given age at that time.[1]

Age	Survivors
0	100
6	64
16	40
26	25
36	16
46	10
56	6
66	3
76	1

Question 1 *According to Graunt's London Life Table, what percentage of the population did not survive to age 16?*

Solution 60%

With the kind of information like that found in Graunt's table, a prospective insurer could calculate how likely they would be to face a life-insurance claim from a 16-year old client in the next 10 years, for example.

Question 2 *According to Graunt's table, what percentage of 16-year olds lived to be 26-years old?*

Learning outcomes:

[1] Burton, D. M. (2011). *The history of mathematics: An introduction* (7th ed.). New York, NY: McGraw-Hill.

Solution

Hint: *Out of 100 people, how many lived to 16? to 26?*

Hint: *What percentage of 40 is 25?*

62.5%

Modern insurance companies collect and analyze a vast array of information so that they can accurate predictions about the number and size of the claims they will face. Here is a small example:

Phoebe's neighborhood has been plagued by bicycle thefts. As a result, some bicycle owners have bought bike locks and all are interested in insuring against theft. After conducting some research, Phoebe estimates that for every 40 unlocked bicycles in the neighborhood, one is stolen each month, while the rate of stolen locked bicycles is only 1 in 100 per month. Furthermore, Phoebe calculates the average value of the bicycles in her neighborhood at \$200 each.

Using the above data, and the fact that her neighborhood contains 300 bicycles, half of which are unlocked, Pheobe creates the following table:

	number of bikes	$\times$	probability of theft	$=$	number of thefts
Unlocked	150	$\times$	$\frac{1}{40}$	$=$	3.75
Locked	150	$\times$	$\frac{1}{100}$	$=$	1.5
Total					**5.25**

An expectation of 5.25 thefts corresponds to an expected loss in bicycle value of $5.25 \times \$200 = \$1,050$. Since $\$1,050 \div 300 = \$3.50$, Phoebe convinces each bike owner to pay her \$5 per month to insure their bike against theft. If Phoebe's calculations are close to accurate, then she stands to make a profit of $\$1,500 - \$1,000 = \$500$ if 5 bikes are stolen and $\$1,500 - \$1,200 = \$300$ if 6 bikes are stolen.

Of course, the Phoebe could even more shrewdly charge those who leave their bikes unlocked more for insurance (a reasonable practice considering unlocked bikes are more likely to be stolen) and make an even larger profit.

Question 3 *Suppose Phoebe charges owners of unlocked bicycles \$7.50 and owners of locked bicycles \$3 for insurance. How much would Phoebe earn if their were 6 thefts that month?*

Solution

Hint: *How much will Phoebe collect in premiums from the owners of unlocked bicycles? How much from the owners of locked bicycles?*

Hint: *How much will 6 bicycle thefts cost Phoebe?*

\$375

Nice work!!

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9 Taxes

Basic learning objectives

These are the tasks you should be able to perform with reasonable fluency **when you arrive at our next class meeting**. Important new vocabulary words are indicated *in italics*.

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Advanced learning objectives

In addition to mastering the basic objectives, here are the tasks you should be able to perform **after class, with practice**:

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Learning outcomes:

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10 Taxes

> Our new Constitution is now established, and has an appearance that promises permanency; but in this world nothing can be said to be certain, except death and taxes. — Benjamin Franklin, 1789

Taxes, in some form or another, are familiar to all of us. Local, state, and federal government agencies all levy taxes of various types and most have taken pains to ensure that collection of taxes is automatic, meaning that the calculations involved in taxation are often hidden from view.

Some taxes are simple percentage rates, such as most sales tax. The 2014 sales tax rate in Henderson is 8.1%, meaning that for every \$100 in taxable purchases an additional \$8.10 is collected in tax.

Other taxes, like income tax, combine percentage calculations with more complicated delineations of income into brackets. For example, a single taxpayer with \$25,000 in taxable income will be charged a federal tax rate of 10% while for an equivalent filer with \$125,000 in taxable income the rate of taxation is 28%.

Still other taxes involve more complicated computations involving a wide variety of factors.

A subsidy can be thought of as a negative tax — money is paid from the government to an entity for a particular economic, social, or political purpose. For example, during the Great Depression food prices dropped considerably because of farms producing more crop than the market demanded. This resulted in a large number of farmers falling into poverty and defaulting on their farm (and home!) loans. To counteract this, Congress passed the Agriculture Adjustment Act in 1933 that offered farmers money for *not* growing certain crops. This money offset the farmers' debt and reduced the quantity of certain crops, leading to an increase in food prices that further benefited farming communitities.

The practice of offering subsidies, including for agriculture, continues in large measure today despite substantial controversy surrounding the practice.

A tariff is a type of tax that is levied on goods produced in one location and sold in another location, such as imported or exported goods. Tariffs are sometimes imposed so that goods manufactured cheaply in another part of the world cost the same as or more than goods produced in another part of the world. This is done so that the more expensively produced goods are not undersold by foreign competitors' goods.

Question 1 *Suppose that when a consumer in coastal California is faced with purchasing either of two lightbulbs that differ in price by \$d, the probability they will buy the cheaper lightbulb is* $\frac{1}{1+2^{-4d}}$*. For example, if one lightbulb is \$0.50 more than the other, the consumer will purchase the cheaper lightbulb* $\frac{1}{1+2^{-4\times 0.50}} = 0.8 = 80\%$ *of the time. If consumers in coastal California buy 1,500,000 lightbulbs in a given month and one of them is \$0.10 more expensive than the other, about how many of the cheaper lightbulbs will be bought?*

Learning outcomes:

Solution

(a) 850 thousand ✓

(b) 750 thousand

(c) 700 thousand

(d) 1 million

(e) 500 thousand million

Hint: *Evaluate* $\frac{1}{1+2^{-4\times 0.10}}$.

Hint: *How many lightbulbs, out of 1,500,000 does* $\frac{1}{1+2^{-4\times 0.10}}$ *represent?*

Question 2 *Tom operates a lightbulb factory in coastal California. The costs associated with the factory are such that Tom must sell $1,000,000 worth of lightbulbs per month to break even. Javier operates a similar factory in rural Nevada. The costs of business are lower for Javier than Tom; Javier can break even by selling $800,000 worth of lightbulbs per month.*

Suppose Javier prices his lightbulbs at $0.80 each and ships them to coastal California for an additional $0.05 per lightbulb, where Tom's are currently priced at $1.00 each. In response to the cheap lightbulbs, authoritites in coastal California put a tariff of $0.45 each on Javier's lightbulbs. If customers buy 1,500,000 lightbulbs per month, what will be the outcome of the tarrif on Tom's business? (Assume that Tom's and Javier's lightbulbs are the only lightbulbs for sale.)

Solution

(a) Tom will make a large profit (over $100,000/month)

(b) Tom will make a small profit (under $100,000/month) ✓

(c) Tom will break even

(d) Tom will suffer a small loss (less than $100,000/month)

(e) Tom will suffer a large loss (more than $100,000/month)

Hint: *What will the total price of Javier's imported lightbulbs be, including his $0.80 cost, shipping charges, and the tariff?*

Hint: *Given the price of the lightbulbs, use* $\frac{1}{1+2^{-4d}}$ *to find out how likely a customer is to buy Tom's lightbulbs.*

Hint: *How many of Tom's lightbulbs will the consumers purchase at the price of $1.00 each?*

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11 Mortgages

We introduce the procedures for calculating interest.

Basic learning objectives

These are the tasks you should be able to perform with reasonable fluency **when you arrive at our next class meeting**. Important new vocabulary words are indicated *in italics*.

- Be able to navigate the website `http://www.mortgagecalculator.org/` and obtain relevant output.
- Know the meaning of terms *mortgage*, *PMI*, and *principal*.
- Know how the interest rate and term of the mortgage affect the payments.

Advanced learning objectives

In addition to mastering the basic objectives, here are the tasks you should be able to perform **after class, with practice**:

- Be able to use the affordability guidelines to determine whether you can afford a home.
- Know how to use points to help pay for a home.

Learning outcomes:

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12 Mortgage Payments

We learn to calculate mortgage payments.

In this activity we will explore a tool to help us calculate the value of a mortgage payment under various conditions. We need to be familiar with the following terms.

A mortgage is a loan that finances the purchase of a home. A mortgage can be repaid under a variety of conditions. The balance of the mortgage is the amount of money that must be repaid. The interest rate is the portion of the balance that must be paid to the lender each year. The term of the loan is the number of years until the loan must be repaid.

A loan with a down-payment of less than 20% of the cost typically involves a charge called principal mortgage insurance, or PMI. The PMI will usually be dropped after 20% of the mortgage has been paid off.

Open the website `http://www.mortgagecalculator.org/` in a separate window.

Question 1 *Calculate the mortgage under the following conditions:*

Home Value: $\$200,000$

Credit profile: *Good*

Loan amount: $\$190,000$

Loan Purpose: *New Purchase*

Interest rate: 4%

Loan term: 30 *years*

Start Date: *Jan 2017*

Property tax: 1.25%

PMI: 0.5%

The monthly payment for this loan is $1115.42.
The year when the PMI is paid off is 2024.
The first year when the amount of principal paid excedes the amount of interest paid is 2030

Hint: *To see the amortization schedule click on the link for output parameters and check the box next to "Show annual amortization table".*

Question 2 *Which of the following conditions will result in the lowest monthly premium?*

Solution

Learning outcomes:

(a) 4% interest for a 30-year term ✓

(b) 4% interest for a 20-year term

(c) 5% interest for a 20-year term

(d) 5% interest for a 30-year term

Hint: *Try running all four scenarios keeping all the other conditions the same.*

Question 3 *Which of the following conditions will result in the lowest total payout for the loan?*

Solution

(a) 4% interest for a 20-year term ✓

(b) 4% interest for a 30-year term

(c) 5% interest for a 20-year term

(d) 5% interest for a 30-year term

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13 Basic Probability

We introduce the basic concept of probability and work through some simple examples.

Basic learning objectives

These are the tasks you should be able to perform with reasonable fluency **when you arrive at our next class meeting**. Important new vocabulary words are indicated *in italics*.

- List the *sample space* of all possible *events* of an *experiment*.
- Be able to determine the *probability* of an event in simple situations.

Advanced learning objectives

In addition to mastering the basic objectives, here are the tasks you should be able to perform **after class, with practice**:

- Use a *probability tree* to display the possible outcomes of an experiment and to compute the probability of events.
- Know how to properly apply the addition and multiplication rules for probabilities.
- Understand what the *complement* of an event is and how to calculate its probability.

Learning outcomes:

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14 What is probability?

We define the concept of proability.

The basic idea of probability is that it measures the chance that something will happen. You may be familiar with the joke that everything is 50-50 because it either happens or it doesn't! Embedded in this joke is an application of the basic formula for probability (and an example of a common error that is made).

An *experiment* in probability theory is some repeatable procedure that will result in a randomized outcome. Examples of experiments include flipping a coin, rolling dice, drawing from a deck of cards, or picking balls out of an urn (any type of container will do, but mathematicians usually use urns).

If we roll a standard 6-sided die, there are six possible outcomes, which we will denote with the following notation:

$$\{1, 2, 3, 4, 5, 6\}$$

The list of all possible outcomes is known as the *sample space* of an experiment. Notice that we have used curly brackets around the list and that the outcomes are separated by a comma. (This is known as a set.) The individual outcomes are known as the *events*.

