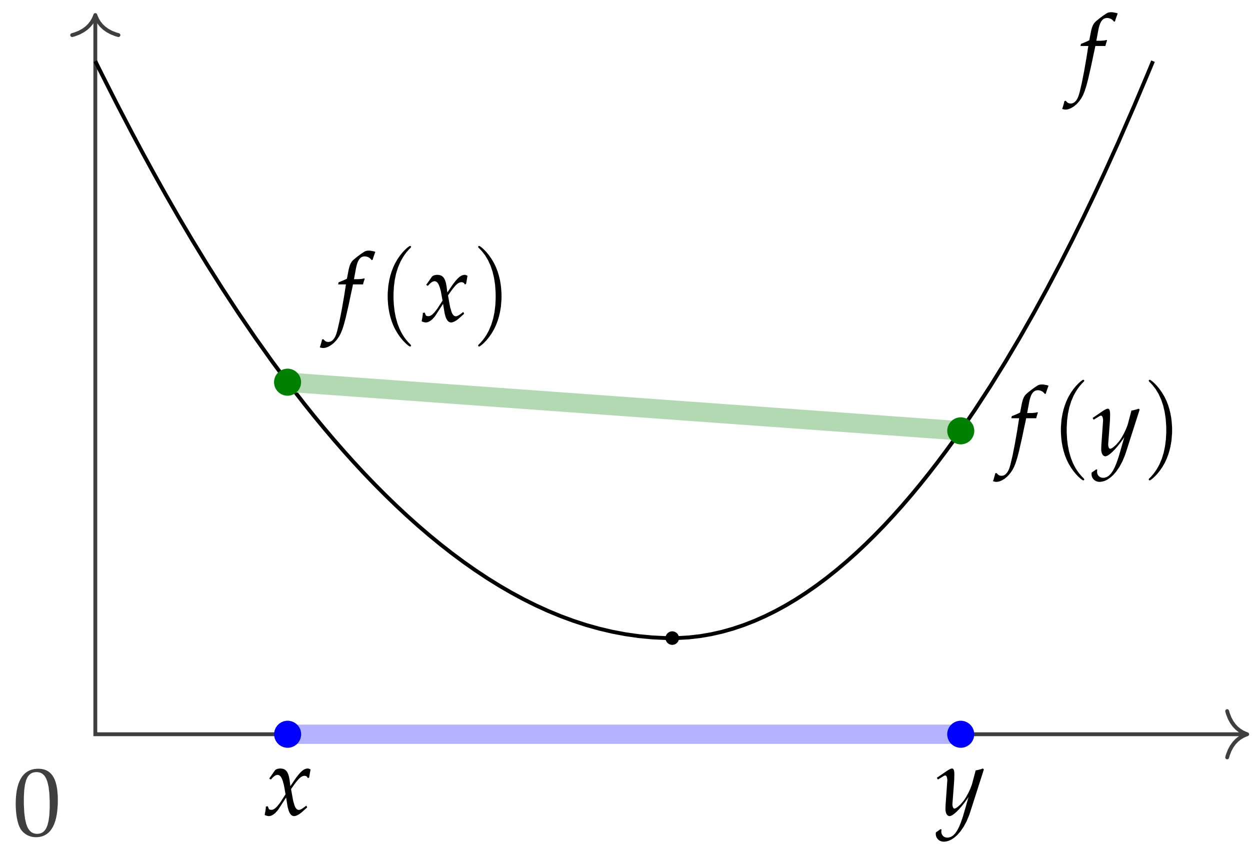
Linear Programming

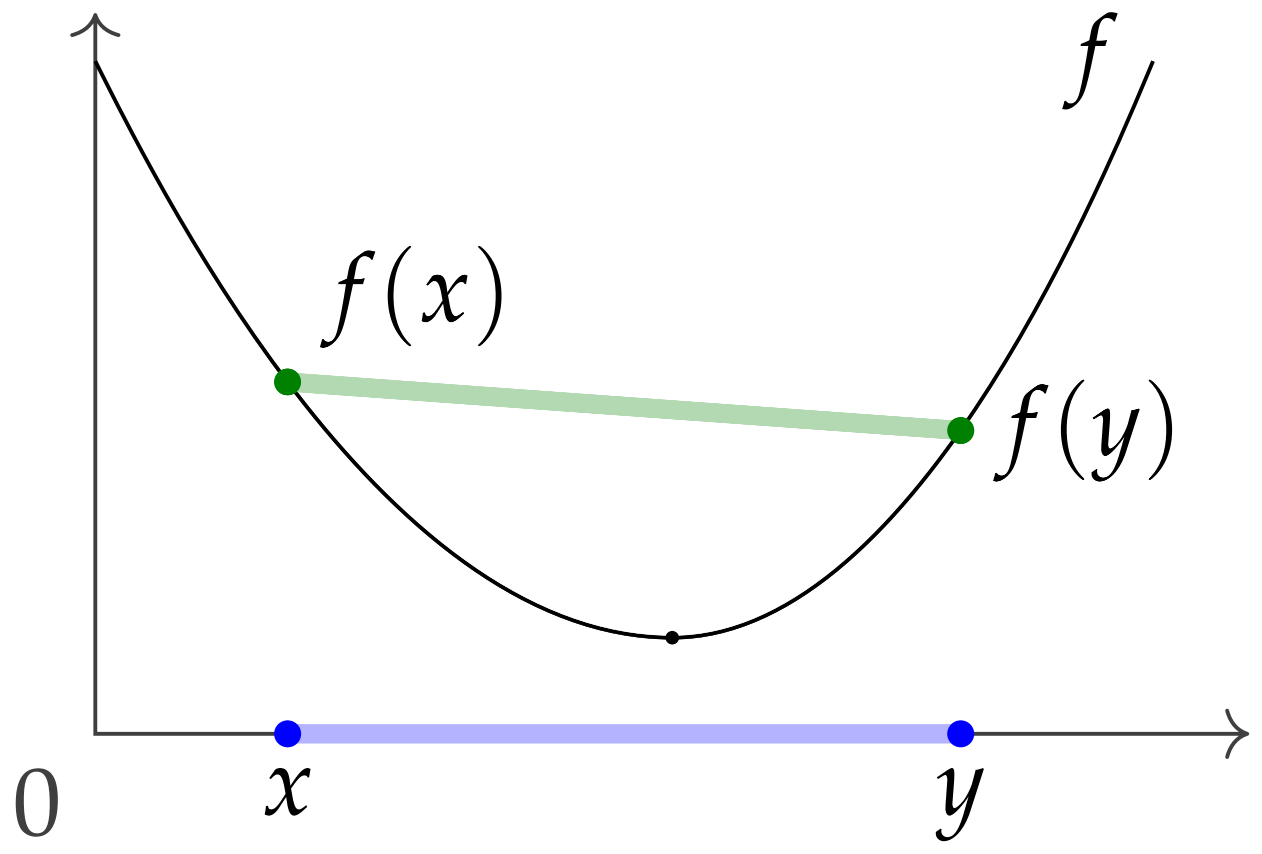
RampUp Discrete Optimization

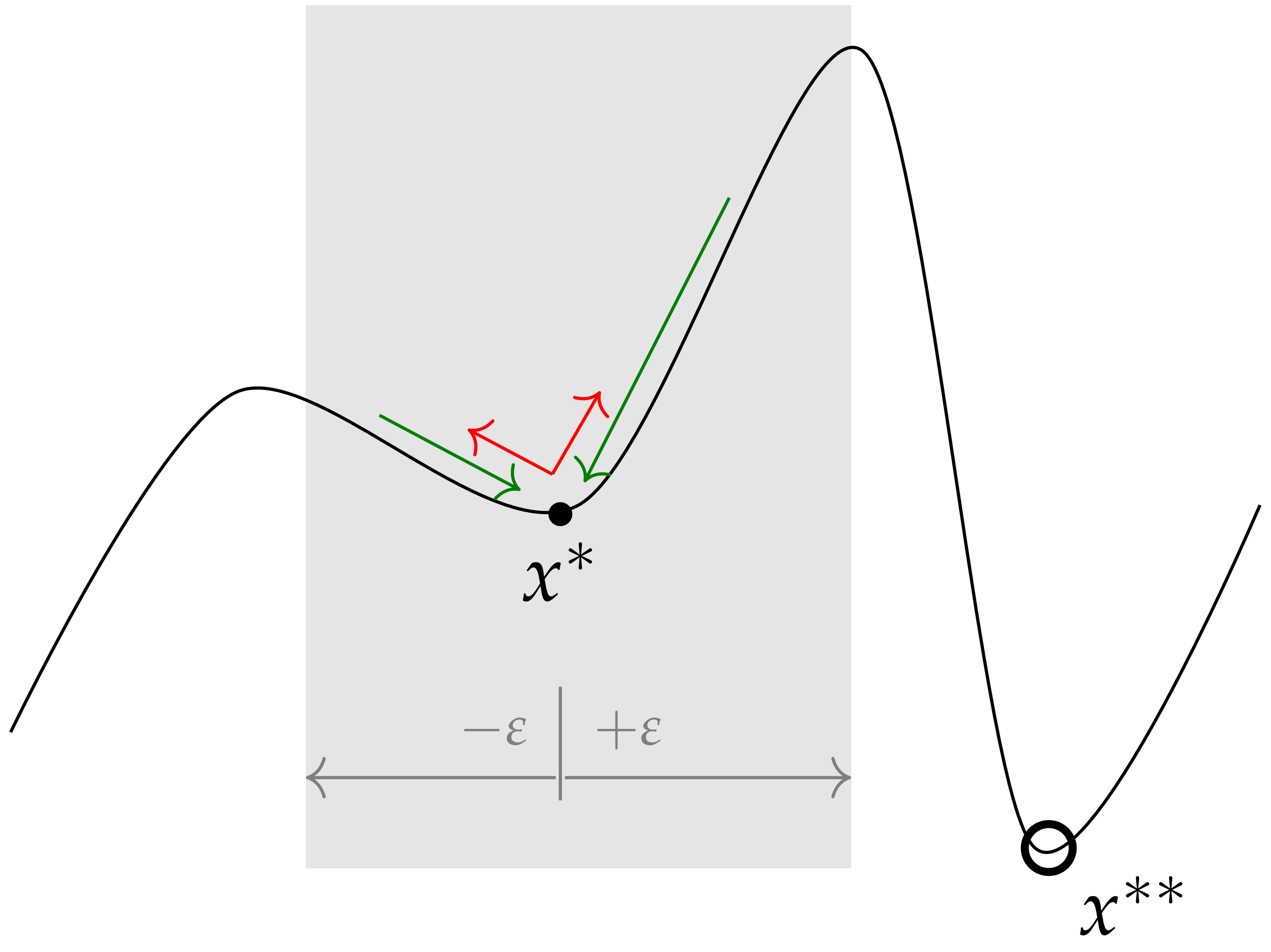
$$\underbrace{\lambda f(x) + (1-\lambda) f(y)} \geq f(\underbrace{\lambda x + (1-\lambda) y}).$$

