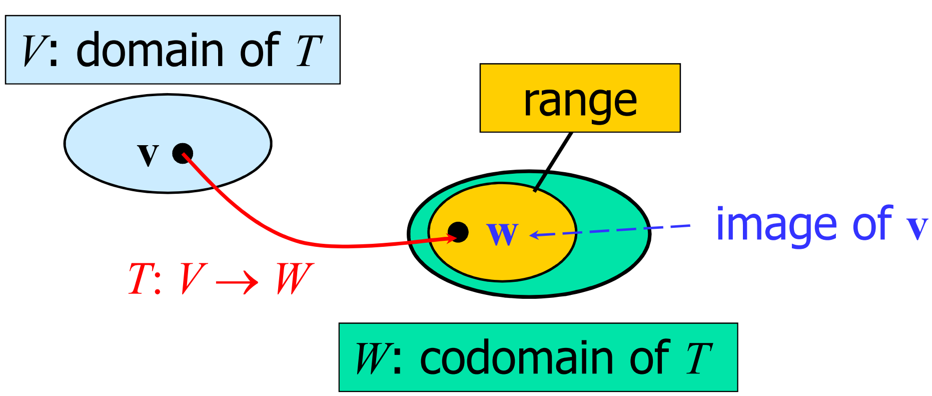
Chap. 6
Linear Transformations

Linear Algebra

Ming-Feng Yeh
Department of Electrical Engineering
Lunghwa University of Science and Technology

6.1 Introduction to Linear Transformation

- Learn about functions that **map** a vector space V into a vector space W --- $T: V \to W$



Map

- If $\mathbf{v}$ is in V and $\mathbf{w}$ is in W s.t. $T(\mathbf{v}) = \mathbf{w}$, then $\mathbf{w}$ is called the **image** of $\mathbf{v}$ under T.
- The set of *all images* of vectors in V is called the **range** of T.
- The set of *all* $\mathbf{v}$ in V s.t. $T(\mathbf{v}) = \mathbf{w}$ is called the **preimage** of $\mathbf{w}$.

Ex. 1: A function from R^2 into R^2

For any vector $\mathbf{v} = (v_1, v_2)$ in R^2, and let $T: R^2 \rightarrow R^2$ be defined by $T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2)$

- Find the image of $\mathbf{v} = (-1, 2)$
  $T(-1, 2) = (-1 - 2, -1 + 2 \cdot 2) = (-3, 3)$
- Find the preimage of $\mathbf{w} = (-1, 11)$

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2) = (-1, 11)$$
$$\Rightarrow v_1 - v_2 = -1;\ v_1 + 2v_2 = 11$$
$$\Rightarrow v_1 = 3;\ v_2 = 4$$

Linear Transformation

- Let V and W be vector spaces. The function $T: V \to W$ is called a **linear transformation** of V into W if the following two properties are true for all **u** and **v** in V and for any scalar c.
  1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
  2. $T(c\mathbf{u}) = cT(\mathbf{u})$
- A linear transformation is said to be *operation reserving* (the operations of ***addition*** and ***scalar multiplication***).

Ex. 2: Verifying a linear transformation from R^2 into R^2

Show that the function $T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2)$ is a linear transformation from R^2 into R^2.

Let $\mathbf{v} = (v_1, v_2)$ and $\mathbf{u} = (u_1, u_2)$

- Vector addition: $\mathbf{v} + \mathbf{u} = (v_1 + u_1, v_2 + u_2)$
  $$\begin{aligned} T(\mathbf{v} + \mathbf{u}) &= T(v_1 + u_1, v_2 + u_2) \\ &= \big((v_1 + u_1) - (v_2 + u_2), (v_1 + u_1) + 2(v_2 + u_2) \big) \\ &= (v_1 - v_2, v_1 + 2v_2) + (u_1 - u_2, u_1 + 2u_2) \\ &= T(\mathbf{v}) + T(\mathbf{u}) \end{aligned}$$
- Scalar multiplication: $c\mathbf{v} = c(v_1, v_2) = (cv_1, cv_2)$
  $T(c\mathbf{v}) = (cv_1 - cv_2, cv_1 + 2cv_2) = c(v_1 - v_2, v_1 + 2v_2) = cT(\mathbf{v})$

Therefore T is a linear transformation.

Ex. 3: Not linear transformation

- $f(x) = \sin(x)$
  In general, $\sin(x_1 + x_2) \neq \sin(x_1) + \sin(x_2)$
- $f(x) = x^2$
  In general, $(x_1 + x_2)^2 \neq x_1^2 + x_2^2$
- $f(x) = x + 1$
  $f(x_1 + x_2) = x_1 + x_2 + 1$
  $f(x_1) + f(x_2) = (x_1 + 1) + (x_2 + 1) = x_1 + x_2 + 2$
  Thus, $f(x_1 + x_2) \neq f(x_1) + f(x_2)$

Linear Operation & Zero / Identity Transformation

- A linear transformation $T: V \to V$ from a vector space into *itself* is called a **linear operator**.
- **Zero transformation** ($T: V \to W$):
  $T(\mathbf{v}) = \mathbf{0}$, for all $\mathbf{v}$
- **Identity transformation** ($T: V \to V$):
  $T(\mathbf{v}) = \mathbf{v}$, for all $\mathbf{v}$

Thm 6.1: Linear transformations

Let T be a linear transformation from V into W, where **u** and **v** are in V. Then the following properties are true.

1. $T(\mathbf{0}) = \mathbf{0}$
2. $T(-\mathbf{v}) = -T(\mathbf{v})$
3. $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$
4. If $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$, then $T(\mathbf{v}) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \ldots + c_nT(\mathbf{v}_n)$

Proof of Theorem 6.1

1. Note that $0\mathbf{v} = \mathbf{0}$. Then it follows that $T(\mathbf{0}) = T(0\mathbf{v}) = 0T(\mathbf{v}) = \mathbf{0}$
2. Follow from $-\mathbf{v} = (-1)\mathbf{v}$, which implies that $T(-\mathbf{v}) = T[(-1)\mathbf{v}] = (-1)T(\mathbf{v}) = -T(\mathbf{v})$
3. Follow from $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$, which implies that
$$T(\mathbf{u} - \mathbf{v}) = T[\mathbf{u} + (-1)\mathbf{v}] = T(\mathbf{u}) + (-1)T(\mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$$
4. Left to you

Remark of Theorem 6.1

- A linear transformation $T: V \to W$ is determined completely by its ***action*** on a basis of V.

