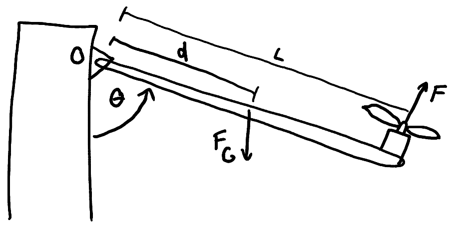
Cascaded Control

System Model:



$u: F \qquad x_1: \theta \qquad x_2: \dot{\theta} \qquad y: \theta$

$$J_0 \cdot \ddot{\theta} = \sum_i M_{0,i} = L \cdot F(t) - d \cdot m \cdot g \cdot \sin(\theta(t))$$

$$\rightarrow \ddot{\theta}(t) = 1/J_0 \left(L \cdot F(t) - d \cdot m \cdot g \sin(\theta(t)) \right)$$

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \dot{\theta} \\ \ddot{\theta} \end{pmatrix} = f(x,u) = \begin{pmatrix} \dot{\theta} \\ \frac{L}{J_0} \cdot F(t) - \frac{dmg}{J_0} \cdot \sin(\theta(t)) \end{pmatrix} = \begin{pmatrix} x_2 \\ \frac{L}{J_0} \cdot u - \frac{dmg}{J_0} \cdot \sin(x_1) \end{pmatrix}$$

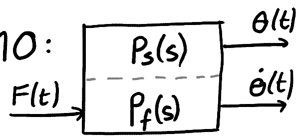
operating point: $\theta_{op} = \pi/2$, $\dot{\theta}_{op} = 0 \rightarrow F_{op} = \frac{dmg}{J_0}$

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} \Bigg|_{op} = \begin{pmatrix} 0 & 1 \\ -\frac{dmg}{J_0} \cos(\theta_{op}) & 0 \end{pmatrix} \Bigg|_{op} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{pmatrix} = \begin{pmatrix} 0 \\ L/J_0 \end{pmatrix} \qquad \begin{array}{l} C = (1 \quad 0) \\ D = 0 \end{array}$$

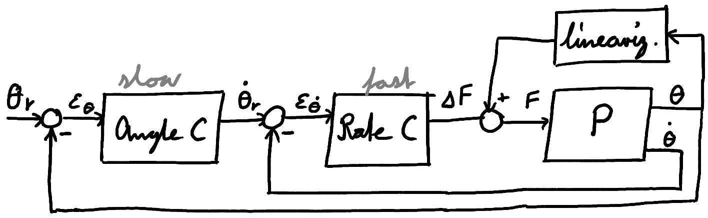
$$\rightarrow P(s) = \left(c \cdot (s \cdot I - A)^{-1} \cdot b + d \right) = \frac{L/J_0}{s^2}$$

Control Design:

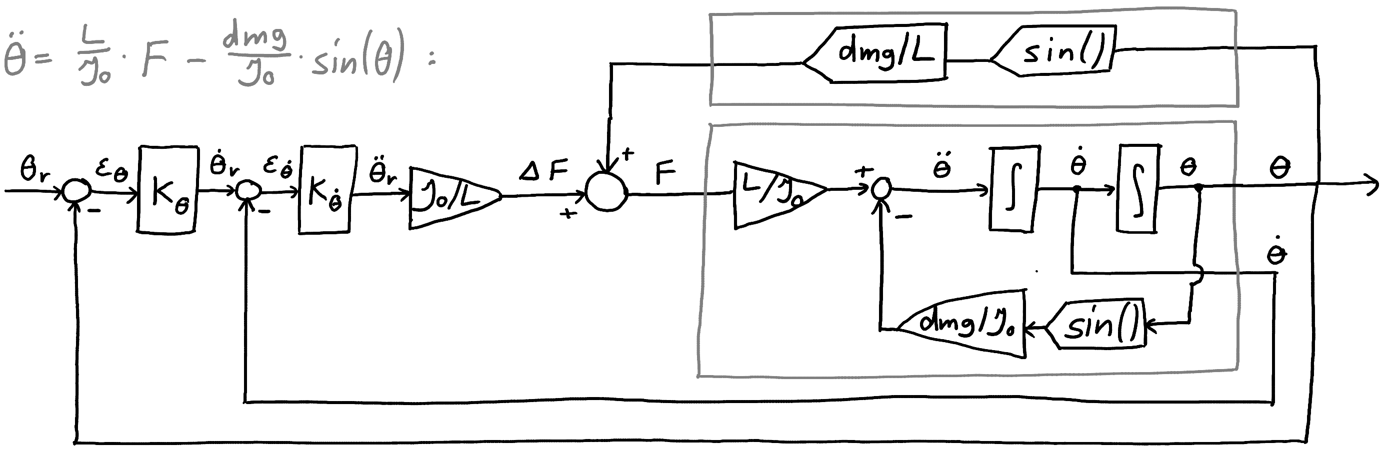
SIMO:



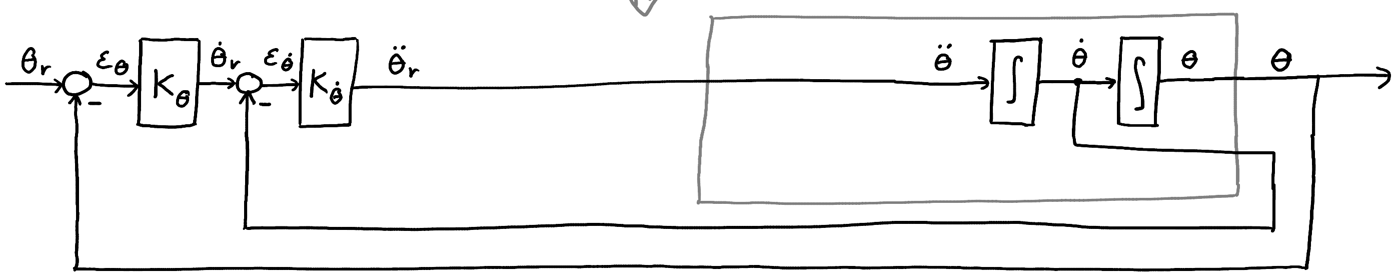
$\Rightarrow$



$\ddot{\theta} = \frac{L}{J_0} \cdot F - \frac{dmg}{J_0} \cdot \sin(\theta)$:



$\Downarrow$



$$\Rightarrow \frac{\theta(s)}{\theta_r(s)} = T(s) = \frac{K_\theta K_{\dot\theta}}{s^2 + K_{\dot\theta} s + K_\theta K_{\dot\theta}} = \frac{K\,\omega_0^2}{s^2 + 2\delta\omega_0 s + \omega_0^2}$$