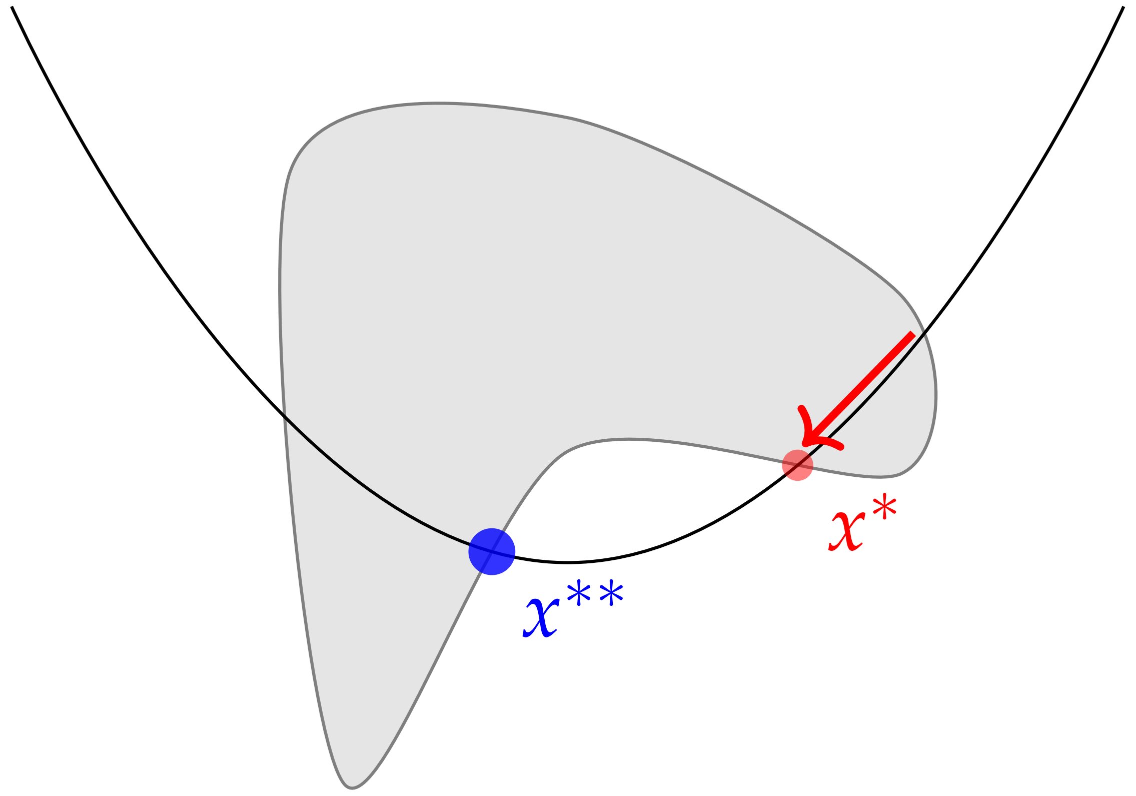


Convexity and optimization

$$\lambda f(x) + (1-\lambda) f(y) \geq f(\lambda x + (1-\lambda) y).$$

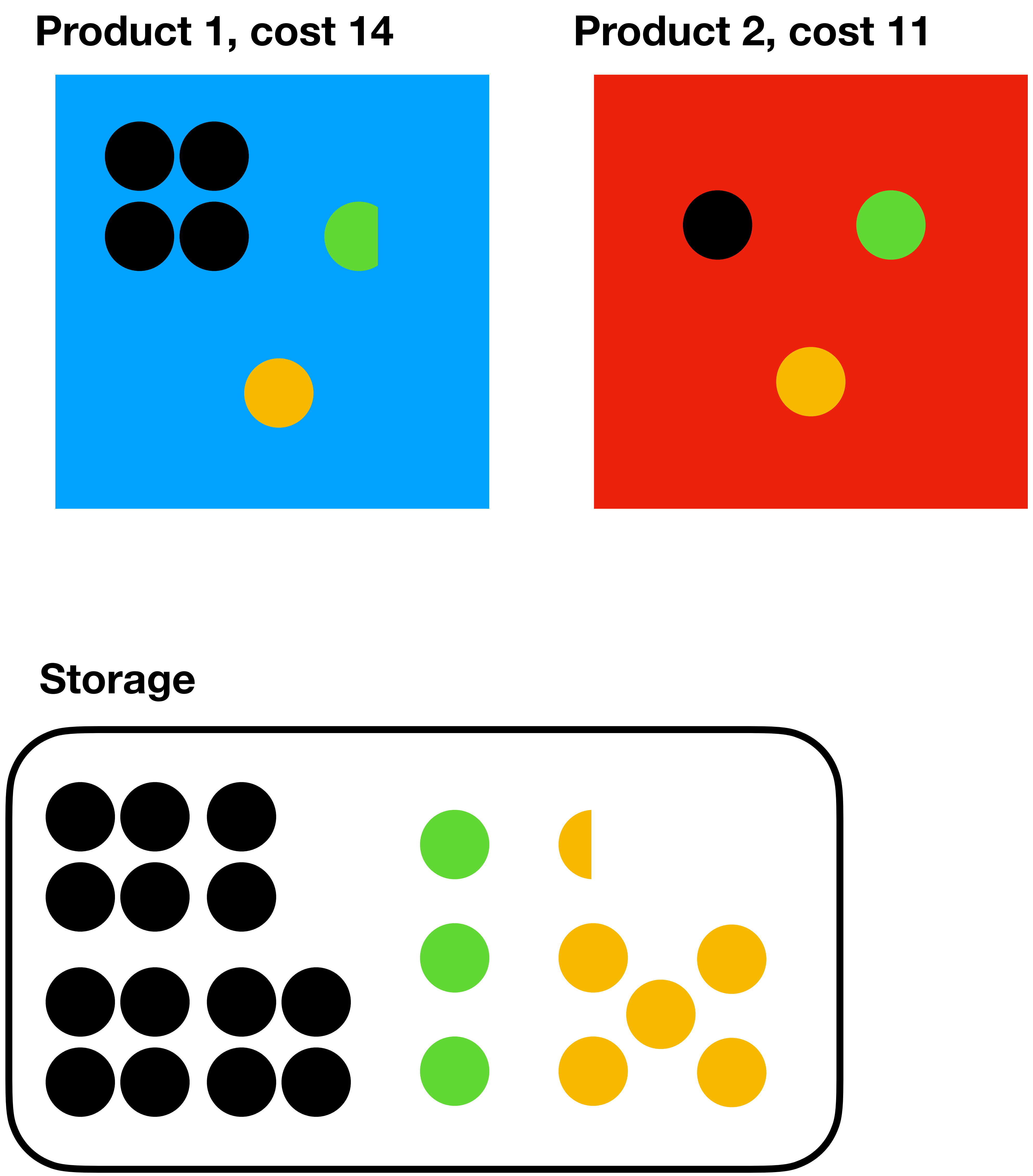






For convex objective function and convex sets of feasible solutions local optima are globally optimal!

Linear programs: Example



$$
\begin{array}{lrcrcr}
\max & 14x_1 & + & 11x_2 & & \\
\text{s. d.} & 4x_1 & + & x_2 & \leq & 14 \\
 & \frac{2}{3}x_1 & + & x_2 & \leq & 3 \\
 & x_1 & + & x_2 & \leq & 5{,}5 \\
 & x_1 & & & \geq & 0 \\
 & & & x_2 & \geq & 0 \quad .
\end{array}
$$

Linear programs: Example

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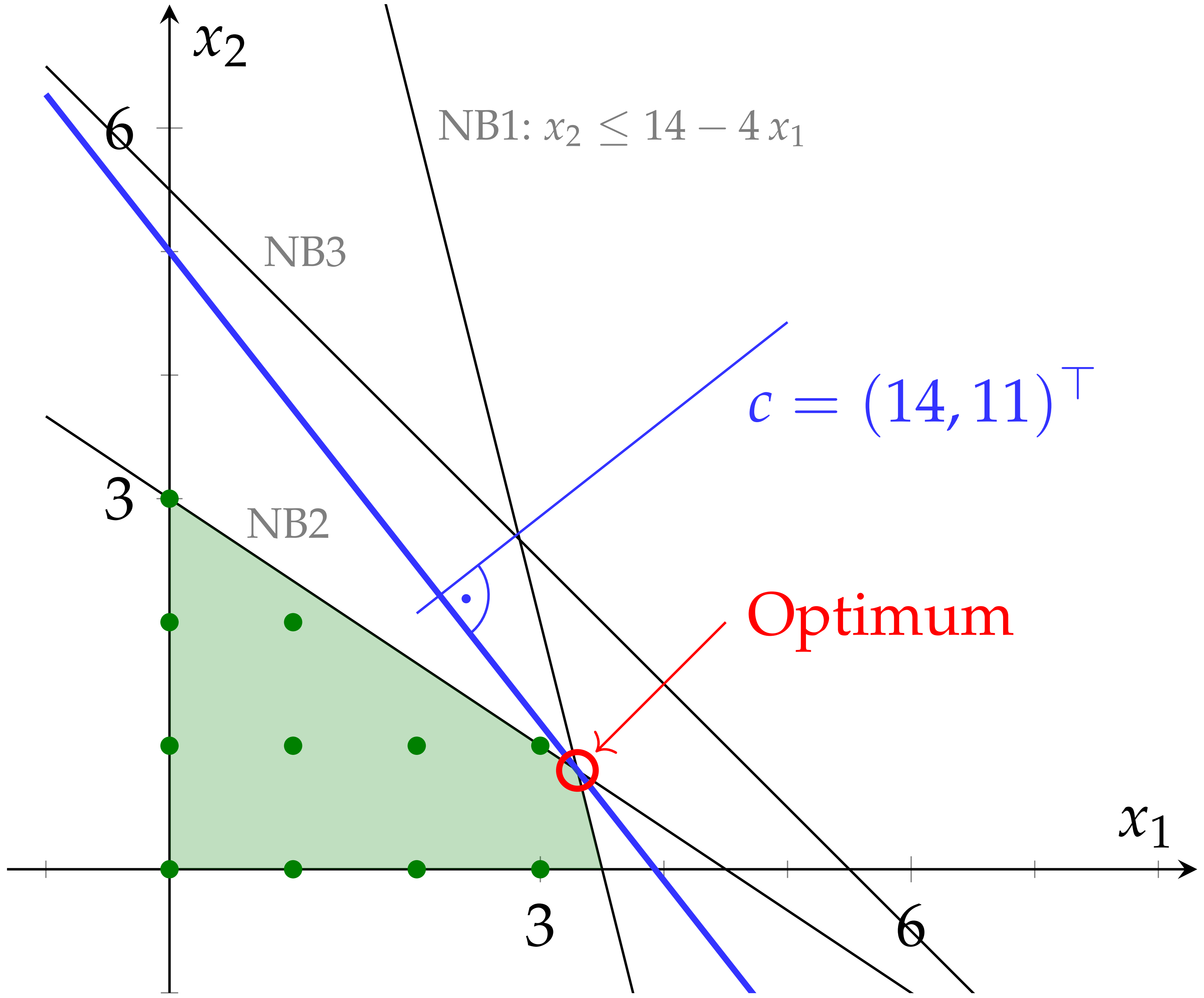
$$\begin{array}{ll} \max & c^\top x \\ \text{s. d.} & Ax \leq b \\ & x \geq 0 \end{array}$$

$$c = \begin{pmatrix} 14 \\ 11 \end{pmatrix} \qquad A = \begin{pmatrix} 4 & 1 \\ \frac{2}{3} & 1 \\ 1 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 14 \\ 1 \\ \frac{11}{2} \end{pmatrix}$$

