Orbital dynamics

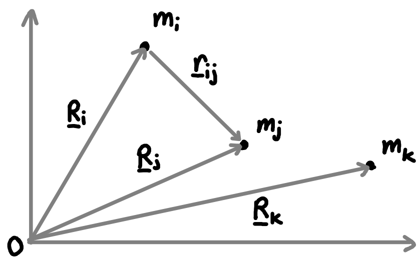
pointmass assumption

point-mass assumption is good as long as bodies are spherical with symmetric densities! (see script)

proof?

n-body problem

EoM: $\underbrace{m_i \cdot \frac{d}{dt}\dot{\underline{R}}_i = m_i \ddot{\underline{R}}_i = \underline{F}_i}_{\text{2nd law of motion}} = \underbrace{G \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{m_i m_j}{r_{ij}^2} \cdot \frac{\underline{r}_{ij}}{\|\underline{r}_{ij}\|}}_{\text{law of gravitation}}$



O : inertial frame
$m_{(\cdot)}$: mass of body
$\underline{R}_{(\cdot)}$: position of body in O
$\underline{r}_{(\cdot)(\cdot)}$: rel. pos. of body in O

- $\sum_{i=1}^{n} m_i \ddot{\underline{R}}_i = .. = \underline{0}$ (sum of forces is 0)
- $\sum_{i=1}^{n} m_i \dot{\underline{R}}_i = .. = \underline{C}_1$ (linear momentum is constant)
- $\sum_{i=1}^{n} m_i \underline{R}_i = .. = \underline{C}_1 t + \underline{C}_2$ (center of mass moves linearly: $R_{cm} = \sum_{i=1}^{n} m_i R_i / \sum_{i=1}^{n} m_i$)
- $\sum_{i=1}^{n} R_i \times (m_i \ddot{R}_i) = .. = \underline{0}$ (sum of torques is 0)
- $\sum_{i=1}^{n} R_i \times (m_i \dot{R}_i) = .. = \underline{C}_3$ (angular momentum is constant)
- $T + V = C_4$ (total energy is constant) $\begin{cases} T = \frac{1}{2} \sum_{i=1}^{n} m_i \dot{\underline{R}}_i \cdot \dot{\underline{R}}_i & \text{(kinetic energy)} \\ V = -\frac{G}{2} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{m_i m_j}{\|\underline{r}_{ij}\|} & \text{(potential energy } -\nabla V = F) \end{cases}$

- solve EoM system of differential equations for $\underline{R}_i(t)$ in closed form (by quadrature):

$$\left.\begin{array}{lll} d/dt\ \underline{R}_1(t) = \dot{\underline{R}}_1(t) & ; & d/dt\ \dot{\underline{R}}_1(t) = 1/m_1 \cdot \text{law\_of\_gravitation}_1(\underline{R}_1(t), \dots, \underline{R}_n(t)) \\ \vdots & & \vdots \\ d/dt\ \underline{R}_n(t) = \dot{\underline{R}}_n(t) & ; & d/dt\ \dot{\underline{R}}_n(t) = 1/m_n \cdot \text{law\_of\_gravitation}_n(\underline{R}_1(t), \dots, \underline{R}_n(t)) \end{array}\right\} \boxed{\text{6n differential equations}}$$

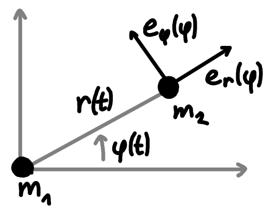
↳ rewrite as $d\underline{x}(t)/dt = \underline{Y}(\underline{x}(t), t)$ with $\underline{x}(t) = [R_1(t), .., R_n(t), \dot{R}_1(t), .., \dot{R}_n(t)]^T$

find 6n "integrals" ≙ independent equations of the form: $\underline{Z}(\underline{x}(t), t) = \begin{pmatrix} Z_1(\underline{x}(t), t) \\ \vdots \\ Z_{6n}(\underline{x}(t), t) \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_{6n} \end{pmatrix} = \underline{a}$ ← const.!
where $\underline{a}$ is const. for all $\underline{x}(t)$ that solve $d\underline{x}(t)/dt = \underline{Y}(\underline{x}(t), t)$

solve $\underline{Z}(\underline{x}(t), t) = \underline{a}$ for $\underline{x}(t)$ → $\underline{x}(t) = \underline{\Phi}(\underline{a}, t)$

10 "integrals" are shown above with $\underline{a} = [\underline{C}_1, \underline{C}_2, \underline{C}_3, C_4]^T$, $\underline{Z}(x(t), t) = \dots$ ← NO more than 10 exist!
↳ no closed form solution exists! not even for n=2 since 10 "integrators" < 12 = 6n

2-body problem (relative motion)



EoM: $\ddot{\underline{r}}(t) = \ddot{\underline{R}}_2(t) - \ddot{\underline{R}}_1(t) = \frac{Gm_1}{r(t)^3}\cdot -\underline{r}(t) - \frac{Gm_2}{r(t)^3}\cdot \underline{r}(t) = -\frac{\mu}{\underline{r}(t)^2}\cdot \underline{r}(t)$ $\quad (\mu = G(m_1+m_2))$

"integrals": $\underline{r}(t)\times\ddot{\underline{r}}(t) = \ldots = 0 \rightarrow \underline{h} = \underline{r}(t)\times\dot{\underline{r}}(t) = \text{const.}$ $\rightarrow$ motion constrained to plane $\underline{h}$: orbital plane!

orbital equation $r(\varphi)$ (solve EoM):

$$\left(\begin{aligned} \underline{e}_r(\varphi(t)) &= [\cos(\varphi(t)), \sin(\varphi(t))] \\ \underline{e}_\varphi(\varphi(t)) &= [-\sin(\varphi(t)), \cos(\varphi(t))] \end{aligned}\right)$$

general relative motion in polar coord. (on $\underline{h}$ plane):

- relative position: $\underline{r}(t) = \underline{R}_2(t) - \underline{R}_1(t) = r(t)\cdot \underline{e}_r(\varphi(t))$
- relative speed: $\dot{\underline{r}}(t) = \dot{\underline{R}}_2(t) - \dot{\underline{R}}_1(t) = \dot{r}(t)\cdot\underline{e}_r(\varphi(t)) + r(t)\dot\varphi(t)\,\underline{e}_\varphi(\varphi(t))$
- relative accel.: $\ddot{\underline{r}}(t) = \ddot{\underline{R}}_2(t) - \ddot{\underline{R}}_1(t) = (\ddot{r}(t) - r(t)\dot\varphi(t)^2)\,\underline{e}_r(\varphi(t)) + \frac{1}{r(t)}\frac{d}{dt}(r(t)^2\dot\varphi(t))\,\underline{e}_\varphi(\varphi(t))$

$$\underbrace{-\mu\cdot\frac{1}{r(t)^2}\,\underline{e}_r(\varphi(t)) = \ddot{\underline{r}}(t)}_{\text{EoM}} \underbrace{= (\ddot{r}(t) - r(t)\dot\varphi(t)^2)\,\underline{e}_r(\varphi(t)) + \frac{1}{r(t)}\frac{d}{dt}(r(t)^2\dot\varphi(t))\,\underline{e}_\varphi(\varphi(t))}_{\text{polar vector derivative for } \ddot{r}}$$