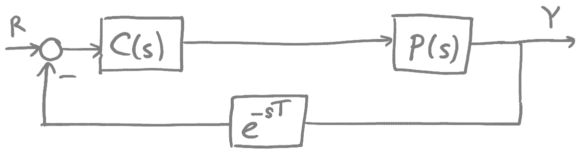
$K = 1$ (static gain) $\quad \omega_0 = \sqrt{K_\theta K_{\dot\theta}}$ (nat. freq) $\quad \frac{1}{2}\sqrt{\frac{K_{\dot\theta}}{K_\theta}}$ (damping)

$\omega_0 = 1\,\mathrm{rad/s}$ (aggressiveness) $\quad \delta = 1$ (critically damped)

$\hookrightarrow K_\theta = 1/2 \;;\; K_{\dot\theta} = 2$

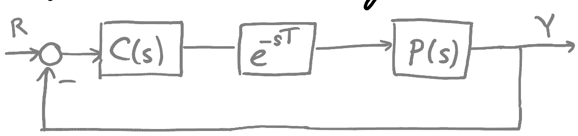
Smith Predictor (Control design for time-delay)

system output delay:



$$Y(s) = \frac{P(s)C(s)}{1 + P(s)C(s)e^{-sT}} \cdot R(s)$$

system input delay:

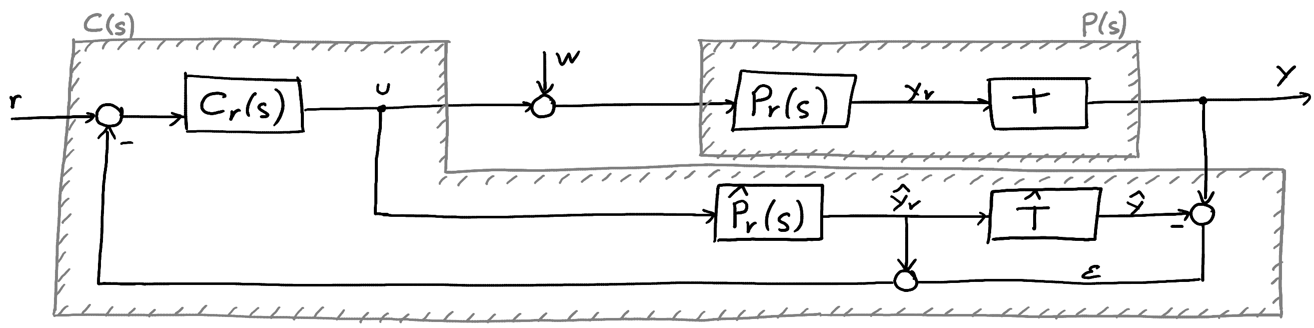


$$Y(s) = \frac{P(s)C(s)e^{-sT}}{1 + P(s)C(s)e^{-sT}} \cdot R(s)$$

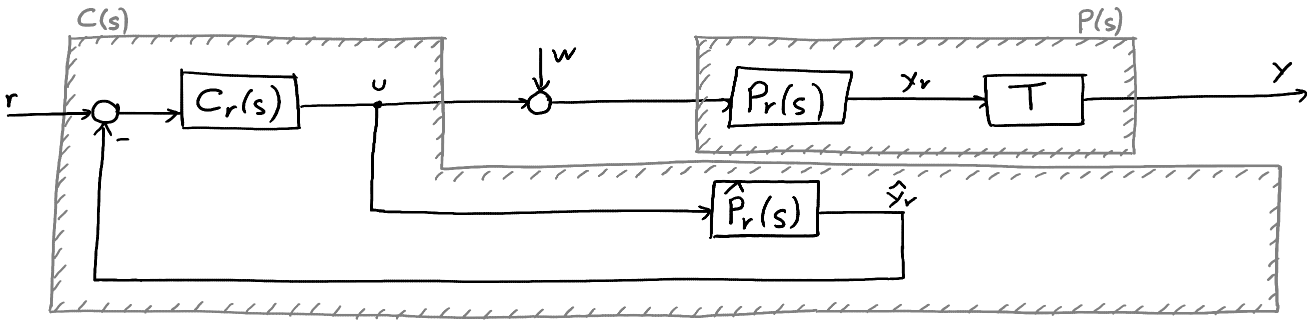
Requirements:

- $T/(T+\tau) > 0.3 \qquad \left(P(s) \approx \frac{K}{1+\tau s} \cdot e^{-Ts}\right)$
- $P(s)$ is assymptotically stable.
- very good model of the plant.

Control Design:

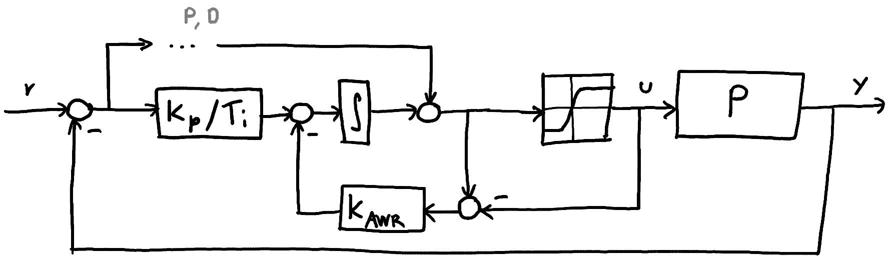


$w = 0 \,;\; \hat{P}_r(s) = P_r(s) \,;\; \hat{T} = T \;\Rightarrow\; \Downarrow$



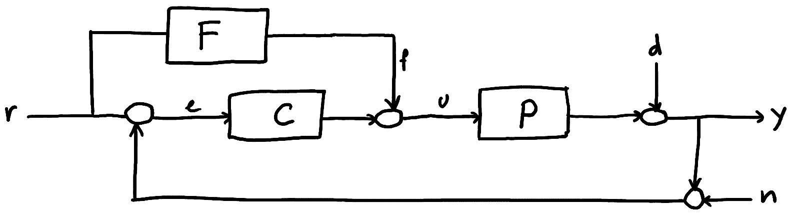
$$\hookrightarrow Y(s) = e^{-sT} \cdot \frac{\hat{P}_r(s)C(s)}{1 + \hat{P}_r(s)C(s)}$$

Input saturation / Anti reset-windup



Prevent integrators from filling up because of saturation (e.g. I in PID)

Feedforward



Feedforward:
- No influence on stability
- No effect on disturbances
- Better Tracking performance

$$Y = T\cdot R + P\cdot S\cdot F\cdot R + S\cdot D - T\cdot N \qquad \left(T = \frac{PC}{1+PC} \quad S = \frac{1}{1+PC} \right)$$

if: $F = P^{-1} \rightarrow Y = (T+S)\cdot R + \ldots = R + \ldots$ often impossible b.c. P^{-1} not causal.