If the die has not been tampered with, then all of the outcomes are *equiprobable*, meaning that the chance of each number coming up is the same. In this case, there is a simple formula that we can use to determine the probability:

$$P(X) = \frac{\#\text{ of successes}}{\#\text{ of possibilities}}$$

In this formula, "$P(X)$" is read "The probability of event X." The symbol X represents a description of a type of event, so that X could be phrases like "rolling a 1" or "rolling an even number."

A *success* simply means that the outcome of the experiment matches described event, and the # of possibilities is simply a count of all of the possible outcomes of the experiment.

Consider the following examples for the dice rolling experiment above:

- If $X =$ "rolling a 1" then $P(X) = \dfrac{1}{6} \approx 16.7\%$ because there is only one way to roll a 1 and six possible outcomes when rolling a die.
- If $X =$ "rolling an even number" then $P(X) = \dfrac{3}{6} = \dfrac{1}{2} = 50\%$ because there are three ways to roll an even number and six possible outcomes when rolling a die.

Sometimes we convert the fraction to a decimal or a percent, but there are many situations in which we leave it as a fraction. You will need to be able to work with all of these situations.

Learning outcomes:

Now let's consider an urn that contains 999 red balls and 1 black ball. The sample space for this experiment is $\{\text{red}, \text{black}\}$. If we're not careful, this may make it look as if there are only two possible outcomes for the experiment. If that were true, then the probability of drawing a black ball and a red ball would be

$$P(\text{black}) = \frac{1}{2} = 50\% \qquad \text{and} \qquad P(\text{red}) = \frac{1}{2} = 50\%.$$

But intuitively, we know that the chance of drawing a red ball is much larger than the chance of drawing a black ball. What is going wrong?

The mistake is that the sample space does not include any information of the probabilities themselves, just the outcomes. Drawing a red ball and drawing a black ball are not equiprobable. When we count the number of possible outcomes, we need to count each individual ball (so that there are 1000 possible outcomes) and not just the colors. Once we do this, we get numbers that match our expectations.

$$P(\text{black}) = \frac{1}{1000} = 0.1\% \qquad \text{and} \qquad P(\text{red}) = \frac{999}{1000} = 99.9\%.$$

And this explains why not every event is 50-50.

Question 1 *Suppose you flip a coin. What are possible representations of the sample space of the experiment? (There are multiple correct answers.)*

Solution

(a) 50% heads, 50% tails

(b) $\{\text{heads}, \text{tails}\}$ ✓

(c) $\{\text{T}, \text{H}\}$ ✓

(d) $\{50\% \text{ heads}, 50\% \text{ tails}\}$

(e) $P(\text{tails}) = \frac{1}{2}, P(\text{heads}) = \frac{1}{2}$

(f) Heads, Tails

Hint: *The sample space of an experiment is a list of all the possible outcomes.*

Hint: *The sample space does not include information about the probabilities.*

Hint: *Sample spaces are sets, so they require the use of set notation.*

Question 2 *Suppose you roll a standard 6-sided die. What is the probability of rolling a number greater than 4? (There are multiple correct answers.)*

Solution

(a) $\frac{2}{3}$

(b) $\frac{1}{3}$ ✓

(c) $\frac{1}{2}$

(d) $\frac{3}{6}$

(e) $\frac{2}{6}$ ✓

(f) $\frac{2}{4}$

(g) $\frac{4}{6}$

Hint: $P(X) = \frac{\#\text{ of successes}}{\#\text{ of possibilities}}$

Hint: *Did you try reducing the fraction?*

Question 3 *Suppose you have an urn containing 30 red balls and 20 black balls. What is the probability of drawing a red ball?*

Solution

(a) 50%

(b) $\frac{3}{5}$ ✓

(c) $\frac{30}{20}$

(d) $\frac{30}{50}$ ✓

(e) $\frac{1}{2}$

(f) 30 : 20

(g) 60% ✓

Hint: $P(X) = \frac{\#\text{ of successes}}{\#\text{ of possibilities}}$

Hint: *Did you try reducing the fraction?*

Hint: *Did you try converting the fraction to a percent?*

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15 Dependent and Independent Events

We introduce the basic concept of probability and work through some simple examples.

Basic learning objectives

These are the tasks you should be able to perform with reasonable fluency **when you arrive at our next class meeting**. Important new vocabulary words are indicated *in italics*.

- Understand the distinction between *dependent* and *independent* events and perform calculations of each type.
- Understand *the gambler's fallacy*.

Advanced learning objectives

In addition to mastering the basic objectives, here are the tasks you should be able to perform **after class, with practice**:

- Understand the *birthday paradox* and its associated calculations.
- Understand the importance of precise language and specifying conditions in advance when discussing probabilities as they pertain to *coincidences*.

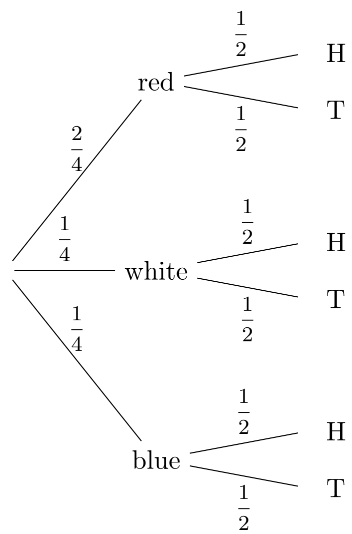
When talking about probability, two events are *independent* if the outcome of one event does not influence the outcome of the other event. For example, if you were to flip a coin and roll a die, you would not expect that the outcome of the coin flip will change the probabilities on the die roll. This would make the events independent.

Two events are *dependent* if the outcome of one event does have an influence on the outcome of the other event. Imagine you have an urn containing 2 red balls and 2 black balls, and you are planning to draw two balls from the urn (one at a time). The probability of getting a red ball on your second draw depends on what happened during your first draw. If you drew a black ball first, then there would be two red balls and one black ball in the urn, so that $P(\text{2nd draw red}) = \dfrac{2}{3}$. But if you had drawn a red ball the first time, then there would be one red ball and two black balls in the urn, and $P(\text{2nd draw red}) = \dfrac{1}{3}$. Since the first event influences the probability of the second event, these events are dependent.

An easy way to determine whether two events are dependent or independent is to look at the probability tree. If all of the branching at the second level is identical (including the probabilities), then the events are independent. If there is any difference in any of the branches, then the events are dependent.

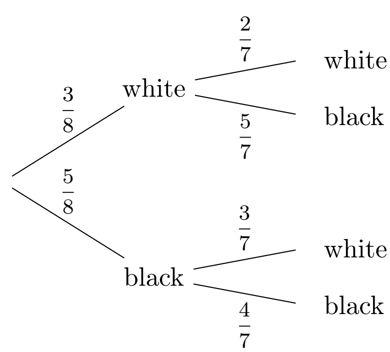
For example, the following is the probability tree for drawing a ball from an urn containing two red balls, one white ball, and one blue ball, followed by flipping a coin.

Learning outcomes:



If you focus on the part to the right of the red, white, and blue, you can see that they are completely identical. This means that drawing the ball and flipping the coin are indepdnent.

The following is an example of a probability tree for drawing two balls without replacement from an urn containing 3 white balls and 5 black balls.



Even though the outcomes are the same, the probabilities of each outcome changes depending on which color was drawn first.

The human mind is wired in a way that makes it good at spotting patterns. However, the idea of randomness is that there are no patterns. The desire to see patterns in randomness, and to believe that independent events are actually dependent events. If you flip a fair coin and get heads 5 times in a row, you might think that you're due for a tail or that head is on a hot streak. Neither of these effects are real if the coin is fair.

This effect is known as the Gambler's Fallacy, and this concept extends beyond mathematics into cognitive psychology. The following video explains this concept in more detail: [1]

[1] YouTube link: `https://www.youtube.com/watch?v=p14icOB2iac`

Question 1 *Which of the following are examples of dependent events?*

Solution

(a) Removing two balls from an urn one at a time without replacement ✓

(b) Removing two balls from an urn one at a time, but replacing the ball after each draw

(c) Flipping the same coin twice

(d) Flipping two different coins

(e) Dealing two cards from a standard deck of cards. ✓

(f) Dealing two cards from a deck of cards that contains no aces. ✓

Hint: *Dependent events are when the outcome of one event has an influence on the outcome of the other event. Independent events are when the outcome of one event does not influence the outcome of the other event.*

Hint: *Imagine that you're performing the experiment and think about what happens after the first event. Is this connected to what is about to happen in the next part?*

Question 2 *The Gambler's Fallacy is the belief that...*

Solution

(a) — you can beat the house when you gamble.

(b) — the chances of future events change based on previous events ✓

(c) — superstitious acts can influence future outcomes

Hint: *The answer is in the video. The wording is slightly different, but the concept is the same.*

Question 3 *Which of the following are examples of the Gambler's Fallacy?*

Solution

(a) The last 4 coin flips have been heads. We're due to see a tail! ✓

(b) Four aces have been dealt from this deck of cards. There's no way the next card will be an ace!

(c) Our first child is going to be a girl because my mom's first child was a girl and my grandma's first child was a girl. ✓

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16 Conditional Probability

We introduce the idea of conditional probability and work through some simple examples.

Basic learning objectives

These are the tasks you should be able to perform with reasonable fluency **when you arrive at our next class meeting**. Important new vocabulary words are indicated *in italics*.

- Understand the definition of a *conditional probability* and its relationship to the general concept of probability.
- Be able to compute a conditional probability by counting objects in a sample space.

Advanced learning objectives

In addition to mastering the basic objectives, here are the tasks you should be able to perform **after class, with practice**:

- Compute conditional probabilities using a probability tree.
- Apply the concept of conditional probability to the example of the *false positive paradox*.

When working with probabilities, we sometimes want to make certain types of restrictions to our sample space. This can sometimes lead to interesting situations that are not intuitive. A conditional probability is a probability calculation in which there is a restriction on the sample space. The notation for conditional probabilities is the following:

$P(X|Y) =$ "The probability that event X occurs *given than* event Y occurred."

In the case that all events are equiprobable, we can use a formula that is similar to the one we used for basic probabilities (which we will repeat here for reference).

$$P(X) = \frac{\#\text{ of successes (for } X)}{\#\text{ of possibilities}}$$

$$P(X|Y) = \frac{\#\text{ of successes for } X \text{ where } Y \text{ also happens}}{\#\text{ of possibilities where } Y \text{ happens}}$$

The best way to understand these situations is to think through a couple examples. Suppose that you are doing an experiment of rolling two standard 6-sided

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Learning outcomes:

dice. The sample space contains 36 elements:

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

What is the probability of rolling a 3 given that the total is 6 or greater? Notice that our successes (X) are described by rolling a 3, but the extra condition (Y) is that the total is 6 or greater.

The first task is to restrict the sample space to combinations that add up to 6 or greater. We will indicate the rejected outcomes being by blacking them out in the table below:

[redacted]	[redacted]	[redacted]	[redacted]	(1,5)	(1,6)
[redacted]	[redacted]	[redacted]	(2,4)	(2,5)	(2,6)
[redacted]	[redacted]	(3,3)	(3,4)	(3,5)	(3,6)
[redacted]	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

We will remove the blackened combinations because we are not interested in them at all. Within the remaining sample space, we need to count the events that have a 3 in them:

				(1,5)	(1,6)
			(2,4)	(2,5)	(2,6)
		(3,3)	(3,4)	(3,5)	(3,6)
	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

We can see that there are 7 combinations that contain a 3 and a total of 26 combinations that met the condition. This means that the probability is $\frac{7}{26} \approx 26.9\%$. (For comparison, the probability of rolling a 3 when rolling two dice is $\frac{11}{36} \approx 30.6\%$.)