$\hookrightarrow -\mu\cdot\frac{1}{r(t)^2} = \ddot{r}(t) - r(t)\dot\varphi(t)^2$

$= \ddot{r}(t) - h^2/r(t)^3$

$$\left(h = |\underline{h}| = |\underline{r}(t)\times\dot{\underline{r}}(t)| = \underbrace{|r(t)\underline{e}_r(\varphi(t)) \times (\dot{r}(t)\underline{e}_r(t) + r(t)\dot\varphi(t)\underline{e}_\varphi(\varphi(t)))|}_{\text{polar vector derivative for } r, \dot{r}} = r(t)^2\cdot\dot\varphi(t) = h \right)$$

$\rightarrow \ddot{r}(t) = h^2\cdot 1/r(t)^3 - \mu\cdot 1/r(t)^2$

reformulate $r(t)$ as $r(\varphi(t))$: $\ddot{r}(\varphi(t)) = \frac{d}{dt}\left(\frac{d}{dt}(r(\varphi(t))\right) = \frac{d}{dt}\left(\frac{dr}{d\varphi}\cdot\frac{d\varphi}{dt}\right) = \frac{d}{d\varphi}\left(\frac{dr}{d\varphi}\cdot\frac{d\varphi}{dt}\right)\frac{d\varphi}{dt} = \frac{d}{d\varphi}\left(\frac{dr}{d\varphi}\cdot\frac{h}{r^2}\right)\cdot\frac{h}{r^2} = \left(\frac{d^2r}{d\varphi^2}\cdot\frac{h}{r^2} - \frac{dr}{d\varphi}\cdot\frac{2h}{r^3}\frac{dr}{d\varphi}\right)\cdot\frac{h}{r^2}$

$$\left(\frac{d^2r}{d\varphi^2}\cdot\frac{h}{r^2} - \frac{dr}{d\varphi}\cdot\frac{2h}{r^3}\frac{dr}{d\varphi}\right)\cdot\frac{h}{r^2} = \frac{h^2}{r^3} - \frac{\mu}{r^2}$$

$$\longrightarrow \frac{d^2r}{d\varphi^2}\cdot\frac{h}{r^2} - \left(\frac{dr}{d\varphi}\right)^2\cdot\frac{2h}{r^3} = \frac{h}{r} - \frac{\mu}{h}$$

substitute $u = 1/r$: $du/dr = -1/r^2 \rightarrow dr = -r^2 du$

$$\frac{d^2r}{d\varphi^2}\cdot\frac{h}{r^2} - \left(\frac{dr}{d\varphi}\right)^2\cdot\frac{2h}{r^3} = \frac{h}{r} - \frac{\mu}{h} \leftarrow \left(\frac{d^2r}{d\varphi^2} = \frac{d}{d\varphi}\frac{dr}{d\varphi} = \frac{d}{d\varphi}\left(-r^2\frac{du}{d\varphi}\right) = -2r\frac{dr}{d\varphi}\frac{du}{d\varphi} - r^2\frac{d^2u}{d\varphi^2} = 2r^3\left(\frac{du}{d\varphi}\right)^2 - r^2\frac{d^2u}{d\varphi^2}\right)$$

$$\left(2r^3\left(\frac{du}{d\varphi}\right)^2 - r^2\frac{d^2u}{d\varphi^2}\right)\frac{h}{r^2} - \left(-r^2\frac{du}{d\varphi}\right)^2\frac{2h}{r^3} = \frac{h}{r} - \frac{\mu}{h}$$

$$2hr\left(\frac{du}{d\varphi}\right)^2 - h\frac{d^2u}{d\varphi^2} - 2hr\left(\frac{du}{d\varphi}\right)^2 = \frac{h}{r} - \frac{\mu}{h}$$

$$-h\frac{d^2u}{d\varphi^2} = hu - \frac{\mu}{h}$$

$$\frac{d^2u}{d\varphi^2} + u = \frac{\mu}{h^2}$$

solve

$$u(\varphi) = 1/r(\varphi) = \frac{\mu}{h^2} + C_1\cos(\varphi - C_2)$$

$$r(\varphi) = \frac{h^2/\mu}{1 + (\mu/h^2)C_1\cos(\varphi + C_2)} \quad \left(e = C_1\,\mu/h^2,\ C_2 = 0\right)$$

$$r(\varphi) = \frac{h^2/\mu}{1 + e\cos(\varphi)} = \frac{p}{1 + e\cos(\varphi)}$$

φ: true anomaly	e: eccentricity
$\varphi = 0$: apoapsis	$p = h^2/\mu$: semilatus rectum
$\varphi = \pi$: periapsis	

orbital velocity $v_r(\varphi), v_\varphi(\varphi)$ and $v_r(r), v_\varphi(r)$

$\underline{v} = \dot{\underline{r}} = \dot{r}\,\underline{e}_r + r\dot\varphi\,\underline{e}_\varphi = \frac{dr}{d\varphi}\frac{d\varphi}{dt}\underline{e}_r + r\frac{d\varphi}{dt}\underline{e}_\varphi$

$\left(\frac{d\varphi}{dt} = \dot\varphi = h/r^2;\ \frac{dr}{d\varphi} = \frac{ep\sin\varphi}{(1+e\cos\varphi)^2}\right)$

- $v_r(\varphi) = \frac{h}{p}\cdot e\cdot\sin(\varphi) \rightarrow v_r(r) = ?$
- $v_\varphi(\varphi) = \frac{h}{p}(1 + e\cdot\cos(\varphi)) \rightarrow v_\varphi(r) = h/r$

energy preservation:

$\mathcal{E} = \frac{1}{2}\underline{v}^2 - \mu/\|\underline{r}\| = \text{const.}$ = kinetic + potential specific E

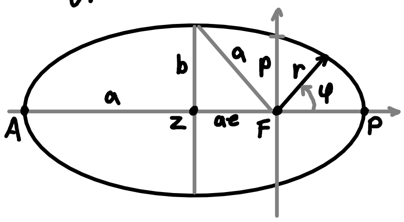
at periapsis ($\varphi = 0$): $\mathcal{E} = -\mu/2p\,(1 - e^2)$

$\uparrow (\varphi = 0 \rightarrow \|\underline{r}\| = r(\varphi),\ \dot{\underline{r}}^2 = v_r(\varphi)^2 + v_\varphi(\varphi)^2)$

vis-viva equation: $\|\underline{v}(r)\| = \sqrt{\mu(2/r - 1/a)}$

(from $\mathcal{E}$ preserv.: $\mathcal{E} = \mathcal{E}(\varphi = 0)$ solved for $\|\underline{v}\|$)

ellyptic orbit $(0<e<1)$:



