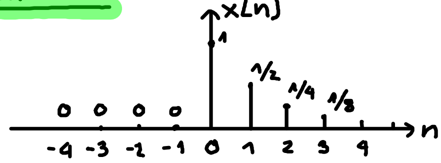
# Signals and Systems

## Notation

| | |
|---|---|
| $x$ | signal |
| $x[n]$ | signal value at DT $n$. |
| $x(t)$ | signal value at CT $t$. |
| $\{x[n]\}$ | entire DT signal |
| $\{x(t)\}$ | entire CT signal |

## Representation

Graph:

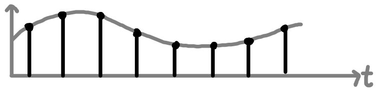



Rule: $x[n] := \begin{cases} (1/2)^n & n \geq 0 \\ 0 & n < 0 \end{cases}$

Sequence: $\{x[n]\} = \{\dots, 0, \underset{\uparrow}{1}, 1/2, 1/4, \dots\}$

## CT → DT (uniform samp.)

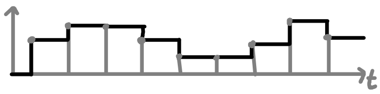
$x[n] = x(n \cdot T_s)$




## DT → CT (0-Order Hold)

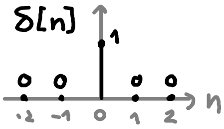
$x(t) = x[n]$

$(nT_s \leq t < (n+1)T_s)$



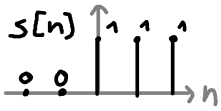

## Common Signals

Dirac delta:




$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$

Heaviside:




$s[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$

## DT Systems


$\{y[n]\} = G\{u[n]\}$

**Impulse Response**: $\{h[n]\} = G\{\delta[n]\}$

**General Response**: $\{y[n]\} = G\{u[n]\} = \underbrace{\sum_{k=-\infty}^{\infty} u[k] \cdot \{h[n-k]\}}$

Convolution: $\{u[n]\} * \{h[n]\}$

**Step Response**: $\{r[n]\} = \{s[n]\} * \{h[n]\} = \{\sum_{k=-\infty}^{n} h[k]\}$

**CT → DT Statespace**

$$\begin{aligned} \dot{x}(t) &= A_c x(t) + B_c u(t) \\ y(t) &= C_c x(t) + D_c u(t) \end{aligned} \Rightarrow \begin{aligned} x[n+1] &= A_d x[n] + B_d [n] \\ y[n] &= C_d x[n] + D_d [n] \end{aligned} \qquad (x(0) = x[0])$$

$$\left[\begin{array}{c:c} A_d & B_d \\ 0 & 0 \end{array}\right] = e^{\wedge}\left(\left[\begin{array}{c:c} A_c & B_c \\ 0 & 0 \end{array}\right] \cdot T_s\right) = \sum_{k=0}^{\infty} \left[\begin{array}{c:c} A_c & B_c \\ 0 & 0 \end{array}\right]^k \cdot \frac{T_s^k}{k!} \qquad \begin{aligned} C_d &= C_c \\ D_d &= D_c \end{aligned}$$

**Memoryless**: $\{y[n]\} = G\{\cancel{u[n-k]}, u[n], \cancel{u[n+k]}\}$

**Linear**: $G\{\alpha_1 u_1[n] + \alpha_2 u_2[n]\} = \alpha_1 G\{u_1[n]\} + \alpha_2 G\{u_2[n]\}$

**Causal**: $\{y[n]\} = G\{u[n-k], u[n], \cancel{u[n+k]}\}$

**T-invar.**: $u_2[n] = u_1[n-k] \longrightarrow y_2[n] = y_1[n-k]$

**LTI Causal**: $h[n] = 0 \ \forall n < 0$

**Finite Impulse Response FIR**: $\exists N$ with $h[n] = 0 \ \forall n \geq N$

**LTI Stable**: $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$

**Infinite Impulse Response IIR**: $\nexists N$ with $h[n] = 0 \ \forall n \geq N$

LCCDE: (?) Lecture 2.

