System Modeling

Definitions

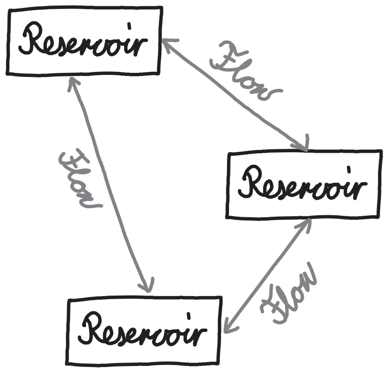
relevant | not relevant

- System Modeling (grey/white Box) vs. System Identification (black Box)
- Parametric Model (ODE, PDE, transfer func., ...) vs. Non-Parametric Model (lists, tables, ...)
- Forward Formulation (cause → effect) vs. Backward Formulation (effect → cause)
- Relevant Dynamics (need model) vs. Fast Dynamics (algebraic) vs. Slow Dynamics (constant)

Reservoir-Based Approach

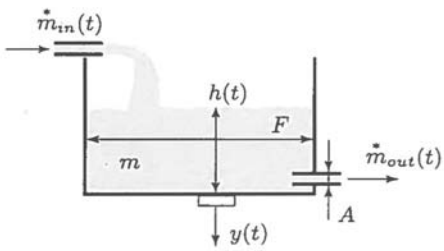
- Reservoir: thermal / kinetic energy, mass, information, ... (tracked by state variable)
- Flows: heat, mass, etc. flowing between reservoirs (typically driven by reservoir difference)

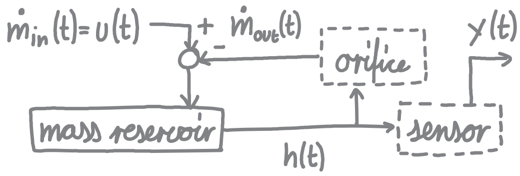
GUIDELINES:



1. define system boundaries (inputs, outputs)
2. identify relevant reservoirs (state variable)
3. formulate conservation law:
   d/dt (reservoir content) $= \Sigma$(inflows) $- \Sigma$(outflows)
4. formulate algebraic relations expressing flows between reservoirs
5. simplify as much as possible
6. identify unknown system parameters with experiments
7. use other experiments to validate model

Reservoir-Based Approach Example





1. system input: $\dot{m}_{in}(t)$; system output: $h(t)$
2. system reservoir: $m(t)$
3. mass balance: $d/dt\ m(t) = \dot{m}_{in}(t) - \dot{m}_{out}(t)$
4. relations: $\dot{m}_{out}(t) = A\rho\sqrt{2gh(t)}$ (Bernoulli)
   $m(t) = \rho F\, h(t)$
5. simplify: $\rho F \, d/dt\, h(t) = \dot{m}_{in}(t) - A\rho\sqrt{2gh(t)}$

Basic Modeling Elements

Mechanical Systems (2D, reservoir: energy, 1DOF)

$$T_t(t) = \frac{1}{2} m \left(v_{x,cg}^2(t) + v_{y,cg}^2(t)\right) \quad \text{(translational energy)}$$

$$T_r(t) = \frac{1}{2} \Theta\, \omega^2(t) \quad \text{(rotational energy)}$$

$$U(t) = U(x(t), y(t)) = mgx \quad \text{(potential energy)}$$

$$\rightarrow E(t) = T_t(t) + T_r(t) + U(t) \quad \text{(total energy)}$$

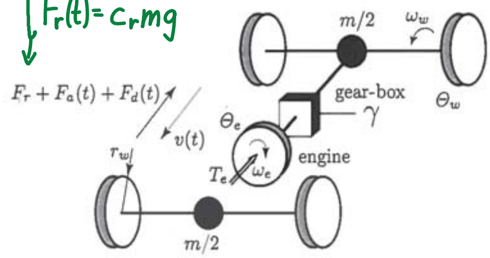
$$\frac{d}{dt} E(t) = \underbrace{\sum_i P_i(t)}_{\text{mech. powers acting on body}} \quad \text{(conservation law)}$$

Example 1:

input: $T_e(t)$, output: $y(t) \sim v(t)$, state: $v(t)$

$$F_a(t) = \frac{1}{2} \rho_a c_w A v^2(t)$$

$$F_r(t) = c_r m g$$



$$E_{tot} = \frac{1}{2} m v^2(t) + 4 \frac{1}{2} \Theta_w \omega_w^2(t) + \frac{1}{2} \Theta_e \omega_e^2(t)$$

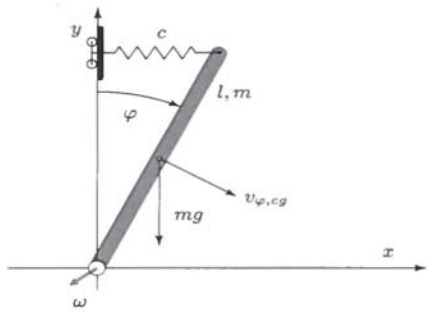
$$= \frac{1}{2}\left(m + \frac{4\Theta_w}{r_w^2} + \frac{\Theta_e}{\gamma^2 r_w^2}\right) v^2(t) \qquad \left(\begin{aligned} v(t) &= r_w \omega_w(t) \\ &= r_w \gamma\, \omega_e(t) \end{aligned}\right)$$

$$\frac{d}{dt} E_{tot}(t) = T_e(t)\omega_e(t) - F_r v(t) - F_a(t) v(t) - F_d(t) v(t)$$

$$\hookrightarrow \frac{d}{dt} v(t) = \text{func}(T_e(t), v(t))$$

Example 2:

input: none, output: $y(t) \sim \varphi(t)$, state: $\varphi(t), \dot{\varphi}(t)$



$$E_{tot}(t) = \underbrace{\frac{1}{2} m \left(\frac{L\dot{\varphi}(t)}{2}\right)^2}_{T_{trans}} + \underbrace{\frac{1}{2} \frac{mL^2}{12} \dot{\varphi}^2(t)}_{T_{rot}} + \underbrace{mg\frac{L}{2}\cos(\varphi(t))}_{U_{grav}} + \underbrace{\frac{1}{2} c \left(L \sin(\varphi(t))\right)^2}_{U_{spring}}$$

$$\frac{d}{dt} E_{tot}(t) = 0 \quad \rightarrow \quad \frac{1}{3} m L^2 \ddot{\varphi}(t) = \left(\frac{L}{2} mg - cL^2 \cos(\varphi(t))\right) \sin(\varphi(t))$$

Example 3:

input: T_1, T_2 output: ω_1

state: $\omega_1, \omega_2, \varphi$

reservoir: compressor, turbine, shaft (mech. energy)

$$\frac{d}{dt}\left(\frac{1}{2}\Theta_1 \omega_1^2(t)\right) = -P_1(t) - P_2(t) + P_3(t) \quad \text{(compressor balance)}$$

$$\frac{d}{dt}\left(\frac{1}{2}\Theta_2 \omega_2^2(t)\right) = -P_4(t) - P_5(t) + P_6(t) \quad \text{(turbine balance)}$$

$$\frac{d}{dt}\left(\frac{1}{2} k \varphi^2(t)\right) = -P_3(t) + P_4(t) \quad \text{(shaft balance)}$$

$P_1 = T_1 \omega_1$ (input) | $P_4 = \omega_2 k \varphi$ (spring)

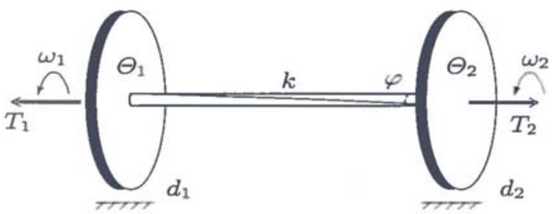
$P_2 = d_1 \omega_1^2$ (fric. loss) | $P_5 = d_2 \omega_2^2$ (fric. loss)

$P_3 = \omega_1 k \varphi$ (spring) | $P_6 = T_2 \omega_2$ (input)

$$\Theta_1 \frac{d}{dt} \omega_1(t) = -T_1(t) - d_1 \omega_1(t) + k\varphi(t)$$

$$\Theta_2 \frac{d}{dt} \omega_2(t) = T_2(t) - d_2 \omega_2(t) - k\varphi(t)$$

$$\frac{d}{dt} \varphi(t) = \omega_2(t) - \omega_1(t)$$



Mechanical Systems (2D, reservoir: energy, DOF = n) (Lagrange Method)

