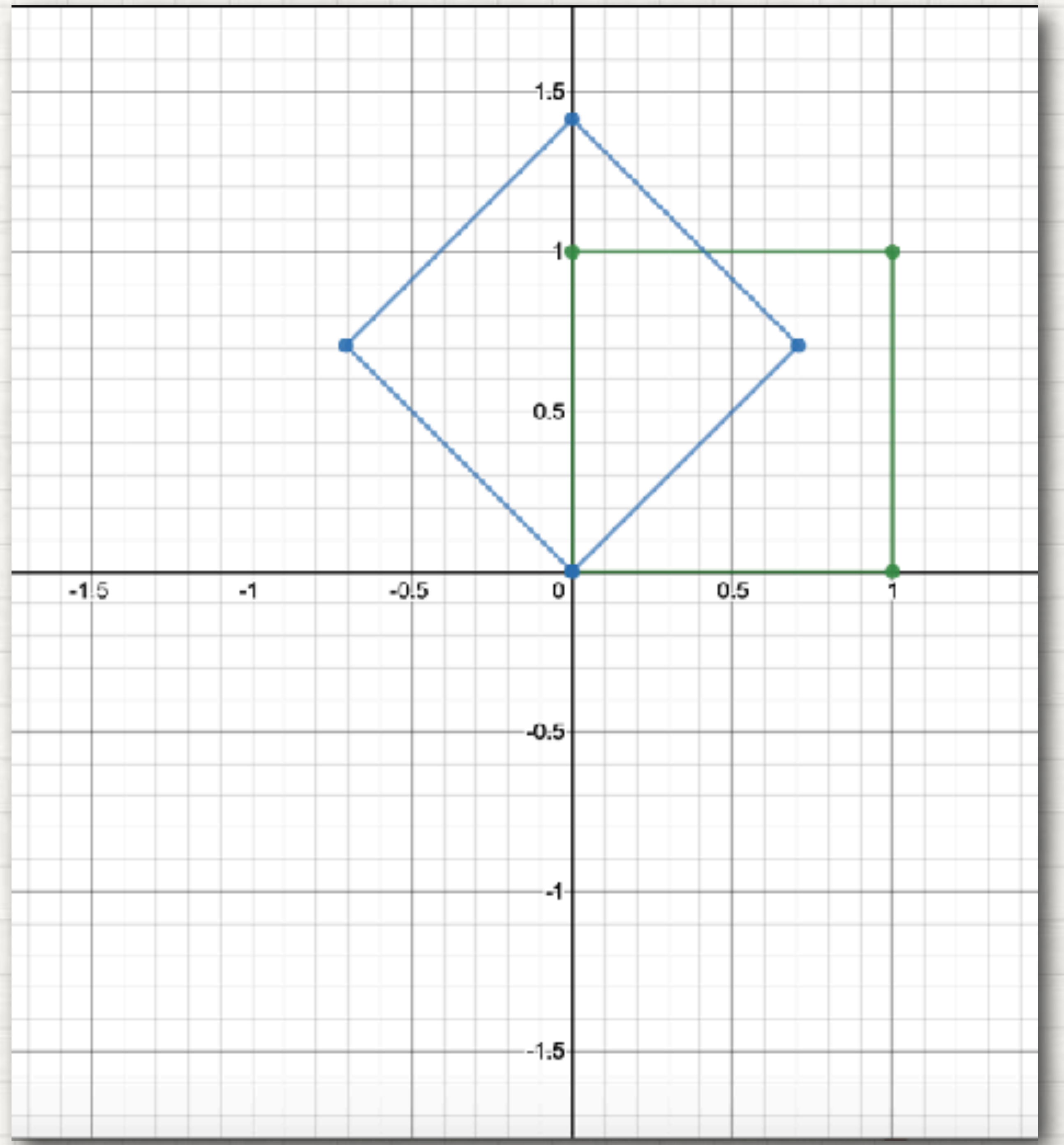
ROTATIONS

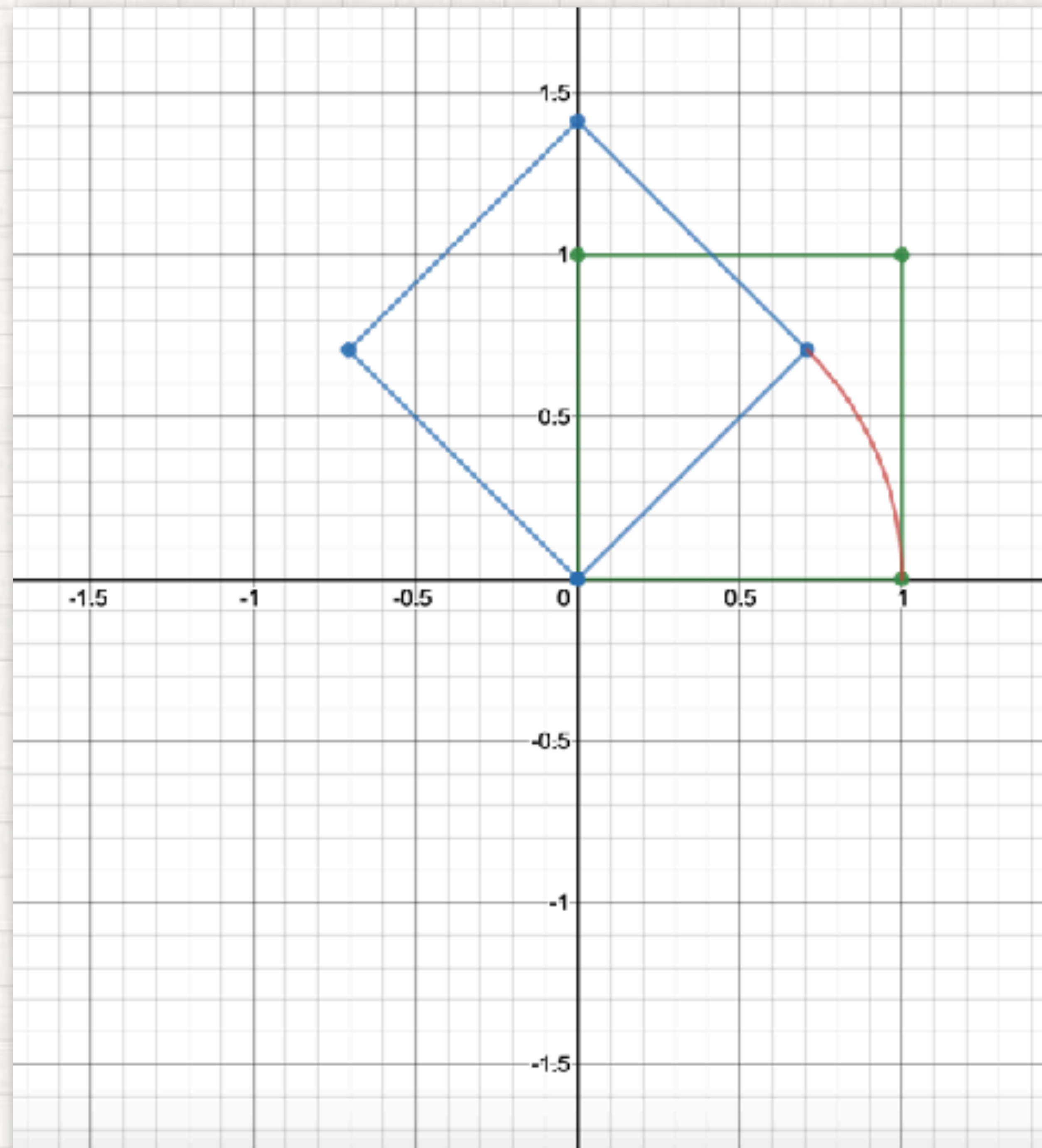
THE ROTATION PROBLEM

- The vertices of the green square are (0, 0), (0, 1), (1, 1), and (1, 0).
- If you rotate the green square by 45 degrees counterclockwise, then we get the blue square.
- We are given the vertices of the green square. How do you compute the vertices of the blue square from the vertices of the green square?



