Computational Mechanics I

(general problem; mechanical problem)

mechanical problem

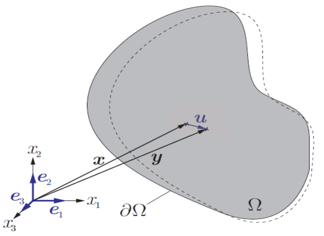
point description: (Lagrangian)

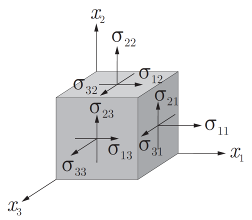
$y(x,t) = x + u(x,t)$

$v(x,t) = {}^{d}/_{dt}\, y(x,t) = \dot{y}(x,t) = \dot{u}(x,t)$

$a(x,t) = {}^{d^2}/_{dt^2}\, y(x,t) = \ddot{y}(x,t) = \ddot{u}(x,t) = \dot{v}(x,t)$

Ω: body ; $\partial\Omega$: surface
x: undeformed point
$y(x,t)$: deformed point
$u(x,t)$: displacement field
$v(x,t)$: velocity field
$a(x,t)$: acceleration field





$\underline{\varepsilon}$: strain tensor
$\underline{\sigma}$: stress tensor
$\mathbb{C}$: elasticity tensor
W: strain energy density
$\underline{t}$: pressure

stress/strain:

$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad\leftrightarrow\quad \underline{\varepsilon} = \frac{1}{2}(\mathrm{grad}(\underline{u}) + \mathrm{grad}(\underline{u})^T)$

$\partial\sigma_{ij}(\varepsilon) = \mathbb{C}_{ijkl}(\varepsilon)\,\partial\varepsilon_{kl} \quad\leftrightarrow\quad \partial\underline{\sigma}(\underline{\varepsilon}) = \mathbb{C}(\underline{\varepsilon})\,\partial\underline{\varepsilon}$

$\partial W(\varepsilon) = \sigma_{ij}(\varepsilon)\,\partial\varepsilon_{ij} \quad\leftrightarrow\quad \partial W(\underline{\varepsilon}) = \underline{\sigma}(\underline{\varepsilon})\cdot\partial\underline{\varepsilon}$

$t_i = \sigma_{ij} n_j \quad\leftrightarrow\quad \underline{t} = \underline{\sigma}\cdot\underline{n}$ ← surface normal on $\partial\Omega$

linear elastic ($\mathbb{C}$ = const.):

$\sigma_{ij} = \sum_{k,l} \mathbb{C}_{ijkl}\,\varepsilon_{kl} = \mathbb{C}_{ijkl}\,\varepsilon_{kl} \quad\leftrightarrow\quad \underline{\sigma} = \mathbb{C}\cdot\underline{\varepsilon}$

↳ $\sigma_{ij} = \mathbb{C}_{ijkl}\, u_{k,l}$ (b.c. $\mathbb{C}$ sym.!)

$W = \frac{1}{2}\varepsilon_{ij}\,\mathbb{C}_{ijkl}\,\varepsilon_{kl} \quad\leftrightarrow\quad W = \frac{1}{2}\underline{\varepsilon}\cdot\mathbb{C}\cdot\underline{\varepsilon}$

1D + linear-elastic:

$\varepsilon = \partial u/\partial x = u_{,x}$; $\sigma = E\varepsilon = E u_{,x}$; $W = \frac{1}{2}E\varepsilon^2 = \frac{1}{2}E\, u_{,x}^2$

balance of linear momentum: ($F = m\cdot a$)

$\underbrace{\int_{\partial\Omega} \underline{\sigma}\,\underline{n}\,dS}_{\text{ext. forces}} + \underbrace{\int_\Omega \rho\,\underline{b}\,dV}_{\text{int. forces}} = \underbrace{\int_\Omega \rho\,\underline{a}\,dV}_{m\cdot a_{CG}} \xrightarrow[\text{(since valid for every }\Omega)]{\text{from div theorem}} \mathrm{div}(\underline{\sigma}) + \rho\underline{b} = \rho\underline{a} \;\leftrightarrow\; \sigma_{ij,j} + \rho b_i = \rho a_i$

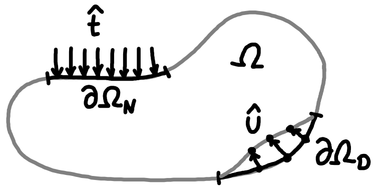
example: $\sigma_{ij,j} + \rho b_i = \rho a_i \xrightarrow[\rho = \text{const.},\ \underline{b} = \underline{0}]{} \rho a_i = \sigma_{ij,j}$

3D: $\xrightarrow{a_i = \ddot{u}_i,\ \sigma_{ij} = \mathbb{C}_{ijkl}\varepsilon_{kl},\ \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})} \boxed{\rho\ddot{u}_i = \mathbb{C}_{ijkl}\, u_{k,jl}}$

1D: $\rho a = \sigma_{,x} \xrightarrow[a = \ddot{u},\ \sigma = E\varepsilon,\ \varepsilon = \frac{1}{2}(u_{,x} + u_{,x})]{} \boxed{\rho\ddot{u} = E\, u_{,xx}}$

boundary/initial conditions

- **initial conditions:** cond. at $t=0$ (1 per t deriv.!) $\quad u(x,t=0) = u_0(x)$; $\dot{u}(x,t=0) = \dot{u}_0(x)$
- **Dirichlet** (essential) **cond.:** fix cond. at $\partial\Omega_D \quad u(x=\partial\Omega_D,t) = \hat{u}(x,t)$
- **Neumann** (natural) **cond.:** deriv. cond. at $\partial\Omega_N \quad t(x=\partial\Omega_N,t) = \sigma(x,t)\cdot n(x,t) = \hat{t}(x,t)$



strong form: (static)

$\sigma_{ij,j}(\underline{u}) + \rho b_i = 0$ in Ω
$u_i = \hat{u}_i$ on $\partial\Omega_D$
$\sigma_{ij} n_j = \hat{t}_i$ on $\partial\Omega_N$

$\xrightarrow{\text{linear elasticity}}$ $\sigma_{ij} = \mathbb{C}_{ijkl}\,\varepsilon_{kl}$, $\varepsilon_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k})$, ↳ $\sigma_{ij} = \mathbb{C}_{ijkl}\cdot u_{k,l}$ ← (b.c. $\mathbb{C}_{ijkl}$ sym.!)

$(\mathbb{C}_{ijkl}\, u_{k,l})_{,j} + \rho b_i = 0$ in Ω $\xrightarrow{\text{linear form}}$ $\mathcal{L}[u] = (\mathbb{C}_{ijkl}\, u_{k,l})_{,j} = L = -\rho b_i$
$u_i = \hat{u}_i$ on $\partial\Omega_D$ $\longrightarrow$ $B_D[u] = u_i = B_D = \hat{u}_i$
$(\mathbb{C}_{ijkl}\, u_{k,l})\, n_j = \hat{t}$ on $\partial\Omega_N$ $\longrightarrow$ $B_N[u] = (\mathbb{C}_{ijkl}\, u_{k,l})\, n_j = B_N = \hat{t}_i$

weak form: (static)

$\min I[u] = \underbrace{\int_\Omega W(\underline{u})\,dV}_{\text{int. E.}} \underbrace{- \int_\Omega \rho b_i u_i\,dV - \int_{\partial\Omega_N} \hat{t}_i u_i\,dS}_{\text{pot. E.}}$ (tot. E; ext. W)
subj. to: $u_i = \hat{u}_i$ on $\partial\Omega_D$

linear elasticity: $W(\underline{\varepsilon}) = \frac{1}{2}\varepsilon_{ij}\,\mathbb{C}_{ijkl}\,\varepsilon_{kl}$, $W(\underline{u}) = \frac{1}{2}u_{i,j}\,\mathbb{C}_{ijkl}\,u_{k,l}$

$\min I[u] = \int_\Omega \frac{1}{2}u_{i,j}\,\mathbb{C}_{ijkl}\,u_{k,l}\,dV - \int_\Omega \rho b_i u_i\,dV - \int_{\partial\Omega_N} \hat{t}_i u_i\,dS$
subj. to: $u_i = \hat{u}_i$ on $\partial\Omega_D$

linear/bilinear form: $I[u] = \frac{1}{2}B[u,u] - \mathcal{L}[u]$
$\mathcal{L}[u] = \int \rho b_i u_i\,dV + \int_{\partial\Omega_N} \hat{t}_i u_i\,dS$
$B[u,v] = \int_\Omega u_{i,j}\,\mathbb{C}_{ijkl}\,v_{k,l}\,dV$

variation:

$0 = \delta I[u] = \underbrace{A[u,\delta u]}_{}$ ← not bilinear $\quad -\mathcal{L}[\delta u]$

$0 = \delta I[u] = \int_\Omega \sigma_{ij}(\underline{u})\cdot\delta u_{i,j}\,dV - \int_\Omega \rho b_i\,\delta u_i\,dV - \int_{\partial\Omega_N} \hat{t}_i\,\delta u_i\,dS \quad \forall\delta u$

$\delta W = \frac{\partial W}{\partial\varepsilon_{ij}}\cdot\delta\varepsilon_{ij} = \sigma_{ij}\cdot\delta u_{i,j}$ $\left(\partial W/\partial\varepsilon_{ij} = \sigma_{ij} = \sigma_{ji};\ \delta\varepsilon_{ij} = \frac{1}{2}(\delta u_{i,j} + \delta u_{j,i})\right)$

subj. to: $u_i = \hat{u}_i$ on $\partial\Omega_D$ and $\delta u_i = 0$ on $\partial\Omega_D$

$\rightarrow$

$0 = \delta I[u] = B[u,\delta u] \quad -\mathcal{L}[\delta u]$

$0 = \delta I[u] = \int_\Omega u_{i,j}\,\mathbb{C}_{ijkl}\,\delta u_{k,l} - \int_\Omega \rho b_i\,\delta u_i\,dV - \int_{\partial\Omega_N} \hat{t}_i\,\delta u_i\,dS \quad \forall\delta u$

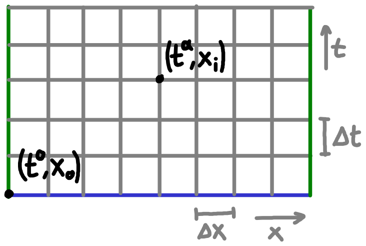
subj. to: $u_i = \hat{u}_i$ on $\partial\Omega_D$ and $\delta u_i = 0$ on $\partial\Omega_D$

differences strong/weak form: ($C^k(\Omega)$: k-times differentiable in Ω ; $H^k(\Omega)$: k-times diff. + square integrable)

strong form: $u \in C^2(\Omega)$ $\neq$ weak form: $u \in H^1(\Omega)$ ↔ not the same set of u can be used to approx. solution!

Finite Differences method

(approx strong form solution by replacing derivatives with num. approx.)



$t^\alpha = \alpha \cdot \Delta t$
$x_i = i \cdot \Delta x$
$u_i^\alpha = u(x_i, t^\alpha)$
■ : initial cond.
■ : boundary cond.

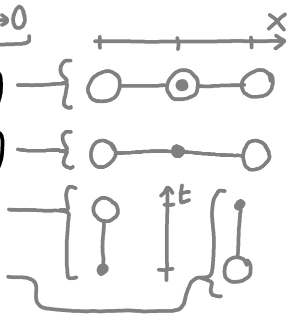
(all work for t and x derivatives! derived from Taylor expansions:
$u(x_{i+1}, t^\alpha) = u_i^\alpha + \frac{\Delta x}{1!}\frac{\partial u}{\partial x}\Big|_{x_i,t^\alpha} + \frac{\Delta x^2}{2!}\frac{\partial^2 u}{\partial x^2}\Big|_{x_i,t^\alpha} + \frac{\Delta x^3}{3!}\frac{\partial^3 u}{\partial t^3}\Big|_{x_i,t^\alpha} + \mathcal{O}(\Delta x^4)$
→ e.g.: $u_{i+1}^\alpha + u_{i-1}^\alpha = \ldots$ = 2nd order central diff.)

2nd order central diff.: $\frac{\partial^2 u}{\partial x^2}(x_i, t^\alpha) = \frac{u_{i+1}^\alpha - 2u_i^\alpha + u_{i-1}^\alpha}{\Delta x^2} + \underbrace{\mathcal{O}(\Delta x^2)}_{\text{to 0 if } \Delta x \to 0}$

1st order central diff.: $\frac{\partial u}{\partial x}(x_i, t^\alpha) = \frac{u_{i+1}^\alpha - u_{i-1}^\alpha}{2\Delta x} + \mathcal{O}(\Delta x^2)$

forward euler: $\frac{\partial u}{\partial t}(x_i, t^\alpha) = \frac{u_i^{\alpha+1} - u_i^\alpha}{\Delta t} + \mathcal{O}(\Delta t)$

backward euler: $\frac{\partial u}{\partial t}(x_i, t^\alpha) = \frac{u_i^\alpha - u_i^{\alpha-1}}{\Delta t} + \mathcal{O}(\Delta t)$

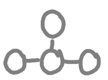


stencils indicate what data is necessary for using an eq.
- explicit: only 1 at newest t
- implicit: multiple at newest t

● example: heat equation (2nd centr. space, euler fwd. time)

$\dot{T} = \kappa T_{,xx} + s \;\rightarrow\; \frac{T_i^{\alpha+1} - T_i^\alpha}{\Delta T} = \kappa \frac{T_{i+1}^\alpha - 2T_i^\alpha + T_{i-1}^\alpha}{\Delta x^2} + s_i^\alpha + \mathcal{O}(\Delta t, \Delta x^2)$

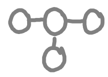
$\rightarrow T_i^{\alpha+1} = T_i^\alpha + \frac{\kappa \Delta t}{\Delta x^2}(T_{i+1}^\alpha - 2T_i^\alpha + T_{i-1}^\alpha) + \Delta t\, s$


→ explicit!
↳ solve iteratively

● example: heat equation (2nd centr. space, euler bkw. time)

$\dot{T} = \kappa T_{,xx} + s \;\rightarrow\; \frac{T_i^\alpha - T_i^{\alpha-1}}{\Delta T} = \kappa \frac{T_{i+1}^\alpha - 2T_i^\alpha + T_{i-1}^\alpha}{\Delta x^2} + s_i^\alpha + \mathcal{O}(\Delta t, \Delta x^2)$

$\rightarrow -c\,T_{i+1}^\alpha + (1+2c)T_i^\alpha - c\,T_{i-1}^\alpha \approx \Delta t\, s_i^\alpha + T_i^{\alpha-1} \qquad \left(c = \frac{\kappa \Delta t}{\Delta x^2}\right)$


→ implicit!
↳ solve sys. of eq.

$$\begin{bmatrix} \ddots & & & \\ -c & (1+2c) & -c & \\ & -c & (1+2c) & -c \\ & & & \ddots \end{bmatrix} \cdot \begin{bmatrix} T_1^\alpha \\ \vdots \\ T_n^\alpha \end{bmatrix} = \begin{bmatrix} \Delta t\, s_1^\alpha + T_1^{\alpha-1} \\ \vdots \\ \Delta t\, s_n^\alpha + T_1^{\alpha-1} \end{bmatrix}$$

● **consistency**: method matches solution for $\Delta x \to 0, \Delta t \to 0$
(true if using central diff, euler, etc.)

● **boundary/initial conditions** (for implicit formulations)

Dirichlet: $\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_0 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} \hat{u}_0 \\ \vdots \\ \hat{u}_n \end{bmatrix}$

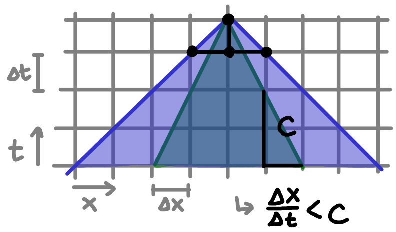
Neumann: $\begin{bmatrix} -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & & & \ddots & & & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_{-1} \\ u_0 \\ \vdots \\ u_n \\ u_{n+1} \end{bmatrix} = \begin{bmatrix} \hat{u}_0 \cdot \Delta x \\ \vdots \\ \hat{u}_n \cdot \Delta x \end{bmatrix}$ ← ghost nodes (u_{-1}, u_{n+1})

● **stability with CFL condition** (necessary but not sufficient!)

domain of dependence: points that can affect the value of another point

numerical domain of dependence > **mathematical domain of dependence**
(from stencil shape) (encompasses) (from diff. eq.)

ex.: wave eq. $\ddot{u} = c^2 u_{,xx}$



↳ $\frac{\Delta x}{\Delta t} < c$
(else unstable!)

● **stability with von Neumann analysis** (usually only "good guess")

necessary + sufficient if scheme of form: $u_j^{\alpha+1} = \sum_{k=-?}^{+?} C_k u_{j+k}^\alpha$ (C_k: const.)

↳ then solutions are additive!

assume error ϵ at $t=0$: $\epsilon_j^0 = e^{-ik\Delta x j}$
- any error func possible by adding these up as Fourier series
- amplitude = 1 since we only care for amp. changes

assume ϵ propagates with: $\epsilon_j^\alpha = z^\alpha e^{-ik\Delta x j}$ (only true for above scheme, but generally good approx.)