1) Define inputs and outputs $u(t), y(t)$
2) Define generalized coordinates $q(t) = (q_1(t), q_2(t), \dots, q_n(t))$, $\dot{q} = (\dots)$
3) Build the Lagrange function $L(q,\dot{q}) = \sum_{i=0}^{m} T_i(q,\dot{q}) - U_i(q)$
4) Build n system dynamics equations $\frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{q}_k}\right\} - \frac{\partial L}{\partial q_k} = Q_k \quad (k=1,\dots,n)$

4+) If system has constraints: replace 4 $\frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{q}_k}\right\} - \frac{\partial L}{\partial q_k} - \sum_{j=1}^{v} \mu_j \alpha_{j,k} = Q_k$

$$\sum_{k=1}^{n} \alpha_{j,k} \cdot \dot{q}_k(t) = 0 \leftarrow (j=1,\dots,v) \uparrow (k=1,\dots,n)$$

5) Solve equation system for $\ddot{q}_k$ $(k=1,\dots,n)$ and μ_j $(j=1,\dots,v)$

→ or inertia Matrix form: $M(q)\,\ddot{q} = f(q,\dot{q},Q)$

n: # DOF
m: # Bodies
q_k: angle or distance coordinate
Q_k: force or torque on q_k
v: # constraints
μ: Lagrange multiplier
α: constants

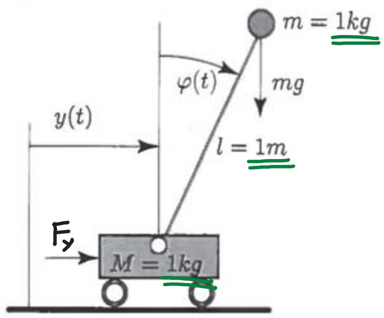
Example 1:

$m=2$, $n=2$; $q_1 = y$, $q_2 = \varphi$; input: F_y output: ?

$$L(q,\dot{q}) = \underbrace{\tfrac{1}{2}\dot{y}^2(t)}_{T_1(t)} - \underbrace{0}_{U_1(t)} + \underbrace{\tfrac{1}{2}\dot{y}^2(t) + \cos(\varphi(t))\dot{\varphi}(t)\dot{y}(t) + \tfrac{1}{2}\dot{\varphi}^2(t)}_{T_2(t)} \underbrace{- \cos(\varphi(t))g}_{U_2(t)}$$

$$\to 2\ddot{y}(t) + \ddot{\varphi}(t)\cos(\varphi(t)) - \dot{\varphi}^2(t)\sin(\varphi(t)) = F_y(t) \quad \to \ddot{y}(t) = \dots$$

$$\to \ddot{\varphi}(t) + \ddot{y}(t)\cos(\varphi(t)) - g\sin(\varphi(t)) = 0 \quad \to \ddot{\varphi}(t) = \dots$$



Example 2:

$n=3$, $m=2$, $v=1$; $q_1 = \psi$, $q_2 = \chi$, $q_3 = \varphi$; input: $u = T$, output: $y = (R+r)\sin(\chi)$

$$L(q,\dot{q}) = T_{tot}(q,\dot{q}) - U_{tot}(q,\dot{q}) = \tfrac{1}{2}m(R+r)^2\dot{\chi}^2(t) + \tfrac{1}{2}\vartheta\dot{\varphi}^2(t) + \tfrac{1}{2}\Theta\dot{\psi}^2(t) - (-mg(1-\cos(\chi(t)))\cdot(R+r))$$

$Q_1 = u(t)$, $Q_2 = Q_3 = 0$ ($u(t)$: input torque)

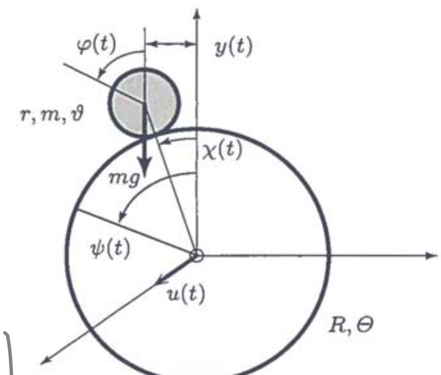
$\underbrace{R}_{\alpha_1}\dot{\psi}(t) \underbrace{-(R+r)}_{\alpha_2}\dot{\chi}(t) + \underbrace{r}_{\alpha_3}\dot{\varphi}(t) = 0$ (no-slip condition)

$\alpha_1\dot{q}_1 + \alpha_2\dot{q}_2 + \alpha_3\dot{q}_3 = 0 \to \alpha_1 = R, \alpha_2 = -R-r, \alpha_3 = r$

→ system equations: (for $\ddot{\psi}, \ddot{\chi}, \ddot{\varphi}, \mu$)

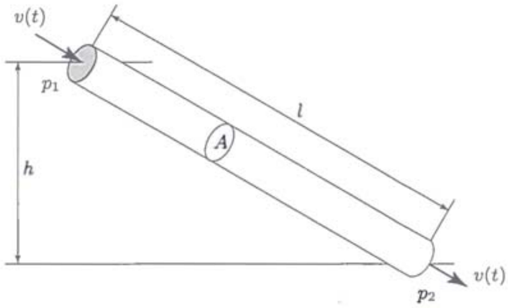
→ simplify: (eliminate $\ddot{\varphi}, \mu$)

$$\begin{pmatrix} \Theta + \vartheta\frac{R^2}{r^2} & -\vartheta\frac{R(R+r)}{r^2} \\ -\vartheta\frac{R(R+r)}{r^2} & m(R+r)^2 + \vartheta\frac{(R+r)^2}{r^2} \end{pmatrix} \begin{pmatrix} \ddot{\psi} \\ \ddot{\chi} \end{pmatrix} = \begin{pmatrix} u \\ mg(R+r)\sin(\chi) \end{pmatrix}$$



Hydraulic Systems

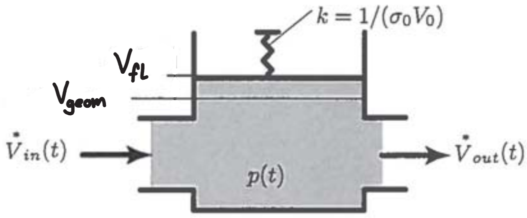
duct element: (momentum conservation)



$$\underbrace{\rho l A}_{m} \, d/dt \, v(t) = A\left(p_1(t) - p_2(t)\right) + A \rho g h - F_f(t)$$

$$F_f(t) = A \cdot \underbrace{\lambda(v(t))}_{\lambda:\ \text{from Moody-Diagram}} \cdot \frac{l}{d} \frac{\rho}{2} \, \text{sign}(v(t)) \cdot v^2(t)$$

compress. tank: (mass conservation)



$$d/dt \, m(t) = \dot{m}_{in}(t) - \dot{m}_{out}(t)$$

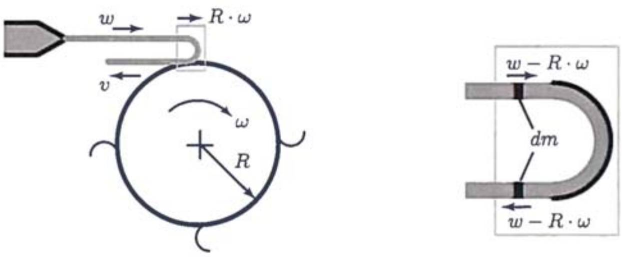
$\cdot \frac{1}{\rho}$

$$d/dt \, V_{fl}(t) = \dot{V}_{in}(t) - \dot{V}_{out}(t) \quad \left(\dot{V} = \frac{\dot{m}}{\rho},\ V_{fl} \neq V_{geom}\right)$$

$$\hookrightarrow p(t) = \frac{1}{\sigma_0} \frac{V_{fl}(t) - V_{geom}}{V_{geom}} + p_{stat,geom}$$

(σ_0: compressibility)

pelton turbine: (momentum conservation)



$$d(mv) = 2(w - R\omega)\,dm = 2(w - R\omega)\,\dot{V}\rho\,dt$$

$$\hookrightarrow T_T = \frac{d(mv)}{dt} R = 2\rho R (w - R\omega)\dot{V}$$

$\hookrightarrow$ ($T_{T\,max}$ for $R = \frac{1}{2}\, w/\omega$)

