Analysis III

Laplace Transform: $(\mathcal{L}f)(s) = \int_0^\infty e^{-st} f(t)\,dt$

$\mathcal{L}(f(x)+g(x))(s) = \mathcal{L}(f(x))(s) + \mathcal{L}(g(x))(s)$ (auch für $\mathcal{L}^{-1}$)

$\mathcal{L}(\alpha f(x))(s) = \alpha\,\mathcal{L}(f(x))(s)$

$f(t)$	$\mathcal{L}(f(t))(s)$	$f(t)$	$\mathcal{L}(f(t))(s)$	$f(t)$	$\mathcal{L}(f(t))(s)$	$f(t)$	$\mathcal{L}(f(t))(s)$
1	$1/s$	$\sin(kt)$	$k/(s^2+k^2)$	$\cos(kt)$	$s/(s^2+k^2)$	$\sinh(kt)$	$k/(s^2-k^2)$
t^n	$n!/s^{n+1}$	$\sin^2(kt)$	$2k^2/(s(s^2+4k^2))$	$\cos^2(kt)$	$(s^2+2k^2)/(s(s^2+4k^2))$	$\cosh(kt)$	$s/(s^2-k^2)$
e^{at}	$1/(s-a)$	$t\sin(kt)$	$2ks/(s^2+k^2)^2$	$t\cos(kt)$	$(s^2-k^2)/(s^2+k^2)^2$	$t\sinh(kt)$	$2ks/(s^2-k^2)^2$
$t^n e^{at}$	$n!/(s-a)^{n+1}$	$e^{at}\sin(kt)$	$k/((s-a)^2+k^2)$	$e^{at}\cos(kt)$	$(s-a)/((s-a)^2+k^2)$	$t\cosh(kt)$	$(s^2-k^2)/(s^2-k^2)^2$

$\mathcal{L}(f(t))(s)^{(n)} = (-1)^n \mathcal{L}(t^n \cdot f(t))(s)$ $(\mathcal{L}(f)'(s) = -\mathcal{L}(t\cdot f(t))(s))$

$\mathcal{L}(f^{(n)})(s) = s^n \mathcal{L}(f)(s) - \sum_{j=0}^{n-1} s^{n-1-j} f^{(j)}(0)$ $(\mathcal{L}(f')(s) = s\mathcal{L}(f)(s) - f(0))$

$\mathcal{L}\left(\int_0^t f(x)\,dx\right)(s) = \frac{1}{s}\mathcal{L}(f(t))(s)$

$\mathcal{L}(f)\cdot\mathcal{L}(g) = \mathcal{L}(f * g)$ convolution $\left((f*g)(t) = \int_0^t f(r)\,g(t-r)\,dr\right)$

$\mathcal{L}(e^{at} f(t))(s) = (\mathcal{L}f)(s-a)$ $(s>a)$

$\mathcal{L}(u(t-a)f(t-a))(s) = e^{-as}\mathcal{L}(f)(s)$ $\left(u(t) = \begin{cases} 1; & t>0 \\ 0; & t\le 0 \end{cases}\right)$

$\hookrightarrow \mathcal{L}^{-1}(e^{-as}\mathcal{L}(f)(s))(t) = u(t-a)f(t-a)$

$\mathcal{L}(u(t-a))(s) = e^{-as}/s$

$\mathcal{L}(\delta(t-a))(s) = e^{-as}$ $\left(\delta(t) = \frac{du(t)}{dt};\ \int_0^\infty \varphi(t)\delta(t-a)\,dt = \varphi(a)\right)$

$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\cdot Y(s)$ $\lim_{t\to 0} y(t) = \lim_{s\to\infty} s\cdot Y(s)$

Fourier Series:

$$f(x) = a_0 + \sum_{n=1}^{N} a_n \cos\left(n\frac{\pi}{L}x\right) + \sum_{n=1}^{N} b_n \sin\left(n\frac{\pi}{L}x\right)$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(n\frac{\pi}{L}x\right)$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(n\frac{\pi}{L}x\right)$$

if $f(x)$ even $\rightarrow b_n = 0\ \forall n \rightarrow a_0 = \frac{1}{L}\int_0^L f(x)dx \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(n\frac{\pi}{L}x\right)dx$

if $f(x)$ odd $\rightarrow a_n = 0\ \forall n \rightarrow b_n = \frac{2}{L}\int_0^L f(x)\sin\left(n\frac{\pi}{L}x\right)dx$

$$f(x) = \sum_{n=-N}^{N} c_n e^{in\pi x/L}$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-in\pi x/L}dx \qquad \left(c_0 = a_0\ ;\ c_n = \frac{a_n}{2} - i\frac{b_n}{2}\ ;\ c_{-n} = \frac{a_n}{2} + i\frac{b_n}{2}\right)$$