F: focus
Z: center
P: periapsis
A: apoapsis

a: semi-major axis
b: semi-minor axis
p: semi latus rectum

$p = h^2/\mu = a(1-e^2) = b\sqrt{1-e^2}$

$a = p/(1-e^2) \qquad b = p/\sqrt{1-e^2}$

$r_p = p/(1+e) \qquad \overline{AP} = 2a$

$r_a = p/(1-e) \qquad \overline{FZ} = e\cdot a$

$\mathcal{E} = -\mu/2a \qquad e = \sqrt{1-b^2/a^2}$

$r_p = \overline{FP}$: periapsis dist.
$r_a = \overline{FA}$: apoapsis dist.
$\overline{AP}$: line of apsides
$\overline{FZ}$: linear excentricity

$\mathcal{E}$: specific Energy
T: period time

Kepler's laws of planetary motion:

I: orbit of planets is ellipse with sun at focus

II: planet-sun line sweeps equal area in equal time

$dA = r\,dr\,d\varphi = \frac{1}{2}r^2 d\varphi \qquad$ (integrate $\int_0^r$)

$dA/dt = \frac{1}{2}r^2\, d\varphi/dt = h/2 \qquad (d\varphi/dt = \dot\varphi = h/r^2)$

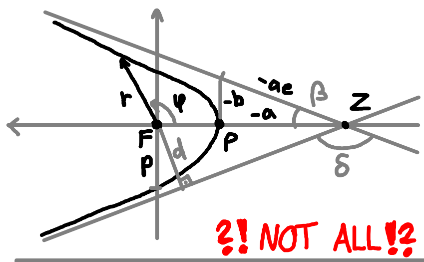
III: T^2 is proportional to a^3

$T = \frac{\text{ellipse\_area}}{dA/dt} = \frac{\pi ab}{h/2} = \ldots = 2\pi\left(\frac{a^3}{\mu}\right)^{1/2}$

$\rightarrow T^2 = \frac{4\pi^2}{\mu}\cdot a^3$

$\rightarrow \left(\frac{T_1}{T_2}\right)^2 \underbrace{\frac{m_0+m_2}{m_0+m_1}}_{\approx 1} = \left(\frac{a_1}{a_2}\right)^3 \qquad$ (T proportions for 2 bodies)

hyperbolic orbit $(e>1)$



p: semi latus rectum
$-a = \overline{ZP}$: semi-major axis
$r_p = \overline{FP}$: periapsis dist.

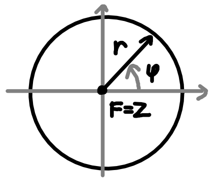
δ: turning angle
d: aiming radius

?! NOT ALL !?

formulas for ellipse are still valid ! but $a<0$!

$\beta = \arccos(1/e) \qquad \delta = \pi - 2\beta = 2\arcsin(1/e)$

circular orbit $(e=0)$



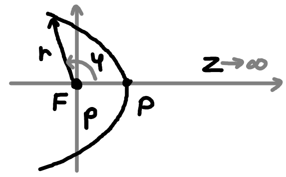
$r = a = b = p = r_p = r_a$

$\mathcal{E} = -\mu/2r$

$V_\varphi = \sqrt{\mu/r}\;, V_r = 0$

$T = 2\pi/\sqrt{\mu}\cdot r^{3/2}$

parabolic orbit $(e=1)$



$r_p = 1/2\, p \;;\; a = \infty$

$\mathcal{E} = 0$

$\|\underline{V}\| = V_{esc} = \sqrt{2\mu/r}$

(V_{esc}: escape velocity)

Position in orbit as func. of time (for 2-body-problem)

elliptic orbit:

area $FPR/t = \pi ab/T \qquad$ (II Kepler law)

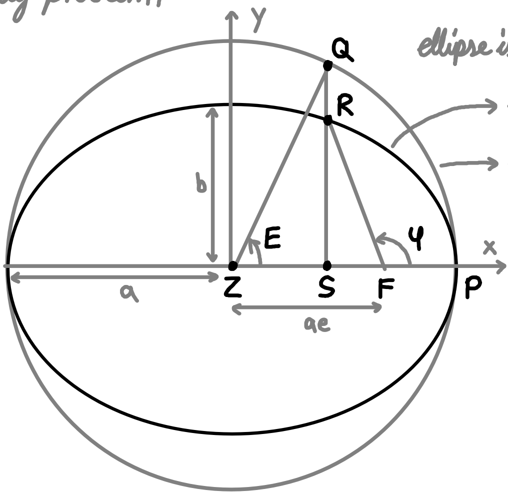
↳ area $FPR(E)$ = area $RSP(E)$ − area $RSF(E)$

area $RSF(E) = \frac{1}{2}(ae - a\cos E)(a\sin E)\frac{b}{a}$

area $RSP(E) = \frac{b}{a}$ area $QSP(E)$
$= \frac{b}{a}\left(\frac{1}{2}a^2E - \frac{1}{2}a\cos E\, a\sin E\right)$

$\rightarrow t(E)\cdot\frac{2\pi}{T} = (E - e\cdot\sin E) = M$

$E(t)$ = only numerical solution!



ellipse is circle squished by $\frac{b}{a}$!

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \rightarrow y = \frac{b}{a}\sqrt{a^2-x^2}$

$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1 \rightarrow y = \sqrt{a^2-x^2}$

E: eccentric anomaly
M: mean anomaly
φ: true anomaly

t: time passed since periapsis pass
T: periode time

$E \leftrightarrow \varphi$ relations: $\cos E = \frac{e+\cos\varphi}{1+e\cos\varphi} \leftrightarrow \tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}}\cdot\tan\frac{\varphi}{2} \qquad$ (from $\overline{ZS} = a\cdot\cos E = ae + r\cos\varphi$)

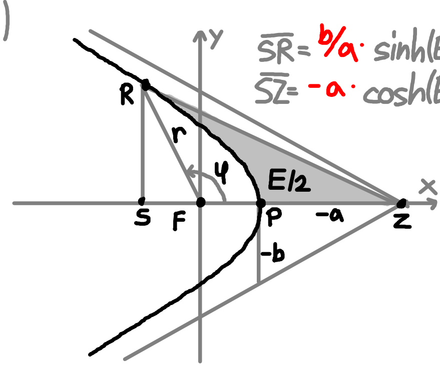
new orbital eq.: $r(E) = a(1 - e\cos(E)) \qquad$ (from orbital equation with $\varphi(E)$)

hyperbolic orbit: (derivation analogous to eliptic orbit!)

$t(E)\cdot\frac{h}{ab} = (e\sinh E - E) = M \quad ; \quad E(t)$ = only numerical solution!