↳ plug in ϵ_j^α in scheme → solve for z
- $|z| < 1$: stable (noise vanishes)
- $|z| = 1$: stable (noise stays)
- $|z| > 1$: unstable (noise amplifies)

→ pick $\Delta x, \Delta t$ so scheme is stable!
(z depends on $k, \Delta x, \Delta t$)

● example: wave equation (2nd order central in space ; 2nd order central in time)

$\frac{u_j^{\alpha+1} - 2u_j^\alpha + u_j^{\alpha-1}}{\Delta t^2} = c^2 \frac{u_{j+1}^\alpha - u_j^\alpha + u_{j-1}^\alpha}{\Delta x^2} + \mathcal{O}(\Delta x^2, \Delta t^2) \xrightarrow{u_j^\alpha = \epsilon_j^\alpha = z^\alpha e^{ik\Delta x j}} z - 2 + 1/z = \left(\frac{c\Delta t}{\Delta x}\right)^2 (\cos(k\Delta x) - 1) \rightarrow |z| \le 1$ stable if: $\left|c\frac{\Delta t}{\Delta x}\right| \le 1$

● phase error analysis: like von Neuman, but $z = e^{i\omega\Delta t \alpha}$ (interpret z^α as a wave in time)

Weighted Residuals Method (approx. strong form solution with some weighted basis functions)

$u^h(x) = \sum_{a=1}^{n} c^a \phi^a(x) \approx u(x)$ approximate $u(x)$ with $u^h(x)$

↳ ϕ^a: n shape functions (e.g.: $\phi^a(x) = x^a$, $\phi^a(x) = \sin(\pi a x)$, ...)

↳ c^a: n coefficients to be determined such that $u^h(x) \approx u(x)$

u^h has to satisfy all Dirichlet BC!

choose ϕ^a with $\phi^a(x) = 0 \;\; \forall a \in [1,..,n], x \in \delta\Omega_D$

↳ e.g.: $u^h(x) = \hat{u}_{\partial\Omega_D}(x) + \sum_{a=1}^{n} c^a \phi^a(x)$

linear operator notation for PDE:

PDE: $\mathcal{L}[u] = L$

BC: $\mathcal{B}[u] = B$

linear means:
$\mathcal{L}[\alpha f + \beta g] = \alpha \mathcal{L}[f] + \beta \mathcal{L}[g]$
$\forall \alpha, \beta \in \mathbb{R}$, $\forall f, g$ functions

1D PDE:
- wave equation ($\ddot{u} = c^2 u_{,xx}$): $\mathcal{L} = \partial^2/\partial t^2 - c^2 \partial^2/\partial x^2$; $L = 0$
- heat equation ($\rho c_v \dot{T} = \kappa T_{,xx} + s$): $\mathcal{L} = \rho c_v \partial/\partial t - \kappa \partial^2/\partial x^2$; $L = s$
- advection equation ($\dot{u} + v u_{,x} = 0$): $\mathcal{L} = \partial/\partial t + v \partial/\partial x$; $L = 0$

1D BC:
- boundary traction ($\hat{t} = \sigma n = E u_{,x}$): $\mathcal{B} = E \partial/\partial x$; $B = \hat{t}$
- boundary flux ($\hat{q} = -\kappa T_{,x}$): $\mathcal{B} = -\kappa \partial/\partial x$; $B = \hat{q}$
- Dirichlet ($u = \hat{u}$): $\mathcal{B} = 1$; $B = \hat{u}$

residuals: $r_\Omega = \mathcal{L}[u^h] - L$ (for $x \in \Omega$) (error to solution of PDE inside Ω)

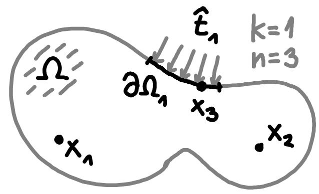
(0 if $u^h = u$) $r_{\delta\Omega_j} = \mathcal{B}_j[u^h] - B_j$ (for $x \in \delta\Omega_j$) (error to Neumann BCs ($j = 1,..,k$) on $\partial\Omega_j$)

total residuals: $R_a = \int_\Omega r_\Omega W_a \, dV + \xi^2 \sum_{j=1}^{k} \int_{\partial\Omega} r_{\partial\Omega_j} w_a \, dS$ ($a = 1,..,n$ weighted sums of residuals; W_a, w_a: weighting func. ; ξ: arbitrary const.)

↳ use $R_a = 0 \;\; \forall a \in [1,..,n]$ to get n eq. and solve for c^a!

- **collocation method**:

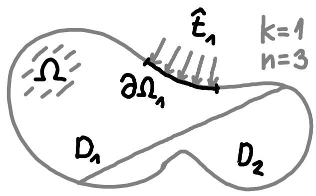
$x_a = \begin{cases} a \leq n-k: & \text{some } x \in \Omega \\ \text{else}: & \text{some } x \in \partial\Omega_{a-(n-k)} \end{cases} \rightarrow W_a = \begin{cases} a \leq n-k: & \delta(x - x_a) \\ \text{else}: & 0 \end{cases} ; \; w_a = \begin{cases} a \leq n-k: & 0 \\ \text{else}: & \delta(x - x_a) \end{cases}$



↳ $r_\Omega(x_i) = 0$ (for $i = 1,..,n-k$) ; $r_{\partial\Omega_{i-(n-k)}}(x_i) = 0$ (for $i = n-k+1,..,n$) → solve for c^a

- **subdomain method**:

$D_a = \begin{cases} a \leq n-k: & \text{some } D \subset \Omega \\ \text{else}: & \partial\Omega_{a-(n-k)} \end{cases} \rightarrow W_a = \begin{cases} a \leq n-k: & 1 \text{ on } x \in D_a \\ \text{else}: & 0 \end{cases} ; \; w_a = \begin{cases} a \leq n-k: & 0 \\ \text{else}: & 1 \text{ on } x \in D_a \end{cases}$



↳ $\int_{D_i} r_\Omega \, dV = 0$ (for $i = 1,..,n-k$) ; $\int_{\partial\Omega_j} r_{\partial\Omega_j} \, dS = 0$ (for $j = 1,..,k$) → solve for c^a

- **least square method**:

$E = \int_\Omega r_\Omega^2 \, dV + \xi^2 \sum_{j=1}^{k} \int_{\partial\Omega_j} r_{\partial\Omega_j}^2 \, dS$ (square error) → $c^a = \text{argmin}\, E$ (least square coeff.)

↳ $\partial E / \partial c^a = \int_\Omega r_\Omega \underbrace{\partial r_\Omega / \partial c^a}_{= \mathcal{L}[\phi^a] \, (=W_a)} dV + \xi^2 \sum_{j=1}^{k} \int_{\partial\Omega_j} r_{\partial\Omega_j} \underbrace{\partial r_{\partial\Omega_j} / \partial c^a}_{= \mathcal{B}_j[\phi^a] \, (=w_a)} dS = 0 = R_a$

$$\rightarrow \sum_{b=1}^{n} \underbrace{\left(\int_\Omega \mathcal{L}[\phi^a] \mathcal{L}[\phi^b] \, dV + \xi^2 \sum_{j=1}^{k} \int_{\partial\Omega_j} \mathcal{B}_j[\phi^a] \mathcal{B}_j[\phi^b] \, dS \right)}_{K^{ab}} \cdot c^b = \underbrace{\int_\Omega L \cdot \mathcal{L}[\phi^a] \, dV + \xi^2 \sum_{j=1}^{k} \int_{\partial\Omega_j} B_j \mathcal{B}_j[\phi^a] \, dS}_{F^a} \quad (\text{for } a = 1,..,n)$$

$\sum_{b=1}^{n} K^{ab} \cdot c^b = F^a$ (for $a = 1,..,n$)

↳ $\underline{\underline{K}}\, \underline{c} = \underline{F} \rightarrow \underline{c} = \underline{\underline{K}}^{-1} \underline{F}$

- **Galerkin's method** (for linear momentum eq.):

principle of virtual work: $\int_\Omega \sigma_{ij} v_{i,j} \, dV = \int_\Omega \rho b_i v_i \, dV + \int_{\partial\Omega} \hat{t}_i v_i \, dS$ ($v(x)$: any admissible virtual displacement)

$\xrightarrow{\text{rewrite}}$ $\int_\Omega \underbrace{(\sigma_{ij,j} + \rho b_i)}_{r_\Omega} \underbrace{v_i}_{W_a} dV + \underbrace{1}_{\xi} \cdot \int_{\partial\Omega} \underbrace{(\hat{t}_i - \sigma_{ij} n_j)}_{r_{\partial\Omega}} \underbrace{v_i}_{w_a} dS = 0$

(for this to be true for all allowable $v(x)$: $\sigma_{ij,j} = \rho b_i$ and $\sigma_{ij} n_j = \hat{t}_i$ → static lin. mom. bal.!)

- **higher dimensions**: $u \rightarrow \underline{u}$ e.g. (u_x, u_y, u_z) $\quad x \rightarrow \underline{x}$ e.g. (x, y, z, t) $\quad c^a \phi^a \rightarrow \underline{c}^a \underline{\phi}^a$

• *example*: static bar under centripetal accel. $\omega^2 x$

EQ: $\sigma_{ij,j} + \varrho b_i = \varrho a_i \xrightarrow{1D,\, b=\omega^2 x,\, a=0} \sigma_{,x} + \varrho\omega^2 x = 0 \xrightarrow{\sigma = E\varepsilon = E u_{,x}} u_{,xx} = -\varrho\omega^2/E \cdot x \;\rightarrow\; \mathcal{L} = E \cdot \partial^2/\partial x^2 \,;\; L = -\varrho\omega^2 x$

BC: $u(0)=0$ and $F(L) = AE\,u_{,x}(L) = m\omega^2 L \qquad \rightarrow \mathcal{B}_1 = AE\,\partial/\partial x;\; B_1 = m\omega^2 x$

approx func.: $u(x) \approx u^h(x) = \sum_{a=0}^{3} c^a x^a = c^1x^1 + c^2x^2 + c^3x^3$ (e.g. polynom 3rd order) ($c^0 = 0$ for BC $u(0)=0$)

↳ residuals: $r_\Omega = \mathcal{L}[u^h] - L = E(2c^2 + 6c^3 x) + \varrho\omega^2 x$

$r_{\partial\Omega_1} = \mathcal{B}[u^h] - B = AE(c^1 + 2c^2 x + 3c^3 x^2) - m\omega^2 x$

- collocation method: $x_1 = L/3,\; x_2 = 2L/3,\; x_3 = L$

$r_\Omega(x_1) = 0,\; r_\Omega(x_2) = 0,\; r_{\partial\Omega_1}(x_3) = 0$

$$\begin{bmatrix} 0 & 2E & 2EL \\ 0 & 2E & 2EL \\ EA & 2EAL & 3EAL^2 \end{bmatrix} \cdot \begin{bmatrix} c^1 \\ c^2 \\ c^3 \end{bmatrix} = \begin{bmatrix} -\varrho\omega^2 L/3 \\ -2\varrho\omega^2 L/3 \\ m\omega^2 L \end{bmatrix}$$

→ solve for $[c^1, c^2, c^3]^T$ ....

- subdomain method: $\begin{cases} D_1 = \{x : 0 \le x < L/2\} \\ D_2 = \{x : L/2 \le x < L\} \\ D_3 = \partial\Omega_1 = \{x : x = L\} \end{cases}$

$\int_{D_1} r_\Omega\, dx = \left[E(2c^2x + 3c^3x^2) - \tfrac{1}{2}\varrho\omega^2 x^2\right]_0^{L/2} = 0$

$\int_{D_2} r_\Omega\, dx = \left[E(2c^2x + 3c^3x^2) - \tfrac{1}{2}\varrho\omega^2 x^2\right]_{L/2}^{L} = 0$

$\int_{D_3} r_{\partial\Omega_1}\, d\Gamma = AE(c^1 + 2c^2 L + 3c^3 L^2) - m\omega^2 L = 0$

→ solve for $[c^1, c^2, c^3]^T$ ....

- least square method: ($\xi^2 = 1/LA$ so units match)

$$\underline{K} = \begin{bmatrix} AE^2/L & 2AE^2 & 3AE^2L \\ 2AE^2 & 8AE^2L & 12AE^2L^2 \\ 3AE^2L & 12AE^2L^2 & 21AE^2L^3 \end{bmatrix} \quad \underline{F} = \begin{bmatrix} Em\omega^2 \\ EL\omega^2(2m - AL\varrho) \\ EL^2\omega^2(3m - 2AL\varrho) \end{bmatrix}$$

→ solve $\underline{K} \cdot (c^1, c^2, c^3)^T = \underline{F}$ for $(c^1, c^2, c^3)^T$

- Galerkin's method: same as variational method!

Variational Method

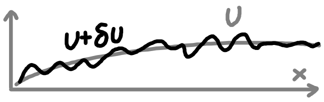
(approx. by finding function with minimal "energy" = weak form)

energy functional: $I[u] : \mathcal{U} \rightarrow \mathbb{R}$ (assigning "energy" $\in \mathbb{R}$ to all functions $u \in \mathcal{U}$)

(1D example: $I[u] = \int_0^1 u_{,x}^2\, dx \,;\; \mathcal{U} = H^1(0,1)$ — u has to be square integrable in $(0,1)$ | 1D Sobolev space: $H^k(\Omega) = \{u(x) : \Omega \rightarrow \mathbb{R}$ such that $\|u(x)\|_{H^k(\Omega)} < \infty\}$; 1D Sobolev norm: $\|u(x)\|_{H^k(\Omega)} = \sqrt{\int_\Omega u^2 dx + \int_\Omega (du/dx)^2 dx + \ldots + \int_\Omega (d^k u/dx^k)^2 dx}$)

variation: $\delta u \in \mathcal{U}_0 = \{\mathcal{U}$ with $u = 0$ on $\partial\Omega_{\text{Dirichlet}}\}$ (arbitrary perturbation)

first variation: $\delta I[u] = \lim_{\varepsilon \to 0} \frac{I[u + \varepsilon\cdot\delta u] - I[u]}{\varepsilon} = \frac{d}{d\varepsilon} I[u + \varepsilon\cdot\delta u]\big|_{\varepsilon = 0}$ (analog to first deriv.)



usefull rules: $\delta(\alpha_1 I_1 + \alpha_2 I_2) = \alpha_1 \delta I_1 + \alpha_2 \delta I_2 \;;\; \delta(I_1 I_2) = \delta I_1 \cdot I_2 + I_1 \delta I_2 \;;\; \delta \frac{du}{dx} = \frac{d}{dx}(\delta u)$

$\delta \int_\Omega u\, dx = \int_\Omega \delta u\, dx \;;\; \delta I[u, v, \ldots] = \frac{d}{d\varepsilon} I[u + \varepsilon\delta u, v + \varepsilon\delta v, \ldots]\big|_{\varepsilon \to 0} \;;\; I[u + \delta u] = \sum_{i=0}^{\infty} \frac{1}{i!} \delta^i I[u]$

common form: $I[u] = \int_\Omega f(u, \nabla u, \nabla^2 u, \ldots)\, dV \;\rightarrow\; \delta I[u] = \int_\Omega \left(\frac{\partial f}{\partial u}\delta u + \frac{\partial f}{\partial \nabla u}\delta\nabla u + \frac{\partial f}{\partial \nabla^2 u}\delta\nabla^2 u + \ldots\right) dV$

$= \int_\Omega f(u_i, u_{i,j}, u_{i,jk}, \ldots)\, dV \;\rightarrow\; = \int_\Omega \left(\frac{\partial f}{\partial u_i}\delta u_i + \frac{\partial f}{\partial u_{i,j}}\delta u_{i,j} + \frac{\partial f}{\partial u_{i,jk}}\delta u_{i,jk} + \ldots\right) dV$

general recipe: find $I[u]$ → calculate $\delta I[u], \delta^2 I[u]$ → find all u_s with $\delta I[u] = 0\; \forall \delta u$ → keep only u_s with $u_s \in \mathcal{U}$ → keep only u_s with $\delta^2 I[u_s] > 0\; \forall \delta u$ (true u can be any of u_s or some u on border of $\mathcal{U}$!)

bilinear + linear form: if $I[u]$ can be written as $I[u] = \frac{1}{2}B[u,u] + \mathcal{L}[u]$ then $\delta I[u] = B[u, \delta u] - \mathcal{L}[\delta u]$

↳ no need to check $\delta^2 I[u] > 0$

(• linear form $\mathcal{L}[a]$: $\mathcal{L}[\alpha_1 a_1 + \alpha_2 a_2] = \alpha_1 \mathcal{L}[a_1] + \alpha_2 \mathcal{L}[a_2] \quad \forall \alpha_1, \alpha_2 \in \mathbb{R}\; \forall a_1, a_2 \in$ func.
• bilinear form $B[a,b]$: linear in a, b and $B[a+c, b] = B[a,b] + B[c,b]$)

• *func. discretization*: (u^a, v^a: const. ; $N^a(x)$: shape func.)