- If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a **basis** for the vector space V and if $T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)$ are given, then $T(\mathbf{v})$ is determined for *any* $\mathbf{v}$ in V.

Ex 4: Linear transformations and bases

Let $T: R^3 \to R^3$ be a linear transformation s.t.
$T(1, 0, 0) = (2, -1, 4)$; $T(0, 1, 0) = (1, 5, -2)$;
$T(0, 0, 1) = (0, 3, 1)$. Find $T(2, 3, -2)$.

- $(2, 3, -2) = 2(1, 0, 0) + 3(0, 1, 0) - 2(0, 0, 1)$

$$\begin{aligned} T(2, 3, -2) &= 2T(1, 0, 0) + 3T(0, 1, 0) - 2T(0, 0, 1) \\ &= 2(2, -1, 4) + 3(1, 5, -2) - 2(0, 3, 1) \\ &= (7, 7, 0) \end{aligned}$$

Ex 5: Linear transformation defined by a matrix

The function T: $R^2 \to R^3$ is defined as follows

$$T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

- Find $T(\mathbf{v})$, where $\mathbf{v} = (2, -1)$

$$T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}$$

Therefore, $T(2, -1) = (6, 3, 0)$

Example 5 (cont.)

- Show that T is a linear transformation from R^2 to R^3.

1. For any $\mathbf{u}$ and $\mathbf{v}$ in R^2, we have
   $T(\mathbf{u} + \mathbf{v}) = A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = T(\mathbf{u}) + T(\mathbf{v})$
2. For any $\mathbf{u}$ in R^2 and any scalar c, we have
   $T(c\mathbf{u}) = A(c\mathbf{u}) = c(A\mathbf{u}) = cT(\mathbf{u})$

Therefore, T is a linear transformation from R^2 to R^3.

Thm 6.2: Linear transformation given by a matrix

Let A be an $m \times n$ matrix. The function T defined by

$T(\mathbf{v}) = A\mathbf{v}$

is a ***linear transformation*** from R^n into R^m.

In order to **conform** to matrix multiplication with an $m \times n$ matrix, the vectors in R^n are represented by $m \times 1$ matrices and the vectors in R^m are represented by $n \times 1$ matrices.

Remark of Theorem 6.2

- The $m \times n$ matrix **zero matrix** corresponds to the ***zero transformation*** from R^n into R^m.
- The $n \times n$ matrix **identity matrix** I_n corresponds to the ***identity transformation*** from R^n into R^n.
- An $m \times n$ matrix A defines a ***linear transformation*** from R^n into R^m.

$$A\mathbf{v} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{\boxed{R^n}} = \underbrace{\begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \cdots a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \cdots a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \cdots a_{mn}v_n \end{bmatrix}}_{\boxed{R^m}}$$

Ex 7: Rotation in the plane

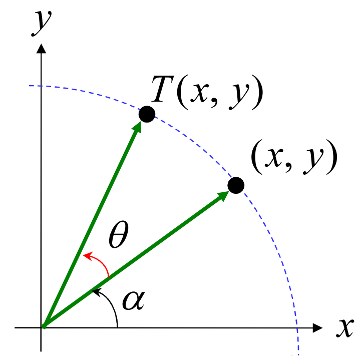
Show that the linear transformation $T: R^2 \to R^2$ given by the matrix $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ has the property that it rotates every vector in ***counterclockwise* about the origin** through the angle θ.

Sol: Let $\mathbf{v} = (x, y) = (r\cos\alpha, r\sin\alpha)$

$$T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} r\cos\alpha \\ r\sin\alpha \end{bmatrix}$$

$$= \begin{bmatrix} r\cos\theta\cos\alpha - r\sin\theta\sin\alpha \\ r\sin\theta\cos\alpha + r\cos\theta\sin\alpha \end{bmatrix}$$

$$= \begin{bmatrix} r\cos(\theta+\alpha) \\ r\sin(\theta+\alpha) \end{bmatrix}$$



Ex 8: A projection in R^3

The linear transformation $T: R^3 \to R^3$ given by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

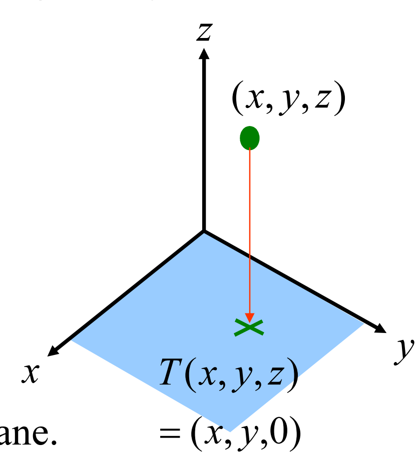
is called a **projection** in R^3.

If $\mathbf{v} = (x, y, z)$ is a vector in R^3,

then $T(\mathbf{v}) = (x, y, 0)$. In other words,

T maps every vector in R^3 to its

orthogonal projection in the xy - plane.