$E \leftrightarrow \varphi$ relations: $\cosh E = \frac{e+\cos\varphi}{1+e\cos\varphi} \leftrightarrow \tanh\frac{E}{2} = \sqrt{\frac{1-e}{1+e}}\cdot\tan\frac{\varphi}{2}$

new orbital eq.: $r(E) = a(1 - e\cosh(E))$

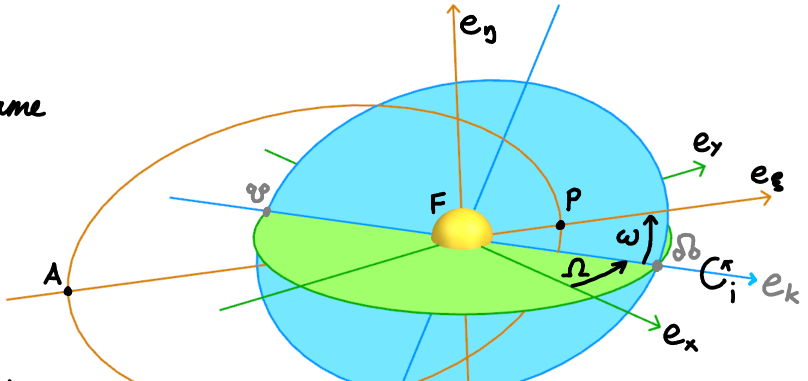


$\overline{SR} = b/a\cdot\sinh(E)$
$\overline{SZ} = -a\cdot\cosh(E)$

Position of Orbit rel. to a reference frame

e_x, e_y, e_z: reference frame ; e_ξ, e_η, e_ζ: orbit frame

(e.g.:
- equator plane earth
- e_x vernal equinox direction
- satellite orbit plane
- satellite trajectory
)



Ω: longitude of the ascending node $= \sphericalangle(e_x, e_k)$

i: inclination $= \sphericalangle(e_z, e_\zeta) = \sphericalangle(e_z, h)$

ω: argument of the periapsis $= \sphericalangle(e_k, e_\xi) = \sphericalangle(e_k, r_p)$

(☊: ascending node; ☋: descending node)

Rocket Dynamics

DO NOT TRY $\frac{d}{dt}(mv) = \dot{m}v + m\dot{v} = F$ — 2nd Newton law is only for const. m !!

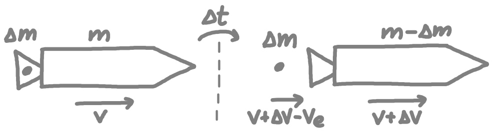
1D equation of motion:

momentum preservation: $P(t+\Delta t) - P(t) = F_{ext} \cdot \Delta t$

$(m-\Delta m)(v+\Delta v) + \Delta m(v+\Delta v - v_e) - mv = F_{ext} \cdot \Delta t$

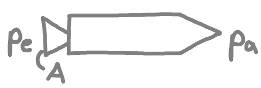
$m\Delta v - \Delta m v_e = F_{ext} \cdot \Delta t$

$\rightarrow m\dot{v} = -\dot{m} v_e + F_{ext}$



(v_e: exhaust velocity rel. to rocket)

→ with exhaust pressure: $m\dot{v} = \underbrace{-\dot{m} v_e + (p_e - p_a)A}_{-\dot{m}c} + F_{ext} = \underbrace{-\dot{m}c}_{S} + F_{ext} = S + F_{ext}$



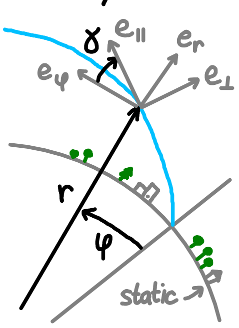
($c = v_e + (p_e - p_a)A/\dot{m}$ effective exhaust vel.)

rocket thrust: $S =$

rocket equation: $\Delta v = c \cdot \ln\left(\frac{m_0}{m_1}\right) \Leftrightarrow \frac{m_1}{m_0} = e^{-(v_1 - v_0)/c}$ (integral $t_0 \to t_1$; $F_{ext} = 0$)

↳ with gravity: $\Delta v = c \cdot \ln\left(\frac{m_0}{m_1}\right) - g\Delta t$ (integral $t_0 \to t_1$; $F_{ext} = mg$) (gravity loss ∝ burn time!)

2D equation of motion:



r, φ: polar pos of rocket

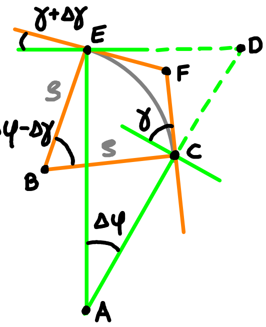
γ: flight path angle

ϱ: flight path curvature

v: abs. rocket velocity

$a_\parallel, a_\perp$: rocket accel. along $e_\parallel, e_\perp$

$(v_\parallel = v, v_\perp = 0)$

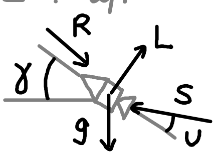


$\sphericalangle EDA = \frac{\pi}{2} - \Delta\varphi$; $\sphericalangle DEF = \gamma + \Delta\gamma$; $\sphericalangle DCF = \frac{\pi}{2} - \gamma$

$\sphericalangle EFC = 2\pi - \sphericalangle EDA - \sphericalangle DEF - \sphericalangle DCF$
$= 2\pi - \pi + \frac{\pi}{2} + \Delta\varphi - \gamma - \Delta\gamma - \frac{\pi}{2} + \gamma$
$= \pi + \Delta\varphi - \Delta\gamma$

$\sphericalangle EBC = 2\pi - \sphericalangle BEF - \sphericalangle BCF - (2\pi - \sphericalangle EFC)$
$= 2\pi - \frac{\pi}{2} - \frac{\pi}{2} - 2\pi + \pi + \Delta\varphi - \Delta\gamma$
$= \underline{\underline{\Delta\varphi + \Delta\gamma}}$

$S = -c\dot{m}$: thrust
$R = \frac{1}{2} v^2 \varrho A C_D$: drag
$g = M/r^2$: gravity
$L = ?$: lift



$$a_\perp = \frac{S}{m} \cdot \sin\upsilon + \frac{L}{m} - g\cos\gamma = a_\perp = v^2/\varrho = v \cdot (\dot{\varphi} - \dot{\gamma}) = -v\dot{\gamma} + \frac{v^2}{r}\cos\gamma$$

$$a_\parallel = \frac{S}{m} \cdot \cos\upsilon - \frac{R}{m} - g \cdot \sin\gamma = a_\parallel = \dot{v}$$

$\boxed{\text{ext. forces}}$ $\boxed{\text{accel. description}}$

$\dot{\varphi} = v_\varphi / r = v \cdot \cos\gamma / r$
$\dot{r} = v_r = v \sin\gamma$
$v = \varrho(\dot{\varphi} - \dot{\gamma})$

(how to deal with rotating earth?
γ angle then no longer rel. to atmosphere...)

staged rocket:

mass of stage i: $m_i = m_{pi} + m_{si}$

mass of rocket at stage i: $m_{0i} = m_i + \ldots + m_n + m_L$

mass ratio at burnout: $Z_i = m_{0i} / m_{si} + m_{0(i+1)}$

structure coefficient: $\sigma_i = m_{si}/m_i = m_{si}/m_{si} + m_{pi} = m_{si}/m_{0i} - m_{0(i+1)}$

payload ratio: $\nu_i = m_{0(i+1)}/m_i = m_{0(i+1)}/m_{si} + m_{pi} = m_{0(i+1)}/m_{0i} - m_{0(i+1)}$

