Dimensionsanalyse (Bsp. Umströmte Kugel)

$$F_D = f(\mu, \rho, U, D) \rightarrow \tilde{f}(F_D, \mu, \rho, U, D) = 0 \rightarrow \tilde{f}(\Pi_1, \dots, \Pi_k) = 0$$

$$[F_D] = [\mu]^\alpha \cdot [\rho]^\beta \cdot [U]^\gamma \cdot [D]^\delta \rightarrow [\mu]^\alpha \cdot [\rho]^\beta \cdot [U]^\gamma \cdot [D]^\delta \cdot [F_D]^\varepsilon = [1]$$

Bezugssystem: Masse, Länge, Zeit, Temperatur, Ladung (oder Kraft statt Masse)

$$\begin{array}{c} \\ kg \rightarrow \\ m \rightarrow \\ s \rightarrow \end{array} \begin{array}{c} \begin{array}{ccccc} \mu & \rho & U & D & F_D \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ -1 & -3 & 1 & 1 & 1 \\ -1 & 0 & -1 & 0 & -2 \end{pmatrix} \end{array} \cdot \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \\ \varepsilon \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Parameter $n = 5$
Masseinheiten $r = 3$
Unabh. Lsg. $k = n - r = 2$

□ **Bezugsgrössen / Referenzvar.:** $n-k$ linear unabhängige Parameter (μ, ρ, D)

■ **Normierungsgrössen:** k gesuchte oder schlecht Messbare Parameter wählen (U, F_D).

$$\begin{pmatrix} 1 & 1 & 0 \\ -1 & -3 & 1 \\ -1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \\ \delta \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad \begin{array}{l} \alpha = 1 \\ \beta = -1 \\ \delta = -1 \end{array}, \quad \begin{array}{l} \gamma \doteq -1 \\ \varepsilon \doteq 0 \end{array}$$

$$\rightarrow \mu^1 \rho^{-1} U^{-1} D^{-1} \rightarrow \Pi_1 = D \cdot \frac{U\rho}{\mu}$$

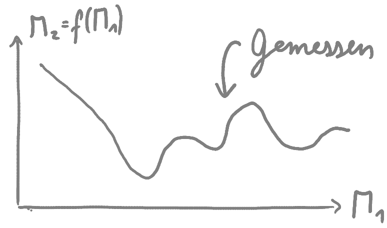
$$\begin{pmatrix} 1 & 1 & 0 \\ -1 & -3 & 1 \\ -1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \\ \delta \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \quad \begin{array}{l} \alpha = 2 \\ \beta = -1 \\ \delta = 0 \end{array}, \quad \begin{array}{l} \gamma \doteq 0 \\ \varepsilon \doteq -1 \end{array}$$

$$\rightarrow \mu^2 \cdot \rho^{-1} \cdot F_D^{-1} \rightarrow \Pi_2 = F_D \cdot \frac{\rho}{\mu^2}$$

Sonderfall $k = 1$: $\tilde{f}(\Pi_1) = 0 \rightarrow \Pi_1 = \text{Konst.}$

$k = 2$: $\tilde{f}(\Pi_1, \Pi_2) = 0 \rightarrow \Pi_2 = f(\Pi_1) \rightarrow$

$k > 2$: $\tilde{f}(\Pi_1, \dots, \Pi_k) = 0 \rightarrow \Pi_1 = f(\dots, \Pi_k)$



	Grösse	F	L	T	ϑ	M	L	T	ϑ	Einheit
l	Länge	0	1	0	0	0	1	0	0	m
F	Kraft	1	0	0	0	1	1	-2	0	N
m	Masse	1	-1	2	0	1	0	0	0	kg
t	Zeit	0	0	1	0	0	0	1	0	s
ϑ	Temperatur	0	0	0	1	0	0	0	1	K
u	Geschwindigkeit	0	1	-1	0	0	1	-1	0	m/s
a	Beschleunigung	0	1	-2	0	0	1	-2	0	m/s^2
p	Druck, E-Modul	1	-2	0	0	1	-1	-2	0	Pa = N/m^2
σ	Oberfl. Spannung	1	-1	0	0	1	0	-2	0	N/m
E	Energie, Moment, Arbeit	1	1	0	0	1	2	-2	0	J = Ws = Nm
P	Leistung, Energiestrom	1	1	-1	0	1	2	-3	0	W = Nm/s
ρ	Dichte	1	-4	2	0	1	-3	0	0	kg/m^3
$\dot{m}$	Massenstrom	1	-1	1	0	1	0	-1	0	kg/s
μ	Dynamische Viskosität	1	-2	1	0	1	-1	-1	0	Pa·s = Ns/m^2
ν	Kinematische Viskosität	0	2	-1	0	0	2	-1	0	m^2/s
c	Spezifische Wärmekapazität	0	2	-2	-1	0	2	-2	-1	J/kgK
λ	Wärmeleitfähigkeit	1	0	-1	-1	1	1	-3	-1	W/mK
R	Spezielle Gaskonstante	0	2	-2	-1	0	2	-2	-1	J/kgK