Ex 9: Linear transformation from $M_{m,n}$ to $M_{n,m}$

Let T: $M_{m,n} \rightarrow M_{n,m}$ be the function that maps $m \times n$ matrix A to its **transpose**. That is, $T(A) = A^T$
Show that T is a linear transformation.

pf: Let A and B be $m \times n$ matrix.

$$\because T(A+B) = (A+B)^T = A^T + B^T = T(A) + T(B)$$

$$\text{and } T(cA) = (cA)^T = c(A^T) = cT(A)$$

$\therefore T$ is a linear transformation form $M_{m,n}$ into $M_{n,m}$

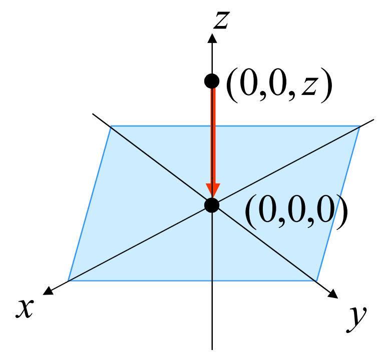
6.2 The Kernel and Range of a Linear Transformation

- Definition of ***Kernel*** of a Linear Transformation
  Let T:$V \rightarrow W$ be a ***linear transformation***. Then the set of all vectors $\mathbf{v}$ in V that satisfy $T(\mathbf{v}) = \mathbf{0}$ is called the **kernel** of T and is denoted by ker(T).
- The kernel of the **zero transformation** T: $V \rightarrow W$ consists of all of V because $T(\mathbf{v}) = 0$ for every $\mathbf{v}$ in V. That is, ker(T) = V.
- The kernel of the **identity transformation** T: $V \rightarrow V$ consists of the single element $\mathbf{0}$. That is, ker(T) = $\{\mathbf{0}\}$.

Ex 3: Finding the kernel

Find the kernel of the projection $T: R^3 \rightarrow R^3$ given by
$T(x, y, z) = (x, y, 0)$.

Sol: This linear transformation projects the vector (x, y, z) in R^3 to the vector $(x, y, 0)$ in xy-plane. Therefore,
$\ker(T) = \{ (0, 0, z) : z \in R \}$



Section 6-2

Ex 4: Finding the kernel

Find the kernel of $T: R^2 \rightarrow R^3$ given by
$T(x_1, x_2) = (x_1 - 2x_2, 0, -x_1)$.

Sol: The kernel of T is the set of all $\mathbf{x} = (x_1, x_2)$ in R^2 s.t. $T(x_1, x_2) = (x_1 - 2x_2, 0, -x_1) = (0, 0, 0)$.
Therefore, $(x_1, x_2) = (0, 0)$.
$\Rightarrow \ker(T) = \{ (0, 0) \} = \{ \mathbf{0} \}$

Ex 5: Finding the kernel

Find the kernel of $T: R^3 \to R^2$ defined by $T(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix}$$

Sol: The kernel of T is the set of all $\mathbf{x} = (x_1, x_2, x_3)$ in R^3 s.t. $T(x_1, x_2, x_3) = (0, 0)$. That is,

$$\begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -t \\ t \end{bmatrix}, t \in R$$

Therefore, $\ker(T) = \{t(1, -1, 1) : t \in \mathrm{R}\} = \mathrm{span}\{(1, -1, 1)\}$

Section 6-2

Thm 6.3: Kernel is a subspace

The ***kernel*** of a linear transformation $T: V \to W$ is a subspace of the domain V.

pf: 1. $\ker(T)$ is a nonempty subset of V.

2. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\ker(T)$. Then
   $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) = \mathbf{0} + \mathbf{0} = \mathbf{0}$ (vector addition)
   Thus, $\mathbf{u}+\mathbf{v}$ is in the kernel

3. If c is any scalar, then $T(c\mathbf{u}) = cT(\mathbf{u}) = c\mathbf{0} = \mathbf{0}$
   (scalar multiplication), Thus, $c\mathbf{u}$ is in the kernel.

- The ***kernel*** of T sometimes called the **nullspace** of T.

Ex 6: Finding a basis for kernel

Let $T: R^5 \rightarrow R^4$ be defined by $T(\mathbf{x}) = A\mathbf{x}$, where $\mathbf{x}$ is in R^5

$$\text{and } A = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix}.$$

Find a basis for $\ker(T)$ as a subspace of R^5.

Example 6 (cont.)

$$\begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 2 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

Thus ***one*** basis for the kernel T is given by

$B = \{ (-2, 1, 1, 0, 0), (1, 2, 0, -4, 1) \}$

Solution Space

- A basis for the **kernel** of a linear transformation $T(\mathbf{x}) = A\mathbf{x}$ was found by solving the homogeneous system given by $A\mathbf{x} = \mathbf{0}$.

- It is the same produce used to find the ***solution space*** of $A\mathbf{x} = \mathbf{0}$.

Thm 6.4: Range is a subspace

The ***range*** of a linear transformation $T: V \rightarrow W$ is a subspace of the domain W.

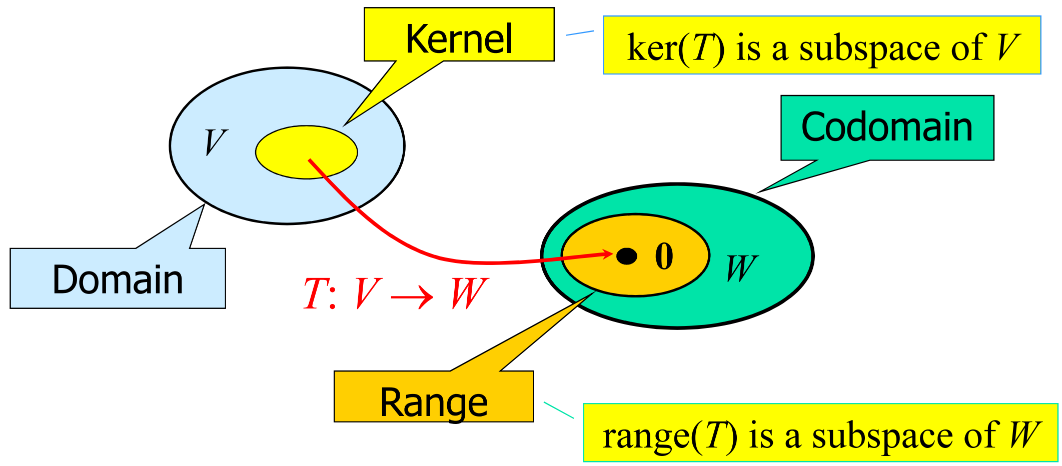
- $\text{range}(T) = \{ T(\mathbf{v}): \mathbf{v} \text{ is in } V \}$
- $\ker(T)$ is a subspace of V.

pf: 1. range(T) is a nonempty because $T(\mathbf{0}) = \mathbf{0}$.