This may not seem that complicated, but the next video will show some highly non-intuitive results that happen with conditional probabilities. Consider the following question: A man has two children. At least one of those children is a boy. What is the probability that the other child is also a boy? (Think for a moment!)

You probably guessed 50%. And you would be wrong. Knowing that one of the children is a boy actually *reduces* the chances of the second child being a boy. Almost everybody is tricked by this problem.

This is known as the boy-boy paradox (or sometimes the boy-girl paradox, if the question is asking for the probability that the other child is a girl). The following video will explain the paradox and give you more experience with conditional probabilities: [1].

[1] YouTube link: `https://www.youtube.com/watch?v=RA6POOOx9h8`

Question 1 *What does $P(X|Y)$ mean?*

Solution

(a) The probability that events X and Y both happen.

(b) The probability that event X happened, given that event Y has also happened. ✓

(c) The probability that event Y happened, given that event X has also happened.

(d) The probability that event Y happened, but event X didn't happen.

Question 2 *When rolling two dice, determine the probability of rolling a 1 given that the total is 6 or greater.*

Solution

(a) $\frac{4}{26}$ ✓

(b) $\frac{11}{36}$

(c) $\frac{1}{6}$

(d) $\frac{2}{13}$ ✓

(e) $\frac{12}{36}$

(f) $\frac{1}{3}$

(g) $\frac{6}{36}$

Hint: *Use the restricted sample space from the example that is similar to this one.*

Hint: *Did you reduce your fraction?*

Question 3 *A man has two children. At least one of those children is a boy. What is the probability that the other child is also a boy?*

Solution

(a) $\frac{1}{3}$ ✓

(b) $\frac{1}{2}$

(c) $\frac{2}{3}$

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17 Expected value

We introduce the idea of the expected value of a situation and compute some examples.

Basic learning objectives

These are the tasks you should be able to perform with reasonable fluency **when you arrive at our next class meeting**. Important new vocabulary words are indicated *in italics*.

- Understand the definition of the *expected value* of a situation.
- Understand the interpretation of the expected value.
- Compute the expected value of a simple probabilistic game.

Advanced learning objectives

In addition to mastering the basic objectives, here are the tasks you should be able to perform **after class, with practice**:

- Compute the expected value of more complex probabilistic games.
- Compute the *house edge* of simple gambling games.

Imagine you have a fair coin and that you're going to play the following gambling game with a "friend" that we will name "the casino." The rules are simple:

- If the coin comes up heads, the casino pays you \$1.
- If the coin comes up tails, you pay the casino \$1.

Setting aside any moral qualms you might have against gambling, you should be able to recognize that this game is fair. But what makes it fair? Most people respond with the idea that you have an equal chance of winning and losing.

But let's change the game a bit. What if you were playing with the following rules:

- If the coin comes up heads, the casino pays you \$0.99.
- If the coin comes up tails, you pay the casino \$1.

Something about this should feel unfair. The unfairness doesn't have anything to do with the frequency with which you win and lose, but how much you win or lose when you win or lose.

Up to think point, we've really only been talking about probabilities. But when you include some form of score-keeping like money, the problem becomes more complicated. This added feature leads us to the concept of *expected value*. The

Learning outcomes:

expected value of a game is how much you expect to win or lose on average when playing the game once.

For example, if the expected value of a game is \$1, then you expect to win \$1 *on average*. This doesn't mean that you will win \$1 every single time, and it may turn out that it's impossible for you to win exactly \$1. But if you play the game many times and average the results, you expect to average a \$1 win for each game.

The formula for expected value looks complicated, but it's not that bad:

$$E[X] = \sum P(X_i) \cdot V(X_i).$$

What does this mean?

- The symbol $\sum$ is called a summation symbol and it indicates that you are to add up several terms, described by the symbols to the right of the summation sign.
- The symbol $E[X]$ is the expected value of the game X.
- The X_i is a notation that represents all the possible outcomes of the game X.
- The $P(X_i)$ is the probability of the event X_i, just as before.
- The $V(X_i)$ is the *value* of the event X_i. This is the amount of points or money the player wins when event X_i happens.

That's a lot of notation to take in, but after working through an example, it will make more sense. We will compute the expected value of the second game above (where you only win \$0.99 when you win).

For this game, we have the following events:

$$X_1 = \text{“H''} \quad X_2 = \text{“T''}$$

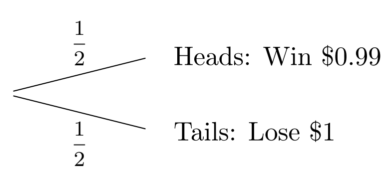
Since we know the coin is fair, we know the probability of each event is the same:

$$P(X_1) = \frac{1}{2} \quad P(X_2) = \frac{1}{2}$$

And we can read off the value of each event from the description of the game. Notice that when the casino wins, we have a negative value because we're measuring this game relative to the player. We also drop the dollar sign because it clutters the notation.

$$V(X_1) = 0.99 \quad V(X_2) = -1$$

This information can be put into a probability tree. The values are listed at the end of the branches.



We can now compute the expected value:

$$\begin{aligned} E[X] &= \sum P(X_i) \cdot V(X_i) \\ &= P(X_1) \cdot V(X_1) + P(X_2) \cdot V(X_2) \\ &= \frac{1}{2} \cdot (0.99) + \frac{1}{2} \cdot (-1) \\ &= -0.005 \end{aligned}$$

What does this mean? It means that on average, each time you play this game you will lose half of a cent. This may not seem like much, but if you play this game over and over again, the casino will eventually have all of your money.

Question 1 *What is the expected value of a game?*

Solution

(a) How often you expect to win the game.

(b) How much you expect to win or lose on average when playing the game once. ✓

(c) How much you win when you win the game.

Question 2 *The symbol $\sum$ represents which arithmetic operation?*

Solution

(a) Addition ✓

(b) Subtraction

(c) Multiplication

(d) Division

Question 3 *Determine the expected value of the following game: You flip a fair coin. If it comes up heads, you win \$100. If it comes up tails, you lose \$80.*

Solution

(a) \$100

(b) -\$80

(c) \$20

(d) \$10 ✓

(e) -\$20

Hint: *Draw the probability tree and label it like the example.*

Hint: *Match up the symbols in the expected value formula with the parts of the problem to make sure everything matches.*

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18 Introduction to Statistics

We introduce the idea of conditional probability and work through some simple examples.

Basic learning objectives

These are the tasks you should be able to perform with reasonable fluency **when you arrive at our next class meeting**. Important new vocabulary words are indicated *in italics*.

- Understand the difference between *probability* and *statistics*.
- Understand how to organize data into a frequency table and to graph the distribution.
- Understand how to calculate the three *measures of central tendency* of a list of data: *mean*, *median*, and *mode*.

Advanced learning objectives

In addition to mastering the basic objectives, here are the tasks you should be able to perform **after class, with practice**:

- Understand the distinction between having a sample of a population and having complete information on a population, and the dangers associated to inferring from a sample.
- Be able to calculate the three measures of central tendency from a frequency table.
- Be able to identify three typical shapes of distributions: normal, bimodal, and skewed.

Probability and statistics are closely related concepts. They both often involve a sense of randomness about the information, but they start from two different places.

Probability starts with the assumption of a complete knowledge of all the possibilities and the chances of each event. For example, with a coin flipping experiment, we usually assume that the coin is fair, so that each side has a 50% chance of landing face up. From that assumption, we can try to draw conclusions about what we expect to happen.

Statistics does not assume such information in advance. Instead, it looks at a *sample* of the data (obtained by experiment, such as physically flipping a coin several times) and draws conclusions based on that information.

Suppose we have an urn that contains balls with numbers on them, but we don't know how many balls are in the urn nor what numbers are on the balls. Furthermore, we are restricted to only pulling out ten balls.

Learning outcomes:

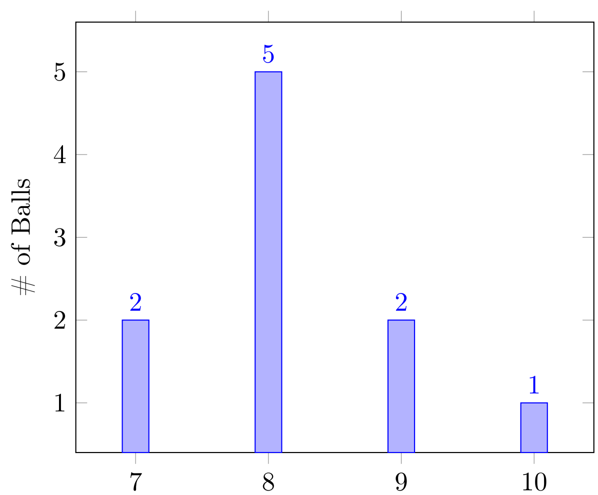
So we draw out the 10 balls one at a time (without replacement) and make a list of the numbers we see. The first ball has a 7, the second a 9, the third and fourth have an 8, the fifth is 10, and so on. We can turn this into a list.

$$\{7, 9, 8, 8, 10, 7, 9, 8, 8, 8\}$$

As a long list like this, the information is difficult to work with. We can turn the list into a *frequency table* by keeping track of how many times we see each number.

Number	Frequency
7	2
8	5
9	2
10	1

This information can also be turned into a bar graph:



All of these are different ways of presenting the *data set* we obtained by drawing balls from the urn. And now that we have the data, we can start to classify the information that we've obtained.

There are three pieces of information about numerical data that are collectively known as the *measures of central tendency*: the mean, the median, and the mode.

The mean is what people generally mean when they talk about an average. It is calculated by adding up the values in the data set and dividing by the number of objects in the data set. Starting from the original list, we would get

$$\text{mean} = \frac{7 + 9 + 8 + 8 + 10 + 7 + 9 + 8 + 8 + 8}{10} = 8.2.$$

The median is the number in the middle when arranged in order. When there is an odd number of data points, such as $[2, 5, 9]$, this is unambiguous (the median is 5). If there is an even number of data points, there is no middle number. In this case, we take the mean of the two middle-most numbers. This is the situation we have with the original data set:

$$7, 7, 8, 8, \underline{8}, \underset{\uparrow}{,} \underline{8}, 8, 9, 9, 10 \rightarrow \text{median} = \frac{8+8}{2} = 8.$$

The mode is the number with the highest frequency. If there are multiple numbers that appear with the highest frequency, then we say that there is no mode. For the data set above, the mode is 8.

Question 1 *Which pair of statements is the correct?*

Solution

(a) Probability uses observed information and statistics uses assumed information.

(b) Both probability and statistics use observed information.

(c) Probability uses assumed information and statistics uses observed information. ✓

(d) Both probability and statistics use assumed information.