(m_{pi}: propellant of stage i
m_{si}: structure of stage i
m_L: payload)

characteristic velocity:

$\Delta v = \sum_{i=1}^{n} c_i \cdot \ln Z_i$

(vel. change with $F_{ext} = 0$ from rocket equation)

optimal staging: given m_L, σ_i, ΔV_{tot} (desired ΔV) $\rightarrow$ choose Z_i that minimize m_{01}

$$\min_{Z_i} m_{01} \quad \text{subj. to: } \sum_{i=1}^{n} c_i \ln(Z_i) - \Delta V_{tot} = 0$$

m_{01} as func of Z_i, σ_i, m_L:

$$m_{01} = m_L \cdot \frac{m_{01}}{m_{02}} \cdot \frac{m_{02}}{m_{03}} \cdots \frac{m_{0n}}{m_L} = m_L \cdot \prod_{i=1}^{n} \frac{m_{0i}}{m_{0(i+1)}}$$

$$\underline{\underline{m_{01} = m_L \cdot \prod_{i=1}^{n} \frac{(1-\sigma_i) Z_i}{1-\sigma_i Z_i}}}$$

$$\frac{m_{0i}}{m_{0(i+1)}} = \frac{m_{0i}}{m_{0i}-m_i} = \frac{Z_i(\sigma_i-1)/(1-Z_i)\cdot m_i}{Z_i(\sigma_i-1)/(1-Z_i)\cdot m_i - m_i} = \frac{Z_i(\sigma_i-1)}{Z_i(\sigma_i-1)-(1-Z_i)}$$

$$\frac{m_{0i}}{m_{0(i+1)}} = \frac{(1-\sigma_i) Z_i}{1-\sigma_i Z_i}$$

$$Z_i = \frac{m_{0i}}{m_{si}+m_{0(i+1)}}$$
$$m_{0i} = Z_i (m_{si} + m_{0(i+1)})$$
$$m_{0i} = Z_i (\sigma_i m_i + m_{0i} - m_i)$$
$$m_{0i} = Z_i m_i (\sigma_i - 1) + Z_i m_{0i}$$
$$m_{0i}(1-Z_i) = Z_i m_i (\sigma_i - 1)$$
$$m_{0i} = \frac{Z_i(\sigma_i-1)}{1-Z_i} \cdot m_i$$

$$\underbrace{\min_{Z_i} m_{01} = \min_{Z_i} \frac{m_{01}}{m_L} = \min_{Z_i} \ln\left(\frac{m_{01}}{m_L}\right)}_{\text{equivalent minimization problems}} = \min_{Z_i} \ln\left(\prod_{i=1}^{n} \frac{(1-\sigma_i) Z_i}{1-\sigma_i Z_i}\right) = \min_{Z_i} \sum_{i=1}^{n} \ln(1-\sigma_i) + \ln(Z_i) - \ln(1-\sigma_i Z_i)$$

subj. to: $\sum_{i=1}^{n} c_i \ln(Z_i) - \Delta V_{tot} = 0$

Lagrangian func: $L(Z_i, \lambda) = \sum_{i=1}^{n} \ln(1-\sigma_i) + \ln(Z_i) - \ln(1-\sigma_i Z_i) + \lambda \cdot \sum_{i=1}^{n} c_i \ln(Z_i) - \Delta V_{tot}$

dual problem: $\max_{\lambda} \min_{Z_i} L(Z_i, \lambda)$

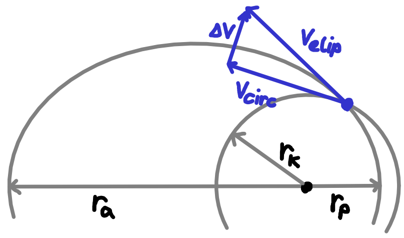
$$0 = \frac{\partial L(Z_i,\lambda)}{\partial Z_i} = \frac{1}{Z_i} + \frac{\sigma_i}{1-\sigma_i Z_i} + \lambda \cdot \frac{c_i}{Z_i} = 1 + \lambda c_i (1-\sigma_i Z_i) \rightarrow Z_i^*(\lambda) = \frac{1+\lambda c_i}{\lambda c_i \sigma_i}$$

$$0 = \frac{\partial L(Z_i = Z_i^*, \lambda)}{\partial \lambda} = \sum_{i=1}^{n} c_i \ln\left(\frac{1+\lambda c_i}{\lambda c_i \sigma_i}\right) - V_{tot} \rightarrow \lambda^* = \text{...calc...}$$

optimal solution: $Z_i^* = Z_i^*(\lambda^*)$

impulsive orbital manovers (instant change of vel.)

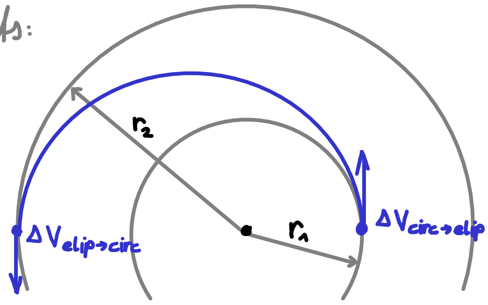
direct: 1 manover transfer between circular (r_K) and elliptic (p,e) orbit:



$$\Delta V^2 = V_{circ}^2 + V_{elip}^2(r_K) - 2\, V_{circ} \cdot V_{elip,\varphi}(r_K) = \mu/r_K + \mu\left(\frac{2}{r_K} - \frac{1-e^2}{p}\right) - 2\sqrt{\mu/r_K} \cdot \sqrt{\mu p}/r_K$$

orbits must intersect! $\rightarrow r_p = p/(1+e) \leq r_K \leq p/(1-e) = r_a$

Hohmann: 2 manover transfer between circular (r_1) and circular (r_2) orbits:

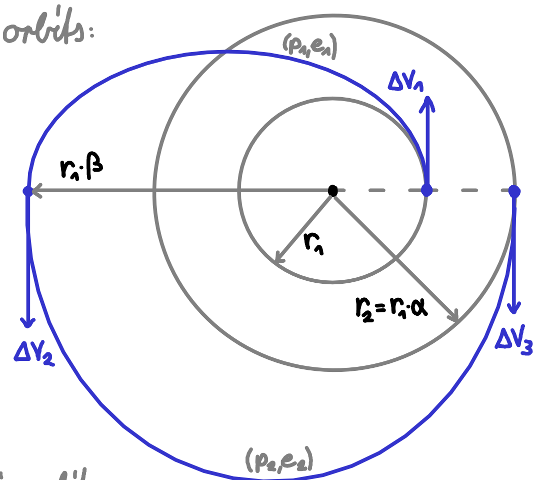


$$\min_{p,e} \underbrace{\Delta V_{circ\rightarrow elip} + \Delta V_{elip\rightarrow circ}}_{\text{minimize } \Delta V_{tot}} \quad \text{subj. to: } \underbrace{r_p = p/(1+e) \leq r_1 \;;\; r_2 \leq p/(1-e) = r_a}_{\text{intersect both circles}}$$