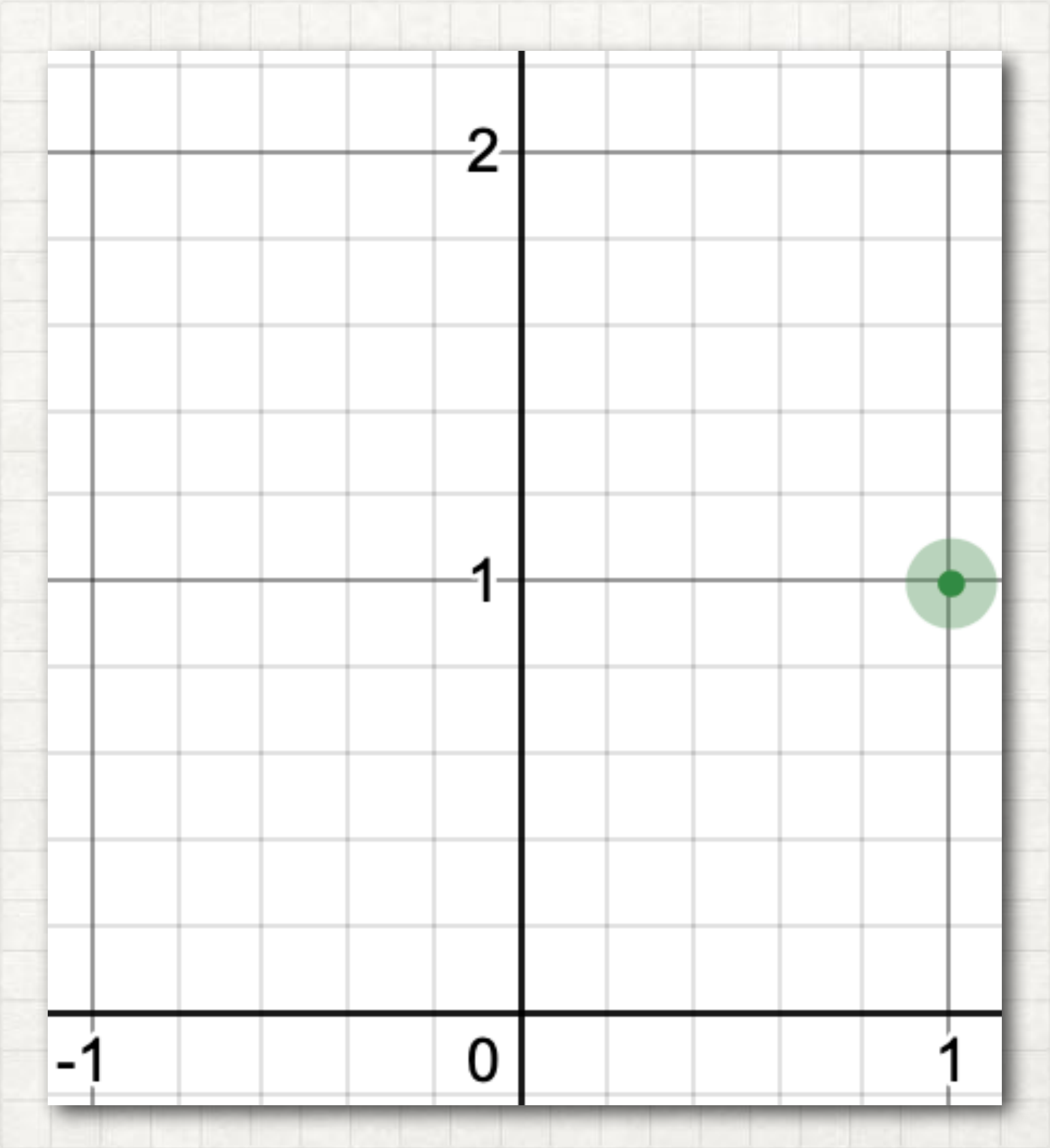
THE ROTATION PROBLEM

- To rotate the green square by 45 degrees counterclock-wise, you rotate each vertex by 45 degrees.



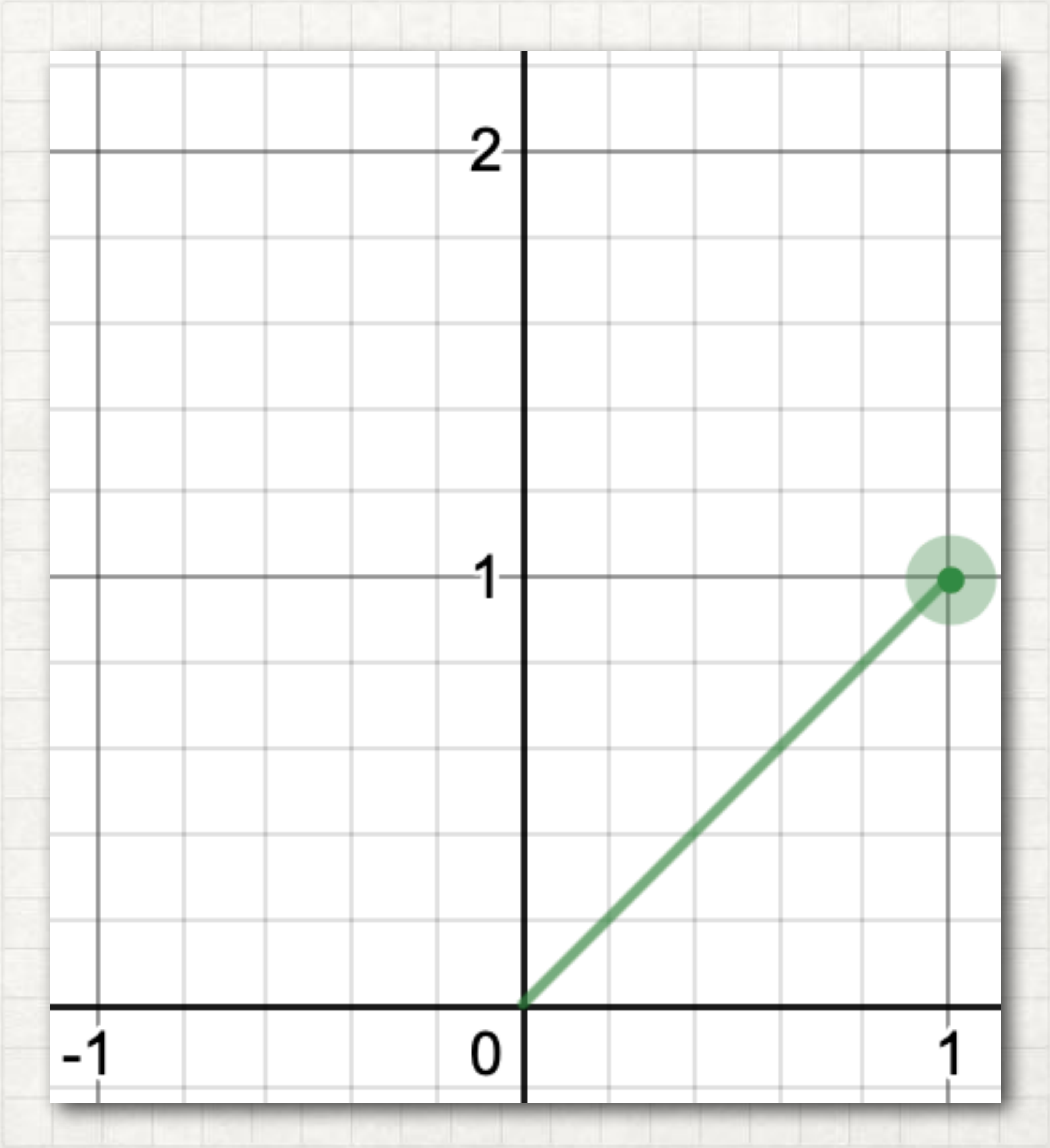
ROTATING A POINT

- Here is the point (1, 1).
- How do you rotate it by 45 degrees counterclockwise?



