u: input
y: output
x: state (memory)

$y = \Sigma(u)$

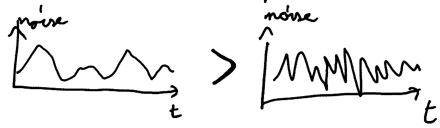
system order: Dim(x) = highest derivative in diff. eq.

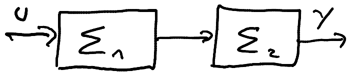
time variant/invariant: (independent) of starting time
static/dynamic system: with/without state var. x (memory)

SISO: single input single output
MIMO: multiple input multiple output

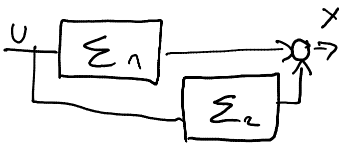
infinite gain: e.g. $\Sigma = \frac{\Sigma_2}{1-\Sigma_1\Sigma_3}$ inf. for $\Sigma_1\Sigma_2 = 1$

autocorrelation:

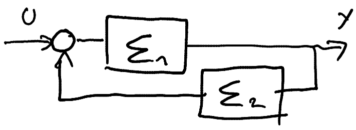




$\Sigma = \Sigma_1 \cdot \Sigma_2$



$\Sigma = \Sigma_1 + \Sigma_2$



$\Sigma = \Sigma_1 \cdot (1 - \Sigma_2 \cdot \Sigma_1)^{-1}$

$u_1 = u + y_2$ $\quad y_1 = \Sigma_1(u_1)$ $\quad \rightarrow u_1 = u + y_2$

$y_1 = u_2 = y$ $\quad y_2 = \Sigma_2(u_2)$ $\quad u_1 = u + \Sigma_2(u_2)$

$y = \Sigma(u)$ $\quad u_1 = u + \Sigma_2(y)$

$\Sigma_1^{-1}(y_1) = u + \Sigma_2 \Sigma(u)$

$\Sigma_1^{-1}\Sigma(u) = u + \Sigma_2\Sigma(u)$

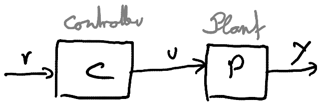
$\Sigma(u) = \Sigma_1(u) + \Sigma_1\Sigma_2\Sigma(u)$

$\Sigma(u) - \Sigma_1\Sigma_2\Sigma(u) = \Sigma_1(u)$

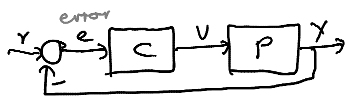
$\Sigma(u)(1 - \Sigma_1\Sigma_2(u)) = \Sigma_1(u)$

$\Sigma(u) = \Sigma_1(u)(1 - \Sigma_1\Sigma_2(u))^{-1}$

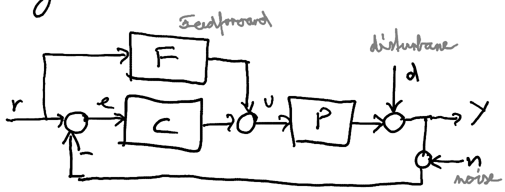
Feedforward:



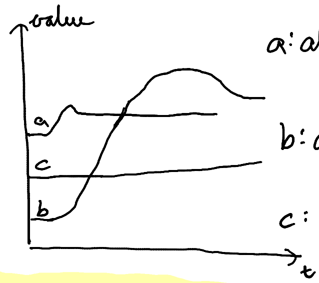
Feedback:



General:



Modelling



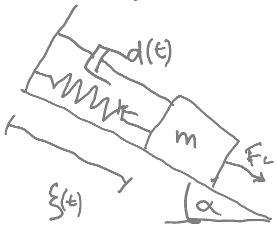
a: algebraic
b: dynamic
c: static

nonparametric: model from measurements (black box)
parametric: model from physical principles (grey box)

Modelling steps: 1. identify inputs, outputs, reservoirs (level variables)

2. $\frac{d}{dt}$(reservoirs) = Σ inflow $-$ Σ outflow

Example:



input: $d(t)$
output: $\xi(t)$

$$\frac{d}{dt}\Big(\underbrace{\tfrac{1}{2}k\,\xi(t)^2}_{E_{FEDER}} + \underbrace{\tfrac{1}{2}m\,\dot{\xi}(t)^2}_{E_{KIN}} \underbrace{- mg\sin(\alpha)\xi(t)}_{E_{POT}}\Big) = \underbrace{F_c \cdot \dot{\xi}(t)}_{P_{F_c}} \underbrace{- d(t)\cdot\dot{\xi}(t)\cdot\dot{\xi}(t)}_{P_{DAMPF}}$$

System Representation

$$\dot{z}(t) = f(z(t), v(t))$$
$$w(t) = g(z(t), v(t))$$

z: state
v: input
w: output

Example: from $\left(\frac{dE}{dt} = \sum P_{IN} - \sum P_{OUT}\right)$

$$m\ddot{\xi}(t) = mg\sin(\alpha) - k_3\,\xi(t)^3 - d(t)\dot{\xi}(t)$$

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \xi \\ \dot{\xi} \end{pmatrix} \qquad v = d \qquad (w = \xi)$$

$$\hookrightarrow \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} z_2 \\ g\sin(\alpha) - \frac{k_3}{m}\xi(t)^3 - \frac{1}{m}d(t)\dot{\xi}(t) \end{pmatrix} \qquad w = \xi(t)$$

Normalization

$$z_i(t) = z_{i0}\cdot x_i(t) \quad v(t) = v_0 \cdot u(t) \quad w(t) = w_0\, y(t)$$

so that x, u, y unitless and +/- 0-1

$$\rightarrow \dot{x}(t) = f_0(x(t), u(t)) \quad y(t) = g_0(x(t), u(t))$$

Linearization

$$f_0(x_e, u_e) = 0 \quad \rightarrow \quad \{x_e, u_e, y_e\}: \text{equilibrium point}$$

$$\dot{x}(t) = Ax + bu$$
$$y(t) = cx + du$$

$$\left.\begin{aligned} x(t) &= x^*(t) - x_e \\ u(t) &= u^*(t) - u_e \\ y(t) &= y^*(t) - u_e \end{aligned}\right\} x^*, u^*, y^* = x, u, y \text{ from Normalization!}$$

$$A = \left.\frac{\partial f_0}{\partial x}\right|_{\substack{x=x_e\\u=u_e}} \in \mathbb{R}^{n\times n} \qquad b = \left.\frac{\partial f_0}{\partial u}\right|_{\substack{x=x_e\\u=u_e}} \in \mathbb{R}^{n\times 1}$$

$$c = \left.\frac{\partial g_0}{\partial x}\right|_{\substack{x=x_e\\u=u_e}} \in \mathbb{R}^{1\times n} \qquad d = \left.\frac{\partial g_0}{\partial u}\right|_{\substack{x=x_e\\u=u_e}} \in \mathbb{R}^{1\times 1}$$

$$A = \begin{bmatrix} \left.\frac{\partial f_{0,1}}{\partial x_1}\right|_{\substack{x=x_e\\u=u_e}} & \cdots & \left.\frac{\partial f_{0,1}}{\partial x_n}\right|_{\substack{x=x_e\\u=u_e}} \\ \vdots & & \vdots \\ \left.\frac{\partial f_{0,n}}{\partial x_1}\right|_{\substack{x=x_e\\u=u_e}} & \cdots & \left.\frac{\partial f_{0,n}}{\partial x_n}\right|_{\substack{x=x_e\\u=u_e}} \end{bmatrix}$$

$$b = \begin{bmatrix} \left.\frac{\partial f_{0,1}}{\partial u}\right|_{x=x_e} & \cdots & \left.\frac{\partial f_{0,n}}{\partial u}\right|_{\substack{x=x_e\\u=u_e}} \end{bmatrix}^T$$

$$c = \begin{bmatrix} \left.\frac{\partial g_0}{\partial x_1}\right|_{\substack{x=x_e\\u=u_e}} & \cdots & \left.\frac{\partial g_0}{\partial x_n}\right|_{\substack{x=x_e\\u=u_e}} \end{bmatrix}$$

$$d = \begin{bmatrix} \left.\frac{\partial g_0}{\partial u}\right|_{\substack{x=x_e\\u=u_e}} \end{bmatrix}$$

Revert x, u, y to z, v, w

$$V = V_0\cdot(U + U_e) \quad \text{input}$$
$$W = W_0(y + y_e) \quad \text{output}$$
$$Z = Z_0(x + x_e) \quad \text{state variable}$$

Solution of linear control systems

$$\left(e^{At} = I + \frac{1}{1!}(At) + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \dots + \frac{1}{n!}(At)^n \right.$$
$$\left. = \text{Transition Matrix} \right)$$

$$x(t) = e^{At} \cdot x(0) + \int_0^t e^{A(t-\xi)} \cdot b \cdot u(\xi)\, d\xi$$

$$y(t) = \underbrace{c \cdot e^{At} \cdot x(0)}_{\text{Initial condition}} + \underbrace{\int_0^t \underbrace{c \cdot e^{A(t-\xi)} \cdot b}_{= \sigma(t-\xi):\ \text{Impulse resp.}} \cdot u(\xi)\, d\xi}_{\sigma(t) * u(t)} + \underbrace{d \cdot u(t)}_{\text{Feedtrough}}$$

