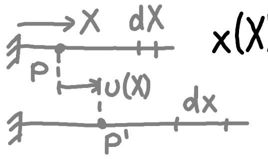
Computational Mechanics II (nonlinear FEA)

nonlinearity sources
- **geometric**: large displacements, large strains, instability
- **material**: hyperelasticity, plasticity, damage
- **BCs**: contact, follower loads

strain measures

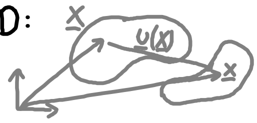
[x: deformed coord. X: reference coord. u: displacement]

- **1D**:



$x(X) = X + u(X)$

- **2D**:



$\underline{x}(\underline{X}) = \underline{X} + \underline{u}(\underline{X})$

deformation gradient: $F = dx/dX = 1 + du/dX$ —— $\underline{\underline{F}} = \frac{d\underline{x}}{d\underline{X}} = \underline{\underline{I}} + \frac{d\underline{u}}{d\underline{X}} = \underline{\underline{I}} + \nabla_X \underline{u}$

linear strain: $\varepsilon = \frac{dx - dX}{dX} = \frac{du}{dX} = F - 1$ —— $\underline{\underline{\varepsilon}} = \frac{1}{2}(\nabla_X \underline{u} + \nabla_X \underline{u}^T) = \nabla_X^s \underline{u}$

$\varepsilon_{11} = \partial u_1/\partial X_1$; $\varepsilon_{22} = \partial u_2/\partial X_2$

$\varepsilon_{12} = \varepsilon_{21} = \frac{1}{2}(\partial u_1/\partial X_2 + \partial u_2/\partial X_1)$

Green-Lagrange strain: $\varepsilon_G = \frac{1}{2}\frac{dx^2 - dX^2}{dX^2} = \frac{1}{2}(F^2 - 1) = \varepsilon + \frac{1}{2}\varepsilon^2$

logarithmic strain: $\varepsilon_L = \ln(dx/dX) = \ln(F) = \ln(1+\varepsilon)$

$\underline{\underline{E}} = \frac{1}{2}(\underline{\underline{F}}^T\underline{\underline{F}} - \underline{\underline{I}}) = \ldots = \frac{1}{2}(\nabla_X \underline{u} + \nabla_X \underline{u}^T + \nabla_X \underline{u} \nabla_X \underline{u}^T)$

$E_{11} = \partial u_1/\partial X_1 + \frac{1}{2}[(\partial u_1/\partial X_1)^2 + (\partial u_2/\partial X_1)^2]$

$E_{22} = \partial u_2/\partial X_2 + \frac{1}{2}[(\partial u_1/\partial X_2)^2 + (\partial u_2/\partial X_2)^2]$

$E_{12} = E_{21} = \frac{1}{2}\left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} + \frac{\partial u_1}{\partial X_1}\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\frac{\partial u_2}{\partial X_2}\right)$

[motivation: $\underline{x} = \underline{\underline{R}}\,\underline{X} \rightarrow \underline{\underline{F}} = \frac{d\underline{x}}{d\underline{X}} = \underline{\underline{R}}$ ($\underline{\underline{R}}$: rigid body rotation)

$\rightarrow \underline{\underline{\varepsilon}} = \ldots = \frac{1}{2}(\underline{\underline{R}} + \underline{\underline{R}}^T - 2\underline{\underline{I}}) \neq 0$ (BAD! rotation → 0 strain irl)

$\rightarrow \underline{\underline{E}} = \ldots = \frac{1}{2}(\underline{\underline{R}}^T\underline{\underline{R}} - \underline{\underline{I}}) = 0$ (BETTER strain measure!)]

variational calculus

(if $\delta\mathcal{F}(u)[\delta u] = 0 \ \forall \delta u \rightarrow u$: extremum of $\mathcal{F}(u)$)

functional: $\mathcal{F}: \mathcal{U} \rightarrow \mathbb{R}$, $\mathcal{F}(u)$ (function of functions assigning "energy" $\in \mathbb{R}$ to all functions $u \in \mathcal{U}$)

[conditions on function $u \in \mathcal{U}$:
- sufficiently regular $\leftrightarrow \mathcal{F}(u \in \mathcal{U}) \neq \pm\infty$ —— { e.g.: square integrable $\mathcal{U} \in L^2(\Omega) \leftrightarrow \int_\Omega u^2(X)dX < \infty$; Sobolev space $\mathcal{U} \in H^k(\Omega) \leftrightarrow \int_\Omega u^2 + (\frac{du}{dX})^2 + \ldots + (\frac{d^k u}{dX^k})^2 dX < \infty$; ⋮ }
- satisfy Dirichlet BCs $\leftrightarrow u(X \in \Omega_D) = \hat{u}(X)$]

admissible variation of u: $\delta u \in \mathcal{U}_0$ (any func. as regular as u and with $\delta u(X \in \Omega_D) = 0$)

variation of $\mathcal{F}$ at u in dir. δu: $\delta\mathcal{F} = \delta\mathcal{F}(u)[\delta u] = \lim_{\substack{\epsilon \to 0 \\ \epsilon \in \mathbb{R}}} \frac{\mathcal{F}(u + \epsilon\delta u) - \mathcal{F}(u)}{\epsilon} = \frac{d}{d\epsilon}\mathcal{F}(u + \epsilon\delta u)\Big|_{\epsilon=0}$

rules:

$\delta(\mathcal{F}_1 + \mathcal{F}_2) = \delta\mathcal{F}_1 + \delta\mathcal{F}_2$

$\delta(\alpha\mathcal{F}) = \alpha\delta\mathcal{F}$

$\delta(\mathcal{F}_1 \cdot \mathcal{F}_2) = \delta\mathcal{F}_1 \cdot \mathcal{F}_2 + \mathcal{F}_1 \cdot \delta\mathcal{F}_2$

$\delta(u_{,x}) = (\delta u)_{,x}$

$\delta\int_\Omega u\,dX = \int_\Omega \delta u\,dX$

if $\mathcal{F}(u) = \mathcal{F}_1(\mathcal{F}_2(u))$:

$\delta\mathcal{F}(u)[\delta u] = \delta\mathcal{F}_1(\mathcal{F}_2(u))[\delta\mathcal{F}_2(u)[\delta u]]$

linearization

- $f(x): \mathbb{R} \rightarrow \mathbb{R}$

$f(\bar{x} + \Delta x) \simeq f(\bar{x}) + \underbrace{df/dx|_{x=\bar{x}} \cdot \Delta x}_{Df(\bar{x})[\Delta x]}$

- $f(\underline{x}): \mathbb{R}^n \rightarrow \mathbb{R}$

$f(\underline{\bar{x}} + \Delta\underline{x}) \simeq f(\underline{\bar{x}}) + \underbrace{\nabla f(\underline{\bar{x}}) \cdot \Delta\underline{x}}_{Df(\underline{\bar{x}})[\Delta\underline{x}]}$

- $\mathcal{F}(u): \mathcal{U} \rightarrow \mathbb{R}$ (Functional)

$\mathcal{F}(u + \Delta u) \simeq \mathcal{F}(u) + \underbrace{\frac{d}{d\epsilon}\mathcal{F}(u + \epsilon\Delta u)\Big|_{\epsilon=0}}_{D\mathcal{F}(u)[\Delta u]}$

linearized increment of f at $\bar{x}$ in dir. Δx ← $Df(\bar{x})[\Delta x]$ —— $Df(\underline{\bar{x}})[\Delta\underline{x}]$ —— $D\mathcal{F}(u)[\Delta u]$

$\boxed{\text{same as variation but with } \delta u \rightarrow \Delta u\text{! same rules apply}}$

[ex.:

$F(u) = 1 + u_{,x}$ $\quad$ $\varepsilon_G(u) = \frac{1}{2}(F(u)^2 - 1)$

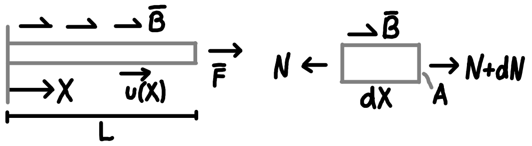
$\delta F = \delta u_{,x}$ $\quad$ $\delta\varepsilon_G = F(u)\delta u_{,x} = (1 + u_{,x})\delta u_{,x}$

$DF = \Delta u_{,x}$ $\quad$ $D\varepsilon_G = (1 + u_{,x})\delta u_{,x}$

$D\delta F = 0$ $\quad$ $D\delta\varepsilon_G = \Delta u_{,x}\delta u_{,x}$]

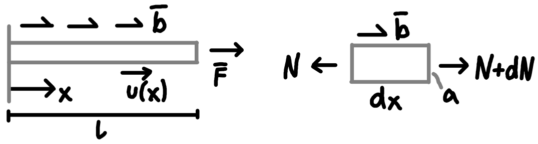
Nonlinear elastic bar (hyperelastic material + const ext. forces)

reference config.:



B: force/ref. volume $\Omega=(0,L)$: ref domain
A: ref. area $\partial\Omega_D=\{0\}$, $\partial\Omega_N=\{L\}$

current config.:



b: force/curr. volume $\omega=(0,l)$: curr. domain
a curr. area $\partial\omega_D=\{0\}$, $\partial\omega_N=\{l\}$

- **first Piola-Kirchhoff stress:** $P=N/A$ (force/ref. area)

$-N+N+dN+\bar{B}A\,dX=0 \Rightarrow N_{,X}+\bar{B}A=0 \Rightarrow (PA)_{,X}+\bar{B}A=0$

$\hookrightarrow (PA)_{,X}+\bar{B}A=0$ in Ω, $u=\bar{u}$ on $\partial\Omega_D$, $PA=\bar{F}$ on $\partial\Omega_N$ **strong form**

$\int_0^L (PA)_{,X}+\bar{B}A\,dX\cdot\delta u=0 \quad \forall\delta u\in\mathcal{U}_0$

$\int_0^L (PA\cdot\delta u_{,X}-\bar{B}A\delta u)dX - PA\cdot\delta u\big|_0^L=0$

$\int_0^L (P\delta F-\bar{B}\delta u)A\,dX-\bar{F}\delta u(L)=0$ **weak form**

$\delta\Pi=\underbrace{\int_0^L P\delta F A\,dX}_{\delta\Pi_{int}}-\underbrace{\left(\int_0^L\bar{B}\delta u A\,dX+F\delta u(L)\right)}_{\delta\Pi_{ext}}=0 \quad \forall\delta u$

$\delta\Pi=\delta\Pi_{int}-\delta\Pi_{ext}=0 \quad \forall\delta u$

P and F are work conjugate!
$\hookrightarrow$ constit. law. $P=\text{func}(F)$ is practical!

- **Cauchy stress:** $\sigma=N/a$ (force/curr. area)

$-N+N+dN+\bar{b}a\,dx=0 \Rightarrow N_{,x}+\bar{b}a=0 \Rightarrow (\sigma a)_{,x}+\bar{b}a=0$

skipped derivation but is analogous to P

σ and $u_{,x}$ are work conjugate!
$\hookrightarrow$ constit. law. $\sigma=\text{func}(u_{,x})$ is practical!

- **second Piola-Kirchhoff stress:** S work conjugate to ε_G!

$P\delta F\overset{!}{=}S\delta\varepsilon_G\rightarrow\ldots\rightarrow S=F^{-1}P\rightarrow\delta\Pi_{int}=\int_0^L S\,\delta\varepsilon_G\,A\,dx$
$\hookrightarrow$ constit. law. $S=\text{func}(\varepsilon_G)$ is practical!

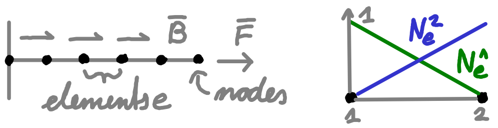
- **elastic strain energy density:** Ψ (in elastic mat.)

$\Pi_{int}=\int_0^L\Psi A\,dX\rightarrow\delta\Pi_{int}=\int_0^L\delta\Psi A\,dX\rightarrow\delta\Psi=P\delta F=S\delta\varepsilon_G$

$\hookrightarrow P=\partial\Psi/\partial F$, $S=\partial\Psi/\partial\varepsilon_G$, $(\sigma=\partial\Psi/\partial u_{,x})$

- **St. Venan-Kirchhoff material:** $S=E\varepsilon_G$ (1D)
(linear elastic model that's good for hyperelasticity)

- **finite element discretization**



$u_e(X)\simeq u_e^h(X)=\sum_{a=1}^2 N_e^a(X)\cdot u_e^a$
$\delta u_e(X)\simeq\delta u_e^h(X)=\sum_{a=1}^2 N_e^a(X)\,\delta u_e^a$

$\begin{pmatrix} N_e^1(X)=1-X/L_e\,; & N_e^2(X)=X/L_e \\ N_{e,X}^1(X)=-1/L_e\,; & N_{e,X}^2(X)=1/L_e \end{pmatrix}$

$F_e^h=1+u_{e,X}^h=1-u_e^1/L_e+u_e^2/L_e$; $\varepsilon_{Ge}^h=\frac{1}{2}(F_e^{h^2}-1)$
$\delta F_e^h=-\delta u_e^1/L_e+\delta u_e^2/L_e$; $\delta\varepsilon_{Ge}^h=F_e^h/L_e\,(\delta u_e^2-\delta u_e^1)$

$\delta\Pi_{int,e}^h=\int_{\Omega_e}S_e^h\,\delta\varepsilon_{Ge}^h A_e\,dX=\ldots=\delta U_e^T\cdot F_{int,e}$
$\delta\Pi_{ext,e}^h=\int_{\Omega_e}\bar{B}_e\,\delta u_e^h A_e\,dX=\ldots=\delta U_e^T\cdot F_{ext,e}$

$F_{int,e}=\begin{bmatrix}-S_e^hF_e^hA_e\\ S_e^hF_e^hA_e\end{bmatrix}$, $F_{ext,e}=\begin{bmatrix}\int_{\Omega_e}N_e^1\bar{B}_eA_e\,dX\\ \int_{\Omega_e}N_e^2\bar{B}_eA_e\,dX\end{bmatrix}$ $\delta U_e^T=\begin{bmatrix}\delta u_e^1\\ \delta u_e^2\end{bmatrix}$