HARD $\rightarrow \Delta V^* = \Delta V_H = \sqrt{\mu}\left[\left(\sqrt{\frac{2}{r_1} - \frac{2}{r_1+r_2}} - \sqrt{\frac{1}{r_1}}\right) + \left(\sqrt{\frac{2}{r_2} - \frac{2}{r_1+r_2}} - \sqrt{\frac{1}{r_2}}\right)\right]$

$\rightarrow e^* = e_H = r_2 - r_1 / r_1 + r_2 \qquad p^* = p_H = r_1(1+e_H) \quad \rightarrow r_p = r_1 \;;\; r_a = r_2$

Bi-elliptic: 3 manover transfer between circular (r_1) and circular (r_2) orbits:

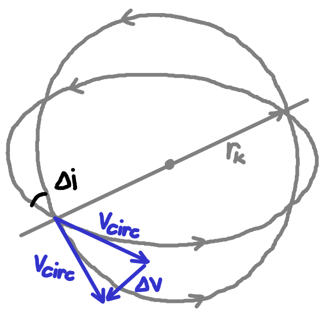


$$\min_{p_1,e_1,p_2,e_2} \Delta V_1 + \Delta V_2 + \Delta V_3 \quad \text{subj. to: intersect } r_1 \rightarrow \beta r_1 \rightarrow \alpha r_1$$

HARD $\rightarrow \Delta V^* = \Delta V_{bi} = \sqrt{\frac{\mu}{r_1}} \cdot \left(\sqrt{\frac{2}{\beta(\beta+1)}}(\beta-1) + \sqrt{2\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)} - \sqrt{\frac{1}{\alpha}} - 1\right)$

$\rightarrow p_1^*, e_1^*, p_2^*, e_2^* = \ldots \leftarrow r_{p1} = r_1; r_{a1} = \beta r_1 = r_{a2}; r_{p2} = r_2$

(optimal β^* one of: $\beta^* = \alpha$ ($\rightarrow$ Hohmann)
$\beta^* = \infty$ ($\rightarrow$ parabola to inf and back)
if $\beta^* = \infty$, then some $\alpha < \beta < \infty$ can be better than Hohmann!)



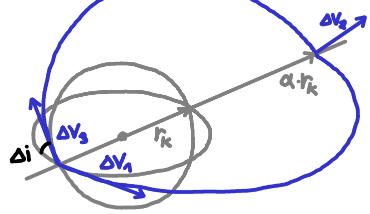
inclination change Δi: 1 manover between 2 circ. orbits

$$\Delta V = V_{circ} \cdot 2\sin\left(\frac{\Delta i}{2}\right) = \sqrt{\mu/r_K} \cdot 2\sin\left(\frac{\Delta i}{2}\right) \quad (r_K = r_{K1} = r_{K2})$$

bi-elliptic inclination change Δi: same but 3 manover

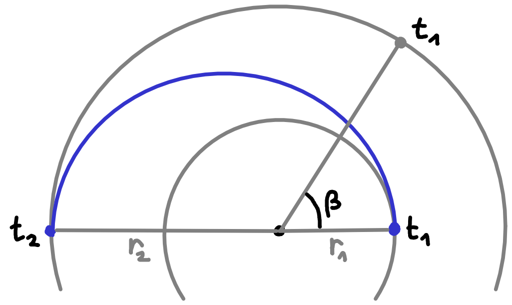
$$\Delta V = \sqrt{\mu/r_K} \cdot \left(\sqrt{\frac{2}{\alpha}\left(\frac{(\alpha-1)}{(\alpha+1)}+1\right)} - 2 + 2\sqrt{\frac{1}{\alpha}}\sin\left(\frac{\Delta i}{2}\right)\right)$$

(out to $\alpha \cdot r_K \rightarrow$ inclin change $\Delta i \rightarrow$ back to r_K)



- **rendevouz** (Hohmann transfer, but arrive when target is at arrival location)

synodic period: $S = 2\pi/(\omega_1-\omega_2) = 1/\left(\frac{1}{T_1}-\frac{1}{T_2}\right)$ ($\omega = 2\pi/T$ angular velocity)
(time before same relative pos. happens again)



probe travel → ← target travel

$$\Delta t = \frac{T_H}{2} = \frac{\pi}{\sqrt{\mu}}\cdot a^{3/2} = \frac{\pi}{\sqrt{\mu}}\left(\frac{r_1+r_2}{2}\right)^{3/2} = \frac{\pi}{\sqrt{\mu}}\left(\frac{\pi-\beta}{\pi}\cdot r_2^{3/2}\right) = \frac{\pi-\beta}{2\pi}\cdot\frac{2\pi}{\sqrt{\mu}}\, r_2^{3/2} = \Delta t$$

$$\hookrightarrow \left(\frac{r_1+r_2}{2}\right)^{3/2} = \frac{\pi-\beta}{\pi}\cdot r_2^{3/2} \to \beta = \pi - \pi\left(\frac{r_1+r_2}{2r_2}\right)^{3/2} = \pi\left(1-\left(\frac{1+r_1/r_2}{2}\right)^{3/2}\right)$$

Continuous orbital manovers (const. accelleration)

assumptions: orbit always circular ($\dot{\underline{r}} = \sqrt{\mu/r}\, e_\varphi$) and thrust $S \parallel \dot{\underline{r}}$ (only valid for realy small thrust S)

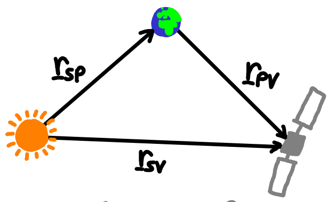
circ orbit: $\dot{\underline{r}} = \sqrt{\frac{\mu}{r}}\, e_\varphi \to \ddot{\underline{r}} = -\sqrt{\frac{\mu}{r}}\,\dot\varphi\, e_r - \frac{1}{2}\sqrt{\frac{\mu}{r}}\,\frac{\dot r}{r}\, e_\varphi \to \ddot r_\varphi = \frac{1}{m}S = -\frac{1}{2}\sqrt{\frac{\mu}{r}}\,\frac{\dot r}{r}$

EoM with S: $\ddot{\underline{r}} = \underbrace{-\frac{\mu}{r^2}\, e_r}_{\text{gravity}}\ \underbrace{+\frac{1}{m}S\, e_\varphi}_{\text{thrust}}$

$\hookrightarrow \Delta V = \int_{t_0}^{t_1}\frac{1}{m}S\,dt = \int_{t_0}^{t_1} -\frac{1}{2}\sqrt{\frac{\mu}{r}}\,\frac{\dot r}{r}\,dt = \sqrt{\frac{\mu}{r_0}} - \sqrt{\frac{\mu}{r_1}} = \Delta V$