**Complex exp. sequences** $x[n] = z^n = (|z| e^{i\Omega})^n = |z|^n (\cos(\Omega n) + i \sin(\Omega n))$ $\quad (z \in \mathbb{C} \ ; \ \Omega = \omega T_s)$

**Periodic signals** $x[n] = x[n + m \cdot N]$ (N: fundamental period, $T_s$ sampling t)

if $x(t)$ is $T$ periodic $\Rightarrow$ $x[n]$ periodic iff: $T_s/T = m/N$

**z-Transform** $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$

$a_1\{x_1[n]\} + a_2\{x_2[n]\} \longleftrightarrow a_1 X_1(z) + a_2 X_2(z)$

$\{x[n+k]\} \longleftrightarrow z^k \cdot X(z)$

$\{x_1[n]\} * \{x_2[n]\} \longleftrightarrow X_1(z) \cdot X_2(z)$

$\{\sum_{k=-\infty}^{n} x[k]\} \longleftrightarrow z/_{z-1} \cdot X(z)$

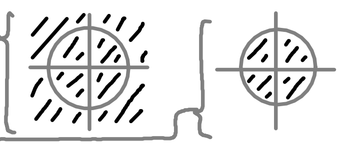
LTI General: $H(z) = Y(z)/U(z)$

LCCDE: $H(z) = \dfrac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \dots + a_N z^{-N}}$

State Space: $H(z) = C_d (zI - A_d)^{-1} B_d + D_d$

**z-Dom. Stability**: all poles $p$ of $H(z)$ with $|p| \neq 1$

**z-Dom Stable+Causal**: all poles $p$ of $H(z)$ with $|p| < 1$



## DT Fourier transform ($\Omega$-Transform) (only stable $x[n]$!)

$$X(\Omega) = \mathcal{F}(x[n])(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-i\Omega n} = X(z)\big|_{z=e^{i\Omega}}$$

$$\{x[n]\} = \mathcal{F}^{-1}(X(\Omega))(n) = \tfrac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega)\, e^{i\Omega n}\, d\Omega$$

$\left( -\pi < \Omega < \pi \quad X(\Omega) = |X(\Omega)| \cdot e^{\theta_X(\Omega)} \right)$

$a_1\{x_1[n]\} + a_2\{x_2[n]\} \rightarrow a_1 X_1(\Omega) + a_2 X_2(\Omega)$

$\{x_1[n]\} * \{x_2[n]\} \rightarrow X_1(\Omega) \cdot X_2(\Omega)$

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\Omega)|^2\, d\Omega$$

LTI General: $Y(\Omega) = U(\Omega) \cdot H(\Omega)$ $\left\{ \begin{array}{l} |Y(\Omega)| = |U(\Omega)| \cdot |H(\Omega)| \\ + \\ \theta_Y(\Omega) = \theta_U(\Omega) + \theta_H(\Omega) \end{array} \right.$

LCCDE: $X(\Omega) = \dfrac{\sum_{k=0}^{M} b_k \cdot e^{-i\Omega k}}{\sum_{k=0}^{N} a_k \cdot e^{-i\Omega k}}$

$\{u[n]\} = \{z^n\} = \{e^{i\Omega_0 n}\} \overset{Re}{=} \cos(\Omega_0 n)$

$y[n] = |H(\Omega_0)| \cdot e^{i(\Omega_0 n + \theta_H(\Omega_0))} \overset{Re}{=} |H(\Omega_0)| \cdot \cos(\Omega_0 n + \theta_H(\Omega_0))$

## Discrete Fourier series (k-Transform) (only periodic $x[n]$!)

$$X[k] = \mathcal{F}_S(x[n])[k] = \sum_{n=0}^{N-1} x[n]\, e^{-ik2\pi n/N}$$

$$x[n] = \mathcal{F}_S^{-1}(X[k])[n] = 1/N \cdot \sum_{k=0}^{N-1} X[k]\, e^{ik2\pi n/N}$$

$\left( \begin{array}{c} x[n] = x[n+N] \\ \updownarrow \\ X[k] = X[k+N] \end{array} \right)$

$a_1\{x_1[n]\} + a_2\{x_2[n]\} \rightarrow a_1 X_1[k] + a_2 X_2[k]$

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2$$

## Discrete Fourier Transform (same as series, but $x[n]$ finite)

- $x[n] \in \mathbb{R} \rightarrow X[N-k] = \mathrm{Conj}(X[k])$
- $H[k] = H(z)\big|_{z=e^{ik2\pi/N}} = H(\Omega)\big|_{\Omega = k2\pi/N}$
- $X(\Omega) = \frac{2\pi}{N} \sum_{k=0}^{N-1} X[k]\, \delta(\Omega - k\frac{2\pi}{N})$