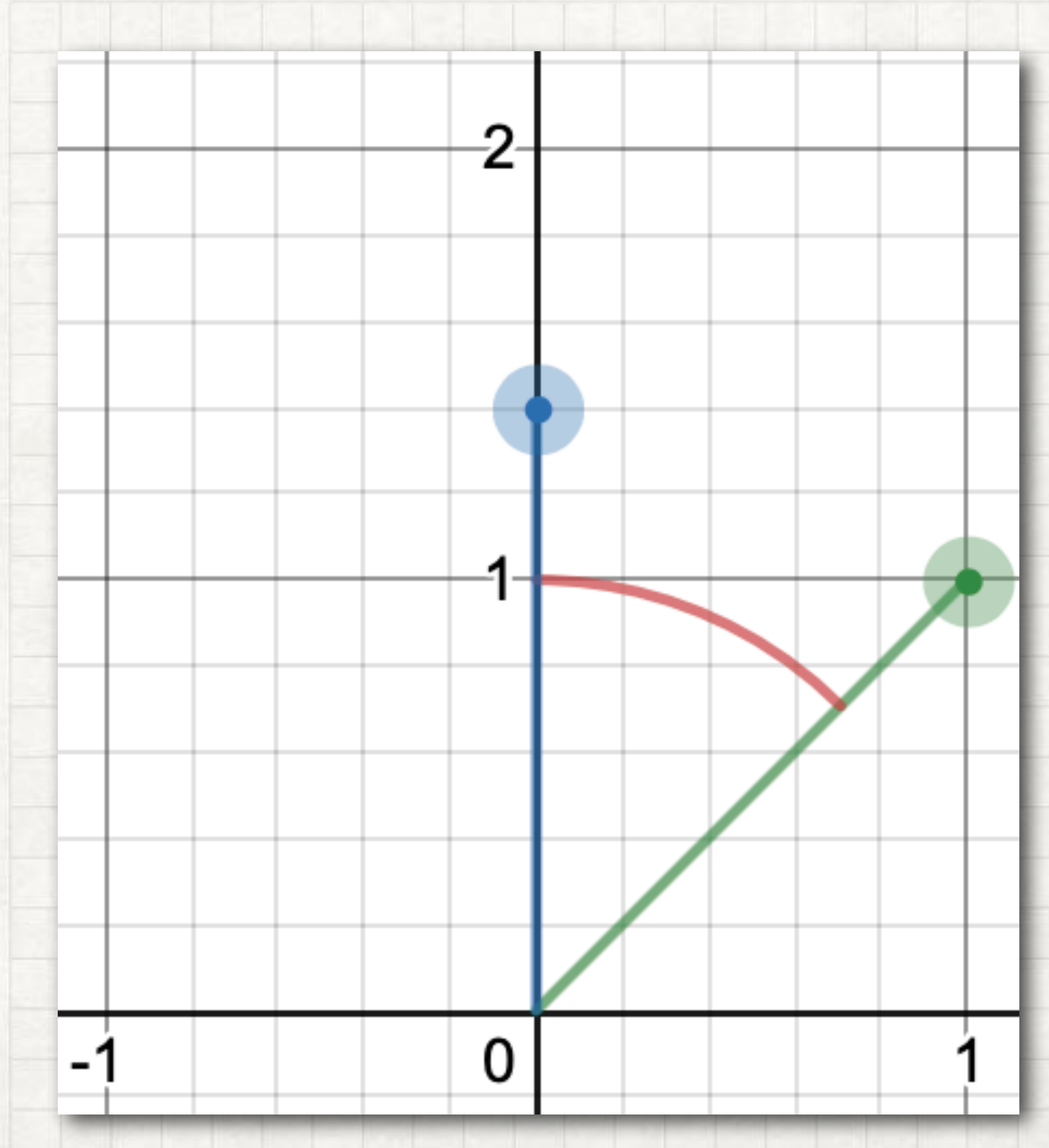
ROTATING A POINT

- First, draw a line from (0, 0) to the point you want to rotate.
- In this case, we draw a line from (0, 0) to (1, 1).



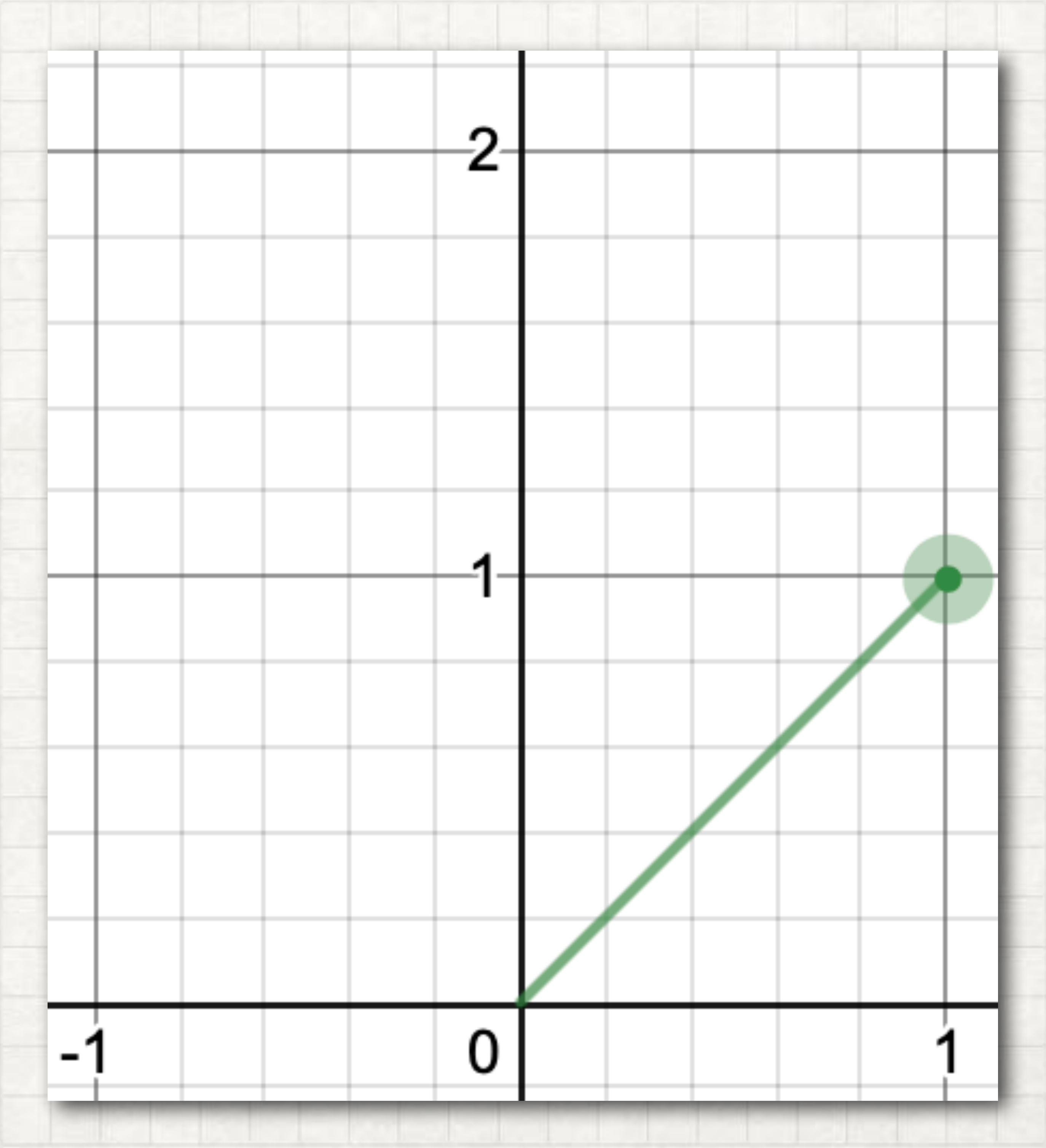
ROTATING A POINT

- Then, you drag the line by 45 degrees counterclock-wise.
- If we rotate the green line by 45 degrees counterclock-wise, then we get the blue line.
- The angle between the green line and the blue line is 45 degrees.
- The length of the green line is equal to the length of the blue line.