2. Let $T(\mathbf{u})$ and $T(\mathbf{v})$ be vectors in range(T). Because $\mathbf{u}$ and $\mathbf{v}$ are in V, it follows that $\mathbf{u} + \mathbf{v}$ is also in V. Hence the sum $T(\mathbf{u}) + T(\mathbf{v}) = T(\mathbf{u} + \mathbf{v})$ is in the range of T. (vector addition)

3. Let $T(\mathbf{u})$ be a vector in the range of T and let c be a scalar. Because $\mathbf{u}$ is in V, it follows that $c\mathbf{u}$ is also in V. Hence, $cT(\mathbf{u}) = T(c\mathbf{u})$ is in the range of T. (scalar multiplication)

Figure 6.6



Column Space

- To find a basis for the **range** of a linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$, observe that the range consists of all vectors $\mathbf{b}$ such that the system $A\mathbf{x} = \mathbf{b}$ is ***consistent***.

$$A\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = \mathbf{b}$$

- $\mathbf{b}$ is in the range of T if and only if $\mathbf{b}$ is a linear combination of the ***column vectors*** of A.

Corollary of Theorems 6.3 & 6.4

Let $T: R^n \to R^m$ be the linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$.

[Theorem 6.3] The ***kernel*** of T is equal to the **solution space** of $A\mathbf{x} = \mathbf{0}$.

[Theorem 6.4] The **column space** of A is equal to the ***range*** of T.

Ex 7: Finding a basis for range

Let $T: R^5 \to R^4$ be the linear transform given in Example 6. Find a basis for the range of T.

Sol: The row echelon of A:

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

One basis for the range of T is

$B = \{ (1, 2, -1, 0), (2, 1, 0, 0), (1, 1, 0, 2) \}$

Rank and Nullity

Let $T: V \to W$ be a linear transformation.

- The dimension of the ***kernel*** of T is called the **nullity** of T and is denoted by nullity(T).
- The dimension of the ***range*** of T is called the **rank** of T and is denoted by rank(T).

Thm 6.5: Sum of rank and nullity

- Let $T: V \to W$ be a linear transformation from an n-dimension vector space V into a vector space W. Then *the sum of the dimensions of the* ***range*** *and the* ***kernel*** is equal to *the dimension of the* ***domain***. That is,
  $\text{rank}(T) + \text{nullity}(T) = n$
  or
  dim(range) + dim(kernel) = dim(domain)

Proof of Theorem 6.5

The linear transformation from an n-dimension vector space into an m-dimension vector space can be represented by a matrix, i.e., $T(\mathbf{x}) = A\mathbf{x}$ where A is an $m \times n$ matrix. Assume that the matrix A has a rank of r. Then,

rank(T) = dim(range of T) = dim(column space) = rank(A) = r

From Thm 4.7, we have

nullity(T) = dim(kernel of T) = dim(solution space) = $n - r$

Thus,

rank(T) + nullity(T) = $n + (n - r) = n$

Ex 8: Finding the rank & nullity

Find the rank and nullity of $T: R^3 \to R^3$ defined by

the matrix $A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Sol: Because rank(A) = 2, the rank of T is 2.

The nullity is dim(domain) – rank = 3 – 2 = 1.

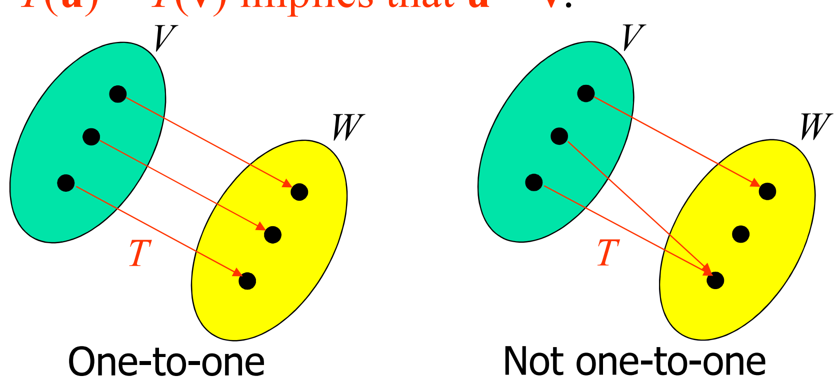
Ex 8: Finding the rank & nullity

Let $T: R^5 \rightarrow R^7$ be a linear transformation

- Find the dimension of the kernel of T if the dimension of the range is 2.
  dim(kernel) = n – dim(range) = 5 – 2 = 3
- Find the rank of T if the nullity of T is 4
  rank(T) = n – nullity(T) = 5 – 4 = 1
- Find the rank of T if ker(T) = {**0**}
  rank(T) = n – nullity(T) = 5 – 0 = 5

One-to-One Mapping

- A linear transformation $T: V \rightarrow W$ is said to be **one-to-one** if and only if for all **u** and **v** in V, $T(\mathbf{u}) = T(\mathbf{v})$ implies that $\mathbf{u} = \mathbf{v}$.



Thm 6.6: One-to-one Linear transformation

Let $T: V \to W$ be a linear transformation. Then T is **one-to-one** if and only if $\ker(T) = \{\mathbf{0}\}$.

pf: $\Rightarrow$ Suppose T is one-to-one. Then $T(\mathbf{v}) = \mathbf{0}$ can have only one solution: $\mathbf{v} = \mathbf{0}$. In this case, $\ker(T) = \{\mathbf{0}\}$.

$\Leftarrow$ Suppose $\ker(T) = \{\mathbf{0}\}$ and $T(\mathbf{u}) = T(\mathbf{v})$. Because T is a linear transformation, it follows that

$T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v}) = \mathbf{0}$

This implies that $\mathbf{u} - \mathbf{v}$ lies in the kernel of T and must therefore equal $\mathbf{0}$. Hence $\mathbf{u} - \mathbf{v} = \mathbf{0}$ and $\mathbf{u} = \mathbf{v}$, and we can conclude that T is one-to-one.