$u^h(x) = \sum_{a=1}^{n} u^a N^a(x) \approx u(x)$ (subj. to: $u^h = \hat{u}$ on $\delta\Omega_D$)

$v^h(x) = \sum_{a=1}^{n} v^a N^a(x) \approx \delta u(x)$ (subj. to: $v^h = 0$ on $\delta\Omega_D$)

→ solve $\partial I[u^h] = B[u^h, v^h] - \mathcal{L}[v^h] = 0 \quad \forall v^h$

↳ $\sum_{a=1}^{n} u^a B[N^a, N^b] = \mathcal{L}[N^b] \quad \forall b = 1, .., n$

↳ $\underline{K} \cdot \underline{u} = \underline{F}$ with $K^{ab} = B[N^a, N^b]$, $F^a = \mathcal{L}[N^a]$

mech. prob.: static 2D/3D linear elasticity (function discretization)

3D Voigt notation: (write $\underline{\sigma}, \underline{\varepsilon}$ as vectors $\tilde{\underline{\sigma}}, \tilde{\underline{\varepsilon}}$)

$$\underbrace{\sigma_{ij}}_{} = \mathbb{C}_{ijkl} \cdot \underbrace{\varepsilon_{kl}}_{} \quad \leftrightarrow \quad \underbrace{\tilde{\underline{\sigma}}}_{} = \underline{\underline{E}} \cdot \underbrace{\tilde{\underline{\varepsilon}}}_{}$$

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \underline{\underline{\underline{\underline{\mathbb{C}}}}} \cdot \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \qquad \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{bmatrix} = \underline{\underline{E}} \cdot \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{xy} \end{bmatrix}$$

from: $\varepsilon_{xx} = \frac{1}{E} \cdot (\sigma_{xx} - \nu(\sigma_{yy}+\sigma_{zz})), \dots, \varepsilon_{yz} = \frac{1+\nu}{E}\sigma_{yz}, \dots$

with: $\underline{\underline{E}} = \frac{E}{(1+\nu)(1-2\nu)} \cdot \begin{bmatrix} 1-\nu & \nu & \nu & \cdot & \cdot & \cdot \\ \nu & 1-\nu & \nu & \cdot & \cdot & \cdot \\ \nu & \nu & 1-\nu & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1/2-\nu & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1/2-\nu & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1/2-\nu \end{bmatrix}$

(E: Young's modulus, ν: Poisson's ratio)

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j}+u_{j,i}) \approx \tfrac{1}{2}(u^h_{i,j}+u^h_{j,i}) \quad \leftrightarrow \quad \tilde{\underline{\varepsilon}} \approx \underline{\underline{B}} \cdot \underline{U}^h$$

(u_i^h approx of u_i with: $u_i \approx u_i^h = \sum_{a=1}^n u_i^a N^a$; u_i^a: constants; N^a: basis functions)

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{xy} \end{bmatrix} \approx \underline{\underline{B}} \cdot \begin{bmatrix} u_x^1 \\ u_y^1 \\ u_z^1 \\ \vdots \\ u_x^n \\ u_y^n \\ u_z^n \end{bmatrix}$$

with: $\underline{\underline{B}} = \left[\begin{array}{ccc|ccc|ccc} N^1_{,x} & \cdot & \cdot & N^2_{,x} & \cdot & \cdot & \dots & N^n_{,x} & \cdot & \cdot \\ \cdot & N^1_{,y} & \cdot & \cdot & N^2_{,y} & \cdot & \dots & \cdot & N^n_{,y} & \cdot \\ \cdot & \cdot & N^1_{,z} & \cdot & \cdot & N^2_{,z} & \dots & \cdot & \cdot & N^n_{,z} \\ \hline \cdot & N^1_{,z} & N^1_{,y} & \cdot & N^2_{,z} & N^2_{,y} & \dots & \cdot & N^n_{,z} & N^n_{,y} \\ N^1_{,z} & \cdot & N^1_{,x} & N^2_{,z} & \cdot & N^2_{,x} & \dots & N^n_{,z} & \cdot & N^n_{,x} \\ N^1_{,y} & N^1_{,x} & \cdot & N^2_{,y} & N^2_{,x} & \cdot & \dots & N^n_{,y} & N^n_{,x} & \cdot \end{array}\right]$

$$W = \tfrac{1}{2}\varepsilon_{ij}\mathbb{C}_{ijkl}\varepsilon_{kl} \quad \leftrightarrow \quad W = \tfrac{1}{2}\tilde{\underline{\varepsilon}}^T \underline{\underline{E}}\, \tilde{\underline{\varepsilon}} \approx \tfrac{1}{2}\underline{U}^{hT}\underline{\underline{B}}^T\underline{\underline{E}}\,\underline{\underline{B}}\,\underline{U}^h$$

$$u_i \approx u_i^h \quad \leftrightarrow \quad \underline{u} \approx \underline{u}^h = \underline{\underline{C}}\,\underline{U}^h \leftarrow \text{with: } \underline{\underline{C}} = \begin{bmatrix} N^1 & \cdot & \cdot & \dots & N^n & \cdot & \cdot \\ \cdot & N^1 & \cdot & \dots & \cdot & N^n & \cdot \\ \cdot & \cdot & N^1 & \dots & \cdot & \cdot & N^n \end{bmatrix}$$

2D Voigt notation: (analog to 3D)

$\tilde{\underline{\varepsilon}} = [\varepsilon_{xx}, \varepsilon_{yy}, 2\varepsilon_{xy}]^T$; $\tilde{\underline{\sigma}} = [\sigma_{xx}, \sigma_{yy}, \sigma_{xy}]^T$

$\underline{U}^h = [u_x^1, u_y^1, u_x^2, u_y^2, \dots, u_x^n, u_y^n]^T$

$$\underline{\underline{B}} = \left[\begin{array}{cc|cc|cc} N^1_{,x} & \cdot & N^2_{,x} & \cdot & \dots\ N^n_{,x} & \cdot \\ \cdot & N^1_{,y} & \cdot & N^2_{,y} & \dots\ \cdot & N^n_{,y} \\ \hline N^1_{,y} & N^1_{,x} & N^2_{,y} & N^2_{,x} & \dots\ N^n_{,y} & N^n_{,x} \end{array}\right]$$

↳ $W \approx \frac{1}{2}\underline{U}^{hT}\underline{\underline{B}}^T\underline{\underline{E}}\,\underline{\underline{B}}\,\underline{U}^h$; $\underline{u} \approx \underline{\underline{C}}\,\underline{U}^h$

- plane strain: $(\varepsilon_{zz}=0, \sigma_{zz}\neq 0)$ $(\varepsilon_{yz}=\varepsilon_{xz}=\sigma_{yz}=\sigma_{xz}=0)$

$$\underline{\underline{E}} = \frac{E}{(1+\nu)(1-2\nu)} \cdot \begin{bmatrix} 1-\nu & \nu & \cdot \\ \nu & 1-\nu & \cdot \\ \cdot & \cdot & 1/2-\nu \end{bmatrix} \quad \text{(from 3D } \underline{\underline{E}} \text{ and above cond.)}$$

↳ recover σ_{zz} from $\tilde{\underline{\varepsilon}}$: $\sigma_{zz} = \frac{E}{(1+\nu)(1-2\nu)} \cdot (\nu\varepsilon_{xx} + \nu\varepsilon_{yy})$

- plane stress: $(\varepsilon_{zz}\neq 0, \sigma_{zz}=0)$ $(\varepsilon_{yz}=\varepsilon_{xz}=\sigma_{yz}=\sigma_{xz}=0)$

$$\underline{\underline{E}} = \frac{E}{1-\nu^2} \cdot \begin{bmatrix} 1 & \nu & \cdot \\ \nu & 1 & \cdot \\ \cdot & \cdot & \frac{1-\nu}{2} \end{bmatrix} \quad \left(\text{from 3D } \underline{\underline{E}} \text{ and above cond. and substitute: } \varepsilon_{zz} = \dots \leftarrow \sigma_{zz} = \underline{E}_3 \tilde{\underline{\varepsilon}} = 0\right)$$

↳ recover ε_{zz} from $\tilde{\underline{\varepsilon}}$: $\varepsilon_{zz} = -\frac{\nu}{\nu-1}\varepsilon_{xx} - \frac{\nu}{\nu-1}\varepsilon_{yy}$

approx.: $\underline{u} \approx \underline{u}^h(\underline{x}) = \sum_{a=1}^n \underline{u}^a N^a(\underline{x}) = \underline{\underline{C}}\,\underline{U}^h$ (subj. to: $u^h = \hat{u}$ on $\delta\Omega_D$)

$\delta\underline{u} \approx \underline{v}^h(\underline{x}) = \sum_{a=1}^n \underline{v}^a N^a(\underline{x}) = \underline{\underline{C}}\,\underline{V}^h$ (subj. to: $v^h = 0$ on $\delta\Omega_D$)

($\underline{u}^a, \underline{v}^a$: constants; N^a: shape functions)

$$\min I[\underline{u}^h] = \int_\Omega W(\underline{u}^h)\,dV - \int_\Omega \rho\,\underline{u}^{hT}\underline{b}\,dV - \int_{\partial\Omega_N} \underline{u}^{hT}\hat{\underline{t}}\,dS = \tfrac{1}{2}\underline{U}^{hT}\int_\Omega \underline{\underline{B}}^T\underline{\underline{E}}\,\underline{\underline{B}}\,dV \cdot \underline{U}^h - \underline{U}^{hT}\left(\int_\Omega \rho\,\underline{\underline{C}}^T\underline{b}\,dV + \int_{\partial\Omega_N}\underline{\underline{C}}^T\hat{\underline{t}}\,dS\right)$$

$$0 = \delta I[\underline{u}^h] = \sim\sim\sim = \underline{V}^{hT}\underbrace{\underbrace{\int_\Omega \underline{\underline{B}}^T\underline{\underline{E}}\,\underline{\underline{B}}\,dV}_{\underline{\underline{K}}} \cdot \underline{U}^h}_{\underline{F}_{int}} - \underline{V}^{hT}\underbrace{\left(\int_\Omega \rho\,\underline{\underline{C}}^T\underline{b}\,dV + \int_{\partial\Omega_N}\underline{\underline{C}}^T\hat{\underline{t}}\,dS\right)}_{\underline{F} = \underline{F}_{ext}} \quad \forall \underline{V}^h$$

$$\boxed{\text{solve } \underline{\underline{K}}\,\underline{U}^h = \underline{F}}$$

mech. prob.: static 2D/3D non-linear elasticity (function discretization)

same approx $\underline{u}^h(\underline{x}), \underline{v}^h(\underline{x})$. $\underline{\underline{B}}, \underline{\underline{C}}$ of Voigt notation also valid for non-linear!

$\left\{\begin{array}{ll} \underline{u} \approx \underline{u}^h = \underline{\underline{C}}\,\underline{U}^h ; & \tilde{\underline{\varepsilon}} \approx \underline{\underline{B}}\,\underline{U}^h \\ \delta\underline{u} \approx \underline{v}^h = \underline{\underline{C}}\,\underline{V}^h ; & \delta\tilde{\underline{\varepsilon}} \approx \underline{\underline{B}}\,\underline{V}^h \end{array}\right.$

weak form: $\min I[\underline{u}^h] = \int_\Omega W(\underline{u}^h)\,dV - \int_\Omega \rho\,\underline{u}^{hT}\underline{b}\,dV - \int_{\partial\Omega_N}\underline{u}^{hT}\hat{\underline{t}}\,dS$

variational: $0 = \delta I[\underline{u}^h] = \int_\Omega \delta\underline{\underline{\varepsilon}}(\underline{v}^h) \cdot \underline{\underline{\sigma}}(\underline{u}^h)\,dV - \int_\Omega \rho\,\delta\underline{u}^{hT}\underline{b}\,dV - \int_{\partial\Omega_N}\delta\underline{u}^{hT}\hat{\underline{t}}\,dS \quad \forall \underline{v}^h$ $\quad (\delta W = \delta\underline{\underline{\varepsilon}} \cdot \partial W/\partial\underline{\underline{\varepsilon}} = \delta\underline{\underline{\varepsilon}} \cdot \underline{\underline{\sigma}})$

$$0 = \underline{V}^{hT}\underbrace{\int_\Omega \underline{\underline{B}}^T\tilde{\underline{\sigma}}(\underline{U}^h)\,dV}_{\underline{F}_{int}(\underline{U}^h)} - \underline{V}^{hT}\underbrace{\left(\int_\Omega \rho\,\underline{\underline{C}}^T\underline{b}\,dV - \int_{\partial\Omega_N}\underline{\underline{C}}^T\hat{\underline{t}}\,dS\right)}_{\underline{F}_{ext}} \quad \forall \underline{V}^h$$

$(\delta\underline{\underline{\varepsilon}} \cdot \underline{\underline{\sigma}} = \delta\tilde{\underline{\varepsilon}}^T\tilde{\underline{\sigma}} = \underline{V}^{hT}\underline{\underline{B}}^T\tilde{\underline{\sigma}})$ $(\delta\underline{u} = \underline{\underline{C}}\,\underline{V}^h)$

↳ solve $\underline{f}(\underline{U}^h) = \underline{F}_{int}(\underline{u}^h = \underline{\underline{C}}\,\underline{U}^h) - \underline{F}_{ext} = 0$ using numerical solvers!

$\underline{\underline{T}}$: incremental stiffness matrix

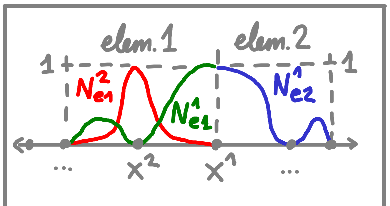
$\tilde{\underline{\underline{\mathbb{C}}}}(\tilde{\underline{\varepsilon}})$ is non-linear equivalent of $\underline{\underline{E}}$

$$\underline{\underline{T}}(\underline{U}^h) = \partial\underline{f}/\partial\underline{U}^h\,(\underline{U}^h) = \int_\Omega B^T\,\partial\tilde{\underline{\sigma}}/\partial\tilde{\underline{\varepsilon}} \cdot \partial\tilde{\underline{\varepsilon}}/\partial\underline{U}^h\,dV = \int_\Omega B^T\,\tilde{\underline{\underline{\mathbb{C}}}}(\tilde{\underline{\varepsilon}} = \underline{\underline{B}}\,\underline{U}^h)\,B\,dV$$

Finite Element Method FEM (= variational method with piecewise-polynomial shape functions)

1) split a body into elements with some space filling shape (can be distorted) → {1D: line segment; 2D: squares, triangle; 3D: cubes, tetrahedra,..}
2) choose a polynomial interpolation order q and # of shape functions n → {1D: $n \geq q+1$; 2D: $n \geq (q+1)(q+2)/2$; 3D: $n \geq (q+1)(q+2)(q+3)/6$}
2) pick n node points x^a per element on vertices, edges, surfaces or inside element (nodes not inside element can be shared with adjacent elements to enforce continuity)
3) pick n polynomial shape functions $N_e^a(x)$ of order p on every element, such that:

$N_e^a(x^b) = \begin{cases} 1 : \text{if } a=b \\ 0 : \text{if } a \neq b \end{cases} \quad \forall a,b \in \{1,..n\} \rightarrow \boxed{u^h(x^a) = u^a : \text{coeff. for } N^a = \text{value at } x^a \,!}$



$N^1 = \sum N_e^1 = N_{e1}^1 + N_{e2}^1$
$N^2 = \sum N_e^2 = N_{e1}^2$
...