Linear programs: Example



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\end{array}
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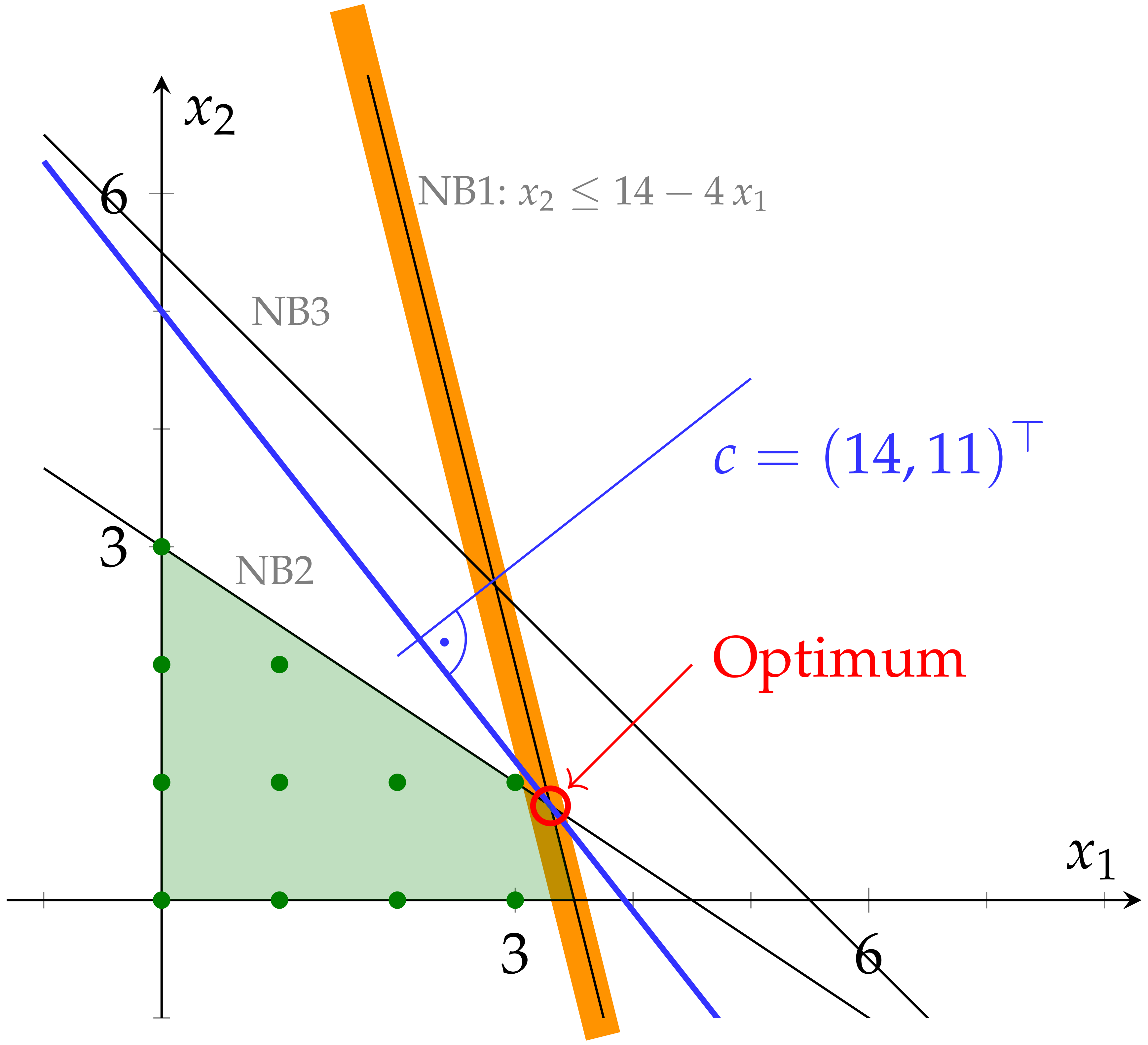
Linear programs: Example



$$\begin{array}{lrcrcr}
\max & 14x_1 & + & 11x_2 & & \\
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 & \frac{2}{3}x_1 & + & x_2 & \leq & 3 \\
 & x_1 & + & x_2 & \leq & 5,5 \\
 & x_1 & & & \geq & 0 \\
 & & & x_2 & \geq & 0 \quad .
\end{array}$$

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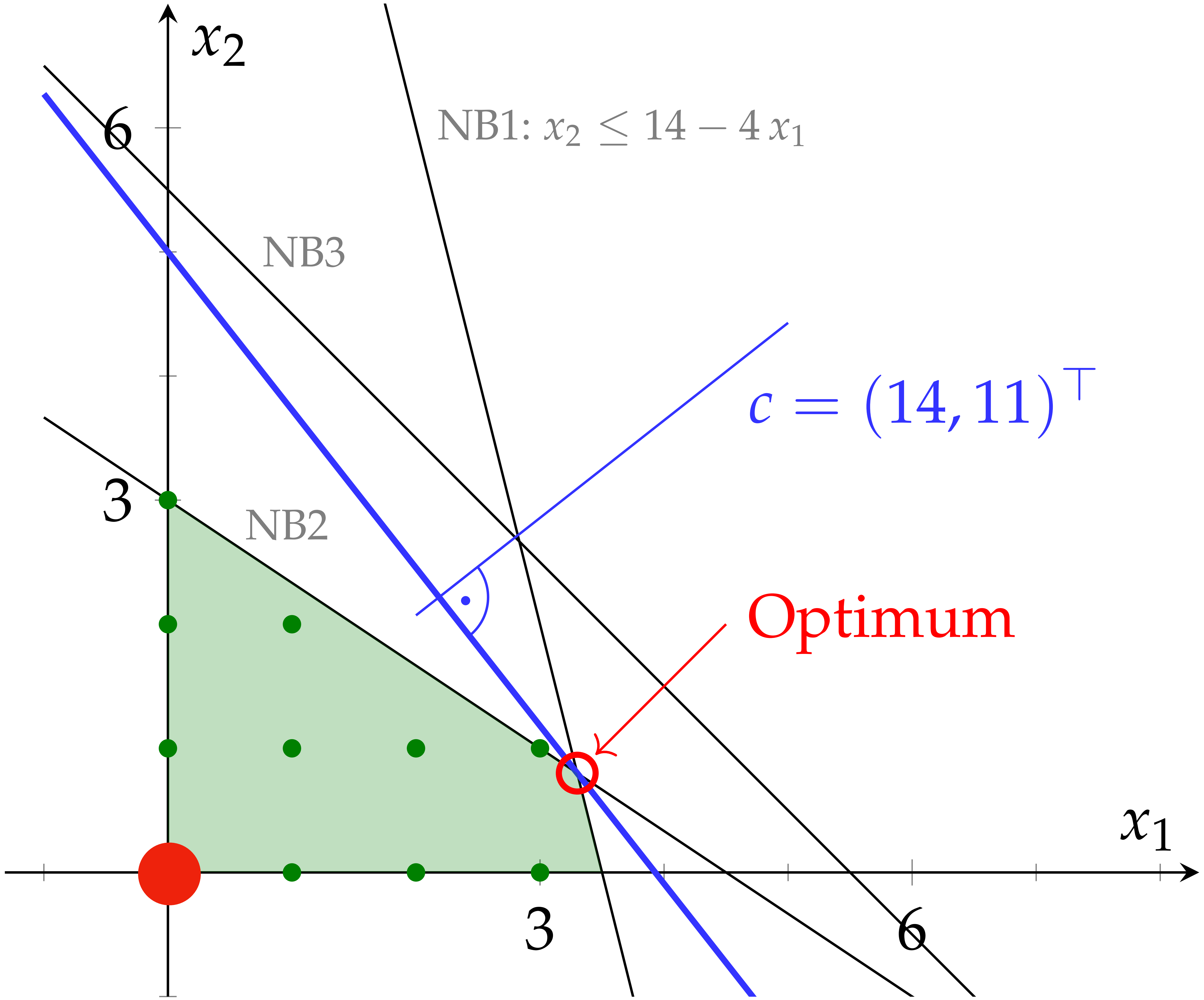
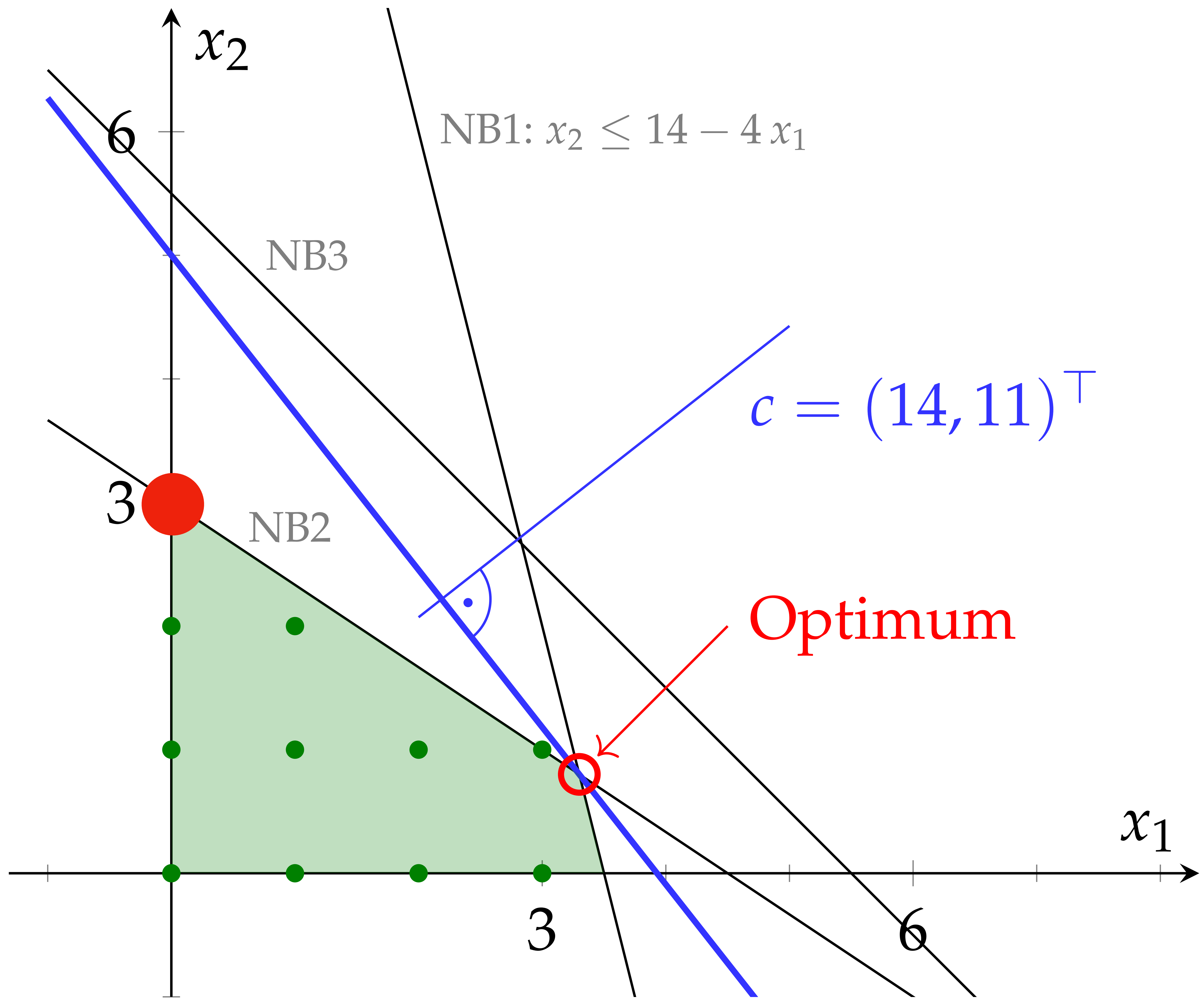
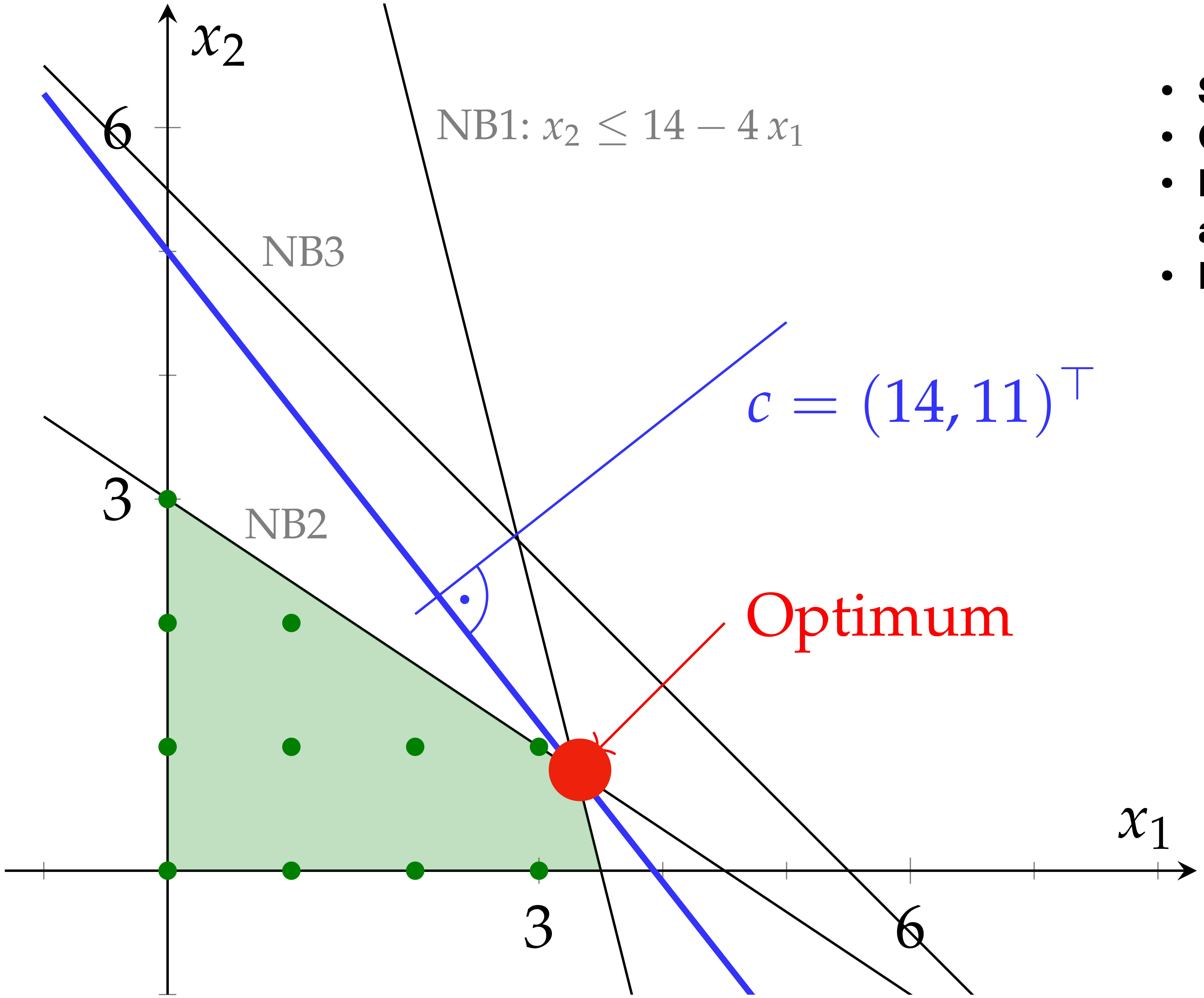
Linear programs: algorithmic idea