First order systems

$$\dot{x} = \underbrace{-\frac{1}{\tau}}_{A} \cdot x(t) + \underbrace{\frac{k}{\tau}}_{b} \cdot u(t) \qquad y = x(t) \quad c = 1 \quad d = 0$$

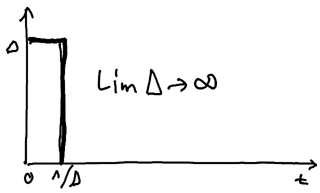
τ : time constant

k : gain

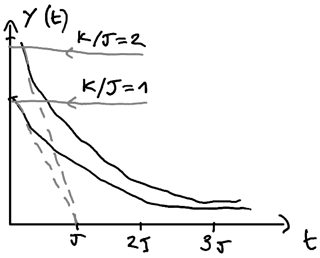
$$y(t) = e^{-t/\tau} \cdot x(0) + \int_0^t e^{-\frac{(t-\xi)}{\tau}} \cdot \frac{k}{\tau} \cdot u(\xi)\, d\xi$$

Test Signals: (for first order)

$\delta(t)$ Impulse func. :

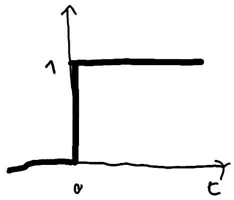


$$\delta(t) \approx \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$$

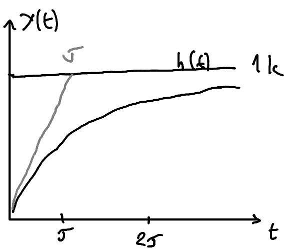


$$y(t) = e^{-t/\tau} \cdot (x_0 + k/\tau)$$

$h(t)$ step func:

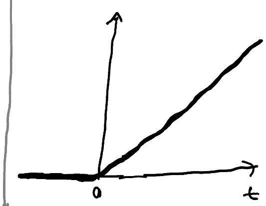


$$h(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

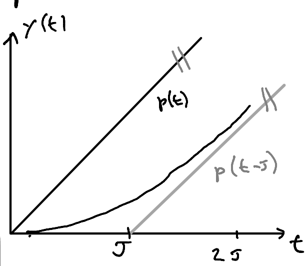


$$y(t) = e^{-t/\tau} \cdot x_0 + k(1 - e^{-t/\tau})$$

$p(t)$ ramp func:

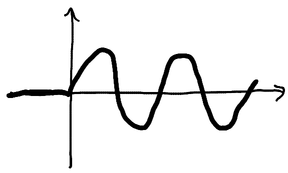


$$p(t) = t \cdot h(t)$$

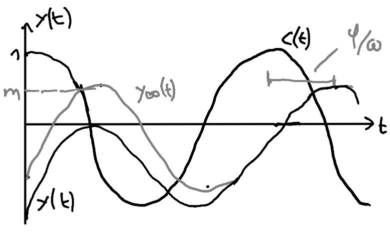


$$y(t) = e^{-t/\tau} x_0 + k \cdot (t + (e^{-t/\tau} - 1) \cdot \tau)$$

$c(t)$ harmonic func:



$$c(t) = \cos(\omega t) \cdot h(t)$$



$$y(t) = e^{-t/\tau} \cdot x_0 + \left(\cos(\omega t) + \omega\tau \cdot \sin(\omega t) - e^{-t/\tau}\right) \cdot k / 1 + \omega^2 \cdot \tau^2$$

$$m(\omega) = k / \sqrt{1 + \omega^2 \cdot \tau^2}$$
$$\varphi(\omega) = -\arctan(\omega\tau)$$

$$y_\infty(t) = m(\omega) \cdot \cos(\omega t + \varphi(\omega))$$

System Stability

Lyapunov Stability:

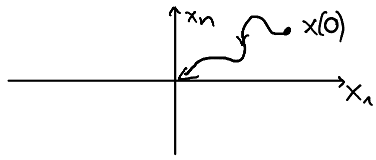
$u(t)=0,\ x(0)\neq 0$ is stable at isolated x_e?

$$\dot{x}(t) = A\cdot x(t) \rightarrow x(t) = e^{At}\,x(0)$$

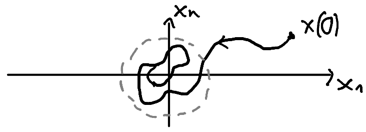
$A \rightarrow \lambda_1, \lambda_2, \dots, \lambda_n$ Eigenvalues $(\det(\lambda_i \mathbb{I} - A) = 0)$

- asymptotically stable: $Re(\lambda_i) < 0 \quad \forall i$
- stable (*) (**) : $Re(\lambda_i) \leq 0 \quad \forall i$
- unstable : else

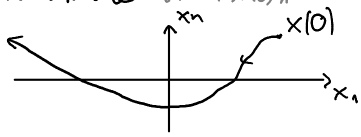
asymptotically stable: $\lim_{t\to\infty} \|x(t)\| = 0$



stable: $\lim_{t\to\infty} \|x(t)\| < \infty$



unstable: $\lim_{t\to\infty} \|x(t)\| = \infty$



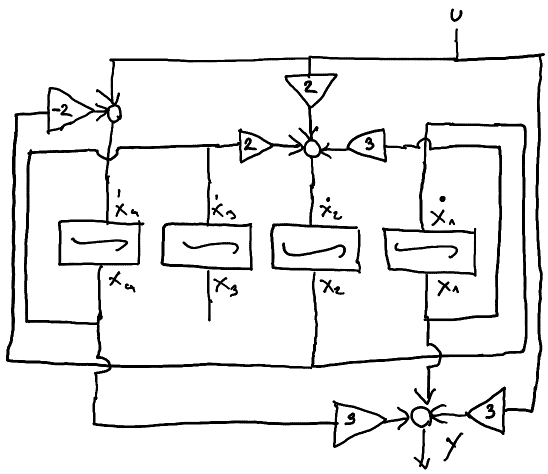
(*) Exception: if $\exists\, i \neq j$ with $\lambda_i = \lambda_j$ and $Re(\lambda_i) = Re(\lambda_j) = 0$ then could be unstable!

(**) Linear system stable $\neq$ Real system stable

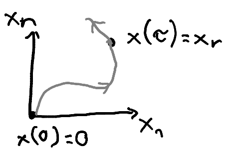
Block Diagram:

$$\underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \end{pmatrix}}_{A} \quad \underbrace{\begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}}_{b} \quad \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 3 \end{pmatrix}^T}_{c} \quad \underbrace{(3)}_{d}$$

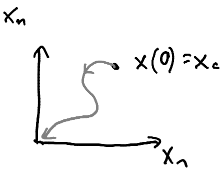
$$\dot{x} = Ax + bu$$
$$y = cx + du$$



Reachability/Controllability



$x_r \in \mathbb{R}^n$ reachable if $x(\tau) = x_r$; $\tau < \infty$, $x(0) = 0$, u = something
system completely reachable if $\forall x_r$ reachable.



$x_c \in \mathbb{R}^n$ controllable if $x(\tau) = 0$; $\tau < \infty$, $x(0) = x_c$, u = something
system completely controllable if $\forall x_c$ controllable

Condition: $\underset{n}{\mathcal{R}'} = [b, Ab, A^2b, A^3b, \dots, A^{n-1}b] \in \mathbb{R}^{n\times n}$ has full Rank n

Potentially stabilizable: all not controllable state var. are assymptotically stable

Observability: possible to uniquely reconstruct $x(0)$ from y with $u(t)=0 \ \forall t$

Condition: $O_n = [c, cA, cA^2, \dots cA^{n-1}]^T \in \mathbb{R}^{n \times n}$ has full Rank n.

Detectable: all non observable state var. are assymptotically stable

Stabilizable: potentially stabilizable and Detectable

$A = \begin{bmatrix} 0.067 & 1.616 \\ 0.616 & -0.067 \end{bmatrix}$ $b = \begin{bmatrix} 0.866 \\ 0.500 \end{bmatrix}$ $c = \begin{bmatrix} 0.376 \\ 1.237 \end{bmatrix} \rightarrow x = T\tilde{x}$ $\tilde{A} = T^{-1}AT$ $\tilde{b} = T^{-1}b$ $\tilde{c} = cT$ $\tilde{d} = d$

$\tilde{A} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ $\tilde{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\tilde{c} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

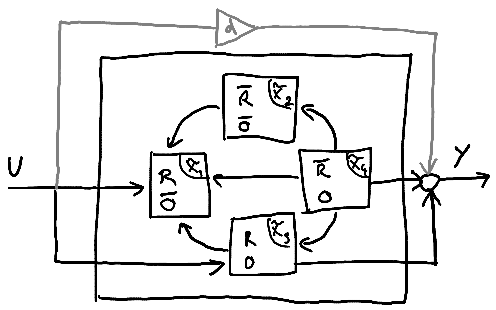
$\lambda_1 = 1 > 0 \rightarrow x_1 =$ unstable $\quad b_1 \neq 0 \rightarrow x_1 =$ controllable $\quad c_1 = 0 \rightarrow x_1 =$ unobservable

$\lambda_2 = -1 < 0 \rightarrow x_2 =$ stable $\quad b_2 = 0 \rightarrow x_2 =$ uncontrollable $\quad c_2 \neq 0 \rightarrow x_2 =$ observable

Decoupled system: A in Jordan normal form $\begin{pmatrix} \lambda_1 & 1 & & & \\ & \lambda_1 & & & \\ & & \lambda_2 & & \\ & & & \lambda_3 & 1 \\ & & & & \lambda_3 & 1 \\ & & & & & \lambda_3 \end{pmatrix}$ (or just so that λ_n on diag.)