Reynolds-Zahl	$\mathrm{Re} = \frac{u \cdot l}{\nu}$	Weber-Zahl	$\mathrm{We} = \frac{\rho \cdot u^2 \cdot l}{\sigma}$
Froude-Zahl	$\mathrm{Fr} = \frac{u}{\sqrt{g \cdot l}}$	Strouhal-Zahl	$\mathrm{Str} = \frac{l}{u \cdot t}$
Mach-Zahl	$\mathrm{Ma} = \frac{u}{a} = u/\sqrt{\gamma R T}$	Eckert-Zahl	$\mathrm{Ec} = \frac{u^2}{C_p \cdot \Delta T}$
Euler-Zahl	$\mathrm{Eu} = \frac{p}{\rho \cdot u^2} = (\gamma \cdot \mathrm{Ma}^2)^{-1}$	Fourier-Zahl	$\mathrm{Fo} = \frac{l^2}{k \cdot t}$
Knudsen-Zahl	$\mathrm{Kn} = \frac{\mathrm{Ma}}{\mathrm{Re}}$	Grashof-Zahl	$\mathrm{Gr} = g \cdot l^3 \cdot \alpha \cdot (T_W - T_\infty) \cdot \nu^{-2}$

Nusselt-Zahl	$\mathrm{Nu} = \frac{\alpha \cdot l}{\lambda}$	Reibungskoeffizient	$c_f = \frac{2 \tau_W}{\rho \cdot u^2}$
Péclet-Zahl	$\mathrm{Pe} = \frac{u \cdot l}{k}$	Druckkoeffizient	$c_p = \frac{2 \cdot \Delta p}{\rho \cdot u^2}$
Prandtl-Zahl	$\mathrm{Pr} = \frac{\nu}{k}$	Auftriebsbeiwert	$c_A = \frac{2 \cdot F_A}{\rho \cdot u^2 \cdot l^2}$
Rayleigh-Zahl	$\mathrm{Ra} = \frac{\mathrm{Gr}}{\mathrm{Pr}}$	Widerstandsbeiwert	$c_W = \frac{2 \cdot F_W}{\rho \cdot u^2 \cdot l^2}$
Stokes-Zahl	$\mathrm{St} = \frac{p \cdot l}{\mu \cdot u}$	Momentenbeiwert	$\zeta_M = \frac{2 \cdot \mathrm{Ma}}{\rho \cdot R^5 \cdot \omega^2}$

Nabla Operator

Carthesian:

$$\nabla \cdot a = \mathrm{grad}(a) = \begin{pmatrix} \partial a/\partial x \\ \partial a/\partial y \\ \partial a/\partial z \end{pmatrix}$$

$$\nabla^2 \cdot a = \Delta \cdot a = \frac{\partial^2 a}{\partial x^2} + \frac{\partial^2 a}{\partial y^2} + \frac{\partial^2 a}{\partial z^2}$$

$$\nabla \cdot \underline{a} = \mathrm{div}(\underline{a}) = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$$

$$\nabla^2 \cdot \underline{a} = \Delta \cdot \underline{a} = \begin{pmatrix} \partial^2 a_x/\partial x^2 + \partial^2 a_x/\partial y^2 + \partial^2 a_x/\partial z^2 \\ \partial^2 a_y/\partial x^2 + \partial^2 a_y/\partial y^2 + \partial^2 a_y/\partial z^2 \\ \partial^2 a_z/\partial x^2 + \partial^2 a_z/\partial y^2 + \partial^2 a_z/\partial z^2 \end{pmatrix}$$

$$\nabla \times \underline{a} = \mathrm{rot}(\underline{a}) = \begin{pmatrix} \partial a_z/\partial y - \partial a_y/\partial z \\ \partial a_x/\partial z - \partial a_z/\partial x \\ \partial a_y/\partial x - \partial a_x/\partial y \end{pmatrix}$$

$$(\underline{a} \cdot \nabla) \cdot \underline{b} = \begin{pmatrix} a_x \cdot \partial b_x/\partial x + a_y \cdot \partial b_x/\partial y + a_z \, \partial b_x/\partial z \\ a_x \cdot \partial b_y/\partial x + a_y \cdot \partial b_y/\partial y + a_z \, \partial b_y/\partial z \\ a_x \cdot \partial b_z/\partial x + a_y \cdot \partial b_z/\partial y + a_z \, \partial b_z/\partial z \end{pmatrix}$$

$$(\underline{a} \cdot \nabla) \cdot b = a_x \frac{\partial b}{\partial x} + a_y \frac{\partial b}{\partial y} + a_z \frac{\partial b}{\partial z}$$

Zylindrical:

$$\nabla \cdot a = \mathrm{grad}(a) = \begin{pmatrix} \partial a/\partial r \\ \frac{1}{r} \cdot \partial a/\partial \varphi \\ \partial a/\partial z \end{pmatrix}$$

$$\nabla^2 \cdot a = \Delta \cdot a = \frac{1}{r} \frac{\partial}{\partial r}\left(r \cdot \frac{\partial a}{\partial r}\right) + \frac{1}{r^2} \cdot \frac{\partial^2 a}{\partial \varphi^2} + \frac{\partial^2 a}{\partial z^2}$$

$$\nabla \cdot \underline{a} = \mathrm{div}(\underline{a}) = \frac{1}{r} \cdot \frac{\partial (r \cdot a_r)}{\partial r} + \frac{1}{r} \cdot \frac{\partial a_\varphi}{\partial \varphi} + \frac{\partial a_z}{\partial z}$$