Linear programs: algorithmic idea



Linear programs: algorithmic idea



- Start at an edge.
- Consider all neighboring edges.
- If any of them is better, go there and iterate.
- Else, we are at an optimum.

Linear programs: forms

$$\begin{aligned} \min\ & c^\top x \\ \text{s. d.}\ & a_i^\top x \geq b_i \quad \forall i \in M_1 \\ & a_i^\top x = b_i \quad \forall i \in M_2 \\ & a_i^\top x \leq b_i \quad \forall i \in M_3 \\ & x_j \geq 0 \quad \forall j \in N_1 \\ & x_j \leq 0 \quad \forall j \in N_2 \end{aligned}$$

General form

$$\begin{aligned} \min \quad & c^\top x \\ \text{s. d.} \quad & Ax \geq b. \end{aligned}$$

Canonical form

$$\begin{aligned} \min \quad & c^\top x \\ \text{s. d.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Standard form

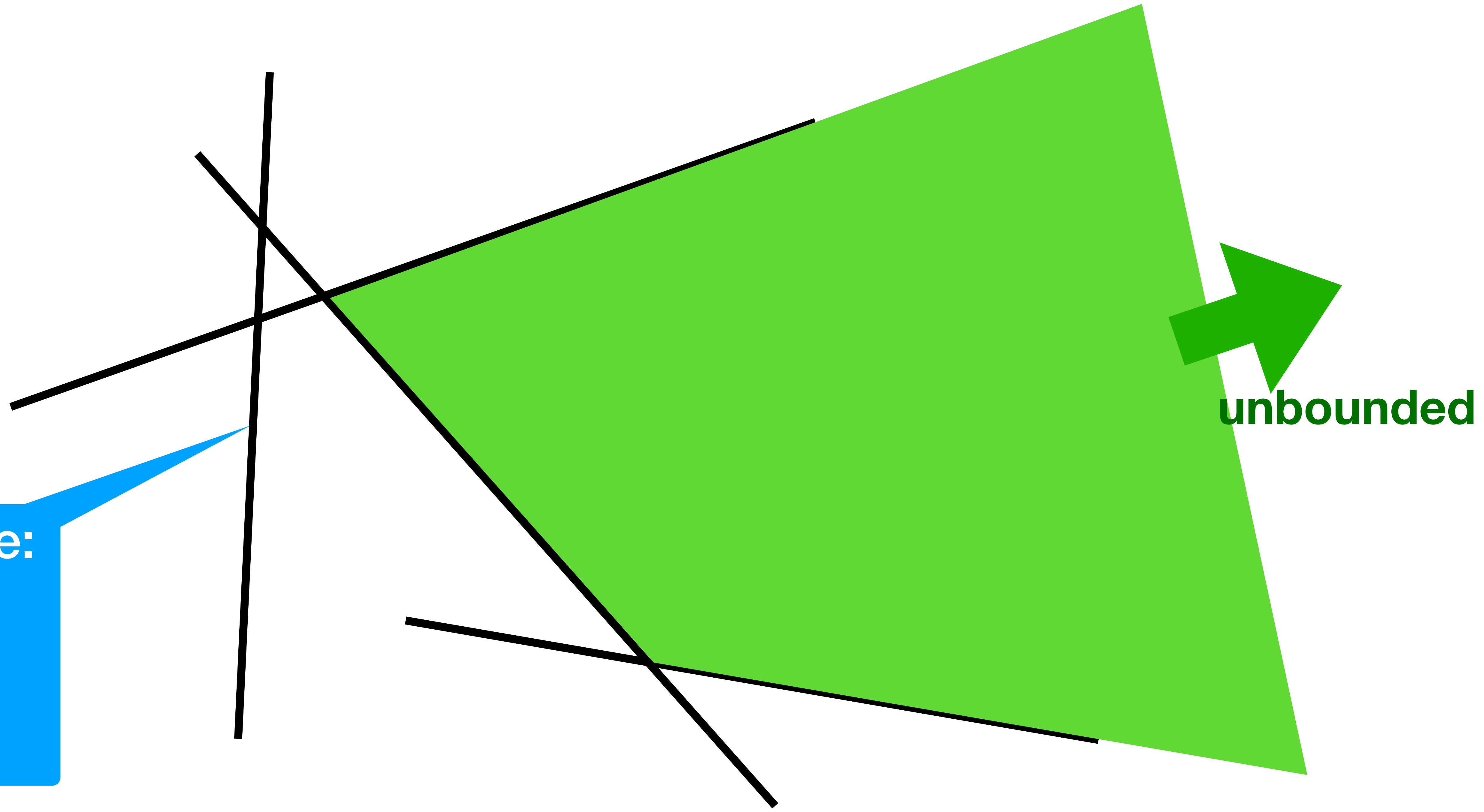
All three forms are equivalent!

Linear programs: polyhedra

$$\begin{aligned} \min \quad & c^\top x \\ \text{s. d.} \quad & Ax \geq b. \end{aligned}$$

Constraints define a polyhedron.

