Summer term 2024

Ramp-up Mathematics — Analysis
Extra Homework Sheet

Here are some more exercises you can do to prepare for the exam.

1. Recall that for a square matrix $M \in \mathbb{R}^{n\times n}$ it holds that $\mathrm{Tr}(M) := \sum_{k=i}^{n} M_{kk}$ and that the inner product for matrices $A, B \in \mathbb{R}^{m\times n}$ is defined by $\langle A, B\rangle := \mathrm{Tr}(AB^T)$.
   Prove or disprove that for this inner product it holds that $\langle A, B\rangle = \langle A^T, B^T\rangle$

2. Define $F : \mathbb{R}^{m\times n} \to \mathbb{R}$ by $F(R) = \mathrm{Tr}(RR^T)$.
   (a) Calculate the derivative $DF(R)[H]$! (You can equip $\mathbb{R}^{m\times n}$ with any of your favorite norms. Recall that all norms on finite-dimensional vector spaces are equivalent to each other.)
   (b) What is the gradient $\nabla F(R) \in \mathbb{R}^{m\times n}$?

3. Let $C = \{x \in \mathbb{R}^n \mid x_i > 0 \forall i\}$ and define $d : C \times C \to [0, \infty[$ by
$$d(x, y) := \sum_{i=1}^{n} \frac{(x_i - y_i)^2}{x_i y_i}.$$
   Which of the axioms of a metric are fulfilled and which are not?

4. Let
$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \vdots \\ 1 & \cdots & \cdots & 1 \end{pmatrix} \in \mathbb{R}^{n\times n}.$$
   Calculate $\|A\|_{1\to 1}$ (i.e. the operator norm of A when $\mathbb{R}^n$ is equipped with the 1-norm).

5. Let $A \in \mathbb{R}^{n\times n}$ and define $F : \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times m}$ by $F(R) = RAR^T$. Calculate the derivative $DF(R)[H]$!

6. Give a definition of a norm $\|\cdot\|_{\mathrm{mynorm}}$ on $\mathbb{R}^2$ which fulfills $\|\binom{1}{0}\|_{\mathrm{mynorm}} = 10$ and $\|\binom{0}{1}\|_{\mathrm{mynorm}} = 1$.

7. Let $F : \mathbb{R}^n \to \mathbb{R}$ be defined by $F(x) = \sqrt{\langle a, x\rangle}$ for $a \in \mathbb{R}^n$. Calculate $\nabla F(x)$ at some x with $\langle a, x\rangle > 0$.

8. We define a map from $\mathbb{R}^2$ to $\mathbb{R}$ by $\|(x, y)\|_* := |x - y|$. Argue why this is *not* a norm on $\mathbb{R}^2$.

9. Let $a_i \in \mathbb{R}^n$, $i = 1, \ldots, m$ and $F : \mathbb{R}^n \to \mathbb{R}$ be defined by $F(x) = \sum_{i=1}^{m} (\langle a_i, x\rangle)^3$. Calculate $\nabla F(x)$.

10. Let $n \in \mathbb{N}$ and consider the map

$$\begin{aligned} \mathbb{R}^{n\times n} &\to [0,\infty) \\ A &\mapsto F(A) := \big(\mathrm{Tr}(A^T A)\big)^{1/2}. \end{aligned}$$

(a) Show that $F(A)$ is positive definite.

Hint: You may freely use that there are an invertible matrix $P \in \mathbb{R}^{n\times n}$ and $\lambda_1, \lambda_2, ..., \lambda_n \in [0,\infty)$ such that $A^T A = P^{-1} D P$ with the diagonal matrix

$$D = \mathrm{diag}(\lambda_1, ..., \lambda_n) = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \lambda_n \end{pmatrix} \in \mathbb{R}^{n\times n}.$$

(b) Show that $F(A)$ is homogeneous.

(c) Show the triangle inequality $F(A+B) \le F(A) + F(B)$.

Recall also the Cauchy–Schwarz inequality

$$\left|\sum_{j=1}^n a_j b_j\right| \le \left(\sum_{j=1}^n a_j^2\right)^{1/2} \left(\sum_{j=1}^n b_j^2\right)^{1/2}$$

for all $a_1, ..., a_n, b_1, ..., b_n \in \mathbb{R}$.

(d) Let $x, y \in \mathbb{R}^n$ and recall the notation $\|x\|_2 := (\sum_{j=1}^n |x_j|^2)^{1/2}$. Show that $F(x\,y^T) = \|x\|_2\,\|y\|_2$ of the matrix $x\,y^T \in \mathbb{R}^{n\times n}$.

11. Let $n = 2$ and consider the map

$$\begin{aligned} \mathbb{R}^{2\times 2} &\to \mathbb{R} \\ A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} &\mapsto G(A) := a_{11}a_{22} - a_{21}a_{12}. \end{aligned}$$

Note that $G(A)$ is just the determinant of A.

(a) Show that for $h \in \mathbb{R}$ the derivative $DG(A)[h\mathbf{1}_2]$ along $h\mathbf{1}_2 = \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}$ is

$$DG(A)[h\mathbf{1}_2] = h\,\mathrm{Tr}(A) = h(a_{11} + a_{22}).$$

(b) Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Compute $DG(A)[h\mathbf{1}_2]$.