Example 10

- The linear transformation $T: M_{m,n} \to M_{n,m}$ given by $T(A) = A^T$ is **one-to-one** because its kernel consists of ***only*** the $m \times n$ zero matrix.

- The ***zero transformation*** $T: R^3 \to R^3$ is not **one-to-one** because its kernel is ***all*** of R^3.

Onto Linear Transformation

- A linear transformation $T: V \to W$ is said to be **onto** if every element in W has a preiamge in V.
- T is onto W when W is equal to the range of T.
- **[Thm 6.7]** Let $T: V \to W$ be a linear transformation, where W is finite dimensional. Then T is **onto** if and only if the rank of T is equal to the dimension of W, i.e., $\text{rank}(T) = \dim(W)$.
- One-to-one: $\ker(T) = \{\mathbf{0}\}$ or $\text{nullity}(T) = 0$

Thm 6.8: One-to-one and onto linear transformation

- Let $T: V \to W$ be a linear transformation with vector spaces V and W *both* of dimension n. Then T is one-to-one if and only if it is onto.

pf: $\Rightarrow$ ⌟ If T is one-to-one, then $\ker(T) = \{\mathbf{0}\}$ and $\dim(\ker(T)) = 0$. In this case, $\dim(\text{range of } T) = n - \dim(\ker(T)) = n = \dim(W)$. By Theorem 6.7, T is onto.

$\Leftarrow$ ⌟ If T is onto, then $\dim(\text{range of } T) = \dim(W) = n$. Which by Theorem 6.5 implies that $\dim(\ker(\text{T})) = 0$ By Theorem 6.6, T is onto-to-one.

Example 11

The linear transformation $T{:}R^n \to R^m$ is given by $T(\mathbf{x}) = A\mathbf{x}$. Find the nullity and rank of T and determine whether T is one-to-one, onto, or either.

$$(a)\ A=\begin{bmatrix}1&2&0\\0&1&1\\0&0&1\end{bmatrix},(b)\ A=\begin{bmatrix}1&2\\0&1\\0&0\end{bmatrix},(c)\ A=\begin{bmatrix}1&2&0\\0&1&-1\end{bmatrix},(d)\ A=\begin{bmatrix}1&2&0\\0&1&1\\0&0&0\end{bmatrix}$$

$T{:}R^n \to R^m$	dim(domain)	rank(T)	nullity(T)	one-to-one	onto
(a) $T{:}R^3 \to R^3$	3	3	0	Yes	Yes
(b) $T{:}R^2 \to R^3$	2	2	0	Yes	No
(c) $T{:}R^3 \to R^2$	3	2	1	No	Yes
(d) $T{:}R^3 \to R^3$	3	2	1	No	No

Isomorphism

Def: A linear transformation $T: V \to W$ that is ***one-to-one*** and ***onto*** is called ***isomorphism***. Moreover, if V and W are vector spaces such that there exists an isomorphism from V to W, then V and W are said to be ***isomorphic*** to each other.

Theorem 6.9: *Isomorphism Spaces & Dimension*

Two finite-dimensional vector spaces V and W are **isomorphic** if and only if they are of the same dimension.

Ex 12: Isomorphic Vector Spaces

The following vector spaces are isomorphic to each other.

- R^4 = 4-space
- $M_{4,1}$ = space of all 4×1 matrices
- $M_{2,2}$ = space of all 2×2 matrices
- P_3 = space of all polynomials of degree 3 or less
- $V = \{(x_1, x_2, x_3, x_4, 0): x_i$ is a real number$\}$ (subspace of R^5)

6.3 Matrices for Linear Transformation

- Which one is better?

$$T(x_1, x_2, x_3) = (2x_1 + x_2 - x_3, -x_1 + 3x_2 - 2x_3, 3x_2 + 4x_3)$$

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 3 & -2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Simpler to write. Simpler to read, and more adapted for computer use.

- The key to representing a linear transformation $T: V \to W$ by a ***matrix*** is to determine how it acts on a **basis** of V.
- Once you know *the **image** of every vector in the basis*, you can use the properties of linear transformations to determine $T(\mathbf{v})$ for *any* $\mathbf{v}$ in V.

Thm 6.10: Standard matrix for a linear transformation

Let $T: R^n \to R^m$ be a ***linear transformation*** such that

$$T(\mathbf{e}_1)=T\left(\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}\right)=\begin{bmatrix}a_{11}\\a_{21}\\\vdots\\a_{m1}\end{bmatrix}, T(\mathbf{e}_2)=T\left(\begin{bmatrix}0\\1\\\vdots\\0\end{bmatrix}\right)=\begin{bmatrix}a_{12}\\a_{22}\\\vdots\\a_{m2}\end{bmatrix}, \cdots, T(\mathbf{e}_n)=T\left(\begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix}\right)=\begin{bmatrix}a_{1n}\\a_{2n}\\\vdots\\a_{mn}\end{bmatrix}$$

Then the $m \times n$ matrix whose n columns corresponds to $T(\mathbf{e}_i)$, is such that $T(\mathbf{v}) = A\mathbf{v}$ for every $\mathbf{v}$ in R^n. A is called the **standard matrix** for T.

$$A=\begin{bmatrix}a_{11} & a_{12} & \cdots & a_{n1}\\ a_{21} & a_{22} & \cdots & a_{n2}\\ \vdots & \vdots & \ddots & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{bmatrix}$$

Proof of Theorem 6.10

Let $\mathbf{v} = [v_1 \quad v_2 \quad \cdots \quad v_n]^T = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n \in R^n$.

