Ramp-up Mathematics — Analysis
Homework Sheet 1

Exercise 1.1 Let $1 \le p \le \infty$ and $\|x\|_p := \left(\sum_{k=1}^{d} |x_k|^p\right)^{1/p}$ on $\mathbb{R}^d$. Show that for a sequence $x_n \in \mathbb{R}^d$ it holds that $x_n \to x^*$ (with respect to the norm $\|\cdot\|_p$) exactly if for any $k = 1, \dots, d$ it holds that $(x_n)_k \to x^*_k$. (In other words: Convergence with respect to $\|\cdot\|_p$ is equivalent to convergence of the components.)

Proof. This follows once we show that for all $p, q \in [1, \infty]$, there is a constant $c = c_{d,p,q} \ge 1$ such that

$$c^{-1}\|x\|_q \le \|x\|_p \le c\|x\|_q, \quad x \in \mathbb{R}^d.$$

Indeed, we have

$$\begin{aligned}\|x\|_q = \left(\sum_{j=1}^{d} |x_j|^q\right)^{1/q} &\le \sup_{j=1,\dots,d} |x_j| \left(\sum_{j=1}^{d} 1\right)^{1/q} = d^{1/q}\|x\|_\infty \le d^{1/q}\|x\|_p \\ &\le d^{1/q+1/p}\|x\|_\infty \le d^{1/q+1/p}\|x\|_q,\end{aligned}$$

as desired. □

Exercise 1.2
Show that if $\|\cdot\|$ is a norm on a vector space V, then $d(x, y) := \|x - y\|$ is a metric on V.

Proof. We check the three properties of a metric.

(i) Symmetry: $d(x, y) = \|x - y\| = \|y - x\|$ follows from $\|-x\| = |-1| \cdot \|x\| = \|x\|$ by property (ii) for norms.

(ii) $d(x, y) = \|x - y\| = 0$ if and only if $x = y$ follows from property (i) for norms.

(iii) $d(x, z) \le d(x, y) + d(y, z)$, i.e., $\|x - z\| \le \|x - y\| + \|y - z\|$ follows from the triangle inequality (iii) for norms since $\|x - z\| = \|(x - y) + (y - z)\| \le \|x - y\| + \|y - z\|$.

□

Exercise 1.3

1. Consider the set

$$M = \left\{ \frac{n}{m + n} \mid m, n = 1, 2, \dots \right\}.$$

Calculate $\inf M$ and $\sup M$. Are infimum and supremum in fact minimum and maximum, respectively?

2. Calculate

$$\inf_{x>0} e^{-x}, \quad \sup_{x>0} e^{x} \quad \inf_{x>0} e^{x}, \quad \sup_{x>0} e^{x}.$$

Proof. 1. Since $0 < n/(n+m) < 1$, we see that 0 and 1 are a lower and upper bound, respectively. We show that they are in fact infimum and supremum, respectively. Since neither 0 nor 1 belong to M, the infimum is not a minimum and the supremum is not a maximum.

Let us now show that 1 is the supremum, i.e., smallest upper bound for M. That means that we have to show that every $b < 1$ cannot be an upper bound of the set M. To that end, we find $n_b, m_b \in \mathbb{N}$ so that $n_b/(n_b + m_b) > b$. We take $m_b = 1$. Then, by an elementary computation, one finds that $n_b := 1 + [b/(1-b)]$ has the desired property, i.e., that $n_b/(n_b + m_b) > b$. [Here, for $x \in \mathbb{R}$, we denote the integer part of x by $[x] := \max\{k \in \mathbb{Z} : k \leq x\}$]. Using similar arguments, one finds that 0 is indeed the infimum of M.

2. $\inf_{x>0} e^{-x} = 0$ (as $x \to \infty$), $\sup_{x>0} e^{-x} = 1$ (as $x \to 0$), $\inf_{x>0} e^{x} = 1$ (as $x \to 0$), $\sup_{x>0} e^{x} = \infty$ (as $x \to \infty$).

□

Exercise 1.4

Are the following maps inner products on $\mathbb{R}^2$?

$$a)\ \langle x, y\rangle_a := x_1 y_1 - x_2 y_2$$
$$b)\ \langle x, y\rangle_b := x_1 y_2 + x_2 y_1$$

Proof. a) Although symmetry and linearity are satisfied, $\langle \cdot, \cdot \rangle_a$ is not an inner product since property (iii), i.e., $\langle x, x\rangle = 0$ if and only if $x = 0$ is violated. Take, e.g., $x = (x_1, x_1)^T$ with any $x_1 \neq 0$ as a counterexample since $\langle x, x\rangle_a = x_1^2 - x_1^2 = 0$.

b) Property (iii) is again violated. Take, e.g., $x = (x_1, 0)$. Then $\langle x, x\rangle_b = x_1 \cdot 0 + 0 \cdot x_1 = 0$.

□

Exercise 1.5

1. Give an example of a set which is neither open nor closed.
2. Show that the set $[0, \infty[$ is closed.
3. Show that the so-called *open balls* $B_\varepsilon(x) := \{y \in X \mid d(x, y) < \varepsilon\}$ in any metric space (X, d) are indeed open sets. (This shows in particular that the open intervals in $\mathbb{R}$ are open sets.)

Proof. 1. The set $[0, 1) \subseteq \mathbb{R}$ is neither open nor closed. It is not open since for every $\delta > 0$ the ball $(-\delta, \delta)$ is not contained in $[0, 1)$. The set $[0, 1)$ is not closed since all elements of the sequence $(a_n)_{n\in\mathbb{N}} := (1 - 1/n)_{n\in\mathbb{N}}$ belong to $[0, 1)$, but the limit $\lim_{n\to\infty} a_n = 1$ does not belong to $[0, 1)$ anymore.

2. This follows since the complement $(-\infty, 0)$ is open.

3. Let $z \in B_\varepsilon(x)$. Let $\delta \in (d(x, z), \varepsilon)$ and $r = \delta - d(x, z)$. Then $B_r(z) \subseteq B_\varepsilon(x)$ since for every point $w \in B_r(z)$, we have $d(w, x) \leq d(w, z) + d(z, x) = \delta - d(x, z) + d(z, x) < \varepsilon$, as required.

□