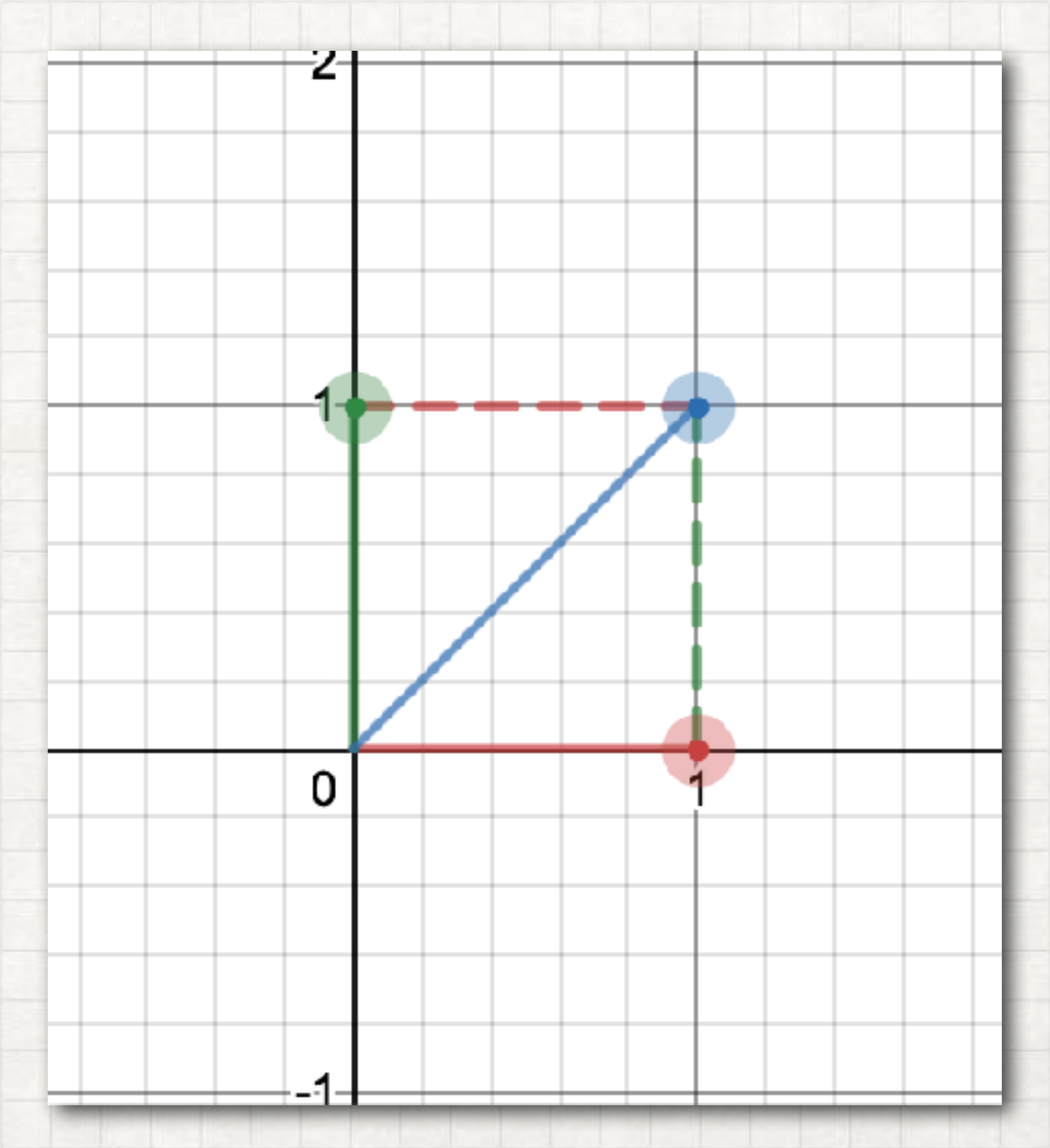
VECTORS

- The line from (0, 0) to a point is useful.
- Such a thing is called a vector.
- If you know the end point of a vector, then you know the vector, since you just draw a line from (0, 0) to the end point. Because of this, we refer to vectors by their end points.
- For example, here, we have the vector (1, 1).



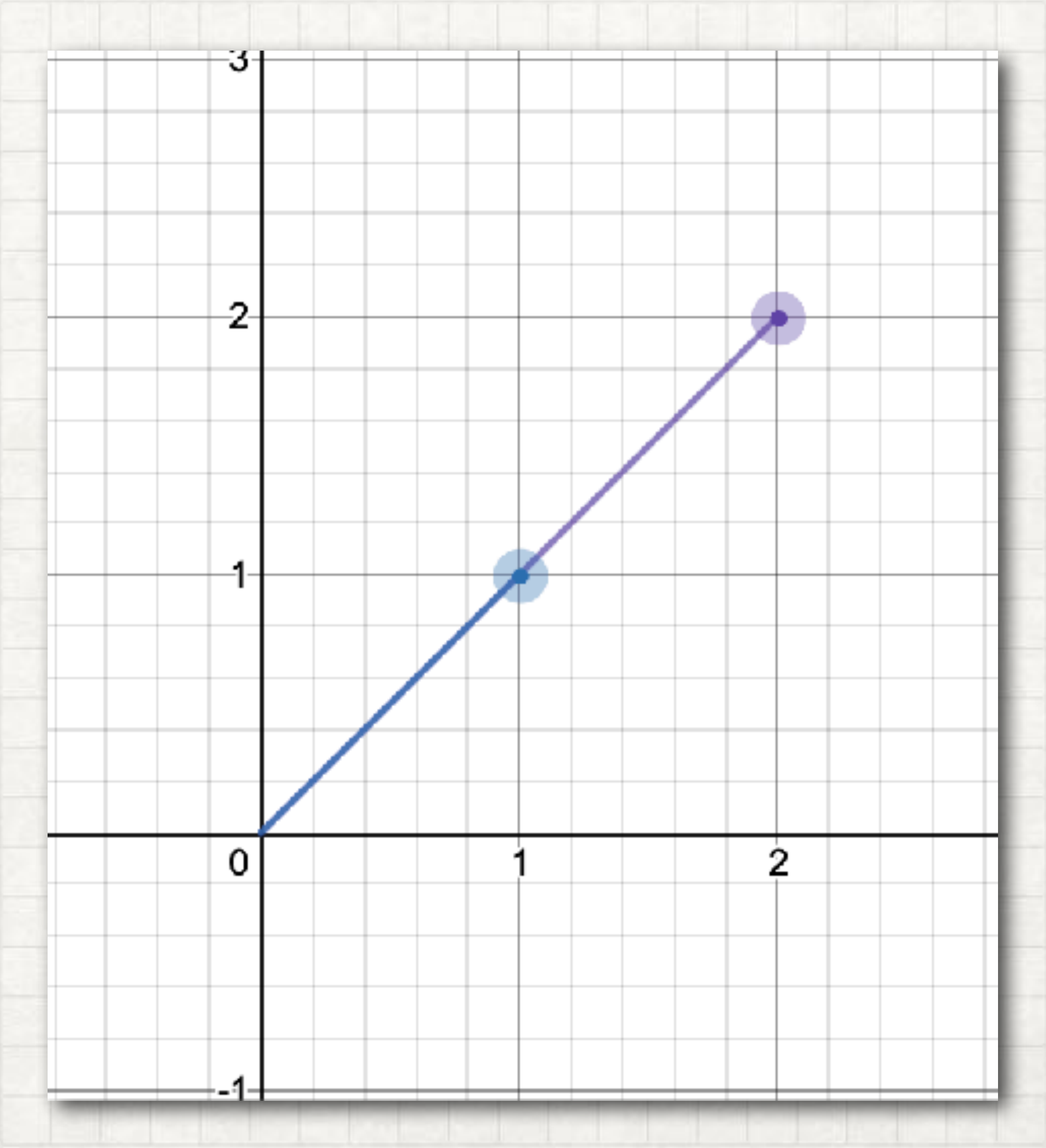
VECTORS

- You can add two vectors.
- Here, the red vector is (1, 0), and the green vector is (0, 1).
- If you add them, you get the blue vector, (1, 1).
- $(x_0, y_0) + (x_1, y_1) = (x_0 + x_1, y_0 + y_1)$.