Because T is a linear transformation, we have

$$\begin{aligned}T(\mathbf{v}) &= T(v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n)\\ &= T(v_1\mathbf{e}_1) + T(v_2\mathbf{e}_2) + \cdots + T(v_n\mathbf{e}_n)\\ &= v_1T(\mathbf{e}_1) + v_2T(\mathbf{e}_2) + \cdots + v_nT(\mathbf{e}_n)\end{aligned}$$

On the other hand,

$$A\mathbf{v}=\begin{bmatrix}a_{11} & a_{12} & \cdots & a_{n1}\\ a_{21} & a_{22} & \cdots & a_{n2}\\ \vdots & \vdots & \ddots & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{bmatrix}\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}=v_1\begin{bmatrix}a_{11}\\a_{21}\\\vdots\\a_{m1}\end{bmatrix}+v_2\begin{bmatrix}a_{12}\\a_{22}\\\vdots\\a_{m2}\end{bmatrix}+\cdots+v_n\begin{bmatrix}a_{1n}\\a_{2n}\\\vdots\\a_{mn}\end{bmatrix}$$

$$= v_1T(\mathbf{e}_1) + v_2T(\mathbf{e}_2) + \cdots + v_nT(\mathbf{e}_n) = T(\mathbf{v})$$

Example 1

Find the standard matrix for the linear transformation $T: R^3 \rightarrow R^2$ defined by $T(x, y, z) = (x - 2y, 2x + y)$

Sol:

$$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 2) \Rightarrow T(\mathbf{e}_1) = T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\end{bmatrix}$$

$$T(\mathbf{e}_2) = T(0, 1, 0) = (-2, 1) \Rightarrow T(\mathbf{e}_2) = T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}-2\\1\end{bmatrix}$$

$$T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0) \Rightarrow T(\mathbf{e}_3) = T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\0\end{bmatrix}$$

Example 1 (cont.)

$$A = \left[T(\mathbf{e}_1) \vdots T(\mathbf{e}_2) \vdots T(\mathbf{e}_3)\right] = \begin{bmatrix}1 & -2 & 0\\2 & 1 & 0\end{bmatrix}$$

Note that

$$A\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1 & -2 & 0\\2 & 1 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}x - 2y\\2x + y\end{bmatrix}$$

which is equivalent to $T(x, y, z) = (x - 2y, 2x + y)$.

Example 2

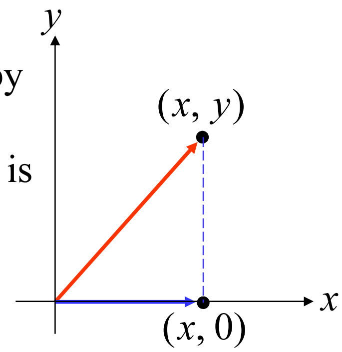
The linear transformation T: $R^2 \rightarrow R^2$ is given by projecting each point in R^2 onto to the x-axis. Find the standard matrix for T.

Sol: This linear transformation is given by

$T(x, y) = (x, 0)$.

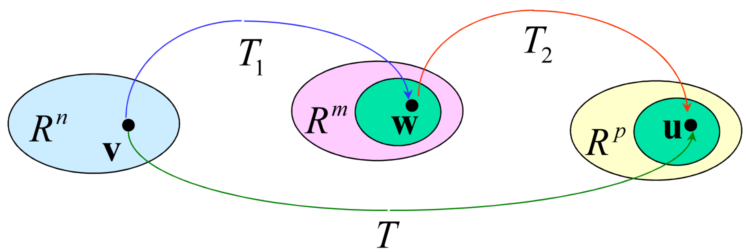
Therefore, the standard matrix for T is

$$A = [T(1,0) \vdots T(0,1)] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$



Composition of linear transformation

- The composition T, of T_1: $R^n \rightarrow R^m$ with T_2: $R^m \rightarrow R^p$ is defined by $T(\mathbf{v}) = T_2(\, T_1(\mathbf{v})\,) = T_2 \circ T_1$ where $\mathbf{v}$ is a vector in R^n.
- The domain of T is defined to the domain of T_1.
- The composition is ***not*** defined ***unless*** the range of T_1 lies within the domain of T_2.



Theorem 6.11

Let T_1: $R^n \rightarrow R^m$ and T_2: $R^m \rightarrow R^p$ be linear transformation with standard matrix A_1 and A_2.

The **composition** T: $R^n \rightarrow R^p$, defined by $T(\mathbf{v}) = T_2(\ T_1(\mathbf{v})\ )$, is *linear transformation*.

Moreover, the ***standard matrix*** of A for T is given by the matrix product $A = A_2 A_1$.

Proof of Theorem 6.11

1. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in R^n and let c be any scalar. Because T_1 and T_2 are linear transformation,

$$T(\mathbf{u}+\mathbf{v}) = T_2(T_1(\mathbf{u}+\mathbf{v})) = T_2(T_1(\mathbf{u}) + T_1(\mathbf{v}))$$
$$= T_2(T_1(\mathbf{u})) + T_2(T_1(\mathbf{v})) = T(\mathbf{u}) + T(\mathbf{v}).$$
$$T(c\mathbf{v}) = T_2(T_1(c\mathbf{v})) = T_2(cT_1(\mathbf{v})) = cT_2(T_1(\mathbf{v})) = cT(\mathbf{v}).$$

Thus, T is a linear transformation.

2. $T(\mathbf{v}) = T_2(T_1(\mathbf{v})) = T_2(A_1\mathbf{v}) = A_2(A_1\mathbf{v}) = A_2A_1\mathbf{v}$

- In general, the composition $T_2 \circ T_1$ is ***not*** the same as $T_1 \circ T_2$.

Example 3

Let T_1 and T_2 be linear transformation R^3 from R^3 such that $T_1(x, y, z) = (2x + y, 0, x + z)$ and $T_2(x, y, z) = (x - y, z, y)$
Find the standard matrices for the compositions $T = T_2 \circ T_1$ and $T' = T_1 \circ T_2$.

Sol: The standard matrices for T_1 and T_2 are

$$A_1 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$T \Rightarrow A = A_1 A_2 = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad T' \Rightarrow A' = A_2 A_1 = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Inverse Linear Transformation

- One benefit of matrix representation is that it can represent the **inverse** of a linear transformation.

[Definition] If $T_1: R^n \rightarrow R^n$ and $T_2: R^n \rightarrow R^n$ are linear transformations such that $T_2(T_1(\mathbf{v})) = \mathbf{v}$ and $T_1(T_2(\mathbf{v})) = \mathbf{v}$, then T_2 is called the ***inverse*** of T_1 and T_1 is said to be ***invertible***.