$\leftarrow (\lambda_1 \, (2x), \lambda_2 \, (1x), \lambda_3 \, (3x)$ EW of A)

State space decomposition

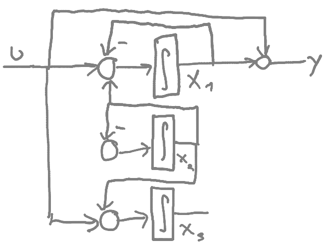


R: controllable $\bar{R}$: not controllable

O: observable $\bar{O}$: not observable

$\tilde{x}_{1,2,3,4}$: state var. in their resp. group.

$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $c = [1 \ 0 \ 0]$ $d = 1$ $\rightarrow \left[\begin{array}{c|c} A & b \\ \hline c & d \end{array}\right] = \left[\begin{array}{ccc|c} -1 & 1 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 \end{array}\right]$



$x_1 = RO \leftarrow b_1 \neq 0 \quad c_1 \neq 0$

$x_2 = \bar{R}\bar{O} \leftarrow b_2 = 0 \quad c_2 = 0$

$x_3 = R\bar{O} \leftarrow b_3 \neq 0 \quad c_3 = 0$

$\boxed{\text{for decoupled system only!}}$

I/O Description (input/output)

Describes system input-output relation (only RO state variables)

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \dots + a_1 y^{(1)}(t) + a_0 y(t) = b_m u^{(m)}(t) + \dots + b_1 u^{(1)}(t) + b_0 u(t)$$ (normalize → $a_n = 1$)

$m < n \rightarrow$ usual ↳ strictly proper $\quad m = n \rightarrow$ feedtrough $d = b_n$ ↳ proper $\quad m > n \rightarrow$ non-causal ↳ not proper

- $I/O \rightarrow \left[\begin{array}{c|c} A & b \\ \hline c & d \end{array}\right]$

$$\left[\begin{array}{c|c} A & b \\ \hline c & d \end{array}\right] = \left[\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -a_0 & -a_1 & -a_2 & -a_3 & -a_4 & 1 \\ \hline b_0 & b_1 & b_2 & 0 & 0 & 0 \end{array}\right]$$

if $m = n$; $d = b_n$ then:
↳ $b_i = b_i - b_n \cdot a_i$ for $0 \leq i \leq n-1$

- $\left[\begin{array}{c|c} A & b \\ \hline c & d \end{array}\right] \rightarrow \left[\begin{array}{c|c} A & b \\ \hline c & d \end{array}\right]$ minimal realization = I/O behaviour

decoupled!

$$\left[\begin{array}{c|c} A & b \\ \hline c & d \end{array}\right] = \left[\begin{array}{ccc|c} -1 & 1 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ \hline 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \begin{array}{l} x_1 = RO \\ x_2 = \overline{RO} \\ x_3 = R\overline{O} \end{array} \rightarrow \text{clear non RO} \rightarrow \left[\begin{array}{c|c} -1 & 1 \\ \hline 1 & 1 \end{array}\right]$$

- $\left[\begin{array}{c|c} A & b \\ \hline c & d \end{array}\right] \rightarrow I/O$

Control systems 2

30 October 2016

19:57

Laplace Transformation / Transfer Function

$$\begin{array}{ccc} \text{TD Eq.} & \xrightarrow{\text{hard}} & \text{TD Sol.} \\ \downarrow \mathcal{L} & & \uparrow \mathcal{L}^{-1} \\ \text{FD Eq.} & \xrightarrow{\text{easy}} & \text{FD Sol.} \end{array}$$

	TD	FD
SP	$\dot{x}=xA+bu$ $y=xc+du$	$Y(s)=(c(sI-A)^{-1}b+d)U(s)$
I/O	$y^{(n)}+a_{n-1}y^{(n-1)}\dots$ $=b_m u^{(m)}+b_{m-1}u^{(m-1)}$	$Y(s)=\dfrac{b_m s^m+\dots+b_0}{s^n+\dots+a_1 s^1+a_0}\,U(s)$

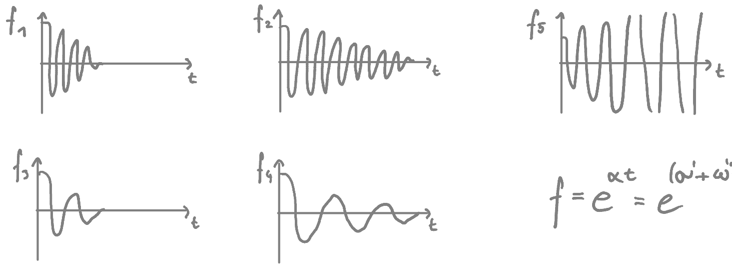
$$FD: \mathcal{L}(y)(s)=\mathcal{L}(\sigma)(s)\cdot\mathcal{L}(u)(s)\to Y(s)=\Sigma(s)\cdot U(s)$$

Transfer Function

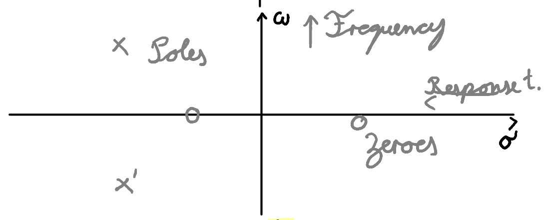
- $SP: \Sigma(s)=c(sI-A)^{-1}b+d$ or $\begin{array}{l}\dot{x}=Ax+bu\\ y=cx+du\end{array} \xrightarrow{\mathcal{L}} \begin{array}{l}sX=AX+bU\\ Y=cX+dU\end{array}\to Y(U)=\Sigma\cdot U$
- $I/O: \Sigma(s)=\dfrac{b_m\cdot s^m+\dots+b_1 s+b_0}{s^n+a_{n-1}s^{n-1}+\dots+a_1 s+a_0}$

Frequency Domain analysis

$$Y(s)=(\mathcal{L}y)(s)=\int_0^\infty y(t)\,e^{-st}\,dt=\frac{\prod(s-s_{ZEROES})}{\prod(s-s_{POLES})} \qquad (s=\sigma+\omega i)$$



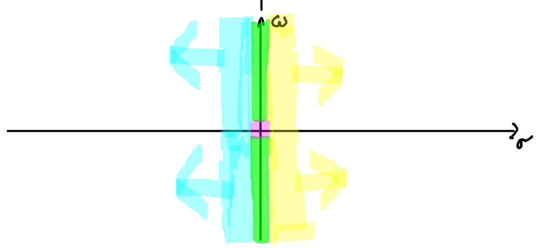
$f=e^{\alpha t}=e^{(\sigma+\omega i)t}$



$y(t)=\sum_n k_n e^{\sigma_n+\omega_n i}=\sum_n k_n e^{\sigma_n}(\cos(\omega_n)+\sin(\omega_n)i)$ (for ODE)

Poles: if $y(t)$ contains $e^{\alpha t}$ then $(\mathcal{L}y)(\alpha)=\infty\to\alpha$ pole

Zeroes: α zero of $\Sigma\to u(t)=ke^{\alpha t}\to y(t)=0\ \forall t$

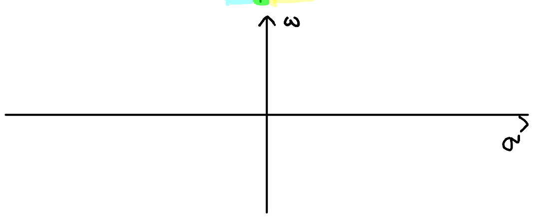


Pole interpretation:

- $\sigma>0$: $\lim_{t\to\infty}(y(t)=\blacktriangle+\bullet e^{\sigma+\blacksquare i})=\infty\to$ unstable
- $\sigma<0$: if all poles $<0\to\lim_{t\to\infty}(y(t))=0\to$ stable
- $\sigma=0,\ \omega\neq0$: $y(t)=\blacktriangle+\bullet e^{\omega i}=\blacktriangle+\bullet\cos(\omega+\varphi)\to$ oscillating
- $\sigma=0,\ \omega=0$: $y(t)=\blacktriangle+\bullet\cdot1\to$ constant if single Pole

Zeroes interpretation:

- close to a pole: reduces its effect on $y(t)$
- in $\sigma>0$: non-min-phase zero → undershoot
- in $\sigma<0$: min phase zero → overshoot



Initial/Final Value Theorem $\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\cdot Y(s)$ / $\lim_{t\to 0} y(t) = \lim_{s\to\infty} s\cdot Y(s)$ (valid if $\forall s_{POLE} \in$ ▫, ▫)

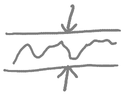
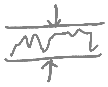
Static gain : $K = \Sigma(0) = y(\infty)$ for step resp. ($= \frac{b_0}{a_0}$ in I/O FD notation)