4) write down variational method formula $K_e U = F_e$ for every element
5) combine element formulas into global formula $K = \sum K_e$, $F = \sum F_e$ (b.c.: actual shape func. of shared node $N^a = \sum N_e^a$ union of shape funcs. from sharing elements)
6) enforce Dirichlet BC by fixing some U^a in U 7) solve $KU = F$

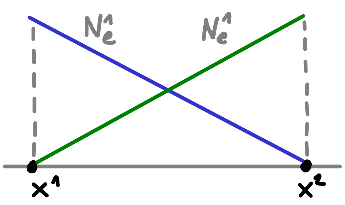
• polynomial basis functions:

completeness order	1D ($n=q+1$)	2D ($n=(q+1)(q+2)/2$)	3D ($n=(q+1)(q+2)(q+3)/6$) (n: # of basis func.)
$q=0$	$\{1\}$	$\{1\}$	$\{1\}$
$q=1$	$\{1,x\}$	$\{1,x,y\}$	$\{1,x,y,z\}$
$q=2$	$\{1,x,x^2\}$	$\{1,x,y,x\cdot y,x^2,y^2\}$	$\{1,x,y,z,x\cdot y,x\cdot z,y\cdot z,x^2,y^2,z^2\}$
⋮			

sufficient completeness: $q \geq p$ (p: order of highest derivative in equation) → mech prob. weak form: $p=1 \rightarrow q \geq 1$

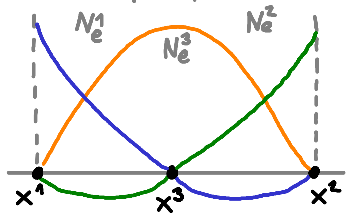
• 1D polynomial shape functions:

$n=2$, $q=1$, $N_e^a \in \{1,x\}$



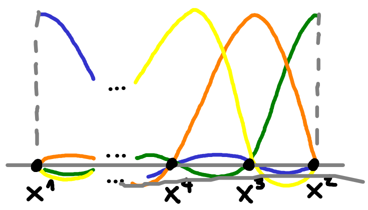
$N_e^1 = 1 - x/\Delta x$; $N_e^2 = x/\Delta x$

$n=3$, $q=2$, $N_e^a \in \{1,x,x^2\}$



$N_e^1(x) = ?$; $N_e^2(x) = ?$; $N_e^3(x) = ?$

$q=?$, $n=q+1$, $N_e^a \in \{1,x,x^2,..\}$

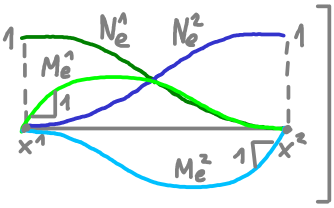


$$N_e^a(x) = \frac{(x-x^1)\cdot\ldots\cdot(x-x^{a-1})(x-x^{a+1})\cdot\ldots\cdot(x-x^n)}{(x^a-x^1)\cdot\ldots\cdot(x^a-x^{a-1})(x^a-x^{a+1})\cdot\ldots\cdot(x^a-x^n)}$$

[Hermite polynomials: (places nodes on derivatives of u) e.g. $q=3$, $n=4$:

$u^h = u^1 N_e^1 + u^2 N_e^2 + \theta^1 M_e^1 + \theta^2 M_e^2$
↳ $u^h(x^1) = u^1$; $u^h(x^2) = u^2$; $u^h_{,x}(x^1) = \theta^1$; $u^h_{,x}(x^2) = \theta^2$

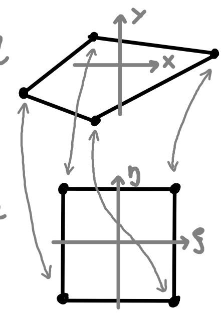
$N_e^1 = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3$; $N_e^2 = 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3$
$M_e^1 = x\left(1-\frac{x}{L}\right)^2$; $M_e^2 = \frac{x^2}{L}\left(\frac{x}{L}-1\right)$



]

• 2D polynomial shape functions:

[irregular element shapes → isoparametric mapping:



point: $\underline{x} = (x,y)$
node: $\underline{x}^a = (x_x^a, x_y^a)$ ✓
shape func: $N_e^a(\underline{x})$?

point: $\underline{\xi} = (\xi, \eta)$
node: $\underline{\xi}^a = (\xi_\xi^a, \xi_\eta^a)$ ✓
shape func: $N_e^a(\underline{\xi})$ ✓

• define an "easy" reference element in ref. space $\underline{\xi}$ with easy node pos $\underline{\xi}^a$ and easy shape func $N_e^a(\underline{\xi})$
• write a mapping $\underline{x} = \phi(\underline{\xi})$ such that the known node points $\underline{\xi}^a \leftrightarrow \underline{x}^a$ match: $\underline{x} = \phi(\underline{\xi}) = \sum_{a=1}^n \underline{x}^a N_e^a(\underline{\xi})$

↳ approx. solution of phys. elem.: $\underline{u}^h(\underline{x}) = \sum_{a=1}^n \underline{u}^a N_e^a(\underline{x})$ becomes:

$\underline{u}^h(\underline{x})\big|_{\underline{x}=\phi(\underline{\xi})} = \sum_{a=1}^n \underline{u}^a N_e^a(\underline{x})\big|_{\underline{x}=\phi(\underline{\xi})} = \sum_{a=1}^n \underline{u}^a N_e^a(\underline{\xi}) = \underline{u}^h(\underline{\xi})$

↳ **same $\underline{u}^a$ in both elements → solve for ref. elem., since easier!**

⋮]

when variational method asks for derivates of N_e^a. (e.g.: $N^a_{e,x}(\underline{\xi})=?$):

coeff from $\underline{x}=\phi(\underline{\xi})$

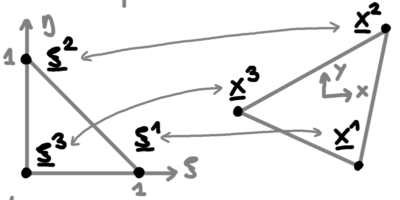
$$\begin{bmatrix} N^a_{e,\xi}(\underline{\xi}) \\ N^a_{e,\eta}(\underline{\xi}) \end{bmatrix} = \begin{bmatrix} N^a_{e,x}(\underline{\xi})\cdot x_{x,\xi}(\underline{\xi}) + N^a_{e,y}(\underline{\xi})\cdot x_{y,\xi}(\underline{\xi}) \\ N^a_{e,x}(\underline{\xi})\cdot x_{x,\eta}(\underline{\xi}) + N^a_{e,y}(\underline{\xi})\cdot x_{y,\eta}(\underline{\xi}) \end{bmatrix} = \underbrace{\begin{bmatrix} x_{x,\xi}(\underline{\xi}) & x_{y,\xi}(\underline{\xi}) \\ x_{x,\eta}(\underline{\xi}) & x_{y,\eta}(\underline{\xi}) \end{bmatrix}}_{J} \begin{bmatrix} N^a_{e,x}(\underline{\xi}) \\ N^a_{e,y}(\underline{\xi}) \end{bmatrix} = J\cdot \begin{bmatrix} N^a_{e,x}(\underline{\xi}) \\ N^a_{e,y}(\underline{\xi}) \end{bmatrix} \rightarrow \begin{bmatrix} N^a_{e,x}(\underline{\xi}) \\ N^a_{e,y}(\underline{\xi}) \end{bmatrix} = J^{-1}\cdot \begin{bmatrix} N^a_{e,\xi}(\underline{\xi}) \\ N^a_{e,\eta}(\underline{\xi}) \end{bmatrix}$$

when variational method asks for an integral dA:

$$dA = \|d\underline{x}\times d\underline{y}\| = \|(\frac{\partial x}{\partial \xi}\cdot d\xi\cdot \underline{e}_x + \frac{\partial x}{\partial \eta}\cdot d\eta\cdot \underline{e}_y)\times(\frac{\partial y}{\partial \xi}\cdot d\xi\cdot \underline{e}_x + \frac{\partial y}{\partial \eta}\cdot d\eta\cdot \underline{e}_y)\| \Rightarrow dA = \det(J)\cdot d\xi\cdot d\eta\cdot \underbrace{\|\underline{e}_x\times\underline{e}_y\|}_{\text{basis vec. of phys. frame}}$$

- reference constant strain triangle (T3):

$n=3$, $q=1$, $N^a_e\in\{1,\xi,\eta\}$



$N^1_e(\underline{\xi})=\xi$

$N^2_e(\underline{\xi})=\eta$

$N^3_e(\underline{\xi})=1-\xi-\eta$

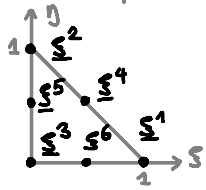
isoparametric mapping: $\underline{x}=\phi(\underline{\xi})=\underline{x}^1\xi+\underline{x}^2\eta+\underline{x}^3(1-\xi-\eta)$

$$\underline{\underline{J}}=\begin{bmatrix} x^1_x-x^3_x & x^1_y-x^3_y \\ x^2_x-x^3_x & x^2_y-x^3_y \end{bmatrix} \qquad \underline{\underline{J}}^{-1}=\frac{1}{2A_e}\cdot\begin{bmatrix} x^2_y-x^3_y & x^3_y-x^1_y \\ x^3_x-x^2_x & x^1_x-x^3_x \end{bmatrix}$$

$$\det(\underline{\underline{J}})=(x^1_x-x^3_x)(x^2_y-x^3_y)-(x^2_x-x^3_x)(x^1_y-x^3_y)=2A_e$$

- reference linear strain triangle (T6)

$n=6$, $q=2$, $N^a_e\in\{1,\xi,\eta,\xi\eta,\xi^2,\eta^2\}$ $\qquad(\theta=1-\xi-\eta)$



$N^1_e(\underline{\xi})=\xi(2\xi-1)$ $\qquad N^4_e(\underline{\xi})=4\xi\eta$

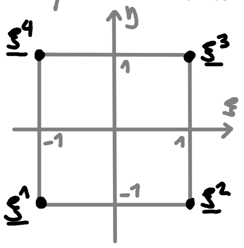
$N^2_e(\underline{\xi})=\eta(2\eta-1)$ $\qquad N^5_e(\underline{\xi})=4\eta\theta$

$N^3_e(\underline{\xi})=\theta(2\theta-1)$ $\qquad N^6_e(\underline{\xi})=4\xi\theta$

isoparametric mapping: $\underline{x}=\phi(\underline{\xi})=\sum_{a=1}^{n}\underline{x}^a N^a_e(\underline{\xi})$

$\underline{J}(\underline{\xi})=\dots$ (here $\underline{J}$ func of $\underline{\xi}$!), $\underline{J}^{-1}(\underline{\xi})=\dots$, $\det(\underline{J})(\underline{\xi})=\dots$

- reference bilinear quadrilateral element (Q4)



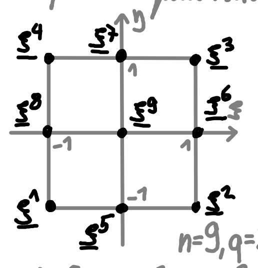
$n=4$, $q=1$, $N^a_e\in\{1,\xi,\eta,\xi\eta\}$

$N^1_e(\underline{\xi})=\frac{1}{4}(1-\xi)(1-\eta)$; $N^3_e(\underline{\xi})=\frac{1}{4}(1+\xi)(1+\eta)$

$N^2_e(\underline{\xi})=\frac{1}{4}(1+\xi)(1-\eta)$; $N^4_e(\underline{\xi})=\frac{1}{4}(1-\xi)(1+\eta)$

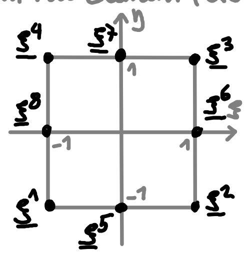
↳ $\underline{u}^h$: linear along edges !

- reference quadratic quadrilateral element (Q9 or Q8)



$n=9$, $q=2$

$N^a_e\in\{1,\xi,\eta,\xi\eta,\xi\eta^2,\xi^2\eta^2\}$



$N^a_e\in\{1,\xi,\eta,\xi\eta,\xi\eta^2\}$ $n=8$, $q=2$

3D polynomial shape functions:

irregular element shapes → isoparametric mapping

same as 2D case, just now with: $\underline{J}=\begin{bmatrix} x_{,\xi}(\underline{\xi}) & y_{,\xi}(\underline{\xi}) & z_{,\xi}(\underline{\xi}) \\ x_{,\eta}(\underline{\xi}) & y_{,\eta}(\underline{\xi}) & z_{,\eta}(\underline{\xi}) \\ x_{,\zeta}(\underline{\xi}) & y_{,\zeta}(\underline{\xi}) & z_{,\zeta}(\underline{\xi}) \end{bmatrix}$, $\begin{bmatrix} N^a_{e,x}(\underline{\xi}) \\ N^a_{e,y}(\underline{\xi}) \\ N^a_{e,z}(\underline{\xi}) \end{bmatrix} = J^{-1}\cdot\begin{bmatrix} N^a_{e,\xi}(\underline{\xi}) \\ N^a_{e,\eta}(\underline{\xi}) \\ N^a_{e,\zeta}(\underline{\xi}) \end{bmatrix}$, $dV=\det(\underline{J})\,d\xi\,d\eta\,d\zeta$

- constant strain tetrahedron (T4)

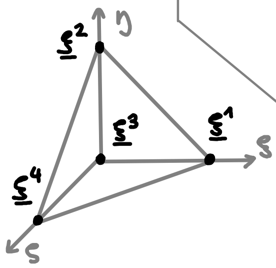
$n=4$, $q=1$, $N^a_e\in\{1,\xi,\eta,\zeta\}$

$N^1_e(\underline{\xi})=\xi$

$N^2_e(\underline{\xi})=\eta$

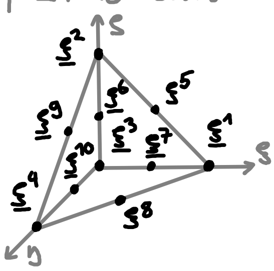
$N^3_e(\underline{\xi})=\zeta$

$N^4_e(\underline{\xi})=1-\xi-\eta-\zeta$



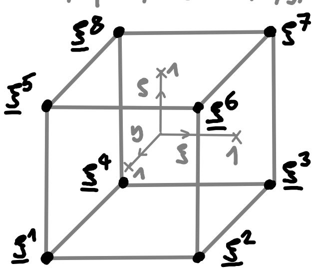
- linear strain tetrahedron (T10)

$n=10$, $q=2$, $N^a_e\in\{\dots\}$

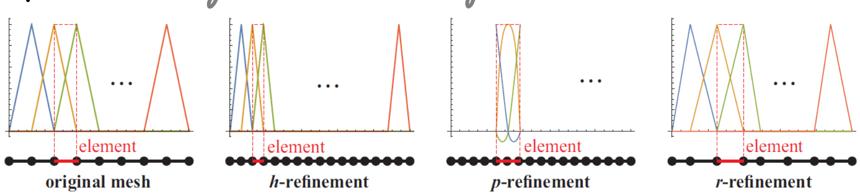


- brick element

$n=8$, $q=1$, $N^a_e\in\{1,\xi,\eta,\zeta,\xi\eta,\xi\zeta,\eta\zeta\}$



refinements (ways to increase accuracy of FEM solution)



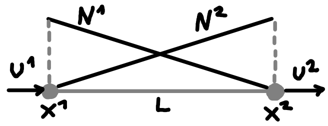
h-refinement: decrease element sizes

p-refinement: increase poly. interp. order p

r-refinement: move nodes more where func. is not smooth

FEM bar example (1D bar under axial load) (static + lin. elastic)

- **2-node bar element in 1D:**



$u^h(x) = N^1(x)u^1 + N^2(x)u^2$

$N^1(x) = 1 - x/L$; $N^2(x) = x/L$

$\hookrightarrow N^1_{,x} = -\frac{1}{L}$ $\quad \hookrightarrow N^2_{,x} = \frac{1}{L}$

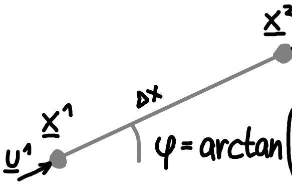
1D mech. problem

$K^{ab} = \mathcal{B}[N^a, N^b] = \int_0^{\Delta x} E N^1_{,x} N^2_{,x} A\, dx = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$F^a = \mathcal{L}[N^a] = ?$

$\hookrightarrow \underline{\underline{K}} \cdot \underline{U} = \underline{F} \rightarrow \underline{U} = \begin{pmatrix} U^1 \\ U^2 \end{pmatrix} = ?$

- **2-node bar element in 2D:**



$\varphi = \arctan\left(\frac{x^2_y - x^1_y}{x^2_x - x^1_x}\right)$

$\underline{\underline{R}}(\varphi) = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 & 0 \\ 0 & 0 & \cos\varphi & \sin\varphi \end{pmatrix}$ (project on bar, so 1D formula can be used)

$\hookrightarrow \underline{\tilde{U}} = (\tilde{U}^1, \tilde{U}^2)^T = \underline{\underline{R}} \cdot (U^1_x, U^1_y, U^2_x, U^2_y)^T = \underline{\underline{R}}\,\underline{U}$

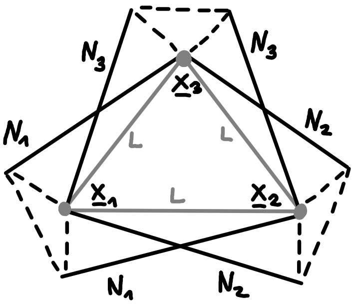
$\hookrightarrow \underline{\tilde{F}} = (\tilde{F}^1, \tilde{F}^2)^T = \underline{\underline{R}} \cdot (F^1_x, F^1_y, F^2_x, F^2_y)^T = \underline{\underline{R}}\,\underline{F}$

$\underline{\underline{R}}^T \underline{\underline{\tilde{K}}}\,\underline{\underline{R}} \cdot \underline{U} = \underline{F} \rightarrow \underline{U} = ?$

$\uparrow \underline{\underline{\tilde{K}}} = \frac{EA}{\Delta x}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ (like 1D)

- **2-node bar element in 3D:** $\underline{\underline{R}} = \begin{pmatrix} \tilde{x}\cdot e^x & \tilde{x}\cdot e^y & \tilde{x}\cdot e^z & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{x}\cdot e^x & \tilde{x}\cdot e^y & \tilde{x}\cdot e^z \end{pmatrix}$ ($\tilde{x}$: unit vector along bar; e^x, e^y, e^z: cartesian basis) **rest same as 2D!**