Fluiddynamic Systems

- **Valves:**

$$\dot m(t) = c_d A \sqrt{2\rho\,(p_{in}(t) - p_{out}(t))} \quad \text{(incompressible / Bernoulli)}$$

$$\dot m(t) = c_d A \, \frac{p_{in}(t)}{\sqrt{R T_{in}(t)}} \cdot \Psi\left(\Pi = \frac{p_{out}(t)}{p_{in}(t)}\right) \quad \text{(compressible / isenthalpic} \rightarrow T_{in} = T_{out}\text{)}$$

real:

$$\Psi = \begin{cases} \sqrt{\left(\frac{2\kappa}{\kappa+1}\right)^{\wedge}\left(\frac{\kappa+1}{\kappa-1}\right)} & \text{for } \Pi < \Pi_{cr} \\ \Pi^{1/\kappa} \cdot \sqrt{\frac{2\kappa}{\kappa-1} \cdot \left(1 - \Pi^{\wedge}\left(\frac{\kappa-1}{\kappa}\right)\right)} & \text{for } \Pi \geq \Pi_{cr} \end{cases}$$

$$\Pi_{cr} = \left(\frac{2}{\kappa+1}\right)^{\wedge}\left(\frac{\kappa}{\kappa-1}\right)$$

aprox.:

$$\Psi = \begin{cases} 1/\sqrt{2} & \text{for } \Pi < 1/2 \\ \sqrt{2\Pi \cdot (1-\Pi)} & \text{for } \Pi_{tr} > \Pi \geq 1/2 \\ a \cdot (\Pi-1)^3 + b(\Pi-1) & \text{for } \Pi > \Pi_{tr} \end{cases}$$

$$a = \frac{\Psi'(\Pi_{tr}) \cdot (\Pi_{tr}-1) - \Psi(\Pi_{tr})}{2(\Pi_{tr}-1)^3} \; ; \; b = \Psi'(\Pi) - 3a(\Pi_{tr}-1)^2$$

- **Gas Turbines:**

$p_3 = p_{in},\ p_4 = p_{out},\ T_3 = T_{in},\ \omega_t \rightarrow T_4 = T_{out},\ \dot m_t,\ T_{t,torq}$

$$T_4 = T_3\left(1 - \eta_t\left(1 - \Pi_t^{\wedge}\left(\frac{1-\kappa}{\kappa}\right)\right)\right) \qquad \Pi_t = p_3/p_4$$

$$\dot m_t = \dot\mu_t \, \frac{p_3/p_{3ref}}{\sqrt{T_3/T_{3ref}}} \qquad \eta_t = \text{table}(c_u) \leftarrow c_u = \frac{r_t\,\omega_t}{c_{us}} \leftarrow c_{us} = \sqrt{2 c_p T_3 \left(1 - \Pi_t^{\wedge}\left(\frac{1-\kappa}{\kappa}\right)\right)}$$

$$T_{t,torq} = \frac{\eta_t \dot m_t c_p T_3}{\omega_t}\left(1 - \Pi_t^{\wedge}\left(\frac{1-\kappa}{\kappa}\right)\right) \qquad \dot\mu_t = \text{table}(\Pi_t, \text{nozle\_pos}, \omega_t)$$

- **Gas Compressor:**

$p_1 = p_{in},\ p_2 = p_{out},\ T_1 = T_{in},\ \omega_c \rightarrow T_2 = T_{out},\ \dot m_c,\ T_{c,torq}$

$$T_2 = T_1\left(1 - \frac{1}{\eta_c}\left(\Pi_c^{\wedge}\left(\frac{1-\kappa}{\kappa}\right) - 1\right)\right) \qquad \Pi_c = p_2/p_1$$

$$\dot m_c = \dot\mu_c \, \frac{p_1/p_{1ref}}{\sqrt{T_1/T_{1ref}}} \qquad \eta_c = \text{table}(\Pi_c, \omega_c)$$

$$T_{c,torq} = \frac{\dot m_c c_p T_1}{\eta_c \omega_t}\left(\Pi_c^{\wedge}\left(\frac{1-\kappa}{\kappa}\right) - 1\right) \qquad \dot\mu_c = \text{table}(\Pi_c, \omega_c)$$

Thermodynamic systems (reservoir: internal energy U)

$$\frac{d}{dt} U(t) = \dot{Q} - \dot{W} + \dot{H}_{in} - \dot{H}_{out}$$

$$U(T) = m \cdot \int_0^T c_v(T)\,dT = c_v \cdot m \cdot T \quad \text{(for liquid/solid, } c = c_p = c_v\text{)}$$

$$\dot{H}(t) = \dot{Q}(t)_{isobar} = c_p \cdot \dot{m} \cdot T \quad \text{(for ideal gass, } R = c_p - c_v,\ p = \varrho R T\text{)}$$

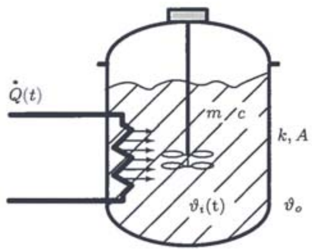
$$pV = nRT$$

- conduction: $\dot{Q} = \kappa A / l \, (T_1 - T_2)$
- convection: $\dot{Q} = kA\,(T_1 - T_2)$
- radiation: $\dot{Q} = \varepsilon \sigma A\,(T_1^4 - T_2^4)$

<u>Example 1:</u>

input: $\dot{Q}_{in}(t)$ reservoir: $U(t)$ level: $T(t) = T_i(t) - T_0$

$$\frac{d}{dt} U(t) = mc \frac{d}{dt} T(t) = \dot{Q}_{in}(t) - \underbrace{kAT(t)}_{\dot{Q}_{out}(t)}$$



<u>Example 3:</u>

inputs: $\dot{m}_{in/out}$, $\dot{H}_{in/out}$, reservoirs: $m(t), U(t)$

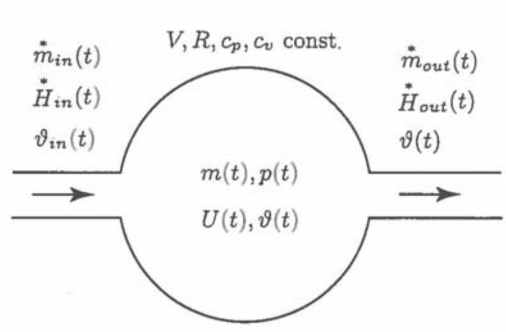
$$\frac{d}{dt} U(t) = \dot{H}_{in}(t) - \dot{H}_{out}(t)$$

$$\frac{d}{dt} m(t) = \dot{m}_{in}(t) - \dot{m}_{out}(t)$$

using $U = c_v m T$, $\dot{H} = c_p \dot{m} T$, $m = pV/RT$ and combining prev. eq.:

$$\frac{d}{dt} T = \frac{TR}{pVc_v} \left(c_p \dot{m}_{in} T_{in} - c_p \dot{m}_{out} T - (\dot{m}_{in} - \dot{m}_{out}) c_v T \right)$$

$$\frac{d}{dt} p = \frac{c_p R}{c_v V} \left(\dot{m}_{in} T_{in} - \dot{m}_{out} T \right)$$



Chemical systems

$\alpha A + \beta B \leftrightarrow \gamma C + \delta D$ ($[\cdot] = \frac{mol}{m^3}$) ($k$: factor E: activation E)

$d/dt\,[A] = \alpha \cdot (r^- \cdot [C]^\gamma \cdot [D]^\delta - r^+ \cdot [A]^\alpha \cdot [B]^\beta) + \dot m_A \cdot \frac{1}{VM_A} + \ldots$

$d/dt\,[C] = \gamma (r^+ [A]^\alpha [B]^\beta - r^- [C]^\gamma [D]^\delta) + \dot m_C \cdot \frac{1}{VM_C} + \ldots$

$\begin{cases} r^+ = k^+(T,p,\ldots) \cdot e^{-E^+/(RT)} \\ r^- = k^-(T,p,\ldots) \cdot e^{-E^-/(RT)} \end{cases}$ if $\rightarrow$ then $r^+ = 0$; if $\leftarrow$ then $r^- = 0$

$d/dt\,U = Q - \dot W + H_A V \frac{d}{dt}[A] + H_B V \frac{d}{dt}[B] + H_C V \frac{d}{dt}[C] + H_D V \frac{d}{dt}[D] + \underbrace{c_p T_{in}}_{H} \dot m_{in} - \underbrace{c_p T_{out}}_{H} \dot m_{out}$

Example 1:

(assumptions: reaction $A+B \to C$, $[B]=$ const., no dissociation $C \to A+B$, ρ & m const.)

