release date: 24.06.2024

"Ramp Up Course Mathematics — Numerical Analysis Part"
2. Exercise Assignment

Task 1 [Newton Raphson Method]
Consider the function

$$F(x,y,z) = \begin{pmatrix} \frac{x^2}{2} - 2 \\ x + y^2 + 1 \\ 2x + y + z^2 + 1 \end{pmatrix}.$$

Given $(x^{(0)}, y^{(0)}, z^{(0)})^\top = (1,1,1)^\top$ calculate the next iterate $(x^{(1)}, y^{(1)}, z^{(1)})^\top$ of the Newton Raphson method for computing a zero of F.
If you like, you may compute further iterates using, say, e.g., Octave or MATLAB or your most favoured programming environment.

weight 3

Task 2 [Linear Least Squares]
Consider the following data points:

$$(x_1,y_1) = (-2,0),\ (x_2,y_2) = (-1,0),\ (x_3,y_3) = (0,1),\ (x_4,y_4) = (1,0),\ (x_5,y_5) = (2,0).$$

We want to model these data points by a surrogate function $f(t) = at^2 + bt + c$ such that $\left(\sum_{i=1}^5 (y_i - f(x_i))^2\right)^{1/2}$ is minimized.

1. Set up the associated linear least squares problem $\|b - Ax\|_2 = \min$ for determining the coefficients a, b, c of $f(t)$. State the matrix A and the vector b.
2. Set up the normal equations
$$A^T A x = A^T b$$
for solving the LLS problem in 1. Verify that $A^T A$ is positive definite.
3. Solve the normal equations using Cholesky decomposition and state the surrogate function $f(t)$ explicitly.
4. Solve the normal equations using PCG with diagonal preconditioning with tolerance 10^{-6}, zero initial guess and compare the solution with that in 3.
5. Solve $\|b - Ax\|_2 = \min$ using the QR decomposition.

weight 5

Task 3 [Low-rank approximation]
Given $X \in \mathbb{R}^{m,n}$ we consider two given full-rank matrices $U \in \mathbb{R}^{m,r}$, $V \in \mathbb{R}^{n,r}$ such that $r \leqslant \min\{m,n\}$. We are seeking for $Z \in \mathbb{R}^{r,r}$ such that $\|X - UZV^T\|_F^2$ is minimized. State Z!

weight 3

Task 4 [Properties of the SVD]
Let $A \in \mathbb{R}^{m,n}$ and let $A = U\Sigma V^T$ be a singular value decomposition of A and suppose that $\operatorname{rank} A = r$. Show the following properties.

1. $\|A\|_2 = \sigma_1$, $\|A\|_F = \sqrt{\sigma_1^2 + \cdots + \sigma_r^2}$
2. If $r = m = n$, then $\|A^{-1}\|_2 = \frac{1}{\sigma_n}$

weight 2

Task 5 [Householder Transformation]
Let $\vec{x} \in \mathbb{R}^n \setminus \{0\}$ be given. Show that using $\vec{v} := \vec{x} + \operatorname{sign}(x_1)\|\vec{x}\|_2 \cdot \vec{e}_1$, the Householder transformation satisfies

$$H\vec{x} = \gamma \cdot \vec{e}_1$$

where $\gamma \in \mathbb{R}$ is suitably chosen and $\vec{e}_1 \in \mathbb{R}^n$ is the first unit vector and $H = I - \frac{2}{v^T v} v v^T$ is the Householder transform.

weight 2

To be discussed on Thursday, 27.06.2024, 11:30 am.