TD solution from Y(s) poles

$$Y(s) = \xi(s) / (s-\pi_1)^{\phi_1} \cdot (s-\pi_2)^{\phi_2} \cdot \ldots \cdot (s-\pi_n)^{\phi_n}$$

π_i: poles ϕ_i: # of poles at π_i

$$\hookrightarrow y(t) = \sum_{i=1}^{n} \sum_{k=1}^{\phi_i} \frac{g_{i,k}}{(k-1)!} \cdot t^{k-1} \cdot e^{\pi_i t} \cdot h(t) \qquad g_{i,k} = \lim_{s\to\pi_i} \frac{1}{(\phi_i-k)!} \cdot \frac{d^{(\phi_i-k)}}{ds^{(\phi_i-k)}} \left(Y(s)\cdot(s-\pi_i)^{\phi_i}\right)$$

Bibo stability $u(t) =$  $\xrightarrow{?} y(t) =$ 

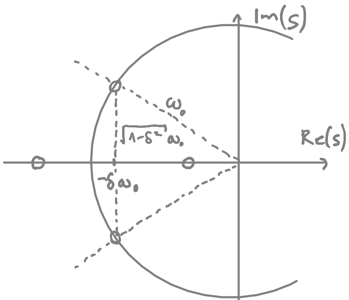
$$\text{stable if: } \int_0^\infty |o(t)|\,dt < \infty \;\hat{=}\; Re(\pi_i) < 0 \quad \forall \pi_i \text{ poles of } \Sigma(s)$$

Poles of 1^{st} order systems

$$\Sigma(s) = \frac{k}{s-\pi} = \frac{k}{\tau\cdot s+1}$$

(see cap. test signals for 1^{st} order systems)

Poles of 2^{nd} order systems (no zeroes)



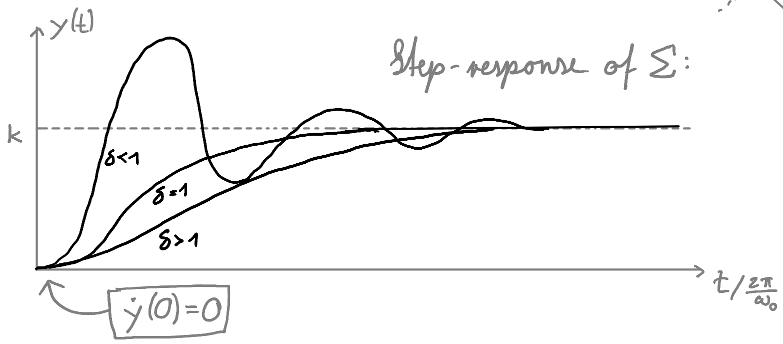
$$\Sigma(s) = k\cdot\omega_0^2 / s^2 + 2\delta\omega_0 s + \omega_0^2$$

ω_0: natural frequency
δ : damping
k : gain

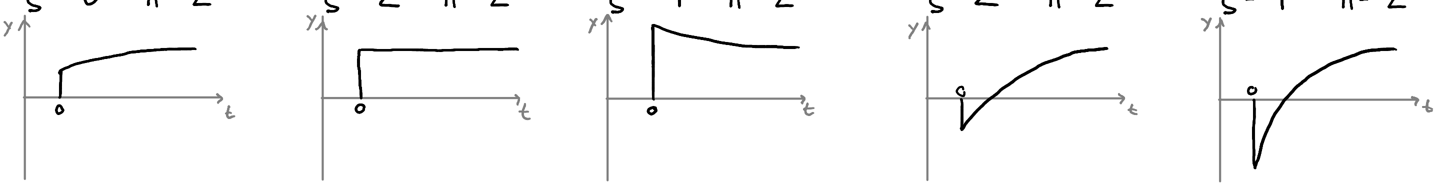
$$\pi_{1,2} = \left(-\delta \pm \sqrt{\delta^2-1}\right)\omega_0$$

$\delta > 1$: Over-damping
$\delta = 1$: critical damping
$0 < \delta < 1$: Under-damping
$\delta = 0$: no damping
$\delta < 0$: unstable

Step-response of Σ:

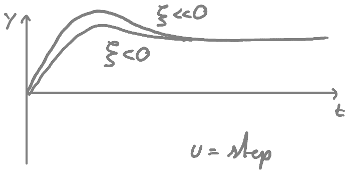


Zeroes of 1^{st} order systems $\Sigma(s) = \frac{s+\zeta}{s+\pi}$ ← zero, ← pole $\quad U(s) = h(s)$ (step-responses)

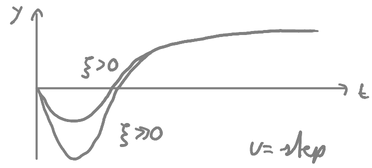


Zeroes of 2^{nd} order systems

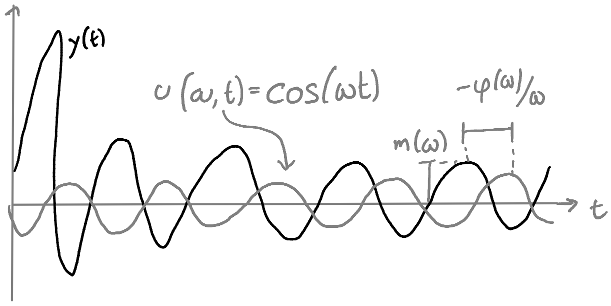
$Re(\zeta) < 0$
minimumphase zero
→ overshoot



$Re(\zeta) > 0$
nonminimumphase zero
→ undershoot



Frequency Response



$$y_\infty(\omega, t) = m(\omega)\cos(\omega t + \varphi(\omega))$$

$$m(\omega) = |\Sigma(\omega i)| \qquad \alpha = Re(\Sigma(\omega i))$$
$$\varphi(\omega) = \sphericalangle\, \Sigma(\omega i) \qquad \beta = -Im(\Sigma(\omega i))$$

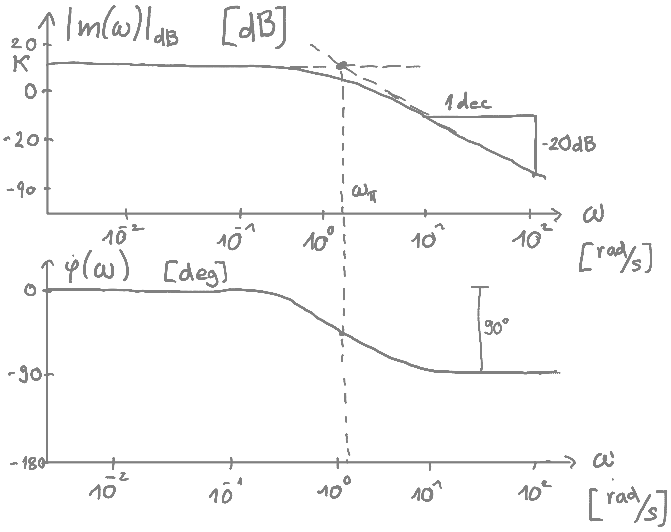
$$m = \sqrt{\alpha^2 + \beta^2} \qquad \alpha = m \cdot \cos(\varphi)$$
$$\varphi = \arctan(-\beta/\alpha) \qquad \beta = -m \cdot \sin(\varphi)$$

static gain : $K = \Sigma(0)$

cutoff-frequencies: $\omega_\pi = |\pi|$ and $\omega_\xi = |\xi|$ (poles, zeros)

$$|m(\omega)|_{dB} = 20 \cdot \log_{10}(m(\omega)); \; m(\omega) = 10 \wedge (|m(\omega)|_{dB}/20)$$

- Bode Diagram

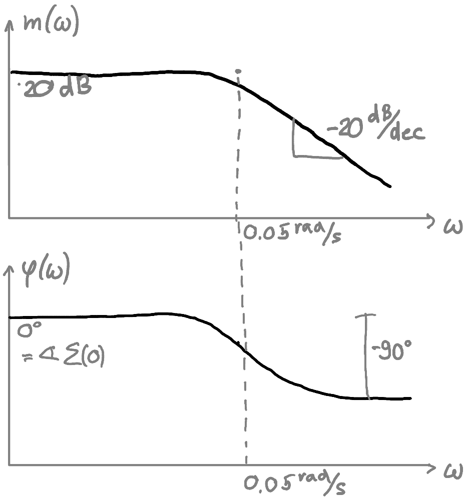


	Magnitude Change	Phase Change
stable pole	-20 dB/dec	$-90°$
unstable pole	-20 dB/dec	$+90°$
min-phase zero	$+20$ dB/dec	$+90°$
nonmin. zero	$+20$ dB/dec	$-90°$
time delay	0 dB/dec	$-\omega T$

First-Order Poles:

$\Sigma(s)=\frac{k}{\tau\cdot s+1}$ (with $k=10, \tau=20$)