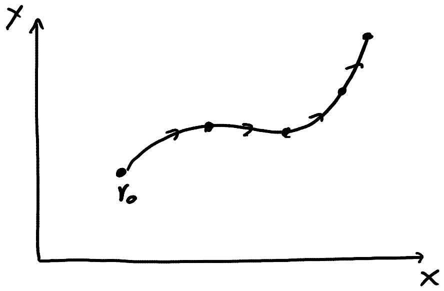
$$\nabla^2 \cdot \underline{a} = \Delta \cdot \underline{a} = \begin{pmatrix} \nabla^2 a_r - \frac{a_r}{r^2} - \frac{2}{r^2} \cdot \frac{\partial a_\varphi}{\partial \varphi} \\ \nabla^2 a_\varphi - \frac{a_\varphi}{r^2} + \frac{2}{r^2} \cdot \frac{\partial a_r}{\partial \varphi} \\ \nabla^2 a_z \end{pmatrix}$$

$$\nabla \times \underline{a} = \mathrm{rot}(\underline{a}) = \begin{pmatrix} \frac{1}{r} \frac{\partial a_z}{\partial \varphi} - \frac{\partial a_\varphi}{\partial z} \\ \frac{\partial a_r}{\partial z} - \frac{\partial a_z}{\partial r} \\ \frac{1}{r} \cdot \frac{\partial (r \cdot a_\varphi)}{\partial r} - \frac{1}{r} \cdot \frac{\partial a_r}{\partial \varphi} \end{pmatrix}$$

$$(\underline{a} \cdot \nabla) \cdot \underline{b} = \begin{pmatrix} a_r \cdot \frac{\partial b_r}{\partial r} + \frac{a_\varphi}{r} \cdot \frac{\partial b_r}{\partial \varphi} + a_z \cdot \frac{\partial b_r}{\partial z} - \frac{a_\varphi b_\varphi}{r} \\ a_r \cdot \frac{\partial b_\varphi}{\partial r} + \frac{a_\varphi}{r} \cdot \frac{\partial b_\varphi}{\partial \varphi} + a_z \cdot \frac{\partial b_\varphi}{\partial z} + \frac{a_\varphi b_r}{r} \\ a_r \cdot \frac{\partial b_z}{\partial r} + \frac{a_\varphi}{r} \cdot \frac{\partial b_z}{\partial \varphi} + a_z \cdot \frac{\partial b_z}{\partial z} \end{pmatrix}$$

$$(\underline{a} \cdot \nabla) \cdot b = a_r \frac{\partial b}{\partial r} + \frac{a_\varphi}{r} \frac{\partial b}{\partial \varphi} + a_z \frac{\partial b}{\partial z}$$

Lagrange Beschreibung: (ein Teilchen fliegt herum)

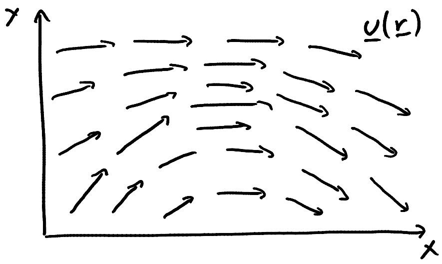


$\underline{r}^L(t,\underline{r}_0) = \dots$ (Position Teilchen)

$\underline{u}^L(t,\underline{r}_0) = \frac{\partial}{\partial t}\underline{r}^L(t,\underline{r}_0)$ (Geschw. ")

$\underline{a}^L(t,\underline{r}_0) = \frac{\partial^2}{\partial t^2}\underline{r}^L(t,\underline{r}_0)$ (Besch. ")

Eulersche Beschreibung (veränderung Strömungsgrössen an Punkt)



z.B.:

$\underline{u}(t,\underline{r})$: (Geschw.)
$\rho(t,\underline{r})$: (Dichte)
$T(t,\underline{r})$: (Temp.)
$p(t,\underline{r})$: (Druck)

- **Lagrange → Euler:** $\left(\underline{r}^L(t,\underline{r}_0) \longrightarrow \underline{u}^E(t,\underline{r})\right)$

from $\underline{r} = r^L(t,\underline{r}_0)$ get $\underline{r}_0 = (r^L)^{-1}(t,\underline{r}) \longrightarrow \underline{u}^E(t,\underline{r}) = \underbrace{\left(\frac{\partial}{\partial t}\underline{r}^L(t,\underline{r}_0)\right)}_{\underline{u}^L(t,r_0)}\Bigg|_{\underline{r}_0 = (r^L)^{-1}(t,\underline{r})}$

- **Euler → Lagrange:** $\left(\underline{u}^E(t,\underline{r}) \longrightarrow \underline{r}^L(t,r_0)\right)$

solve diff.eq. $\frac{\partial \underline{r}^{L*}(t)}{\partial t} = \underline{u}^E(t,\underline{r}^{L*}(t))$ for $\underline{r}^{L*}(t,\overbrace{C_1,C_2,\dots}^{\text{integration constants}})$

$\longrightarrow \underline{r}^{L*}(t,C_1,C_2,\dots)\big|_{t=0} = \underline{r}_0 \Rightarrow$ express $C_1, C_2, \dots$ as func. of $\underline{r}_0 \Rightarrow \underline{r}^L(t,\underline{r}_0)$