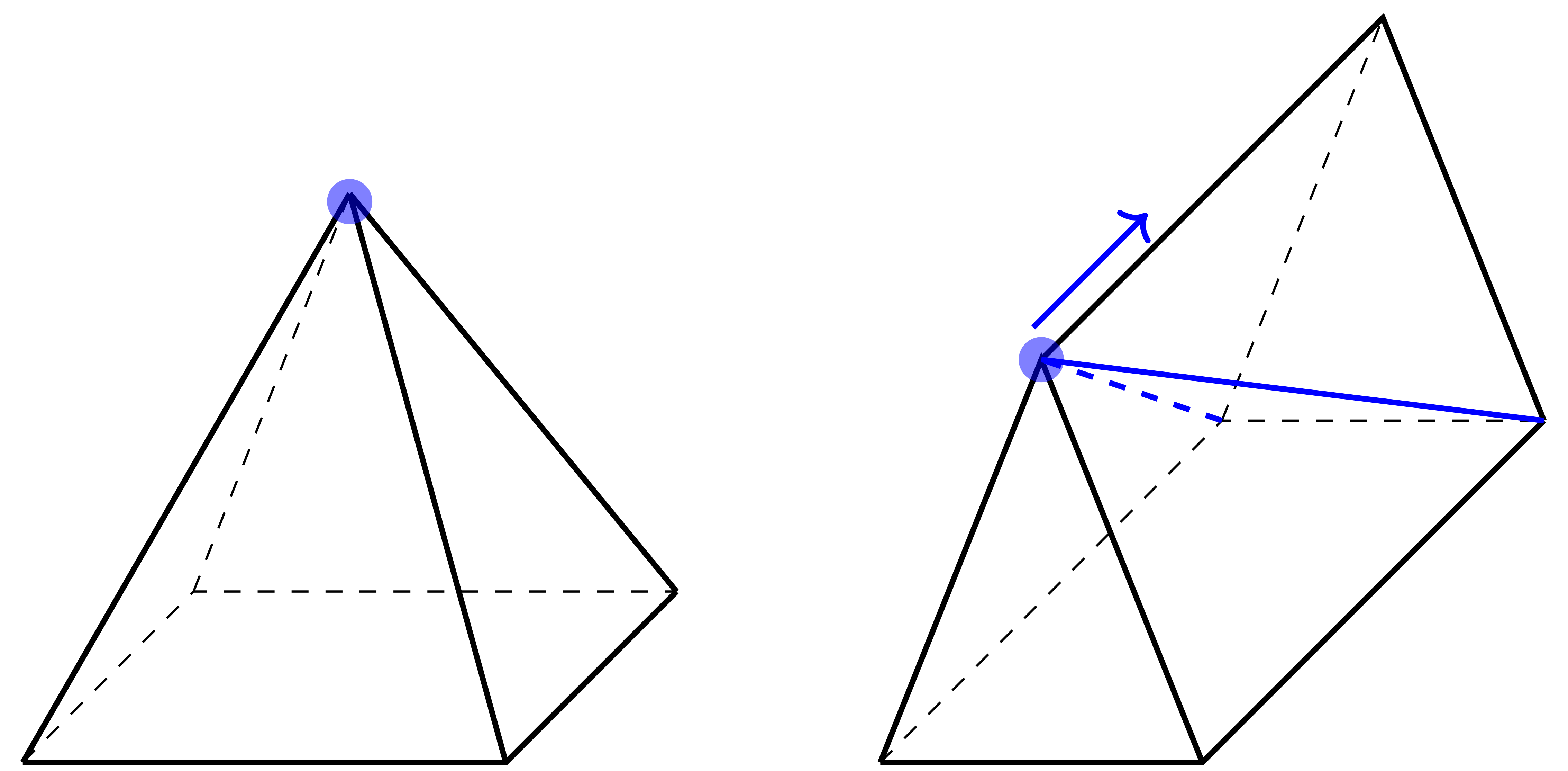
Definition:
A *polyhedron* is the intersection of finitely many halfspaces.
A *polytop* is a bounded polyhedron.



Constraint/hyperplane:

$$a_i' x = \sum_j a_{i,j} x_j = b_i$$

n linearly independent hyperplanes intersect in a point



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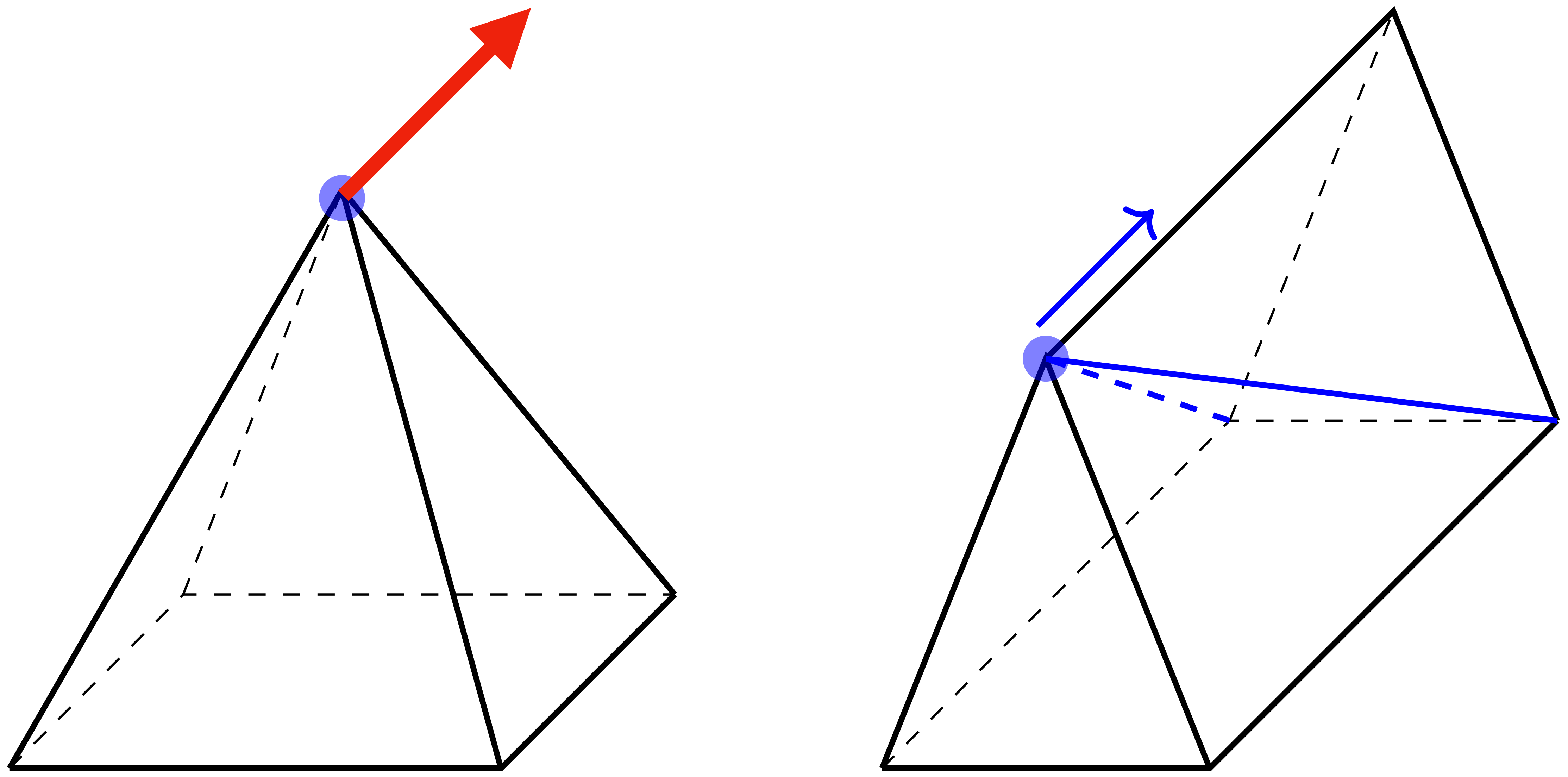

Edge / extrem point of a set:
- Cannot be convex combined by other points of the set.
- There is an (linear) objective function for which it is the unique optimum in the set.

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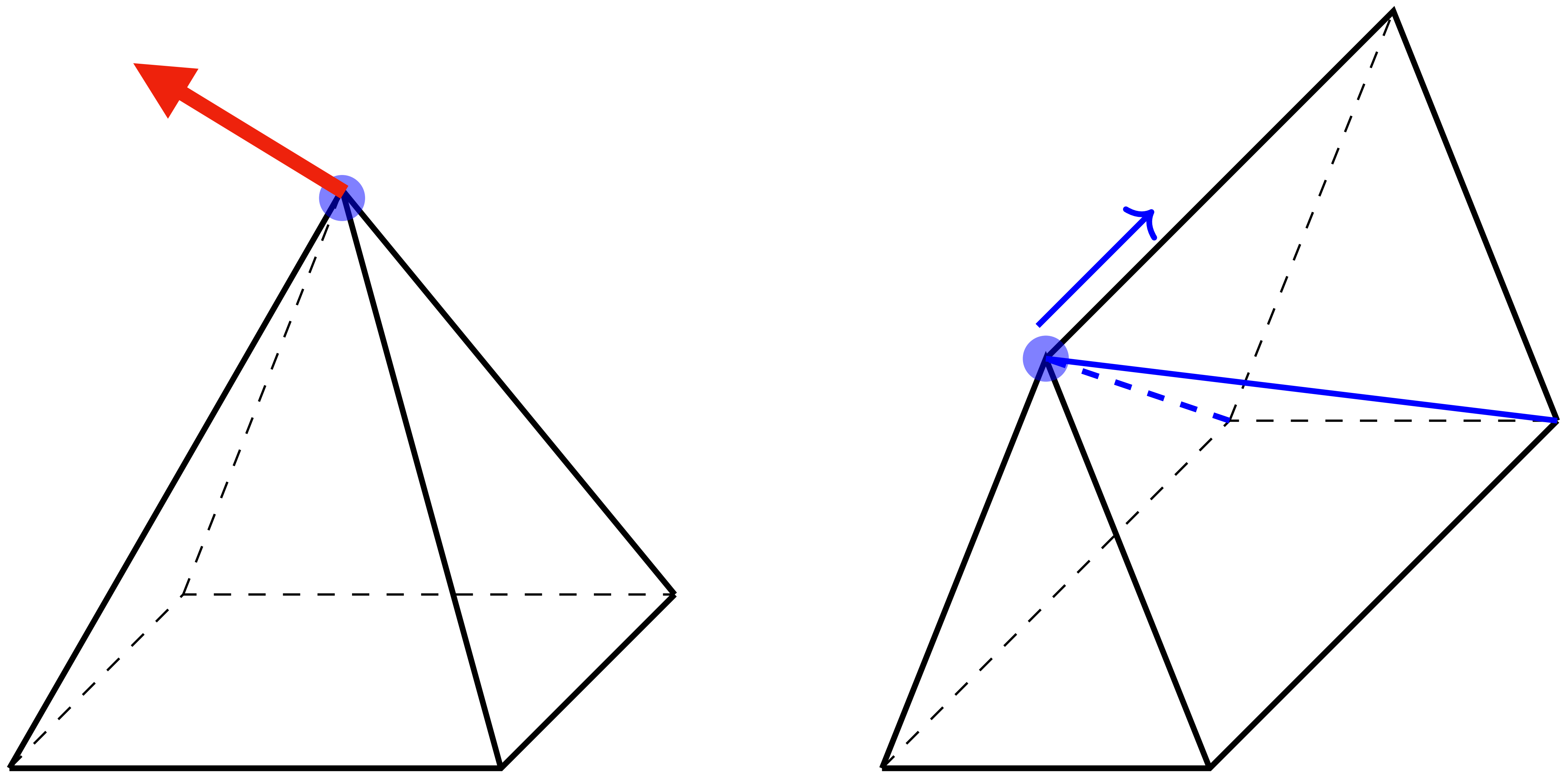
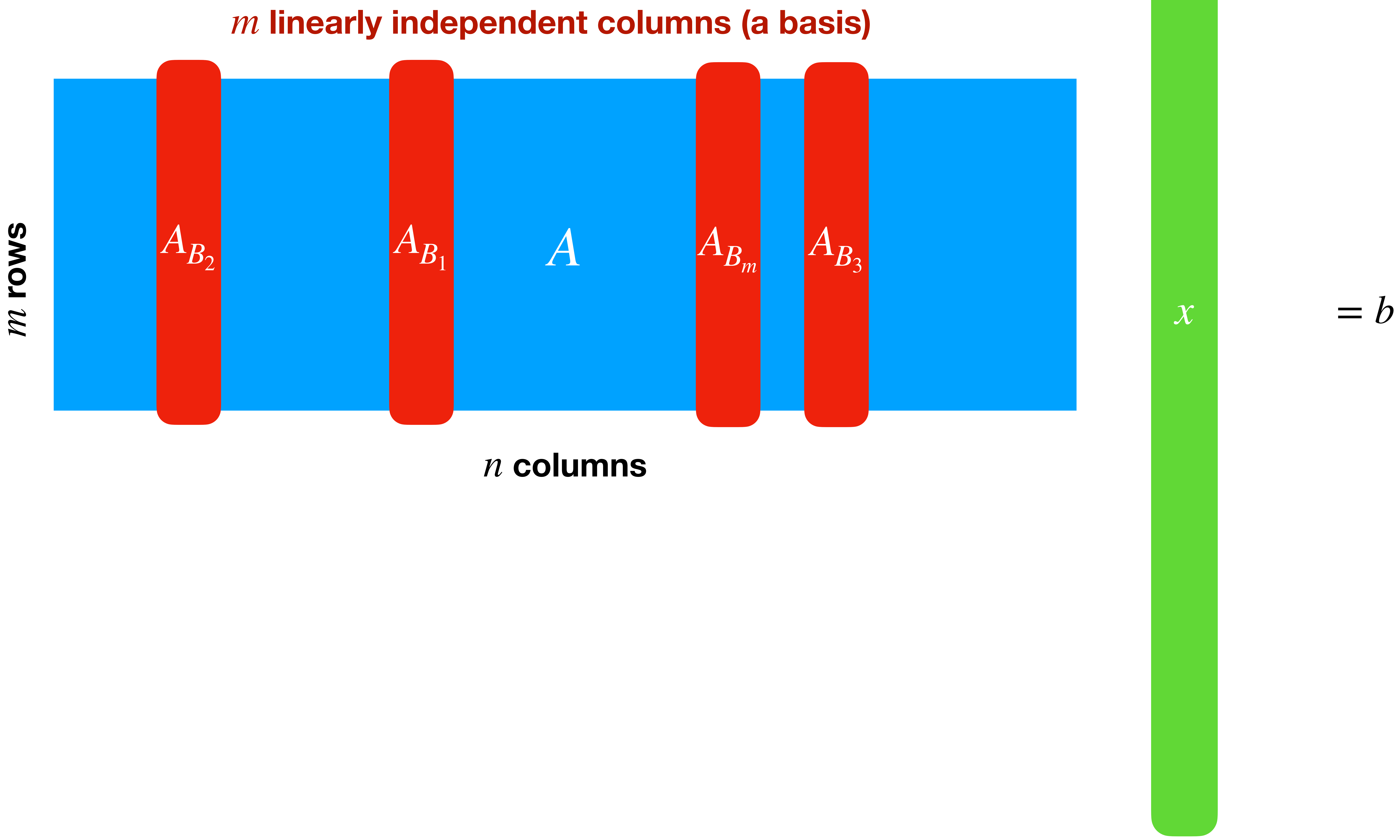