$\delta\Pi^h=\delta\Pi_{int}^h-\delta\Pi_{ext}^h=\sum_{e=1}^{n_e}\delta\Pi_{int,e}^h-\delta\Pi_{ext,e}^h=\sum_{e=1}^{n_e}\delta U_e^T(F_{int,e}(U_e^h)-F_{ext,e})=\delta U^T(F_{int}(U)-F_{ext})=0 \quad \forall\delta U$

$\hookrightarrow F_{int}(U)-F_{ext}=0$ for $U\in\mathcal{U}$ $\left(U=\mathsf{A}_{e=1}^{n_e}U_e\,;\ \delta U=\mathsf{A}_{e=1}^{n_e}\delta U_e\,;\ F_{int}=\mathsf{A}_{e=1}^{n_e}F_{int,e}\,;\ F_{ext}=\mathsf{A}_{e=1}^{n_e}F_{ext,e}\right)$

- **linearization:** $R(U)=F_{int}(U)-F_{ext}\rightarrow R(U+\Delta U)\simeq R(U)+DR(U)[\Delta U]=R(U)+DF_{int}(U)[\Delta U]$

$DF_{int,e}^1=-DF_{int,e}^2=-DS_e^hF_e^hA_e-S_e^hDF_e^hA_e=-ED\varepsilon_{Ge}^hF_e^hA_e-E\varepsilon_{Ge}^hDF_e^hA_e$

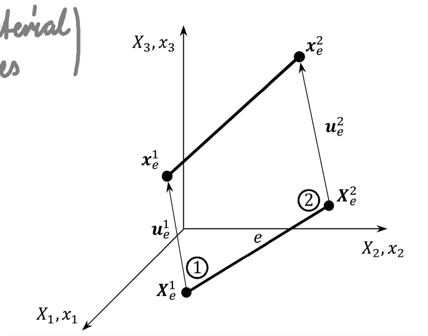
$\left\{\begin{array}{l} S_e^h=E\varepsilon_{Ge}^h\\ DF_e^h=1/L_e(\Delta u_e^2-\Delta u_e^1)\\ D\varepsilon_{Ge}^h=F_e^h/L_e(\Delta u_e^2-\Delta u_e^1)\end{array}\right.$

$\hookrightarrow DF_{int,e}=\underbrace{\frac{EA_e}{L_e}(F_e^{h^2}+\varepsilon_{Ge}^h)\begin{bmatrix}1&-1\\-1&1\end{bmatrix}}_{K_{t,e}(U_e)}\cdot\underbrace{\begin{bmatrix}\Delta u_e^1\\ \Delta u_e^2\end{bmatrix}}_{\Delta U_e}\rightarrow DF_{int}=\mathsf{A}_{e=1}^{n_e}DF_{int,e}\,,\ K_t=\mathsf{A}_{e=1}^{n_e}K_{t,e}$

[here discretization → linearization or linearization → discretization produce same result!
This is not generally the case. e.g. not true with plasticity]

nonlinear elastic trusses

(hyperelastic material, const. ext. forces)



$\underline{x}_e^1 = \underline{X}_e^1 + \underline{u}_e^1 \; ; \; \underline{x}_e^2 = \underline{X}_e^1 + \underline{u}_e^2$

$\underline{v}_e = \underline{x}_e^2 - \underline{x}_e^1$

$l_e = \|\underline{v}_e\| = \sqrt{\underline{v}_e \cdot \underline{v}_e}$

$\lambda_e = l_e / L_e$

$\varepsilon_{Ge} = \frac{1}{2}(\lambda_e^2 - 1)$

$S_e = E\,\varepsilon_{Ge}$

auxiliary matrices:

$\underline{V}_e = \begin{bmatrix} -\underline{v}_e \\ \underline{v}_e \end{bmatrix} \qquad \underline{\underline{M}} = \begin{bmatrix} \mathbb{I} & -\mathbb{I} \\ -\mathbb{I} & \mathbb{I} \end{bmatrix}$

$\underline{U}_e = \begin{bmatrix} \underline{u}_e^1 \\ \underline{u}_e^2 \end{bmatrix}$

variations/linearizations:

$\delta \underline{v}_e = \delta \underline{x}_e^1 - \delta \underline{x}_e^2 = \delta \underline{u}_e^1 - \delta \underline{u}_e^2 \; ; \; D\underline{v}_e = \Delta \underline{u}_e^2 - \Delta \underline{u}_e^2 \; ; \; D\delta \underline{v}_e = 0$

$\delta l_e = \frac{1}{2\sqrt{\underline{v}_e \cdot \underline{v}_e}}(\delta \underline{v}_e \cdot \underline{v}_e + \underline{v}_e \cdot \delta \underline{v}_e) = \frac{1}{l_e} \underline{v}_e \cdot \delta \underline{v}_e = \frac{1}{l_e} \underline{v}_e (\delta \underline{u}_e^2 - \delta \underline{u}_e^1) = \delta \underline{U}_e^T \frac{1}{l_e} \underline{V}_e \; ; \; D l_e = \frac{1}{l_e} \underline{V}_e^T \Delta \underline{U}_e$

$D\delta l_e = D(\frac{1}{l_e} \underline{v}_e \cdot \delta \underline{v}_e) = -\frac{1}{l_e^2} D l_e \, \underline{v}_e \cdot \delta \underline{v}_e + \frac{1}{l_e} D\underline{v}_e \cdot \delta \underline{v}_e + \frac{1}{l_e} \underline{v}_e \cdot \cancelto{0}{D\delta \underline{v}_e} = -\frac{1}{l_e^3} \underline{V}_e^T \Delta \underline{U}_e \cdot \delta \underline{U}_e^T \underline{V}_e + \frac{1}{l_e} \delta \underline{U}_e^T \underline{\underline{M}} \Delta \underline{U}_e$

$\delta \lambda_e = \frac{1}{L_e} \delta l_e = \frac{1}{(L_e l_e)} \cdot \delta \underline{U}_e^T \underline{V}_e \; ; \; D\lambda_e = \frac{1}{(L_e l_e)} \cdot \underline{V}_e^T \Delta \underline{U}_e \; ; \; D\delta\lambda_e = \ldots = \delta \underline{U}_e^T \left(-\frac{1}{(L_e l_e^3)} \underline{V}_e \underline{V}_e^T + \frac{1}{(L_e l_e)} \underline{\underline{M}}\right) \Delta \underline{U}_e$

$\delta \varepsilon_{Ge} = \lambda_e \delta \lambda_e = \frac{l_e}{L_e} \cdot \frac{1}{L_e l_e} \cdot \delta \underline{U}_e^T \underline{V}_e = \frac{1}{L_e^2} \delta \underline{U}_e^T \underline{V}_e \; ; \; D\varepsilon_{Ge} = \frac{1}{L_e^2} \underline{V}_e^T \Delta \underline{U}_e \; ; \; D\delta\varepsilon_{Ge} = D\lambda_e \delta\lambda_e + \lambda_e D\delta\lambda_e = \ldots = \delta \underline{U}_e^T \frac{1}{L_e^2} \underline{\underline{M}} \Delta \underline{U}_e$

$\delta \Pi_{int,e} = S_e \, \delta\varepsilon_{Ge} A_e L_e = \frac{S_e A_e}{L_e} \cdot \delta \underline{U}_e^T \underline{V}_e = \delta \underline{U}_e^T \cdot \underline{F}_{int,e} \rightarrow \underline{F}_{int,e} = \frac{S_e A_e}{L_e} \cdot \underline{V}_e$

$D\delta\Pi_{int,e} = DS_e \delta\varepsilon_{Ge} A_e L_e + S_e D\delta\varepsilon_{Ge} A_e L_e = \delta \underline{U}_e^T \Big(\underbrace{\frac{EA_e}{L_e^3} \underline{V}_e \underline{V}_e^T + \frac{EA_e}{L_e} \varepsilon_{Ge} \underline{\underline{M}}}_{} \Big) \Delta \underline{U}_e$

$DS_e \delta\varepsilon_{Ge} A_e L_e = E D\varepsilon_{Ge} \delta\varepsilon_{Ge} A_e L_e = \delta \underline{U}_e^T \left(\frac{EA_e}{L_e^3} \underline{V}_e \underline{V}_e^T\right) \Delta \underline{U}_e$

$S_e D\delta\varepsilon_{Ge} A_e L_e = E\varepsilon_{Ge} D\delta\varepsilon_{Ge} A_e L_e = \delta \underline{U}_e^T \left(\frac{EA_e}{L_e} \varepsilon_{Ge} \underline{\underline{M}}\right) \Delta \underline{U}_e$

$\underline{\underline{K}}_{te} = \frac{EA_e}{L_e^3} \underline{V}_e \underline{V}_e^T + \frac{EA_e}{L_e} \varepsilon_{Ge} \underline{\underline{M}}$

$0 = \delta\Pi = \delta\Pi_{int} - \delta\Pi_{ext} = \delta \underline{U}^T (\underline{F}_{int} - \underline{F}_{ext}) \rightarrow R = F_{int} - F_{ext} = 0$

$\hookrightarrow D\delta\Pi = D\delta\Pi_{int} = \delta \underline{U}^T \underline{\underline{K}}_t \Delta \underline{U} \rightarrow DR = DF_{int} = K_t \Delta U$

$\delta\Pi_{int} = \sum_{e=1}^{n_e} \delta\Pi_{int,e} \qquad F_{int} = \mathop{\mathsf{A}}_{e=1}^{n_e} F_{int,e}$

$\delta\Pi_{ext} = \sum_{e=1}^{n_e} \delta\Pi_{ext,e} \qquad K_t = \mathop{\mathsf{A}}_{e=1}^{n_e} K_{te}$

iterative solvers

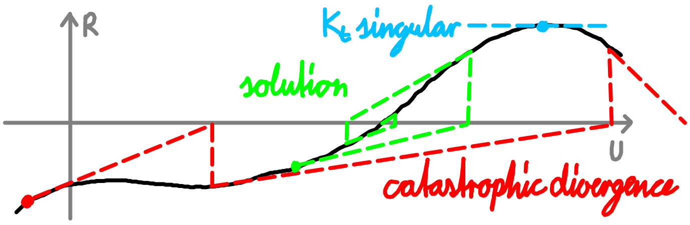
(solve eq. $R(U) = 0$ by starting from initial guess $U^{(0)}$ and improving the guess untill convergeance)

- **Newton-Raphson**

$R(U^{(i)} + \Delta U^{(i)}) \simeq R(U^{(i)}) + \underbrace{DR(U^{(i)})[\Delta U^{(i)}]}_{= \partial R / \partial U (U^{(i)}) \cdot \Delta U^{(i)}} = 0$, $\; \partial R / \partial U (U^{(i)}) = K_t(U^{(i)})$: tangent stiffness matrix

$\hookrightarrow \Delta U^{(i)} = -K_t^{-1}(U^{(i)}) R(U^{(i)}) \; ; \; U^{(i+1)} = U^{(i)} + \Delta U^{(i)}$

initial guess must be close and $R(U)$ be smooth!
↳ if not: catastrophic divergeance.



break criteria: $\|R(U^{(i+1)})\| < tol_R = $ e.g. $10^{-6}, 10^{-7}, 10^{-8}, \ldots$
(or $\|R(U^{(i+1)})\| / \|R(U^{(i)})\| < tol_R^{rel}$ or $\|U^{(i+1)} - U^{(i)}\| < tol_U$)

impose dirichlet BC: set $U^{(0)}$ such that Dirichlet BC is fulfilled → for all NR iterations, set row of BC in K_t and R to 0 and row/col index of BC in K_t to 1. (so NR iteration won't move that node anymore!)

- **incremental iterative solver**: apply loading in steps, starting with small load and $U^{(0)} = 0$ to ensure that $U^{(0)}$ is close to solution. → solve with e.g. above method → increase load and use prev. sol. as $U^{(0)}$ → continue untill full load is applied! (mitigates catastrophic divergence)

- **modified Newton-Raphson method**:

always reuse first K_t instead of recomputing it at every timestep
+ cheaper iterations − slow convergeance

- **line search**:

stability of elastic structures

(assumptions: elastic mat., conservative loads, quasistatic approach)

• **static stability**: given an equilibrium U for a load $\lambda \cdot F_{ext}$, can it be perturbed by δU and still be at equilibrium?

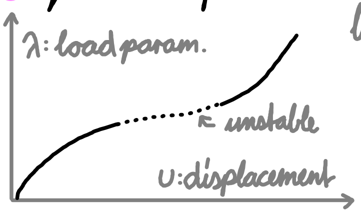
$\Pi^h(\underline{U}+\delta\underline{U}) \simeq \Pi^h(\underline{U}) + \delta\Pi^h(\underline{U})[\delta\underline{U}] + \frac{1}{2}\delta\delta\Pi^h(\underline{U})[\delta\underline{U}][\delta\underline{U}]$ (2nd expansion of total potential energy)

$\delta\Pi^h(\underline{U})[\delta\underline{U}] = \delta\underline{U}^T R(\underline{U}) = \delta\underline{U}^T(\underline{F}_{int}(U)-\lambda \underline{F}_{ext}) = 0$ (since equilibrium was assumed)

$\delta\delta\Pi^h(\underline{U})[\delta\underline{U}][\delta\underline{U}] = \delta U^T K_t \delta U$ (derived analogous to $D\delta\Pi^h \triangleq \delta\delta\Pi^h$)

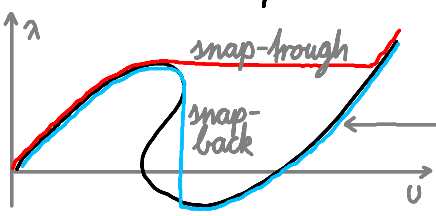
- **strongly stable**: if $\delta\underline{U}^T \underline{K}_t \delta\underline{U} > 0 \; \forall \delta U$ ⇔ energy would increase ⇔ all $EW(K_t) > 0$ ⇔ K_t positive definite
- **neutrally stable**: if $\delta\underline{U}^T \underline{K}_t \delta\underline{U} = 0$ for some $\delta\underline{U}$ ⇔ can move along δU without changing energy
  ↳ some $EW(K_t)=0$ ⇔ K_t positive semi definite → $\det(\underline{K}_t)=0$
- **unstable**: if $\exists \delta\underline{U}$ s.t. $\delta\underline{U}^T \underline{K}_t \delta U < 0$ ⇔ energy decreases in δU dir. ⇔ some $EW(K_t)<0$ ⇔ K_t indefinite

• **equilibrium path**:



location of equilibrium force vs. displacement (NOT a process!)