- **multiple bar elements in 2D:**



bar 1→2: $\underbrace{\frac{EA}{L} \cdot \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{K^{loc}_{12} = R^T(0)\tilde{K}R(0)} \cdot \begin{bmatrix} U^1_x \\ U^1_y \\ U^2_x \\ U^2_y \end{bmatrix} \longrightarrow \underbrace{\frac{EA}{L} \cdot \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{K^{glob}_{12}} \cdot \underbrace{\begin{bmatrix} U^1_x \\ U^1_y \\ U^2_x \\ U^2_y \\ U^3_x \\ U^3_y \end{bmatrix}}_{\underline{U}}$

bar 2→3: $K^{loc}_{23} = R^T\left(\frac{2\pi}{3}\right)\tilde{K}R\left(\frac{2\pi}{3}\right) \longrightarrow K^{glob}_{23}$

bar 3→1: $K^{loc}_{31} = R^T\left(\frac{4\pi}{3}\right)\tilde{K}R\left(\frac{4\pi}{3}\right) \longrightarrow K^{glob}_{23}$

} analog to K^{glob}_{12}

$\rightarrow \underline{\underline{K}} = K^{glob}_{12} + K^{glob}_{23} + K^{glob}_{31} \rightarrow$ $\boxed{\underline{\underline{K}} \cdot \underline{U} = \underline{F}}$

(combine eq. of shared nodes)

$\boxed{\begin{array}{c}\underline{F} \text{ can be interpreted as forces on nodes!} \\ \underline{F} = [F^1_x\ F^1_y\ F^2_x\ F^2_y\ F^3_x\ F^3_y]^T\end{array}}$

(- **Dirichlet BC**: write force in F^a_i
- **Neumann BC**: rewrite row of $\underline{\underline{K}}\,\underline{U} = \underline{F}$ with F^a_i so it reads $U^a_i = \hat{U}^a_i$)

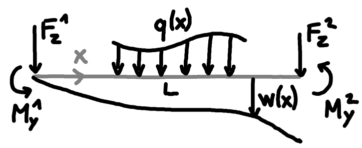
FEM euler beam example (1D beam under transversal load) (static + lin. elastic)

strong form: $EI_y \frac{\partial^4 w}{\partial x^4}(x) = q(x)$ $\quad$ BCs: $F^1_z = EI_y \frac{\partial^3 w}{\partial x^3}(0)$; $F^2_z = -EI_y \frac{\partial^3 w}{\partial x^3}(L)$; $M^1_y = EI_y \frac{\partial^2 w}{\partial x^2}(0)$; $M^2_y = -EI_y \frac{\partial^2 w}{\partial x^2}(L)$

weak form: $\min I[w] = \int_0^L EI_y w^2_{,xx} - qw\, dx - F^1_z w(0) - F^2_z w(L) + M^1_y w_{,x}(0) + M^2_y w_{,x}(L)$

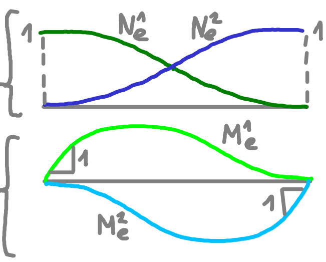
$\left(I[w] = \mathcal{B}[w,w] - \mathcal{L}[w] : \ \mathcal{B}[w,v] = \int_0^L EI_y w_{,xx} v_{,xx}\, dx \ ; \ \mathcal{L}[v] = \int_0^L qv\, dx + F^1_z v(0) + F^2_z v(L) - M^1_y v_{,x}(0) - M^2_y v_{,x}(L) \right)$

$\hookrightarrow 0 = \delta I[w] = \mathcal{B}[w, \delta w] - \mathcal{L}[\delta w] \quad \forall \delta w \in \mathcal{U}_0$



poly. order: $q = 3 \rightarrow n = 4$ so that we have 4 nodes (same as # of Neumann BC)

nodes: pick nodes on ●●●● so one can replace Neumann BC with Dirichlet BC (●● fix points $w^1_e = w^h_e(0)$, $w^2_e = w^h_e(L)$; ●● fix angles $\theta^1_e = w^h_{e,x}(0)$, $\theta^2_e = w^h_{e,x}(L)$)



shape func: $w^h_e = w^1_e N^1_e + w^2_e N^2_e + \theta^1_e M^1_e + \theta^2_e M^2_e$ (Hermitian polynomials)

$\left\{ \begin{array}{ll} N^1_e = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3 ; & N^2_e = 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3 \\ M^1_e = x\left(1 - \frac{x}{L}\right)^2 & ; \ M^2_e = \frac{x^2}{L}\left(\frac{x}{L} - 1\right) \end{array} \right.$

var. method: $\underline{U} = \begin{bmatrix} w^1_e \\ \theta^1_e \\ w^2_e \\ \theta^2_e \end{bmatrix}$ $\quad \underline{\underline{K}} = \frac{2EI_y}{L^3}\begin{bmatrix} 6 & 3L & -6 & 3L \\ 3L & 2L^2 & -3L & L^2 \\ -6 & -3L & 6 & -3L \\ 3L & L^2 & -3L & 2L^2 \end{bmatrix}$ $\quad \underline{F} = \begin{bmatrix} F^1_e \\ M^1_e \\ F^2_e \\ M^2_e \end{bmatrix}$ ($\underline{F}$ = forces and moments on nodes)

FEM beam example (1D euler beam AND 1D bar)

- 2-node beam element in 1D:

$$\underline{U} = \begin{bmatrix} U_e^1 \\ W_e^1 \\ \theta_e^1 \\ U_e^2 \\ W_e^2 \\ \theta_e^2 \end{bmatrix} ; \quad \underline{\underline{K}} = \underbrace{\frac{2EI_y}{L^3} \cdot \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 3L & 0 & -6 & 3L \\ 0 & 3L & 2L^2 & 0 & -3L & L^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -6 & -3L & 0 & 6 & -3L \\ 0 & 3L & L^2 & 0 & -3L & 2L^2 \end{bmatrix}}_{\underline{\underline{K}} \text{ of 1D euler beam}} + \underbrace{\frac{EA}{L} \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\underline{\underline{K}} \text{ of 1D bar}} ; \quad \underline{F} = \begin{bmatrix} F^1_{axial} \\ F^1_{trans} \\ M^1 \\ F^2_{axial} \\ F^2_{trans} \\ M^2 \end{bmatrix}$$

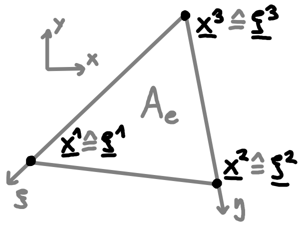
- 2-node beam element in 2D:

$$\underline{U} = \begin{bmatrix} U_x^1 \\ U_y^1 \\ \theta^1 \\ U_x^2 \\ U_y^2 \\ \theta^2 \end{bmatrix} ; \quad \underline{\underline{R}} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 & 0 & 0 & 0 \\ -\sin\varphi & \cos\varphi & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\varphi & \sin\varphi & 0 \\ 0 & 0 & 0 & -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \underline{\underline{K}} = \underline{\underline{R}}^T \tilde{\underline{\underline{K}}} \underline{\underline{R}} ; \quad \underline{F} = \begin{bmatrix} F_x^1 \\ F_y^1 \\ M^1 \\ F_x^2 \\ F_y^2 \\ M^2 \end{bmatrix}$$

(rest analog to FEM bar example)

$(\tilde{\underline{\underline{K}}} \leftarrow \underline{\underline{K}}$ from 1D)

FEM plane stress/strain on T3 element (triangle element of fixed depth t) (static + lin. elastic)



$\underline{\xi}^1 = \binom{1}{0}$ $\quad N_e^1(\underline{\xi}) = \xi$

$\underline{\xi}^2 = \binom{0}{1}$ $\quad N_e^2(\underline{\xi}) = \eta$

$\underline{\xi}^3 = \binom{0}{0}$ $\quad N_e^3(\underline{\xi}) = 1 - \xi - \eta$

mapping: $\underline{x} = \underline{\phi}(\underline{\xi}) = \sum_{a=1}^{n} \underline{x}^a N_e^a(\underline{\xi}) = \underline{x}^1 \cdot \xi + \underline{x}^2 \cdot \eta + \underline{x}^3 (1 - \xi - \eta)$

jacobian: $\underline{\underline{J}} = \begin{bmatrix} x_{,\xi}(\underline{\xi}) & y_{,\xi}(\underline{\xi}) \\ x_{,\eta}(\underline{\xi}) & y_{,\eta}(\underline{\xi}) \end{bmatrix} = \begin{bmatrix} x_x^1 - x_x^3 & x_y^1 - x_y^3 \\ x_x^2 - x_x^3 & x_y^2 - x_y^3 \end{bmatrix}$

$\det \underline{\underline{J}} = (x_x^1 - x_x^3)(x_y^2 - x_y^3) - (x_x^2 - x_x^3)(x_y^1 - x_y^3) = \|(\underline{x}^1 - \underline{x}^3) \times (\underline{x}^2 - \underline{x}^3)\| = 2A_e$

$$\begin{bmatrix} N^a_{e,x}(\underline{\xi}) \\ N^a_{e,y}(\underline{\xi}) \end{bmatrix} = \underline{\underline{J}}^{-1} \begin{bmatrix} N^a_{e,\xi}(\underline{\xi}) \\ N^a_{e,\eta}(\underline{\xi}) \end{bmatrix} = \frac{1}{2A_e} \cdot \begin{bmatrix} x_y^2 - x_y^3 & x_y^3 - x_y^1 \\ x_x^3 - x_x^2 & x_x^1 - x_x^3 \end{bmatrix} \cdot \begin{bmatrix} N^a_{e,\xi}(\underline{\xi}) \\ N^a_{e,\eta}(\underline{\xi}) \end{bmatrix}$$

$N^1_{e,x}(\underline{\xi}) = \frac{1}{2A_e} x_y^2 - x_y^3$; $N^2_{e,x}(\underline{\xi}) = \frac{1}{2A_e} x_y^3 - x_y^1$; $N^3_{e,x}(\underline{\xi}) = \frac{1}{2A_e} x_y^1 - x_y^2$

$N^1_{e,y}(\underline{\xi}) = \frac{1}{2A_e} x_x^3 - x_x^2$; $N^2_{e,y}(\underline{\xi}) = \frac{1}{2A_e} x_x^1 - x_x^3$; $N^3_{e,y}(\underline{\xi}) = \frac{1}{2A_e} x_x^2 - x_x^1$

from variational method example: $\underline{\underline{K}} \cdot \underline{U}^h = \underline{F}$

$\underline{U}^h = [U_x^1, U_y^1, U_x^2, U_y^2, U_x^3, U_y^3]^T$

$\underline{\underline{K}}_e = \int_{\Omega_e} \underline{\underline{B}}^T \underline{\underline{E}} \underline{\underline{B}} \, dV = A_e \cdot t \cdot \underline{\underline{B}}^T \underline{\underline{E}} \underline{\underline{B}}$ (since B, E const.)

$\underline{F}_e = \int_{\Omega_e} \varrho \underline{\underline{C}}^T \underline{b} \, dV + \int_{\partial\Omega_e} \underline{\underline{C}}^T \hat{\underline{t}} \, dS = \ldots$

$\underline{\underline{B}} = $ (see Voigt notation) $= \frac{1}{2A_e} \begin{bmatrix} x_y^2 - x_y^3 & 0 & x_y^3 - x_y^1 & 0 & x_y^1 - x_y^2 & 0 \\ 0 & x_x^3 - x_x^2 & 0 & x_x^1 - x_x^3 & 0 & x_x^2 - x_x^1 \\ x_x^3 - x_x^2 & x_y^2 - x_y^3 & x_x^1 - x_x^3 & x_y^3 - x_y^1 & x_x^2 - x_x^1 & x_y^1 - x_y^2 \end{bmatrix}$

$\underline{\underline{E}} = $ (see Voigt notation) $-$ pick the one for plane stress or plane strain!

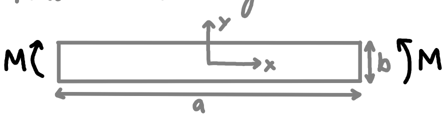
$dV = t \cdot dA = t \cdot \det(\underline{\underline{J}}) \, d\xi \, d\eta = 2A_e t \, d\xi \, d\eta$ $\quad$ (t: element thickness)

assemble problem $\underline{\underline{K}} = \sum_e \underline{\underline{K}}_e$; $\underline{F} = \sum_e \underline{F}_e$ ↳ solve $\underline{\underline{K}} \underline{U} = \underline{F}$ for $\underline{U}$

($\underline{F} \triangleq$ forces on nodes $= [F_{x}^1, F_y^1, F_x^2, F_y^2, \ldots]^T$)

Element defects: euler beam as Q4 element (static + lin. elastic)

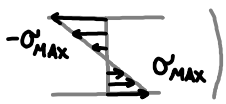
real beam bending:



$\sigma_{xx} = -\frac{2y}{b} \cdot \sigma_{MAX}$; $\sigma_{yy} = 0$; $\sigma_{xy} = 0$

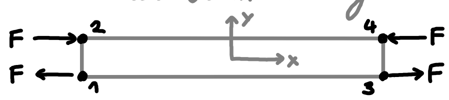
$\varepsilon_{xx} = \sigma_{xx}/E$; $\varepsilon_{yy} = -\nu \, \sigma_{xx}/E$; $\varepsilon_{xy} = 0$

(crosssection:



)

Q4 element beam bending:



$\underline{\underline{K}} \underline{U} = \underline{F} \rightarrow \underline{U} = \ldots$ (solve)

↳ $U_y = U_y^1 = U_y^2 = U_y^3 = U_y^4 = 0$

↳ $U_x = -U_x^1 = U_x^2 = -U_x^3 = U_x^4 = 2 \frac{a \sigma_{MAX}}{E} \cdot \frac{(1-\nu)(1-2\nu)}{2(1-\nu) + (a/b)^2 (1-2\nu)}$

$\tilde{\underline{\varepsilon}} = \underline{\underline{B}} \underline{U} \rightarrow \varepsilon_{xx} = -U_x \cdot \frac{2y}{ab}$; $\varepsilon_{yy} = 0$; $\varepsilon_{xy} = -U_x \frac{2x}{ab}$

DON'T MATCH!

F: forces on nodes from real stresses at ends

$$F^1 = \int_{-b/2}^{b/2} \underbrace{\left(-\frac{2y}{b} \sigma_{MAX}\right)}_{\sigma_{xx\,real}} \underbrace{\left(1 - \frac{2y}{b}\right)}_{N^1(-a/2, y)} dy = \frac{1}{3} b \, \sigma_{MAX}$$

$F = F^1 = -F^2 = F^3 = -F^4 = \frac{1}{3} b \, \sigma_{MAX}$ (analog)

$$\boxed{\frac{\theta_{Q4}}{\theta_{real}} = \frac{1 - \nu^2}{1 + \frac{1-\nu}{2} \left(\frac{a}{b}\right)^2}}$$

→ if a/b is large (slender beam) then Q4 predicts much lower displacements than real beam → "locking"

shape function requirements

$u(x) \approx u^h(x) = \sum_{a=1}^{n} u^a N^a(x)$ (u^a: constants ; N^a: shape functions) ($u \in \mathcal{U}$; $u^h \in \mathcal{U}^h$: function spaces)

requirements:
1) $\forall x \in \Omega \;\; \exists a$ s.t. $N^a(x) \neq 0$ (N^a covers Ω)
2) simple way to enforce Dirichlet BC
3) $\sum_{a=1}^{n} u^a N^a = 0 \Leftrightarrow u^a = 0 \; \forall a$ (linearly indep.)
4) satisfy requirements of equation (e.g. $N^a \subset C^2(\Omega)$)
5) $u^h \to u$ for $n \to \infty$ and any u (completeness)

(support: $\text{supp}\, N^a = \{x \in \Omega : N^a(x) \neq 0\}$
↳ global basis func: if $\text{supp}\, N^a = \Omega \;\; \forall a$
↳ local basis func: if $\text{supp}\, N^a \ll \Omega \;\; \forall a$)

Numerical quadrature = numerical integration

usefull if e.g. $K = \int \ldots d\xi$ or $F = \int \ldots d\xi$ is hard to compute analytically!

- **Gauss-Legendre quadrature** (good for line/square/cube element and polynomial func.)

1D: $\int_{-1}^{1} f(\xi) d\xi \approx \sum_{i=0}^{n_Q - 1} W_i f(\xi_i)$ ($f(\xi)$: function to be integrated, ξ_i: integration points, n_Q: # of integration points, W_i: integration weights)

↳ with the correct W_i, ξ_i this method produces exact integration for $f(\xi)$ = polynomial of order $< q_Q = 2n_Q - 1$!

↳ solve: $\int_{-1}^{1} \xi^k d\xi = \sum_{i=0}^{n_Q-1} W_i \xi_i^k \;\; \forall k = 0, \ldots, q_Q$ → e.g.: $q_Q = 1$; $n_Q = 1$ → $W_0 = 2, \xi_0 = 0$
→ e.g.: $q_Q = 3$; $n_Q = 2$ → $W_0 = W_1 = 1, \xi_0 = \frac{-1}{\sqrt{3}}, \xi_1 = \frac{1}{\sqrt{3}}$

(proof: $\int_{-1}^{1} f(\xi) d\xi = \int_{-1}^{1} \sum_{j=0}^{q_Q} a_j \xi^j d\xi = \sum_{j=0}^{q_Q} a_j \int_{-1}^{1} \xi^j d\xi = \sum_{j=0}^{q_Q} a_j \sum_{i=0}^{n_Q-1} W_i \xi_i^j = \sum_{i=0}^{n_Q-1} W_i \sum_{j=0}^{q_Q} a_j \xi_i^j = \sum_{i=0}^{n_Q-1} W_i f(\xi_i)$ ■)

2D: $\int_{-1}^{1}\int_{-1}^{1} f(\boldsymbol{\xi}) d\xi d\eta \approx \sum_{k=0}^{n_Q-1} W_k f(\boldsymbol{\xi}_k)$: exact integration for $f(\boldsymbol{\xi})$ = polynomial of order $< q_Q = 2\sqrt{n_Q} - 1$!