Question 2 *Determine the mean of the following data set:* $\{1, 1, 2, 5, 6\}$

Solution

(a) 1

(b) 2

(c) 3 ✓

(d) 4

(e) 5

Hint: *The mean is calculated by adding the numbers together and dividing by the number of data points.*

Hint: *This data set has 5 data points.*

Question 3 *Determine the median of the following data set:* $\{32, 27, 40, 13, 18\}$

Solution

(a) 13

(b) 27 ✓

(c) 18

(d) 13

(e) 26

(f) 40

Hint: *Arrange the numbers in order.*

Hint: *The median is the middle number of the data set when the data set is put in order.*

Question 4 *Determine the median of the following data set:*

Number	*Frequency*
10	*3*
15	*8*
20	*6*
25	*5*

Solution

(a) 10

(b) 15 ✓

(c) 20

(d) 25

Hint: *The mode is the number with the highest frequency.*

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19 Normal Curves

We will delve more deeply into the statistics of normal curves.

Basic learning objectives

These are the tasks you should be able to perform with reasonable fluency **when you arrive at our next class meeting**. Important new vocabulary words are indicated *in italics*.

- Know the general shape of the normal curve.
- Understand the interpretation of standard deviation as a measure of how much the data is clustered around the mean.
- Use Wolfram Alpha to compute the standard deviation of a small data set.
- Use the Geogebra app to graphically understand z-scores.

Advanced learning objectives

In addition to mastering the basic objectives, here are the tasks you should be able to perform **after class, with practice**:

- Compute z-scores of data and the values of data points corresponding to specific z-scores.
- Understand the meaning of *percentile*.
- Use a z-table to obtain percentiles for normal curves.
- Determine the percent of data that falls between certain parts of the normal curve.

Learning outcomes:

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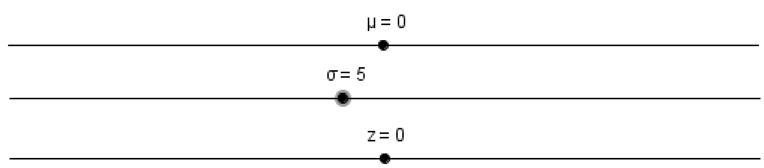
20 Playing With Normal Curves

We will delve more deeply into the statistics of normal curves.

In the LINK-TO-BE-INSERTED-HERE, you will find a Geogebra App. Geogebra is a free math program that can be used to make online interactive programs, and it can be used for a wide range of mathematical topics. At the bottom of the page, you will find three sliders, which allow you to change the values of different variables.



Start with the μ slider. (The symbol μ is the Greek letter m and is read "mu.") You should be able to slide the dot back and forth and see the graph shift to the left and to the right. As you do this, look at the specific value of μ and you should see that it corresponds to a certain part of the graph.

Question 1 *What phrase most accurately describes the relationship between μ and the graph?*

Solution

(a) As μ increases, the graph becomes squeezed together.

(b) As μ increases, the graph shifts to the right. ✓

(c) As μ increases, graph becomes more spread out.

(d) As μ increases, the graph shifts to the left.

Set $\mu = 0$, and then play with the σ slider. (The symbol σ is the Greek letter s and is read "sigma.").

Question 2 *What phrase most accurately describes the relationship between σ and the graph?*

Solution

(a) As σ increases, the graph becomes squeezed together. ✓

(b) As σ increases, the graph shifts to the right.

(c) As σ increases, graph becomes more spread out.

(d) As σ increases, the graph shifts to the left.

Learning outcomes:

As you continue to play with the value of σ, notice how there are four marks that change positions and change values. The values are listed above the curve at the top of the marks. You should see that the distance between those bars is equal to σ. This works regardless of the value of μ. (Try it!)

Set $\mu = 0$ and $\sigma = 5$. Now play with the z slider. You will notice that this doesn't change the graph, but creates a new vertical line that slides back and forth depending on the value of z. For example, when $z = 2.5$, the location of the vertical bar is 12.5.

Our next task will be to understand the meaning of the symbols μ, σ, and z, and their relationship with the vertical line.

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21 Playing With Normal Curves

We will delve more deeply into the statistics of normal curves.

The graphs that you were working with in the LINK-TO-BE-INSERTED-HERE are normal curves. These represent a very wide class of distributions that can be used to describe real life phenomena (at least approximately). For example, people's heights and weights can be modeled with a normal distribution.

The value μ is the mean of the distribution. Graphically, it corresponds to the middle of the curve. We have already talked about the mean as one of the measures of central tendency.

The value σ is known as the standard deviation. It measures how spread out the curve is. When σ is small, the curve is very narrow and very sharp. When σ is large, the curve is wider and flatter.

Calculating σ is a little bit complicated, and we work through the details of the calculation in class. But most of the time, we use a computer to get the standard deviation because of the number of individual steps. One program that was can use is Wolfram Alpha. Wolfram Alpha is an online version of a program called Mathematica, which is used by mathematicians, engineers, computer scientists, and others.

To use Wolfram Alpha to compute the standard deviation, simply pull it up in a web browser and use the `standard deviation` command. In the previous section, we had used the following distribution obtained from drawing numbered balls from an urn:

$$\{7, 9, 8, 8, 10, 7, 9, 8, 8, 8\}$$

To compute the standard deviation of this data using Wolfram Alpha, we need to type out the individual data points inside of set brackets. Specifically, we would type `standard deviation{7, 9, 8, 8, 10, 7, 9, 8, 8, 8}`. (This gives $\sigma \approx 0.919$.)

Both μ and σ are used to describe the normal curve itself. The value of z is used to talk about a particular data point within a data set. Specifically, the value of z measures how close or far the data point is to the mean using standard deviations as the measurement. We call this the z-score of the data point.

Using the LINK-TO-BE-INSERTED-HERE, set $\mu = 20$ and $\sigma = 10$. As you change the z-score, you can see how it changes the arrow that measures outward from μ. Notice that when $z = 1$, the vertical bar (which represents the data point) is located at 30. This corresponds to the idea of being one standard deviation above the mean. If $z = 2$, then the bar would be at 40 and this would mean the data point is two standard deviations above the mean. To get positions below the mean, we would use negative values of z.

The importance of the z-score is that it gives a better relative sense of how the data point is fits in the overall picture of the distribution. For example, if you set $\mu = 10$ and $z = 0.5$ and start changing σ, you'll see that the arrow stretches and compresses as σ increases and decreases. This turns out to be extremely useful for statistics.

Learning outcomes:

For the following problems, you will need to use the LINK-TO-BE-INSERTED-HERE or Wolfram Alpha.

Question 1 *The value of μ corresponds to the ______ of the data.*

Solution

(a) mean ✓

(b) median

(c) mode

(d) standard deviation

(e) z-score

Question 2 *Determine the standard deviation of the data set* $\{10, 12, 13, 13, 18, 20, 21\}$ *(rounded to two decimals).*

Solution 4.79.

Question 3 *Suppose that $\mu = 10$ and $\sigma = 5$. If a particular data point is located at 3, what is its z-score?*

Solution -1.4.

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22 Detecting Fraud

We apply statistical observations to data in order to find fraudulent data.

Basic learning objectives

These are the tasks you should be able to perform with reasonable fluency **when you arrive at our next class meeting**. Important new vocabulary words are indicated *in italics*.

- Understand how conditional probabilities can be used to detect hidden information.
- Understand how to use a known distribution to investigate the legitimacy of a data point.
- Understand the limits of such an investigation.

Advanced learning objectives

In addition to mastering the basic objectives, here are the tasks you should be able to perform **after class, with practice**:

- Perform explicit calculations to get an approximate quantification of the chances of fraud.

Learning outcomes:

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23 The Two-Headed Coin

We use probability to infer information about a coin.

Suppose your friend is flipping a coin and telling you the results. On the first flip, he gets a head. Then he flips it again and gets another head. And then another, and another, and another. As this keeps happening, at a certain point you will start to wonder whether your friend is faking the results or if the coin is biased because with each head it becomes increasingly unlikely that the outcome is truly a result of chance.

This line of reasoning is exactly the same line of reasoning used in fraud detection. There are many situations in which certain types of patterns in randomness stand out as being unusual.

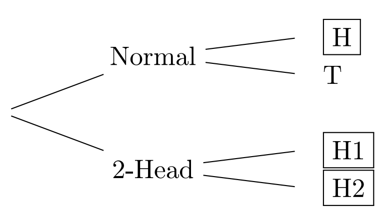
To simplify the matter, let's start over and change the situation slightly. Your friend truthfully tells you that he has a red box and a blue box, and inside each box is a coin. One coin is a normal coin and the other coin has a head on both sides, but you don't know which is in which. You pick the coin in the blue box.

Question 1 *What is the probability that you have chosen the two-headed coin? (Give your answer as a decimal.)*

Solution 0.5

Now she flips the coin and truthfully announces that it's a head. What can you say about the probability that the coin is two-headed? At first, you might think that this doesn't change the probability. But suppose she said that she flipped a tail. You would say that there's a 0% chance that the coin is two-headed. So we do have the ability to update our probabilities based on this information.

We will use a probability tree to describe this situation. Notice that for the two-headed coin, we still have two sides, so that when we draw out the tree we will draw two branches and label them "H1" and "H2." The boxed outcomes are all the ways that we would see a head. (All the branches in the picture are 50-50, so we will leave that out to avoid clutter.)



Notice that there are three outcomes where we see a head and two of them correspond to the 2-headed coin. This means that the probability that we have picked the two-headed coin is $\frac{2}{3}$.

Learning outcomes:

Question 2 *If your friend flips the coin and gets another head, what's the probability that you chose the two-headed coin? (Give your answer as decimal.)*

Solution 0.8

Hint: *Draw the probability tree. Start from the one above (draw it without boxes) and extend it one more level.*

Hint: *The ways you can see two heads in a row are H-H, H1-H1, H1-H2, H2-H1, and H2-H2.*

So you can see that the information that you are receiving from your friend is giving you higher levels of confidence in your conclusion. It's important to note that you can never conclusively conclude that the coin is two-headed using this technique. There will always be a small chance that the flips of a fair coin will randomly land on heads many heads in row.

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24 Prior Distributions

We learn how knowing information in advance can help us to identify fraudulent data.

When using real-world data, we often have past data that can be used as a reference point for making observations about new data. Specifically, this can be used to detect fraud.

Benford's Law is one type of distribution. Benford's Law applies to the first digit of randomly generated data as long as that data spans a very wide range of values. In this context, randomly generated data simply means information in the world that has no particular reason to cluster together. The following video explains Benford's Law: [1].

Question 1 *When looking at data that conforms to Benford's law, which of the following digits will be the most common leading digit?*

Solution

(a) 4 ✓

(b) 7

(c) 9

Hint: *Think about the raffle ticket example from the video, starting around 3:30.*

This type of analysis is used very often in forensic accounting. Within larger organizations, numbers corresponding to things like inventory or the value of purchase orders tend to span a wide range of values and data points are not subject to clustering. This process can be very complex and technical in practice, so we will use a simpler example in class to demonstrate how this type process can accurately identify fraud.

Learning outcomes:

[1] Link: `https://www.youtube.com/watch?v=XXjlR2OK1kM`

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25 Prior Distributions

We discuss the limits of statistically-based inferences.

One of the difficulties of drawing conclusions based on statistics is that it is possible for rare random events to occur. In a problem you worked out previously, you found that the probability of flipping 10 heads in a row is $\frac{1}{1024}$. So for any particular person to get 10 heads in 10 tries is quite rare. But if you had 1024 people running this experiment, you would not be surprised if you found someone with that outcome.

So the mere fact that someone claims to have flipped 10 heads in a row is not sufficient to make a meaningful accusation that the person faked the data. We need to think about things in a broader context.