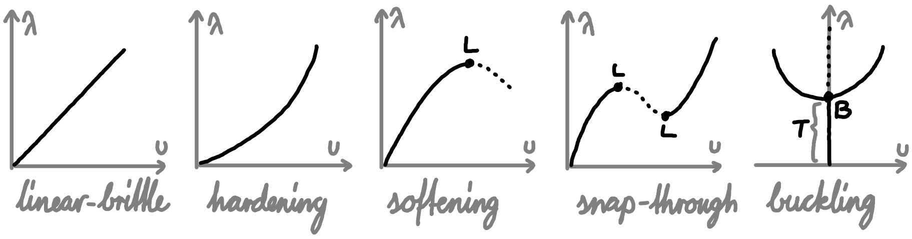
• **load control vs. displacement control**:



solve $R(U,\lambda)=0$ with variable λ and one const. u^i

• **critical points**:

limit point L: horizontal tangent
bifurcation point B: two or more brances
turning point T: vertical tangent



• **determination of L,B critical points**:

- **indirect method**: follow equilibrium path and track $\det(K_t)$ if sign flips, then inbetween there was an eq. with $\det(K_t)=0$ → neutrally stable → crit. point.
- **direct method**: assume K_t with one $EW=0$ and $EV=\Phi$

solve: $\begin{pmatrix} R(U,\lambda) \\ K_t\Phi \\ \|\Phi\|-1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ ← equilibrium ← Φ is EV of K_t ← $\Phi \neq 0$

?
$\boxed{\begin{array}{l} \Phi^T F_{ext} = 0 \to B \\ \text{else} \to L \end{array}}$

• **branch switching**: when at a bifurcation point, let the next NR initial guess be $U^{(0)} = U_{crit} + \xi \, \Phi / \|\Phi\|$ $(\xi \in \mathbb{R})$

• **path following**: allows the whole equilibrium path to be tracked, by searching over U and λ

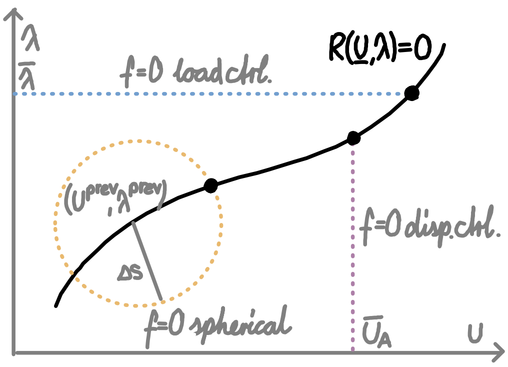
$$\begin{pmatrix} R(\underline{U},\lambda) \\ f(\underline{U},\lambda) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \xrightarrow{NR} \begin{pmatrix} \partial R/\partial U & \partial R/\partial \lambda \\ \partial f/\partial U & \partial f/\partial \lambda \end{pmatrix}\begin{pmatrix} \Delta U \\ \Delta\lambda \end{pmatrix} = \begin{pmatrix} K_t & -F_{ext} \\ K_{\lambda u} & K_{\lambda\lambda} \end{pmatrix}\begin{pmatrix} \Delta U \\ \Delta\lambda \end{pmatrix} = -\begin{pmatrix} R(\underline{U},\lambda) \\ f(\underline{U},\lambda) \end{pmatrix}$$

f: some constraint to compensate for added variable λ

partitioned solution: $K_t \Delta U - F_{ext}\Delta\lambda = -R \to \Delta U = -K_t^{-1}R + K_t^{-1}F_{ext}\Delta\lambda \to \Delta U = \Delta U_u + \Delta U_\lambda \Delta\lambda$ ③ ($\Delta U_u = -K_t^{-1}R$ ①; $\Delta U_\lambda = K_t^{-1}F_{ext}$)

$K_{\lambda u}\Delta U + K_{\lambda\lambda}\Delta\lambda = -f \to K_{\lambda u}\Delta U_u + K_{\lambda u}\Delta U_\lambda \Delta\lambda + K_{\lambda\lambda}\Delta\lambda = -f \to \Delta\lambda = -(f + K_{\lambda u}\Delta U_u)/(K_{\lambda u}\Delta U_\lambda + K_{\lambda\lambda})$ ②

- **load control**: $f(\underline{U},\lambda) = \lambda - \bar{\lambda} = 0$ ($\bar{\lambda}$: prescribed load)
  ↳ $K_{\lambda u}=0$, $K_{\lambda\lambda}=1$ → $\Delta\lambda = -f$; $\Delta U = \Delta U_u$
- **displ. control**: $f(\underline{U},\lambda) = u^A - \bar{u}^A$ ($\bar{u}^A$: prescribed displ. on node u^A)
  ↳ $K_{\lambda u} = [0..0,1,0..0]$, $K_{\lambda\lambda}=0$
  ↳ $\Delta\lambda = -\left(u_A - \bar{u}_A + \Delta U_{uA}\right)/\Delta U_{\lambda A}$; $\Delta U = \Delta U_u + \Delta U_\lambda \Delta\lambda$
- **spherical**: $f(\underline{U},\lambda) = \sqrt{(\underline{U}-\underline{U}^{prev})^T(\underline{U}-\underline{U}^{prev}) + (\lambda-\lambda^{prev})^2 \cdot \psi} - \Delta s$
  ↳ $K_{\lambda u} = \frac{1}{\sqrt{\square}}(\underline{U}-\underline{U}^{prev})^T$ $\quad K_{\lambda\lambda} = \frac{1}{\sqrt{\square}}(\lambda-\lambda^{prev})$

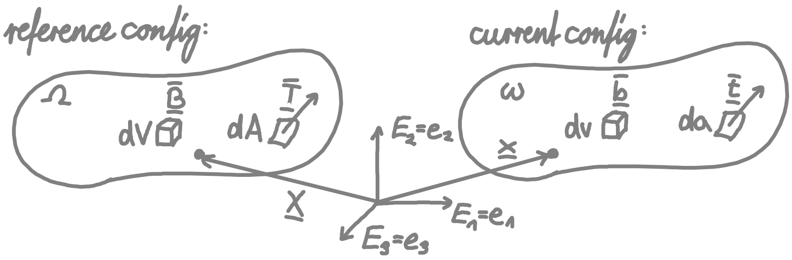


nonlinear continuum mechanics in 3D

$\underline{X}$: ref. coord. $\quad$ $\underline{x}$: curr. coord $\quad$ $\underline{x} = \underline{X} + \underline{u}(\underline{X})$

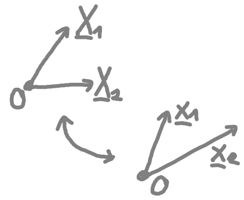
$\underline{\bar{B}}$: force/ref. vol $\quad$ $\underline{\bar{b}}$: force/curr. vol $\quad$ $\underline{\bar{B}}dV = \underline{\bar{b}}dv$

$\underline{\bar{T}}$: force/ref. surf. $\quad$ $\underline{\bar{t}}$: force/curr. surf. $\quad$ $\underline{\bar{T}}dA = \underline{\bar{t}}da$



• deformation measures:

deformation gradient $\underline{\underline{F}}$: $\underline{\underline{F}} = \partial\underline{x}/\partial\underline{X} = \underline{\underline{I}} - \partial\underline{u}/\partial\underline{X} = \underline{\underline{I}} + \nabla_X \underline{u}$ $\rightarrow$ $d\underline{x} = \underline{\underline{F}}d\underline{X}$; $d\underline{X} = \underline{\underline{F}}^{-1}d\underline{x}$



right Cauchy-Green deformation tensor $\underline{\underline{C}}$: $d\underline{x}_1^T d\underline{x}_2 = d\underline{X}_1^T \underline{\underline{F}}^T\underline{\underline{F}} d\underline{X}_2 = d\underline{X}_1^T \underline{\underline{C}} d\underline{X}_2$ $\leftarrow$ $\underline{\underline{C}} = \underline{\underline{F}}^T\underline{\underline{F}}$

left Cauchy-Green deformation tensor $\underline{\underline{b}}$: $dX_1^T dX_2 = dx_1 F^{-T}\underline{\underline{F}}^{-1} dx_2 = dx_1 \underline{\underline{b}}^{-1} dx_2$ $\leftarrow$ $b = \underline{\underline{F}}\underline{\underline{F}}^T$

Green-Lagrange strain tensor $\underline{\underline{E}}$: $\frac{1}{2}(d\underline{x}_1^T d\underline{x}_2 - d\underline{X}_1^T d\underline{X}_2) = \ldots = d\underline{X}_1^T \frac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}}) d\underline{X}_2 = d\underline{X}_1^T \underline{\underline{E}} d\underline{X}_2$ $\leftarrow$ $\underline{\underline{E}} = \frac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}})$ $\leftrightarrow$ $\underline{\underline{E}} = \underline{\underline{F}}^T \underline{\underline{e}} \underline{\underline{F}}$

Euler-Almans strain tensor $\underline{\underline{e}}$: $\frac{1}{2}(d\underline{x}_1^T d\underline{x}_2 - d\underline{X}_1^T d\underline{X}_2) = \ldots = d\underline{x}_1^T \frac{1}{2}(\underline{\underline{I}} - \underline{\underline{b}}^{-1}) d\underline{x}_2 = d\underline{x}_1^T \underline{\underline{e}} d\underline{x}_2$ $\leftarrow$ $\underline{\underline{e}} = \frac{1}{2}(\underline{\underline{I}} - \underline{\underline{b}}^{-1})$ $\leftrightarrow$ $\underline{\underline{e}} = \underline{\underline{F}}^{-T} \underline{\underline{E}} \underline{\underline{F}}^{-1}$

($\underline{\underline{E}}$ and $\underline{\underline{e}}$ are good strain measures, b.c. $\underline{\underline{E}} = 0$ & $\underline{\underline{e}} = 0$ if $\underline{x} = \underline{\underline{R}}\underline{X}$ rigid body rotation)

variations and linearizations:

$\delta\underline{\underline{F}}[\delta\underline{u}] = \nabla_X(\delta\underline{u}) = \nabla_x(\delta\underline{u})\underline{\underline{F}}$ $\qquad$ $\delta\underline{\underline{E}}[\delta\underline{u}] = \frac{1}{2}(\delta\underline{\underline{F}}^T\underline{\underline{F}} + \underline{\underline{F}}^T\delta\underline{\underline{F}}) = \frac{1}{2}(\nabla_X^T(\delta\underline{u})\underline{\underline{F}} + \underline{\underline{F}}^T\nabla_X(\delta\underline{u}))$

$D\underline{\underline{F}}[\Delta\underline{u}] = \nabla_X(\Delta\underline{u}) = \nabla_x(\Delta\underline{u})\underline{\underline{F}}$ $\qquad$ $D\underline{\underline{E}}[\Delta\underline{u}] = \frac{1}{2}(\nabla_X^T(\Delta\underline{u})\underline{\underline{F}} + \underline{\underline{F}}^T\nabla_X(\Delta\underline{u}))$

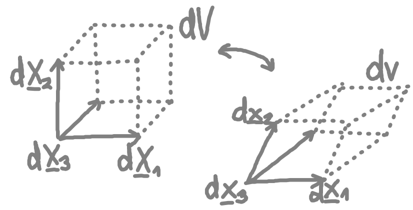
$D\delta\underline{\underline{F}}[\delta\underline{u}][\Delta\underline{u}] = 0$ $\qquad$ $D\delta\underline{\underline{E}}[\delta\underline{u}][\Delta\underline{u}] = \frac{1}{2}(\nabla_X^T(\delta\underline{u})\nabla_X(\Delta\underline{u}) + \nabla_X^T(\Delta\underline{u})\nabla_X(\delta\underline{u}))$

• volume change:

$d\underline{X}_1 = [dX_1\ 0\ 0]^T$ $\qquad$ $d\underline{x}_1 = \underline{\underline{F}}d\underline{X}_1 = \partial\underline{x}/\partial X_1\, dX_1$ $\qquad$ $dV = d\underline{X}_1^T(d\underline{X}_2 \times d\underline{X}_3) = dX_1 dX_2 dX_3$

$d\underline{X}_2 = [0\ dX_2\ 0]^T$ $\rightarrow$ $d\underline{x}_2 = \underline{\underline{F}}d\underline{X}_2 = \partial\underline{x}/\partial X_2\, dX_2$ $\rightarrow$ $dv = d\underline{x}_1^T(d\underline{x}_2 \times d\underline{x}_3) = \ldots = dV \det\underline{\underline{F}}$

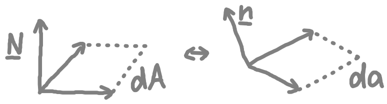
$d\underline{X}_3 = [0\ 0\ dX_3]^T$ $\qquad$ $d\underline{x}_3 = \underline{\underline{F}}d\underline{X}_3 = \partial\underline{x}/\partial X_3\, dX_3$ $\qquad$ ↳ $J = \det\underline{\underline{F}} = dv/dV$



• area change:

$d\underline{A} = dA\,\underline{N}$ $\quad$ $(\|\underline{N}\| = 1)$ $\qquad$ $dv = d\underline{L}\cdot d\underline{a} = \underline{\underline{F}}d\underline{L}\cdot d\underline{a} = JdV = Jd\underline{L}\cdot d\underline{A}$