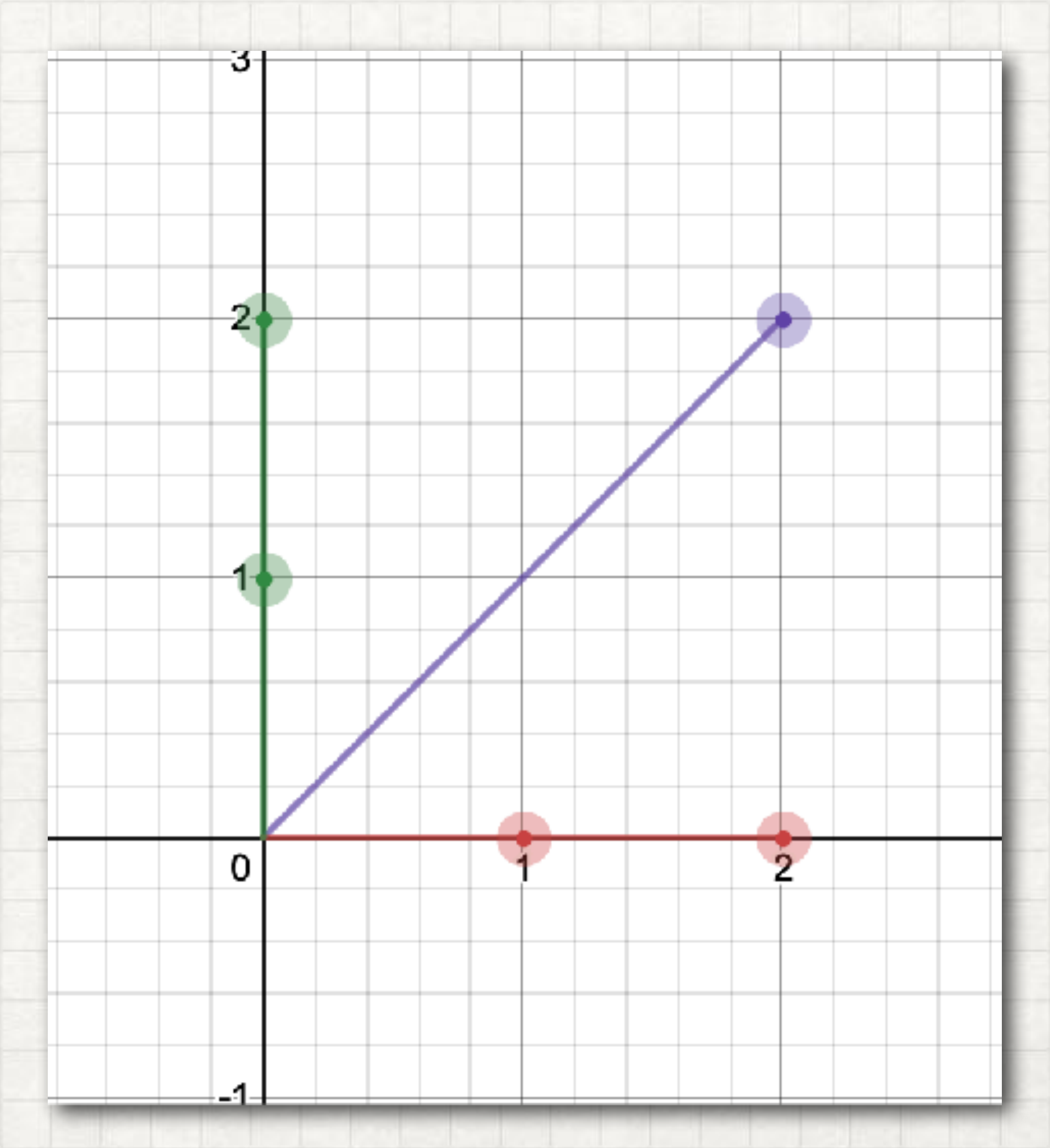
VECTORS

- You can add multiply a vector by a number.
- Here, the blue vector is (1, 1).
- If you multiply the blue vector by 2, then you get the purple vector, (2, 2).
- The purple vector is two times the length of the blue vector.
- $k \cdot (x, y) = (k \cdot x, k \cdot y)$.



VECTORS

- Any vector (x, y) can be written as x · (1, 0) + y · (0, 1).
- Here, (2, 2) = 2 · (1, 0) + 2 · (0, 1).



ROTATING A VECTOR

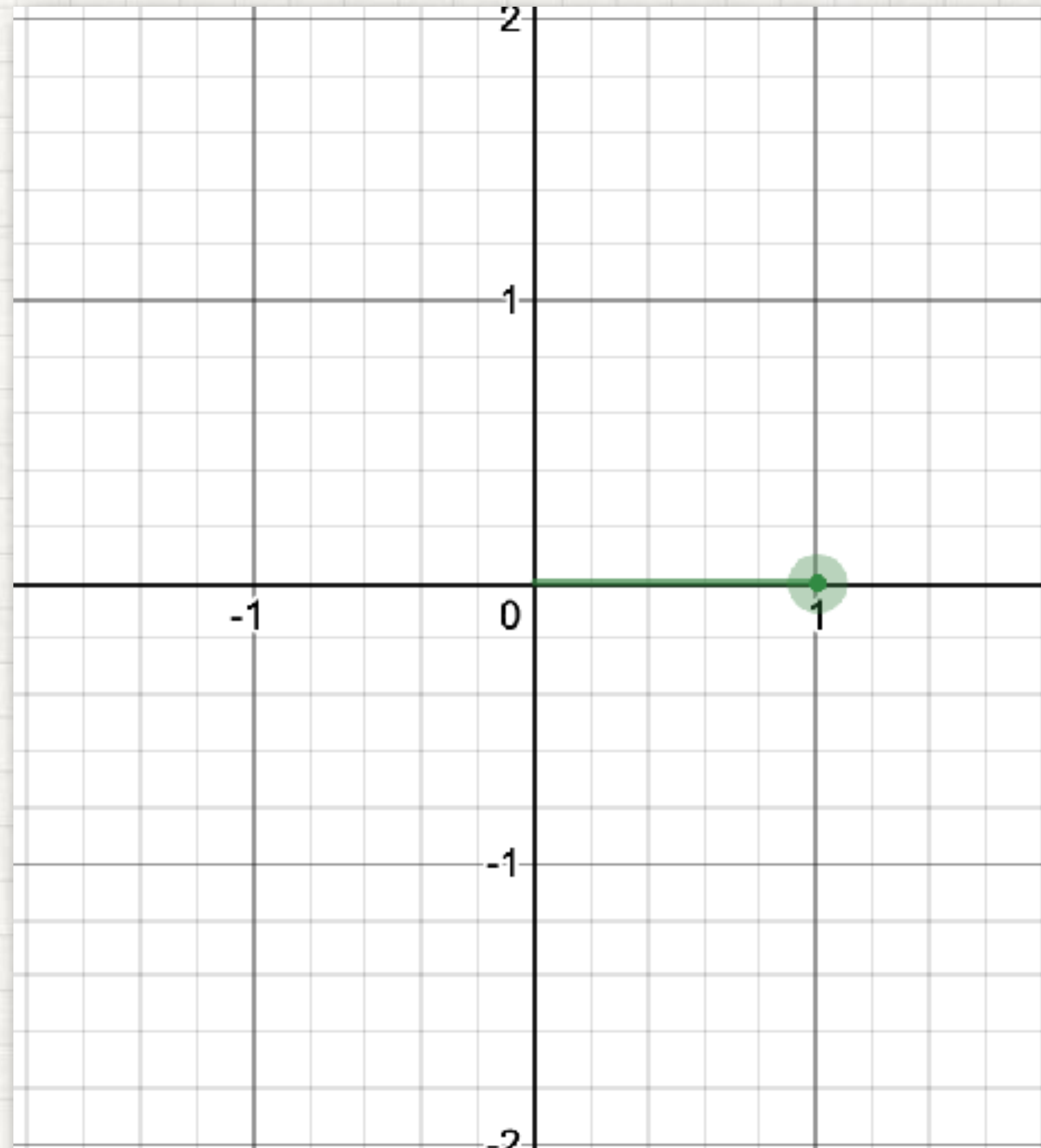
- Suppose that you are given a vector (x_0, y_0). Let (x_1, y_1) be the vector you get from rotating (x_0, y_0) by 45 degrees counterclockwise.
- How do you compute (x_1, y_1)?

ROTATING A VECTOR

- There are two facts about rotations that will help us.
- If v is a vector, then let R(v) be the result of rotating v by 45 degrees counterclockwise.
- First, rotating and then multiplying is the same as multiplying and then rotating.
    - $R(v) \cdot k = R(k \cdot v)$, where k is a number and v is a vector.
- Second, rotating and then adding is the same as adding and then rotatinig.
    - $R(u) + R(v) = R(u + v)$, where u and v are vectors.