- **Substantielle Ableitung:** Veränderung von Str.grös. φ aus sicht von Partikel
(z.B. $\varphi = \underline{u}, \rho, \dots$)

$$\frac{D\varphi}{Dt} = \underbrace{\frac{\partial \varphi}{\partial t}}_{\text{lokale Abl.}} + \underbrace{(\underline{u}\cdot\underline{\nabla})\,\varphi}_{\text{Konvektive Abl.}} = \begin{pmatrix} \frac{\partial \varphi_1}{\partial t} \\ \frac{\partial \varphi_2}{\partial t} \end{pmatrix} + \begin{pmatrix} u_1 \cdot \frac{\partial \varphi_1}{\partial x} + u_2 \frac{\partial \varphi_1}{\partial y} \\ u_1 \cdot \frac{\partial \varphi_2}{\partial x} + u_2 \frac{\partial \varphi_2}{\partial y} \end{pmatrix}$$

- **Reinolds Transporttheorem:**

$\Psi(t) = \int_{\tilde V(t)} \varphi(\underline{x},t)\, d\tilde V$ Ψ: $m = \int_V \rho\, dV$, $W = \int_V p\, dV$

$\frac{d\Psi}{dt} = \int_{V(t)} \frac{\partial \varphi}{\partial t}\, dV + \int_{S(t)} \varphi\, \underline{u}\cdot\underline{n}\, dS$ $\cancel{\xrightarrow{V\ \text{konst.}} \frac{d\Psi}{dt} = \frac{d}{dt}\int_V \varphi\, dV + \int_S \varphi\, \underline{u}\cdot\underline{n}\, dS}$

Kinematische Eigenschaften von Strömungen

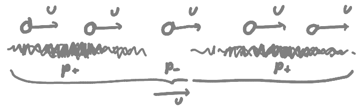
- Inkompressibles Fluid: $\rho(t,\underline{r}) = \text{Konst.}$ { Alle Partikel erfahren dieselbe Dichte in ihrem Leben

- Inkompressible Strömung: $\frac{D\rho}{Dt} = 0$ $(\nabla \underline{u} = 0)$ { Jedes Partikel erfährt eine eigene Konst. Dichte in seinem Leben

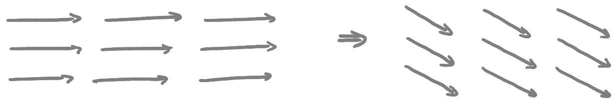
- Stationäre Strömung: $\frac{\partial \varphi}{\partial t} = 0$ $\varphi = \underline{u}, \rho, T, p, \dots$

$\frac{\partial \varphi}{\partial t} = 0 \overset{\not\Leftrightarrow}{\longrightarrow} \frac{D\varphi}{Dt} = 0$ $\qquad$ $\frac{D\varphi}{Dt} = 0 \overset{\not\Leftrightarrow}{\longrightarrow} \frac{\partial \varphi}{\partial t} = 0$

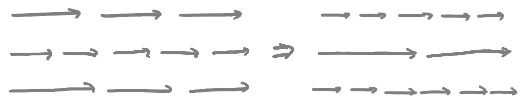
Partikel Beschl., aber Geschw. an Punkt Konst.



- Gleichförmige Strömung: $\frac{\partial u}{\partial r} = 0$ $\leftarrow x, y, z$



- Ausgebildete Strömung: $\frac{\partial u}{\partial v} = 0$ $\leftarrow$ Stromrichtung



- Wirbelstärke: $\underline{\omega} = \underline{\nabla} \times \underline{u} = \text{rot}(\underline{u}) = \left(\frac{\delta u_3}{\delta y} - \frac{\delta u_2}{\delta z} ; \frac{\delta u_1}{\delta z} - \frac{\delta u_3}{\delta x} ; \frac{\delta u_2}{\delta x} - \frac{\delta u_1}{\delta y} \right)$

- barotrop: $\rho = \rho(p)$ $\qquad$ baroklin: $\rho = \rho(p, T)$

Ausgezeichnete Linien

- Stromlinie: Linie Tangential zu $\underline{u}$ von momentaner Zeit t.

2D Karth: $u(t,x,y)\big|_{t=t_0} = \begin{pmatrix} u_x(x,y) \\ u_y(x,y) \end{pmatrix} \rightarrow \frac{dx}{u_x(x,y)} = \frac{dy}{u_y(x,y)} \xrightarrow{\text{sep.}} f(x)\,dx = g(y)\,dy \xrightarrow{\text{int.}} F(x) = G(y) + C$

2D Zyl.: $u(t,r,\varphi)\big|_{t=t_0} = \begin{pmatrix} u_r(r,\varphi) \\ u_\varphi(r,\varphi) \end{pmatrix} \rightarrow \frac{dr}{u_r(r,\varphi)} = \frac{r \cdot d\varphi}{u_\varphi(r,\varphi)} \xrightarrow{\text{sep.}} f(r)\,dr = g(\varphi)\,d\varphi \xrightarrow{\text{int.}} F(r) = G(\varphi) + C$

3D Karth: $\frac{dx}{u_x(x,y,z)} = \frac{dy}{u_y(x,y,z)} = \frac{dz}{u_z(x,y,z)}$

- Bahnlinie: Linie die ein einzelnes Partikel begeht.

(Euler → Lagrange -darstellung Verfahren)

- Streichline: Pfad aller Partikel die durch einen Punkt gehen. "Kerze im Wind"

- Zeitlinie : Linie von Partikeln die früher auf anderer Linie waren.