Edge / extrem point of a set:

- Cannot be convex combined by other points of the set.
- There is an (linear) objective function for which it is the unique optimum in the set.

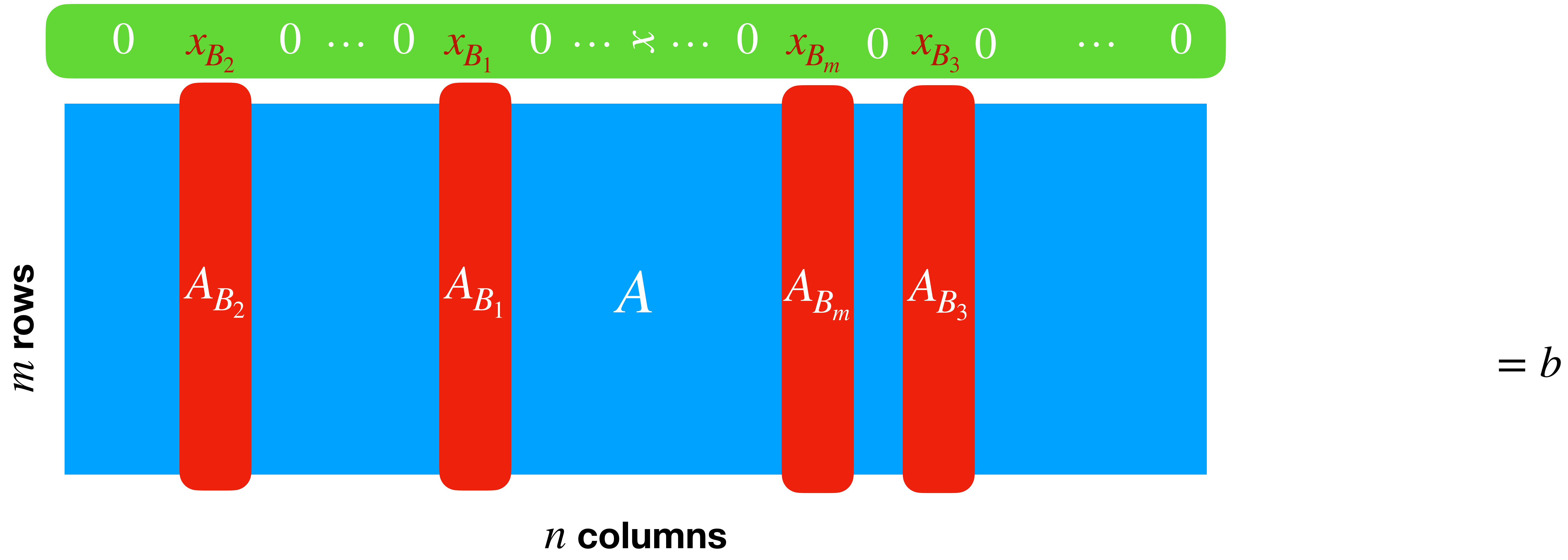
Simplex algorithm

$$\begin{aligned} \min \quad & c^\top x & \\ \text{s. d.} \quad & Ax &= b \\ & x &\geq 0 \end{aligned}$$



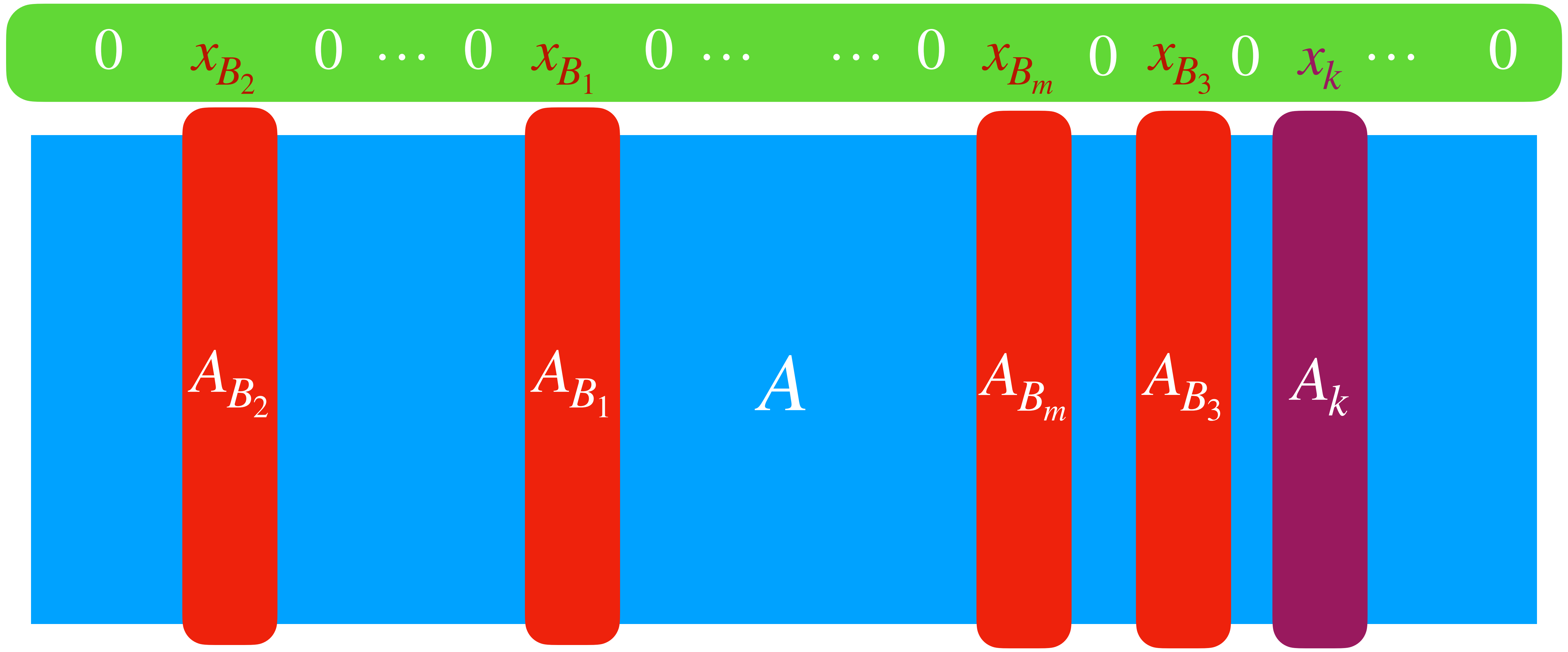
Basic solutions

$$\begin{array}{lrl} \min & c^\top x & \\ \text{s. d.} & Ax & = b \\ & x & \geq 0 \end{array}$$



$$Ax = b \;\Leftrightarrow\; Bx_B + \sum_{j \notin \{B_1,\ldots,B_m\}} A_j x_j = b \;\Leftrightarrow\; \boxed{x_B = B^{-1}b - \sum_{j \notin \{B_1,\ldots,B_m\}} B^{-1} A_j x_j}.$$

Changing the basis



We have to maintain $Ax = b$.

How do we adjust $x_{B_1}, \ldots, x_{B_m}$, if $x_k = 1$?