$d\underline{a} = da\,\underline{n}$ $\quad$ $(\|\underline{n}\| = 1)$ $\qquad$ ↳ $d\underline{a} = J\underline{\underline{F}}^{-1}d\underline{A}$ $\qquad$ ($d\underline{L}$, $d\underline{L}$: arb. vector)

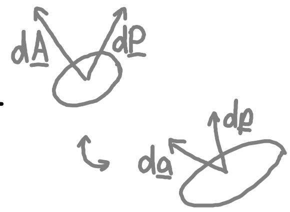


• stress measures:

Cauchy's stress tensor $\underline{\underline{\sigma}}$: $d\underline{p} = \underline{t}da = \underline{\underline{\sigma}}\underline{n}da = \underline{\underline{\sigma}}d\underline{a}$ $\rightarrow$ $d\underline{p} = \underline{\underline{\sigma}}d\underline{a}$

first Piola-Kirchhoff stress tensor $\underline{\underline{P}}$: $d\underline{p} = \underline{\underline{\sigma}}d\underline{a} = \underline{\underline{\sigma}}J\underline{\underline{F}}^{-T}d\underline{A} = \underline{\underline{P}}d\underline{A}$ $\rightarrow$ $d\underline{p} = \underline{\underline{P}}d\underline{A}$ $\leftarrow$ $\underline{\underline{P}} = J\underline{\underline{\sigma}}\underline{\underline{F}}^{-T}$

second Piola-Kirchhoff stress tensor $\underline{\underline{S}}$: $d\underline{P} = \underline{\underline{F}}^{-1}d\underline{p} = \underline{\underline{F}}^{-1}\underline{\underline{P}}d\underline{A} = \underline{\underline{S}}d\underline{A}$ $\rightarrow$ $d\underline{P} = \underline{\underline{S}}d\underline{A}$ $\leftarrow$ $\underline{\underline{S}} = \underline{\underline{F}}^{-1}\underline{\underline{P}}$



• balance of mass:

$dm = \rho_0 dV = \rho dv = \rho J dV$ $\rightarrow$ $\rho_0 = \rho J$

• balance of linear momentum:

$\int_\omega \underline{\bar{b}}dv + \int_{\partial\omega}\underline{t}da = \int_\omega \underline{\bar{b}} + \nabla_x\underline{\underline{\sigma}}\,dv = \underline{0}$ $\xrightarrow{\forall\omega}$ $\nabla_x\underline{\underline{\sigma}} + \underline{\bar{b}} = \underline{0}$

$(\underline{\bar{b}}dv = \underline{\bar{B}}dV \rightarrow \underline{\bar{B}} = \underline{\bar{b}}J)$ $\quad$ $\int_\Omega \underline{\bar{B}}dV + \int_{\partial\Omega}\underline{\bar{T}}dA = \int_\Omega \underline{\bar{B}} + \nabla_X\underline{\underline{P}}\,dV = \underline{0}$ $\xrightarrow{\forall\Omega}$ $\nabla_X\underline{\underline{P}} + \underline{\bar{B}} = \underline{0}$

• strong form: (in ref. config)

$\nabla_X\cdot\underline{\underline{P}} + \underline{\bar{B}} = 0$ $\quad$ in Ω

$\underline{u} = \underline{\bar{u}}$ $\quad$ on $\partial\Omega_D$

$\underline{\underline{P}}\underline{N} = \underline{\bar{T}}$ $\quad$ on $\partial\Omega_N$

with $\underline{\underline{P}} = \text{func}(\underline{\underline{F}})$, $\underline{\underline{F}} = \underline{\underline{I}} + \nabla_X\underline{u}$

• weak form: (in ref. config)

$\int_\Omega(\nabla_X\underline{\underline{P}} + \underline{\bar{B}})\cdot\delta\underline{u}\,dV = \int_{\partial\Omega}\overbrace{\underline{\underline{P}}\underline{N}}^{\underline{\bar{T}}}\cdot\delta\underline{u}\,dA - \int_\Omega \underline{\underline{P}}\cdot\overbrace{\nabla_X(\delta\underline{u})}^{\delta\underline{\underline{F}}} - \underline{\bar{B}}\cdot\delta\underline{u}\,dV = 0 \quad \forall\delta\underline{u}\in\mathcal{U}_0$

↳ $\int_\Omega \underbrace{\underline{\underline{P}}\cdot\delta\underline{\underline{F}}}\,dV = \int_\Omega\underline{\bar{B}}\cdot\delta\underline{u}\,dV + \int_{\partial\Omega_N}\underline{\bar{T}}\cdot\delta\underline{u}\,dA \quad \forall\delta\underline{u}\in\mathcal{U}_0$

or $\underline{\underline{S}}\cdot\delta\underline{\underline{E}}$ if more convenient for material model

• alternative weak form derivation in ref. config. (only for hyperelastic mat.):

$\Pi(\underline{u}) = \int_\Omega \Psi(\underline{u}) - \underline{\bar{B}}\cdot\underline{u}\,dV - \int_{\partial\Omega_N}\underline{\bar{T}}\cdot\underline{u}\,dA$

$\delta\Pi(\underline{u})[\delta\underline{u}] = \int_\Omega \delta\Psi - \underline{\bar{B}}\cdot\delta\underline{u}\,dV - \int_{\partial\Omega_N}\underline{\bar{T}}\cdot\delta\underline{u}\,dA = \underbrace{\int_\Omega \overbrace{\underline{\underline{P}}\cdot\delta\underline{\underline{F}}}^{\text{or } \underline{\underline{S}}\cdot\delta\underline{\underline{E}} \text{ if more convenient for material model}}dV}_{\delta\Pi_{int}} - \underbrace{\left(\int_\Omega\underline{\bar{B}}\cdot\delta\underline{u}\,dV + \int_{\partial\Omega_N}\underline{\bar{T}}\cdot\delta\underline{u}\,dA\right)}_{\delta\Pi_{ext}} = 0$

- **Hyperelastic constitutive laws:** $\leftarrow \exists$ elastic strain energy density Ψ, ψ

$\underline{\underline{\sigma}} = \partial\psi/\partial(\nabla_x \underline{u})$ or $\underline{\underline{P}} = \partial\Psi/\partial\underline{\underline{F}}$ or $\underline{\underline{S}} = \partial\Psi/\partial\underline{\underline{E}}$ $\leftarrow$ $\delta\Psi = \underline{\underline{P}} \cdot \delta\underline{\underline{F}} = \underline{\underline{S}} \cdot \delta\underline{\underline{E}}$; $\delta\psi = \underline{\underline{\sigma}} \cdot \delta(\nabla_x \underline{u})$ work conjugates!

$\hookrightarrow \underline{\underline{\sigma}} = \text{func}(\nabla_x \underline{u})$ $\hookrightarrow \underline{\underline{P}} = \text{func}(\underline{\underline{F}})$ $\hookrightarrow \underline{\underline{S}} = \text{func}(\underline{\underline{E}})$ $\quad \Pi_{int} = \int_\Omega \Psi dV = \int_\omega \psi dv \rightarrow \delta\Pi_{int} = \int_\Omega \delta\Psi dV = \int_\omega \delta\psi dv$

requirements: • no change for changing observer (given if $\Psi = \Psi(\underline{\underline{C}})$) • $\Psi(\underline{\underline{C}} = \underline{\underline{I}}) = 0$ • $\Psi \rightarrow +\infty$ as $\det \underline{\underline{C}} \rightarrow +\infty$

• (optional) material symmetry (e.g. $\Psi = \Psi(\text{inv. of } \underline{\underline{C}})$) • $\Psi(\underline{\underline{C}}) \geq 0$ • $\Psi \rightarrow +\infty$ as $\det \underline{\underline{C}} \rightarrow 0^+$

- **St. Venant-Kirchoff material:** $\Psi(\underline{\underline{E}}) = \frac{1}{2}\lambda \, tr(\underline{\underline{E}})^2 + \mu \underline{\underline{E}} \cdot \underline{\underline{E}} \rightarrow \underline{\underline{S}} = \partial\Psi/\partial\underline{\underline{E}} = \lambda tr(\underline{\underline{E}}) \underline{\underline{I}} + 2\mu\underline{\underline{E}}$ $\quad (J^2 = \det(\underline{\underline{C}}))$
- **neo-Hooke material:** $\Psi(\underline{\underline{C}}) = \frac{1}{2}\mu(tr(\underline{\underline{C}}) - 3) - \mu \ln(J) + \frac{1}{2}\lambda \ln(J)^2 \rightarrow \underline{\underline{S}} = \partial\Psi/\partial\underline{\underline{E}} = 2\partial\Psi/\partial\underline{\underline{C}} = \mu(\underline{\underline{I}} - \underline{\underline{C}}^{-1}) + \lambda \ln(J) \underline{\underline{C}}^{-1}$

- **linearization of constitutive laws:**

$DS_{IJ}[\Delta\underline{u}] = \frac{d}{d\epsilon} S_{IJ}(E_{KL}(\underline{u} + \epsilon\Delta\underline{u}))|_{\epsilon=0} = \partial S_{IJ}/\partial E_{KL} \, \frac{d}{d\epsilon} E_{KL}(\underline{u}+\epsilon\Delta\underline{u})|_{\epsilon=0} = \partial S_{IJ}/\partial E_{KL} \, DE_{KL}[\Delta\underline{u}] = \mathbb{C}_{IJKL} DE_{KL}[\Delta\underline{u}]$

$D\underline{\underline{S}}[\Delta\underline{u}] = \frac{d}{d\epsilon} \underline{\underline{S}}(\underline{\underline{E}}(\underline{u}+\epsilon\Delta\underline{u}))|_{\epsilon=0} = \partial\underline{\underline{S}}/\partial\underline{\underline{E}} : \frac{d}{d\epsilon} \underline{\underline{E}}(\underline{u}+\epsilon\Delta\underline{u})|_{\epsilon=0} = \partial\underline{\underline{S}}/\partial\underline{\underline{E}} : D\underline{\underline{E}}[\Delta\underline{u}] = \underline{\underline{\mathbb{C}}} : D\underline{\underline{E}}[\Delta\underline{u}]$

material tangent constitutive tensor: $\underline{\underline{\mathbb{C}}} = \partial\underline{\underline{S}}/\partial\underline{\underline{E}}$ $\quad (\mathbb{C}_{IJKL} = \partial S_{IJ}/\partial E_{KL})$

if hyperelastic: $\mathbb{C}_{IJKL} = \partial^2\Psi/\partial E_{IJ}\partial E_{KL}$; $\underline{\underline{\mathbb{C}}} = \partial^2\Psi/\partial\underline{\underline{E}}\partial\underline{\underline{E}}$

- **St. Venant-Kirchoff material:** $C_{IJKL} = \lambda\delta_{IJ}\delta_{KL} + \mu(\delta_{IK}\delta_{JL} + \delta_{IL}\delta_{JK})$ $\quad$ - **neo-Hooke material:** $C_{IJKL} = ?$

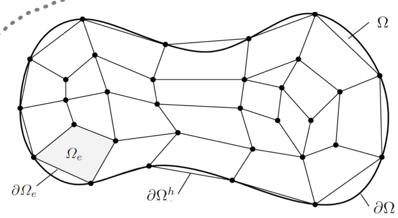
- **linearization of weak form:** (in ref config.)

$D\delta\Pi = D\delta\Pi_{int} - D\delta\Pi_{ext} = D\delta\Pi_{int} = \int_\Omega D\underline{\underline{S}} \cdot \delta\underline{\underline{E}} + \underline{\underline{S}} \cdot D\delta\underline{\underline{E}} \, dV$ $\quad$ ($D\delta\Pi_{ext} = 0$ as const ext. forces assumed)

$\delta\underline{\underline{E}} = \frac{1}{2}(\nabla_X^T(\delta\underline{u})\underline{\underline{F}} + \underline{\underline{F}}^T \nabla_X(\delta\underline{u}))$ $\quad D\delta\underline{\underline{E}} = \frac{1}{2}(\nabla_X^T(\delta\underline{u})\nabla_X(\Delta\underline{u}) + \nabla_X^T(\Delta\underline{u})\nabla_X(\delta\underline{u}))$

$\underline{\underline{S}} \cdot D\delta\underline{\underline{E}} = \underline{\underline{S}} \cdot \nabla_X^T(\delta\underline{u})\nabla_X(\Delta\underline{u}) = \nabla_X(\delta\underline{u}) \cdot \nabla_X(\Delta\underline{u})\underline{\underline{S}}$ $\quad$ (because $\underline{\underline{S}}$ is symmetric)

$\rightarrow D\delta\Pi = \int_\Omega \delta\underline{\underline{E}} \cdot \underline{\underline{\mathbb{C}}} : D\underline{\underline{E}} + \nabla_X(\delta\underline{u}) \cdot \nabla_X(\Delta\underline{u})\underline{\underline{S}} \, dV$



- **FE discretization of weak form:** (in ref config.) $\quad n_e$: element count; n_{ee}: nodes per element

discretized ref. config element: $\underline{u}_e^h(\underline{X}) = \sum_{a=1}^{n_{ee}} N_e^a(\underline{X})\underline{u}_e^a$; $\underline{X}_e^h(\underline{X}) = \sum_{a=1}^{n_{ee}} N_e^a(\underline{X})\underline{X}_e^a$; $\underline{x}_e^h(\underline{X}) = \sum_{a=1}^{n_{ee}} N_e^a(\underline{X})(\underline{X}_e^a + \underline{u}_e^a)$

$\delta\Pi^h = \int_{\Omega^h} \underline{\underline{S}}^h \cdot \delta\underline{\underline{E}}^h - \underline{\bar{B}} \cdot \delta\underline{u}^h dV - \int_{\partial\Omega_N^h} \underline{\bar{T}} \cdot \delta\underline{u}^h dA = \sum_{e=1}^{n_e} \int_{\Omega_e} \underline{\underline{S}}_e^h \cdot \delta\underline{\underline{E}}_e^h - \underline{\bar{B}}_e \cdot \delta\underline{u}_e^h dV - \int_{\partial\Omega_e} \underline{\bar{T}}_e \cdot \delta\underline{u}_e^h dA = 0$ $\quad \forall \delta\underline{u}_e^h$ admiss.