→ Stationäre Strömung: Stromlinie = Bahnlinie = Streichlinie

Massenerhaltung $\left(\frac{dm}{dt} = 0\right)$

$$\frac{dm}{dt} = \frac{d}{dt}\int_V \rho\, dV + \int_S \rho \underline{u}\cdot\underline{n}\, dS = 0$$

$$\frac{\partial \rho}{\partial t} + \nabla(\rho\underline{u}) = 0$$

$$\sum_i A_i \rho u_i = 0$$

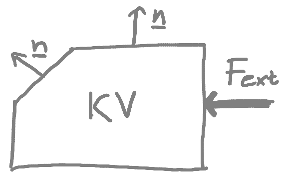
Massenänderung im Gebiet | Massenfluss über Gebietsrand

(Kontinuitätsgl.)

Energieerhaltung

$$\frac{DE}{Dt} = \frac{d}{dt}\int_V \rho\left(e+\frac{u^2}{2}\right)dV + \int_S \rho\left(e+\frac{u^2}{2}\right)(\underline{u}\cdot\underline{n})\,dS = P_{OUT}$$

$\llcorner_{2D}\rightarrow$ $$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho\cdot u_x)}{\partial x} + \frac{\partial(\rho\cdot u_y)}{\partial y} = 0$$



Impulserhaltung $\left(\frac{d\underline{P}}{dt} = \sum \underline{F} \;\Leftrightarrow\; \underline{P} = \int_V \rho\underline{u}\,dV\right)$

$$\frac{d\underline{P}}{dt} = \frac{d}{dt}\int_V \rho\cdot\underline{u}\,dV + \int_S \rho\cdot\underline{u}(\underline{u}\cdot\underline{n})\,dS = -\int_S p\cdot\underline{n}\,dS + \int_V \rho\cdot\underline{f}\,dV + \int_S \underline{\underline{\tau}}\cdot\underline{n}\,dS + \underline{F}_{ext}$$

$$\rho\frac{D\underline{u}}{Dt} = \frac{\partial}{\partial t}\rho\cdot\underline{u} + \nabla(\rho\underline{u}\cdot\underline{u}) = -\nabla p + \rho\cdot\underline{f} + \nabla\underline{\underline{\tau}}$$

$$\sum A_i \rho u_i^2 = -\sum A_i p_i + f\cdot\rho\cdot V + F_{ext}$$

Impulsänderung im Gebiet | Impulsabfluss über Gebietsrand | Druck-Kräfte | Volumen-Kräfte | Oberfl. scher-Kräfte | äussere Kraft

Volumenkräfte:

Schwerkraft: $f = -g \qquad U = g\cdot z \qquad (f = -\nabla U)$

Fliehkraft: $f = \omega^2 r \qquad U = -\frac{\omega^2}{2}\cdot r^2$

Eulergleichung: $(\underline{\underline{\tau}} = 0,\ \rho = \text{Konst.})$

$$\frac{D\underline{u}}{Dt} = \frac{\partial\underline{u}}{\partial t} + (\underline{u}\cdot\nabla)\underline{u} = -\frac{1}{\rho}\nabla p + \underline{f}$$

$\llcorner_{2D}\rightarrow$ $$\begin{pmatrix}\partial u_x/\partial t\\ \partial u_y/\partial t\end{pmatrix} + \begin{pmatrix}u_x\cdot\partial u_x/\partial x + u_y\cdot\partial u_x/\partial y\\ u_x\cdot\partial u_y/\partial x + u_y\cdot\partial u_y/\partial y\end{pmatrix} = -\frac{1}{\rho}\begin{pmatrix}\partial p/\partial x\\ \partial p/\partial y\end{pmatrix} + \begin{pmatrix}f_x\\ f_y\end{pmatrix}$$

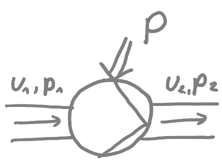
Bernoulligl.: $\left(\underline{\underline{\tau}} = 0,\ \rho = \text{Konst.},\ \underline{f} = -\nabla U,\ \begin{bmatrix}\text{rot}(\underline{u}) = 0 \text{ oder}\\ ds \text{ auf Stromlin.}\end{bmatrix}\right)$

$$\int_{①}^{②}\frac{\partial u}{\partial t}ds + \left[\frac{u_2\cdot u_2}{2} + \frac{p_2}{\rho} + U\right] = \left[\frac{u_1\cdot u_1}{2} + \frac{p_1}{\rho} + U\right]$$

(wenn idealgas: $\frac{p}{\rho} = h$)

Pumpe: $$P = \frac{\dot V}{\eta_P}\left(p_2 - p_1 + \rho\,\frac{u_2^2 - u_1^2}{2}\right)$$

$$p_2 - p_1 = \Delta p = \eta_P P/\dot V - \rho\,\frac{u_2^2 - u_1^2}{2}$$



Turbine: $\eta_P \Rightarrow \frac{1}{\eta_T}$

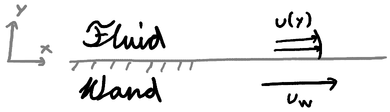
Reibungsbehaftete Strömung

Reibungs-Konstanten:

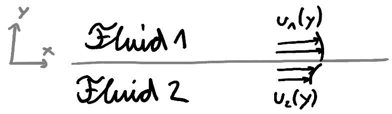
$$\mu = \nu \cdot \rho \qquad \tau = \mu \cdot \frac{\partial u_x}{\partial y} = \mu \cdot \left(\frac{1}{r}\frac{\partial u_r}{\partial \varphi} + r \cdot \frac{\partial}{\partial r}\left(\frac{u_\varphi}{r}\right)\right) \qquad F = A \cdot \tau \, ; \, M = 2\pi R^2 L \cdot \tau$$