(input: $\dot Q$, output: $[C], T$, disturb: $[A_i], T_i$, reservoirs: n_A, n_C, U)

$d/dt\, n_A(t) = \dot V \cdot [A_i](t) - \dot V [A](t) - V k^- e^{-E/(RT)} \cdot [A](t) \cdot [B]$

$d/dt\, n_C(t) = -\dot V [C](t) + V k^- e^{-E/(RT)} [A](t) \cdot [B]$

$d/dt\, U(T, n_A, n_B, n_C) = \dot H_i(T_i(t)) - \dot H(T(t)) + \dot Q(t)$ ($\dot H_i(T_i) = \dot m \cdot c_p \cdot T_i$; $\dot H(t) = \dot m \cdot c_p \cdot T$)

$dU(T, n_A, n_B, n_C) = \frac{\partial U}{\partial T} dT + \frac{\partial U}{\partial n_A} dn_A + \frac{\partial U}{\partial n_B} dn_B + \frac{\partial U}{\partial n_C} dn_C$

$= \rho V c_v\, dT + H_A\, dn_A + H_B\, dn_B + H_C\, dn_C$ ($H_{A,B,C}$: enthalpy of formation)

$\tau\, d/dt\,[A](t) = [A_i](t) - (1 + \tau k e^{-E/(RT(t))}) \cdot [A](t)$

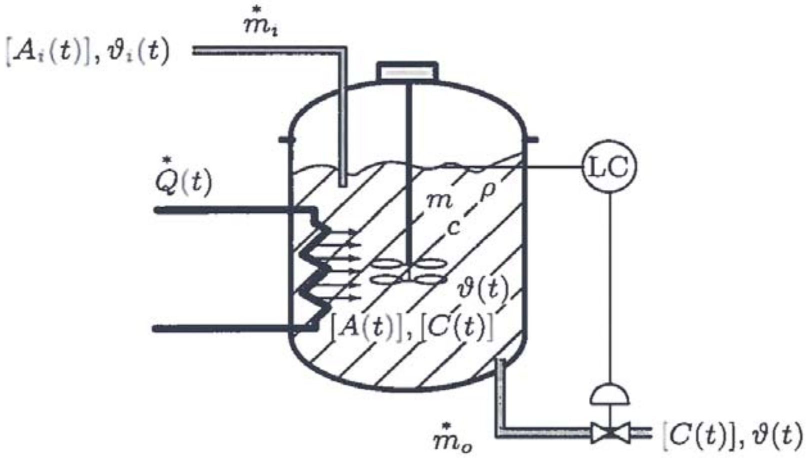
$\tau\, d/dt\,[C(t)] = -[C](t) + \tau k e^{-E/(RT(t))} \cdot [A](t)$

$\tau\, d/dt\, T(t) = T_i(t) - T(t) + \frac{1}{\rho c_v} \cdot \frac{\dot Q(t)}{\dot V} + \tau H_0 \frac{k}{\rho c_v} e^{-E/(RT(t))} \cdot [A](t)$

$\begin{pmatrix} \tau = V/\dot V \\ k = k^- \cdot [B] \\ H_0 = H_A + H_B - H_C \end{pmatrix}$

steady state ($\dot Q = 0$, $d/dt[\cdot] = 0$):

$\dot H_{flow} + \dot Q_{chem} = \dot m c_p (T_i - T) + H_0 \frac{V k e^{\wedge}(-E/(RT))}{1 + \tau k e^{\wedge}(E/(RT))} = 0$

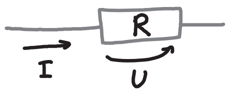


Distributed Parameter Systems (missing!)

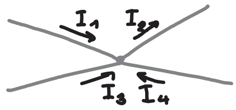
Electromagnetic Systems (RLC Networks)

	capacitance C	inductance L	resistance R
energy	$W_E = \frac{1}{2} C U(t)^2$	$W_M = \frac{1}{2} L I(t)^2$	$W = 0$
level variable	$U(t)$	$I(t)$	–
conservation law	$C \, d/dt \, U(t) = I(t)$	$L \, d/dt \, I(t) = U(t)$	–

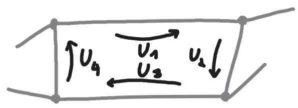
Ohms law: $U = R \cdot I$



1st Kirchoff's law: $\sum_i I_i = 0$

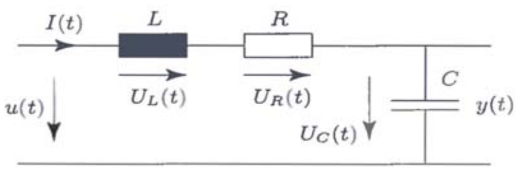


2nd Kirchoffs law: $\sum_i U_i = 0$



<u>Example:</u>

input: $u(t)$, output: $U_c(t)$, reservoirs: W_E & W_M



$$\left.\begin{array}{l} U_L(t) = L \cdot d/dt \, I(t) \\ I(t) = C \cdot d/dt \, U_c(t) \\ U_R(t) = R \cdot I(t) \end{array}\right\} \rightarrow \left\{\begin{array}{l} U_L(t) = L \cdot C \cdot d^2/dt^2 \, U_c(t) \\ U_c(t) = U_c(t) \\ U_R(t) = R \cdot C \cdot d/dt \, U_c(t) \end{array}\right.$$

$$U_L(t) + U_R(t) + U_c(t) = u(t)$$

$$\rightarrow LC \, d^2/dt^2 \, U_c(t) + RC \, d/dt \, U_c(t) + U_c(t) = u(t)$$

Electromechanical systems

Lorenz law:

$$\underline{F} = q\underline{E} + q\underline{v}\times\underline{B} = I\underline{L}\times\underline{B}$$

Faraday law:

$$U = -\tfrac{d}{dt}\, B\cdot A = -v\cdot(\underline{L}\times\underline{B})$$

Example 1:

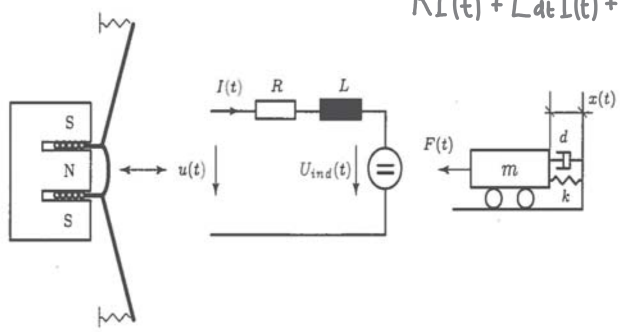
$F(t) = Bnd\pi\cdot I(t) = \kappa\cdot I(t)$ (motor law)

$U_{ind}(t) = Bnd\pi\cdot \dot{x}(t) = \kappa\cdot\dot{x}(t)$ (generator law)

$F(t) = m\ddot{x}(t) + d\dot{x}(t) + kx(t)$ (membrane dynamics)

$u(t) = U_R(t) + U_L(t) + U_{ind}(t)$ (RLC dynamics)

$\quad\;\; = RI(t) + L\tfrac{d}{dt}I(t) + U_{ind}(t)$



Example 2:

input: $u(t), T_L(t)$ output: $\omega(t)$ reservoirs: W_{ROT}, W_M

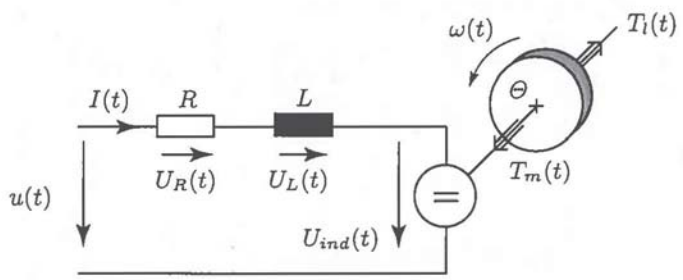
$L\cdot \tfrac{d}{dt} I(t) = u(t) - RI(t) - U_{ind}(t)$ (W_m conservation)

$\Theta\, \tfrac{d}{dt}\omega(t) = T_m(t) - T_L(t) - \underbrace{d\cdot\omega(t)}_{\text{loss}}$ (W_{ROT} conservation)

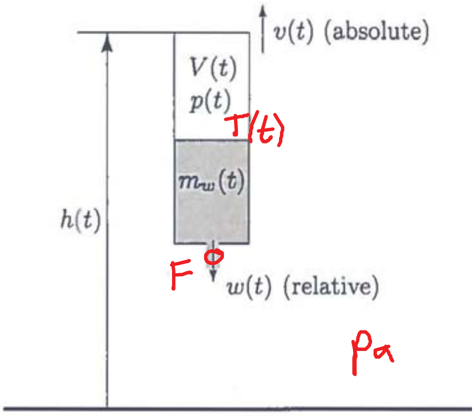
$T_m(t) = Bnd^2\tfrac{\pi}{2}\cdot I(t) = \kappa\cdot I(t)$ (motor law)