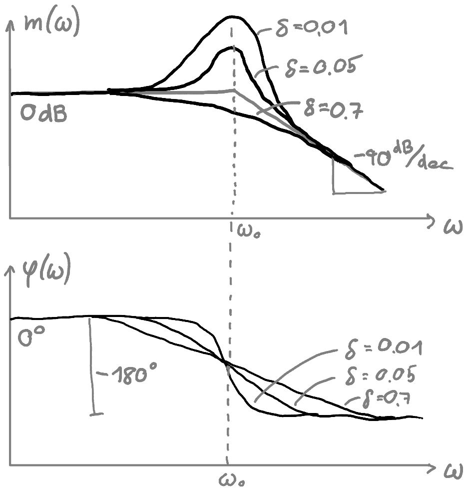
Static gain: $|\Sigma(0)|=k=10=20\,dB$
Cutoff-freq: $|\pi|=|-1/\tau|=0.05\,rad/s$
Mag. chg.: $Re(\pi)<0 \to -20\,dB/dec$
Phase chg.: $\hookrightarrow -90°$



Second Order Poles:

$\Sigma(s)=k\,\frac{\omega_0^2}{s^2+2\delta\omega_0 s+\omega_0^2}$ $(k=1)$

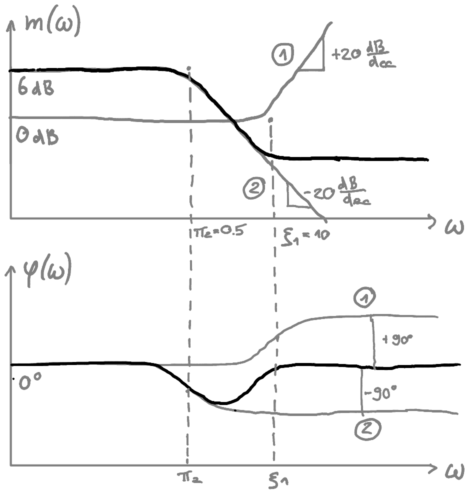
Static gain: $|\Sigma(0)|=k=1=0\,dB$
Cutoff-freq: $|\pi_1|=|\pi_2|=\omega_0$
Mag. chg: $Re(\pi_{1,2})<0 \to 2\cdot -20\,dB$
Phase chg: $\hookrightarrow 2\cdot -90°$



n-Order System:

$\Sigma_1(s)\cdot\Sigma_2(s)=(0.1s+1)\cdot\frac{1}{s+0.5}$

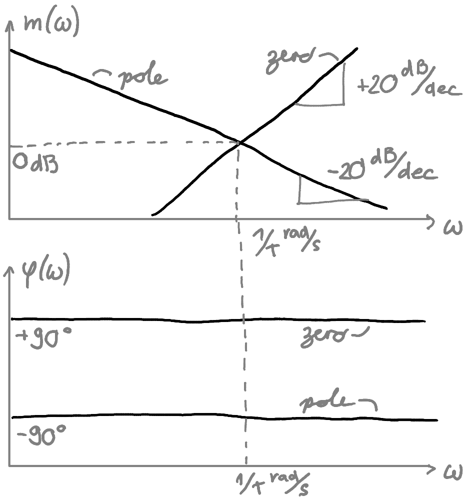
Static gain: $|\Sigma_1(0)|=0\,dB$ $|\Sigma_2(0)|=6\,dB$
Cutoff-freq: $|\xi_1|=10\,\frac{rad}{s}$ $|\pi_2|=0.5\,\frac{rad}{s}$
Mag. chg: $\xi_1\to+20\,dB$ $\pi_2\to-20\,dB$
Phase chg: $\xi_1\to+90°$ $\pi_2\to-90°$



Pole/Zero in Origin:

$\Sigma(s)=\frac{1}{Ts}$ $\Sigma(s)=T\cdot s$

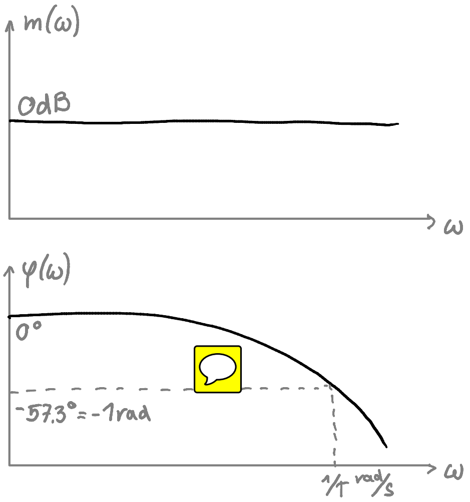
Static gain: $|\Sigma(0)|=\pm\infty\,dB$
Cutoff-freq: $|\pi|=0\,rad/s$
Mag. chg.: $\mp 20\,dB/dec$
Phase chg.: $\mp 90°$



Time delay Element:

$\Sigma(s)=e^{-sT}$ $(y(t)=u(t-T))$

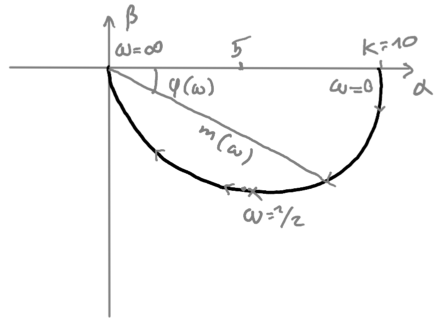
Static gain: $|\Sigma(0)|=1=0\,dB$
Cutoff-freq: -
Mag chg.: $0\,dB/dec$
Phase chg: $-\omega T\,rad/s$



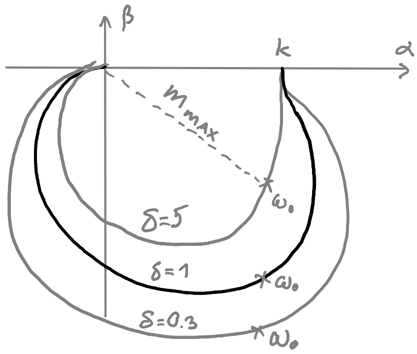
First Order Zero:

• Nyquist Diagram (for above TF)

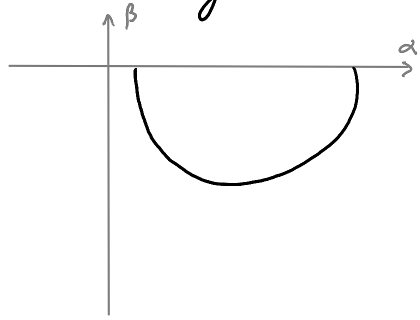
First Order Pole



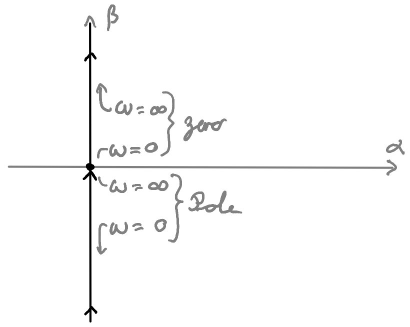
Second Order Pole



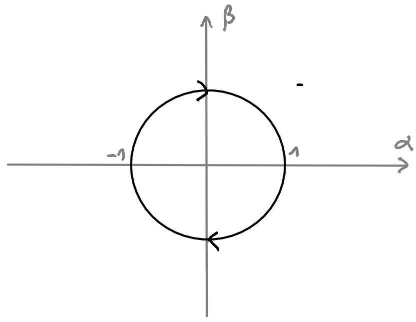
n-Order System



Pole/Zero in Origin



Time delay Element



First order zero

Asymptotic properties:

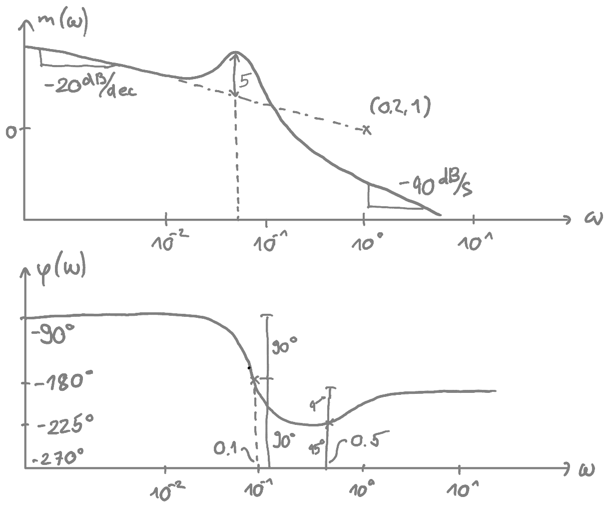
$$\Sigma(s) = \frac{b_m s^m + \ldots + b_1 s + b_0}{s^k(s^{n-k} + a_{n-1-k}\cdot s^{n-1-k} + \ldots + a_1 s + a_0)}$$

usually

k = # of Poles at 0. (Type) $\longrightarrow \varphi(0) = \begin{cases} -k\cdot 90° & \text{if } \frac{b_0}{a_0} \geq 0 \\ -180° - k\cdot 90° & \text{else} \end{cases}$

$r = n - m$ (Relative degree) $\longrightarrow \dot{m}(\infty) = -r\cdot 20\,^{dB}/_{dec}$

Bode diagram → TF



1) Identify k (Type) and r (rel. deg.)
↳ $\varphi(0) = -90° \to k = 1$ ↳ $\dot{m}(\infty) = -40\,^{dB}/_s \to r = 2$

2) From low to high ω add Σ_i's. Gain = 1 for $i \neq 0$.