## Noise

$p(x)$: probability density $\left( \int_{-\infty}^{\infty} p(x)dx \overset{!}{=} 1 \text{ and } p(x) \overset{!}{\geq} 0\ \forall x \right)$

$E(x)$: expected value $\left( E(x) = \int_{-\infty}^{\infty} x \cdot p(x)\, dx \right)$

$Var(x)$: variance $\left( Var(x) = E((x - E(x))^2) \right)$

White Noise $\left\{ \begin{array}{l} E(x[n]) = 0, \quad E(x[n] \cdot x[l]) = \begin{cases} 1 & n=l \\ 0 & \text{else} \end{cases} \\ E(X[k]) = 0, \quad E(X^*[k] X[q]) = \begin{cases} N & k=q \\ 0 & \text{else} \end{cases} \end{array} \right.$

## Non-causal filter

General: $\{u[n]\} \xrightarrow{DFT} \{U[k]\} \xrightarrow{filt.} \{Y[k]\} \xrightarrow{DFT} \{y[n]\}$

from causal: $H_1(z) \rightarrow H_2(z) = H_1(z) H_1(z^{-1})$ $\left( \begin{array}{l} \text{better non-causal} \\ \rightarrow \measuredangle H = 0 \end{array} \right)$

Median: $y[n] = \text{median}(u[n - M/2], \dots, u[n], \dots, u[n + M/2])$

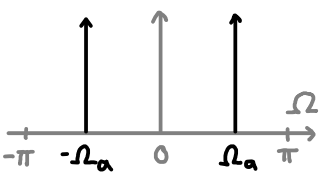
## Transforms of Cos / Sin

$\omega_a = 2\pi f = 2\pi/T$ ; $\Omega_a = \omega_a T_s$ ; $k_a = \Omega_a \frac{N}{2\pi}$

$x(t) = a \cdot \cos(\omega_a \cdot t)$

$x[n] = a \cdot \cos(\omega_a T_s n) = a \cdot \cos(\Omega_a n)$

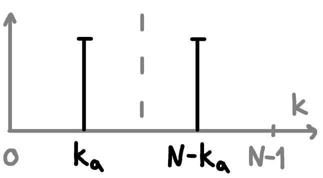
$X(\Omega)$:




$X(\Omega_a) = \infty$

$X(-\Omega_a) = \infty$

$X[k]$:




$X[k_a] = \frac{a \cdot N}{2}$

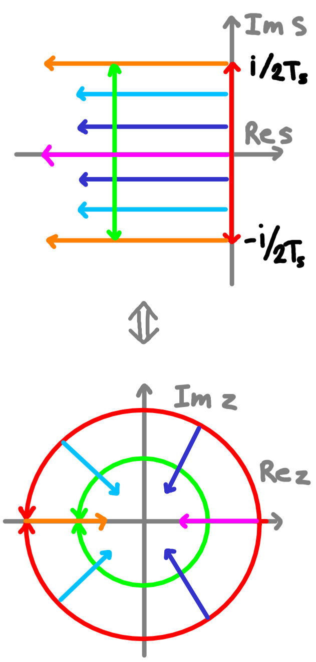
$X[N-k_a] = \frac{a \cdot N}{2}$

$X[0] = aN$

$X[N/2] = aN$

cos → sin: $X[k_a] = -i\frac{aN}{2}$, $X[N-k_a] = i\frac{aN}{2}$

## S-Z Mapping




## CT→DT Transform $\boxed{s = i\omega,\ z = e^{i\Omega}}$

direct: $z = e^{sT_s}$ ; $s = \ln(z)/T_s$

$\Omega = \omega T_s$

bilinear = Tustin: $z = \frac{1 + sT_s/2}{1 - sT_s/2}$ ; $s = \frac{2}{T_s} \frac{z-1}{z+1}$

$\Omega = 2 \cdot \tan^{-1}(\omega \frac{T_s}{2})$

| $\omega$: | 0 | $2/T_s$ | $\infty$ |
|---|---|---|---|
| $\Omega$: | 0 | $\pi/2$ | $\pi$ |

e. fwd.: $z = sT_s + 1$ ; $s = \frac{z-1}{T_s}$

e. bwd.: $z = 1/(1 - sT_s)$ ; $s = \frac{1 - z^{-1}}{T_s}$

## FIR Filter (Casual)

General: $y[n]=\sum_{k=0}^{M-1} b_k \cdot u[n-k]$ $\quad \begin{cases} h=\{b_0,b_1,\dots,b_{M-1}\} \\ H(z)=\sum_{k=0}^{M-1} b_k z^{-k} \; ; \; H(\Omega)=\sum_{k=0}^{M-1} b_k e^{-i\Omega k} \end{cases}$ $\quad \begin{cases} M: \text{filter lenght} \\ M-1: \text{filter order} \end{cases}$