$U_{ind}(t) = Bnd^2\tfrac{\pi}{2}\cdot\omega(t) = \kappa\cdot\omega(t)$ (generator law)

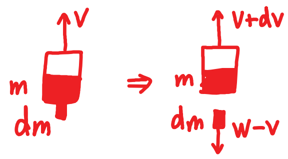
$$\left\{\begin{array}{l} L\,\tfrac{d}{dt} I(t) = u(t) - RI(t) - \kappa\omega(t) \\ \Theta\,\tfrac{d}{dt}\omega(t) = \kappa I(t) - T_L(t) - d\,\omega(t) \end{array}\right.$$



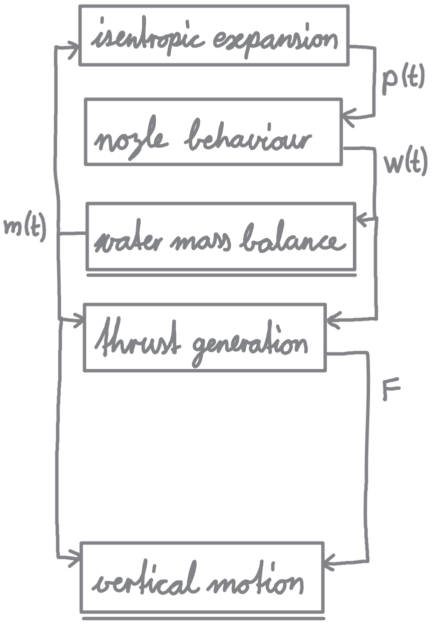
Case Study: Rocket



(assumptions: no drag, isentropic air expansion
air mass neglected)



• Water Thrust Phase



$p(t) = \left({}^{V(0)}/_{V_L - m_w(t)/\rho} \right)^{\kappa} \cdot p(0)$ (isentropic relation)

$w(t) = \sqrt{{}^{2}/_{\rho}\,(p(t) - p_a)}$ (bernoulli eq.)

${}^{d}/_{dt}\, m(t) = \rho F \cdot w(t)$ (mass balance)

$dP = m(v+dv) - dm(w-v) - (m+dm)v = m\,dv - dm\,w$

$dP = -gm\,dt$ $\quad dm = \rho F w\,dt$

$\hookrightarrow m(t)\, {}^{d}/_{dt}\, v(t) = F(t) = -g\,m(t) + \rho F w^2(t)$ (momentum balance)

${}^{d}/_{dt}\, v(t) = F(t)/m(t)$; ${}^{d}/_{dt}\, h(t) = v(t)$

• Air Thrust Phase

$p(t) = m_{air}(t)\, R\, T(t) / V_L$ $\leftarrow$ ${}^{d}/_{dt}\, U(t) = c_v\, m_{air}(t)\, {}^{d}/_{dt}\, T(t) = \dot{H}_{out} = -\dot{m}_{out}(t)\, c_p\, T(t)$

$T(t_1) = p(t_1) \cdot V_L / R\, m_{air}(t_1)$

$m_{air}(t_1) = m_{air}(0) = \frac{p(0)\, V(0)}{R\, T(0)}$

w(t) = see valve formula for gas.

rest is the same as for water phase!

• Balistic Phase

$F(t) = -g\,m(t)$; $m(t) = m(t_2)$; only "vertical motion" block needed.

Model Parametrization

considerations:
- linear or nonlinear system?
- frequency content of exitation?
- noise at input and output?
- safety issues?

data purpose:
- identify system and parameters
- validate result of modeling
  ↳ don't use same dataset!!

Least Square Method for linear Systems

$$y_k = h^T(u_k)\cdot\pi + e_k$$

$k\in[1,\dots,r]$: index of measurement
$u_k\in\mathbb{R}^m$: k-th input vector
$y_k\in\mathbb{R}$: k-th output measurement
$e_k\in\mathbb{R}$: k-th measurement error
$h(u_k)=h_k\in\mathbb{R}^q$: regressor function. nonlinear
$\pi\in\mathbb{R}^q$: parameters of system. $q\ll r$

$\tilde{y}=[y_1,\dots,y_r]^T \in\mathbb{R}^r$
$\tilde{e}=[e_1,\dots,e_r]^T \in\mathbb{R}^r$
$H=[h_1,\dots,h_r]^T \in\mathbb{R}^{r\times q}$

$\tilde{e}=\tilde{y}-H\cdot\pi$
↳ minimize: $\epsilon=\tilde{e}^T W\cdot\tilde{e}$
↳ $\pi_{LS}=[H^TWH]^{-1}H^TW\tilde{y}$

($W=I$ or $W=\mathrm{diag}\{w_i\}$ if not all measurements reliable)

- **Iterative Solution** $W=I \rightarrow \pi_{LS}(r)=\left(\sum_{k=1}^{r}h_k\cdot h_k^T\right)^{-1}\cdot\sum_{k=1}^{r}h_k\cdot y_k$

$$\pi_{LS(r+1)}=\pi_{LS(r)}+\underbrace{\delta_{r+1}}_{\text{correction direction}}\cdot\underbrace{(y_{r+1}-h^T_{r+1}\cdot\pi_{LS(r)})}_{\text{prediction error}}$$

(new π_{LS} after adding one measurement)

$\pi_{LS(0)}=$ estim.

$\Omega_0=$??

$\delta_{r+1}=1/(1+c_{r+1})\cdot\Omega_r\cdot h_{r+1}$

$\Omega_{r+1}=\Omega_r-1/(1+c_{r+1})\cdot\Omega_r\cdot h_{r+1}\cdot h^T_{r+1}\cdot\Omega_r$ $\left(\Omega_r=\left[\sum_{k=1}^{r}h_k\cdot h_k^T\right]^{-1}\right)$

$c_{r+1}=h^T_{r+1}\cdot\Omega_r\cdot h_{r+1}$

- **Exponential Forgetting**

$\epsilon_{(r)}=\sum_{k=1}^{r}\lambda^{r-k}\cdot(y_k-h_k^T\cdot\pi_{LS(k)})^2$ $(\lambda<1)$ (old errors get less important than new)

↳ $\delta_{r+1}=1/(\lambda+c_{r+1})\cdot\Omega_r h_{r+1}$
↳ $\Omega_{r+1}=\frac{1}{\lambda}\Omega_r(I-1/(\lambda+c_{r+1})h_{r+1}h^T_{r+1}\cdot\Omega_r$

(rest is the same as iterative LS)

- **Simplified recursive LS**

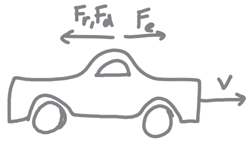
$\delta_{r+1}=(\gamma\cdot h_{r+1})/(\lambda+h^T_{r+1}h_{r+1})$ ($0<\gamma<2$: convergence, $0<\lambda<1$: forgetting) (rest is the same as iterative LS)

Example 1

output $y=\dot{v}$, input $u=F/m$, state $x=v$, parameter $\pi_1=-k_0/m$; $\pi_2=-k_1/m$

model: $m\cdot \frac{d}{dt} v(t) = -(k_0+k_1 v(t)^2)+F(t)$

↳ $\dot{x}(t)=\pi_1+\pi_2\cdot x(t)^2+u(t)$; $y(t)=x(t)$



experiment: $u(t)=0\ \forall t$, set $x(0)=?$

$$\begin{pmatrix}\dot{x}_1\\ \vdots\\ \dot{x}_r\end{pmatrix}=\begin{pmatrix}1 & x_1\\ \vdots & \vdots\\ 1 & x_r\end{pmatrix}\cdot\begin{pmatrix}\pi_1\\ \pi_2\end{pmatrix}=H\cdot\pi \rightarrow \pi_{LS}=[H^TWH]^{-1}\cdot H^TW\dot{x} \qquad (W=I)$$

Nonlinear LS Methods (SISO)

$\frac{d}{dt}\hat{x}(t)=f(\hat{x}(t),u(t),\hat{\pi})$ $\quad \hat{y}(t)=h(\hat{x}(t),u(t),\hat{\pi})$ $\quad$ ($\hat{x}\in\mathbb{R}^n$ state, $u\in\mathbb{R}$ input, $\hat{y}\in\mathbb{R}$ output)

$k\in[1,\dots,r]$ measurements, $t_i=[t_1,\dots,t_r]^T$ time, $u_i=u(t_i)$ etc.

minimize: $\epsilon=\sum_{i=1}^{r}\underset{\text{weight}}{g_i}\cdot(\underset{\text{real}}{y_i(\pi)}-\underset{\text{model}}{\hat{y}_i(\hat{\pi})})^2$ → use comp methods!