$\Sigma_0 = \frac{1}{Ts}$ $\qquad 0.2 = 1/T \to T = 5$

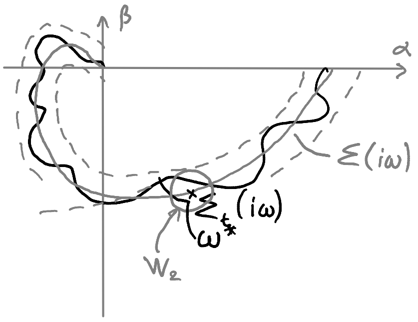
$\Sigma_1 = k\cdot\frac{\omega_0^2}{s^2 + 2\delta\omega_0 s + \omega_0^2}$ $\qquad k = 1;\ \omega_0 = 0.1;\ \Delta m = 5 \to \delta = 0.1$

$\Sigma_2 = k\cdot(\tau\cdot s + 1)$ $\qquad k = 1;\ 0.5 = 1/\tau \to \tau = 2$

$$\Sigma = \Sigma_0\cdot\Sigma_1\cdot\Sigma_2 = \frac{1}{5s}\cdot\frac{0.01}{s^2 + 0.02s + 0.01}\cdot(2s+1)$$

$\left(\Delta m(\omega_{MAX}) = 1/2\delta\sqrt{1-\delta^2}\right)$

Model uncertainty

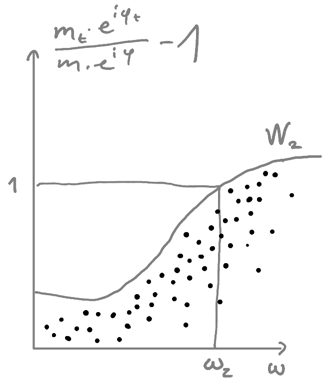


$$\text{Uncert.} = \mathcal{E}(i\omega^*) + \overset{\text{every } |\Delta|<1}{\Delta}\, \mathcal{E}(i\omega^*) \cdot \overset{\text{error}}{W_2(i\omega^*)}$$

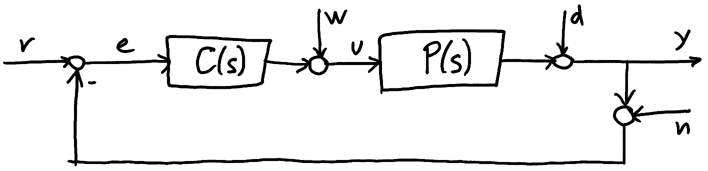
$$\mathcal{E}_t(i\omega) = m_t \cdot e^{i\varphi_t} \qquad \mathcal{E}(i\omega) = m \cdot e^{i\varphi}$$

$$\left(\frac{m_t \cdot e^{i\varphi_t}}{m \cdot e^{i\varphi}} - 1\right) \leq |W_2(i\omega)| \quad \forall \omega$$

$\omega > \omega_2 \rightarrow$ uncertainty $> 100\%$



Analysis Feedback System



r: reference
y: output
e: error

w: input disturbance
d: output disturbance
n: measurement noise

- **Loop Gain**: $L(s) = C(s) \cdot P(s)$ $\quad (e \rightarrow y : Y = L(s) \cdot E$ if $w = d = 0)$
- **Sensitivity**: $S(s) = 1/1 + L(s)$ $\quad \left(\begin{array}{lll} d \rightarrow y : & Y = S(s) \cdot D & \text{if } r = w = n = 0 \\ r \rightarrow e : & E = S(s) \cdot R & \text{if } w = d = n = 0 \end{array}\right)$
- **Complementary Sensitivity**: $T(s) = L(s)/1 + L(s)$ $\quad \left(\begin{array}{lll} -n \rightarrow y : & Y = -T(s) \cdot N & \text{if } w = d = r = 0 \\ r \rightarrow y : & Y = T(s) \cdot R & \text{if } w = d = n = 0 \end{array}\right)$

$$Y(s) = S(s) \cdot (D(s) + P(s) \cdot W(s)) + T(s) \cdot (R(s) - N(s))$$

$$S(s) + T(s) = 1$$

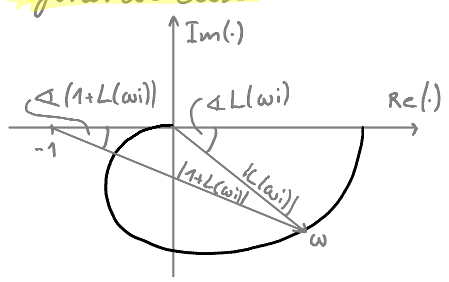
$Y(s) = S(s) \cdot D(s) - T(s) N(s)$ $\qquad (W(s) = R(s) = 0)$

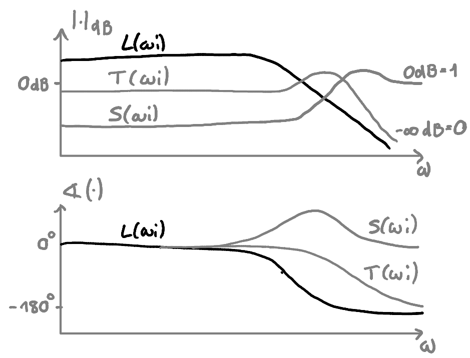
$|L(s)| \rightarrow \infty \Rightarrow S(s) \rightarrow 0,\ T(s) \rightarrow 1$ $\quad \rightarrow$ disturbance has no effect

$|L(s)| \rightarrow 0 \Rightarrow T(s) \rightarrow 0,\ S(s) \rightarrow 1$ $\quad \rightarrow$ noise has no effect

Graphical interpretation of L, S, T for frequency response

- **General case**:

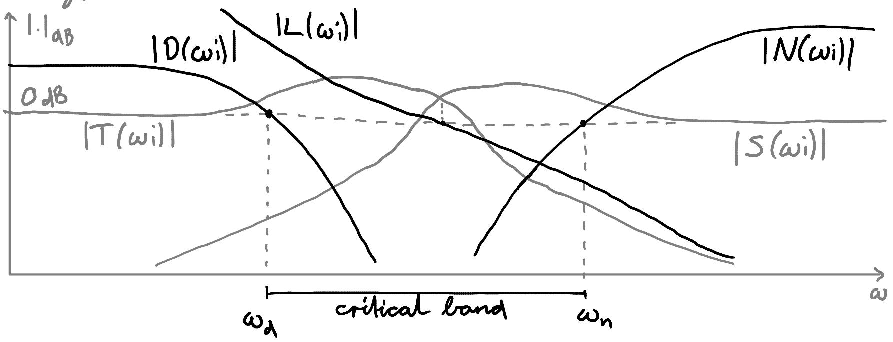




$$|S(\omega i)| = 1/|1 + L(i\omega)|$$
$$\sphericalangle S(\omega i) = -\sphericalangle(1 + L(i\omega))$$

$$|T(\omega i)| = |L(i\omega)| / |1 + L(i\omega)|$$
$$\sphericalangle T(\omega i) = \sphericalangle L(i\omega) - \sphericalangle(1 + L(i\omega))$$

- Typical case:



$N(\omega i)$: "fast" $D(\omega i)$: "slow"

$Y(\omega i) = S(\omega i)\, D(\omega i) - T(\omega i)\, N(\omega i)$

$\downarrow$

ω "slow": $S=0, T=1 \Rightarrow Y=0$

ω "fast": $S=1, T=0 \Rightarrow Y=0$

Closed-Loop System Stability

- Method 1:

$$\begin{pmatrix} U \\ Y \\ E \end{pmatrix} = \begin{pmatrix} S & -S\cdot C & S\cdot C \\ S\cdot P & S & T \\ -S\cdot P & -S & S \end{pmatrix} \cdot \begin{pmatrix} W \\ D \\ R \end{pmatrix}$$

all 9 Func. in the Mat. are assymptotically stable $\Rightarrow$ system stable

- Method 2:

$S(s)$ is assymptotically stable and $S(\xi_i^+) = 1$, $S(\pi_i^+) = 0$ $\Rightarrow$ system stable

of $L(s)$

- Method 3:

$1 + L(s)$ has no zero in $Re(s) > 0$ and no canc. of ξ_i / π_i of L in S, T $\Rightarrow$ system stable

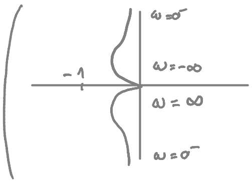
Nominal Closed-Loop System Stability (Nyquist Theorem)

n_+ : # of Poles of $L(s)$ with $Re(s) > 0$

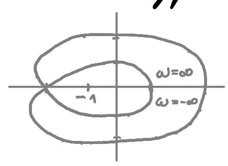
n_0 : # of Poles of $L(s)$ with $Re(s) = 0$

$n_c = n_0/2 + n_+ \Rightarrow$ system stable

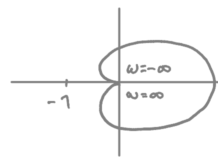
n_c : # of CCW encirclements of -1 in the Nyquist diag. of $L(s)$ for $-\infty < \omega < \infty$ $\left(L(-\omega i) = \overline{L}(\omega i) \right)$