if Neumann BC

$\underline{\underline{F}}_e^h = \partial\underline{x}_e^h/\partial\underline{X}_e^h = \sum_{a=1}^{n_{ee}} (\underline{X}_e^a + \underline{u}_e^a) \otimes \nabla_X N_e^a$ $\quad F_{e\,iI}^h = \partial x_{e\,i}^h/\partial X_{e\,I}^h = \sum_{a=1}^{n_{ee}} \partial N_e^a/\partial X_I \cdot (\underline{X}_e^a + \underline{u}_e^a)_i$

$\delta\underline{\underline{F}}_e^h = \sum_{a=1}^{n_{ee}} \delta\underline{u}_e^a \otimes \nabla_X N_e^a$ $\quad \delta F_{iI}^h = \sum_{a=1}^{n_{ee}} \partial N_e^a/\partial X_I \cdot \delta u_{e\,i}^a$

$\delta\underline{\underline{E}}_e^h = \frac{1}{2}(\delta\underline{\underline{F}}_e^{hT}\underline{\underline{F}}_e^h + \underline{\underline{F}}_e^{hT}\delta\underline{\underline{F}}_e^h) = \sum_{a=1}^{n_{ee}} \frac{1}{2}(\nabla_X N_e^a \otimes \delta\underline{u}_e^a \underline{\underline{F}}_e^h + \underline{\underline{F}}_e^{hT}\delta\underline{u}_e^a \otimes \nabla_X N_e^a)$ $\quad \delta E_{e\,IJ}^h = \sum_{a=1}^{n_{ee}} \frac{1}{2}(N_{e,I}^a F_{e\,iJ}^h + F_{e\,iI}^h N_{e,J}^a) \cdot \delta u_{e\,i}^a$

Voigt notation: $\quad$ (omitted "e")

$\delta\hat{\underline{E}}_e = [\delta E_{e11}^h \;\; \delta E_{e22}^h \;\; \delta E_{e33}^h \;\; 2\delta E_{e12}^h \;\; 2\delta E_{e23}^h \;\; 2\delta E_{e31}^h]^T$

$\hat{\underline{S}}_e = [S_{11}^h \;\; S_{22}^h \;\; S_{33}^h \;\; S_{12}^h \;\; S_{23}^h \;\; S_{31}^h]^T$

$\delta\underline{U}_e = [\delta\underline{u}_e^{1T} \cdots \delta\underline{u}_e^{n_{ee}T}]^T$; $\underline{\underline{B}}_e = [\underline{\underline{B}}_e^1 \cdots \underline{\underline{B}}_e^{n_{ee}}]$; $\underline{\underline{C}}_e = [\underline{\underline{C}}_e^1 \cdots \underline{\underline{C}}_e^{n_{ee}}]$

$$\underline{\underline{B}}_e^a = \begin{bmatrix} F_{11}^h N_{,1}^a & F_{21}^h N_{,1}^a & F_{31}^h N_{,1}^a \\ F_{12}^h N_{,2}^a & F_{22}^h N_{,2}^a & F_{32}^h N_{,2}^a \\ F_{13}^h N_{,3}^a & F_{23}^h N_{,3}^a & F_{33}^h N_{,3}^a \\ F_{11}^h N_{,2}^a + F_{12}^h N_{,1}^a & F_{21}^h N_{,2}^a + F_{22}^h N_{,1}^a & F_{31}^h N_{,2}^a + F_{32}^h N_{,1}^a \\ F_{12}^h N_{,3}^a + F_{13}^h N_{,2}^a & F_{22}^h N_{,3}^a + F_{23}^h N_{,2}^a & F_{32}^h N_{,3}^a + F_{33}^h N_{,2}^a \\ F_{13}^h N_{,1}^a + F_{11}^h N_{,3}^a & F_{23}^h N_{,1}^a + F_{21}^h N_{,3}^a & F_{33}^h N_{,1}^a + F_{31}^h N_{,3}^a \end{bmatrix}$$

$\hookrightarrow \delta\underline{u}_e^h = \sum_{a=1}^{n_{ee}} N_e^a \delta\underline{u}_e^a = \sum_{a=1}^{n_{ee}} \underline{\underline{C}}_e^a \delta\underline{u}_e^a = \underline{\underline{C}}_e \delta\underline{U}_e$

$\hookrightarrow \delta\hat{\underline{E}}_e = \sum_{a=1}^{n_{ee}} \underline{\underline{B}}_e^a \delta\underline{u}_e^a = \underline{\underline{B}}_e \delta\underline{U}_e$

$$\underline{\underline{C}}_e^a = \begin{bmatrix} N_e^a & 0 & 0 \\ 0 & N_e^a & 0 \\ 0 & 0 & N_e^a \end{bmatrix}$$

$\underline{\underline{S}}_e^h \cdot \delta\underline{\underline{E}}_e^h = \delta\hat{\underline{E}}_e^T \hat{\underline{S}}_e = \delta\underline{U}_e \cdot \underline{\underline{B}}_e^T \hat{\underline{S}}_e$; $\underline{\bar{B}}_e \cdot \delta\underline{u}_e^h = \delta\underline{U}_e \cdot \underline{\underline{C}}_e^T \underline{\bar{B}}_e$; $\underline{\bar{T}}_e \cdot \delta\underline{u}_e^h = \delta\underline{U}_e \cdot \underline{\underline{C}}_e^T \underline{\bar{T}}_e$

$\rightarrow \delta\Pi^h = \sum_{e=1}^{n_e} \delta\underline{U}_e \cdot [\underbrace{\int_{\Omega_e} \underline{\underline{B}}_e^T \hat{\underline{S}}_e \, dV}_{\underline{F}_{int,e}} - \underbrace{(\int_{\Omega_e} \underline{\underline{C}}^T \underline{\bar{B}} dV + \int_{\partial\Omega_e} \underline{\underline{C}}^T \underline{\bar{T}} dA)}_{\underline{F}_{ext,e}}] = 0$ $\quad \forall \delta\underline{U}_e$ admissible

$\underline{R} = \sum_{e=1}^{n_e} [\underline{F}_{int,e} - \underline{F}_{ext,e}] = \underline{0}$

$\underline{R} = [\underline{F}_{int} - \underline{F}_{ext}] = \underline{0}$ $\quad (\underline{F}_{int} = \mathsf{A}_{e=1}^{n_e} \underline{F}_{int,e}$; $\underline{F}_{ext} = \mathsf{A}_{e=1}^{n_e} \underline{F}_{ext,e})$

- **FE discretization of linearized weak form:** (in ref. config.)

$$D\delta\Pi^h = \int_\Omega \delta \underline{\underline{E}}^h \cdot \underline{\underline{\underline{\underline{C}}}}^h : D\underline{\underline{E}}^h + \nabla_X(\delta \underline{u}^h)\cdot \nabla_X(\Delta \underline{u}^h)\underline{\underline{S}}^h \, dV = \sum_{e=1}^{n_e} \int_{\Omega_e} \delta \underline{\underline{E}}_e^h \cdot \underline{\underline{\underline{\underline{C}}}}_e^h : D\underline{\underline{E}}_e^h + \nabla_X(\delta \underline{u}_e^h)\cdot \nabla_X(\Delta \underline{u}_e^h)\underline{\underline{S}}_e^h \, dV$$

$$\delta \underline{\underline{E}}_e^h \cdot \underline{\underline{\underline{\underline{C}}}}_e^h : D\underline{\underline{E}}_e^h = \delta \hat{\underline{E}}_e^T \underline{\underline{D}}_e D\hat{\underline{E}}_e = \delta \underline{U}_e \cdot \underline{\underline{B}}_e^T \underline{\underline{D}} \underline{\underline{B}}_e \Delta \underline{U}_e$$

(omitted "e")

$$\underline{\underline{D}}_e = \begin{bmatrix} C^h_{1111} & C^h_{1122} & C^h_{1133} & C^h_{1112} & C^h_{1123} & C^h_{1131} \\ & C^h_{2222} & C^h_{2233} & C^h_{2212} & C^h_{2223} & C^h_{2231} \\ & & C^h_{3333} & C^h_{3312} & C^h_{3323} & C^h_{3331} \\ & \text{sym} & & C^h_{1212} & C^h_{1223} & C^h_{1231} \\ & & & & C^h_{2323} & C^h_{2331} \\ & & & & & C^h_{3131} \end{bmatrix}$$

$$\nabla_X(\delta \underline{u}_e^h)\cdot \nabla_X(\Delta \underline{u}_e^h)\underline{\underline{S}}_e^h = \sum_{a=1}^{n_{ee}} (\delta \underline{u}_e^a \otimes \nabla_X N_e^a) \cdot \sum_{b=1}^{n_{ee}} (\Delta \underline{u}_e^b \otimes \nabla_X N_e^b \underline{\underline{S}}_e^h)$$

$$= \sum_a \sum_b \delta u_{e_i}^a N^a_{e,j} \, \Delta u^b_{e_i} N^b_{e,k} S^h_{e_{kj}} = \sum_a \sum_b \delta \underline{u}_e^a \cdot (\nabla_X N_e^{b\,T} \underline{\underline{S}}_e^h \nabla_X N_e^a) \underline{\underline{I}} \, \Delta \underline{u}_e^b$$

$$= \delta \underline{U}_e \cdot ((\underline{\underline{A}}_e^T \underline{\underline{S}}_e^h \underline{\underline{A}}_e) \otimes \underline{\underline{I}}) \, \Delta \underline{U}_e$$

$$\underline{\underline{A}}_e = [\nabla_X N_e^1 \; \cdots \; \nabla_X N_e^{n_{ee}}]$$

kron tensorp.

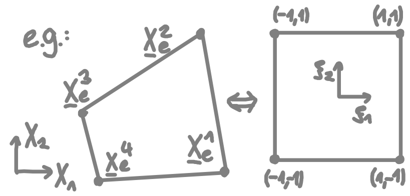
$$\hookrightarrow D\delta\Pi^h = \sum_{e=1}^{n_e} \delta \underline{U}_e \cdot \underbrace{\left(\int_{\Omega_e} \underline{\underline{B}}_e^T \underline{\underline{D}} \underline{\underline{B}}_e + (\underline{\underline{A}}_e^T \underline{\underline{S}}_e^h \underline{\underline{A}}_e) \otimes \underline{\underline{I}} \, dV\right)}_{} \Delta \underline{U}_e$$

$$DR = \sum_{e=1}^{n_e} \left(\quad \underline{\underline{K}}_e \quad \right) \Delta \underline{U}_e$$

$$DR = \left(\quad \underline{\underline{K}} \quad \right) \Delta \underline{U} \qquad (\underline{\underline{K}} = \mathsf{A}_{e=1}^{n_e} \underline{\underline{K}}_e)$$

- **parent element:** integrate over parent element instead of ref. config. element, as it's the same for all elements and gives same result!

e.g.:



$$\underline{u}_e^h(\underline{\xi}) = \sum_{a=1}^{n_{ee}} N^a(\underline{\xi}) \underline{u}_e^a \; ; \; \underline{X}_e^h(\underline{\xi}) = \sum_{a=1}^{n_{ee}} N^a(\underline{\xi}) \underline{X}_e^a \; ; \; \underline{x}_e^h(\underline{\xi}) = \sum_{a=1}^{n_{ee}} N^a(\underline{\xi}) (\underline{X}_e^a + \underline{u}_e^a)$$

- if equation asks for $N_e^a(\underline{X})$: replace with $N^a(\underline{\xi})$
- if equation asks for $\nabla_X N_e^a(\underline{X})$: replace with $\underline{\underline{J}}_e^{-T} \nabla_\xi N^a$
- if equation asks for dV: replace with $\det(\underline{\underline{J}}_e) dV$
- if equation asks for dA: surface jacobian?