Express A_k by the columns $B = (A_{B_1}, \ldots, A_{B_m})$

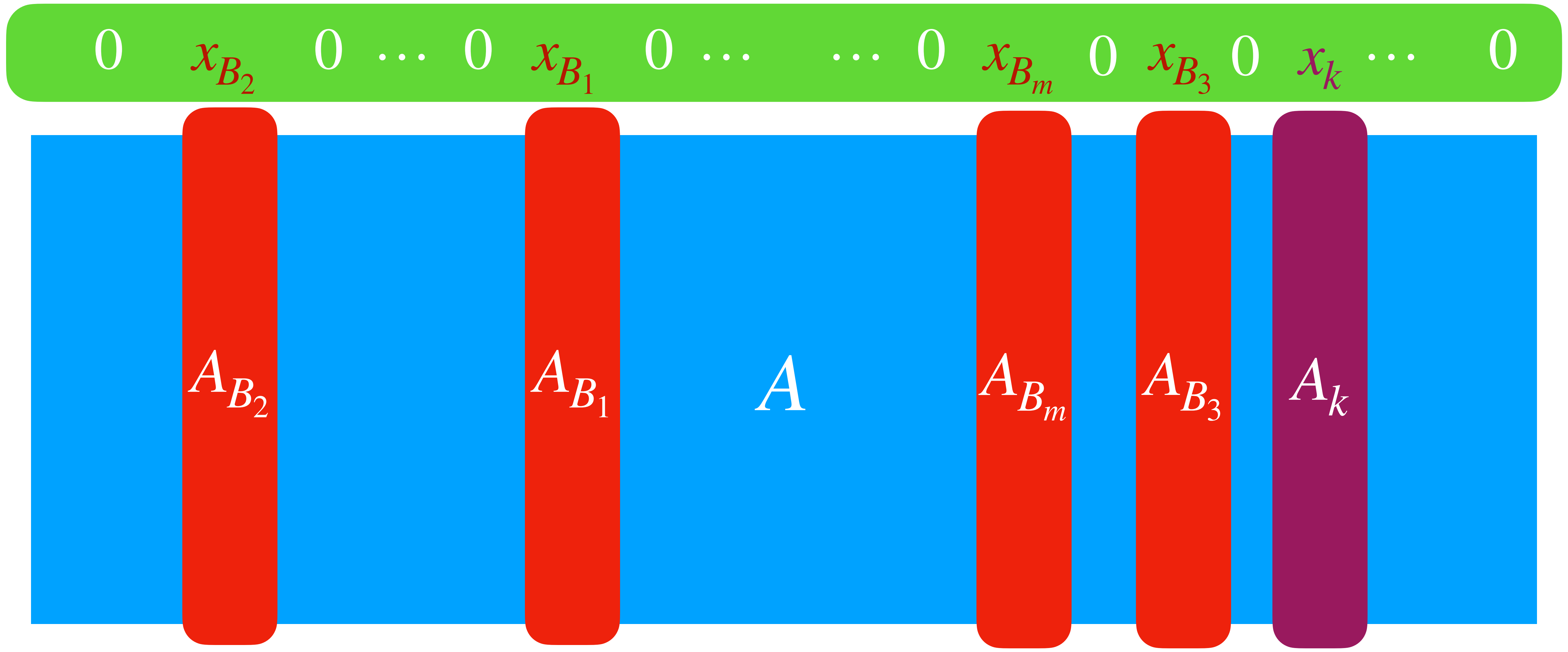
(Remember, B is a basis for $\mathbb{R}^m$!) :

Consider $d_k = -B^{-1}A_k$.

These are the m coefficients

to express A_k by $B = (A_{B_1}, \ldots, A_{B_m})$.

Changing the basis: How does cost change?



Reduced cost: $\bar{c}_k := c_k - c_b'(B^{-1}A_k)$.

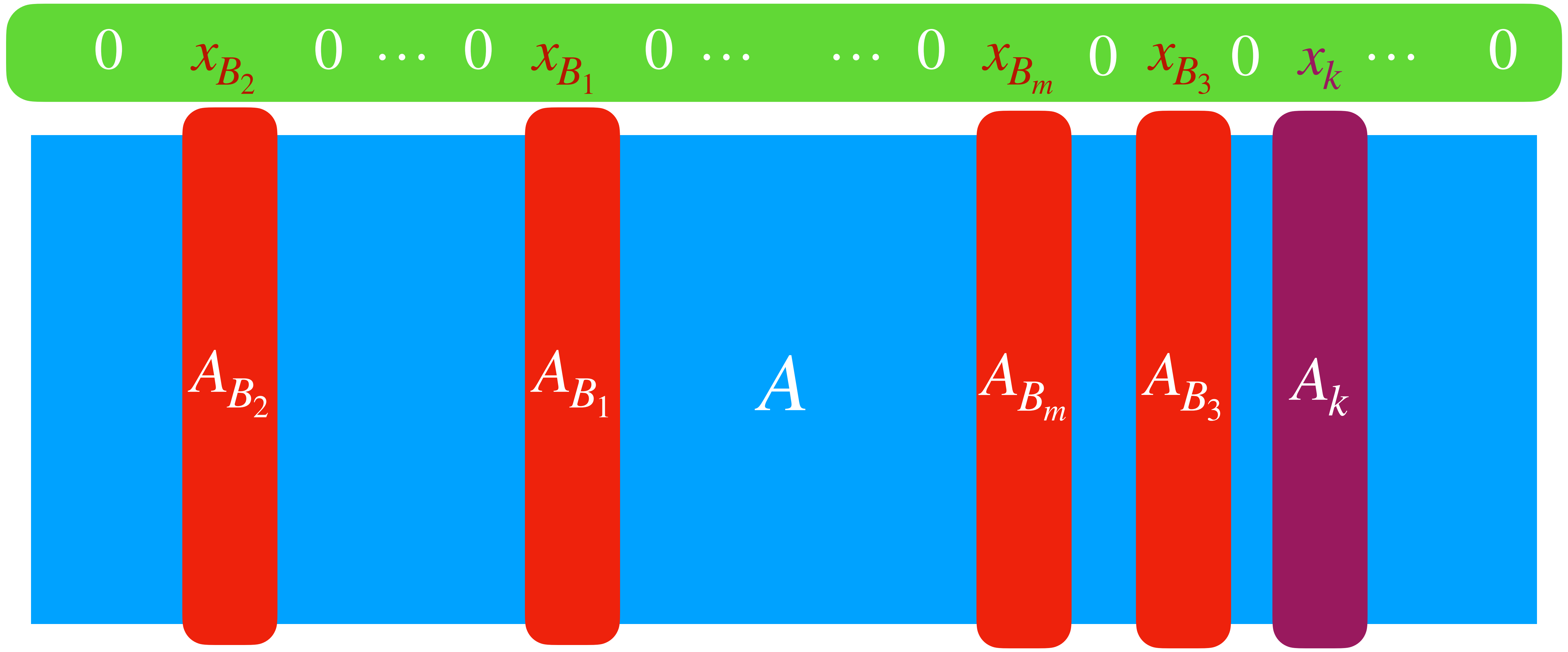
Consider $d_k = -B^{-1}A_k$.

Cost for setting new column to 1, i.e., $x_k = 1$.

Coefficients to express $x_k = 1$ by $B = (A_{B_1}, \ldots, A_{B_m})$.

Multiplied by the cost of the basis columns.

Changing basis: how far in direction d_k?



We have to maintain $Ax = b$ and $x \geq 0$!

How much x_k is possible?

Until the first x_{B_i} goes to zero!

Consider $d_k = -B^{-1}A_k$.

A Criterion for Optimality

Theorem:
If for a feasible, basic solution the reduced cost are non-negative (in a minimization LP) for every column, then the basic solution is an optimal solution.

An Iteration of the Simplex Algorithm

Given a basis.
1. Column with negative **reduced cost**. Else: already optimal!
2. **Basic direction**. If non-negative, then unbounded.
3. Row that hits zero first defines **step length**.
4. **Change basis**.
5. **Iterate** with new basis.

An Iteration of the Simplex Algorithm

Given a basis $B := (A_{B_1}, \dots, A_{B_m})$ with a feasible basic solution x_B.

1. Find a column A_j with negative **reduced cost**. IF there is none, RETURN x_B, the basic solution associated to B as an optimal solution.
2. Define **basic direction** $u := B^{-1}A_j$. IF $u \le 0$, RETURN: „LP is unbounded“.
3. Choose a row index $\ell \in [m]$, that minimizes $\min\limits_{i:u_i>0} \frac{x_{B_i}}{u_i} =: \frac{x_{B_\ell}}{u_\ell} =: \theta^*$. Here, θ^* is the **step length**.
4. Perform **change of basis** by $y_j = \theta^*$ and $y_{B_i} = x_{B_i} - \theta^* u_i$.
5. RETURN FOR NEXT ITERATION new basis $(A_{B_1}, \dots, A_{B_{\ell-1}}, A_j, A_{B_{\ell+1}}, A_{B_m})$ and new basic solution y.

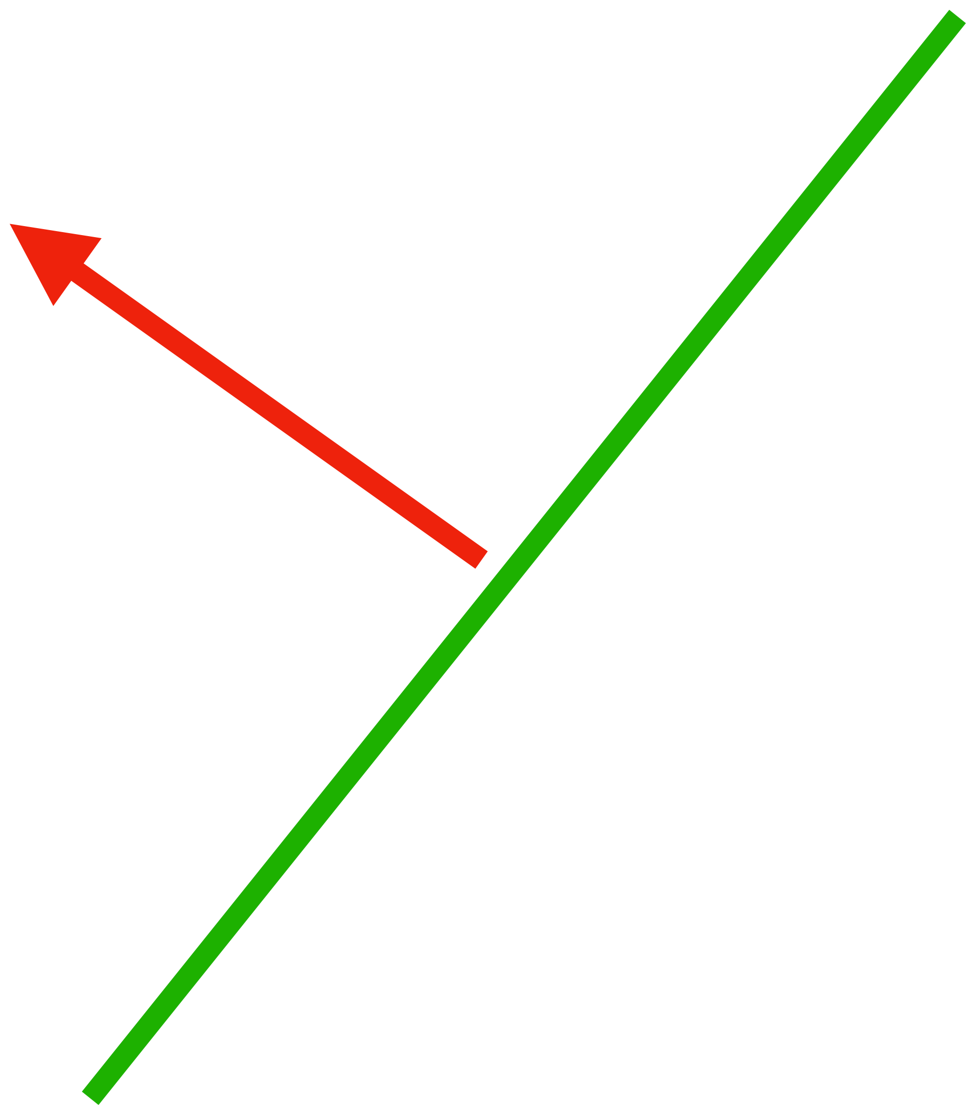
***Remark:**

For 1 note, that the reduced cost of columns in the basis are always 0.