Static feedforward: (Flying arm example)

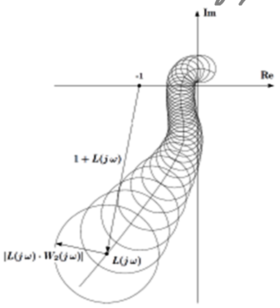
$f = F_0 \rightarrow F = \frac{dmg}{L}\cdot \sin(\theta_0)$ (algebraic term)

Dynamic feedforward:

Design $C \rightarrow T_{fb}, S_{fb} \rightarrow$ pick r-to-y: $T_{ref} = \frac{1}{(0.25s+1)^2} \rightarrow F = \frac{T_{ref} - T_{fb}}{P\cdot S_{fb}} \rightarrow$ add roll-off: $F_{roll} = F\cdot \frac{1}{(1+0.01s)^2}$

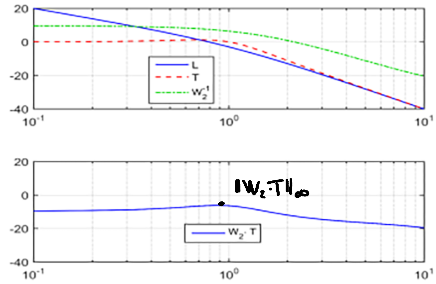
Robustness and performance criteria

Robust Nyquist Theorem:



$W_2(i\omega)$: Model error at freq ω.

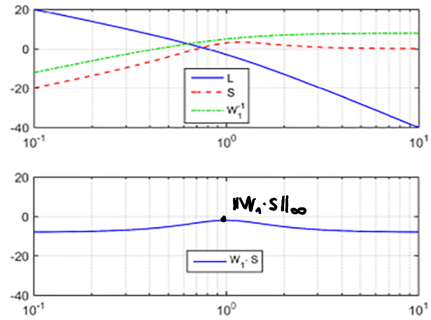
$$|1+L(i\omega)| > |L(i\omega)\cdot W_2(i\omega)|$$
$$\Downarrow$$
$$|T(i\omega)|\cdot|W_2(i\omega)| < 1 \qquad \forall\, \omega\in[0,\infty)$$
$$\Leftarrow\ \|T(i\omega)\cdot W_2(i\omega)\|_\infty < 1$$



Nominal performance:

$W_1(i\omega)$: Upper bound for $S \rightarrow$ good perf.

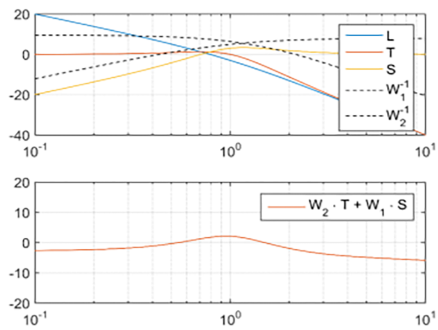
$$|S(i\omega)|\cdot|W_1(i\omega)| < 1 \qquad \forall\, \omega\in[0,\infty)$$
$$\Leftarrow\ \|S(i\omega)\cdot W_1(i\omega)\|_\infty < 1$$



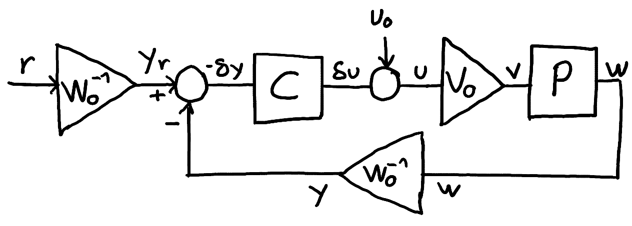
Robust performance:

Robust Nyq. + Nominal perf.

$$|S(i\omega)|\cdot|W_1(i\omega)| + |T(i\omega)|\cdot|W_2(i\omega)| < 1 \qquad \forall\, \omega\in[0,\infty)$$
$$\Leftarrow\ \|S(i\omega)\cdot W_1(i\omega) + T(i\omega)\cdot W_2(i\omega)\|_\infty < 1$$



Controller implementation



Physics / Experiment

Mod.

$$\tfrac{d}{dt}\, z(t) = f(z(t), v(t))$$
$$w(t) = g(z(t), v(t))$$

Plant Model based on Physics or Experiment

Norm.

$$\tfrac{d}{dt}\, x(t) = f_0(x(t), u(t))$$
$$y(t) = g_0(x(t), u(t))$$

Normalize all to 0-1

$x_i = \frac{z_i}{z_{i0}}$; $u = \frac{v}{V_0}$; $y = \frac{w}{W_0}$

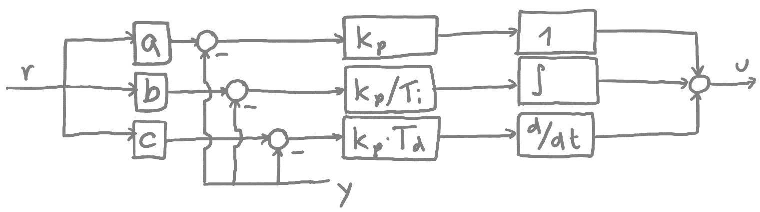
Lin.

$$\tfrac{d}{dt}\,\delta x(t) = A\cdot\delta x(t) + b\cdot\delta u(t)$$
$$\delta y(t) = c\cdot\delta x(t) + d\cdot\delta u(t)$$

linearize arround eq.

$\delta x_i = x_i - x_{ie}$; $\delta u = v - v_e$; $\delta y = y - y_e$

Setpoint weighting for PID



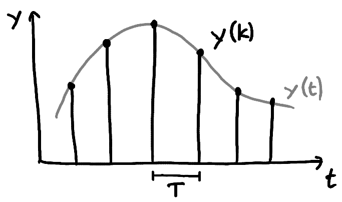
$0 < a < 1$

$b = 1$

$c = 0$

(other options: low-pass before r ; roll-off after u.)