Rand Bedingungen:

$$\boxed{p_1|_{y=0} = p_2|_{y=0}}$$



$$u(y)|_{y=0} = u_w$$



$$u_1(y)|_{y=0} = u_2(y)|_{y=0}$$

$$\mu_1 \cdot \frac{\partial u_1}{\partial y}\bigg|_{y=0} = \mu_2 \cdot \frac{\partial u_2}{\partial y}\bigg|_{y=0} \qquad \left(\Leftarrow \tau_1(y)|_{y=0} = \tau_2(y)|_{y=0}\right)$$

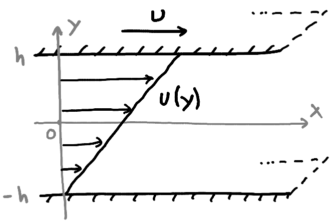
$$\frac{\partial u}{\partial y}\bigg|_{y=0} = 0 \qquad \left(\Leftarrow \tau(y)|_{y=0} = 0\right)$$

Navier-Stokes-Gl.: ($\underline{\tau} \neq 0$, ν = Konst. ρ = Konst.)

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + (u \cdot \nabla) \cdot u = -\frac{1}{\rho} \underline{\nabla} p + \nu \underline{\nabla}^2 \underline{u} + \underline{f}$$

$$\xrightarrow{2D} \begin{pmatrix} \partial u_x/\partial t \\ \partial u_y/\partial t \end{pmatrix} + \begin{pmatrix} u_x \cdot \partial u_x/\partial x + u_y \cdot \partial u_x/\partial y \\ u_x \cdot \partial u_y/\partial x + u_y \cdot \partial u_y/\partial y \end{pmatrix} = -\frac{1}{\rho}\begin{pmatrix} \partial p/\partial x \\ \partial p/\partial y \end{pmatrix} + \nu \begin{pmatrix} \partial^2 u_x/\partial x^2 + \partial^2 u_x/\partial y^2 \\ \partial^2 u_y/\partial x^2 + \partial^2 u_y/\partial y^2 \end{pmatrix} + \begin{pmatrix} f_x \\ f_y \end{pmatrix}$$

- **Ebene Couette Strömung:**

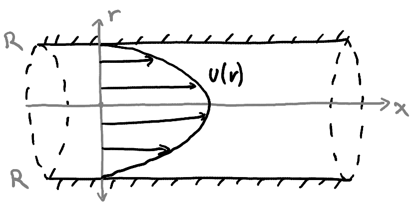


Geg.: $\underline{u} = \begin{pmatrix} u(y) \\ 0 \end{pmatrix}$,

$u(\pm h) = 0$, $\frac{dp}{dx} = 0$

$$\hookrightarrow u(y) = \frac{U}{2}\left(1 + \frac{y}{h}\right)$$

- **Hagen-Poiseuille Strömung:**



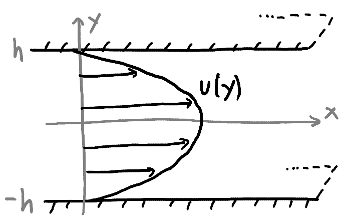
Geg.: $\underline{u} = \begin{pmatrix} u(y) \\ 0 \end{pmatrix}$

$u(R) = 0$, $\frac{dp}{dx} \neq 0$

$$\hookrightarrow u(r) = -\frac{R^2}{4\mu}\frac{dp}{dx}\left(1 - \frac{r^2}{R^2}\right) = u_{MAX}\left(1 - \frac{r^2}{R^2}\right)$$

$(u_{MAX} = u(0))$

- **Poiseuille Strömung:**



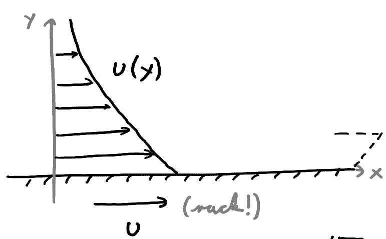
Geg.: $\underline{u} = \begin{pmatrix} u(y) \\ 0 \\ 0 \end{pmatrix}$

$u(\pm h) = 0$, $\frac{dp}{dx} \neq 0$

$$\hookrightarrow u(y) = -\frac{h^2}{2\mu}\frac{dp}{dx}\left(1 - \frac{y^2}{h^2}\right) = u_{MAX}\left(1 - \frac{y^2}{h^2}\right)$$

$(u_{MAX} = u(0))$

- **Stokessches Problem:**



Geg.: $\underline{u} = \begin{pmatrix} u(t,y) \\ 0 \\ 0 \end{pmatrix}$

$u(0,t) = 0$, $\frac{dp}{dx} = 0$

$$\hookrightarrow u(y,t) = U\left(1 - \frac{1}{\sqrt{\pi}}\int_0^{y/\sqrt{\nu t}} e^{(-\xi^2/4)}\, d\xi\right) = U\left(1 - \mathrm{erf}\left(\frac{y}{2\sqrt{\nu t}}\right)\right)$$

Nährungs Lösungen:

$$x_i' = \frac{x_i}{L},\ t' = \frac{U}{L}\cdot t,\ u_i' = \frac{u_i}{U},\ p' = \frac{p}{\mu U}\cdot L \text{ bzw. } p' = \frac{p}{\rho U^2} \rightarrow Re\cdot\left(\frac{\partial \underline{u}'}{\partial t'} + (\underline{u}'\,\underline{\nabla}')\underline{u}'\right) = -\underline{\nabla}' p' + \underline{\nabla}'^2 \underline{u}'$$

- $Re \ll 1$, $Re \rightarrow 0$ (L → Tröpfchen, U → Gletscher, ν → Honig)

  ↳ $\underline{\nabla} p = \mu \underline{\nabla}^2 \underline{u}$ (Stokes Gleichung) $\underline{\nabla}^2 p = 0$, $\underline{\nabla}^2 \omega = 0$ (Laplace Gl.)

