Exercise Sheet — Linear Algebra

Exercise 1

Rephrase the following statements using propositional logic. For this, first define suitable elementary propositions.

1. "You are eligible to be president of the U.S.A. if and only if you are at least 35 years of age, were born in the U.S.A. or if both parents were American citizens at the time of birth, and have been living in the U.S.A. for at least 14 years."
2. "You can upgrade your operating system if you have a 32-bit processor with a speed of 1 GHz or more, at least 1 GB RAM and 16 GB available disk space, or a 64-bit processor with a speed of 2 GHz or more, at least 2 GB RAM and at least 32 GB available disk space."

Exercise 2

Prove the following logical equivalences using truth tables.

1. $\neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B$.
2. $(A \rightarrow B) \wedge (\neg A \rightarrow B) \Leftrightarrow B$.
3. $A \wedge (B \vee C) \Leftrightarrow (A \wedge B) \vee (A \wedge C)$.
4. $(A \rightarrow B) \wedge (B \rightarrow A) \Leftrightarrow \neg A \leftrightarrow \neg B$.

Exercise 3

Let $\mathbb{U}$ be the universe of objects and $A, B \subseteq \mathbb{U}$ sets. Prove that

1. $A \cup A^c = \mathbb{U}$, and
2. $(A \cap B)^c = A^c \cup B^c$.
3. $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$.

Exercise 4

Consider the function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, x \mapsto x^2$.

1. Show that f is bijective and has the inverse function $f^{-1} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, x \mapsto \sqrt{x}$.
2. Does f still admit a (or the above) inverse function if the domain and codomain of f are replaced by $\mathbb{R}$? Explain your answer.

Exercise 5

Consider the relation $\sim$ on $\mathbb{R}$ given by

$$\forall a, b \in \mathbb{R} : a \sim b \;:\Leftrightarrow\; \lceil a \rceil \;=\; \lceil b \rceil.$$

1. Show that $\sim$ is an equivalence relation.
2. Show that every equivalence class of $\sim$ has the same cardinality.

Exercise 6

Write out the system of linear equations that corresponds to the following matrix and solve it:

$$\left[\begin{array}{rrr|r} 2 & 1 & 4 & -1 \\ 4 & -2 & 3 & 4 \\ 5 & 2 & 6 & -1 \end{array}\right]$$

Exercise 7

A *complex number* z is of the form

$$z \;:=\; a + i \cdot b$$

where $a, b \in \mathbb{R}$. i is called the *imaginary unit* satisfying $i^2 = -1$; a is called the *real part*, and b is called the *imaginary part* of a z. One adds complex numbers $z_1 := a_1 + i \cdot b_1$ and $z_2 := a_2 + i \cdot b_2$ as follows:

$$z_1 + z_2 \;:=\; (a_1 + a_2) + i \cdot (b_1 + b_2)$$

Show (using linear algebra methods): There exist two unique complex numbers z_1 and z_2 satisfying the following properties:

1. z_1 has a real part equal to 1, and z_2 has an imaginary part equal to 1.
2. $z_1 + z_2 = 3 + i \cdot \sqrt{5}$.

Exercise 8

For each of the following matrices, find an elementary matrix E such that $EA = B$.

1. $$A \;:=\; \begin{bmatrix} 2 & -1 \\ 5 & 3 \end{bmatrix}, \quad B \;:=\; \begin{bmatrix} -4 & 2 \\ 5 & 3 \end{bmatrix}$$

2. $$A \;:=\; \begin{bmatrix} 2 & 1 & 3 \\ -2 & 4 & 5 \\ 3 & 1 & 4 \end{bmatrix}, \quad B \;:=\; \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 4 \\ -2 & 4 & 5 \end{bmatrix}$$

Exercise 9

Compute the following determinants:

$$\det \begin{bmatrix} 5 & -2 \\ -8 & 4 \end{bmatrix}, \quad \det \begin{bmatrix} 4 & 3 & 0 \\ 3 & 1 & 2 \\ 5 & -1 & -4 \end{bmatrix}, \quad \det \begin{bmatrix} 4 & 0 & 2 & 1 \\ 5 & 0 & 4 & 2 \\ 2 & 0 & 3 & 4 \\ 1 & 0 & 2 & 3 \end{bmatrix}$$