Discrete-time system



$x(k)$: x at $t = k \cdot T$

$$\dot{x}(t) = f(x(t), u(t)) \qquad\qquad x(k+1) = f_k(x(k), u(k))$$
$$y(t) = g(x(t), u(t)) \quad \Rightarrow \quad y(k) = g_k(x(k), u(k))$$

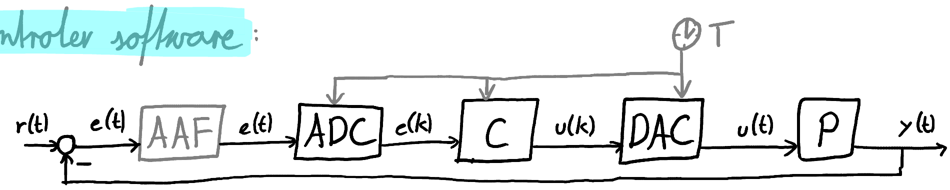
Discrete final value theorem: $\lim\limits_{k \to \infty} w(kT) = \lim\limits_{z \to 1} (z-1) W(z) = \lim\limits_{z \to 1} (1 - z^{-1}) W(\text{"}z^{-1}\text{"})$

synthesis:

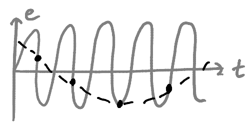
$$P(s) \begin{cases} \xrightarrow{\text{"hold"}} P(z) \xrightarrow{\text{disc. time synth.}} \tilde{C}(z) & (T \text{ small vs. } \tau \text{ time const.}) \\ \qquad\qquad\qquad\qquad\qquad\quad \neq & \\ \xrightarrow{\text{synth.}} C(s) \xrightarrow{\text{Emulation}} C(z) & (T \text{ large vs. } \tau \text{ time const.}) \end{cases}$$

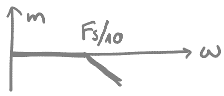
Emulation $(P(s) \to C(s) \to C(z))$

Controler software:

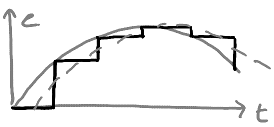


- Antialiasing filter AAF: or $\begin{cases} \text{Sampling Rate } F_s = \frac{1}{T} \overset{!}{\geq} 10 \cdot \frac{\omega_c}{2\pi} \\ \text{Lowpass-filter for } F_s/10 \text{ before } C \end{cases}$

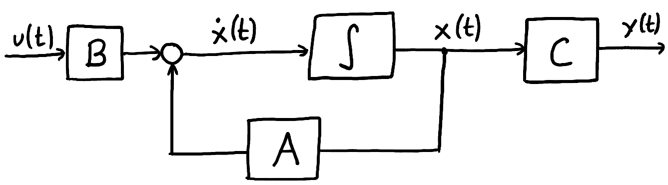




- Digital-Analog conv. DAC: introduces $T/2$ time delay!

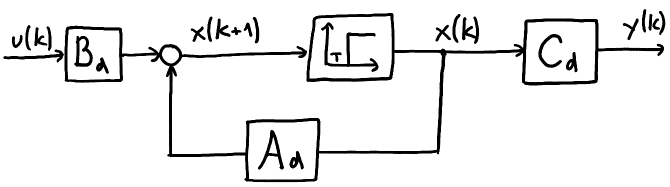


Controller discretization: (in state-space)



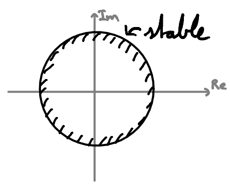
$$\dot{x}(t) = A \cdot x(t) + B \cdot u(t)$$
$$y(t) = C \cdot x(t)$$

$$\Downarrow \qquad \longleftarrow \qquad \begin{cases} A_d = e^{AT} \\ B_d = \int_0^T e^{A\tau} d\tau \cdot B \\ C_d = C \end{cases}$$



$$x(k+1) = A_d \cdot x(k) + B_d \cdot u(k)$$
$$y(k) = C_d \cdot x(k)$$

Stability: $\mathrm{Re}(\lambda_{c,i}) < 0 \ \forall i, \ \lambda_{c,i} = EW(A) \ \Rightarrow \ |\lambda_{d,i}| < 1 \ \forall i; \ \lambda_{d,i} = EW(A_d)$

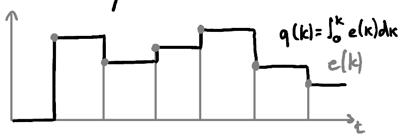


Controller discretization: (frequency domain)

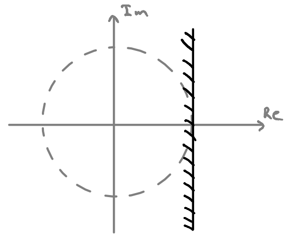
$C(s) \rightarrow C(z)$ mit $z^n \cdot u(k) = u(k+n)$ $\qquad (z \cdot u(k) = u(k+1))$

- Direct: $C(z) = C(s)\big|_{s=\ln(z)/T_s}$ —— C(z) identical to C(s)!

- Euler forward: $C(z) = C(s)\big|_{s=\frac{z-1}{T}}$

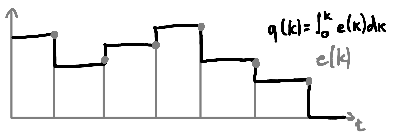


$q(k+1) = q(k) + e(k) \cdot T$

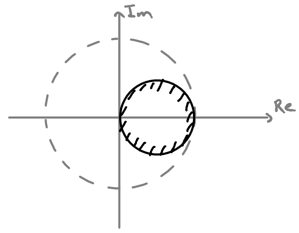


C(z) can become unstable! (bad)

- Euler backward: $C(z) = C(s)\big|_{s=\frac{z-1}{zT}}$

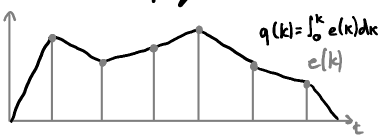


$q(k+1) = q(k) + e(k+1)T$

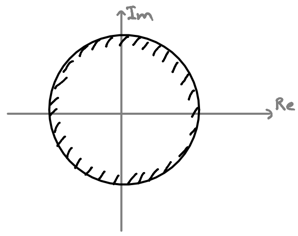


C(z) stable if C(s) stable! (good)

- Tustin (trapez.): $C(z) = C(s)\big|_{s=\frac{2}{T}\cdot\frac{z-1}{z+1}}$



$q(k+1) = q(k) + \frac{T}{2}\left(e(k) + e(k+1)\right)$

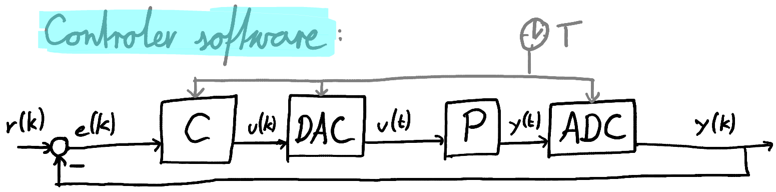


C(z) stable if C(s) stable! (best)