Domain of influence of a planet

(domain inside which less error is made by considering only gravity of planet compared to only considering gravity of sun)



$\ddot{\underline{R}}_V = -\frac{Gm_P}{r_{PV}^3}\cdot\underline{r}_{PV} - \frac{Gm_S}{r_{SV}^3}\cdot\underline{r}_{SV}$

$\ddot{\underline{R}}_P = +\frac{Gm_V}{r_{PV}^3}\cdot\underline{r}_{PV} - \frac{Gm_S}{r_{SP}^3}\cdot\underline{r}_{SP}$

$\ddot{\underline{R}}_S = +\frac{Gm_V}{r_{SV}^3}\cdot\underline{r}_{SV} + \frac{Gm_S}{r_{SP}^3}\cdot\underline{r}_{SP}$

$$\ddot{\underline{r}}_{PV} = \ddot{\underline{R}}_V - \ddot{\underline{R}}_P = \underbrace{-\frac{G(m_P+m_V)}{r_{PV}^3}\cdot\underline{r}_{PV}}_{A_P}\ \underbrace{-\,Gm_S\left(\frac{\underline{r}_{SV}}{r_{SV}^3}+\frac{\underline{r}_{SP}}{r_{SP}^3}\right)}_{P_S}$$

A_P: accel. planet; P_S: perturb. sun
$\hookrightarrow A_P/P_S$: "only planet" error

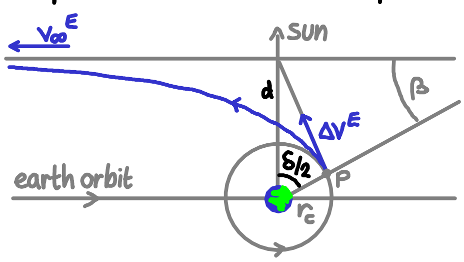
$$\ddot{\underline{r}}_{SV} = \ddot{\underline{R}}_V - \ddot{\underline{R}}_S = \underbrace{-\frac{G(m_P+m_V)}{r_{PV}^3}\cdot\underline{r}_{PV}}_{A_S}\ \underbrace{-\,Gm_S\left(\frac{\underline{r}_{SV}}{r_{SV}^3}+\frac{\underline{r}_{SP}}{r_{SP}^3}\right)}_{P_P}$$

A_S: accel. sun; P_P: perturb. planet
$\hookrightarrow A_S/P_P$: "only sun" error

$\hookrightarrow$ planet domain of influence $= \{ r \mid A_P/P_S \leq A_S/P_P \} \approx$ sphere with $r_{inf} \approx \left(\frac{m_P}{m_S}\right)^{2/5}\cdot\|r_{SP}\|$

Patched conics (calculate only with body of domain of influence + assume to be at inf. when leaving/after entering)

- **departure + Hohmann transfer + reentry** (in 2 manovers) (spheres of influence of planets must be small!)



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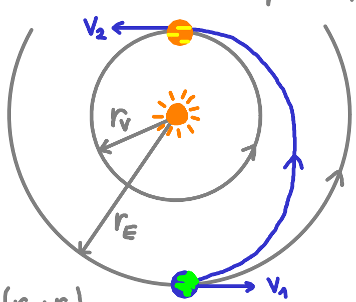

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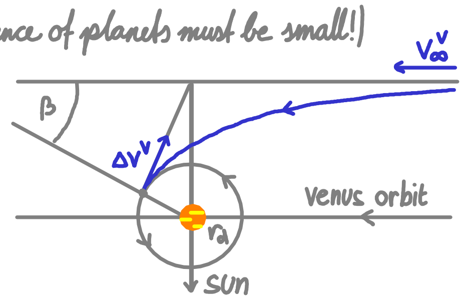


$V_0 = \sqrt{\mu_E/r_c}$

$V_0 + \Delta V^E = \sqrt{\mu_E(2/r_c - 1/a)} = 2V_0 + V_\infty^E$

$V_\infty^E = \sqrt{\mu_E(-1/a)}$

$V_1 = \sqrt{\mu_S/r_E} - V_\infty^E$

$a = \frac{1}{2}(r_E + r_V)$

$V_1 = \sqrt{\mu_S(2/r_E - 1/a)}$ $\quad V_2 = \sqrt{\mu_S(2/r_V - 1/a)}$

$V_\infty^E = \sqrt{\mu_S/r_E} - V_1$ $\quad V_\infty^V = \sqrt{\mu_S/r_V} + V_2$

$\Delta V^E = \sqrt{\mu_E/r_c} + V_\infty^E$ $\quad \Delta V^V = \sqrt{\mu_V/r_a} + V_\infty^V$

$V_f = \sqrt{\mu_V/r_a}$

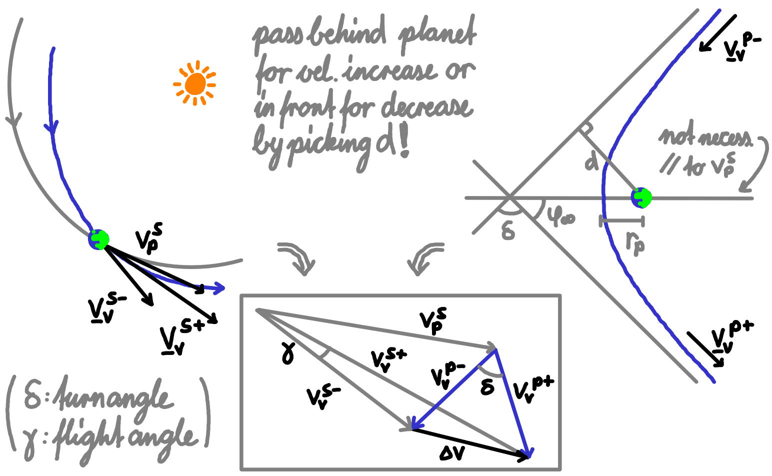
$V_f + \Delta V^V = \sqrt{\mu_V(2/r_a - 1/a)} = 2V_f + V_\infty^V$

$V_\infty^V = \sqrt{\mu_V(-1/a)}$

$V_2 = V_\infty^V - \sqrt{\mu_S/r_V}$

(δ/2 can be calculated from orbit eq. @ periapsis solved for e → $\delta/2 = \arcsin(1/e)$
d can be calculated from angular momentum preservation $V_\infty d = V_p r_p$)

Gravity assist manouer (use planet for "free" ΔV)



$\Delta V = |V_v^{S+} - V_v^{S-}|$ (vel. change in sun frame)

$= |V_v^{P+} - V_v^{P-}|$ (vel. change in planet frame)

$= 2V_\infty \cdot \sin(\delta/2)$ (from geometry with $V_\infty = |V_v^{P-}| = |V_v^{P+}|$)

$= 2V_\infty / e$ (from parabola eq. $\delta = 2\arcsin(1/e)$)

$= 2V_\infty / (v_p^2 r_p/\mu - 1)$ (from solve orb. eq. for e at $\varphi=0$)

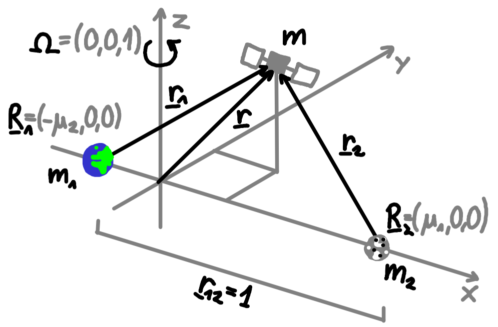
$= 2V_\infty / (V_\infty^2 r_p/\mu + 1)$ (from $\mathcal{E}$ preserv. at $\varphi=0$ and inf.)