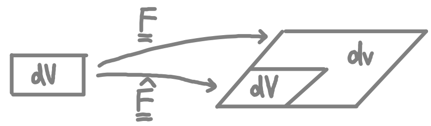
$$\left[\begin{array}{l} \underline{\underline{J}}_e = \partial \underline{X}_e^h / \partial \underline{\xi} = \sum_{a=1}^{n_{ee}} \underline{X}_e^a \otimes \nabla_\xi N^a \\ (J_{e_{I\alpha}} = \partial X^h_{e_I} / \partial \xi_\alpha = \sum_{a=1}^{n_{ee}} \partial N^a / \partial \xi_\alpha \cdot X^a_{e_I}) \\ \nabla_\xi N^a = \partial N^a / \partial \underline{\xi} = \partial N^a / \partial \underline{X} \cdot \partial \underline{X} / \partial \underline{\xi} = \underline{\underline{J}}_e^{-T} \nabla_X N^a \end{array}\right]$$

- **numerical quadrature:** approximate integrals $\int_{\Omega_e}$ with e.g. Gauss-Legendre or simplical quadrature (see comp. mech I notes for details)

- **comment on St. Venant-Kirchhoff material:** only good for small strains, but large rotations → e.g.: plates, shells

incompressible constitutive laws

- **volumetric-deviatoric decomposition:** $* = *_{vol} + *_{dev}$



deformation: $\hat{\underline{\underline{F}}} = J^{-1/3} \underline{\underline{F}} \Leftrightarrow \underline{\underline{F}} = J^{1/3} \hat{\underline{\underline{F}}} \rightarrow \hat{\underline{\underline{C}}} = J^{-2/3} \underline{\underline{C}} \Leftrightarrow \underline{\underline{C}} = J^{2/3} \hat{\underline{\underline{C}}}$ $\quad$ $\hat{\underline{\underline{F}}}, \hat{\underline{\underline{C}}}$: def. without vol. change, $\det \hat{\underline{\underline{F}}} = .. = 1$!

stress: $\underline{\underline{\sigma}} = \underbrace{\tfrac{1}{3}\mathrm{tr}(\underline{\underline{\sigma}}) \underline{\underline{I}}}_{\sigma_{vol}} + \underbrace{\underline{\underline{\sigma}}_{dev}}_{\sigma_{dev}} = p\underline{\underline{I}} + \underline{\underline{\sigma}}_{dev} \rightarrow \underline{\underline{S}} = J \underline{\underline{F}}^{-1} \underline{\underline{\sigma}} \underline{\underline{F}}^{-T} = \ldots = \underbrace{pJ\underline{\underline{C}}^{-1}}_{S_{vol}} + \underbrace{J \underline{\underline{F}}^{-1} \underline{\underline{\sigma}}_{dev} \underline{\underline{F}}^{-T}}_{S_{dev}}$ $\quad$ ($p = \tfrac{1}{3}\mathrm{tr}(\underline{\underline{\sigma}})$: hydrostatic pressure)

strain energy dens.: $\Psi = \Psi_{vol}(J) + \Psi_{dev}(\hat{\underline{\underline{C}}}) \rightarrow \underline{\underline{S}} = 2 \partial \Psi / \partial \underline{\underline{C}} = \overbrace{\frac{\partial \Psi_{vol}}{\partial J} J \underline{\underline{C}}^{-1}} + \overbrace{2 \cdot \frac{\partial \Psi_{dev}(\hat{\underline{\underline{C}}})}{\partial \underline{\underline{C}}}} \rightarrow \partial \Psi_{vol} / \partial J = p$ $\quad$ $\left(\frac{\partial J}{\partial \underline{\underline{C}}} = \frac{J}{2} \underline{\underline{C}}^{-1}\right)$

↳ e.g. neo-Hooke: $\Psi_{vol} = \frac{K}{2}(J-1)^2$; $\Psi_{dev} = \frac{\mu}{2}(\mathrm{tr}(\hat{\underline{\underline{C}}}) - 3) \rightarrow \underline{\underline{S}} = K(J-1)J\underline{\underline{C}}^{-1} + \mu J^{-2/3}(\underline{\underline{I}} - \tfrac{1}{3}\mathrm{tr}(\underline{\underline{C}})\underline{\underline{C}}^{-1}) \rightarrow p = K(J-1)$

full incompressible: $K \to \infty \Leftrightarrow J - 1 \to 0$ $\quad$ (neo-Hooke is incomp. for $K \to \infty$)

- **FE implementation:** (difficult b.c. volumetric locking: FE discret. + incomp. constr. → overconstrain → only 0 sol.)

- **mixed FE:** $\min \Pi(\underline{u})$ subj. to: $J(\underline{u}) - 1 = 0 \rightarrow \Pi_L(\underline{u}, \lambda) = \int_\Omega \Psi_{dev}(\underline{\underline{C}}(\underline{u})) dV - \Pi_{ext}(\underline{u}) + \int_\Omega \lambda (J(\underline{u}) - 1) dV$ (Lagrangian)

$\delta\Pi_L(\underline{u}, \lambda)[\delta\lambda] = \int_\Omega \delta\lambda (J(\underline{u}) - 1) dV \; \forall \delta\lambda \rightarrow J(\underline{u}) - 1 = 0$

$\delta\Pi_L(\underline{u}, \lambda)[\delta\underline{u}] = \ldots = \int_\Omega (\underline{\underline{S}}_{dev} + \lambda J \underline{\underline{C}}^{-1}) \cdot \delta \underline{\underline{E}} \, dV - \delta\Pi_{ext} = 0 \; \forall \delta\underline{u}$ $\quad$ Lagrange multip. $\lambda \hat{=}$ pressure p! (interp. spaces should follow LBB cond.)

- **reduced/selective integration:** use reduced integration for dev parts (= selective) or for dev & vol parts (= reduced)
↳ adds back enough DoF to avoid volumetric locking

1D plasticity

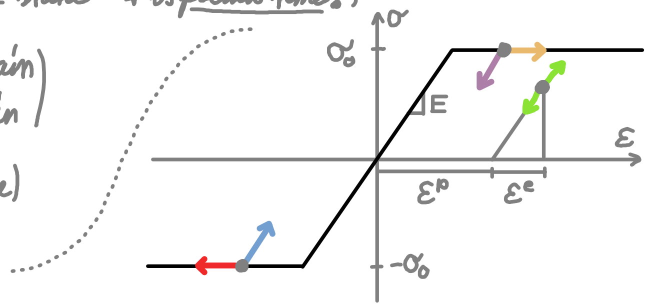
(small strain $\varepsilon = \frac{\partial u}{\partial x}$, rate-independent, quasi-static → $\ast$ is pseudo time!)

$\varepsilon = \varepsilon^e + \varepsilon^p$ (strain decomposition) ← (ε^e: elastic strain, ε^p: plastic strain)

$\sigma = E\varepsilon^e = E(\varepsilon - \varepsilon^p)$ (stress-strain relationship)

$f(\sigma) = \dots$ (yield function) $\quad \dot{\varepsilon}^p = \dot{\gamma}\,\mathrm{sign}(\sigma)$ (flow rule)

$\mathbb{A} = \{\sigma \mid f(\sigma) \le 0\}$; $\partial\mathbb{A} = \{\sigma \mid f(\sigma) = 0\}$; $\mathcal{E} = \{\sigma \mid f(\sigma) < 0\}$



- **perfect plasticity**: $f(\sigma) = |\sigma| - \sigma_0$

$f(\sigma) < 0 \longrightarrow \dot{\varepsilon} = \dot{\varepsilon}^e = \frac{1}{E}\dot{\sigma}$; $\dot{\varepsilon}^p = 0$ (elastic)

$f(\sigma) = 0$:
- $\sigma > 0$:
  - $\dot{\sigma} < 0 \to \dot{\varepsilon} = \dot{\varepsilon}^e = \frac{1}{E}\dot{\sigma}$; $\dot{\varepsilon}^p = 0$ (elastic)
  - $\dot{\sigma} = 0 \to \dot{\varepsilon} = \dot{\varepsilon}^p > 0$; $\dot{\varepsilon}^e = 0$ (plastic)
- $\sigma < 0$:
  - $\dot{\sigma} > 0 \to \dot{\varepsilon} = \dot{\varepsilon}^e = \frac{1}{E}\dot{\sigma}$; $\dot{\varepsilon}^p = 0$ (elastic)
  - $\dot{\sigma} = 0 \to \dot{\varepsilon} = \dot{\varepsilon}^p > 0$; $\dot{\varepsilon}^e = 0$ (plastic)

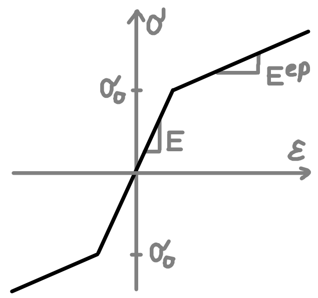
$\dot{\gamma} = 0, f < 0$; $\dot{\gamma} \ge 0, f = 0$ → $\dot{\gamma} \ge 0, f \le 0, f\dot{\gamma} = 0$ (Kuhn-Tucker cond.)

$\dot{\gamma} = 0, \dot{f} < 0$; $\dot{\gamma} > 0, \dot{f} = 0$ → if $f = 0 \to \dot{f}\dot{\gamma} = 0$ (consistency cond.)

- **isotropic linear hardening plasticity**: $f(\sigma, \alpha) = |\sigma| - (\sigma_0 + K\alpha)$ (K: hardening modulus)

$\alpha = \int |\dot{\varepsilon}^p|$ (accumulated plastic strain) $\to \dot{\alpha} = |\dot{\varepsilon}^p| = \dot{\gamma}$

- $\dot{\gamma} > 0$: plastic
  $\dot{f} = \partial f/\partial\sigma\,\dot{\sigma} + \partial f/\partial\alpha\,\dot{\alpha} = \dots = \mathrm{sign}(\sigma)E\dot{\varepsilon} - (E+K)\dot{\gamma} = 0$
  - $\dot{\gamma} = \frac{E}{E+K}\dot{\varepsilon}\,\mathrm{sign}(\sigma)$; $\dot{\varepsilon}^p = \frac{E}{E+K}\dot{\varepsilon}$
  - $\dot{\sigma} = E(\dot{\varepsilon} - \dot{\varepsilon}^p) = E\dot{\varepsilon} - E\dot{\gamma}\,\mathrm{sign}(\sigma) = \underbrace{\frac{EK}{E+K}}\dot{\varepsilon} = \underbrace{E^{ep}}\dot{\varepsilon}$
- elastic $\dot{\gamma} = 0$: $\dot{\sigma} = E\dot{\varepsilon}$



(Kuhn-Tucker + consistency cond. still apply!)

- remarks: plasticity has an incremental constitutive law. no longer $\sigma = \mathrm{func}(\varepsilon)$, but $\dot{\sigma} = \mathrm{func}(\dot{\varepsilon}(,\alpha))$!

- **FEM**

weak form: $\delta\Pi = \delta\Pi_{int} - \delta\Pi_{ext} = 0$ with: $\delta\Pi_{int} = \int_0^L \sigma\,\delta\varepsilon\,A\,dx$; $\delta\Pi_{ext} = \dots$ ($\delta\varepsilon = \frac{d}{dx}(\delta u)$, σ from plastic const. law)

FE discret.: $\delta\Pi^h_{int} = \sum_{e=1}^{ne}\int_{\Omega_e}\sigma_e^h\,\delta\varepsilon_e^h\,A\,dx = \sum_{e=1}^{ne}\delta\underline{U}_e^T\int_{\Omega_e}\underline{B}_e^T\sigma_e^h\,A\,dx = \delta\underline{U}^T \mathop{\mathrm{A}}_{e=1}^{ne}\underline{F}_{int,e} = \delta\underline{U}^T\underline{F}_{int}$; $\delta\Pi^h_{ext} = \dots = \delta\underline{U}^T\underline{F}_{ext}$

lineariz.: $D\delta\Pi^h = \delta\underline{U}^T D\underline{R} = \delta\underline{U}^T D\underline{F}_{int} = \delta\underline{U}^T \mathop{\mathrm{A}}_{e=1}^{ne} D\underline{F}_{int,e} = \delta\underline{U}^T \mathop{\mathrm{A}}_{e=1}^{ne}\underline{K}_{t,e}\Delta\underline{U}_e = \delta\underline{U}^T\underline{K}_t\Delta\underline{U}$ ↳ $\delta\Pi^h = \delta\underline{U}^T(\underline{F}_{int} - \underline{F}_{ext}) = \delta\underline{U}^T\underline{R}$

$D\underline{F}_{int,e} = \int_{\Omega_e}\underline{B}_e^T D\sigma_e^h\,A\,dx = \int_{\Omega_e}\underline{B}_e^T\left(\frac{\partial\sigma_e^h}{\partial\varepsilon_e^h}\right)D\varepsilon_e^h\,A\,dx = \int_{\Omega_e}\underline{B}_e^T D\,\underline{B}_e\,A\,dx\,\Delta\underline{U}_e = \underline{K}_{t,e}\Delta\underline{U}_e$

$\underline{F}_{int,e} = \int_{\Omega_e}\underline{B}_e^T\sigma_e^h A\,dx \to \underline{F}_{int} = \mathop{\mathrm{A}}_{e=1}^{ne}\underline{F}_{int,e} \to \delta\Pi = \delta\underline{U}^T\underline{R} = \delta\underline{U}^T(\underline{F}_{int} - \underline{F}_{ext}) = 0$

$\underline{K}_{t,e} = \int_{\Omega_e}\underline{B}_e^T D\underline{B}_e A\,dx \to \underline{K}_t = \mathop{\mathrm{A}}_{e=1}^{ne}\underline{K}_{t,e} \to D\delta\Pi = \delta\underline{U}^T\underline{K}_t\Delta\underline{U} \to D\underline{R} = \underline{K}_t\Delta\underline{U}$

($\underline{B}_e = [dN_e^1/dx \dots dN_e^{n_{en}}/dx] = [-1/L_e \;\; 1/L_e]$ for 2-node bar; $D = \partial\sigma_e^h/\partial\varepsilon_e^h$)

→ same as elastic, except need to compute σ_e^h, $D = \partial\sigma_e^h/\partial\varepsilon_e^h$ at gauss points (= local problem)

- **local problem**: given $\varepsilon_n, \varepsilon_n^p, \alpha_n$: strain at prev. load step; $\Delta\varepsilon_n$: curr. strain change in NR iteration → compute curr. guess for σ_{n+1}, D_{n+1} and $\varepsilon_{n+1}, \varepsilon^p_{n+1}, \alpha_{n+1}$

$\varepsilon_{n+1} = \varepsilon_n + \Delta\varepsilon_n$

$\sigma_{n+1} = E(\varepsilon_{n+1} - \varepsilon^p_{n+1})$

$f_{n+1} = |\sigma_{n+1}| - (\sigma_0 + K\alpha_{n+1})$

$\dot{\varepsilon}^p = \dot{\gamma}\,\mathrm{sign}(\sigma) \to \varepsilon^p_{n+1} = \varepsilon^p_n + \Delta\gamma\,\mathrm{sign}(\sigma_{n+1})$

$\dot{\alpha} = \dot{\gamma} \to \alpha_{n+1} = \alpha_n + \Delta\gamma$

$\dot{\gamma} \ge 0 \to \Delta\gamma \ge 0$

$f \le 0 \to f_{n+1} \le 0$ (Kuhn-Tucker cond.)