Analysis of linear systems

non linear : $\dot{\underline{z}}(t) = \underline{f}(\underline{z}(t), \underline{v}(t), t)$, $\underline{w}(t) = \underline{g}(\underline{z}(t), \underline{v}(t), t)$

linear : $\dot{\underline{x}}(t) = \underline{\underline{A}} \cdot \underline{x}(t) + \underline{\underline{B}}\, \underline{u}(t)$, $\underline{y}(t) = \underline{\underline{C}}\, \underline{x}(t) + \underline{\underline{D}}\, \underline{u}(t)$

state $z, x \in \mathbb{R}^n$
input $v, u \in \mathbb{R}^m$
output $w, y \in \mathbb{R}^p$

Normalization

$\underline{x}(t) = \underline{z}(t)/z_0 \rightarrow \dot{\underline{x}}(t) = \dot{\underline{z}}(t)/z_0$, $\underline{u}(t) = \underline{v}(t)/\underline{v_0}$, $\underline{y}(t) = \underline{w}(t)/\underline{w_0}$ $z_0, \underline{v_0}, \underline{w_0}$: some reference value

$\hookrightarrow \dot{\underline{x}}(t) = \underline{f_0}(\underline{x}(t), \underline{u}(t), t)$, $\underline{y}(t) = \underline{g_0}(\underline{x}(t), \underline{u}(t), t)$

Linearization

$\underline{f_0}(\underline{x_e}, \underline{u_e}, t) = \underline{0} \rightarrow \tilde{\underline{x}}(t) = \underline{x}(t) - \underline{x_e}$, $\tilde{\underline{u}}(t) = \underline{u}(t) - \underline{u_e}$, $\tilde{\underline{y}}(t) = \underline{y}(t) - \underline{g_0}(x_e, u_e, t)$

$\hookrightarrow \dot{\tilde{\underline{x}}}(t) = \tilde{\underline{f_0}}(\tilde{\underline{x}}(t), \tilde{\underline{u}}(t), t)$, $\tilde{\underline{y}}(t) = \tilde{\underline{g_0}}(\tilde{\underline{x}}(t), \tilde{\underline{u}}(t), t)$

$$\underline{\underline{A}} = \begin{pmatrix} \left.\frac{\partial f_{0,1}}{\partial x_1}\right|_{\substack{x=x_e\\u=u_e}} & \cdots & \left.\frac{\partial f_{0,1}}{\partial x_n}\right|_{\substack{x=x_e\\u=u_e}} \\ \vdots & & \vdots \\ \left.\frac{\partial f_{0,n}}{\partial x_1}\right|_{\substack{x=x_e\\u=u_e}} & \cdots & \left.\frac{\partial f_{0,n}}{\partial x_n}\right|_{\substack{x=x_e\\u=u_e}} \end{pmatrix} (n \times n) \qquad \underline{\underline{B}} = \begin{pmatrix} \left.\frac{\partial f_{0,1}}{\partial u_1}\right|_{\substack{x=x_e\\u=u_e}} & \cdots & \left.\frac{\partial f_{0,1}}{\partial u_m}\right|_{\substack{x=x_e\\u=u_e}} \\ \vdots & & \vdots \\ \left.\frac{\partial f_{0,n}}{\partial u_1}\right|_{\substack{x=x_e\\u=u_e}} & \cdots & \left.\frac{\partial f_{0,n}}{\partial u_m}\right|_{\substack{x=x_e\\u=u_e}} \end{pmatrix} (n \times m)$$

$$\underline{\underline{C}} = \begin{pmatrix} \left.\frac{\partial g_{0,1}}{\partial x_1}\right|_{\substack{x=x_e\\u=u_e}} & \cdots & \left.\frac{\partial g_{0,1}}{\partial x_n}\right|_{\substack{x=x_e\\u=u_e}} \\ \vdots & & \vdots \\ \left.\frac{\partial g_{0,p}}{\partial x_1}\right|_{\substack{x=x_e\\u=u_e}} & \cdots & \left.\frac{\partial g_{0,p}}{\partial x_n}\right|_{\substack{x=x_e\\u=u_e}} \end{pmatrix} (p \times n) \qquad \underline{\underline{D}} = \begin{pmatrix} \left.\frac{\partial g_{0,1}}{\partial u_1}\right|_{\substack{x=x_e\\u=u_e}} & \cdots & \left.\frac{\partial g_{0,1}}{\partial u_m}\right|_{\substack{x=x_e\\u=u_e}} \\ \vdots & & \vdots \\ \left.\frac{\partial g_{0,p}}{\partial u_1}\right|_{\substack{x=x_e\\u=u_e}} & \cdots & \left.\frac{\partial g_{0,p}}{\partial u_m}\right|_{\substack{x=x_e\\u=u_e}} \end{pmatrix} (p \times m)$$

$\hookrightarrow \dot{\underline{x}}(t) = \underline{\underline{A}}\, \underline{x}(t) + \underline{\underline{B}}\, \underline{u}(t)$, $\underline{y}(t) = \underline{\underline{C}}\, \underline{x}(t) + \underline{\underline{D}}\, \underline{u}(t)$

Reversion

$\underline{z}(t) = \underline{z_0}(\underline{x}(t) + \underline{x_e})$, $\underline{v}(t) = \underline{v_0}(\underline{u}(t) + \underline{u_e})$, $\underline{w}(t) = \underline{w_0}(\underline{y}(t) + \underline{y_e})$

Solution

$\underline{x}(t) = \underline{\underline{\Phi}}(t)\, \underline{x}(0) + \int_0^t \underline{\underline{\Phi}}(t-\sigma)\, \underline{\underline{B}}\, \underline{u}(\sigma)\, d\sigma$

$\uparrow\ \underline{\underline{\Phi}}(t) = e^{\underline{\underline{A}}t} = \underline{\underline{I}} + \underline{\underline{A}}t + (\underline{\underline{A}}t)^2/2! + \ldots + (\underline{\underline{A}}t)^n/n! + \ldots$

$y(t) = \underline{\underline{C}}\, \underline{x}(t) + \underline{\underline{D}}\, \underline{u}(t)$

- **Lyapunov Stability** $\left(\underline{u}(t)=\underline{0} \rightarrow \underline{x}(t)= \underline{\underline{\Phi}} \cdot \underline{x}(0) \right.$ *finite for* $\left. t\to\infty ? \right)$

asymptotically stable : $\lim_{t\to\infty} \|\underline{x}(t)\| = 0 \quad \leftrightarrow \quad \forall\, EW(\underline{\underline{A}}) < 0$

stable (*) (**) : $\|\underline{x}(t)\| < \infty \;\; \forall t \in (0,\infty) \quad \leftrightarrow \quad \forall\, EW(\underline{\underline{A}}) \leq 0$

unstable : $\lim_{t\to\infty} \|\underline{x}(t)\| = \infty \quad \leftrightarrow \quad \exists\, EW(\underline{\underline{A}}) > 0$

$\|\underline{x}(t)\| = \sqrt{\sum_{i=1}^{n} x_i^2(t)}$ / $EW(A) = \lambda_i \leftarrow \det(\lambda_i \mathbb{I} - A) = 0$

incomplete!

(*) *if* $\exists\, i \neq j$ *with* $\lambda_i = \lambda_j$ and $Re(\lambda_i) = Re(\lambda_j) = 0$ *then could be unstable!* $\leftarrow$

$$\begin{pmatrix} \lambda_1 & 1 & & & & \\ & \lambda_1 & & & & \\ & & \lambda_2 & & & \\ & & & \lambda_3 & 1 & \\ & & & & \lambda_3 & 1 \\ & & & & & \lambda_3 \end{pmatrix}$$

(**) *linear system stable* $\overset{\times}{\underset{\checkmark}{\rightleftarrows}}$ *real system stable*

- **Reachability** $\left(\forall\, \underline{x} \right.$ *can be reached with some* $u(t)$ *from any* $\left. \underline{x}(0) \right)$

$R_n = [B, AB, A^2B, \ldots, A^{n-1}B] \in \mathbb{R}^{n\times n}$ *has full Rank* n $\quad (\det(R_n) \neq 0)$

- **Observability** (*possible to uniquely reconstruct* $x(0)$ *from* $y(t)$ *wit* $u(t)=0 \;\; \forall t$)

$O_n = [C; CA; CA^2; \ldots; CA^{n-1}]^T \in \mathbb{R}^{n\times n}$ *has full Rank* n $\quad (\det(O_n) \neq 0)$

