Institute for Mathematical Stochastics
Ramp up Course Mathematics
Summer Semester 2024
Prof. Dr. Benedikt Jahnel • Yannic Steenbeck

Technische Universität Braunschweig

Ramp up Course Mathematics - Stochastics - Exercise Sheet 1

Exercise 1 (Distribution of a Random Variable)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $X\colon \Omega \to \mathbb{R}$ be a random variable on this probability space.

Consider the distribution

$$\begin{aligned} &P_X\colon \mathfrak{B}(\mathbb{R}) \to [0,1], \\ &A \mapsto P_X(A) := \mathbb{P}(X \in A) := \mathbb{P}(\{\omega \in \Omega \,|\, X(\omega) \in A\}) = \mathbb{P}(X^{-1}(A)). \end{aligned}$$

of X.

Show that P_X is a probability measure (on $(\mathbb{R}, \mathbb{R})$) according to the definition of probability measure.

Exercise 2 (Variational Properties of Expectation and Variance)

Let X be a random variable (defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$) with existing expectation and variance.

Show that the following properties hold.

a) $\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$,
b) $\mathbb{E}[X] = \mathrm{argmin}_{a\in\mathbb{R}} \mathbb{E}[|X-a|^2]$,
c) $\mathrm{Var}(X) = \min_{a\in\mathbb{R}} \mathbb{E}[|X-a|^2]$.

Exercise 3 (Counterexamples: Uncorrelatedness, Pairwise Independence, Independence)

a) Let X be uniformly distributed on the interval $[-1,1]$. Check that X and $Y := X^2$ are uncorrelated, but not independent.
b) Let X, Y be two independent random variables with

$$\frac{1}{2} = \mathbb{P}(X = -1) = \mathbb{P}(X = +1) = \mathbb{P}(Y = -1) = \mathbb{P}(Y = +1).$$

Define another random variable by $Z := X \cdot Y$. Check that X, Y, Z are pairwise indenpendent (i.e. X, Y are independent, Y, Z are independent and X, Z are independent) but not independent.

Exercise 4 (Conditional probability, Bayes' formula)

Consider a collection of tetrahedra with faces which, in principle, can either be red or green. We randomly draw one of the tetrahedra. The proportion P of green faces of the tetrahedron is actually unknown to us, but

$$\mathbb{P}\left(P = \frac{k}{4}\right) = \frac{1}{5}$$

for $k \in \{0, 1, 2, 3, 4\}$.

Suppose someone throws the (to us still unknown) tetrahedron five times for us and observes the sequence "green - red - red - green - red" as the colors of the faces the tetrahedron had landed on.

Which tetrahedron (i.e. with which proportion of green faces P) was most probably thrown?

Exercise 5 (Higher Moments of the Normal Distribution)

Let X be a normal distributed random variables with expectation $\mu = 0$ and variance $\sigma^2 = 1$. Calculate the moments $\mathbb{E}[X^k]$ for $k \in \mathbb{N}$.

Hint: Use integration by parts.