When making statistically-based inferences, there are two features which help make stronger cases that an event is not the result of chance:

- The size of the sample set: If you flip a coin twice and got heads twice, that's not enough to be concerned that there may be something wrong with the coin. If you flip a coin a hundred times and get heads every single time, then you have a lot of reason to question the coin.

- The number of samples obtained: The more samples you take, the more likely it is that you will see a data point that appears in unlikely places. For example, if there is only a 1% chance of an event, and you've taken thousands of samples, you should not be surprised to see many events in that 1% category. So as you look at more types of events, you need to increase your standards for what becomes a meaningful observation. (This is sometimes referred to as the "look-elsewhere effect.")

Question 1 *A person wants to find out whether people with psychic powers exist by having people predict future events. He announces that he will build an urn that contains three colored balls, and he will draw from 15 times with replacement. If someone predicts the picks correctly, he will certify that person as a true psychic. Based on this measure, if each of the 7 billion people on earth participated, how many true psychics would we expect him to certify?*

Solution

(a) None. Psychic powers don't exist.

(b) About 100.

(c) About 500. ✓

(d) About 1,000.

(e) About 10,000.

Learning outcomes:

Question 2 *The homework included an assignment where you were to make two lists of 75 random coin tosses. In one list, you were supposed to make it up off the top of your head. And in the other list, you were supposed to actually flip a coin 75 times and record the results. We will need this for the upcoming class. Did you do this assignment?*

Solution

(a) Yes ✓

(b) No

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26 Introduction To Functions

We introduce functions, function notation, and various ways to display functions.

Basic learning objectives

These are the tasks you should be able to perform with reasonable fluency **when you arrive at our next class meeting**. Important new vocabulary words are indicated *in italics*.

- Know the difference between functions and other relations.
- Be able to use `http://desmos.com` to enter functions and obtain tables of function values.
- Know how to use function notation to evaluate a function.

Advanced learning objectives

In addition to mastering the basic objectives, here are the tasks you should be able to perform **after class, with practice**:

- Know how to model a function as a graph, a formula, and a table.
- Know how to find functions to algebraically model real-world applications.
- Understand the difference between the dependent variable and the independent variable.

Learning outcomes:

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27 What is a Function?

We define functions and see some examples.

If two variables, x and y, are related to each other, we say that y is a *function* of x if each x value is paired with only one y-value. In other words, each input value, x, results in exactly one output value, y.

For example, if t is the time of day, and s is the speed of your car, then is s a function of t? At any time of day there is only one speed your car is traveling, so s is a function of t. In other words s depends on t.

Is t a function of s? No. There will almost certainly be some some speed that your car reached at two different times. Thus t is not a function of s. The speed could be determined by knowing the time, but the time could not be determined just by knowing the speed.

Another example can be found at a vending machine. Let x be the number of candy bars you buy (for a dollar) and y be the amount you spend on candy bars. Then y is a function of x because the number of candy bars determines the exact amount you spend. Also, x is a function of y because the number of candy bars can be determined by the amount you spent.

Question 1 *Select each option below where the first item is a function of the second item.*

Solution

(a) —the amount on a paycheck and the number of hours worked at an hourly wage job ✓

(b) —a man's height and his father's height

(c) —the number of calories in a meal and the amount you pay for the meal

(d) —the area of a square and the length of its sides ✓

(e) —the length of the sides of a square and its area ✓

(f) —the area of a rectangle and its perimeter

(g) —driverse licence numbers and drivers

Hint: *If the hourly wage is set, then we should only expect one possible payout.*

Hint: *Is it possible for two father's of the same height to have sons of different heights?*

Hint: *Is it possible that two meals that share the same price might have a different number of calories?*

Hint: *Does the formula for the area of a square depend on the side length?*

Hint: *The length of the side of a square is always the square root of the area*

Learning outcomes:

Hint: *A 3×5 rectangle has the same perimeter as a 2×6 rectangle. Do they have the same area?*

Hint: *Does any driver have more than one driver's license numbers?*

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28 Function Notation

We learn about function notation.

Recalling Function Notation

We can think of a function as a machine that takes an input and returns an output. For example, at an arcade a quarter machine accepts dollars and gives out quarters. If x is the number of dollars, and Q is the number of quarters, then we write $Q(x) = 4x$ to denote that when we input x dollars we output $4x$ quarters.

In a nickel arcade the machine might accept x dollars and give out N nickels. In this case the function that governs the machine would be $N(x) = 20x$.

We usually give our functions names like $f(x)$ or $g(x)$ instead of just using the letter y. This is so we can keep track of which formula we are using. The notation is a good one. To evaluate a function such as $f(x) = x^2 + 4x - 7$ we simply replace each x with the same input value. For example,

- $f(0) = (0)^2 + 4(0) - 7 = -7$,
- $f(1) = (1)^2 + 4(1) - 7 = -2$,
- $f(-5) = (-5)^2 + 4(-5) - 7 = -2$.

As long as we replace each x with the same input we will be fine.

Question 1 *Evaluate each value of the function $f(x) = x - \sqrt{x}$.*

Solution

Hint: $f(9) = 9 - \sqrt{9} = 9 - 3 = 6$.

$f(9) = 6$.
$f(1) = 0$.
$f(16) = 12$.

Question 2 *Evaluate each value of the function $g(x) = \dfrac{x(x+1)(x-1)}{6}$.*

Solution

Hint: $g(3) = \dfrac{(3)\,((3)+1)\,((3)-1)}{6}$.

$g(3) = 4$.
$g(1) = 0$.
$g(-2) = -1$.

Learning outcomes:

Technical Details

In higher level math courses we care about the nitty gritty technical details. In this class a working understanding will suffice. We should know that definitions of the terms below.

A *relation* from one set to another is a pairing of the elements in the first set with the elements in the second set. Thus a relation is a set of *ordered pairs*.

A *function* is a relation relating two sets called the *domain* and *range* in which each element of the domain is paired with exactly one element of the range.

The domain is the set of input values, while the range is the set of output values.

For example $f(x) = x^2 - 1$ is a function.

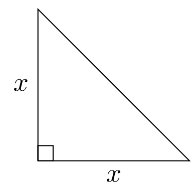
- f is the name of the function.
- x is the input variable or independent variable.
- $x^2 - 1$ is the output expression or dependent variable.
- $f(x)$ denotes the ouput expression or dependent variable.
- In set notation we would write $f = \{(x, x^2 - 1) \mid x \text{ is a real number}\}$.
- The natural domain of f is all real numbers.
- The range of f is $\{y \mid y \geq -1\}$.

29 Visualizing Functions

We learn three methods for presenting functions: graphing, tables, and formulas.

There are three nice methods for presenting a function. We can draw its graph in the plane, create a table, and/or write a formula.

For example, Let $A(x)$ be the the area of the triangle below.

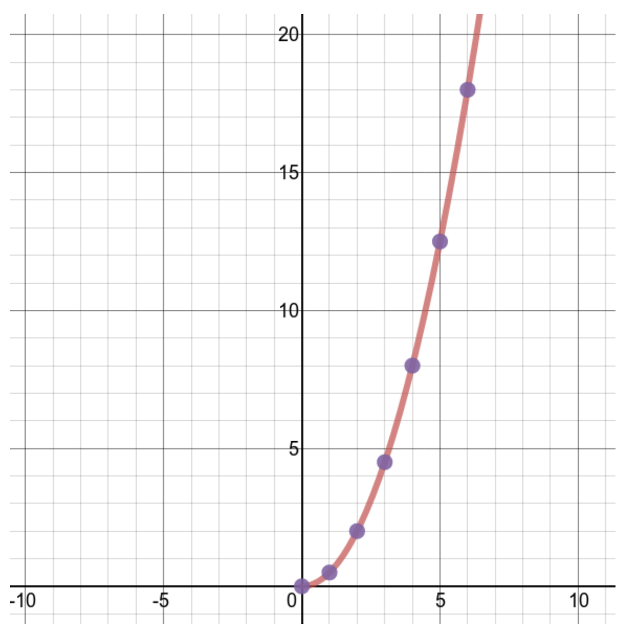


- We can represent $A(x)$ as a table.

x	0	1	2	3	4	5	6
$A(x)$	0	.5	2	4.5	8	12.5	18

- We can represent $A(x)$ as a formula. The area of a triangle is one half base times height, so $A(x) = \frac{1}{2}x^2$.

- We can draw a graph of $y = A(x)$.



Learning outcomes:

All three methods of presenting the function are useful, though some circumstances might make one method more useful than another. The good news is that we will be using a tool to help us obtain these various representations of functions. Watch the video below to learn how to obtain a graph and a table from a function.[1]

Question 1 *Go to* `desmos.com` *and define the three functions*

$$f(x) = x^4 - 2x^3 + x^2 - x, \quad g(x) = x^5 - 4x^4 + 3, \quad \text{and} \quad h(x) = \frac{x^2(x+1)^2}{4}.$$

Create a table that has x-values -1, 0, 1, 2, 3, 4, and 5. Create columns with headings $y = f(x)$, $y = g(x)$, and $y = h(x)$.

Now we will compare the output values of each function in our table. For which x-values below does the greatest output come from the function $f(x)$?

Solution

(a) $x = -1$ ✓
(b) $x = 0$
(c) $x = 1$
(d) $x = 2$
(e) $x = 3$
(f) $x = 4$ ✓
(g) $x = 5$

Hint: *Do the comparison one row at a time.*

For which x-values below does the greatest output come from the function $h(x)$?

(a) $x = -1$
(b) $x = 0$
(c) $x = 1$ ✓
(d) $x = 2$ ✓
(e) $x = 3$ ✓
(f) $x = 4$
(g) $x = 5$

Hint: *Do the comparison one row at a time.*

For which x-values below does the least output come from the function $g(x)$?

(a) $x = -1$ ✓
(b) $x = 0$
(c) $x = 1$
(d) $x = 2$ ✓
(e) $x = 3$ ✓
(f) $x = 4$ ✓
(g) $x = 5$

[1] YouTube link: `https://www.youtube.com/watch?v=kXgb64qcvik`

Hint: *Negative values are less than positive values. Read the problem carefully.*

Question 2 *At* `desmos.com` *define a function* $f(x) = \sqrt[3]{\dfrac{1}{x+4}}$. *Then find* $f(1.6)$ *accurate to three decimal places.*

Solution

Hint: *To get a cube root you can use the input panel at the bottom of the screen. Click on the button labeled "more" to find additional symbols.*

$f(1.6) = 0.563$.

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30 Algebra Of Functions

We list the learning objectives.

Basic learning objectives

These are the tasks you should be able to perform with reasonable fluency **when you arrive at our next class meeting**. Important new vocabulary words are indicated *in italics*.

- Be able to add, subtract, multiply, and divide functions.
- Know how to compose functions.
- Understand how to find an inverse function.

Advanced learning objectives

In addition to mastering the basic objectives, here are the tasks you should be able to perform **after class, with practice**:

- Understand the geometry of adding and subtracting functions.
- Be able to use `desmos.com` to compose functions.
- Interpret the output of a function and its inverse.

Learning outcomes:

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31 Function Arithmetic

We add, subtract, multiply, and divide functions.