↳ $W_k = W_i \cdot W_j$; $\boldsymbol{\xi}_k = (\xi_i, \xi_j)$ (i, j: all comb. of above 1D solutions) → e.g.: $q_Q = 1$, $n_Q = 1$ → $W_0 = 4$, $\boldsymbol{\xi}_0 = (0,0)$
{1D solution values → e.g.: $q_Q = 3$, $n_Q = 4$ → $W_{0-3} = 1$, $\boldsymbol{\xi}_{0-3} = (\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}})$

(proof: $\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta) d\xi d\eta = \int_{-1}^{1} \sum_{i=0}^{N-1} W_i f(\xi_i, \eta) d\eta = \sum_{i=0}^{N-1} W_i \sum_{j=0}^{N-1} W_j f(\xi_i, \eta_j) = \sum_{j=0}^{N-1} \sum_{i=0}^{N-1} W_i W_j f(\xi_i, \eta_j)$ ■ ($N = \sqrt{n_Q}$))

3D: $\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f(\boldsymbol{\xi}) d\xi d\eta d\zeta \approx \sum_{k=0}^{n_Q-1} W_k f(\boldsymbol{\xi}_k)$: exact integration for $f(\boldsymbol{\xi})$ = polynomial of order $< q_Q = 2\sqrt[3]{n_Q} - 1$!

analog to 2D: → e.g.: $q_Q = 1$, $n_Q = 1$ → $W_0 = 8$, $\boldsymbol{\xi}_0 = (0,0,0)$ → e.g.: $q_Q = 3$, $n_Q = 8$ → $W_{0-7} = 1$, $\boldsymbol{\xi}_{0-7} = (\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}})$

(full integration (specific to mech. problem!):
q_Q needed for exact integration of finite element that is undistorted (J = const), homogeneous, linear elastic

Q4 element: $q_Q = 3, n_Q = 4$
Q8/Q9 element: $q_Q = 5, n_Q = 9$
brick element: $q_Q = 3, n_Q = 8$
(from orders of N^a, J, B, E, C))

(example: Q4 element with $q_Q = 1$ quadrature.
↳ below full integration order!

evaluate to same energy! bad, b.c. solver is then unsure which is right. (called **zero energy mode**))

(if $f(\boldsymbol{\xi})$ has order $> q_Q$ or is no polynomial at all, then integration error appears! → still worth using this method because still more efficient than Riemann sums. just increase q_Q if needed. with $q_Q \to \infty$ method is exact for any $f(\boldsymbol{\xi})$!)

- **simplical quadrature** (good for triangle/tetrahedon element and polynomial func.)

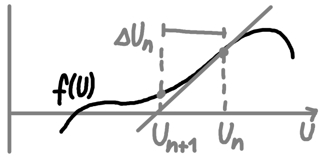
[not discussed in class. see "look-up tables" online.]

iterative solvers

general problem: solve $\underline{f}(\underline{U})=0$ (e.g. $\underline{f}(\underline{U})=F_{int}(\underline{U})-F_{ext}(\underline{U})$ from nonlinear mech prob.)
iterative solver: $\underline{U}_0$ initial guess (e.g. $\underline{U}_0=\underline{0}$) → loop $\underline{U}_{n+1}=\underline{U}_n+\Delta\underline{U}_n$, untill $\Delta\underline{U}_n$< treshold.
$\begin{cases} n: \text{iteration counter} \\ \Delta U_n: \text{update rule} \end{cases}$

- **Newton-Raphson (NR) method** (exact solution after 1 iteration if $f(\underline{U})$ linear)

$$0=\underline{f}(\underline{U}_{n+1})=\underline{f}(\underline{U}_n+\Delta\underline{U}_n)=\underbrace{\underline{f}(\underline{U}_n)+\partial\underline{f}/\partial\underline{U}(\underline{U}_n)\,\Delta\underline{U}_n}_{\text{Taylor expansion}}+\underbrace{\mathcal{O}(|\Delta\underline{U}_n|^2)}_{\text{assumed}\approx 0} \rightarrow \Delta\underline{U}_n=-\big[\underbrace{\partial\underline{f}/\partial\underline{U}(\underline{U}_n)}_{=\underline{T}(\underline{U}_n)}\big]^{-1}\underline{f}(\underline{U}_n)$$



• advantage: quadratic convergeance • disadvantage: always have to recompute $T(\underline{U}_n)$ & T not always invertible

- **Damped NR method**
(like NR, but slower convergence and more stable)
$\Delta\underline{U}_n=-\alpha[\partial\underline{f}/\partial\underline{U}(\underline{U}_n)]^{-1}\underline{f}(\underline{U}_n)=\alpha\Delta\underline{U}_n^{NR}$ $(0<\alpha<1)$

- **Line search method**
(same as dampened NR, but optimize stepsize)
$\Delta\underline{U}_n=\beta_n\Delta\underline{U}_n^{NR} \leftarrow \beta_n=\operatorname{argmin}\|\underline{f}(\underline{U}_n-\beta_n\Delta\underline{U}_n^{NR})\|^2$

- **Conjugate Gradient (CG) method**
(like "line search", but optimize dir. + dist. of next step)
$\Delta U_n=\gamma_n\underline{S}_n \leftarrow \gamma_n,\underline{S}_n=\operatorname{argmin}\|\underline{f}(\underline{U}_n-\gamma_n\underline{S}_n)\|^2$ (→ no $\underline{T}$!)

- **Quasi-Newton (QN) method**
(avoids recomputing and inverting $\underline{T}(\underline{U}_n)$)
done by approx. $\underline{T}(\underline{U}_{n+1})$ based on $\underline{T}(\underline{U}_n)$ and $\underline{f}(\underline{U}_n)$.

- **Nonlinear least squares**
(beneficial for overconstrained systems)
minimize $r(\underline{U})=\|\underline{f}(\underline{U})\|^2$ (residual) ⇒ $\partial r/\partial\underline{U}(\underline{U})=\ldots=\underline{0}$

- **Gradient flow method**
(good because no need to compute $\underline{T}$)
assume $\dot{\underline{U}}=-\underline{C}\cdot\underline{f}(\underline{U}) \rightarrow \underline{U}_{n+1}=\underline{U}_n-\underline{C}\Delta t\,\underline{f}(\underline{U}_n)$

mech problem: boundary conditions (in finite element context)

- **Neumann boundary conditions** are considered in $\underline{F}=\sum_e\underline{F}_e \rightarrow \underline{K}\underline{U}=\underline{F}$ (linear elastic) or $\underline{F}_{ext}=\underline{F}=\sum\underline{F}_e \rightarrow \underline{F}_{int}(\underline{U})-\underline{F}_{ext}=0$ (non-linear elastic)
  - body force: $\underline{F}_e=\int_{\Omega_e}\varrho\,\underline{C}^T\underline{b}\,dV$
  - traction force: $\underline{F}_e=\int_{\partial\Omega_e}\underline{C}^T\hat{\underline{t}}\,dS$
  - point force on node a: $\underline{F}^a=\underline{P}$
  - point force on point $\underline{x}$: $\underline{F}_e=\underline{C}(\underline{x})^T\underline{P}$
  - spring force (small/large strain): $\underline{F}_e$ func of $\underline{U}$! → $F_{ext}(\underline{U})$

- **Dirichlet boundary condition** (for linear problem) are considered in rows of $\underline{K}\underline{U}=\underline{F}$

want to enforce $U^3=\hat{U}^3$ on:
$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix}\cdot\begin{bmatrix} U^1 \\ U^2 \\ U^3 \\ U^4 \end{bmatrix}=\begin{bmatrix} F^1 \\ F^2 \\ F^3 \\ F^4 \end{bmatrix}$$

• substitution:
$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ 0 & 0 & 1 & 0 \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix}\cdot\begin{bmatrix} U^1 \\ U^2 \\ U^3 \\ U^4 \end{bmatrix}=\begin{bmatrix} F^1 \\ F^2 \\ \hat{U}^3 \\ F^4 \end{bmatrix}$$
(+ cheap to do
− matrix ill conditioned)

• condensation:
$$\left[\begin{array}{cc|c} K_{11} & K_{12} & K_{14} \\ K_{21} & K_{22} & K_{24} \\ \hline K_{41} & K_{42} & K_{44} \end{array}\right]\cdot\left[\begin{array}{c} U^1 \\ U^2 \\ \hline U^4 \end{array}\right]=\left[\begin{array}{c} F^1 \\ F^2 \\ \hline F^4 \end{array}\right]-\left[\begin{array}{c} K_{13} \\ K_{23} \\ \hline K_{43} \end{array}\right]\hat{U}_3$$
(+ matrix size smaller
− expensive to do)

FEM mech problem on Q4 element (static + linear elastic + numerical quadrature)

$N_e^a(\underline{\xi})=\ldots \rightarrow \underline{C}(\underline{\xi})=\ldots \rightarrow \underline{J}(\underline{\xi})=\ldots \rightarrow \underline{B}(\underline{\xi})=\ldots$

$$\underline{K}_e=\int_{\Omega_e}\underline{B}(\underline{\xi})^T\underline{\underline{E}}\,\underline{B}(\underline{\xi})\,dV=\int_{-1}^{1}\int_{-1}^{1}\underline{B}(\underline{\xi})^T\underline{\underline{E}}\,\underline{B}(\underline{\xi})\det(J(\underline{\xi}))\,d\xi\,d\eta=\sum_{k=0}^{n_Q-1}W_k\underline{B}(\underline{\xi}_k)^T\underline{\underline{E}}\,\underline{B}(\underline{\xi}_k)\det(\underline{J}(\underline{\xi}_k))$$ ($q_Q=3, n_Q=4$ for Q4)

$\underline{F}_e=$
→ body force: $\int_{\Omega_e}\varrho\,\underline{C}(\underline{\xi})^T\underline{b}(\underline{\xi})\,dV=\int_{-1}^{1}\int_{-1}^{1}\varrho\,\underline{C}(\underline{\xi})^T\underline{b}(\underline{\xi})\det(J(\underline{\xi}))\,d\xi\,d\eta=\sum_{k=0}^{n_Q-1}W_k\,\varrho\,\underline{C}(\underline{\xi}_k)^T\underline{b}(\underline{\xi}_k)\det(J(\underline{\xi}_k))$ ($q_Q=3, n_Q=4$)
→ traction force: $\int_{\partial\Omega_e}\underline{C}(\underline{\xi})^T\hat{\underline{t}}\,dS=\int_{-1}^{1}\underline{C}(*)^T\hat{\underline{t}}(*)\det(J(*))\,d*=\sum_{k=0}^{n_Q-1}W_k\,\underline{C}(*_k)^T\hat{\underline{t}}(*_k)\det(J(*_k))$ ($q_Q=3, n_Q=2$)
($*$ = face $\pm\xi$ or $\pm\eta$)

Dynamic mech. problem (displacement field u: $\underline{u}(\underline{x},t)$)

- **strong form** (dynamic): $\operatorname{div}(\underline{\sigma}(\underline{u}))+\rho\underline{b}=\rho\ddot{\underline{u}}$ in Ω ; $\underline{u}=\hat{\underline{u}}$ on Ω_D $\forall t$; $\underline{\sigma}(\underline{u})\cdot\underline{n}=\hat{\underline{t}}$ on Ω_N $\forall t$

  $\sigma_{ij,j}(\underline{u})+\rho b_i=\rho\ddot{u}_i$ in Ω ; $u_i=\hat{u}_i$ on Ω_D $\forall t$; $\sigma_{ij}(\underline{u})\cdot n_j=\hat{t}_i$ on Ω_N $\forall t$

- **weak form** (dynamic): $\min \underbrace{A[\underline{u}]}_{\text{action}}=\int_{t_1}^{t_2}\underbrace{\int_\Omega W(\underline{u})}_{\text{int } E}-\underbrace{\int_\Omega \frac{\rho}{2}|\dot{\underline{u}}|^2 dV}_{\text{kin } E}-\underbrace{\int_\Omega \rho\underline{b}\,\underline{u}\,dV}_{\text{pot } E}-\underbrace{\int_{\partial\Omega}\hat{\underline{t}}\,\underline{u}\,dS}_{\text{ext } W}\,dt$ subj. to: $\underline{u}=\hat{\underline{u}}$ on $\partial\Omega_D$

  variation: $0=\delta A[\underline{u}]=\int_{t_1}^{t_2}\int_\Omega \sigma_{ij}\delta\varepsilon_{ij}-\rho\dot{u}_i\delta\dot{u}_i\,dV-\int_\Omega \rho b_i\delta u_i dV-\int_{\partial\Omega_N}\hat{t}_i\delta u_i dS\,dt$ subj. to: $u_i=\hat{u}_i$ on $\partial\Omega_D$ $\forall t$

  $=\int_{t_1}^{t_2}\int_\Omega \sigma_{ij}\delta\varepsilon_{ij}+\rho\ddot{u}_i\delta u_i\,dV-\int_\Omega \rho b_i\delta u_i dV-\int_{\partial\Omega_N}\hat{t}_i\delta u_i dS\,dt$ $\quad\delta u_i=0$ on $\partial\Omega_D$ $\forall t$; $\delta u_i=0$ at t_1,t_2 $\forall x$

  func. discretization: $\underline{u}(\underline{x},t)\approx\underline{u}^h(\underline{x},t)=\sum_{a=1}^n \underline{u}^a(t)N^a(\underline{x})=\underline{\underline{C}}(\underline{x})\cdot\underline{U}^h(t)$ (subj. to: $u^h=\hat{u}$ on $\delta\Omega_D$ $\forall t$)

  $\delta\underline{u}(\underline{x},t)\approx\underline{v}^h(\underline{x},t)=\sum_{a=1}^n \underline{v}^a(t)N^a(\underline{x})=\underline{\underline{C}}(\underline{x})\cdot\underline{V}^h(t)$ (subj. to: $v^h=0$ on $\delta\Omega_D$ $\forall t$ and $v^h=0$ at t_1,t_2 $\forall\underline{x}$)

  variation approx: $0=A[u]=\int_{t_1}^{t_2}\int_\Omega \underline{\underline{\sigma}}(\underline{u}^h)\cdot\delta\underline{\underline{\varepsilon}}(\underline{v}^h)+\rho\ddot{\underline{u}}^h\underline{v}^h dV-\int_\Omega\rho\underline{b}\cdot\underline{v}^h dV-\int_{\partial\Omega_N}\hat{\underline{t}}\cdot\underline{v}^h dS\,dt$ $\quad\forall\underline{v}^h$

  $=\int_{t_1}^{t_2}\underline{V}^{hT}\left(\int_\Omega \underline{\underline{B}}^T\tilde{\underline{\sigma}}(u^h)+\rho\underline{\underline{C}}^T\underline{\underline{C}}\ddot{\underline{U}}^h dV-\int_\Omega\rho\underline{\underline{C}}^T\underline{b}\,dV-\int_{\partial\Omega_N}\underline{\underline{C}}^T\hat{\underline{t}}\,dS\right)dt$ $\quad\forall\underline{V}^h$

  $\hookrightarrow \underbrace{\underbrace{\int_\Omega\rho\underline{\underline{C}}^T\underline{\underline{C}}\,dV}_{\underline{\underline{M}}}\ddot{\underline{U}}^h}+\underbrace{\int_\Omega\underline{\underline{B}}^T\tilde{\underline{\sigma}}(u^h)dV}_{\underline{F}_{int}(\underline{u}^h)}-\underbrace{\int_\Omega\rho\underline{\underline{C}}^T\underline{b}\,dV-\int_{\partial\Omega_N}\underline{\underline{C}}^T\hat{\underline{t}}\,dS}_{\underline{F}_{ext}}=0$

  (consistent mass matrix) $\quad\underline{\underline{M}}\cdot\ddot{\underline{U}}^h+\underline{F}_{int}(\underline{u}^h)-\underline{F}_{ext}=0$ $\quad$ ($\underline{F}_{int}=\underline{\underline{K}}\cdot\underline{U}^h$ for linear elastic!)

$\underline{\underline{M}}$ example for FEM bar:

1D: $N_e^1=1-\frac{x}{L}$; $N_e^2=\frac{x}{L}$ $\rightarrow$ $\underline{\underline{C}}=[N_e^1\ \ N_e^2]=[1-\frac{x}{L}\ \ \frac{x}{L}]$

$\underline{\underline{M}}=\int_\Omega\rho\,\underline{\underline{C}}^T\underline{\underline{C}}\,dV=\int_0^L\rho\,\underline{\underline{C}}^T\underline{\underline{C}}\,A\,dx=\rho LA\frac{1}{6}\begin{bmatrix}2&1\\1&2\end{bmatrix}$

2D: $\underline{\underline{M}}=\rho LA\frac{1}{6}\begin{bmatrix}2&\cdot&1&\cdot\\\cdot&2&\cdot&1\\1&\cdot&2&\cdot\\\cdot&1&\cdot&2\end{bmatrix}$ (explaination?)