$f\dot{\gamma} = 0 \to f_{n+1}\Delta\gamma = 0$

compute trial states $\varepsilon^{p\,tr}_{n+1}, \alpha^{tr}_{n+1}, \sigma^{tr}_{n+1}, f^{tr}_{n+1}$ assuming $\Delta\gamma = 0$

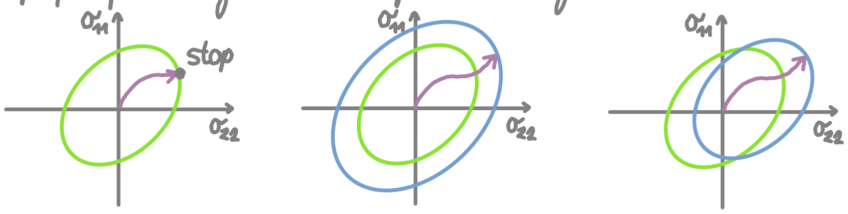
- if $f^{tr}_{n+1} \le 0 \to \ast = \ast^{tr}$ (trial state is admissible) $\longrightarrow D = \partial\sigma/\partial\varepsilon = \dot{\sigma}/\dot{\varepsilon} = E$
- if $f^{tr}_{n+1} > 0 \to f_{n+1} \overset{!}{=} 0 \to \Delta\gamma = f^{tr}_{n+1}/E+K \to$ recalc $\ast$ with new $\Delta\gamma \to D = \partial\sigma/\partial\varepsilon = \dot{\sigma}/\dot{\varepsilon} = E^{ep}$

($\sigma_{n+1} = \sigma^{tr}_{n+1} - E\Delta\gamma\,\mathrm{sign}(\sigma_{n+1})$ ↳ $\mathrm{sign}(\sigma^{tr}_{n+1}) = \mathrm{sign}(\sigma_{n+1})$ ↳ $|\sigma_{n+1}| + E\Delta\gamma = |\sigma^{tr}_{n+1}|$)

3D plasticity

- initial yield surface
- current yield surface
- load path

- perfect plasticity - isotropic hardening - kinematic hardening - real combination + other effects..



$\underline{\underline{\varepsilon}} = \underline{\underline{\varepsilon}}^e + \underline{\underline{\varepsilon}}^p \rightarrow \dot{\underline{\underline{\varepsilon}}} = \dot{\underline{\underline{\varepsilon}}}^e + \dot{\underline{\underline{\varepsilon}}}^p$ $\underline{\underline{\sigma}} = \mathbb{C} : \underline{\underline{\varepsilon}}^e = \mathbb{C} : (\underline{\underline{\varepsilon}} - \underline{\underline{\varepsilon}}^p) \rightarrow \dot{\underline{\underline{\sigma}}} = \mathbb{C} : \dot{\underline{\underline{\varepsilon}}}^e = \mathbb{C} : (\dot{\underline{\underline{\varepsilon}}} - \dot{\underline{\underline{\varepsilon}}}^p)$

($\underline{\underline{\varepsilon}}^e$: elastic-, $\underline{\underline{\varepsilon}}^p$: plastic strain) $= K\,\mathrm{tr}(\underline{\underline{\varepsilon}}^e)\underline{\underline{I}} + 2\mu\,\underline{\underline{\varepsilon}}^e_{dev}$ (neo-Hooke mat.)

yield function: $f(\underline{\underline{\sigma}}) \le 0$ (convex!) $\mathcal{A} = \{\underline{\underline{\sigma}} \mid f(\underline{\underline{\sigma}}) \le 0\}$ $\mathcal{E} = \{\underline{\underline{\sigma}} \mid f(\underline{\underline{\sigma}}) < 0\}$ $\delta\mathcal{A} = \{\underline{\underline{\sigma}} \mid f(\underline{\underline{\sigma}}) = 0\}$ ← yield surface

flow rule: $\dot{\underline{\underline{\varepsilon}}}^p = \dot{\gamma}\, \partial f / \partial \underline{\underline{\sigma}}$ (assume evolution of $\delta\mathcal{A}$ is normal to $\delta\mathcal{A}$. true for standard mat., e.g. metals)

Kuhn-Tucker + consistency cond.: $\dot{\gamma} \ge 0$; $f \le 0$; $\dot{\gamma} f = 0$; $\dot{\gamma}\dot{f} = 0$ if $f = 0$

- **perfect von Mises plasticity**: $f(\underline{\underline{\sigma}}) = \sigma_{eq}(\underline{\underline{\sigma}}) - \sigma_0 = \sqrt{\frac{3}{2}\underline{\underline{\sigma}}_{dev} \cdot \underline{\underline{\sigma}}_{dev}} - \sigma_0 = \sqrt{\frac{3}{2}}\|\underline{\underline{\sigma}}_{dev}\| - \sigma_0 \le 0$

  $\partial f/\partial \underline{\underline{\sigma}} = \sqrt{\frac{3}{2}}\, \underline{\underline{\sigma}}_{dev} / \|\underline{\underline{\sigma}}_{dev}\| = \sqrt{\frac{3}{2}}\, \underline{\underline{n}}$ ($\underline{\underline{n}} = \underline{\underline{\sigma}}_{dev}/\|\underline{\underline{\sigma}}_{dev}\| \rightarrow \|\underline{\underline{n}}\| = 1$)

  ↳ $\dot{\underline{\underline{\varepsilon}}}^p = \sqrt{\frac{3}{2}}\, \dot{\gamma}\, \underline{\underline{n}}$ ($\dot{\underline{\underline{\varepsilon}}}^p$ collinear with $\underline{\underline{\sigma}}_{dev} \rightarrow \dot{\underline{\underline{\varepsilon}}}^p$ is deviatoric → no volume change in plastic strain)
  ↳ volumetric locking?

  $\dot{f} = \partial f/\partial \underline{\underline{\sigma}} \cdot \dot{\underline{\underline{\sigma}}} = \partial f/\partial \underline{\underline{\sigma}} \cdot \mathbb{C} : \dot{\underline{\underline{\varepsilon}}} - \partial f/\partial \underline{\underline{\sigma}} \cdot \mathbb{C} : \dot{\underline{\underline{\varepsilon}}}^p = \sqrt{\frac{3}{2}}\, \underline{\underline{n}} \cdot \mathbb{C} : \dot{\varepsilon} - \frac{3}{2}\dot{\gamma}\, \underline{\underline{n}} \cdot \mathbb{C} : \underline{\underline{n}}$

  - plastic $\dot{\gamma} > 0$: $\dot{f} = 0 \rightarrow \dot{\gamma} = \sqrt{\frac{2}{3}}\, \underline{\underline{n}} \cdot \mathbb{C} : \dot{\underline{\underline{\varepsilon}}} / \underline{\underline{n}} \cdot \mathbb{C} : \underline{\underline{n}} = \ldots$ neo-Hooke-mat. $\ldots = \sqrt{\frac{3}{2}}\, \underline{\underline{n}} \cdot \dot{\underline{\underline{\varepsilon}}}$

    $\dot{\underline{\underline{\sigma}}} = \mathbb{C} : \dot{\underline{\underline{\varepsilon}}} - \mathbb{C} : \dot{\underline{\underline{\varepsilon}}}^p = \ldots$ neo-Hooke-mat. $\ldots = \underbrace{(\mathbb{C} - 2\mu\, \underline{\underline{n}} \otimes \underline{\underline{n}})}_{\mathbb{C}^{ep}} : \dot{\underline{\underline{\varepsilon}}} = \mathbb{C}^{ep} : \dot{\underline{\underline{\varepsilon}}}$

  - elastic $\dot{\gamma} = 0$: $\dot{\underline{\underline{\sigma}}} = \mathbb{C} : \dot{\underline{\underline{\varepsilon}}}$

- **isotropic linear hardening von Mises plasticity**: $f(\underline{\underline{\sigma}}, \alpha) = \sigma_{eq}(\underline{\underline{\sigma}}) - (\sigma_0 + K\alpha) = \sqrt{\frac{3}{2}}\|\underline{\underline{\sigma}}_{dev}\| - (\sigma_0 + K\alpha)$

  ↖ $\alpha = \sqrt{\frac{2}{3}} \int \|\dot{\underline{\underline{\varepsilon}}}^p\|$ (accumulated plastic strain) $\rightarrow \dot{\alpha} = \sqrt{\frac{2}{3}}\|\dot{\underline{\underline{\varepsilon}}}^p\| = \dot{\gamma}$

  - plastic $\dot{\gamma} > 0$: $\dot{\gamma} = \sqrt{\frac{2}{3}}\, \underline{\underline{n}} \cdot \dot{\underline{\underline{\varepsilon}}} / 1 + K/(3\mu)$; $\dot{\underline{\underline{\sigma}}} = \underbrace{(\mathbb{C} - 2\mu\, \underline{\underline{n}} \otimes \underline{\underline{n}} / 1 + K/(3\mu))}_{\mathbb{C}^{ep}} : \dot{\underline{\underline{\varepsilon}}} = \mathbb{C}^{ep} : \dot{\underline{\underline{\varepsilon}}}$
  - elastic $\dot{\gamma} = 0$: $\dot{\underline{\underline{\sigma}}} = \mathbb{C} : \dot{\underline{\underline{\varepsilon}}}$

  } for neo-Hooke.. no derivation..

- **FEM**: ... skipp deriv. ... → $\underline{F}_{int,e} = \int_{\Omega_e} \underline{\underline{B}}_e^T \hat{\underline{\sigma}}_e^h \, dV$ ← $\hat{\underline{\sigma}}_e^h$: Voigt form of $\underline{\underline{\sigma}}_e^h$

  $\underline{\underline{K}}_{t,e} = \int_{\Omega_e} \underline{\underline{B}}_e^T \underline{\underline{D}}\, \underline{\underline{B}}_e \, dV$ ← $\underline{\underline{D}}$: Voigt form of $\mathbb{D} = \partial \underline{\underline{\sigma}}_e^h / \partial \underline{\underline{\varepsilon}}_e^h$

- **local problem**: given $\underline{\underline{\varepsilon}}_n, \underline{\underline{\varepsilon}}_n^p, \alpha_n$: strain at prev. load step → compute curr. guess for $\underline{\underline{\sigma}}_{n+1}, \mathbb{D}_{n+1}$
  $\Delta\underline{\underline{\varepsilon}}_n$: curr. strain change in NR iteration and $\underline{\underline{\varepsilon}}_{n+1}, \underline{\underline{\varepsilon}}_{n+1}^p, \alpha_{n+1}$

$\underline{\underline{\varepsilon}}_{n+1} = \underline{\underline{\varepsilon}}_n + \Delta\underline{\underline{\varepsilon}}_n$
$\underline{\underline{\sigma}}_{n+1} = \mathbb{C} : (\underline{\underline{\varepsilon}}_{n+1} - \underline{\underline{\varepsilon}}_{n+1}^p)$
$f_{n+1} = \sqrt{\frac{3}{2}}\|\underline{\underline{\sigma}}_{dev\,n+1}\| - (\sigma_0 + K\alpha_{n+1})$

$\dot{\underline{\underline{\varepsilon}}}^p = \sqrt{\frac{3}{2}}\, \dot{\gamma}\, \underline{\underline{n}} \rightarrow \underline{\underline{\varepsilon}}_{n+1}^p = \underline{\underline{\varepsilon}}_n^p + \sqrt{\frac{3}{2}}\, \Delta\gamma\, \underline{\underline{n}}_{n+1}$
$\dot{\alpha} = \dot{\gamma} \rightarrow \alpha_{n+1} = \alpha_n + \Delta\gamma$

$\dot{\gamma} \ge 0 \rightarrow \Delta\gamma \ge 0$
$f \le 0 \rightarrow f_{n+1} \le 0$
$f\dot{\gamma} = 0 \rightarrow f_{n+1}\Delta\gamma = 0$

compute trial states $\underline{\underline{\varepsilon}}_{n+1}^{p\,tr}, \alpha_{n+1}^{tr}, \underline{\underline{\sigma}}_{n+1}^{tr}, f_{n+1}^{tr}$ assuming $\Delta\gamma = 0$

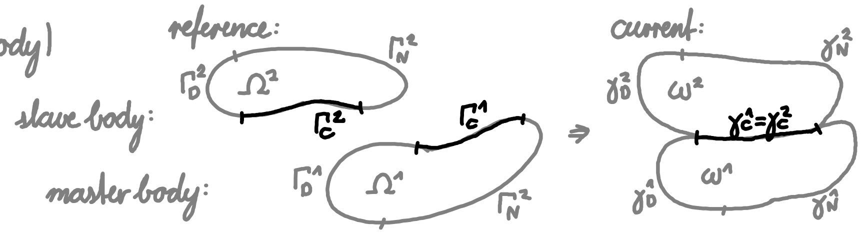
↳ if $f_{n+1}^{tr} \le 0 \rightarrow * = *^{tr}$ (trial state is admissible) ⟶ $\mathbb{D} = \mathbb{C}$

↳ if $f_{n+1}^{tr} > 0 \rightarrow f_{n+1} \overset{!}{=} 0 \rightarrow \Delta\gamma = f_{n+1}^{tr} / 3\mu + K$ → **recalc** $*$ with new $\Delta\gamma$ → $\mathbb{D} = \mathbb{C}^{ep} - \sqrt{\frac{3}{2}}\, \Delta\gamma\, \frac{4\mu^2}{\|\underline{\underline{\sigma}}_{dev}^{tr}\|} (\mathbb{I} - \underline{\underline{n}} \otimes \underline{\underline{n}})$?