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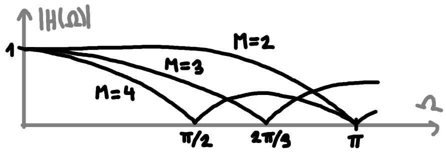
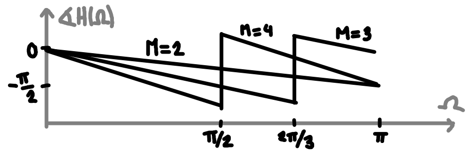
**Moving Average Filter:** $y[n]=\frac{1}{M}\sum_{k=0}^{M-1} u[n-k]$ ; $H(\Omega)=\frac{1}{M}\sum_{k=0}^{M-1} e^{-i\Omega k}=\frac{1}{M}\frac{(1-e^{\wedge}i\Omega M)}{(1-e^{\wedge}i\Omega)}$

Zeros: $\Omega=2\pi k/M$ $(k\in\mathbb{N})$

Aprox (up to first 0): $|H(\Omega)|\approx|\mathrm{sinc}(\frac{\Omega M}{2})|$ ; $\sphericalangle H(\Omega)\approx-\frac{\Omega(M-1)}{2}$

Fast MA filter: $y[n]=y[n-1]+\frac{u[n]-u[n-M]}{M}$ (unstable if num error)

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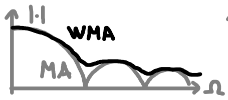
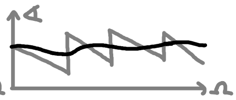
**Weighted MA filter:** $y[n]=\frac{1}{S}\sum_{k=0}^{M-1} w_k \cdot u[n-k]$ $(w_k=M-k \; ; \; S=M\cdot(M+1)/2)$

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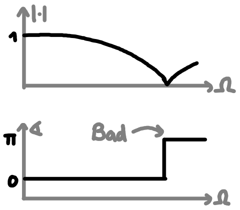
**Non-Causal MA**

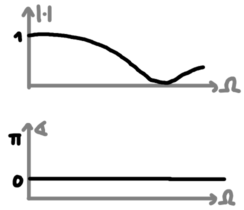
$y[n]=u[n-\frac{M-1}{2}]+\dots+u[n]+\dots+u[n+\frac{M-1}{2}]$

$H(\Omega)=\frac{1}{M}\sum_{k=0}^{M-1} e^{-i\Omega(k-\frac{M-1}{2})}=e^{i\Omega\frac{M-1}{2}}\cdot H_{MA}(\Omega)$




**Non-Causal WMA**

Analog to MA→non C. MA

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**Differentiation using FIR filters** (casual = Euler backward ; anti-casual = Euler forward ; non-causal = ? )

| | | | | |
|---|---|---|---|---|
| Casual : | $y(t)\approx u(t)-u(t-\tau)/\tau$ | $y[n]=\frac{1}{T_s}(u[n]-u[n-1])$ | $H(z)=1-z^{-1}/T_s$ | $H(\Omega)=2i\frac{e^{-i\Omega/2}}{T_s}\cdot\sin(\Omega/2)$ |
| Anti-Causal : | $y(t)\approx u(t+\tau)-u(t)/\tau$ | $y[n]=\frac{1}{T_s}(u[n+1]-u[n])$ | $H(z)=z-1/T_s$ | $H(\Omega)=2i\frac{e^{i\Omega/2}}{T_s}\cdot\sin(\Omega/2)$ |
| Non-Causal : | $y(t)\approx u(t+\tau)-u(t-\tau)/\tau$ | $y[n]=\frac{1}{2T_s}(u[n+1]-u[n-1])$ | $H(z)=z-z^{-1}/2T_s$ | $H(\Omega)=\frac{i}{T_s}\cdot\sin(\Omega)$ |

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## IIR Filters (Casual)