3D: $\underline{\underline{M}}=\rho LA\frac{1}{6}\begin{bmatrix}2&\cdot&\cdot&1&\cdot&\cdot\\\cdot&2&\cdot&\cdot&1&\cdot\\\cdot&\cdot&2&\cdot&\cdot&1\\1&\cdot&\cdot&2&\cdot&\cdot\\\cdot&1&\cdot&\cdot&2&\cdot\\\cdot&\cdot&1&\cdot&\cdot&2\end{bmatrix}$

$\underline{\underline{M}}$ example for CST element:

$N_e^1=\xi$, $N_e^2=\eta$, $N_e^3=1-\xi-\eta$ $\rightarrow$ $\underline{\underline{C}}=\begin{bmatrix}\xi&\cdot&\eta&\cdot&1-\xi-\eta&\cdot\\\cdot&\xi&\cdot&\eta&\cdot&1-\xi-\eta\end{bmatrix}$

$\underline{\underline{M}}=\int_\Omega\rho\underline{\underline{C}}^T\underline{\underline{C}}\,dV=\int_0^1\int_0^{1-\xi}\rho C^TC\det(J)\,d\eta\,d\xi\cdot t_e$ $\longrightarrow$ $\underline{\underline{M}}=\rho A_e t_e\frac{1}{12}\begin{bmatrix}2&\cdot&1&\cdot&1&\cdot\\\cdot&2&\cdot&1&\cdot&1\\1&\cdot&2&\cdot&1&\cdot\\\cdot&1&\cdot&2&\cdot&1\\1&\cdot&1&\cdot&2&\cdot\\\cdot&1&\cdot&1&\cdot&2\end{bmatrix}$

(alternative $\underline{\underline{M}}_{Lump}$: concentrate mass on nodes $\rightarrow$ $\underline{\underline{M}}$ is less dense, but has error (vanishes for small elements)

lumped mass matrix $\hookrightarrow$ $\underline{\underline{M}}=\int_\Omega\rho\underline{\underline{C}}^T\underline{\underline{C}}\,dV=\frac{\rho V}{\#\text{nodes}}\sum_{\text{nodes}}\underline{\underline{C}}(\text{node})^T\underline{\underline{C}}(\text{node})=\frac{\rho V}{\#\text{nodes}}\mathbb{I}$)

Vibrations (small amplitude oscillation about a stable equilibrium)

- **free vibration** ($F_{ext}=$ const.) + linear elasticity:

$\underline{U}(t)=\underline{U}_0+\underline{V}(t)$ ($V(t)$: small perturbation ; $\underline{U}_0$ equilibrium $KU_0=F$)

$\underline{\underline{M}}\ddot{\underline{U}}+\underline{\underline{K}}\underline{U}=\underline{F}$ $\rightarrow$ $\underline{\underline{M}}\ddot{\underline{V}}+\underline{\underline{K}}(\underline{U}_0+\underline{V})=\underline{F}$ $\rightarrow$ $\underline{\underline{M}}\ddot{\underline{V}}+\underline{\underline{K}}\underline{V}=\underline{0}$ $\rightarrow$ real solution: $\underline{V}(t)=\underline{C}_1e^{i\omega t}+\underline{C}_2e^{-i\omega t}=\hat{\underline{U}}\cos(\omega t+\varphi)$ $\rightarrow$

$\rightarrow$ insert solution: $(\underline{\underline{K}}-\omega^2\underline{\underline{M}})\hat{\underline{U}}\,\cancel{\cos(\omega t+\varphi)}=\underline{0}$ $\forall t$ $\rightarrow$ $(\underline{\underline{M}}^{-1}\underline{\underline{K}}-\omega^2)\hat{\underline{U}}=\underline{0}$ $\rightarrow$ $\omega_i^2=EW(\underline{\underline{M}}^{-1}\underline{\underline{K}})$; $\hat{\underline{U}}_i=EV(\underline{\underline{M}}^{-1}\underline{\underline{K}})$ $\rightarrow$

$\rightarrow$ multiple $\omega_i^2\hat{U}_i$ solve equation. True solution is linear comb.: $V(t)=\sum_j c_j\hat{\underline{U}}_j\cos(\omega_j t+\varphi_j)$ (find c_j,φ_j with IC: $V(0),\dot{V}(0)$)

c_j: some const, since any $c_j\hat{\underline{U}}_j$ is still an EV to ω_j^2 $\quad$ φ_j: any phase of the vibration

ω example for FEM bar:

$\underline{\underline{M}}=\frac{m}{6}\begin{bmatrix}2&1\\1&2\end{bmatrix}$ $\rightarrow$ $\hat{\underline{U}}_0=\binom{1}{1}$, $\hat{\underline{U}}_1=\binom{1}{-1}$	$\underline{\underline{M}}=\frac{m}{2}\begin{bmatrix}1&0\\0&1\end{bmatrix}$ $\rightarrow$ $\hat{\underline{U}}_0=\binom{1}{1}$, $\hat{\underline{U}}_1=\binom{1}{-1}$	real bar $\rightarrow$ $\hat{\underline{U}}_0=\binom{1}{1}$, $\hat{\underline{U}}_1=\binom{1}{-1}$
(consistent) $\omega_0=0$, $\omega_1=2\sqrt{3}\sqrt{\frac{EA}{mL}}$	(lumped) $\omega_0=0$, $\omega_1=2\sqrt{\frac{EA}{mL}}$	$\omega_0=0$, $\omega_1=\pi\sqrt{\frac{EA}{mL}}$

- **forced vibrations** ($\underline{F}(t)=\underline{F}_0+\hat{\underline{F}}\cos(\Omega t)$) + linear elasticity:

$\underline{U}(t)=\underline{U}_0+\underline{V}(t)$ ($\underline{V}(t)$: small perturbation; $\underline{U}_0$ equilibrium $\underline{K}\underline{U}_0=\underline{F}_0$)

$\underline{M}\ddot{\underline{U}}+\underline{K}\underline{U}=\underline{F}(t) \rightarrow \underline{M}\ddot{\underline{V}}+\underline{K}\underline{V}=\hat{\underline{F}}\cos(\Omega t) \rightarrow$ particular solution: $\underline{V}_{part}(t)=(\underline{K}-\Omega^2\underline{M})^{-1}\hat{\underline{F}}\cos(\Omega t)$

$\rightarrow \underline{V}(t)=\underline{V}_{homog}(t)+\underline{V}_{part}(t)$ (homogen solution = free vibration)

if $\Omega=\omega_i$ some eigenfreq. then $V_{part}\rightarrow\infty$!

- **modal decomposition** (use vibration solution to easily find solution for any $F(t)$)

idea: $\underline{U}(t)=\sum z_i(t)\hat{\underline{U}}_i$ find coefficients $z_i(t)$ instead of directly solving for $\underline{U}(t)$ (is ok since $\hat{\underline{U}}_i$ is basis of vector-space)

$$\hat{\underline{U}}_j\cdot\underbrace{(\underline{K}-\omega_i^2\underline{M})\hat{\underline{U}}_i}_{=0}-\hat{\underline{U}}_i\cdot\underbrace{(\underline{K}-\omega_j^2\underline{M})\hat{\underline{U}}_j}_{=0}=0 \rightarrow (\hat{\underline{U}}_j\underline{K}\hat{\underline{U}}_i-\hat{\underline{U}}_i\underline{K}\hat{\underline{U}}_j)-(\hat{\underline{U}}_j\omega_i^2\underline{M}\hat{\underline{U}}_i-\hat{\underline{U}}_i\omega_j^2\underline{M}\hat{\underline{U}}_j)=0$$

($\hat{\underline{U}}_i\underline{K}\hat{\underline{U}}_j=\hat{\underline{U}}_j\underline{K}\hat{\underline{U}}_i$; $\hat{\underline{U}}_i\underline{M}\hat{\underline{U}}_j=\hat{\underline{U}}_j\underline{M}\hat{\underline{U}}_i$ because $\underline{K},\underline{M}$ symetrical)

$\rightarrow (\omega_i^2-\omega_j^2)\hat{\underline{U}}_i\underline{M}\hat{\underline{U}}_j=0$
- if $i\neq j \rightarrow \omega_i^2-\omega_j^2\neq 0 \longrightarrow \hat{\underline{U}}_i\underline{M}\hat{\underline{U}}_j=0$
- if $i=j \rightarrow \omega_i^2-\omega_j^2=0 \rightarrow \hat{\underline{U}}_i\underline{M}\hat{\underline{U}}_i\neq 0 \rightarrow$ **scale** $\hat{\underline{U}}_i$ s.t. $\hat{\underline{U}}_i\underline{M}\hat{\underline{U}}_i=1$ ($\hat{\underline{U}}_i=\hat{\underline{U}}_i/\sqrt{\hat{\underline{U}}_i\underline{M}\hat{\underline{U}}_i}$)

$\hat{\underline{U}}_j\cdot\underbrace{(\underline{K}-\omega_i^2\underline{M})\hat{\underline{U}}_i}_{=0}=0 \rightarrow \hat{\underline{U}}_j\cdot\underline{K}\hat{\underline{U}}_i=\omega_i^2\hat{\underline{U}}_j\cdot\underline{M}\hat{\underline{U}}_i$
- if $i\neq j \rightarrow \hat{\underline{U}}_j\underline{M}\hat{\underline{U}}_i=0 \rightarrow \hat{\underline{U}}_j\cdot\underline{K}\hat{\underline{U}}_i=0$
- if $i=j \rightarrow \hat{\underline{U}}_i\underline{M}\hat{\underline{U}}_i=0 \rightarrow \hat{\underline{U}}_i\cdot\underline{K}\hat{\underline{U}}_i=\omega_i^2 \leftarrow$ scaled $\hat{\underline{U}}_i$!

$\boxed{\text{Rayleigh's formula: } \omega_i^2=\hat{\underline{U}}_i\cdot\underline{K}\hat{\underline{U}}_i\,/\,\hat{\underline{U}}_i\cdot\underline{M}\hat{\underline{U}}_i}$

$\underline{M}\ddot{\underline{U}}+\underline{K}\underline{U}=\underline{F}(t) \rightarrow \hat{\underline{U}}_j\sum_i[\ddot{z}_i(t)\underline{M}\hat{\underline{U}}_i+z_i(t)\underline{K}\hat{\underline{U}}_i]=\hat{\underline{U}}_j\underline{F}(t)\ \forall_j \rightarrow \sum_i[\ddot{z}_i(t)\hat{\underline{U}}_j\underline{M}\hat{\underline{U}}_i+z_i(t)\hat{\underline{U}}_j\underline{K}\hat{\underline{U}}_i]=\hat{\underline{U}}_j\underline{F}(t)\ \forall_j$

$\rightarrow \ddot{z}_j(t)+\omega_j z_j(t)=\hat{\underline{U}}_j\underline{F}(t)\ \forall_j \rightarrow$ system of equations becomes decoupled! easier to solve. ($\hat{\underline{U}}_j$ must be scaled!)

Time-stepping

(dynamic mech. prob. with finite difference method in time)

general dynamic mech. prob.: $\underline{M}\ddot{\underline{U}}+\underline{C}\dot{\underline{U}}+\underline{F}_{int}(\underline{U})+\underline{F}_{ext}(t)=0$

($\underline{C}$: damping matrix. e.g.: $\underline{C}=\alpha\underline{M}+\beta\underline{K}$ — not same $\underline{C}$ as in Voigt notation!
α: damps low freq. ; β: damps high freq.)

- **explicit time integration** (1st/2nd order central diff. scheme)

$$\underline{M}\cdot\frac{\underline{U}^{\alpha+1}-2\underline{U}^{\alpha}+\underline{U}^{\alpha-1}}{\Delta t^2}+\underline{C}\cdot\frac{\underline{U}^{\alpha+1}-\underline{U}^{\alpha-1}}{2\Delta t^2}+\underline{F}_{int}(\underline{U}^{\alpha})-\underline{F}_{ext}(t^{\alpha})=0 \Rightarrow \left(\frac{\underline{M}}{\Delta t^2}+\frac{\underline{C}}{2\Delta t}\right)\cdot\underline{U}^{\alpha+1}=\frac{2\underline{M}}{\Delta t^2}\cdot\underline{U}^{\alpha}+\left(\frac{\underline{C}}{2\Delta t}-\frac{\underline{M}}{\Delta t^2}\right)\cdot\underline{U}^{\alpha-1}-\underline{F}_{int}(\underline{U}^{\alpha})+\underline{F}_{ext}(t^{\alpha})$$

stability (without proof): $\Delta t\leq\Delta t_{max}=2/\omega_{max}$ ← highest eigenfreq. ← $\omega_{max}\propto 1/$[smallest element size]

- **Newmark-β method** (implicit FDM + special scheme)

finite difference scheme: (β,γ: some constants)

$$\dot{\underline{U}}^{\alpha+1}=\dot{\underline{U}}^{\alpha}+\Delta t\left(\gamma\ddot{\underline{U}}^{\alpha+1}+(1-\gamma)\ddot{\underline{U}}^{\alpha}\right)$$
$$\underline{U}^{\alpha+1}=\underline{U}^{\alpha}+\Delta t\dot{\underline{U}}^{\alpha}+\frac{\Delta t^2}{2}\left(2\beta\ddot{\underline{U}}^{\alpha+1}+(1-2\beta)\ddot{\underline{U}}^{\alpha}\right)$$

$$\ddot{\underline{U}}^{\alpha+1}=\frac{1}{\beta\Delta t^2}\left(\underline{U}^{\alpha+1}-\underline{U}^{\alpha}-\Delta t\dot{\underline{U}}^{\alpha}\right)-\frac{1-2\beta}{2\beta}\ddot{\underline{U}}^{\alpha}$$
$$\dot{\underline{U}}^{\alpha+1}=\dot{\underline{U}}^{\alpha}+\Delta t(1-\gamma)\ddot{\underline{U}}^{\alpha}+\frac{\gamma}{\beta\Delta t}\left(\underline{U}^{\alpha+1}-\underline{U}^{\alpha}+\Delta t\dot{\underline{U}}^{\alpha}\right)-\gamma\Delta t\frac{1-2\beta}{2\beta}\ddot{\underline{U}}^{\alpha}$$

↳ plug into mech. prob.: $\underline{M}\ddot{\underline{U}}^{\alpha+1}+\underline{C}\dot{\underline{U}}^{\alpha+1}+\underline{F}_{int}(\underline{U}^{\alpha+1})-\underline{F}_{ext}(t^{\alpha+1})=0$

$$\underbrace{\left(\frac{1}{\beta\Delta t^2}\underline{M}+\frac{\gamma}{\beta\Delta t}\underline{C}\right)}_{\underline{M}^*}\underline{U}^{\alpha+1}+\underline{F}_{int}(\underline{U}^{\alpha+1})=\underbrace{\underline{F}_{ext}(t^{\alpha+1})+\underline{M}\left(\frac{1}{\beta\Delta t^2}\underline{U}^{\alpha}+\frac{1}{\beta\Delta t}\dot{\underline{U}}^{\alpha}+\left(\frac{1}{2\beta}-1\right)\ddot{\underline{U}}^{\alpha}\right)+\underline{C}\left(\frac{\gamma}{\beta\Delta t}\underline{U}^{\alpha}+\left(\frac{\gamma}{\beta}-1\right)\dot{\underline{U}}^{\alpha}+\Delta t\left(\frac{\gamma}{2\beta}-1\right)\ddot{\underline{U}}^{\alpha}\right)}_{\underline{F}^*_{ext}}$$

$$\underline{M}^*\,\underline{U}^{\alpha+1}+\underline{F}_{int}(\underline{U}^{\alpha+1})=\underline{F}^*_{ext}(t^{\alpha+1},\underline{U}^{\alpha},\dot{\underline{U}}^{\alpha},\ddot{\underline{U}}^{\alpha})$$

algorithm: ($\alpha=\alpha+1$)
- solve $\underline{M}^*\underline{U}^{\alpha+1}+\underline{F}_{int}=\underline{F}^*_{ext}$ for $\underline{U}^{\alpha+1}$ (linear elastic: $(\underline{M}^*+\underline{K})\underline{U}^{\alpha+1}=\underline{F}^*_{ext}$)
- compute $\ddot{\underline{U}}^{\alpha+1}$ and $\dot{\underline{U}}^{\alpha+1}$ with $\underline{U}^{\alpha+1}$

stability:
- if $\gamma\geq 1/2$ and $\beta\geq 1/2$: stable for any Δt
- if $\gamma\geq 1/2$ and $\beta<\gamma/2$: stable for $\Delta t_{max}\propto 1/\omega_{max}$
- else: unknown