Theorem 6.12

- Let T:$R^n \to R^n$ be linear transformation with standard matrix A. Then the following conditions are equivalent.
  1. T is **invertible**.
  2. T is an **isomorphism**.
  3. A is ***invertible***.

  And, if T is invertible with standard matrix A, then the standard matrix for T^{-1} is A^{-1}.

Example 4

The linear transformation T: $R^3 \to R^3$ is defined by

$$T(x, y, z) = (2x + 3y + z, 3x + 3y + z, 2x + 4y + z)$$

Show that T is invertible, and find its inverse.

Sol:

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \Rightarrow \det(A) \neq 0$$

$\therefore A$ is invertible. Its inverse is $A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}$

Therefore T is invertible and its standard matrix is A^{-1}.

Example 4 (cont.)

Using the standard matrix for the inverse, we can find the rule for T^{-1} by computing the image of an arbitrary vector $\mathbf{v} = (x, y, z)$.

$$T^{-1}(\mathbf{v}) = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -x+y \\ -x+z \\ 6x-2y-3y \end{bmatrix}$$

$$T^{-1}(x, y, z) = (-x+y, -x+z, 6x-2y-3z).$$

Nonstandard Bases

- Finding a matrix for a linear transformation $T:V \rightarrow W$, where B and B' are ordered bases for V and W, respectively.
- The coordinate matrix of $\mathbf{v}$ relative to B is $[\mathbf{v}]_B$.
- To represent the linear transformation T, A must be multiplied by a ***coordinate matrix relative to*** B.
- The result of the multiplication will be a ***coordinate matrix relative to*** B'.
- $[T(\mathbf{v})]_{B'} = A[\mathbf{v}]_B$. A is called the ***matrix of T relative to the bases*** B ***and*** B'.

Transformation Matrix

Let V and W be finite-dimensional vector spaces with bases B and B', respectively, where $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$

If $T{:}V \to W$ is a linear transformation such that

$$[T(\mathbf{v}_1)]_{B'} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix},\ [T(\mathbf{v}_2)]_{B'} = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \cdots, [T(\mathbf{v}_n)]_{B'} = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix},$$

then the $m \times n$ matrix whose n columns correspond to $[T(\mathbf{v})]_{B'}$,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{n1} \\ a_{21} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

is s.t. $[T(\mathbf{v})]_{B'} = A[\mathbf{v}]_B$ for every $\mathbf{v}$ in V.

Example 5

Let T: $R^2 \to R^2$ be a linear transformation defined by $T(x, y) = (x + y, 2x - y)$. Find the matrix of T relative to the bases $B = \{\underset{\mathbf{v}_1}{(1, 2)}, \underset{\mathbf{v}_2}{(-1, 1)}\}$ and $B' = \{\underset{\mathbf{w}_1}{(1, 0)}, \underset{\mathbf{w}_2}{(0, 1)}\}$

Sol:

$$T(\mathbf{v}_1) = T(1, 2) = (3, 0) = 3\mathbf{w}_1 + 0\mathbf{w}_2$$

$$T(\mathbf{v}_2) = T(-1, 1) = (0, -3) = 0\mathbf{w}_1 - 3\mathbf{w}_2$$

Therefore the coordinate matrices of $T(\mathbf{v}_1)$ and $T(\mathbf{v}_2)$ relative to B' are $[T(\mathbf{v}_1)]_{B'} = [3 \quad 0]^{\mathrm{T}}$, $[T(\mathbf{v}_2)]_{B'} = [0 \quad -3]^{\mathrm{T}}$.

The matrix for T relative to B and B' is $A = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$

Example 6

For the linear transformation T: $R^2 \to R^2$ given in Example 5, use the matrix A to find $T(\mathbf{v})$, where $\mathbf{v} = (2, 1)$.

Sol: $B = \{(1, 2), (-1, 1)\}$

$$\mathbf{v} = (2, 1) = 1(1, 2) - 1(-1, 1) \Rightarrow [\mathbf{v}]_B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\therefore [T(\mathbf{v})]_{B'} = A[\mathbf{v}]_B = \begin{bmatrix} 3 & 0 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\because B' = \{(1, 0), (0, 1)\} \Rightarrow T(\mathbf{v}) = 3(1, 0) + 3(0, 1) = (3, 3).$$

$$\boxed{Ex\,5:\ T(x, y) = (x + y, 2x - y) \Rightarrow T(2, 1) = (3, 3)}$$

6.4 Transition Matrix and Similarity

- The matrix of a linear transformation T:$V \to V$ depends on the basis of V.
- The matrix of T relative to a basis B is ***different from*** the matrix of T relative to another basis B'
- Is is possible to find a basis B such that the matrix of T relative to B is diagonal?

Transition Matrices

- A linear transformation is defined by $T{:}V \to V$

Matrix of T relative to B: A
Matrix of T relative to B' : A'
Transition matrix from B' to B: P
Transition matrix from B to B': P^{-1}

$$[T(\mathbf{v})]_B = A[\mathbf{v}]_B$$
$$[T(\mathbf{v})]_{B'} = A'[\mathbf{v}]_{B'}$$
$$[\mathbf{v}]_B = P[\mathbf{v}]_{B'};\ [\mathbf{v}]_{B'} = P^{-1}[\mathbf{v}]_B$$

$$\begin{tikzcd}
{[\mathbf{v}]_B} \arrow[r, "A"] & {[T(\mathbf{v})]_B} \arrow[d, "P^{-1}"] \\
{[\mathbf{v}]_{B'}} \arrow[u, "P"] \arrow[r, "A'"] & {[T(\mathbf{v})]_{B'}}
\end{tikzcd}$$

$$\left.\begin{aligned} [T(\mathbf{v})]_{B'} &= A'[\mathbf{v}]_{B'} \\ [T(\mathbf{v})]_{B'} &= P^{-1}AP[\mathbf{v}]_{B'} \end{aligned}\right\}$$
$$\Rightarrow A' = P^{-1}AP$$

Example 1

Find the matrix A' for T: $R^2 \to R^2$, $T(x, y) = (2x - 2y, -x + 3y)$ relative to the basis $B' = \{(1, 0), (1, 1)\}$.