Recap: Design C(s) → Pick F_s → AAF → Emulate C(z) → Analyse stability → Implement

Discrete time synthesis ($P(s) \longrightarrow P(z) \longrightarrow C(z)$)

Controler software:

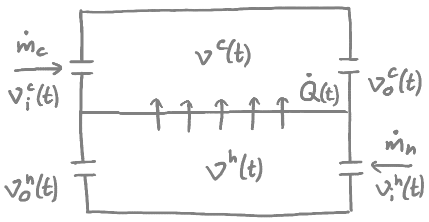


Plant discretization:

$$P(z) = (1-z^{-1}) \cdot \mathcal{Z}\left\{ \mathcal{L}^{-1}\left(\frac{P(s)}{s}\right)(kT) \right\} \qquad \left(W(z) = \mathcal{Z}\{w(k)\} = \sum_{k=0}^{\infty} w(k) \cdot z^{-k} \right)$$

MIMO Time-domain

Modeling:



$x_1(t) = \vartheta^h(t)$: Temp. hot fluid
$x_2(t) = \vartheta^c(t)$: Temp. cold fluid
$u_1(t) = \vartheta_i^h(t)$: Temp. entering hot fluid
$u_2(t) = \vartheta_i^c(t)$: Temp. entering cold fluid
$y_1(t) = \vartheta_o^h(t)$: Temp. exiting hot fluid
$y_2(t) = \vartheta_o^c(t)$: Temp. exiting cold fluid

$$\frac{d}{dt} U^h = \dot{H}_i^h - \dot{H}_o^h - \dot{Q} \qquad U^{h,c} = \rho \cdot V \cdot c \cdot \vartheta^{h,c}$$
$$\frac{d}{dt} U^c = \dot{H}_i^c - \dot{H}_o^c + \dot{Q} \qquad \dot{H}_{i,o}^{h,c} = \dot{m} \cdot c \cdot \vartheta_{i,o}^{h,c}$$
$$\dot{Q} = k \cdot F (\vartheta^h - \vartheta^c)$$

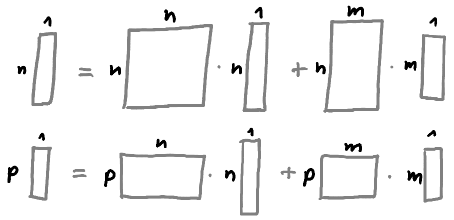
$$\rho V c \frac{dx_1}{dt} = \dot{m} c (u_1 - x_1) - k F (x_1 - x_2)$$
$$\rho V c \frac{dx_2}{dt} = \dot{m} c (u_2 - x_2) - k F (x_1 - x_2)$$

$$\tau \dot{x}_1 = -x_1 + \sigma x_2 + \beta u_1 \qquad \tau = \frac{\rho V c}{\dot{m} c + kF} \qquad \sigma = \frac{kF}{\dot{m} c + kF}$$
$$\tau \dot{x}_2 = -x_2 + \sigma x_1 + \beta u_2 \qquad \beta = \frac{\dot{m} c}{\dot{m} c + kF}$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -1/\tau & \sigma/\tau \\ \sigma/\tau & -1/\tau \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \beta/\tau & 0 \\ 0 & \beta/\tau \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$
$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

State-space:

$$\frac{d}{dt} \underline{x} = \underline{\underline{A}}\, \underline{x} + \underline{\underline{B}}\, \underline{u}$$
$$\underline{y} = \underline{\underline{C}}\, \underline{x} + \underline{\underline{D}}\, \underline{u}$$



$x \in \mathbb{R}^n$
$u \in \mathbb{R}^m$
$y \in \mathbb{R}^p$

Equilibria:

$$0 = \underline{\underline{A}} \cdot \underline{x_e} + \underline{\underline{B}} \cdot \underline{u_e} \rightarrow x_e = -\underline{\underline{A}}^{-1} \underline{\underline{B}} \cdot \underline{u_e}$$

Stability:

$$\det(\lambda \cdot I - A) = 0 \rightarrow \mathrm{Re}(\lambda_n) < 0 \rightarrow \text{asympt. stable}$$

Controllability:

$$R_n = [B, AB, \ldots, A^{n-1} B] \in \mathbb{R}^{n \times (n \cdot m)} \text{ has indep. rows}$$

Observability:

$$O_n = [C^T, A^T C^T, \ldots, (A^{n-1})^T C^T]^T \in \mathbb{R}^{(n \cdot p) \times n} \text{ has indep. columns}$$

MIMO Frequency-domain

Transfer func.: $\underline{\underline{P}}(s) = \underline{\underline{C}} \cdot (s \cdot \underline{\underline{I}} - \underline{\underline{A}})^{-1} \cdot \underline{\underline{B}} + \underline{\underline{D}}$ (or exp. holding U's...)

$$\underline{Y}(s) = \underline{\underline{P}}(s) \cdot \underline{U}(s) \quad \Leftrightarrow \quad \begin{pmatrix} Y_1(s) \\ Y_2(s) \\ \vdots \\ Y_p(s) \end{pmatrix} = \begin{pmatrix} P_{11}(s) & P_{12}(s) & \cdots & P_{1m}(s) \\ P_{21}(s) & P_{22}(s) & \cdots & P_{2m}(s) \\ \vdots & \vdots & \ddots & \vdots \\ P_{p1}(s) & P_{p2}(s) & \cdots & P_{pm}(s) \end{pmatrix} \cdot \begin{pmatrix} U_1(s) \\ U_2(s) \\ \vdots \\ U_m(s) \end{pmatrix}$$

Poles: Roots of the least common denominator of all minors of $P(s)$

Zeros: Roots of the greatest common divisor of the numerators of the maximum minors of $P(s)$

Pole/Zero Example:

$$P(s) = \begin{pmatrix} 2/s+1 & 3/s+2 \\ 1/s+1 & 1/s+1 \end{pmatrix}$$

Minors: $\frac{2}{s+1}, \frac{1}{s+1}, \frac{1}{s+1}, \frac{3}{s+2}, \frac{(-s+1)}{(s+1)^2(s+2)}$

least comm. den.: $(s+1)^2(s+2) \rightarrow \pi_{1,2} = -1,\ \pi_3 = -2$

great. comm. div of max minor: $(-s+1) \rightarrow \zeta_1 = 1$

Zero/Pole cancellation: (... ? ...)