Example 1

$$\dot{x} = A \cdot x + B \cdot u \quad , \quad y = C \cdot x + D \cdot u$$

$$\begin{pmatrix} \delta\dot{\psi} \\ \delta\ddot{\psi} \\ \delta\dot{x} \\ \delta\ddot{x} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & a_2 & 0 \end{pmatrix} \cdot \begin{pmatrix} \delta\psi \\ \delta\dot{\psi} \\ \delta x \\ \delta\dot{x} \end{pmatrix} + \begin{pmatrix} 0 \\ b_1 \\ 0 \\ b_2 \end{pmatrix} \cdot \delta u \quad , \quad \delta y = \begin{pmatrix} 0 \\ 0 \\ c_1 \\ 0 \end{pmatrix}^T \cdot \begin{pmatrix} \delta\psi \\ \delta\dot{\psi} \\ \delta x \\ \delta\dot{x} \end{pmatrix} + (0) \cdot \delta u$$

$$a_1 = \frac{mgR\vartheta}{\Gamma} \qquad b_1 = \frac{mr^2 + \vartheta}{\Gamma} \qquad c_1 = R + r$$

$$a_2 = \frac{mg(R^2\vartheta + r^2\Theta)}{(R+r)\Gamma} \qquad b_2 = \frac{R\vartheta}{(R+r)\Gamma}$$

with some numbers for $R, r, \vartheta, \Theta, g, m, \Gamma$:

$$R_n = \begin{pmatrix} 0 & 2.6 & 0 & 1 \\ 2.6 & 0 & 1 & 0 \\ 0 & 7.8 & 0 & 9.1 \\ 7.8 & 0 & 9.1 & 0 \end{pmatrix} \rightarrow \det(R_n) = 263.4$$

$\hookrightarrow$ *fully controllable*

$$O_n = \begin{pmatrix} 0 & 0 & 0.8 & 0 \\ 0 & 0 & 0 & 0.8 \\ 0 & 0 & 7.4 & 0 \\ 0 & 0 & 0 & 7.4 \end{pmatrix} \rightarrow \det(O_n) = 0$$

$\hookrightarrow \psi, \dot{\psi}$ *not observable.*

$EW(A)$: $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = -\lambda_4 = 3{,}03 \rightarrow$ *unstable!*

Balanced Realization and Order Reduction

Controlability/Observability measure

$$A=\begin{pmatrix}-1&\varepsilon&0\\0&0&1\\0&a_0&a_1\end{pmatrix}\quad B=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

if $\varepsilon\neq0 \rightarrow \det(R_n)\neq0 \rightarrow$ fully controllable

↳ but if ε very small $\rightarrow$ hard to control x_1

controllability measure: $W_R=\int_0^\infty e^{A\sigma}BB^Te^{A^T\sigma}d\sigma \leftrightarrow AW_R+W_RA=-BB^T$

↳ $W_R(\varepsilon=1)=\begin{pmatrix}-1.5&-1.5&0.5\\-1.5&-2&0\\0.5&0&-1\end{pmatrix}$ $\quad W_R(\varepsilon=0.01)=\begin{pmatrix}\approx0&\approx0&\approx0\\\approx0&-2&0\\\approx0&0&-1\end{pmatrix}$ $\rightarrow \det(W_R)\approx0$ $\rightarrow x_1$ hard to control

observability measure: $W_o=\int_0^\infty e^{A^T\sigma}C^TCe^{A\sigma}d\sigma \leftrightarrow A^TW_o+W_oA=-C^TC$

Order Reduction

bad idea: delete system parts that do not contribute to W_R, W_O. (W_R coud compensate W_O)

good idea:

find transformation $Tx_b=x$ so that $W_{R,b}=W_{O,b}=\mathrm{diag}(\sigma_i)$

↳ relatively small σ_i can be neglected, but not their gain contribution!

① $W_R=V_R\Lambda_R^2V_R^T \rightarrow T_R=V_R\Lambda_R$

$(\tilde W_O=T_R^TW_OT_R)=V_O\Lambda_O^2V_O^T \rightarrow T_O=V_O\Lambda_O^{-1/2}$

$\rightarrow T=T_RT_O \rightarrow W_{R,b}=W_{O,b}=\begin{pmatrix}\sigma_1&0&\cdots&0\\0&\sigma_2&\cdots&0\\\vdots&&\ddots&\\0&0&&\sigma_n\end{pmatrix}$

② new system: $\dot x_b=\tilde Ax_b+\tilde Bu$, $y=\tilde Cx_b+\tilde Du$ with $\tilde A=T^{-1}AT$, $\tilde B=T^{-1}B$, $\tilde C=CT$, $\tilde D=D$

③ divide system: $\frac{d}{dt}\begin{pmatrix}x_{b1}\\x_{b2}\end{pmatrix}=\begin{pmatrix}\tilde A_{11}&\tilde A_{12}\\\tilde A_{21}&\tilde A_{22}\end{pmatrix}\begin{pmatrix}x_{b1}\\x_{b2}\end{pmatrix}+\begin{pmatrix}\tilde B_1\\\tilde B_2\end{pmatrix}u \quad y=(\tilde C_1\ \tilde C_2)\begin{pmatrix}x_{b1}\\x_{b2}\end{pmatrix}+\tilde Du$

↑ where $x_{b2}=$ system parts that can be neglected (low σ_i!)

④ new system: $\dot x_{b1}=\tilde A_{11}x_{b1}+\tilde B_1u$, $y=\tilde C_1x_{b1}+\tilde Du$

to match gain: $\tilde A_{11}\rightarrow\tilde A_{11}-A_{12}A_{22}^{-1}A_{21}$ $\quad \tilde C_1\rightarrow C_1-C_2A_{22}^{-1}A_{21}$ (singular perturbation)

$\tilde B_1\rightarrow\tilde B_1-A_{12}A_{22}^{-1}B_2$ $\quad \tilde D\rightarrow\tilde D-C_2A_{22}^{-1}B_2$

Zero Dynamics (SISO)

$\dot{x} = Ax + Bu \qquad y = Cx + Du$

n: system order
k: input gain
r: relative degree

$$P(s) = C(sI-A)^{-1}B + D \rightarrow P(s) = k \cdot \frac{1 \cdot s^{n-r} + b_{n-r-1}s^{n-r-1} + \dots + b_1 s + b_0}{1 \cdot s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0}$$

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{pmatrix} \cdot x + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ k \end{pmatrix} \cdot u \qquad y = \begin{pmatrix} b_0 \\ \vdots \\ b_{n-r-1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} x$$

Zero Dynamics: $u(t) \neq 0$ and $x(0) = ?$ $\rightarrow$ $y(t) = 0$ for finite t $\rightarrow$ problematic for controler.

$$\begin{aligned}
z_1 &= y &&= Cx &&= b_0x_1 + b_1x_2 + \dots + b_{n-r-1}x_{n-r} + x_{n-r+1} \\
z_2 &= \dot{y} &&= C\dot{x} = CAx + CBx = CAx &&= b_0x_2 + b_1x_3 + \dots + b_{n-r-1}x_{n-r+1} + x_{n-r+2} \\
z_3 &= \ddot{y} &&= CA\dot{x} = CA^2x + CABx = CA^2x &&= b_0x_3 + b_1x_4 + \dots + b_{n-r-1}x_{n-r+2} + x_{n-r+3} \\
&\vdots \\
z_r &= y^{(r-1)} &&= CA^{r-1}x + CA^{r-2}Bu = CA^{r-1}x &&= b_0x_r + b_1x_{r+1} + \dots + b_{n-r-1}x_{n-1} + x_n \\
\Big(& \quad y^{(r)} &&= CA^rx + CA^{r-1}Bu = CA^rx + \underline{ku} &&= b_0x_{r+1} + b_1x_{r+2} + \dots + b_{n-r-1}x_n + \underline{\dot{x}_n} \Big) \\
z_{r+1} &= x_1 \\
z_{r+2} &= x_2 \\
&\vdots \\
z_n &= x_{n-r}
\end{aligned}$$

$$\rightarrow \quad z = \Phi x = \begin{pmatrix} \xi = \begin{pmatrix} z_1 \\ \vdots \\ z_r \end{pmatrix} \\ \eta = \begin{pmatrix} z_{r+1} \\ z_n \end{pmatrix} \end{pmatrix}$$

(coordinate transform)

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \left(\begin{array}{ccccc|ccccc} 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ & & r^T & & & & & s^T & & \\ \hline 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 \\ & & p^T & & & & & q^T & & \end{array}\right) \cdot \begin{pmatrix} \xi \\ \eta \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ k \\ \hline 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \cdot u \quad , \quad y = z_1 = \xi(1)$$

$$\left(\begin{array}{c|c} R & S \\ \hline P & Q \end{array}\right)$$

to have a vanishing output: $\xi^*(0)=0$, $u^*(t)=-\frac{1}{k}s^T\eta^*(t)$ ($\eta^*(0)$ arbitrary)

