Summer term 2024

Ramp-up Mathematics — Analysis
Homework Sheet 1

Exercise 1.1 For $x \in \mathbb{R}^d$, let $\|x\|_p := \left(\sum_{k=1}^{d} |x_k|^p\right)^{1/p}$ if $1 \leq p < \infty$ and $\|x\|_\infty := \max_{k=1,\dots,d} |x_k|$.

- Show that for a sequence $x_n \in \mathbb{R}^d$ it holds that $x_n \to x^*$ (with respect to the norm $\|\cdot\|_p$) if and only if for any $k = 1, \dots, d$ it holds that $(x_n)_k \to x^*_k$. (In other words: Convergence with respect to $\|\cdot\|_p$ is equivalent to convergence of the components.)
- How about the case $p = \infty$?

Exercise 1.2
Show that if $\|\cdot\|$ is a norm on a vector space V, then $d(x, y) := \|x - y\|$ is a metric on V.

Exercise 1.3

1. Consider the set
$$M = \left\{ \frac{n}{m+n} \mid m, n = 1, 2, \dots \right\}.$$
Calculate $\inf M$ and $\sup M$. Are infimum and supremum in fact minimum and maximum, respectively?
2. Calculate
$$\inf_{x>0} e^{-x}, \quad \sup_{x>0} e^{x} \quad \inf_{x>0} e^{x}, \quad \sup_{x>0} e^{x}.$$

Exercise 1.4
Are the following maps inner products on $\mathbb{R}^2$?

$$\begin{aligned} a)\ \langle x, y\rangle_a &:= x_1 y_1 - x_2 y_2 \\ b)\ \langle x, y\rangle_b &:= x_1 y_2 + x_2 y_1 \end{aligned}$$

Exercise 1.5

1. Give an example of a set which is neither open nor closed.
2. Show that the set $[0, \infty[$ is closed.
3. Show that the so-called *open balls* $B_\varepsilon(x) := \{y \in X \mid d(x, y) < \varepsilon\}$ in any metric space (X, d) are indeed open sets.