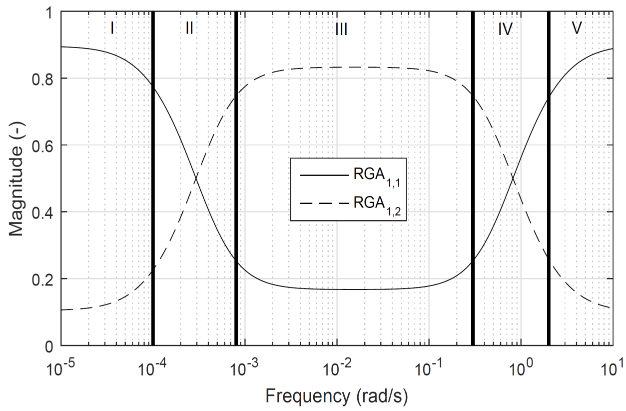
Relative Gain Array (RGA):

$$RGA(s) = P(s) \odot (P(s)^{-1})^T$$

element wise multip.

(if P not square use pseudo-inverse)

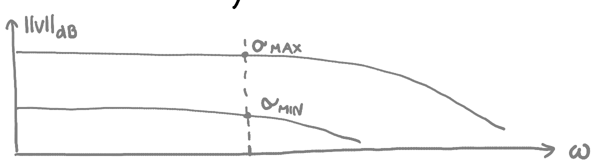
$s = i\omega$:
- $RGA_{1,1} = RGA_{2,2} \approx 1 \Rightarrow$ SISO $U_1 \rightarrow Y_1,\ U_2 \rightarrow Y_2$
- $RGA_{1,1} = RGA_{2,2} \approx 0 \Rightarrow$ SISO $U_1 \rightarrow Y_2,\ U_2 \rightarrow Y_1$
- else $\Rightarrow$ MIMO



Frequency response:

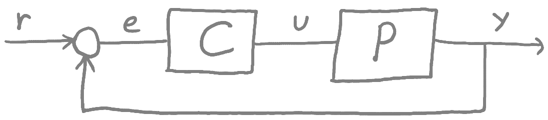
$$u(t) = \begin{pmatrix} \mu_1 \cdot \cos(\omega t + \varphi_1) \cdot h(t) \\ \mu_2 \cdot \cos(\omega t + \varphi_2) \cdot h(t) \\ \vdots \\ \mu_m \cdot \cos(\omega t - \varphi_m) \cdot h(t) \end{pmatrix} \Rightarrow y_\infty(t) = \begin{pmatrix} \nu_1 \cdot \cos(\omega t + \psi_1) \cdot h(t) \\ \nu_2 \cdot \cos(\omega t + \psi_2) \cdot h(t) \\ \\ \nu_m \cdot \cos(\omega t + \psi_m) \cdot h(t) \end{pmatrix}_{p=m}$$

$$\min \sigma_i(P(j\omega)) \leq \frac{\|\nu\|}{\|\mu\|} \leq \max \sigma_i(P(j\omega))$$



$\left(\sigma_i(P(j\omega)) : \text{singular values} = \sqrt{EW(\overline{P}^T(j\omega) \cdot P(j\omega))} \right)$

MIMO Feedback system



$$Y(s) = \underbrace{P(s)\cdot C(s)}_{L_y(s)} \cdot R(s) - \underbrace{P(s)C(s)}_{L_y(s)} \cdot Y(s)$$

$$E(s) = R(s) - \underbrace{P(s)\cdot C(s)}_{L_e(s)} \cdot E(s)$$

$$U(s) = C(s)\cdot R(s) - \underbrace{C(s)\cdot P(s)}_{L_u(s)} \cdot U(s)$$

Loop gain: $L_e(s) = P(s)\cdot C(s)$ $\qquad$ $L_u(s) = P(s)\cdot C(s)$

Return difference: $D_e(s) = I + L_e(s)$ $\qquad$ $D_u(s) = I + L_u(s)$

Sensitivity: $S_e(s) = (I + L_e(s))^{-1}$ $\qquad$ $S_u(s) = (I + L_u(s))^{-1}$

Comp. sensitivity: $T_e(s) = (I + L_e(s))^{-1} \cdot L_e(s)$ $\qquad$ $T_u(s) = (I + L_u(s))^{-1} \cdot L_u(s)$

$T_e(s) + S_e(s) = I$ $\qquad$ $T_u(s) + S_u(s) = I$

Closed-loop stability:

- Nyquist theorem:

$n_c = n_0/2 + n_+ \Rightarrow$ stable

n_0: # of poles of L_e with $Re(\lambda) = 0$
n_+: # of poles of L_e with $Re(\lambda) > 0$
n_c: # of CCW encirclements of -1 by $N(\omega i)$
$N(\omega i) = \det(I + L_e(\omega i))$ $\quad \omega \in [-\infty, \infty]$

- State-space analysis:

Plant: $\frac{d}{dt}\underline{x}(t) = \underline{\underline{A}}\,\underline{x}(t) + \underline{\underline{B}}\,\underline{u}(t)$
$\underline{y}(t) = \underline{\underline{C}}\,\underline{x}(t)$

Controller: $\frac{d}{dt}\underline{z}(t) = \underline{\underline{F}}\cdot\underline{z}(t) + \underline{\underline{G}}\cdot\underline{e}(t)$
$\underline{u}(t) = \underline{\underline{H}}\cdot\underline{z}(t)$

$$\Rightarrow \frac{d}{dt}\underbrace{\begin{pmatrix} \underline{x} \\ \hline \underline{z} \end{pmatrix}}_{\tilde{x}} = \underbrace{\left(\begin{array}{c|c} A & B\cdot H \\ \hline -G\cdot C & F \end{array}\right)}_{\tilde{A}} \cdot \underbrace{\begin{pmatrix} \underline{x} \\ \hline \underline{z} \end{pmatrix}}_{\tilde{x}} + (\ldots)$$

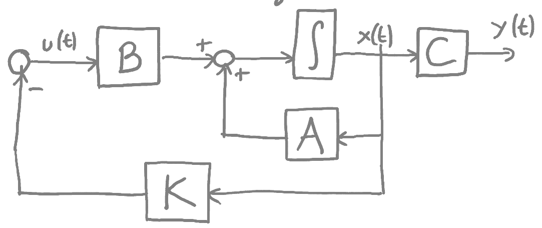
$$\frac{d}{dt}\tilde{x} = \tilde{A}\cdot\tilde{x} + (\ldots)$$

all $Re(EW)$ of $\tilde{A} < 0 \Rightarrow$ stable

Regulators (stabilization, dist. rejection, no ref. track)