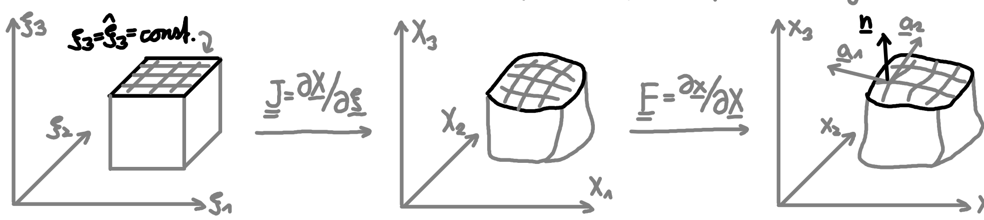
for neo-Hooke.. no derivation..

contact mechanics (3D, frictionless, 2 body)

goal: ω^1 and ω^2 should not intersect and share tractions at contact



- **convective coordinates:** introduce coord frame ξ for Ω_1, such that γ_c is contained in the area with $\xi_3=\hat{\xi}_3=\text{const.}$



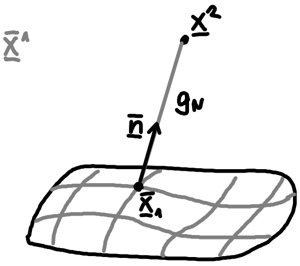
$$\hat{\underline{x}}(\underline{\xi})=\hat{\underline{x}}(\xi_1,\xi_2)=\underline{x}(\xi_1,\xi_2,\hat{\xi}_3)$$
$$\underline{a}_1=\partial\hat{\underline{x}}/\partial\xi_1 \;;\; \underline{a}_2=\partial\hat{\underline{x}}/\partial\xi_2$$
$$\underline{n}=\underline{a}_1\times\underline{a}_2/\|\underline{a}_1\times\underline{a}_2\|$$

- **pairing:** for each slave point $\underline{x}^2$ find signed dist. g_N to closest point on master surface $\bar{\underline{x}}^1$

$$\bar{\underline{\xi}}=\underset{\underline{\xi}}{\operatorname{argmin}}\|\underline{x}^2-\hat{\underline{x}}^1(\underline{\xi})\| \;\rightarrow\; \text{solve}\; \begin{matrix}(\underline{x}^2-\hat{\underline{x}}^1(\bar{\underline{\xi}}))\cdot\underline{a}_1=0\\ (\underline{x}^2-\hat{\underline{x}}^1(\bar{\underline{\xi}}))\cdot\underline{a}_2=0\end{matrix} \;\rightarrow\; \begin{matrix}\bar{\underline{x}}=\hat{\underline{x}}(\bar{\underline{\xi}}) & \bar{\underline{n}}=\underline{n}(\bar{\underline{\xi}})\\ \bar{\underline{a}}_1=\underline{a}_1(\bar{\underline{\xi}}) & \bar{\underline{a}}_2=\underline{a}_2(\bar{\underline{\xi}})\end{matrix}$$

↳ $g_N=(\underline{x}^2-\bar{\underline{x}}^1)\cdot\bar{\underline{n}}$

↳ variation: $\delta g_N=(\delta\underline{x}^2-\delta\bar{\underline{x}}^1-\underline{a}_1\delta\xi_1-\underline{a}_2\delta\xi_2)\cdot\bar{\underline{n}}+(\underline{x}^2-\bar{\underline{x}}^1)\cdot\delta\bar{\underline{n}}=\ldots=(\delta\underline{u}^2-\delta\bar{\underline{u}}^1)\cdot\bar{\underline{n}}$



- **contact constitutive law:** $g_N\geq 0$; $t_N\leq 0$; $\left.\begin{matrix}\text{if } g_N>0 \text{ then } t_N=0\\ \text{if } g_N=0 \text{ then } t_N\leq 0\end{matrix}\right\}\; g_N t_N=0$ (HSM conditions)

- **mech. prob. weak form in reference config.:**

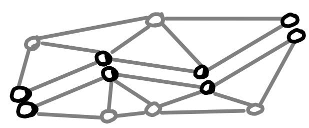
$\delta\Pi=\delta\Pi^1_{int}+\delta\Pi^2_{int}-(\delta\Pi^1_{ext}+\delta\Pi^2_{ext})+\delta\Pi_c \quad \forall\delta\underline{u}$ adm. $\leftarrow \delta\Pi^i_{int}=\int_{\Omega^i}\underline{\underline{P}}^i:\delta\underline{\underline{F}}^i dV$; $\delta\Pi^i_{ext}=\int_{\Omega^i}\underline{B}^i\cdot\delta\underline{u}^i dV+\int_{\Gamma^i_N}\underline{T}^i\cdot\delta\underline{u}^i dA$

↑ $\delta\Pi_c=-\int_{\gamma_c^1}\underline{t}^1\cdot\delta\underline{u}^1 da-\int_{\gamma_c^2}\underline{t}^2\cdot\delta\underline{u}^2 da=\int_{\gamma_c}\underline{t}^2(\delta\underline{u}^2-\delta\bar{\underline{u}}^1)da=\int_{\gamma_c}t_N\,\delta g_N\,da=\int_{\Gamma_c^2}T_N\,\delta g_N\,dA \qquad (T_N=t_N\,da/dA)$

↖ same traction for pairs! ↖ or Γ_c^1

- **Lagrange multiplier method:** solve for $\underline{u}$ and T_N → $\delta\Pi[\delta\underline{u}]=\delta\Pi^1+\delta\Pi^2+\int_{\Gamma_c^2}T_N\,\delta g_N\,dA=0 \quad \forall\delta\underline{u}$ adm.
  $\delta\Pi[\delta T_N]=\int_{\Gamma_c^2}g_N\,\delta T_N\,dA=0 \quad \forall\delta T_N$
  (assumes Γ_c^2 is known → $g_N\leq 0$ on Γ_c^2 ; need discretization for T_N ; exact constraint enforcement)
- **penalty method:** penalize $g_N<0$ with $T_N=\epsilon_N g_N$ → $\delta\Pi[\delta\underline{u}]=\delta\Pi^1+\delta\Pi^2+\int_{\Gamma_c^2}\epsilon_N g_N\,\delta g_N\,dA=0 \quad \forall\delta\underline{u}$ adm.
  (assumes Γ_c^2 is known → $g_N\leq 0$ on Γ_c^2 ; approx. constraint enforcement ; ϵ_N can't be to large or to small)

- **contact discretization**



- **node to node** contact element: slave node + matching master node
  (+ simple to implement, + passes pressure uniformly, − need conforming meshes, − need small defo. & sliding)
- **mortar method:** contact element: **slave surface + master surface**
  (+ suitable for large defo. & sliding, + passes pressure uniformly, − hard to implement)
- **node to surface** contact element: slave node + master surface that contains node

with penalty method: $\delta\Pi_c=\int_{\Gamma_c^2}\epsilon_N g_N\,\delta g_N\,dA$

↳ $\delta\Pi_c^h=\sum_{i=1}^{n_{sn}}\epsilon_N g_{N_i}\,\delta g_{N_i}\,A_i=\sum_{i=1}^{n_{sn}}\delta\underline{U}_i^T\epsilon_N g_{N_i}A_i\underline{N}_i$
$=\sum_{i=1}^{n_{sn}}\delta\underline{U}_i^T\underline{F}_{c_i}=\delta\underline{U}^T\underline{F}_c \quad (\underline{F}_{c_i}=\epsilon_N g_{N_i}A_i\underline{N}_i\;;\;\underline{F}_c=\mathrm{A}_{i=1}^{n_{sn}}\underline{F}_{c_i})$

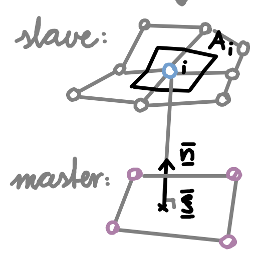
↳ $\delta\Pi^h=\delta\underline{U}^T(\underline{F}_{int}(\underline{U})-\underline{F}_{ext}+\underline{F}_c)=0 \quad \forall\delta\underline{U}$ admissible

(problem if node inside multiple or outside all. Then use node to node with closest master node?)

n_{sn}: # slave nodes in contact
A_i: tributary area of node i

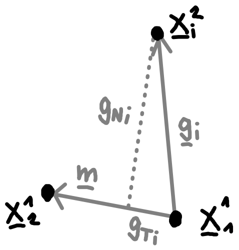
$$\delta\underline{U}_i=\begin{bmatrix}\delta\underline{x}_i^2\\ \delta\underline{x}_1^1\\ \vdots\\ \delta\underline{x}_{n_{ee}}^1\end{bmatrix} \;;\; \underline{N}_i=\begin{bmatrix}\bar{\underline{n}}\\ -N_1(\bar{\underline{\xi}})\bar{\underline{n}}\\ \vdots\\ -N_{n_{ee}}(\bar{\underline{\xi}})\bar{\underline{n}}\end{bmatrix}$$

↳ $\delta g_{N_i}=(\delta\underline{x}_i^2-\sum_{a=1}^{n_{ee}}N_a(\bar{\underline{\xi}})\delta\underline{x}_a^1)\cdot\bar{\underline{n}}=\delta\underline{U}_i^T\underline{N}_i$



(+ simple to implement, + suitable for large defo. & sliding, − doesn't pass pressure uniformly)

2D case:



$\underline{m} = \underline{x}_2^1 - \underline{x}_1^1$; $l_m = \|\underline{m}\|$; $\underline{t} = \frac{1}{l_m} \cdot \underline{m}$; $\underline{n} = \underline{t} \times \underline{e}_3$; $\underline{g}_i = \underline{x}_i^2 - \underline{x}_1^1$; $g_{Ni} = \underline{g}_i \cdot \underline{n}$; $g_{Ti} = \underline{g}_i \cdot \underline{t}$; $\xi_i = \frac{1}{l_m} g_{Ti}$

$\delta\underline{m} = \delta\underline{u}_2^1 - \delta\underline{u}_1^1$; $\delta\underline{g}_i = \delta\underline{u}_i^2 - \delta\underline{u}_1^1$; $\delta l_m = \frac{1}{l_m} \underline{m} \cdot \delta\underline{m} = (\delta\underline{u}_2^1 - \delta\underline{u}_1^1) \cdot \underline{t} = \delta\underline{U}_i^T \underline{T}_0$; $D l_m = \underline{T}_0^T \cdot \Delta\underline{U}_i$

$\delta\underline{t} = \frac{1}{l_m}\delta\underline{m} - \frac{1}{l_m^2}\delta l_m \underline{m} = \frac{1}{l_m}(\delta\underline{u}_2^1 - \delta\underline{u}_1^1) \cdot \underline{n}\,\underline{n} = \frac{1}{l_m}\delta\underline{U}_i^T \underline{N}_0 \underline{n}$; $D\underline{t} = \frac{1}{l_m}\underline{N}_0^T \Delta\underline{U}_i \underline{n}$; $\delta\underline{n} = \delta\underline{t} \times \underline{e}_3 = -\frac{1}{l_m}\delta\underline{U}_i^T \underline{N}_0 \underline{t}$

$D\underline{n} = -\frac{1}{l_m}\underline{N}_0^T \Delta\underline{U}_i \underline{t}$; $D\delta\underline{n} = \ldots = \frac{1}{l_m^2}\delta\underline{U}_i^T \underline{T}_0 \underline{N}_0^T \Delta\underline{U}_i \underline{t} - \frac{1}{l_m^2}\delta\underline{U}_i^T \underline{N}_0 \underline{N}_0^T \Delta\underline{U}_i^T \underline{n} + \frac{1}{l_m^2}\delta\underline{U}_i^T \underline{N}_0 \underline{T}_0 \Delta\underline{U}_i \underline{t}$

$\delta g_{Ni} = \delta\underline{g}_i \cdot \underline{n} + \underline{g}_i \cdot \delta\underline{n} = \delta\underline{U}_i^T \underline{N}_g - \xi_i \delta\underline{U}_i^T \underline{N}_0 = \delta\underline{U}_i^T \underline{N}_s$; $D\delta g_{Ni} = D\delta\underline{g}_i \cdot \underline{n} + \delta\underline{g}_i \cdot D\underline{n} + \delta\underline{n} \cdot D\underline{g}_i + \underline{g}_i \cdot D\delta\underline{n} =$

$= \delta\underline{U}_i^T \left(-\frac{1}{l_m}\underline{T}_s \underline{N}_0^T - \frac{1}{l_m}\underline{N}_0 \underline{T}_s^T - \frac{g_{Ni}}{l_m^2}\underline{N}_0 \underline{N}_0^T\right) \Delta\underline{U}_i$

$\delta\Pi_c^h = \sum_{i=1}^{n_{sn}} \epsilon_N g_{Ni} \delta g_{Ni} A_i = \ldots = \sum_{i=1}^{n_{sn}} \delta\underline{U}_i^T \underline{F}_{ci}$

$D\delta\Pi_c^h = \sum_{i=1}^{n_{sn}} \epsilon_N A_i (\delta g_{Ni} D g_{Ni} + g_{Ni} D\delta g_{Ni}) = \ldots = \sum_{i=1}^{n_{sn}} \delta\underline{U}_i^T \underline{\underline{K}}_{ci} \Delta\underline{U}_i$

$\underline{F}_{ci} = \epsilon_N A_i g_{Ni} \underline{N}_s$

$\underline{\underline{K}}_{ci} = \epsilon_N A_i \left(\underline{N}_s \underline{N}_s^T - \frac{g_{Ni}}{l_m}(\underline{T}_s \underline{N}_0^T + \underline{N}_0 \underline{T}_s^T) - \frac{g_{Ni}^2}{l_m^2}\underline{N}_0 \underline{N}_0^T\right)$

$$\Delta\underline{U}_i = \begin{bmatrix} \delta\underline{u}_i^2 \\ \delta\underline{u}_1^1 \\ \delta\underline{u}_2^1 \end{bmatrix} \quad \underline{T}_0 = \begin{bmatrix} \underline{0} \\ -\underline{t} \\ \underline{t} \end{bmatrix} \quad \underline{N}_0 = \begin{bmatrix} \underline{0} \\ -\underline{n} \\ \underline{n} \end{bmatrix} \quad \underline{T}_g = \begin{bmatrix} \underline{t} \\ -\underline{t} \\ \underline{0} \end{bmatrix} \quad \underline{N}_g = \begin{bmatrix} \underline{n} \\ -\underline{n} \\ \underline{0} \end{bmatrix} \quad \begin{matrix} \underline{T}_s = \underline{T}_g - \xi_i \underline{T}_0 \\ \underline{N}_s = \underline{N}_g - \xi_i \underline{N}_0 \end{matrix}$$

- notes:
  - body with finer mesh should be slave. if similar, then more deformable body should be slave.
  - to avoid bias, one can swap master ⇔ slave at each iter. is more expensive and can cause surface locking
  - "smoothing" the master surface can be used to find unique node-to-surface pairs