We can add or subtract functions to obtain new functions. For example if $f(x) = x^2 - 2$ and $g(x) = x + 3$ then

$$f(x)+g(x) = [x^2-2]+[x+3] = x^2+x+1, \qquad \text{and} \qquad f(x)-g(x) = [x^2-2]-[x+3] = x^2-x-5$$

Question 1 *Let $f(x) = 2 - x$ and $g(x) = x^2 + 2x$. Then*

Solution

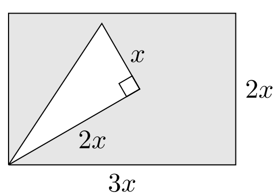
Hint: *Just compute $[2 - x] + [x^2 + 2x]$*

$f(x) + g(x) = x^2 + x + 2$.

Hint: *Just compute $[2 - x] - [x^2 + 2x]$*

$f(x) - g(x) = -x^2 - 3x + 2$.

Question 2 *Let $f(x)$ be the function that gives the area of the rectangle below and $g(x)$ be the function that gives the area of the triangle inside the rectangle.*



Find a formula for each function.

Solution

Hint: *The area of a rectangle is length times width.*

$f(x) = 6x^2$.

Hint: *The area of a triangle is one half base times height.*

$g(x) = x^2$.

What does the function $h(x) = f(x) - g(x)$ represent geometrically?

(a) —the combined perimeter of both figures

(b) —the area of both figures

(c) —the area of the shaded region ✓

(d) —half of the total area

Learning outcomes:

Hint: *What do f and g represent?*

$h(2) = 20$

We can obtain new functions by multiplying and dividing functions as well. For example if $f(x) = x^2 - 2$ and $g(x) = x + 3$ then

$$f(x)g(x) = [x^2 - 2][x + 3] = [x^2 - 2] \cdot x + [x^2 - 2] \cdot 3 = x^3 - 2x + 3x^2 - 6,$$

and

$$\frac{f(x)}{g(x)} = \frac{x^2 - 2}{x + 3}.$$

Question 3 *Let $w(t)$ and $l(t)$ be the functions that gives the width and length of a rectangle at time t.*

What does the function $h(t) = w(t)l(t)$ represent geometrically?

Solution

(a) —the perimeter of the rectangle at time t

(b) —the length of the diagonal of the rectangle at time t

(c) —the area of the rectangle at time t ✓

(d) —the difference between the width and length at time t

Hint: *What do you get when you multiply the length and the width of a rectangle?*

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32 Function Composition

We can compose two functions.

We can obtain a new function by composing two functions. This means that we stick one function inside another. Recall how function notation works. We evaluate $f(x)$ by replacing every x with the same quantity. For example, if $f(x) = x^2 + x + 1$ then

- $f(0) = (0)^2 + (0) + 1 = 1$
- $f(1) = (1)^2 + (1) + 1 = 3$
- $f(b) = b^2 + b + 1$
- $f(2a + 1) = (2a + 1)^2 + (2a + 1) + 1$

Be careful to always replace one expression with something equal to it. Use equal signs where they belong.

For example if $f(x) = x^2 - 2$ and $g(x) = x + 3$ then

$$f(g(x)) = f(x + 3) = (x + 3)^2 - 2.$$

Note that we replaced $g(x)$ with $x + 3$ and then evaluated $f(x)$.

For the same two functions

$$g(f(x)) = g(x^2 - 2) = [x^2 - 2] + 3 = x^2 + 1.$$

We replaced $f(x)$ with $x^2 - 2$ and then evaluated $g(x)$.

Question 1 *Let $f(x) = 2 - x$ and $g(x) = x^2 + 2x$. Then $f(g(x)) = 2 - x^2 - 2x$.*

Solution

Hint: $f(g(x)) = f(x^2 + 2x)$.

Suppose $s(x)$ is the function that gives you your speed while traveling to work when you eat x bowls of oatmeal. Let $T(s)$ be the function that tells how long it takes to get to work when traveling at speed, s. Then $T(s(x))$ is the function that outputs the time to get to work when you eat x bowls of oatmeal.

Question 2 *Let m be the level of traffic at the time of day t. m depends of t so we can write it $m(t)$ for m.*

Let W be the time it takes to get a work. W depends on m, so we can write $W(m)$ for W.

What does the function $W(m(t))$ represent?

Solution

Learning outcomes:

(a) —the total traffic on your way to work at time t

(b) —the time of day when the traffic is heaviest

(c) —the total time it takes to get to work at time t ✓

(d) —the traffic at time t

Hint: *Note that you are inputting time t and outputting travel time W.*

Note that the composition doesn't make sense in the reverse direction. We can't consider $m(W(t))$ because W doesn't accept time as an input. It accepts traffic levels as input.

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33 Inverse Functions

We learn how to find an inverse function.

In some cases two variables are related in such a way that either could determine the value of the other. For example, if A is the area of a square and x is the side length of a square then $A = x^2$ $(x \geq 0)$ and $x = \sqrt{A}$. Thus A is a function of x and x is a function of A.

We call two variables that are each functions of each other *inverse functions*. It would be appropriate to use function notation: $A = A(x) = x^2$ and $x = x(A) = \sqrt{A}$. Note that when we compose these two functions we are just left with the input.

Inverse functions are not hard to find, but we will definitely take advantage of the tools available to us.

Question 1 *We want to find the inverse function of* $f(x) = \dfrac{2x}{x-1}$. *Go to* `wolframalpha.com` *and type in* `inverse function f(x)=2x/(x-1)` *to see the output.*

Solution $f^{-1}(x) = x/(x-2)$

Hint: *By default Wolfram Alpha assumes the input to both functions is x.*

Hint: *Be sure to group the denominator of a fraction inside parentheses.*

Question 2 *Consider the function that converts the temperature from Celcius to Fahrenheit.*

$$F = \frac{9}{5}C + 32.$$

We want to find the function that converts the temperature from Fahrenheit to Celcius.

You can solve for C, but instead please use `wolframalpha.com` *to do it for you.*

Solution $C = 5/9 * (F - 32)$

Hint: *Just enter* `solve for x, y=9/5*x+32` *in* `wolframalpha.com`*. Then substitute $x = C$ and $y = F$. Wolfram Alpha tries to be too clever if you leave in the C and F.*

In summary, we get an inverse function, when we swap the roles of the dependent variable and the independent variable. Not all functions have an inverse—only functions where each output is paired with a single input have an inverse.

Learning outcomes:

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34 Graphs of Parent Functions

The goal of this section is to learn the basic shape of several basic functions that we will refer to as parent functions. These are the basic building block shapes out of which most basic functions are built.

Basic learning objectives

These are the tasks you should be able to perform with reasonable fluency **when you arrive at our next class meeting**. Important new vocabulary words are indicated *in italics.*

- Know the basic shapes of each of the *parent functions* $y = x$, $y = x^2$, $y = x^3$, $y = \sqrt{x}$, $y = |x|$, $y = \dfrac{1}{x}$.
- Be able to use `http://desmos.com` to plot functions. Be able to change the color and appearance of the plots.
- Know the basic characteristics of each parent function, and be able to plot them by hand.

Advanced learning objectives

In addition to mastering the basic objectives, here are the tasks you should be able to perform **after class, with practice**:

- Know the standard form of the equation of a circle $(x-h)^2 + (y-k)^2 = r^2$ and the significance of h, k, and r.
- Be able to find a parent function to match data.

Open up `https://www.desmos.com/calculator/g3l5c4yfkz` in a separate window and experiment with the display and other options. Note that you can select which functions to display by toggling the circle to the left of each function. Each of the functions plotted in this applet are called *parent functions* because they are basic building blocks out of which a wide variety of functions can be built.[1]

Question 1 *Which of the parent functions below go through the origin?*

Solution

(a) $y = x$ ✓

(b) $y = x^2$ ✓

(c) $y = x^3$ ✓

(d) $y = |x|$ ✓

(e) $y = \sqrt{x}$ ✓

Learning outcomes:

[1] YouTube link: `https://www.youtube.com/watch?v=BXLQDhPi6-k`

(f) $y = \dfrac{1}{x}$

Hint: *Plot each function individually by toggling the circle to the left of each function. Look to see which curve includes the point* $(0, 0)$.

Hint: *You can try plugging* $x = 0$ *into each expression to see if the output is* $y = 0$.

Question 2 *Which of the parent functions below go through the point* $(1, 1)$*?*

Solution

(a) $y = x$ ✓

(b) $y = x^2$ ✓

(c) $y = x^3$ ✓

(d) $y = |x|$ ✓

(e) $y = \sqrt{x}$ ✓

(f) $y = \dfrac{1}{x}$ ✓

Hint: *Just look at the graphs.*

Hint: *You can try plugging* $x = 1$ *into each expression to see if the output is* $y = 1$.

Recall that the domain of a function is the set of input values (or the set of x-values).

Question 3 *Which of the parent functions below has as its domain the interval* $(-\infty, \infty)$*?*

Solution

(a) $y = x$ ✓

(b) $y = x^2$ ✓

(c) $y = x^3$ ✓

(d) $y = |x|$ ✓

(e) $y = \sqrt{x}$

(f) $y = \dfrac{1}{x}$

Hint: *You aren't allowed to take square roots of negatives (when working with real numbers) or to divide by zero.*

Hint: *As each curve travels from left to right are there any holes or gaps?*

Question 4 *Which of the parent functions below has a graph with a jagged corner?*

Solution

(a) $y = x$

(b) $y = x^2$

(c) $y = x^3$

(d) $y = |x|$ ✓

(e) $y = \sqrt{x}$

(f) $y = \dfrac{1}{x}$

Question 5 *Which of the parent functions below has some negative y-values as its ouput?*

Solution

(a) $y = x$ ✓

(b) $y = x^2$

(c) $y = x^3$ ✓

(d) $y = |x|$

(e) $y = \sqrt{x}$

(f) $y = \dfrac{1}{x}$ ✓

Hint: *Which curves drop below the x-axis?*

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35 More Parent Functions

In this section we are introduced to the sine and cosine functions and the exponential and logarithmic functions.

Basic learning objectives

These are the tasks you should be able to perform with reasonable fluency **when you arrive at our next class meeting**. Important new vocabulary words are indicated *in italics*.

- Know the basic shapes of each of the *parent functions* $y = sin(x)$, $y = cos(x)$, $y = log(x)$, and $y = 10^x$.
- Be familiar with the constant $e \approx 2.718281828$.
- Know the basic characteristics of each parent function, and be able to plot them by hand.

Advanced learning objectives

In addition to mastering the basic objectives, here are the tasks you should be able to perform **after class, with practice**:

- Be able to evaluate logarithmic expressions.
- Know how to solve equations graphically and how to solve equations using `wolframalpha.com`.
- Be able to find a parent function to match data.

Learning outcomes:

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36 Wave Functions

We meet the sine and cosine functions.

We start off by learning about two wave functions called the *sine* and *cosine* functions. In a trigonometry class you would learn a great deal about these functions. In this class we will just learn some of their important characteristics, and we will use them for some limited applications.

Open up the link [1] in a separate window. Watch the video at ———— Answer each question below.

Exercise 1 *Compute* $\sin\left(\frac{\pi}{6}\right)$.