- **State-feedback Regulator**:

$u(t) = -K\cdot x(t)$

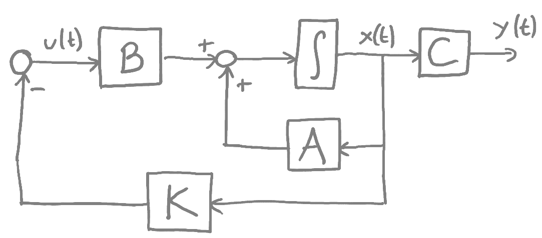


$A_{CL} = A - B\cdot K$ $\quad(\dot{x} = A_{CL}\cdot x)$

$\underset{n\times n}{\square} - \underset{n\times m}{\square}\cdot\underset{m\times n}{\square}$

↳ pick K to move poles of A_{CL}

- **Linear Quadratic Regulator** (LQR):



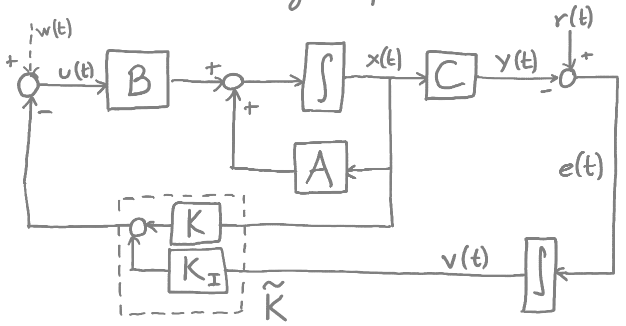
minimize $\int_0^\infty \underbrace{x\cdot Q\cdot x^T}_{Q^T=Q\geq 0} + \underbrace{u\cdot R\cdot u^T}_{R^T=R>0}\, dt$

$K = R^{-1}\cdot B^T\cdot \Phi \xleftarrow{<0} \left(\Phi\cdot B\cdot R^{-1}\cdot B^T\cdot\Phi - \Phi\cdot A - A^T\cdot\Phi - Q = 0\right)$

good choice: $Q = C^T C$ $\quad R = r\cdot \mathbb{I}_{m\times m}$

($Q \gg R \rightarrow$ cheap ctrl. E $\quad Q \ll R$ expensive ctrl. E)

- **LQRI** (yes ref. track!):



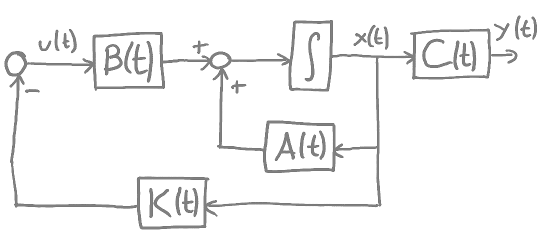
$\tilde{K} = \underbrace{\begin{pmatrix} K & -K_I \end{pmatrix}}_{n \;\; p}\Big\}m \qquad \tilde{x} = \begin{pmatrix} x \\ v \end{pmatrix}\begin{matrix} n \\ p \end{matrix}$ (column width 1)

↳ $\tilde{Q} = \begin{pmatrix} Q & 0 \\ 0 & \gamma\cdot\mathbb{I}_{p\times p} \end{pmatrix}$; $\tilde{R} = R$

→ solve for $\tilde{K}$ like LQR then extract K, K_I

$\left(\frac{d}{dx}\tilde{x}(t) = \underbrace{\tilde{A}}_{\begin{pmatrix} A & 0 \\ -C & 0\end{pmatrix}}\cdot\tilde{x}(t) + \underbrace{\tilde{B}_u}_{\begin{pmatrix} B \\ 0\end{pmatrix}}\cdot u(t) + \underbrace{\tilde{B}_r}_{\begin{pmatrix} 0 \\ I\end{pmatrix}}\cdot r(t) + \underbrace{\tilde{B}_w}_{\begin{pmatrix} B \\ 0\end{pmatrix}}\, w(t) \right)$

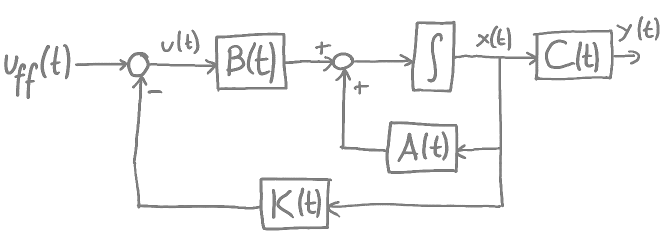
- **Finite horizon LQR**:



minimize $\int_0^T x\cdot Q\cdot x^T + u\cdot R\cdot u^T\, dt + x(T)^T P\, x(T)$

solve for $K(t)$ → store in memory

- **Feedforward LQR** (+ finite horizon):



$\min_{u(t)} \int_0^T \underbrace{L(x(t), u(t))\, dt}_{\text{Stage cost}} + \underbrace{m(x(t))}_{\text{Terminal cost}}$

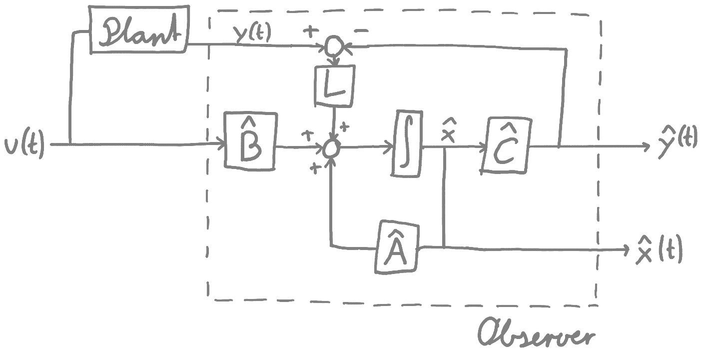
Initial Conditions
Constraints

$\Downarrow$

$u_{ff}(t)$

Linear Quadratic Gaussian (LQG)

State observer:



($\hat{A} = A$, $\hat{B} = B$, $\hat{C} = C$ from Plant)

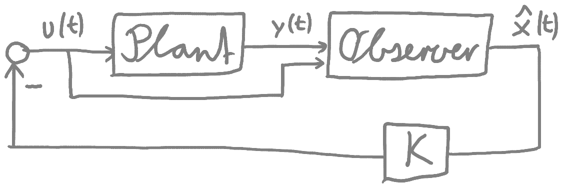
$\bar{x}(t) = x(t) - \hat{x}(t)$ (observation error)