Sol: The standard matrix for T is $A = \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix}$

The transition matrix from B' to the standard basis

$$B = \{(1, 0), (0, 1)\} \text{ is } P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

Therefore the matrix for T relative to B' is

$$A' = P^{-1}AP = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$$

Example 2

Let $B = \{(-3, 2), (4, -2)\}$ and $B' = \{(-1, 2), (2, -2)\}$.
be bases for R^2, and let $A = \begin{bmatrix} -2 & 7 \\ -3 & 7 \end{bmatrix}$ be the matrix for
T: $R^2 \to R^2$ relative to B. Find A'.

Sol: In Example 5 in Section 4.7,

$$P = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix},\ P^{-1} = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$$

The matrix of T relative to B' is given by

$$A' = P^{-1}AP = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$$

Example 3

For the linear transformation T: $R^2 \to R^2$ given in Example 2,
find $[\mathbf{v}]_B$, $[T(\mathbf{v})]_B$, and $[T(\mathbf{v})]_{B'}$ for the vector **v** whose coordinate matrix $[\mathbf{v}]_{B'} = \begin{bmatrix} -3 & -1 \end{bmatrix}^{\mathrm{T}}$

Sol:

$$[\mathbf{v}]_B = P[\mathbf{v}]_{B'} = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} -7 \\ -5 \end{bmatrix}$$

$$[T(\mathbf{v})]_B = A[\mathbf{v}]_B = \begin{bmatrix} -2 & 7 \\ -3 & 7 \end{bmatrix}\begin{bmatrix} -7 \\ -5 \end{bmatrix} = \begin{bmatrix} -21 \\ -14 \end{bmatrix}$$

$$[T(\mathbf{v})]_{B'} = P^{-1}[T(\mathbf{v})]_B = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} -21 \\ -14 \end{bmatrix} = \begin{bmatrix} -7 \\ 0 \end{bmatrix}$$

$$= A'[\mathbf{v}]_{B'}$$

Similar Matrices

[Definition] For square matrices A and A' of order n, A' is said to be ***similar*** to A if there exists an invertible matrix P such that $A' = P^{-1}AP$.

[Theorem 6.13] Let A, B, and C be square matrices of order n, Then the following properties are true.

1. A is similar to A.
2. If A is similar to B, then B is similar to A.
3. If A is similar to B and B is similar to C, then A is similar to C.

Example 5

Suppose $A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ is the matrix for $T: R^3 \to R^3$ relative the standard basis. Find the matrix for T relative to the basis $B' = \{(1, 1, 0), (1, -1, 0), (0, 0, 1)\}$

Sol: The transition matrix from B' to the standard matrix is

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow A' = P^{-1}AP = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

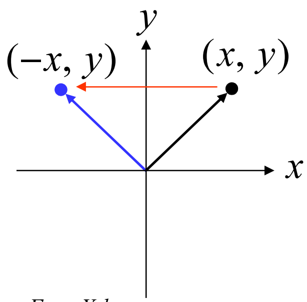
6.5 Applications of Linear Transformation

The geometry of linear transformations in the plane

Reflection in the y-axis

$T(x, y) = (-x, y)$

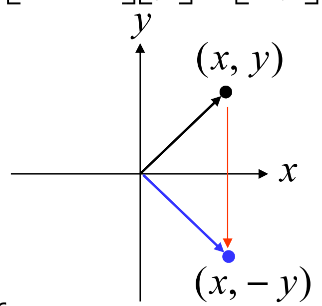
$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$



Reflection in the x-axis

$T(x, y) = (x, -y)$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$

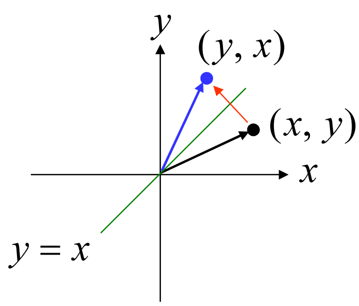


Reflection in the Plane

Reflection in the line $y = x$

$T(x, y) = (y, x)$

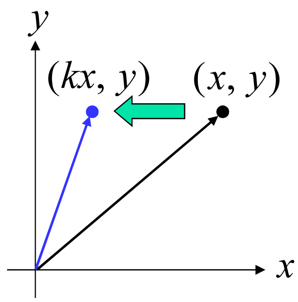
$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$



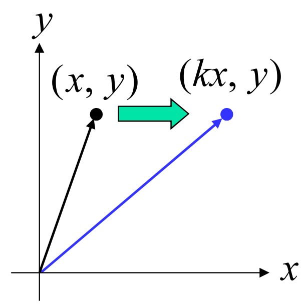
Expansions & Contractions in the Plane -- Horizontal

$$T(x, y) = (kx, y)$$

$$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ y \end{bmatrix}$$



Contraction: $0 < k < 1$

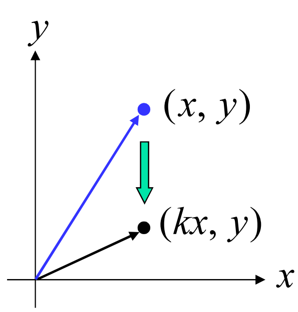


Expansion: $k > 1$

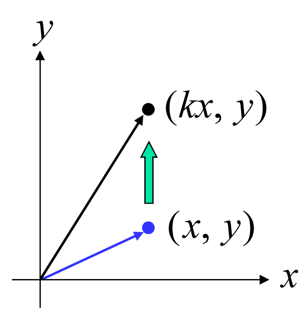
Expansions & Contractions in the Plane -- Vertical

$$T(x, y) = (x, ky)$$

$$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ ky \end{bmatrix}$$



Contraction: $0 < k < 1$



Expansion: $k > 1$

Shears in the Plane

Horizontal shear:

$$T(x, y) = (x + ky, y)$$

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + ky \\ y \end{bmatrix}$$

Vertical shear:

$$T(x, y) = (x, y + kx)$$

$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ kx + y \end{bmatrix}$$