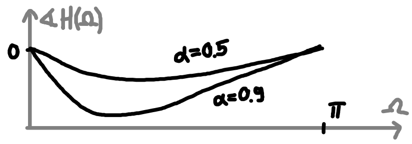
General: $y[n]=\sum_{k=0}^{M-1} b_k\cdot u[n-k]-\sum_{k=1}^{N-1} a_k\cdot y[n-k]$ $\quad \begin{cases} H(z)=\sum_{k=0}^{M-1} b_k z^{-k} / (1+\sum_{k=0}^{N-1} a_k z^{-k}) \\ H(\Omega)=\sum_{k=0}^{M-1} b_k e^{-i\Omega k}/(1+\sum_{k=0}^{N-1} a_k e^{-i\Omega k}) \end{cases}$ $\{\max(M-1,N-1)$ : filter order

**First-Order Low-Pass:** $y[n]=\alpha\cdot y[n-1]+(1-\alpha)\,u[n]$

$H(z)=1-\alpha/1-\alpha z^{-1}$ ; $H(\Omega)=1-\alpha/1-\alpha e^{-i\Omega}$

decay time: $y[0]=1$ and $u[n]=0 \rightarrow \alpha=e^{-T_s/T_0}$ ($T_0$: time to decay to $\frac{1}{2}$)

$\alpha=e^{-\Omega_c}$ $\quad T_0=\tau=1/\omega_c$


**Butterworth filter:** $R(\omega)=1/\sqrt{1+\omega^{2K}}$ (CT function ; $\omega_c=1\,\mathrm{rad}/s$)

transfer func.: $\xrightarrow{|H(\omega i)|=R(\omega)}$ $H(s)=1/\prod_{k=1}^{K}(s-s_k)$ $(s_k=e^{\wedge}\frac{i(2k+K-1)\pi}{2K})$

$\omega_c\neq1$ : $H^*(s)=H(s)|_{s=s/\omega_c}$

DT filter: $\xrightarrow{\text{Tustin}}$ $H(z)=H(s)|_{s=\frac{2}{T_s}\cdot\frac{z-1}{z+1}}$

**First-Order Low-Pass** v2

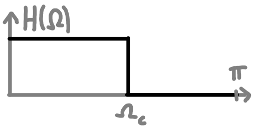
$H(s)=\omega_c/s+\omega_c$ $\quad \}$ bilinear

$H(z)=\frac{(1-\alpha)(1+z^{-1})}{2(1-\alpha z^{-1})}$

$\alpha=\frac{2-T_s\omega_c}{2+T_s\omega_c}\overset{?}{=}\frac{1-\tan(\Omega_c/2)}{1+\tan(\Omega_c/2)}$

# Applied Filters

**Lowpass:**



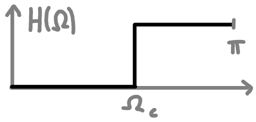

CT: $1^{st}$ $H_{LP}(s) = \omega_c / s+\omega_c$

$2^{nd}$ $H_{LP}(s) = \omega_c^2 / s^2+\sqrt{2}\,\omega_c s+\omega_c^2$

DT: $n^{th}$ $H_{LP}(z) = H_{LP}(s)\big|_{CT\to DT}$

$1^{st}$ $H_{LP}(z)$ ← see $1^{st}$ LP v2

**Highpass:**



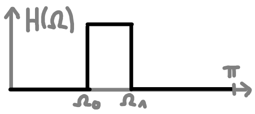

CT: $n^{th}$ $H_{HP}(s) = H_{LP}(s)\big|_{s\to s^{-1};\ \omega_c\to\omega_c^{-1}}$

$1^{st}$ $H_{HP}(s) = s/s+\omega_c$

DT: $n^{th}$ $H_{HP}(z) = H_{LP}(z)\big|_{\Omega_c\to\Omega_c-\pi;\ z\to -z}$

$n^{th}$ $H_{HP}(z) = H_{HP}(s)\big|_{CT\to DT}$

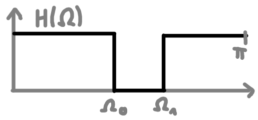
**Bandpass:**




CT: $H_{BP}(s) = H_{LP}(s)\big|_{\omega_c=\omega_1}\cdot H_{HP}(s)\big|_{\omega_c=\omega_0}$ (only $\omega_1/\omega_0 \gg 1$ !)

$H_{BP}(s) = H_{LP}\left(\frac{s^2+\omega_1\cdot\omega_0}{s}\right)\Big|_{\omega_c=\omega_1-\omega_0}$

**Bandstop:**




CT: $H_{BS}(s) = H_{LP}(s)\big|_{\omega_c=\omega_0} + H_{HP}(s)\big|_{\omega_c=\omega_1}$ (only $\omega_1/\omega_0 \gg 1$ !) ($\omega_1=\omega_0=\omega_c$ ↳ Notch-filt.)