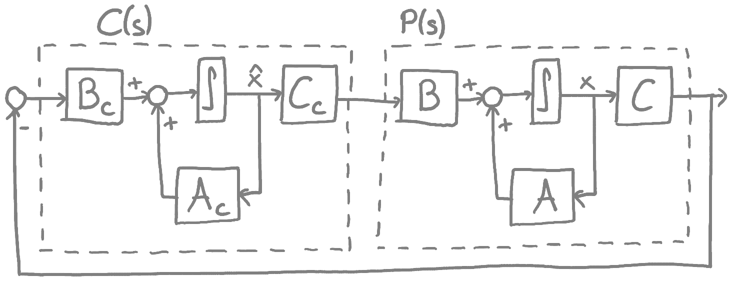
$$\frac{d}{dt}\bar{x}(t) = \underbrace{(A - LC)}_{\text{asym. stab. EW} \to \bar{x}(t\to\infty) = 0} \cdot \bar{x}(t)$$

$$L^T = \frac{1}{q} \cdot C \cdot \Psi \;\xleftarrow{<0}\; \left(\frac{1}{q}\Psi \cdot C^T C \Psi - \Psi \cdot A^T - A\Psi - \bar{B}\cdot\bar{B}^T = 0\right)$$

(q large: fast convergence of $\hat{x}$ to x
q small: noise amplification
$\bar{B}\cdot\bar{B}^T$: pick somehow...)

LQG Controller:

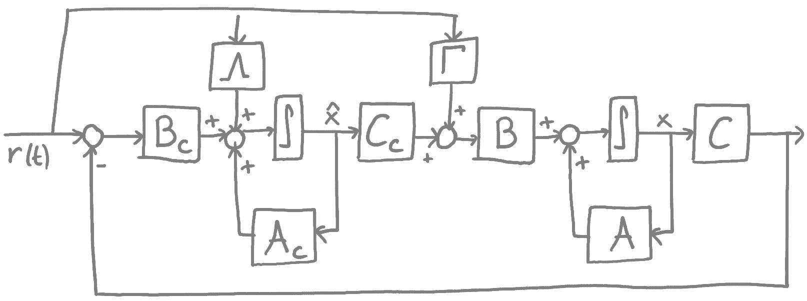




$A_c = A - BK - LC$; $B_c = -L$; $C_c = -K$

$$\tilde{x} = \begin{pmatrix} x \\ \hat{x} \end{pmatrix} \to \tilde{A}_{CL} = \left(\begin{array}{c|c} A & -B\cdot K \\ \hline LC & A - BK - L\cdot C \end{array}\right), \quad \tilde{B}_r = \begin{pmatrix} B \\ B \end{pmatrix}, \quad \tilde{C} = \left(\begin{array}{c|c} C & 0 \end{array}\right)$$

LQG reference tracking:

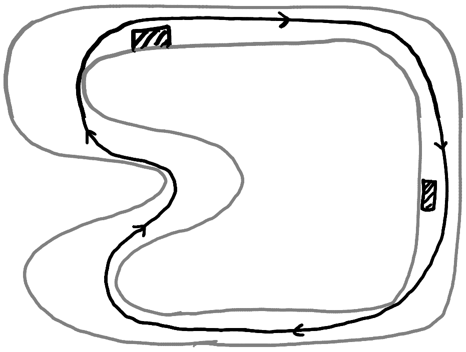


$$\Gamma = -\left(\tilde{C}\cdot\tilde{A}_{CL}^{-1}\cdot\tilde{B}_r\right)^{-1}$$

$$\Lambda = L + B\cdot\Gamma$$

(Γ, Λ counteract observation error in $\hat{x}$ → no static error)

Model Predictive Control (MPC)



1. Measure / Estimate current state
2. Find optimal input trajec. for planning window T
3. Implement first δt part of trajectory

Matrix Minors

$$M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \Rightarrow \text{Submatrices: } \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}, \begin{pmatrix} a & b \\ d & e \end{pmatrix}, \begin{pmatrix} a & c \\ d & f \end{pmatrix}, \dots, (g), (h), (i)$$

Minors: det([Submatrix])

Max. Minors: det([biggest Submatrix])

Singular value decomposition (SVD)

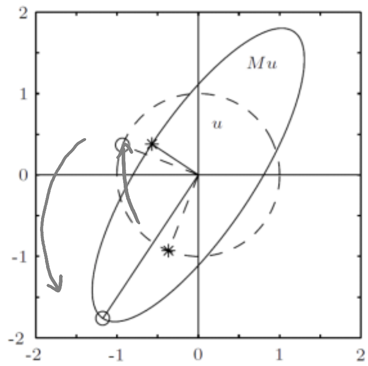
$$M = U \cdot \underset{\text{diag. Mat!}}{\Sigma} \cdot V^T \qquad {}_{p}\overset{m}{\boxed{M}} = {}_{p}\overset{p}{\boxed{U}} \cdot {}_{p}\overset{m}{\boxed{\Sigma}} \cdot {}_{m}\overset{m}{\boxed{V^T}}$$

$$\begin{pmatrix} \det(U) = \det(V) = 1 \\ U \cdot U^T = \mathbb{I} \; ; \; V \cdot V^T = \mathbb{I} \end{pmatrix}$$

e.g. $M = \begin{pmatrix} 1{,}3 & 0{,}1 \\ 1{,}5 & -1 \end{pmatrix} = \begin{pmatrix} -0{,}55 & -0{,}83 \\ -0{,}83 & 0{,}55 \end{pmatrix} \cdot \begin{pmatrix} 2{,}12 & 0 \\ 0 & 0{,}68 \end{pmatrix} \cdot \begin{pmatrix} -0{,}93 & 0{,}36 \\ -0{,}36 & -0{,}93 \end{pmatrix}$

$\sigma_1 = 2{,}12 \qquad M \cdot \begin{pmatrix} -0{,}93 \\ 0{,}36 \end{pmatrix} = 2{,}12 \cdot \begin{pmatrix} -0{,}55 \\ -0{,}83 \end{pmatrix}$

$\sigma_2 = 0{,}68 \qquad M \cdot \begin{pmatrix} -0{,}36 \\ -0{,}93 \end{pmatrix} = 0{,}68 \cdot \begin{pmatrix} -0{,}83 \\ 0{,}55 \end{pmatrix}$



Singular Values: $\sigma_i = \sqrt{EW_i(\overline{M}^T \cdot M)}$ (conj. if complex)

Induced Norm: $\|M\| = \max_i(\sigma_i)$ (maximum amplification)