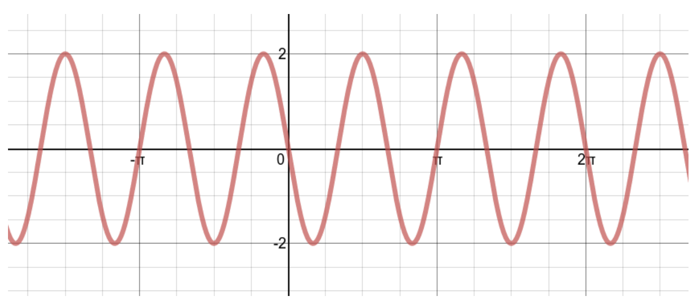
Solution

Hint: *Plug it into desmos or another calculator.*

Hint: *Be careful, because some calculators are in degree mode. You may need to find a mode setting to change the calculator to be in radian mode.*

$\sin\left(\frac{\pi}{6}\right) = .5$.

Question 2 *The graph below shows a function of the form* $f(x) = a\sin(bx)$, *where* $b \geq 0$. *Find the value of* a *and* b *to match the graph below.*



Solution

Hint: *Use Desmos to create the function* $f(x)$ *with sliders for* a *and* b.

Hint: *Remember that you can reach the menu to change the* x*-axis labels in desmos by clicking on the wrench.*

In the picture $a =$ -2 and $b =$ 4 3.

Learning outcomes:

[1]Link: `https://www.desmos.com/calculator/ubhracbupf`

Question 3 *In a function of the form $f(x) = a\sin(bx)$ the constant a does what to the graph of $y = \sin(x)$?*

Solution

(a) —shifts the graph to the right by a units

(b) —shifts the graph upward by a units

(c) —scales the graph horizontally by a factor of a

(d) —scales the graph vertically by a factor of a ✓

Hint: *Use Desmos to create the function $f(x) = a\sin(bx)$ with sliders for a and b. As you change the value of a, what happens to the graph?*

Question 4 *In a function of the form $f(x) = a\sin(bx)$ the constant b does what to the graph of $y = \sin(x)$?*

Solution

(a) —shifts the graph to the right by b units

(b) —shifts the graph upward by b units

(c) —scales the graph horizontally by a factor of $\frac{1}{b}$ ✓

(d) —scales the graph vertically by a factor of $\frac{1}{b}$

Hint: *Use Desmos to create the function $f(x) = a\sin(bx)$ with sliders for a and b. As you change the value of b, what happens to the graph?*

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37 Exponential Functions

We become acquainted with exponential functions and their graphs.

An *exponential function* has the form $y = a^x$, where $a > 0$ and $a \neq 1$. For example, the function $g(x) = 2^x$. We need to know the general shape of an exponential function and its behavior at both extremes. In a separate window go to `desmos.com` and create a function $g(x) = a^x$ along with a slider for a. Next restrict the values of a so that $0.1 < a < 10$.

Question 1 *Select each true statement below about the exponential function $g(x) = a^x$.*

Solution

(a) The point $(0, 1)$ belongs to the graph of $y = g(x)$ for all values of a. ✓

(b) When $a < 1$ the function $g(x)$ increases in value from left to right.

(c) When $a > 1$ the function $g(x)$ increases in value from left to right. ✓

(d) The output values of $g(x)$ are never negative. ✓

(e) The function $g(x) = 2^x$ has a horizontal asymptote on the left. ✓

Hint: *A horizontal asymptote is a horizontal line (or y-value) that the function output approaches when the x-values get large in either the positive or negative direction.*

One mathematical constant of significance is the number $e \approx 2.718281828$. The function $g(x) = e^x$ is just one exponential function among many, but it shows up in so many contexts that we call it the *natural exponential function*. Add the graph of $y = e^x$ to your Desmos window to compare it to $g(x) = a^x$.

Question 2 *Notice that the graph of each exponential function $y = a^x$ is related to the graph of $y = e^x$ by some stretch factor. Is it a horizontal stretch, a vertical stretch, or both?*

Solution

(a) horizontal ✓

(b) vertical

(c) both

Hint: *Do all nonzero x-values on the graph increase as a increases? What about the nonzero y-values?*

Question 3 *Which function below grows the fastest as x-values get large.*

Learning outcomes:

Solution

(a) 2^x

(b) e^x

(c) 3^x

(d) 5^x ✓

Hint: *Does a bigger value of a in $g(x) = a^x$ result in a faster increase in y-value?*

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38 Logarithms

We learn about logarithms.

Now that we know about the function $y = a^x$ we will learn about another equivalent expression: $x = \log_a(y)$. In general the function $h(x) = \log_a(x)$ is the function whose output is the exponent, y, so that $a^y = x$.

For example $\log_2(8) = 3$ because $2^3 = 8$.

Similarly $\log_3(9) = 2$ because $3^2 = 9$. Each exponential equation corresponds to a logarithmic equation.

To compute the value of $\log_5(25)$ we say "5 to what power is 25?" The answer is 2, so $\log_5(25) = 2$.

Question 1 *Evaluate each expression.*

Solution

Hint: *2 to what power is 16?*

$\log_2(16) = 4$.

Hint: *6 to what power is 1?*

$\log_6(1) = 0$.

Hint: *10 to what power is 100?*

$\log_{10}(100) = 2$.

Hint: *3 to what power is* $\frac{1}{3}$*?*

$\log_3\left(\frac{1}{3}\right) =$ -1.

Open up `https://www.desmos.com/calculator/6fpizionn0` in a separate window. You see the graph of $y = 2^x$ and $y = \log_2(x)$. The graphs of these functions are mirror images of each other across the line $y = x$. Notice that to obtain one function from the other, all you need to do is swap the x and y-values.

Question 2 *Identify the functions below that have a vertical asymptote.*

Solution

Hint: *Plot the functions. You may need to use the math input panel for subscripts (or just use the underscore character in Desmos).*

(a) $y = \log_4(x)$ ✓
(b) $y = 3^x$
(c) $y = \sqrt{x}$

Learning outcomes:

(d) $y = \log_7(x)$ ✓

Question 3 *Which function below grows the fastest for large values of x?*

Solution

Hint: *Plot the functions. You may need to use the math input panel for cube roots.*

(a) $y = \log_4(x)$
(b) $y = \sqrt[3]{x}$
(c) $y = \sqrt{x}$ ✓
(d) $y = \log_7(x)$

Question 4 *The natural log function is denoted $y = \ln(x)$. Plot the function $y = \ln(x)$ along with the the function $y = \log_e(x)$ in Desmos. Which function grows the fastest for large x values?*

Solution

Hint: *Plot the functions. You may need to use the math input panel for cube roots.*

(a) $y = \log_e(x)$
(b) $y = \ln(x)$
(c) Neither, they are the same function. ✓

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39 Transformations

We learn how to find formulas that stretch and shift functions.

Basic learning objectives

These are the tasks you should be able to perform with reasonable fluency **when you arrive at our next class meeting**. Important new vocabulary words are indicated *in italics*.

- Know how to shift a function to the right by k units or up by h units.
- Know how to scale a function vertically by a factor of a or horizontally by a factor of b.
- Be able to recognize all the parent functions from the previous two sections.

Advanced learning objectives

In addition to mastering the basic objectives, here are the tasks you should be able to perform **after class, with practice**:

- Find the exact equation of a curve based on one of the parent functions.
- Understand how order matters when shifting and scaling functions.
- Be able to plot curves with a restricted domain and be able to draw a picture using [1].

Now that we have mastered the basic parent functions, we will use these parent functions as building blocks for other functions. We can transform the graph of a function by shifting the graph horizontally or vertically or by scaling (stretching) the graph horizontally or vertically. The rules are as follows.

- To shift $y = f(x)$ to the right by k units we replace x with $x - k$.
- To shift $y = f(x)$ to up by h units we replace y with $y - h$.
- To scale $y = f(x)$ horizontally by a factor of b we replace x with $\dfrac{1}{b}x$.
- To scale $y = f(x)$ vertically by a factor of a we replace y with $\dfrac{1}{a}y$.

For example, suppose we want to scale the parabola $y = x^2$ vertically by a factor of 2, and then shift the parabola 2 units to the right and 3 units down.

Learning outcomes:

[1] Link: `desmos.com`

First replace y with $\frac{1}{2}y$ to get $\frac{1}{2}y = x^2$. Then replace x with $x-2$ and y with $y+3$ to get

$$\frac{1}{2}(y+3) = (x-2)^2.$$

This answer is correct, though we typically solve for y by multiplying both sides by 2 and subtracting 3 from both sides to get

$$y = 2(x-2)^2 - 3.$$

Using function notation we can rewrite the 4 rules for transforming functions as follows.

- To shift $y = f(x)$ to the right by k units use $y = f(x-k)$
- To shift $y = f(x)$ to up by h units use $y = f(x) + h$
- To scale $y = f(x)$ horizontally by a factor of b use $y = f(\frac{1}{b}x)$.
- To scale $y = f(x)$ vertically by a factor of a use $y = af(x)$.

Question 1 *In* `http://desmos.com` *plot the function $f(x) = x^2$. Then enter functions $y = af(x)$ and $y = f(bx)$ and create sliders for a and b. Compare the graphs when $b = 2$ and $a = 4$. Which statement accurately describes the relationship between the resulting graphs?*

Solution

(a) Scaling $y = x^2$ horizontally by a factor of $\frac{1}{2}$ is the same as scaling it vertically by a factor of 4. ✓

(b) Scaling $y = x^2$ horizontally by a factor of 2 is the same as scaling it vertically by a factor of 4.

(c) Scaling $y = x^2$ horizontally by a factor of 2 is the same as scaling it vertically by a factor of $\frac{1}{4}$.

(d) Scaling $y = x^2$ horizontally by a factor of $\frac{1}{2}$ is the same as scaling it vertically by a factor of $\frac{1}{4}$.

Hint: *Note that* $4x^2 = (2x)^2 = \left(\frac{1}{1/2}x\right)^2$.

Which case below results in the narrowest graph?

(a) $a = 2$

(b) $a = 4$ ✓

(c) $b = 2$

(d) $b = 4$

If the graph of the parabola goes through the point $(3, 1)$, then $a = 1/9$ or $b = 3$.

Hint: *Use the sliders to find the values. Remember that you can narrow down the possible values for the parameters a and b by clicking below the sliders.*

Question 2 *In* `http://desmos.com` *plot the function* $f(x) = 2x$. *Then enter functions* $y = f(x-h)$ *and* $y = f(x)+k$ *and create sliders for* h *and* k. *Compare the graphs when* $h = -1$ *and* $k = 2$. *Which statement accurately describes the relationship between the resulting graphs?*

Solution

(a) Shifting $y = 2x$ horizontally by -1 units is the same as shifting the graph vertically by 2 units. ✓

(b) Shifting $y = 2x$ horizontally by 1 units is the same as shifting the graph vertically by 2 units.

(c) Shifting $y = 2x$ horizontally by 1 units is the same as shifting the graph vertically by -2 units.

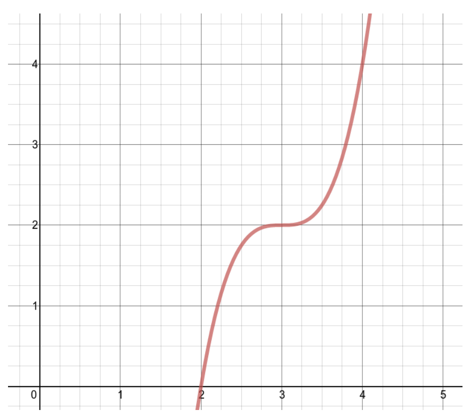
(d) Shifting $y = 2x$ horizontally by -1 units is the same as shifting the graph vertically by -2 units.

Hint: *Note that* $2(x-(-1)) = 2x+2$.