$H_{BS}(s) = H_{HP}\left(\frac{s^2+\omega_1\cdot\omega_0}{s}\right)\Big|_{\omega_c=\omega_1-\omega_0}$

**Pre-Warping** if bilinear CT-DT is used: $\overline{\omega}_c = \frac{2}{T_s}\tan\left(\omega_c \frac{T_s}{2}\right)$ (remove Warping $\Omega=\omega T_s \approx 2\tan^{-1}\left(\omega\frac{T_s}{2}\right)=\Omega$)

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# System Identification


$Y_m = G\cdot U_e + (G\cdot U_d + Y_m)$

- **Impulse Response:** $\{U_e[n]\} = \{\delta[n]\}$ $\to \hat{H}(\Omega_k) = Y_m[k] = \sum_{n=0}^{N-1} e^{-i\frac{2\pi k}{N}n}$ $\left(\Omega_k = \frac{2\pi k}{N}\right)$ ($N\uparrow \to$ noise$\uparrow$ ; $N\downarrow \to \Delta\Omega\uparrow$)

- **Closed Loop:** (if G unstable: add Controler C in feedback → get $H^*$ → remove C from $H^*$ → H)

- **Sinus Response:** $U_e[n] = A\cos(\Omega_L n)$ ($n = 0,\dots,N_T+N-1$) [$N_T$: transient] $\left(\Omega_L = \frac{2\pi L}{N}\right.$ and $\left.L\in[0, N/2]\in\mathbb{N}\right)$

↳ $\hat{H}(\Omega_L) = Y_m(\Omega_L)/U_e(\Omega_L) = \sum_{n=N_T}^{N_T+N-1} Y_m[n]\cdot e^{-i\Omega_L n} \,/\, (N\cdot A/2)$ ($N\uparrow \to$ noise$\downarrow$, $\Delta\Omega\downarrow$ ☺)

↳ get $a_n, b_n$:

$$H(\Omega) = \frac{\sum_{k=0}^{B-1} b_k e^{-i\Omega k}}{1+\sum_{k=1}^{A-1} a_k e^{-i\Omega k}} = \frac{\sum_{k=0}^{B-1} b_k z^{-k}}{1+\sum_{k=1}^{A-1} a_k z^{-k}} \qquad \begin{pmatrix} a_1,\dots,a_{A-1} \\ b_0,\dots,b_{B-1}\end{pmatrix}$$

$$F = \begin{array}{c} \overbrace{\hphantom{R_L\cos(\phi_L-\Omega_L)\ \dots\ R_L\cos(\phi_L-(A-1)\Omega_L)}}^{(A-1)}\ \overbrace{\hphantom{-1\ \ -\cos(\Omega_L)\ \dots\ -\cos((B-1)\Omega_L)}}^{B} \\ \left(\begin{array}{cccccc} R_L\cos(\phi_L-\Omega_L) & \dots & R_L\cos(\phi_L-(A-1)\Omega_L) & -1 & -\cos(\Omega_L) & \dots\ -\cos((B-1)\Omega_L) \\ \big\downarrow_{L=0,\dots,L} & & \big\downarrow_{L=0,\dots,L} & & \big\downarrow_{L=0,\dots,L} & \big\downarrow_{L=0,\dots,L} \\ R_L\sin(\phi_L-\Omega_L) & \dots & R_L\sin(\phi_L-(A-1)\Omega_L) & 0 & \sin(\Omega_L) & \dots\ \sin((B-1)\Omega_L) \\ \big\downarrow_{L=0,\dots,L} & & \big\downarrow_{L=0,\dots,L} & & \big\downarrow_{L=0,\dots,L} & \big\downarrow_{L=0,\dots,L} \end{array}\right) \end{array} \Bigg] 2L \qquad \begin{cases} R_L = |\hat{H}(\Omega_L)| \\ \phi_L = \measuredangle(\hat{H}(\Omega_L)) \end{cases}$$

$$G = \begin{pmatrix} -R_L\cos(\phi_L) \\ \downarrow L \\ -R_L\sin(\phi_L) \\ \downarrow L \end{pmatrix} \Big] 2L \qquad \Theta = (a_1,\dots,a_{A-1}, b_0,\dots,b_{B-1})^T$$

$$\Rightarrow \Theta = (F^T F)^{-1} F^T G$$

possibly
$F \to -F$ :(
$G \to -G$