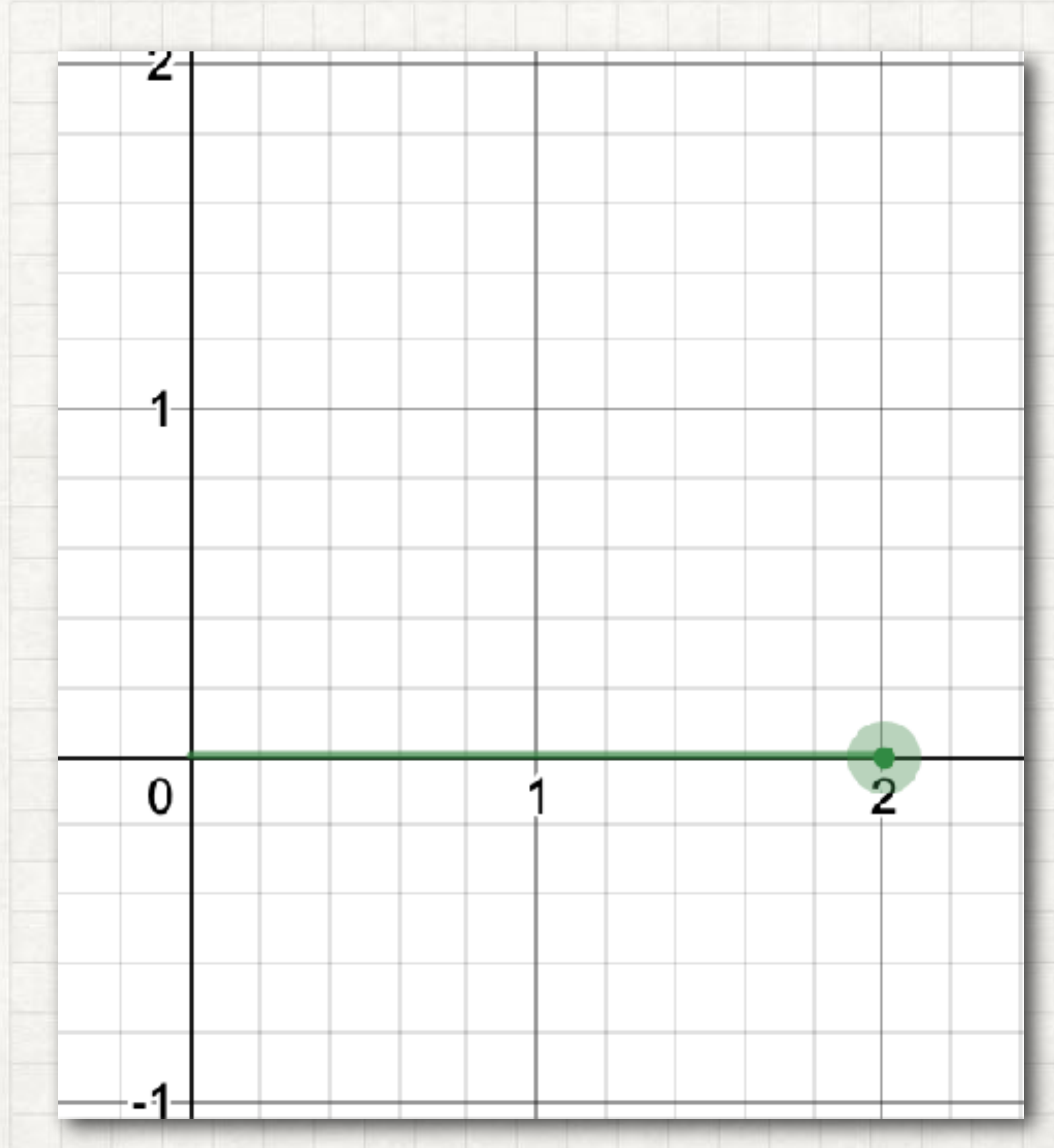
ROTATING AND MULTIPLYING

- Here is the vector (1, 0).



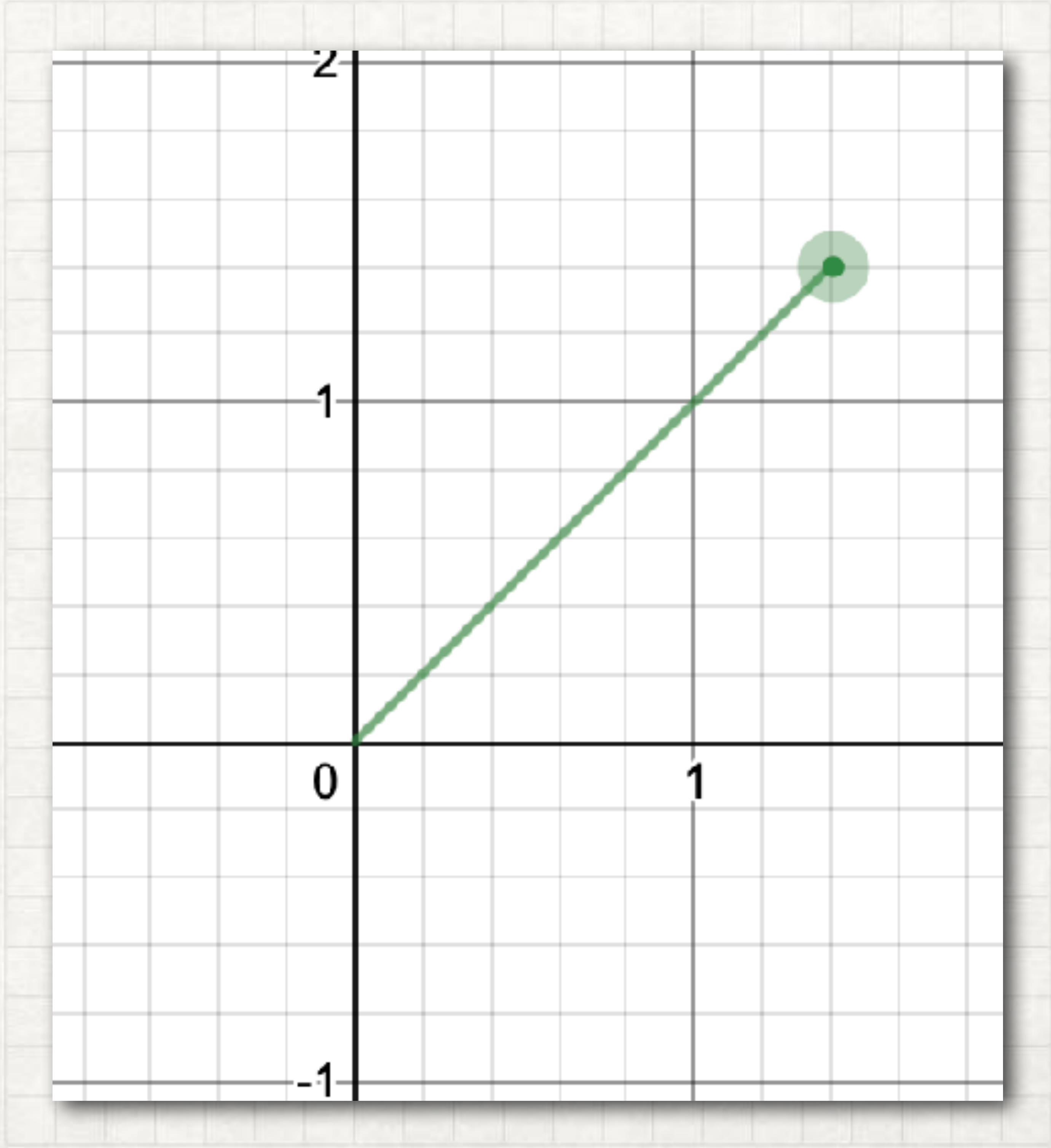
ROTATING AND MULTIPLYING

- If you double its length, then you get the vector (2, 0).



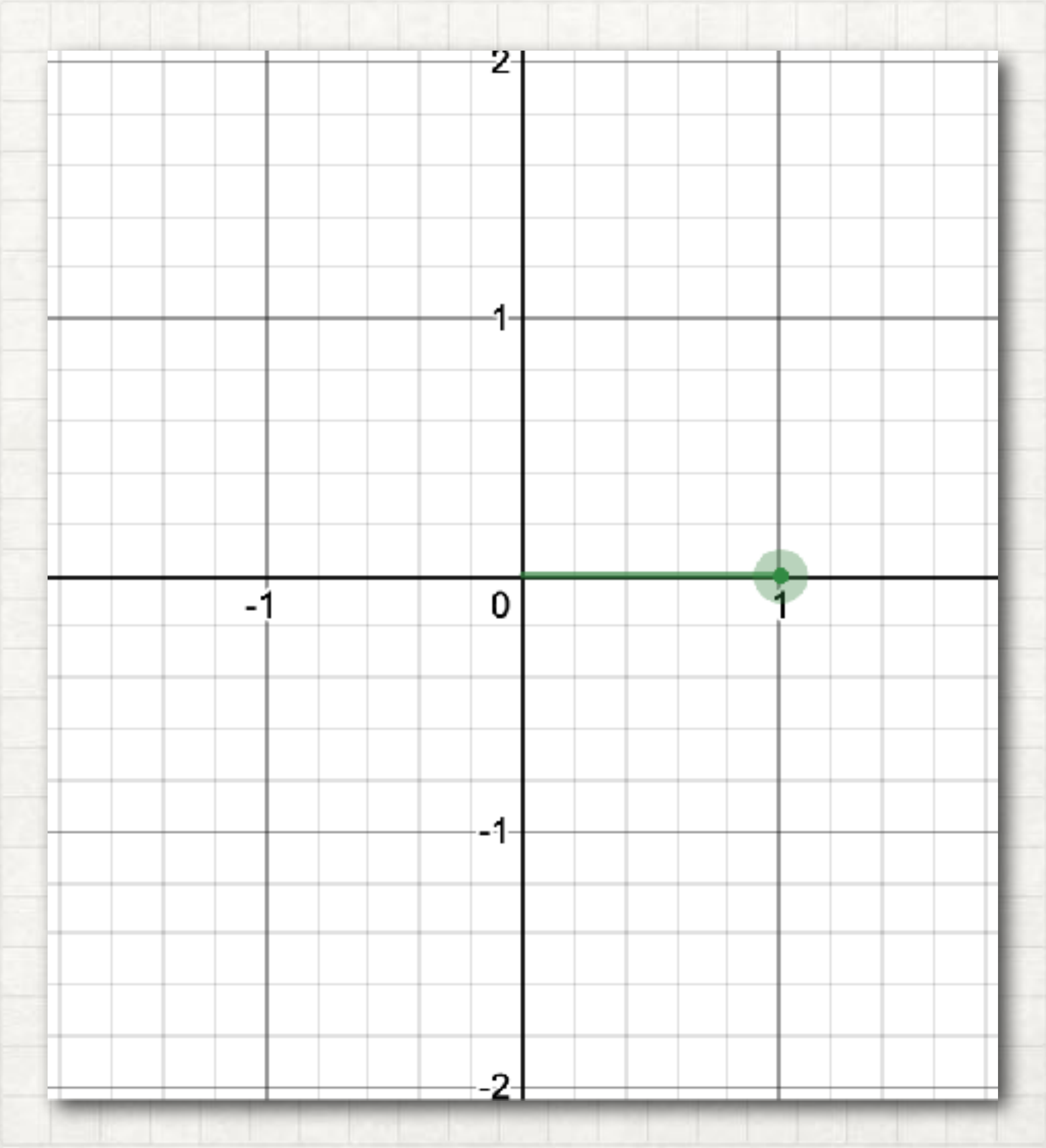
ROTATING AND MULTIPLYING

- If you rotate (2, 0) by 45 degrees counterclockwise, then you get (1.404, 1.404).