- $Re \gg 1$, $Re \rightarrow \infty$ (L → Schiff, U → Flugzeug, ν → )

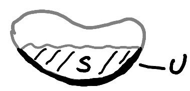
  ↳ $\frac{\partial \underline{u}}{\partial t} + (\underline{u}\,\underline{\nabla})\underline{u} = -\frac{1}{\rho}\underline{\nabla} p$ (Euler Gl.) **mit Grenzschicht!**

Reibungsbehafteter Bernoulli: (im Rohr, $Re_D = \frac{\bar{U}\cdot D}{\nu}$)

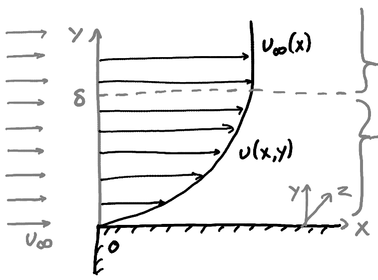
$$\left[\frac{p_1}{\rho} + \frac{u_1^2}{2} + U\right] = \left[\frac{p_2}{\rho} + \frac{u_2^2}{2} + U\right] + \frac{\Delta p_v}{\rho} - \frac{\Delta p_{ext}}{\rho}$$

$$\Delta p_v = \rho\cdot\lambda\cdot\frac{L}{D}\cdot\frac{U^2}{2} \quad (\text{mit } \lambda = 64/Re_D) \quad ; \quad \Delta p_{ext} = \overbrace{\underbrace{\eta_P\cdot\frac{N_P}{\dot V}}_{\text{Pumpe}} - \underbrace{\frac{1}{\eta_T}\cdot\frac{N_T}{\dot V}}_{\text{Turbine}}}^{\text{Leistung}}$$

Nicht runde Röhre: $"D" = D_H = 4\cdot\frac{S}{U}$ ← Querschnitt Fluid, ← Umfang nasse Wand



Grenzschichten



für Reibungsbehaftete Str. mit $Re \gg 1$ (siehe nährungs Lsg.)

Euler Gl.

Nav. Stoke Gl.

$$\left(\begin{aligned} &u(x,y)|_{y=0} = 0 \\ &u(x,y)|_{y\to\infty} = U_\infty(x) \end{aligned}\right)$$

Grenzschicht Dicke:

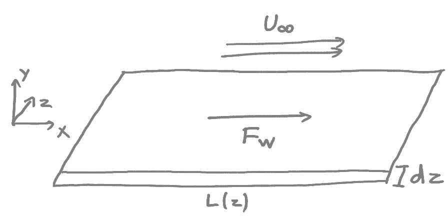
- $\delta_{99} \rightarrow u(x,y)|_{y=\delta_0} = 0{,}99\cdot U_\infty(x)$ (Grenzschichtdicke)
- $\delta_1 \rightarrow \delta_1 = \int_0^\infty \left(1 - \frac{u}{U_\infty}\right) dy$ (Verdrängungsdicke)
- $\delta_2 \rightarrow \delta_2 = \int_0^\infty \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy$ (Impulsverlustdicke)

Blasius-Grenzschicht: (ρ = Konst., Stationär, Ausgebildet, U_∞ = Konst., $Re_x > 100'000$)

$$\delta_{99} = 5\cdot\frac{x}{\sqrt{Re_x}} \propto \sqrt{x} \qquad \delta_1 = 1{,}721\frac{x}{\sqrt{Re_x}} \approx \frac{\delta_0}{3} \qquad \delta_2 = 0{,}664\frac{x}{\sqrt{Re_x}} \approx \frac{\delta_0}{8} \qquad \left(Re_x = \frac{U_\infty\cdot x}{\nu}\right)$$

$$\tau_w = 0{,}332\cdot\rho\, U_\infty^2/\sqrt{Re_x} \qquad \left(c_f = \frac{\tau_w}{\rho/2\ U_\infty^2} = 0{,}664\frac{1}{\sqrt{Re_x}}\right)$$

$$dF_w = dz\cdot\int_0^L \tau_w\, dx = 0{,}664\,\mu\, U_\infty\sqrt{Re_L}\cdot dz$$



Turbulenz $\left(\frac{\partial [\cdot]}{\partial t} \neq 0 \,,\ rot(\underline{u}) \neq 0 \right)$

Transition:

Ebene Platte : $Re_{x\,KRIT} = 3 \cdot 10^5 .. 3 \cdot 10^6$

Zylinder : $Re_{D\,KRIT} = 3 \cdot 10^5 .. 3 \cdot 10^6$

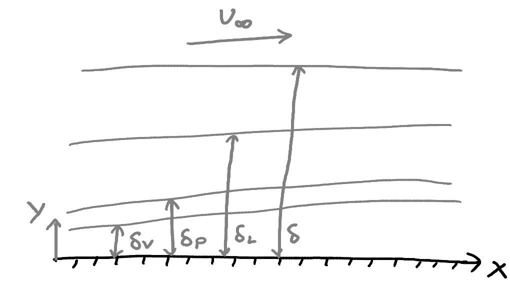
Kugel : $Re_{D\,KRIT} = 1 \cdot 10^5 .. 5 \cdot 10^5$