If the graph of the line goes through the point $(2,0)$, then $h = 2$ or $k =$ -4.

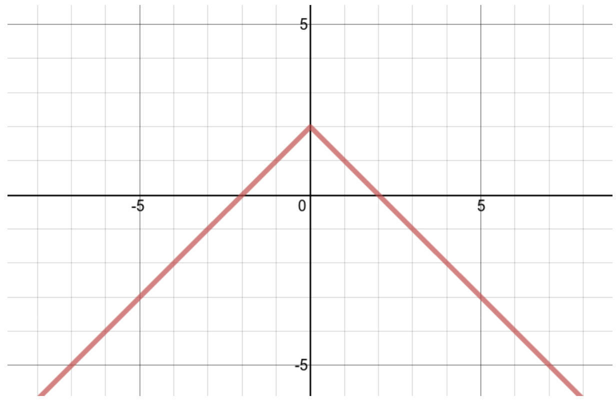
Hint: *Use the sliders to find the values.*

Question 3 *Below is a transformation of the parent function* $y = x^3$.



The formula for this function can be written in the form $f(x) = a(x-h)^3 + k$ *where* $a =$ *2,* $h =$ *3, and* $k =$ *2.*

Question 4 *Below is a transformation of the parent function $y = |x|$.*



The formula for this function can be written in the form $f(x) = a|x - h| + k$ where $a =$ -1, $h =$ 0, and $k =$ 2.

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40 Modeling and Predictions

We model data as functions and use our models to make predictions.

Basic learning objectives

These are the tasks you should be able to perform with reasonable fluency **when you arrive at our next class meeting**. Important new vocabulary words are indicated *in italics*.

- Plot points and recognize which type of curve fits the data.
- Plot a curve to fit data.

Advanced learning objectives

In addition to mastering the basic objectives, here are the tasks you should be able to perform **after class, with practice**:

- Use a curve fitted to a table of values to extrapolate values of a function.
- Be able to recognize whether a model is a good fit.
- Use [1] and [2] to find curves that fit data.

In this section we create a model by fitting a curve to data. Then we will use our model to make predictions.

Suppose we are given the function values

x	0	1	2	3	4	5	6
$f(x)$	0	1	0	1	0	1	0

Open up [3] and create a table containing the data points. A curve that that includes those points must go up and down fairly regularly, so we might try to model the curve with a wave function.

Our first guess is $f(x) = \cos(x)$ which we plot in Desmos. It doesn't work quite right, so we change the function to be $f(x) = a\cos(bx + c) + d$ and create sliders for a, b, c, and d. (Follow along by creating this function in Desmos yourself. Otherwise it won't mean anything to you.)

Our function is twice as tall as it needs to be so we set $a = .5$. The graph needs to be shifted vertically, so that its center line is at .5 so we set $d = .5$. Our function is too wide so we set $b = 2$ or $b = 3$ to make it narrower. In fact if we play with it for a minute we might decide $b = \pi$ is a good value. Then we shift things horizontally, and find that $c = \pi$ is a good choice.

Learning outcomes:
[1]Link: `http:\\wolframalpha.com`
[2]Link: `http:\\desmos.com`
[3]Link: `desmos.com`

Thus our model is $f(x) = .5\cos(\pi x + \pi) + .5$. Perhaps another student would have come up with $f(x) = -.5\cos(\pi x) + .5$, which happens to be the same curve.

Is our model a good one? Probably. However, there are many other curves that fit the data.

Question 1 *Which functions below do not fit the data for the function $f(x)$ that we have been discussing? (Please don't just guess. You won't learn anything that way.)*

Solution

(a) $y = \left(\sin(\frac{3\pi}{2}(x))\right)^2$

(b) $y = \big|\,|\mathrm{mod}\,(x-1,2)| - 1\big|$

(c) $y = \left|\sin\left(\frac{\pi x}{2}\right)\right|$

(d) $y = \sin(2\pi x) - .5\cos(\pi x) + .5$

(e) $y = \dfrac{\sin(\pi x)}{\cos(\pi x) + 2}$ ✓

(f) $y = \sin(\pi(x - 2.5y))$

Hint: *Enter each equation into Desmos. Be careful.*

Hint: *You can use* `abs(x)` *in place of* `|x|` *if you are struggling with the absolute value expressions.*

Question 2 *Plot the data in the table*

x	0	1	2	3	4	5	6
$f(x)$	3	7	3	-9	-29	-571	-93

What is the parent function for the function $f(x)$?

Solution

(a) $y = x^2$ ✓

(b) $y = x$

(c) $y = x^3$

(d) $y = \sqrt{x}$

(e) $y = \sin(x)$

(f) $y = 2^x$

(g) $y = \log(x)$

The formula of the function $f(x)$ is $f(x) = -4x^2 + 8x + 3$.

Hint: *The coefficients are all integers.*

Question 3 *Go to* [4] *and enter the code* `vrt3`. *Use your real name, and complete the activity (about 15 minutes.) We will discuss it in class.*

Solution The number of pennies in the big circle was 663.

[4]Link: `https://student.desmos.com`

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41 Optimization

We minimize or maximize various quantities.

Basic learning objectives

These are the tasks you should be able to perform with reasonable fluency **when you arrive at our next class meeting**. Important new vocabulary words are indicated *in italics*.

- Use Wolfram Alpha or Desmos to solve equations.
- Memorize the basic formulas for area and volume.

Advanced learning objectives

In addition to mastering the basic objectives, here are the tasks you should be able to perform **after class, with practice**:

- Find the formula for an function that we want to minimize or maximize.
- Be able to find a maximum or minimum value of a curve.
- State the conclusion to the optimization problem as a complete, descriptive sentence.

We have used many web tools to help us in our computations for this course. In this unit we are using [1] and [2] to serve as graphing calculators for us.

In this section we are trying to optimize various functions. When we say the word "optimize" we mean that we are trying to find the minimum or maximum output values of the function. For example, we might want to minimize costs or maximize profit.

First we want to show/remind you how useful [3] is at solving equations.

Question 1 *Find the number of solutions to the system of equations:*

$$\begin{cases} y &= x^3 - 4x \\ y &= x \end{cases}$$

Solution The system has 3 solutions

Hint: *Either plot it in Desmos and count the number of intersection points or just type in both equations to wolfram alpha.*

Learning outcomes:

[1] Link: `http://wolframalpha.com`

[2] Link: `http://desmos.com`

[3] Link: `http://wolframalpha.com`

Question 2 *Find the of solutions to the system of equations:*

$$\begin{cases} y &= x^3 - 9x \\ y &= -2x + 6 \end{cases}$$

Solution There are 3 solutions. When $x = -2$, $y = 10$ When $x = -1$, $y = 8$ When $x = 3$, $y = 0$

Hint: *Either plot it in Desmos and count the number of intersection points or just type in both equations to wolfram alpha.*

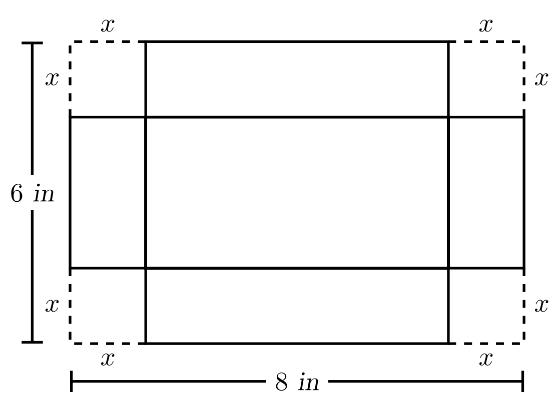
Both Wolfram Alpha and Desmos have the ability to help us find the minimum or maximum value(s) of curves.

Question 3 *Find the y-coordinate of the highest point on the curve $y = 4x^3 - x^4$.*

In [4] you can get the answer by entering `maximize 4x^3-x^4`. *In [5] you can define the curve $y = 4x^3 - x^4$. Then (with that cell still selected) you can click on the point at the top of the curve to get its coordinates.*

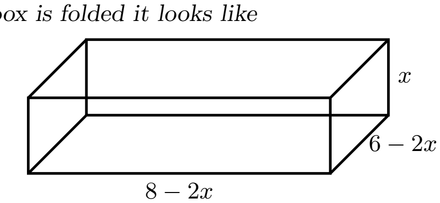
Solution The maximum y value is $y = 27$.

Question 4 *In this problem we consider the problem of creating a box by taking a* 6 in × 8 in *sheet and cutting out square corners and folding up the sides as pictured below.*



Solution We will ignore the units for much of the calculation, but we need to put them back in at the end. The length of the sheet is 8 so the length of the box (after we have removed the corners) is $8 - 2x$.

Hint: *When the box is folded it looks like*



[4]Link: `\http://wolframalpha.com`
[5]Link: `http:desmos.com`

The width of the sheet is 6 so the width of the box is $6 - 2x$. The height of the box is x.

The volume of the box (in in^3) is given by the formula $V(x) = x(8-2x)(6-2x)$.

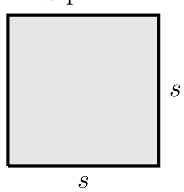
Now go to [6] and enter `maximize x(8-2x)(6-2x)` to find the value of x that maximizes the volume.

Alternatively, use the function $V(x) = x(8-2x)(6-2x)$ in [7] and click the dot at the top of the hump of the curve to see the coordinates of the point at that maximum value.

The side length of each corner square is $x = 1.13$ inches.

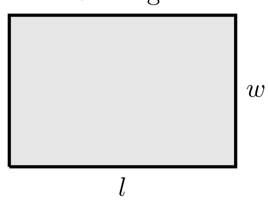
Memorize the formulas below. We will assume you know them in class and on the test.

Square



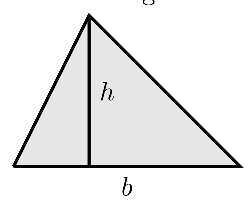
$A = s^2$

Rectangle



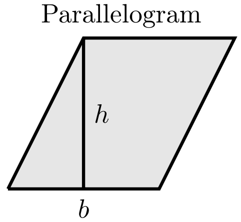
$A = l \cdot w$

Triangle



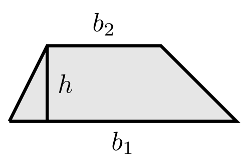
$A = \frac{1}{2} b \cdot h$

Parallelogram



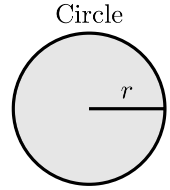
$A = b \cdot h$

Trapezoid



$A = \frac{b_1 + b_2}{2} \cdot h$

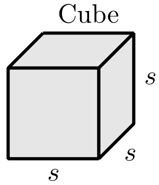
Circle



$A = \frac{b_1 + b_2}{2} \cdot h$

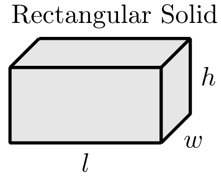
Circumference: $C = 2\pi r$

Cube



$V = s^3$

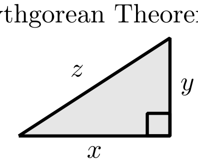
Rectangular Solid



$V = l \cdot w \cdot h$

Surface Area $= 2lw + 2lh + 2wh$

Pythgorean Theorem



$x^2 + y^2 = z^2$

[6]Link: `http://wolframalpha.com`

[7]Link: `http://desmos.com`