(maximal ΔV for a given r_p: **relevance?**

$d\Delta V / dV_\infty = \dots = 0 \rightarrow V_\infty^* = \sqrt{\mu/r_p} \rightarrow \Delta V^* = \sqrt{\mu/r_p}$

↳ $e^* = 2$; $\delta^* = 60° = \pi/3$; $\varphi_\infty^* = \frac{\pi - \delta}{2}$; $\gamma^* = ?$; $d^* = ?$)

Restricted 3-body problem (works when patched conics is not applicable)



set up a rotating frame e_x, e_y, e_z where m_1 and m_2 appear stationary, such that: (must be circular orbit!)

- origin at baricenter ($R_1 m_1 + R_2 m_2 = 0$) and m_1, m_2 on x-axis
- axes scaled such that $r_{12} = R_2 - R_1 = 1$
- masses scaled such that $\mu_{12} = G(m_1 + m_2) = \mu_1 + \mu_2 = 1$ ($\mu_{1,2} = G m_{1,2}$)

↳ $0 = R_1 m_1 + R_2 m_2 = R_1 m_1 + (R_1 - 1)(\frac{1}{G} - m_1) = \dots \rightarrow R_1 = 1 - G m_1 = 1 - \mu_1 = \mu_2$

↳ $0 = R_1 m_1 + R_2 m_2 = (R_2 - 1)(\frac{1}{G} - m_2) + R_2 m_2 = \dots \rightarrow R_2 = 1 - G m_2 = 1 - \mu_2 = \mu_1$

↳ $T = 2\pi \sqrt{a^3/\mu_{12}} = 2\pi \sqrt{r_{12}^3/\mu_{12}} = 2\pi \rightarrow \Omega = 2\pi/T = 1$

position of craft: $\underline{r} = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} \rightarrow \dot{\underline{r}} = \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{bmatrix} + \underline{\Omega} \times \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} \rightarrow \ddot{\underline{r}} = \begin{bmatrix} \ddot{x}(t) \\ \ddot{y}(t) \\ \ddot{z}(t) \end{bmatrix} + 2\underline{\Omega} \times \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{bmatrix} + \underline{\Omega} \times \left(\underline{\Omega} \times \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}\right) = \begin{bmatrix} \ddot{x}(t) - 2\dot{y}(t) - x(t) \\ \ddot{y}(t) + 2\dot{x}(t) - y(t) \\ \ddot{z}(t) \end{bmatrix}$

forces on craft: $m\ddot{\underline{r}} = \underline{F}_1 + \underline{F}_2 = -\frac{G m_1 m}{r_1^3} \underline{r}_1 - \frac{G m_2 m}{r_2^3} \underline{r}_2 \rightarrow \ddot{\underline{r}} = -\frac{\mu_1}{r_1^3} \underline{r}_1 - \frac{\mu_2}{r_2^3} \underline{r}_2$ with $\underline{r}_1 = \begin{bmatrix} x + \mu_2 \\ y \\ z \end{bmatrix}, \underline{r}_2 = \begin{bmatrix} x - \mu_1 \\ y \\ z \end{bmatrix}$

↳ combined EoM:

$\ddot{x} - 2\dot{y} - x = -\frac{\mu_1}{r_1^3}(x + \mu_2) - \frac{\mu_2}{r_2^3}(x - \mu_1) \rightarrow \ddot{x} - 2\dot{y} = \partial U / \partial x$

$\ddot{y} - 2\dot{x} - y = -\frac{\mu_1}{r_1^3} y - \frac{\mu_2}{r_2^3} y \rightarrow \ddot{y} - 2\dot{x} = \partial U / \partial y$

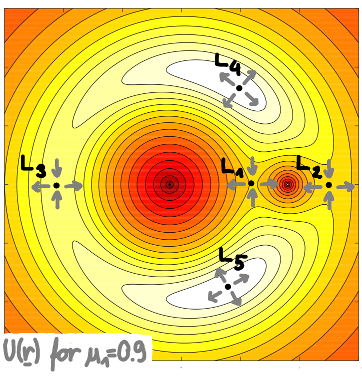
$\ddot{z} = -\frac{\mu_1}{r_1^3} z - \frac{\mu_2}{r_2^3} z \rightarrow \ddot{z} = \partial U / \partial z$

with $U = \frac{1}{2}(x^2 + y^2) + \mu_1/r_1 + \mu_2/r_2$

$r_1 = \sqrt{(x + \mu_2)^2 + y^2 + z^2}$

$r_2 = \sqrt{(x - \mu_1)^2 + y^2 + z^2}$

(U from left side of EoM: $dU/dt = \partial U/\partial x \cdot \dot{x} + \partial U/\partial y \cdot \dot{y} + \partial U/\partial z \cdot \dot{z} = \ddot{x}\dot{x} + \ddot{y}\dot{y} + \ddot{z}\dot{z} \rightarrow U = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 + \overset{\text{const.}}{C})$)



$U(\underline{r})$ for $\mu_1 = 0.9$

for zero vel. in rot. frame ($\dot{x} = \dot{y} = \dot{z} = 0$) one gets $[\ddot{x}, \ddot{y}, \ddot{z}]^T = [\partial U/\partial x, \partial U/\partial y, \partial U/\partial z]^T = \text{grad}(U) = \nabla U$

equilibrium points ($\nabla U = \underline{0} \Leftrightarrow \ddot{x} = \ddot{y} = \ddot{z} = 0$):

$L_1 = \begin{bmatrix} \mu_1 - (\mu_2/3)^{1/3} \\ 0 \\ 0 \end{bmatrix}$ (unstable); $L_2 = \begin{bmatrix} \mu_1 + (\mu_2/3)^{1/3} \\ 0 \\ 0 \end{bmatrix}$ (unstable); $L_3 = \begin{bmatrix} -\mu_1 - 5/12 \cdot \mu_2 \\ 0 \\ 0 \end{bmatrix}$ (unstable); $L_4 = \begin{bmatrix} (\mu_1 - \mu_2)/2 \\ \sqrt{3}/2 \\ 0 \end{bmatrix}$ (stable); $L_5 = \begin{bmatrix} (\mu_1 - \mu_2)/2 \\ -\sqrt{3}/2 \\ 0 \end{bmatrix}$ (stable)

(L4/L5 stable only because of coriolis force when $v \neq 0$ → can't stay, but can orbit!)