- **general:**
  $\ddot{\underline{U}}(t)=\dots$ (for $t\in[t^{\alpha},t^{\alpha+1}]$)
  ↳ $\dot{\underline{U}}^{\alpha+1}=\dot{\underline{U}}^{\alpha}+\int_{t^\alpha}^{t^{\alpha+1}}\ddot{\underline{U}}(t)\,dt$
  ↳ $\underline{U}^{\alpha+1}=\underline{U}^{\alpha}+\int_{t^\alpha}^{t^{\alpha+1}}\dot{\underline{U}}(t)\,dt$
- **average acceleration:**
  $\ddot{\underline{U}}(t)=\frac{1}{2}(\ddot{\underline{U}}^{\alpha+1}+\ddot{\underline{U}}^{\alpha})$
  ↳ $\dot{\underline{U}}^{\alpha+1}=\dot{\underline{U}}^{\alpha}+\frac{\Delta t}{2}(\ddot{\underline{U}}^{\alpha+1}+\ddot{\underline{U}}^{\alpha})$
  ↳ $\underline{U}^{\alpha+1}=\underline{U}^{\alpha}+\Delta t\dot{\underline{U}}^{\alpha}+\frac{\Delta t^2}{4}(\ddot{\underline{U}}^{\alpha+1}+\ddot{\underline{U}}^{\alpha})$
  ↳ $\beta=1/4$, $\gamma=1/2$
- **linear acceleration:**
  $\ddot{\underline{U}}(t)=\ddot{\underline{U}}^{\alpha}+\frac{t-t^{\alpha}}{\Delta t}(\ddot{\underline{U}}^{\alpha+1}-\ddot{\underline{U}}^{\alpha})$
  ↳ $\dot{\underline{U}}^{\alpha+1}=\dot{\underline{U}}^{\alpha}+\frac{\Delta t}{2}(\ddot{\underline{U}}^{\alpha+1}+\ddot{\underline{U}}^{\alpha})$
  ↳ $\underline{U}^{\alpha+1}=\underline{U}^{\alpha}+\Delta t\dot{\underline{U}}^{\alpha}+\frac{\Delta t^2}{6}(\ddot{\underline{U}}^{\alpha+1}+2\ddot{\underline{U}}^{\alpha})$
  ↳ $\beta=1/6$, $\gamma=1/2$
- **explicit central diff.:**
  $\ddot{\underline{U}}(t)=\ddot{\underline{U}}^{\alpha}$
  ↳ $\dot{\underline{U}}^{\alpha+1}=\dot{\underline{U}}^{\alpha}+\frac{\Delta t}{2}(\ddot{\underline{U}}^{\alpha+1}+\ddot{\underline{U}}^{\alpha})$
  ↳ $\underline{U}^{\alpha+1}=\underline{U}^{\alpha}+\Delta t\dot{\underline{U}}^{\alpha}+\frac{\Delta t^2}{2}\ddot{\underline{U}}^{\alpha}$
  ↳ $\beta=0$, $\gamma=1/2$

- **HHT-α method** (variation of Newmark β)

same as Newmark-β, but instead of solving $\underline{M}\ddot{\underline{U}}^{\alpha+1}+\underline{C}\dot{\underline{U}}^{\alpha+1}+\underline{F}_{int}(\underline{U}^{\alpha+1})=\underline{F}_{ext}(t^{\alpha+1})$, solve the following: ($\alpha$: some const)

$$\underline{M}\ddot{\underline{U}}^{\alpha+1}+\underline{C}[(1-\alpha)\dot{\underline{U}}^{\alpha+1}+\alpha\dot{\underline{U}}^{\alpha}]+(1-\alpha)\underline{F}_{int}(\underline{U}^{\alpha+1})+\alpha\underline{F}_{int}(\underline{U}^{\alpha})=(1-\alpha)\underline{F}_{ext}(t^{\alpha+1})+\alpha\underline{F}_{ext}(t^{\alpha})$$

↳ substitute $\ddot{\underline{U}}^{\alpha+1},\dot{\underline{U}}^{\alpha+1}$ with Newmark-β formulas. → solve for $\underline{U}^{\alpha+1}$

stability: if $\alpha\leq 1/3$ and $\beta=(1+\alpha)^2/4$ and $\gamma=\alpha+1/2$ → stable for any Δt (increasing α removes high frequency noise!)

- **remarks**: in general: implicit schemes underpredict ω_i → use w/ consistent $\underline{M}$ (that overpredicts ω_i)
  explicit schemes overpredict ω_i → use w/ lumped $\underline{M}$ (that underpredicts ω_i)

Extensions

- **finite kinematics** (large deformations)

for NOT small deformations classical stress/strain does no longer work! instead use e.g.:

$\underline{x}(\underline{X},t)=\underline{X}+\underline{u}(\underline{X},t)$ ⟶ $\underline{F}=\partial\underline{x}/\partial\underline{X}$ $(F_{ij}=\partial x_i/\partial X_j)$ ⟶ $W=\text{func}(\underline{F})$ ⟶ $\underline{P}=\partial W/\partial\underline{F}$ $(P_{ij}=\partial W/\partial F_{ij})$

(x: deformed, X undeformed) (F: deformation gradient replaces ε) (W: strain energy density) (P: Piola-Kirchoff stress tensor replaces σ)

- **thermal problem**

strong form: $\kappa\nabla^2T+\rho s=\rho c_v\dot{T}$ subj. to: $T=\hat{T}$ on $\partial\Omega_D$; $-\kappa\nabla T\cdot n=\hat{q}$ on $\partial\Omega_N$; $T=T_0$ at $t=0$ (T: temperature, κ: therm. conductivity, s: heat source, c_v: heat capacity)

weak form (static): $\min I[T]=\int_\Omega \frac{\kappa}{2}|\nabla T|^2dV-\int_\Omega\rho sTdV+\int_\Omega\hat{q}TdS$ subj. to: $T\in\mathcal{U}=\{H^1(\Omega):T=\hat{T}\text{ on }\partial\Omega_D\}$

$$0=\delta I[T]=\int_\Omega\kappa T_{,i}\,\delta T_{,i}dV-\int_\Omega\rho s\delta TdV+\int_\Omega\hat{q}\delta TdS$$ subj. to: $T\in\mathcal{U}$, $\delta T\in\mathcal{U}_0=\{H^1(\Omega):T=0\text{ on }\partial\Omega_D\}$

$$\left(T(\underline{x})\approx T^h(\underline{x})=\Sigma_{a=1}^n T^aN^a(\underline{x})\ ;\ \delta T(\underline{x})\approx V^h(\underline{x})=\Sigma_{b=1}^n V^bN^b(\underline{x})\right)$$

$$\underbrace{T^a\left(\int_\Omega\kappa N^a_{,i}N^b_{,i}dV\right)}_{\underline{K}\,\underline{T}^h}V^b-\underbrace{\left(\int_\Omega sN^bdV-\int_{\partial\Omega_N}\hat{q}N^bdS\right)}_{\underline{Q}}V^b=\underbrace{0}_{=0}\quad\forall V^b$$

weak form (dyn.): $\ldots\to\underline{C}\,\dot{\underline{T}}^h+\underline{K}\,\underline{T}^h-\underline{Q}=0$ $(C^{ab}=\int_\Omega\rho c_vN^aN^bdV)$

Error sources (FEM)

- **Discretization error**: mesh only approximates real body Ω
- **Numerical integration error**: if numerical integration is used, error grows with element size and function "rougness"
- **Solution error**: when numerically solving a lin. eq. $\underline{K}\underline{U}=\underline{F}$ error is made if $\underline{K}$ is ill conditioned $(\|\underline{K}\|\cdot\|\underline{K}^{-1}\|=|\lambda_{max}/\lambda_{min}|\gg 1)$
- **Approximation error**: shape func. + coefficients can only approximate real function
- **Truncation error**: computers can only store numbers with finite precision
- **Modelling error**: model vs. reality missmatch

Adaptive mesh refinement

(example with CST elements)

1) **smoothen strains** $=\varepsilon^*$: e.g. assign "nodal strain" to all nodes by averaging strains of adjacent element weighted by volume and interpolate these nodal strains over the elements
2) **ZZ error estimation**: compute error measure as integral over element of difference of smooth ε^* and normal ε squared.
3) **refine mesh**: if ZZ error of an element is over a treshold split that element (e.g.: longest edge bisection)

Common pitfalls

- **poor mesh quality:**
  - appropriate mesh size for scale of expected stress changes
  - element size changes should be gradual
  - there must not be any "hanging" nodes (nodes that could move into an adjacent element)
  - avoid element distortions (elements should be close to reference shape)
- **wrong element type:**
  - completeness order of element must be sufficient
  - numerical integration must have full integration
- **linear vs. nonlinear:** if deformations are large, dont use linear analysis (this course)
- **boundary conditions:**
  - fix rigid body modes (don't allow entire body to rotate/translate)
  - avoid point loads as there $\sigma \to \infty$ and FEA never converges
  - dont exploit symmetries in vibration analysis
- $\boxed{\text{do convergeance study!}}$

Einstein Notation

repeating index: sum over dimensions

index after comma: partial derivative in that dimension

v,w: vectors ; T,S: matrices ; ϕ, ψ: scalar ; e_i: basis vectors ; ϵ_{ij}: Levi-Civita symbol

$\boldsymbol{v}$	$[\boldsymbol{v}]_i = v_i$
$\boldsymbol{v} \cdot \boldsymbol{w}$	$v_i w_i$
$\boldsymbol{v} \otimes \boldsymbol{w}$	$[\boldsymbol{v} \otimes \boldsymbol{w}]_{ij} = v_i w_j$
$\boldsymbol{v} \times \boldsymbol{w}$	$[\boldsymbol{v} \times \boldsymbol{w}]_k = v_i w_j \epsilon_{ijk}$
$\boldsymbol{v} \cdot (\boldsymbol{w} \times \boldsymbol{x})$	$v_i w_j x_k \epsilon_{ijk}$
$\boldsymbol{T}$	$[\boldsymbol{T}]_{ij} = T_{ij}$
$\boldsymbol{T}^\mathsf{T}$	$[\boldsymbol{T}^\mathsf{T}]_{ij} = T_{ji}$
$\boldsymbol{T}\boldsymbol{v}$	$[\boldsymbol{T}\boldsymbol{v}]_i = T_{ij} v_j$
$\boldsymbol{T}^\mathsf{T}\boldsymbol{v}$	$[\boldsymbol{T}^\mathsf{T}\boldsymbol{v}]_i = T_{ji} v_j$
$\boldsymbol{v} \cdot \boldsymbol{T}\boldsymbol{w}$	$v_i T_{ij} w_j$
$\boldsymbol{T}\boldsymbol{S}$	$[\boldsymbol{T}\boldsymbol{S}]_{ik} = T_{ij} S_{jk}$
$\boldsymbol{T}^\mathsf{T}\boldsymbol{S}$	$[\boldsymbol{T}^\mathsf{T}\boldsymbol{S}]_{ik} = T_{ji} S_{jk}$
$\boldsymbol{T}\boldsymbol{S}^\mathsf{T}$	$[\boldsymbol{T}\boldsymbol{S}^\mathsf{T}]_{ik} = T_{ij} S_{kj}$
$\boldsymbol{T} \cdot \boldsymbol{S}$	$T_{ij} S_{ij}$
$\boldsymbol{T} \times \boldsymbol{v}$	$[\boldsymbol{T} \times \boldsymbol{v}]_{il} = T_{ij} v_k \epsilon_{jkl}$
$\boldsymbol{T} \otimes \boldsymbol{v}$	$[\boldsymbol{T} \otimes \boldsymbol{v}]_{ijk} = T_{ij} v_k$
$\boldsymbol{v} \times \boldsymbol{T}$	$[\boldsymbol{v} \times \boldsymbol{T}]_{il} = v_j T_{ki} \epsilon_{jkl}$
$\boldsymbol{v} \otimes \boldsymbol{T}$	$[\boldsymbol{v} \otimes \boldsymbol{T}]_{ijk} = v_i T_{jk}$
$\mathrm{tr}\, \boldsymbol{T}$	T_{ii}
$\det(\boldsymbol{T})$	$\epsilon_{ijk} T_{i1} T_{j2} T_{k3}$
$\mathrm{grad}\, \phi$	$[\mathrm{grad}\, \phi]_i = \phi_{,i}$
$\mathrm{grad}\, \boldsymbol{v}$	$[\mathrm{grad}\, \boldsymbol{v}]_{ij} = v_{i,j}$
$\mathrm{grad}\, \boldsymbol{T}$	$[\mathrm{grad}\, \boldsymbol{T}]_{ijk} = T_{ij,k}$
$\mathrm{div}\, \boldsymbol{v}$	$v_{i,i}$
$\mathrm{div}\, \boldsymbol{T}$	$[\mathrm{div}\, \boldsymbol{T}]_i = T_{ij,j}$
$\mathrm{curl}\, \boldsymbol{v}$	$[\mathrm{curl}\, \boldsymbol{v}]_k = v_{i,j} \epsilon_{jik}$
$\mathrm{curl}\, \boldsymbol{T}$	$[\mathrm{curl}\, \boldsymbol{T}]_{ij} = T_{ik,l} \epsilon_{lkj}$
$\mathrm{div}(\mathrm{grad}\, \phi)$	$[\mathrm{div}(\mathrm{grad}\, \phi)]_i = \phi_{,kk}$
$\mathrm{div}(\mathrm{grad}\, \boldsymbol{v})$	$[\mathrm{div}(\mathrm{grad}\, \boldsymbol{v})]_i = v_{i,kk}$

divergence theorem:

$$\int_\Omega \mathrm{div}(\underline{v})\, dV = \int_{\partial\Omega} \underline{v} \cdot \underline{n}\, dS \leftrightarrow \int_\Omega v_{i,i}\, dV = \int_{\partial\Omega} v_i n_i\, dS$$

$$\int_\Omega \mathrm{div}(\underline{\underline{T}})\, dV = \int_{\partial\Omega} \underline{\underline{T}} \cdot \underline{n}\, dS \leftrightarrow \int_\Omega T_{ij,j}\, dV = \int_{\partial\Omega} T_{ij} n_j\, dS$$

$$\int_\Omega \mathrm{div}(\underline{v}\phi)\, dV = \int_\Omega \mathrm{div}(\underline{v})\phi + \underline{v}\,\mathrm{grad}(\phi)\, dV = \int_{\partial\Omega} \underline{v}\phi\, \underline{n}\, dS$$

$$\int_\Omega (v_i \phi)_{,i}\, dV = \int_\Omega v_{i,i} \cdot \phi + v_i \cdot \phi_{,i}\, dV = \int_{\partial\Omega} v_i \phi\, n_i\, dS$$

$$\int_\Omega \mathrm{div}(\underline{\underline{T}}\underline{v})\, dV = \int_\Omega \mathrm{div}(\underline{\underline{T}})\underline{v} + \underline{\underline{T}}\,\mathrm{grad}(\underline{v})\, dV = \int_{\partial\Omega} \underline{\underline{T}}\,\underline{v}\,\underline{n}\, dS$$

$$\int_\Omega (T_{ij} v_j)_{,i}\, dV = \int_\Omega T_{ij,j} v_i + T_{ij} v_{i,j}\, dV = \int_{\partial\Omega} T_{ij} n_j v_i\, dS$$

divergence product rules:

$$\mathrm{div}(\underline{v} \cdot \phi) = \mathrm{div}(\underline{v}) \cdot \phi + \underline{v} \cdot \mathrm{grad}(\phi) \leftrightarrow (v_i \phi)_{,i} = v_{i,i} \phi + v_i \cdot \phi_{,i}$$

$$\mathrm{div}(\underline{\underline{T}} \cdot \underline{v}) = \mathrm{div}(\underline{\underline{T}}) \cdot \underline{v} + \underline{\underline{T}} \cdot \mathrm{grad}(\underline{v}) \leftrightarrow (T_{ij} v_j)_{,i} = T_{ij,j} v_i + T_{ij} v_{i,j}$$

integration by part:

$$\int_a^b \phi\, \psi_{,x}\, dx = [\phi\psi]_a^b - \int_a^b \phi_{,x} \psi\, dx$$

1D

$$\int_a^b \phi \frac{d\psi}{dx}\, dx = [\phi\psi]_a^b - \int_a^b \frac{d\phi}{dx} \psi\, dx$$

$$\int_\Omega \psi \cdot \phi_{,i}\, dV = \int_{\partial\Omega} \psi \cdot \phi \cdot n\, dS - \int_\Omega \psi_{,i} \cdot \phi\, dV$$

$$\int_\Omega \phi\, \mathrm{div}(\underline{v})\, dV = \int_{\partial\Omega} \phi\, \underline{v}\, \underline{n}\, dS - \int_\Omega \mathrm{grad}(\phi)\, \underline{v}\, dV$$

$$\int_\Omega \phi\, v_{i,i}\, dV = \int_{\partial\Omega} \phi\, v_i n_i\, dS - \int_\Omega \phi_{,i} v_i\, dV$$

$$\int_\Omega \underline{v}\, \mathrm{div}(\underline{\underline{T}})\, dV = \int_{\partial\Omega} \underline{v}\, \underline{\underline{T}}\, \underline{n}\, dS - \int_\Omega \mathrm{grad}(\underline{v})\, \underline{\underline{T}}\, dV$$

$$\int_\Omega v_i T_{ij,j}\, dV = \int_{\partial\Omega} v_i T_{ij} n_j\, dS - \int_\Omega v_{i,j} T_{ij}\, dV$$