$\hookrightarrow$ $y(t)=0$ and $\xi(t)=0\ \forall t$ $\rightarrow$ zerodynamics: $\dot{\eta}^*(t)=Q\cdot\eta^*(t)$

$$Q=\begin{pmatrix}0&1&0&\cdots&0\\0&0&1&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&1\\&&q^T&&\end{pmatrix}\leftarrow q=\begin{pmatrix}-b_0\\-b_1\\\vdots\\-b_{n-r-2}\\-b_{n-r-1}\end{pmatrix}$$

Q assympt. stable $\rightarrow$ system is minimum phase $\leftarrow$ $\forall\ \mathrm{Re}(\text{zero})<0$

Example: ($n=4$, $r=2$)

$$P(s)=\frac{Y(s)}{U(s)}=k\,\frac{b_1s+b_0}{a_3s^3+a_2s^2+a_1s+a_0}\Rightarrow\dot{x}(t)=\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\-a_0&-a_1&-a_2&-a_3\end{pmatrix}\cdot x(t)+\begin{pmatrix}0\\0\\0\\k\end{pmatrix}\cdot u(t)$$

$$y(t)=\begin{pmatrix}b_0&b_1&1&0\end{pmatrix}\cdot x(t)+(0)\cdot u(t)$$

$$\left.\begin{aligned}z_1&=y=b_0x_1+b_1x_2+x_3\\z_2&=\dot{y}=b_0x_2+b_1x_3+x_4\\&\ddot{y}=-a_0x_1-a_1x_2+(b_0-a_2)x_3\\&\qquad+(b_1-a_3)x_4+ku\\z_3&=x_1\\z_4&=x_2\end{aligned}\right\}$$

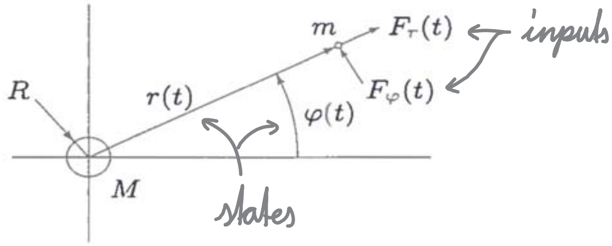
$z=\Phi^{-1}x$ $\qquad$ $x=\Phi z$

$$\downarrow\quad\Phi^{-1}=\begin{pmatrix}b_0&b_1&1&0\\0&b_0&b_1&1\\1&0&0&0\\0&1&0&0\end{pmatrix}\rightarrow\quad\uparrow\quad\Phi=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&-b_0&-b_1\\-b_1&1&b_0b_1&b_1^2-b_0\end{pmatrix}$$

$$\Rightarrow\dot{z}=\Phi^{-1}A\Phi z+\Phi^{-1}Bu\quad,\quad y=C\Phi z$$

$$\frac{d}{dt}\begin{pmatrix}\xi_1\\\xi_2\\\eta_1\\\eta_2\end{pmatrix}=\begin{pmatrix}0&1&0&0\\r_1&r_2&s_1&s_2\\0&0&0&1\\1&0&-b_0&-b_1\end{pmatrix}\cdot\begin{pmatrix}\xi_1\\\xi_2\\\eta_1\\\eta_2\end{pmatrix}+\begin{pmatrix}0\\k\\0\\0\end{pmatrix}\cdot u\leftarrow\begin{cases}r_1=b_0-a_2-b_1(b_1-a_3)\\r_2=b_1-a_3\\s_1=b_0b_1(b_1-a_3)-a_0-b_0(b_0-a_2)\\s_2=(b_1-a_3)(b_1^2-b_0)-a_1-(b_0-a_2)b_1\end{cases}$$

Case Study: Geostationary satellite



Lagrange Method: $L = T - V$

$$T = \tfrac{1}{2} m \dot{r}^2 + \tfrac{1}{2} m (r\dot{\varphi})^2$$

$$V = \int_R^r F(\xi)\,d\xi = \int_R^r G\frac{M\cdot m}{\xi^2}\,d\xi = GMm\left(\frac{1}{R}-\frac{1}{r}\right)$$

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{r}}\right] - \frac{\partial L}{\partial r} = F_r$$

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\varphi}}\right] - \frac{\partial L}{\partial \varphi} = F_\varphi \cdot r$$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad \frac{\partial L}{\partial \dot{\varphi}} = m r^2 \dot{\varphi}$$

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{r}}\right] = m\ddot{r} \qquad \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\varphi}}\right] = m r^2 \ddot{\varphi} + 2 m r \dot{r}\dot{\varphi}$$

$$\frac{\partial L}{\partial r} = m r \dot{\varphi}^2 - GMm\frac{1}{r^2} \qquad \frac{\partial L}{\partial \varphi} = 0$$

$$\to m\ddot{r} = m r \dot{\varphi}^2 - GMm\frac{1}{r^2} + F_R$$

$$\to m r^2 \ddot{\varphi} = -2 m r \dot{\varphi}\dot{r} + F_\varphi r$$

inputs: $u_1 = F_r/m$; $u_2 = F_\varphi/m$

equilibrium: $u_1 = 0,\ \ddot{r} = 0,\ \dot{r} = 0,\ r = r_0$

$u_2 = 0,\ \ddot{\varphi} = 0,\ \dot{\varphi} = \omega_0,\ \varphi = \omega_0 t$

$\omega_0 = 7.29 \cdot 10^{-5}\,\mathrm{rad/s}$ geostationary orbit

$$\to r_0\omega_0^2 - GM\frac{1}{r_0^2} = 0 \ \to r_0 = \left(\frac{GM}{\omega_0^2}\right)^{1/3}$$

states: $x_1 = r$, $x_2 = \dot{r}$, $x_3 = \varphi$, $x_4 = \dot{\varphi}$

$$\frac{d}{dt}\underline{x}(t) = \underline{f}(\underline{x}(t), \underline{u}(t))$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 x_4^2 - GM/x_1^2 + u_1 \\ x_4 \\ -2x_2x_4/x_1 + u_2/x_1 \end{pmatrix}$$

outputs: $y_1 = x_1/r_0$; $y_2 = x_3$

$$\underline{y}(t) = \underline{h}(\underline{x}(t), \underline{u}(t))$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1/r_0 \\ x_3 \end{pmatrix}$$

$$\underline{x}_0(t) = \begin{pmatrix} r_0 \\ 0 \\ \omega_0 t \\ \omega_0 \end{pmatrix}, \quad \underline{u}_0(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\hookrightarrow \underline{\underline{A}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3\omega_0^2 & 0 & 0 & 2r_0\omega_0 \\ 0 & 0 & 0 & 1 \\ 0 & -2\frac{\omega_0}{r_0} & 0 & 0 \end{pmatrix} \qquad \underline{\underline{B}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1/r_0 \end{pmatrix}$$

$$\underline{\underline{C}} = \begin{pmatrix} 1/r_0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \underline{\underline{D}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

stability: $\det(sI - A) = s^2(s^2 + \omega_0^2)$

$\hookrightarrow \lambda_{1,2} = 0 \quad \lambda_{3,4} = \pm i\omega_0 \quad \to$ unstable?

controlability: $R_n = [B, AB, A^2B, \ldots]$ has full rank $\hookrightarrow \det(R_n) \neq 0$

$$R_n = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 & \cdots \\ 1 & 0 & 0 & 2\omega & \cdots \\ 0 & 0 & 0 & \frac{1}{r_0} & \cdots \\ 0 & \frac{1}{r_0} & \frac{-2\omega_0}{r_0} & 0 & \cdots \end{array}\right) \to \text{controllable}$$

observability: $O_n = [C; CA; CA^2; \ldots]$ has full rank $\hookrightarrow \det(O_n) \neq 0$

$$O_n = \left(\begin{array}{cccc} \frac{1}{r_0} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & \frac{1}{r_0} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \end{array}\right) \to \text{observable}$$

transfer function: $P(s) = \begin{pmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{pmatrix} = C(sI-A)^{-1}B = \dfrac{C \cdot \mathrm{Adj}(sI-A)B}{\det(sI-A)}$

$$P(s) = \begin{pmatrix} \dfrac{1}{r_0(s^2+\omega^2)} & \dfrac{2\omega_0}{r_0 s(s^2+\omega_0^2)} \\[2ex] \dfrac{-2}{r_0 s(s^2+\omega_0^2)} & \dfrac{s^2-3\omega_0^2}{r_0 s^2(s^2+\omega_0^2)} \end{pmatrix}$$