For 4 note, that for $i = \ell$ we have $y_{B_\ell} = 0$.

Duality

$$\min \boxed{5x_1 - 3x_2} \quad \textbf{s.t.} \quad \boxed{5x_1 - 3x_2 \geq 7}$$



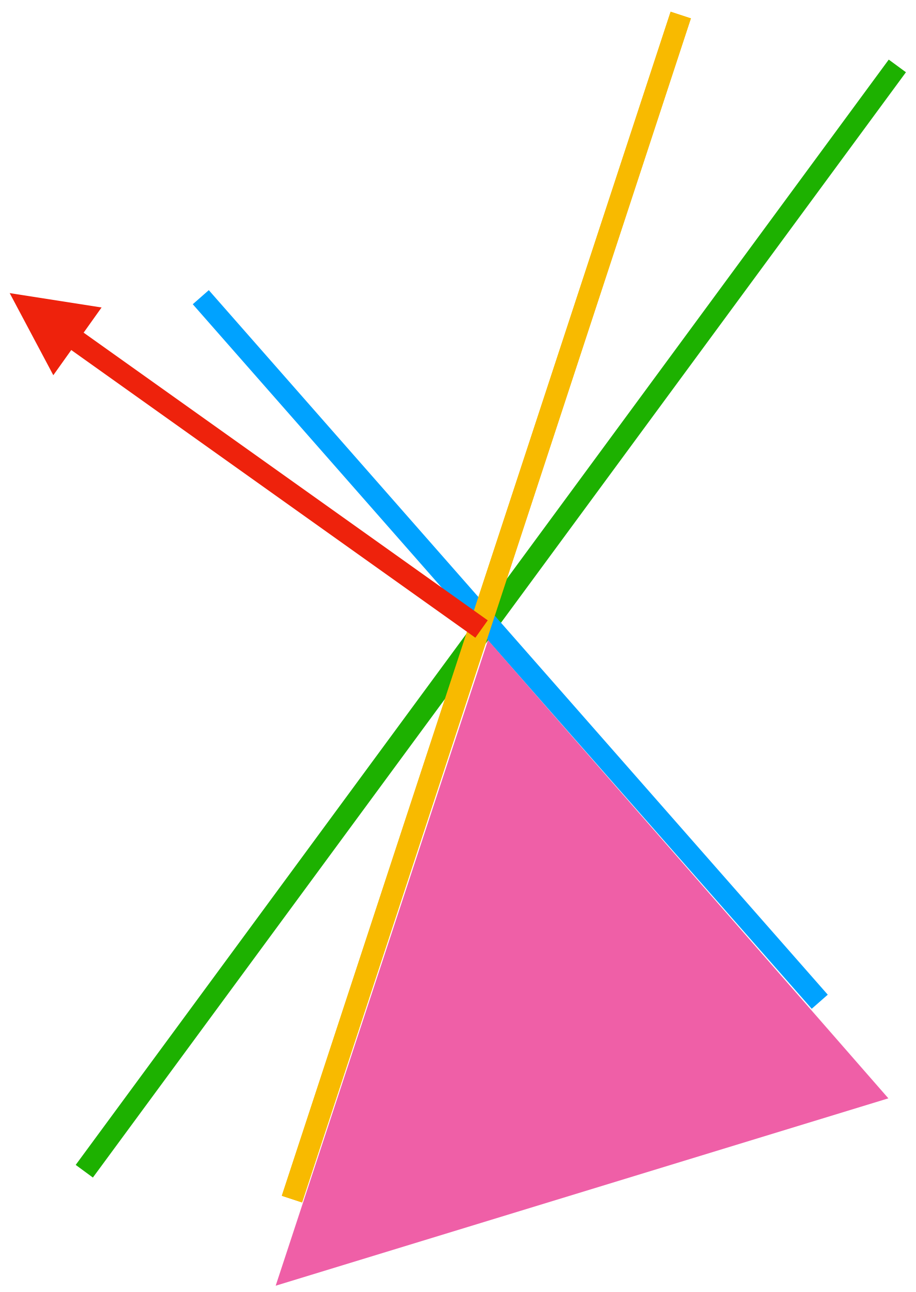
Answer: 7

Duality

$$\min \boxed{5x_1 - 3x_2} \qquad \textbf{s.t.} \quad \boxed{3x_1 - x_2 \geq 5} \quad \cdot 1$$

$$+ \qquad \boxed{x_1 - x_2 \geq 1} \quad \cdot 2$$

$$\boxed{5x_1 - 3x_2 \geq 7}$$



What is the best bound of this form?

$$\begin{array}{llcl} \min & c^\top x & & \\ \text{s. d.} & Ax & \geq & b \\ & x & \geq & 0. \end{array}$$

$$\begin{array}{llcl} \max & p^\top b & & \\ \text{s. d.} & p^\top A & \leq & c^\top \\ & p & \geq & 0 \end{array}$$

Dualizing

$$\begin{aligned} \min\ & c^\top x \\ \text{s.d.}\ & a_i^\top x \geq b_i && \forall i \in M_1 \\ & a_i^\top x \leq b_i && \forall i \in M_2 \\ & a_i^\top x = b_i && \forall i \in M_3 \\ & x_j \geq 0 && \forall j \in M_4 \\ & x_j \leq 0 && \forall j \in M_5 \\ & x_j \ \textbf{free} && \forall j \in M_6 \end{aligned}$$

$$\begin{aligned} \max\ & p^\top b \\ \text{s.d.}\ & p_i \geq 0 && \forall i \in M_1 \\ & p_i \leq 0 && \forall i \in M_2 \\ & p_i \ \textbf{free} && \forall i \in M_3 \\ & p^\top A_j \leq c_j && \forall j \in M_4 \\ & p^\top A_j \geq c_j && \forall j \in M_5 \\ & p^\top A_j = c_j && \forall j \in M_6 \end{aligned}$$

Weak duality

Theorem:

The dual LP for a minimization (maximization) LP is a lower (upper) bound, i.e., every feasible solution to the dual problem has an objective value lessor equal to that of every feasible solution to the primal problem. Formally, we have:

$$c^\top x \geq p^\top b.$$

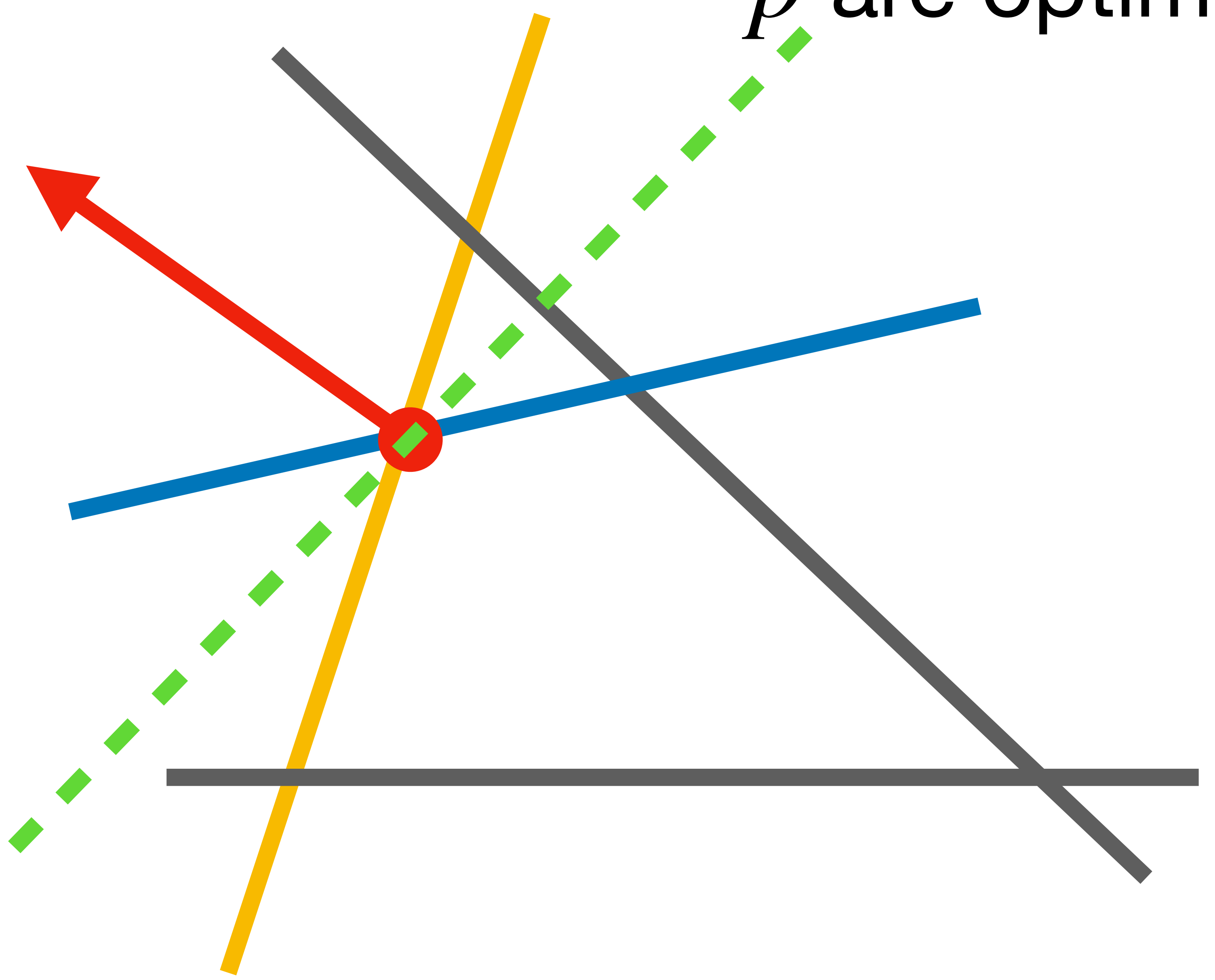
Strong duality

Theorem:

Let x be an optimal solution for an LP and p optimal for its dual LP, then both have equal objective value.

Primal-dual slackness

Theorem:
Let x be a feasible solution for an LP with m constraints in dimension n and let p be feasible for its dual LP. Then x and p are optimal, if and only if the following holds:



$$u_i := p_i(a_i^\top x - b_i) \quad = 0 \quad \forall i \in [m]$$
$$v_j := (c_j - p^\top A_j)x_j = 0 \quad \forall j \in [n]$$