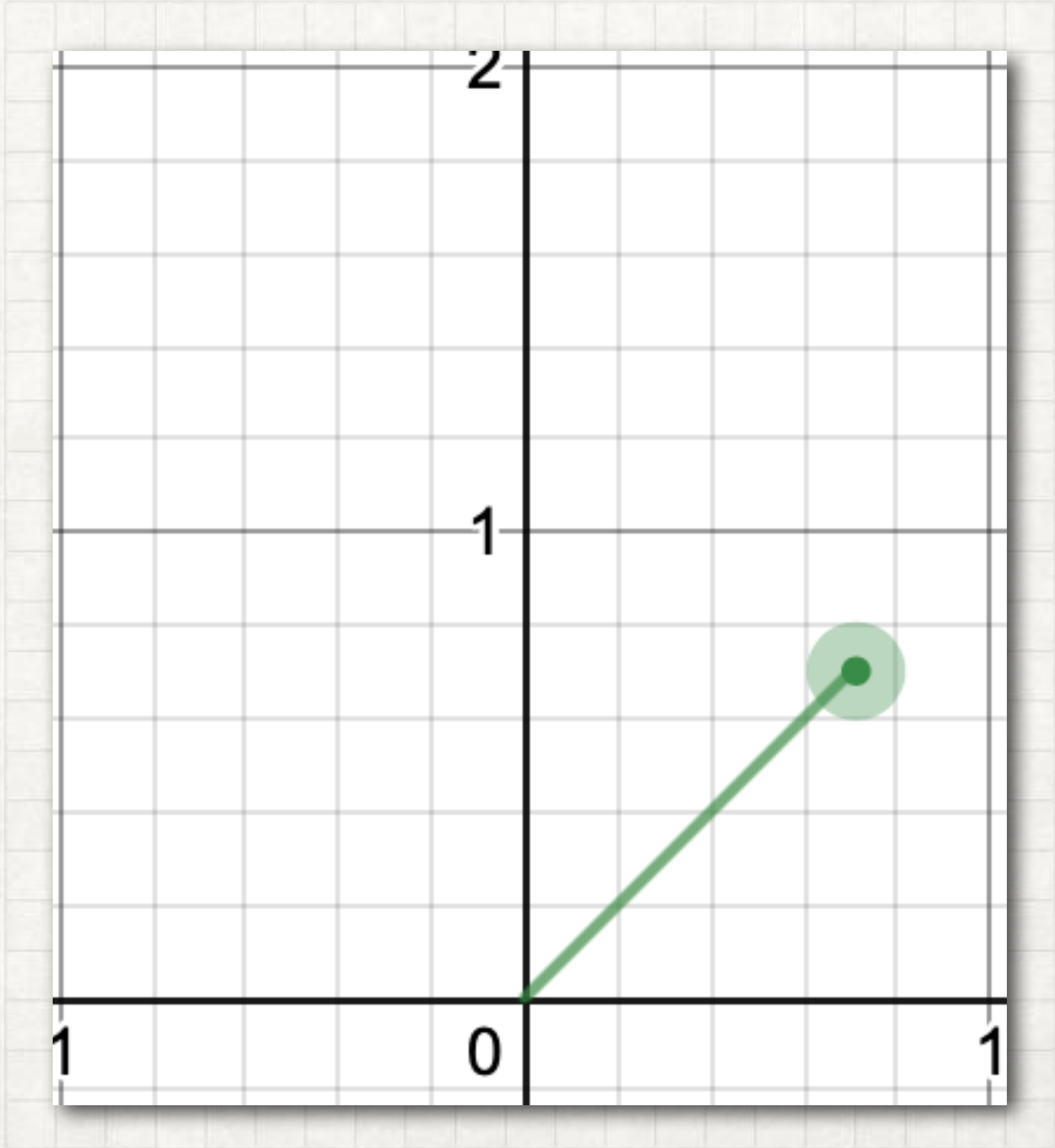
ROTATING AND MULTIPLYING

- Now, let's go back to the start. Here is the vector (1, 0).



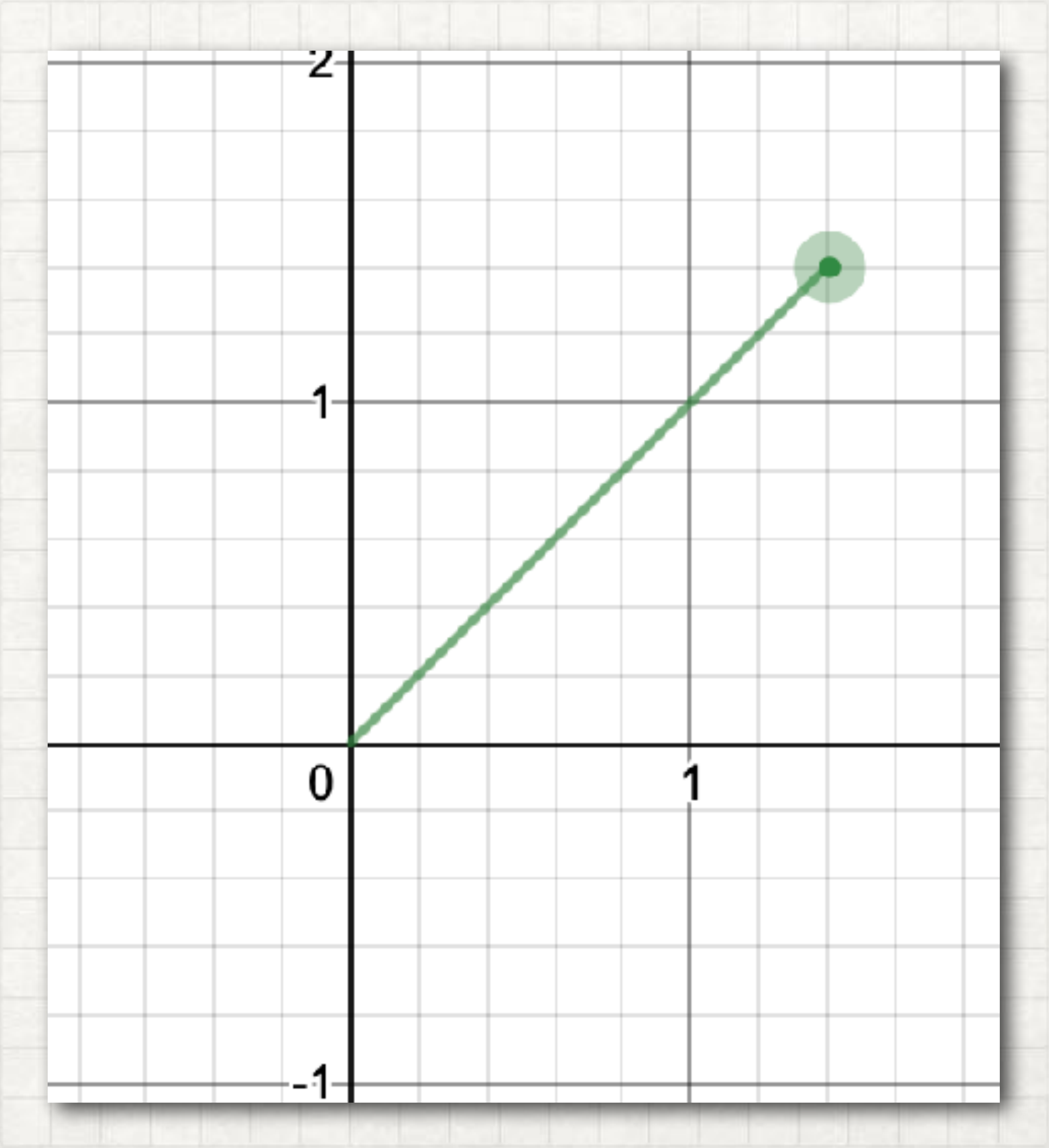
ROTATING AND MULTIPLYING

- If you rotate (1, 0) by 45 degrees counterclockwise, then you get (0.707, 0.707).