Fourier Transform: $\left(F(f)(\omega) = \mathcal{L}(f)(s=i\omega)\right)$

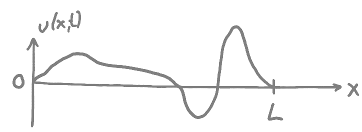
$$F(f)(\omega) = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx \qquad f(x) = \int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega x}d\omega$$

$$F(f')(\omega) = i\omega\, F(f)(\omega)$$

$$F(f * g)(\omega) = F(f)(\omega)\, F(g)(\omega)$$

Partial differential Equations:

- Fourier Series solution of 1D-Wave equation



$U_{tt} = c^2 \cdot U_{xx} \qquad u(0,t) = u(L,t) = 0 \qquad u(x,0) = f(x) \qquad u_t(x,0) = g(x)$

$$u(x,t) = \sum_{n=1}^{\infty}\left(B_n\cos(\lambda_n t) + B_n^*\sin(\lambda_n t)\right)\cdot\sin\left(\frac{n\pi}{L}x\right) \qquad \left(\lambda_n = \frac{cn\pi}{L}\right)$$

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

$$B_n^* = \frac{2}{L\lambda_n}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

- **D' Allembert solution of the 1D-Wave equation**

$$U_{tt} = c^2 \cdot U_{xx} \qquad u(x,0) = f(x) \qquad u_t(x,0) = g(x)$$

$$u(x,t) = \frac{1}{2}\Big(f(x+ct) + f(x-ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

- **D'Alembert solution of Wave Equations**

$$A(x,y)\,u_{xx} + 2B(x,y)\,u_{xy} + C(x,y)\,u_{yy} = F(u,u_x,u_y,x,y)$$

$$A(x,y)(y')^2 - 2B(x,y)\cdot y' + C(x,y) = 0 \qquad \text{(characteristic Eq.)}$$

$\xrightarrow{\text{solve}}$ $\xi(x,y) = c_1 \qquad \zeta(x,y) = c_2 \qquad$ (characteristics)

$$D(x,y) = B^2(x,y) - A(x,y)\cdot C(x,y)$$

- $D(x,y) > 0 \ \forall x,y \longrightarrow v = \xi(x,y) \qquad w = \zeta(x,y)$ (Hyperbolic)
- $D(x,y) = 0 \ \forall x,y \longrightarrow v = x \qquad w = \zeta(x,y)$ (Parabolic)
- $D(x,y) < 0 \ \forall x,y \longrightarrow v = \frac{\xi(x,y)+\zeta(x,y)}{2} \qquad w = \frac{\xi(x,y)-\zeta(x,y)}{2}$ (Elliptic)

Rewrite original PDE as Functions of v,w $\rightarrow$ solve for $u(v,w)$

(if PDE $= u_{vw} = 0 \rightarrow u(v,w) = \varphi(v) + \psi(w) \qquad \varphi,\psi$ arbitrary)

- **Fourier Series solution of 1D-Heat equation**

$$u_t = c^2 u_{xx} \qquad u(0,t) = u(L,t) = 0 \qquad u(x,0) = f(x) \qquad \left(c^2 = \frac{K}{\sigma\, s}\right)$$

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\lambda_n^2 \cdot t} \qquad \left(\lambda_n^2 = \left(\frac{cn\pi}{L}\right)^2\right)$$

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

- **Heat equation on an infinite bar**

$$u_t = c^2 u_{xx} \qquad u(x,0) = f(x)$$

$$u(x,t) = \frac{1}{2c\sqrt{\pi t}}\int_{-\infty}^{\infty} f(v)\exp\left(-\left(\frac{x-v}{2c\sqrt{t}}\right)^2\right)dv$$

- **Wave equation on a rectangular Membrane**

$$u_{tt} = c^2 \nabla^2 u \qquad u(x,y,t) \text{ mit } 0 \le x \le a,\ 0 \le y \le b$$

$$u(x,y,t) = 0 \quad \forall\, (x,y) \in \text{Rand}$$

$$u(x,y,0) = f(x,y) \qquad u_t(x,y,0) = g(x,y)$$

$$u(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(B_{mn} \cos(\lambda_{mn} t) + B^*_{mn} \sin(\lambda_{mn} t) \right) \cdot \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

$$B_{mn} = \frac{4}{ab} \int_0^b \int_0^a f(x,y) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) dx\, dy$$

$$B^*_{mn} = \frac{4}{ab\lambda_{mn}} \int_0^b \int_0^a g(x,y) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) dx\, dy$$

$$\lambda_{mn} = \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$$

- **"Separation of Variables" approach** $(a\, u_{xx} + b\, u_{xy} + c\, u_{yy} + d\, u_x + \ldots = 0)$

assume: $u(x,y) = X(x) \cdot Y(y) \qquad (u_{xx} = X''Y\ ,\ u_{xy} = X'Y', \ldots)$

$\hookrightarrow F(X, X', X'') = \text{Konst.} = G(Y, Y', Y'')$

solve $F(X, X', X'') = K\ ,\ G(Y, Y', Y'') = K$