$\Rightarrow n_c = 1/2$

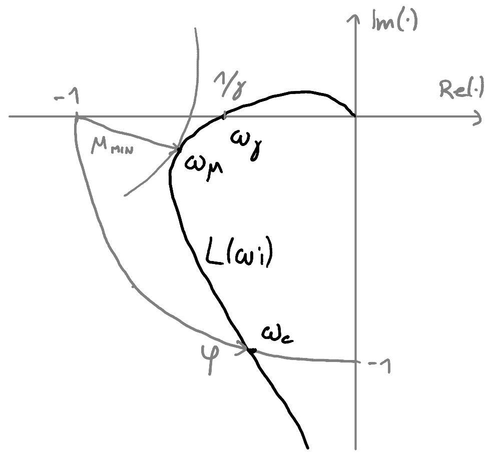


$\Rightarrow n_c = -2$



$\Rightarrow n_c = 0$

Robust Closed-Loop System Stability (Nyquist Theorem)



ω_c : crossover Frequency $|L(\omega_c i)| = 1$ $|S(\omega_c i)| = |T(\omega_c i)|$

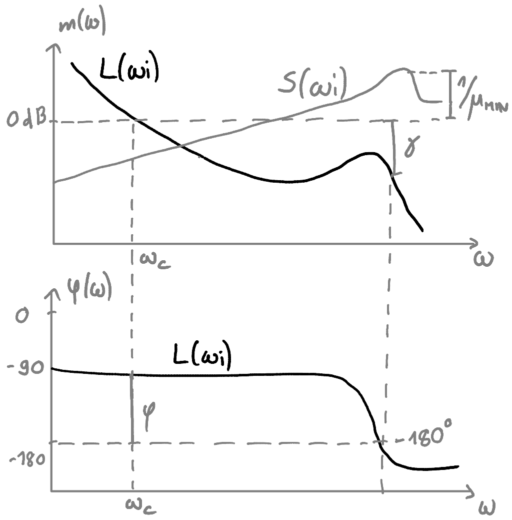
ω_b : bandwidth $|T(\omega_b i)| = -3dB$

$\omega_c \approx \omega_b$ (ω_c, ω_b high $\rightarrow$ fast response in TD)

φ : Phase Margin ($\measuredangle L(\omega_c i) + 180°$)

γ : Gain Margin ($|L(\omega^* i)|$ where $\measuredangle L(\omega^* i) = -180°$)

M_{MIN} : minimum return difference ($\min_\omega |1 + L(\omega i)| = 1 / \max_\omega |S(\omega i)|$)



Robust stability conditions:

- $n_c = n_0/2 + n_+$
- $\left\{\begin{array}{l} \text{if } \exists\ W_2(\omega i) : |L(\omega i) \cdot W_2(\omega i)| < |1 + L(\omega i)| \quad \forall \omega \in [0, \infty) \\ \text{else: } \varphi \geq 60° \quad \gamma \geq 2 \quad \mu_{MIN} \geq 1/2 \end{array}\right.$
- $\max |S(\omega i)| \leq 6\,dB \quad \max |T(\omega i)| \leq 6\,dB$ (same as φ, γ, μ?)

- $\omega_c \in \left(\max(10 \cdot \omega_d, 2|\pi^+|), \min\left(\frac{|\xi^+|}{2}, \frac{\omega_T}{2}, \frac{\omega_2}{5}, \frac{\omega_n}{10}\right) \right)$

of $L(s)$

π^+: "fastest" unstable pole ω_n: noise rejection freq.

ξ^+: "slowest" non-min-phase zero ω_2: uncertainty greater 100%

ω_d: disturbance rejection freq. ω_T: $1/T$ if delay element exists

Control systems 3

09 December 2016
10:37

Specifications for Feedback Systems

• Static Error

$$E(s) = S(s)\cdot\left(R(s) - N(s) - D(s) - P(s)\cdot W(s)\right) \Rightarrow e_\infty = \lim_{t\to\infty} e(t) = \lim_{s\to 0} E(s)$$

• r, n or d = step

$$E(s) = S(s)\tfrac{1}{s} \Rightarrow e_\infty = S(0) = 1/_{1+L(0)}$$

$$e_\infty = 0 \rightarrow P(s) \text{ or } C(s) \text{ has } k \geq 1 \quad (\text{\# of poles at } 0)$$

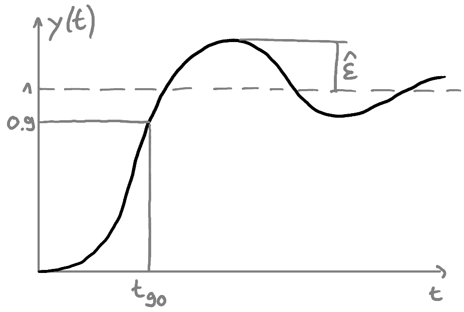
• w = step

$$E(s) = -S(s)P(s)\tfrac{1}{s} \Rightarrow e_\infty = -P(0)/_{1+P(0)C(0)}$$

$$e_\infty = 0 \rightarrow C(s) \text{ has } k \geq 1 \quad (\text{\# of poles at } 0)$$

• Response time / Overshoot

(step. resp; $T(s) \approx$ 2nd-Order)



$$\omega_c \approx 1.7/t_{90}$$

$$\varphi \approx 71° - 117°\,\hat{\varepsilon}$$

for $\hat{\varepsilon} < 0.205$ and no π^+, ζ^+ in L

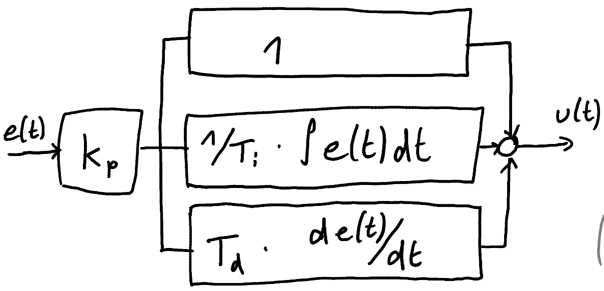
• Robustness

• Reject disturbances/noise

(robust stability cpt. above)

• Limited control action

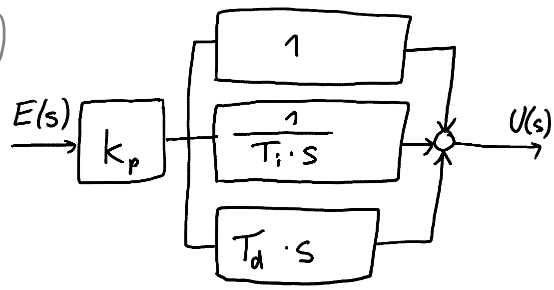
PID Controller



(← Proportional →)

(← Integral →)

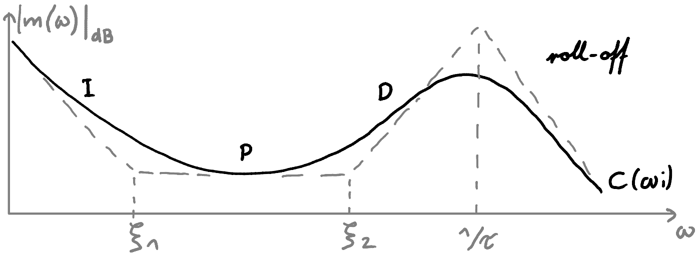
(← Differential →)

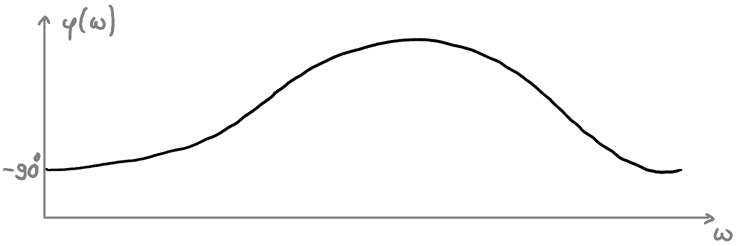


$$u(t) = k_p\left(1\cdot e(t) + 1/T_i \int e(t)\,dt + T_d \frac{de(t)}{dt}\right)$$

$$U(s) = k_p\cdot\left(1 + \frac{1}{T_i\cdot s} + T_d\cdot s\right)\cdot E(s)$$

$$C(s) = k_p \Big(\underbrace{1}_{P} + \underbrace{\frac{1}{T_i \cdot s}}_{I} + \underbrace{T_d \cdot s}_{D}\Big) \cdot \underbrace{\frac{1}{(s\cdot\tau+1)^2}}_{\text{roll-off}} = \frac{k_p + k_p \cdot T_i \cdot s + k_p \cdot T_i \cdot T_d \cdot s^2}{T_i \cdot s} \cdot \frac{1}{(s\cdot\tau+1)^2}$$





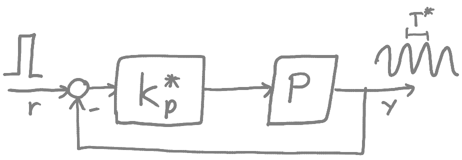
PID Param. determination

$P(s) \approx \frac{k}{\tau \cdot s+1} \cdot e^{-T\cdot s}$ and $\frac{T}{T+\tau} \ll 0.3$ (T: time delay τ: time constant)