Rohrstr. : $Re_{D\,KRIT} = 2300 .. 3000$

Turbulente Grenzschicht an Wand (glatt)

$$U_\tau = \sqrt{\tau_W / \rho} = U_\infty \cdot \sqrt{c_f / 2} \qquad \text{(Schubsp. geschw.)}$$

$$c_f = 0{,}0576\, Re_x^{-1/5}$$



$$\delta = 0{,}37 \cdot x \cdot Re_x^{-1/4} = 0{,}37 \frac{\nu}{U_\infty} \cdot Re_x^{4/5}$$

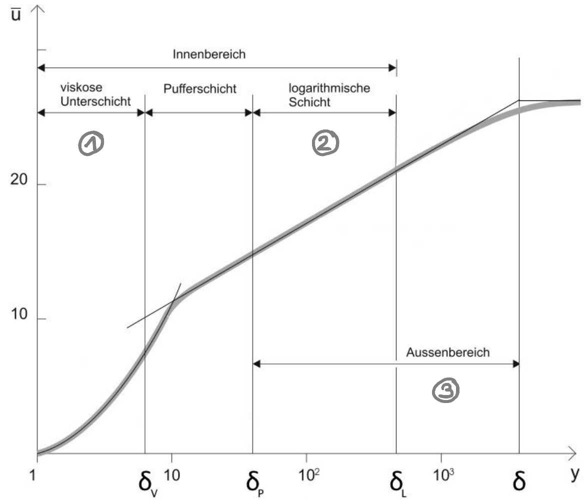
$$\delta_V = 5 \cdot \frac{\nu}{U_\tau} \qquad \delta_P = 30 \cdot \frac{\nu}{U_\tau} \qquad \delta_L = 0{,}2 \cdot \delta$$

① $\bar{U}(y) = \frac{y \cdot U_\tau^2}{\nu}$ $\quad (\tau_V \gg \tau_T \,;\ \tau_{TOT} \approx \tau_W)$

② $\bar{U}(y) = \frac{U_\tau}{0{,}41} \cdot \ln\left(\frac{U_\tau \cdot y}{\nu}\right) + 5$ $\quad (\tau_T \gg \tau_V \,;\ \tau_{TOT} \approx \tau_W)$

③ $\bar{U}(y) = U_\infty \cdot \left(\frac{y}{\delta_T}\right)^{1/7}$ $\quad (\tau_T \gg \tau_V \,;\ \tau_{TOT} < \tau_W)$

$y^+ = y \frac{U_\tau}{\nu} \qquad U^+ = \frac{\bar{U}}{U_\tau} \qquad (y^+, U^+ \text{ Einheitslos})$



Turbulente Rohrströmung ($Re_D = \frac{\bar{U}\cdot D}{\nu}$)

Widerstandszahl λ:

- Moody-Diagram auslesen.
- glatt spezial Blasius : $\lambda = 0{,}3164\cdot(Re_D)^{-1/4}$
- glatt $u_\tau \cdot k_s/\nu \le 5$: $\frac{1}{\sqrt{\lambda}} = 2{,}03\cdot\log(Re_D\cdot\sqrt{\lambda}) - 0{,}8$
- $\downarrow$ $5 < u_\tau k_s/\nu < 70$: $\frac{1}{\sqrt{\lambda}} = -2{,}03\log\left(\frac{2{,}51}{Re_D\cdot\sqrt{\lambda}} + 0{,}27\frac{k_s}{D}\right)$
- rauh $u_\tau \cdot k_s/\nu \ge 70$: $\frac{1}{\sqrt{\lambda}} = 1{,}14 - 2\cdot\log\left(\frac{k_s}{D}\right)$

Schubsp. geschw.: $u_\tau = \sqrt{\frac{\lambda}{8}}\cdot\bar{U} = \sqrt{\frac{\tau_w}{\rho}}$ $\qquad (U_{MAX} = \bar{U} + 3{,}75\, u_\tau)$

Wand Schubsp.: $\tau_w = \rho\cdot u_\tau^2 = -\frac{D}{4}\cdot\frac{dp}{dx}$

Turbulenter Bernoulli:

$$\left[\frac{p_1}{\rho}+\frac{U_1^2}{2}+U\right] = \left[\frac{p_2}{\rho}+\frac{U_2^2}{2}+U\right] + \frac{\Delta p_v}{\rho} - \frac{\Delta p_{ext}}{\rho}$$

Leistung

$$\Delta p_v = \underbrace{\rho\cdot\lambda\cdot\frac{L}{D}\cdot\frac{U^2}{2}}_{\text{Turb. verl.}} + \underbrace{\rho\cdot\sum_i \zeta_i\,\frac{U^2}{2}}_{\text{Einb. verl.}} \quad ; \quad \Delta p_{ext} = \underbrace{\eta_P\cdot\frac{N_P}{\dot{V}}}_{\text{Pumpe}} - \underbrace{\frac{1}{\eta_T}\cdot\frac{N_T}{\dot{V}}}_{\text{Turbine}}$$

Nicht runde Röhre: "D" $= D_H = 4\cdot\frac{S}{U}$ $\leftarrow$ Querschnitt Fluid (S), $\leftarrow$ Umfang nasse Wand (U)