Institute for Mathematical Stochastics
Ramp up Course Mathematics
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Ramp up Course Mathematics - Stochastics - Exercise Sheet 2

Exercise 1 (Point Estimation: Mean Squared Error, Consistency)

Consider a statistical model $(\mathcal{X}, \mathcal{L}, (\mathbb{P}_\theta)_{\theta\in\Theta}))$, $\Theta \subseteq \mathbb{R}$, and a sequence of point estimators $\hat{\theta}_n$, $n \in \mathbb{N}$, for θ.

The quality of a point estimator $\hat{\theta}$ is sometimes assessed by the mean squared error

$$\mathrm{MSE}_\theta(\hat{\theta}) := \mathbb{E}_\theta\left[\left|\theta - \hat{\theta}\right|^2\right]$$

a) Check that

$$\mathrm{MSE}_\theta(\hat{\theta}_n) = \mathrm{bias}_\theta(\hat{\theta}_n)^2 + \mathrm{Var}_\theta(\hat{\theta}_n)$$

where $\mathrm{bias}_\theta(\hat{\theta}) = \mathbb{E}_\theta[\hat{\theta}] - \theta$.

b) Suppose that $\mathrm{bias}_\theta(\hat{\theta}_n) \to 0$ and $\mathrm{Var}_\theta(\hat{\theta}_n) \to 0$ as $n \to \infty$, for all $\theta \in \Theta$. Show that the estimators $\hat{\theta}_n$ are consistent in the sense that

$$\mathbb{P}_\theta\left(\left|\hat{\theta}_n - \theta\right| > \epsilon\right) \xrightarrow[n\to\infty]{} 0$$

for all $\epsilon > 0$ and $\theta \in \Theta$.
Hint: Recall the Chebychev-Markov inequality.

Exercise 2 (Confidence Intervals for Coin Flips)

Consider a statistical model $\left(\mathcal{X}, \mathcal{L}, (\mathbb{P}_p)_{p\in(0,1)})\right)$ of repeatedly and independently flipping a coin and testing if it is fair. Herein, $\mathbb{P}_p$ is the distribution under which the coin has probability p to show "head" and probability $1-p$ to show "tail". Define the estimator $\hat{p}_n = \sum_{i=1}^n X_i$ for p, where X_i is the outcome of the i-th coin flip - it is 1 if the coin shows "head" and 0 if the coin shows "tail".

a) By the so called Hoeffding inequality, it holds that

$$\mathbb{P}_p\left(|\hat{p}_n - p| \geq \epsilon\right) \leq 2e^{-2n\epsilon^2}$$

for any $p \in (0,1), n \in \mathbb{N}, \epsilon > 0$. Use this fact to calculate for all $0 < \alpha < 1$ a confidence interval for p of level $1-\alpha$.

b) How would you test the plausibility of your confidence interval from a) using computer simulation? Write some pseudocode (and maybe inform yourself on how to implement it in your favorite programming environment).

Exercise 3 (Maximum Likelihood Estimatation: Equivariance, Example)

a) Consider a statistical model $(\mathcal{X}, \mathcal{L}, (\mathbb{P}_\theta)_{\theta\in\Theta})$ where every $\mathbb{P}_\theta$ has a density $x \mapsto f(x,\theta)$. Suppose $\tau = g(\theta)$ where g is an invertible function. Then we have
$$(\mathcal{X}, \mathcal{L}, (\mathbb{P}_\theta)_{\theta\in\Theta}) = \left(\mathcal{X}, \mathcal{L}, (\tilde{\mathbb{P}}_\tau)_{\tau\in g(\Theta)}\right)$$
with $\tilde{\mathbb{P}}_\tau = \mathbb{P}_{g^{-1}(\tau)}$.
Show that $g(\hat{\theta})$ is a maximum likelihood estimator of τ when $\hat{\theta}$ is a maximum likelihood estimator of θ.

b) Let $X_1, \dots, X_n$ be i.i.d. $\mathcal{N}(\mu, 1)$ random variables and let τ be the 0.95 percentile, i.e. $\mathbb{P}(X_1 < \tau) = 0.95$. Find the maximum likelihood estimator of τ using a).
Hint: If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\frac{X-\mu}{\sigma} \sim \mathcal{N}(0,1)$. Use this to represent τ as function of μ, using the CDF Φ of the standard normal distribution.

Exercise 4 (Hypothesis Testing: Example)

Let $X_1, \dots, X_n$ be i.i.d. Uniform($(0,\theta)$) random variables and $Y = \max\{X_1, \dots, X_n\}$.
We want to test
$$H_0 : \theta = \frac{1}{2} \quad \text{versus} \quad H_1 : \theta > \frac{1}{2}.$$
We decide to reject H_0 when $Y > c$ for an appropriately chosen c.

a) Calculate the corresponding power function.

b) How should one choose c for the test to be of level α?

Exercise 5 (Linear Regression: Consistency)

Consider the linear regression through the origin model
$$Y_k = \gamma \cdot X_k + \xi_k, \quad 1 \le k \le n.$$
where $X_1, \dots, X_n$ are i.i.d. random variables with existing expectation and existing, non-vanishing variance, γ is an unknown real-valued coefficient and $\xi_1, \dots, \xi_n$ are i.i.d. random variables with $\mathbb{E}[\xi_k] = 0$ and $\mathbb{E}[\xi_k^2] = 1$ which are independent of $X_1, \dots, X_n$.

a) Write down the least squared estimator $\hat{\gamma} = \hat{\gamma}_n$ for γ.

b) Prove consistency of the estimator $\hat{\gamma}_n$ by showing that
$$\mathbb{P}\left(|\hat{\gamma}_n - \gamma| > \epsilon\right) \xrightarrow[n\to\infty]{} 0$$
for all $\epsilon > 0$.
Hint: Recall the (weak) law of large numbers
$$\mathbb{P}\left(\left|\frac{1}{n}\sum_{j=1}^{n} Z_j - \mathbb{E}[Z_1]\right| > \epsilon\right) \xrightarrow[n\to\infty]{} 0$$
for all $\epsilon > 0$ and all i.i.d. random variables $(Z_j)_{j\in\mathbb{N}}$ with existing expectation.