- Set $T_i = \infty$, $T_d = 0$, $k_p = k_p^*$ (P-Controller)
- For $r(t) = \delta(t)$ ⎍, set k_p^* so that $y(t)$ oscillates
- Remember k_p^* and oscillation period T^*
- Pick k_p, T_i, T_d from Table

or

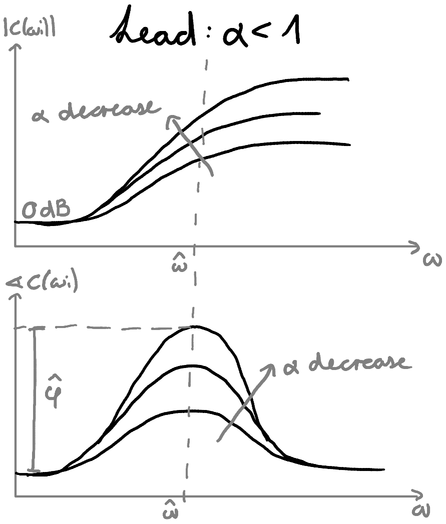
- $\measuredangle(k_p^* \cdot P(\omega^* i)) = -180° \rightarrow \omega^* = \ldots$
- $T^* = 2\pi/\omega^*$
- $|k_p^* \cdot P(\omega^* i)| = 1 \rightarrow k_p^* = \ldots$
- Pick k_p, T_i, T_d from Table

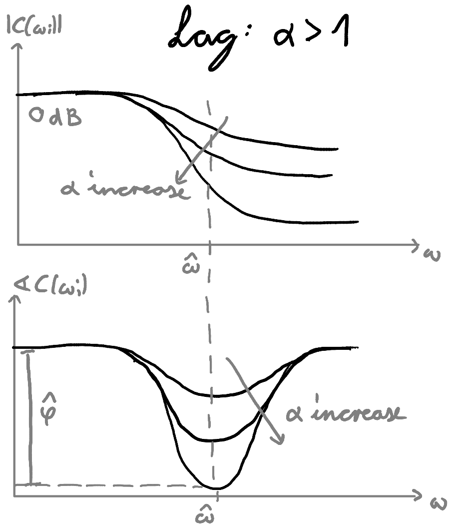


	k_p	T_i	T_d
P	$0{,}50 \cdot k_p^*$	$\infty \cdot T^*$	$0 \cdot T^*$
PI	$0{,}45 \cdot k_p^*$	$0{,}85 \cdot T^*$	$0 \cdot T^*$
PD	$0{,}55 \cdot k_p^*$	$\infty \cdot T^*$	$0{,}15 \cdot T^*$
PID	$0{,}60 \cdot k_p^*$	$0{,}50 \cdot T^*$	$0{,}125 \cdot T^*$

Lead / Lag Element

$C(s) = \frac{T\cdot s+1}{\alpha \cdot T \cdot s+1}$ $(\alpha, T > 0)$

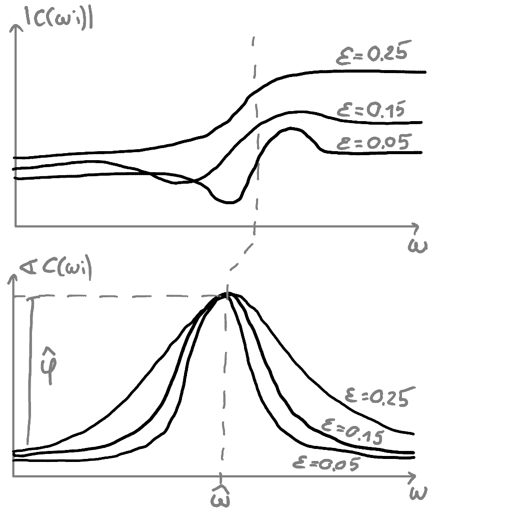




$$\alpha = \left(\sqrt{\tan^2(\hat{\varphi})+1} - \tan(\hat{\varphi})\right)^2$$

$$T = 1/\hat{\omega}\sqrt{\alpha}$$

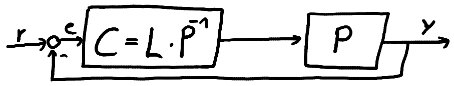
$$C(s) = k \cdot \frac{s^2 + 2\kappa\varepsilon\cdot(1-\varepsilon)\omega_0 s + (1-\varepsilon)^2\omega_0^2}{s^2 + 2\kappa\varepsilon\cdot(1+\varepsilon)\omega_0 s + (1+\varepsilon)^2\omega_0^2}$$

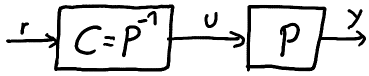


$k = \frac{(1+\varepsilon)^2}{(1-\varepsilon)^2}$ (usually $k=1 \rightarrow \varepsilon=0$)

$\kappa = \frac{\cot(\hat{\varphi}/2)}{\sqrt{1-\varepsilon^2}}$ $\omega_0 = \frac{\hat{\omega}}{\sqrt{1-\varepsilon^2}}$

Plant Inversion (no unstable poles or nonmin zeros in P!)

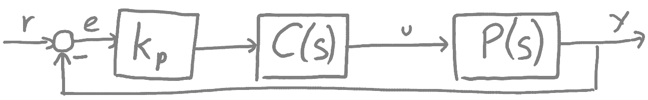
Feedback: 

Feedforward: 

$L(s) = \frac{1}{T_i \cdot s} \cdot \frac{1}{(\tau s+1)^{r-1}}$

- r = rel. deg. of P
- $T_i = 1/\omega_c$

Root-Locus Method

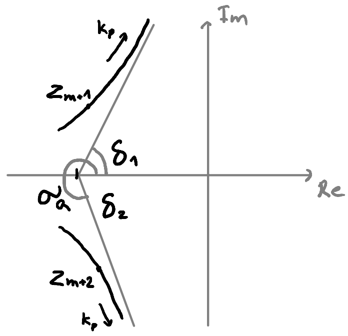


Goal: choose k_p so that poles of $T(s)$ are on π_{des}

$$L(s) = P(s) \cdot C(s) = \frac{b(s)}{a(s)} = \frac{b_m \cdot s^m + \ldots + b_1 s + b_0}{s^n + \ldots + a_1 s + a_0} = b_m \cdot \frac{(s-\zeta_1)\cdot \ldots \cdot (s-\zeta_m)}{(s-\pi_1)\cdot \ldots \cdot (s-\pi_n)} \quad (0 \le m \le n,\ b_m \neq 0)$$

closed loop poles (of $T(s)$): $a(z) + k_p \cdot b(z) = 0 \Rightarrow z_n$ poles for any k_p

- $k_p = 0 \Rightarrow z_n = \pi_n$
- $k_p \to \infty \Rightarrow z_{1\,to\,m} = \zeta_m$, $z_{m\,to\,n} \to \infty$



$$\sigma_a = \frac{1}{n-m} \cdot \left(\sum_{i=1}^{n} Re(\pi_i) - \sum_{i=1}^{m} Re(\zeta_i) \right)$$

$$\delta_i = \frac{\pi}{n-m} \cdot (2(i-1)+1) \quad (i = 1, \ldots, n-m)$$

z is part of the root-locus if:

- some k_p can make z a pole of $T(s)$.
- $\sum_{i=1}^{m} \angle(z-\zeta_i) - \sum_{i=1}^{n} \angle(z-\pi_i) = \pi$

π_{des} from $\hat{\varepsilon}, t_{90}$: $\pi_{des} = -\delta \cdot \omega_0 \pm i\sqrt{1-\delta^2}\,\omega_0$ $\quad \left(\delta = \frac{-\ln(\hat{\varepsilon})}{\sqrt{\pi^2 + \ln^2(\hat{\varepsilon})}},\ \omega_0 = (0.14 + 0.48)\frac{2\pi}{t_{90}}\right)$

Example: (Given: $P(s) = \frac{1}{(s+1)(s+3)}$, $\pi_{des} = -4+4i \to \pi_{des}' = -4-4i$)

- $C_1(s) = 1 \to L(s) = \frac{1}{(s+1)(s+3)} \to \pi_1 = -1,\ \pi_2 = -3$

$\to \angle(\pi_{des} - \pi_1) + \angle(\pi_{des} - \pi_2) \neq \pi \to$ π_{des} not on root-locus!

$\cdot\ C_2(s) = s+a \rightarrow L(s) = \frac{s+a}{(s+1)(s+3)} \rightarrow \pi_1 = -1,\ \pi_2 = -3,\ \zeta_1 = -a$

$\rightarrow \angle(\pi_{des} - \pi_1) + \angle(\pi_{des} - \pi_2) - \angle(\pi_{des} - \zeta_1) = \pi \rightarrow a = 7{,}25$

$\rightarrow a(\pi_{des}) + k_p \cdot b(\pi_{des}) = 0 \rightarrow k_p = 4 \rightarrow$ but C_2 not realizable

$\cdot\ C_3(s) = \frac{s+1}{s+a} \rightarrow L(s) = \frac{1}{(s+3)(s+a)} \rightarrow \pi_1 = -3,\ \pi_2 = -a \rightarrow \dots \rightarrow a = \quad, k_p =$

only if pole is stable