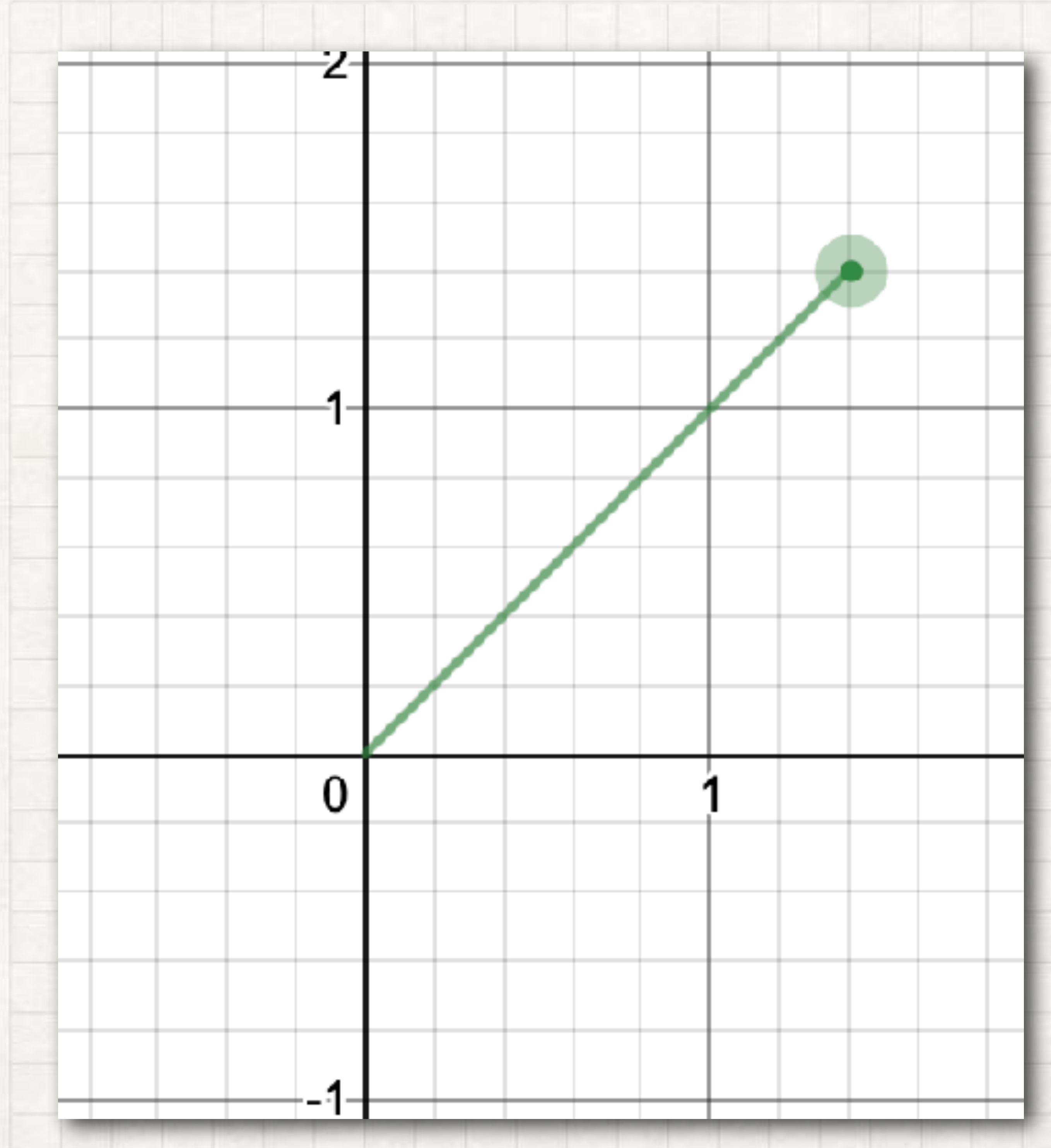
ROTATING AND MULTIPLYING

- If you double the length of (0.707, 0.707), then you get (1.404, 1.404).



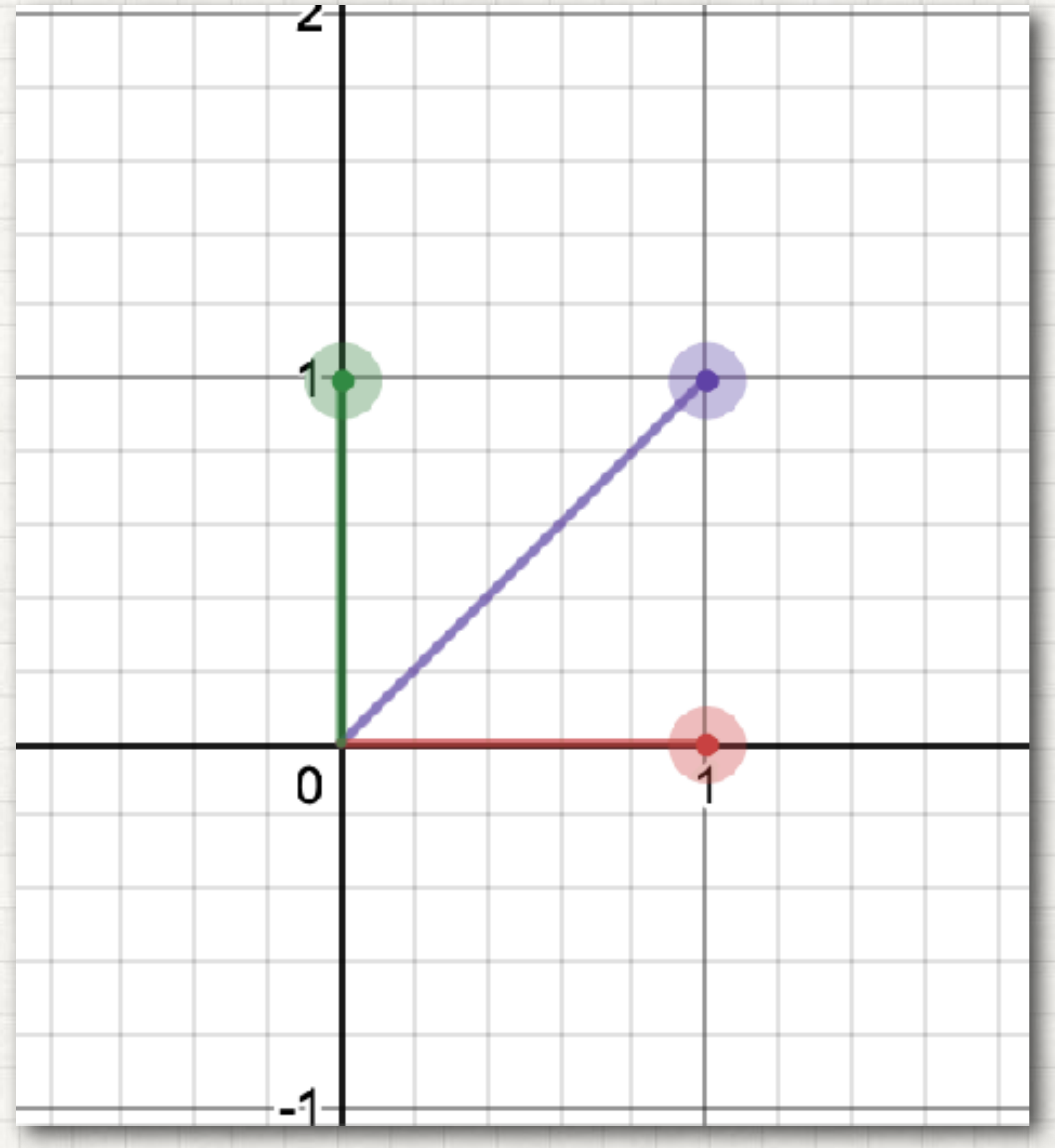
ROTATING AND MULTIPLYING

- The fact about rotating and multiplying was that $R(v) \cdot k = R(k \cdot v)$, where k is a number and v is a vector.
- In the example you just saw, k was 2 and v was (1, 0).



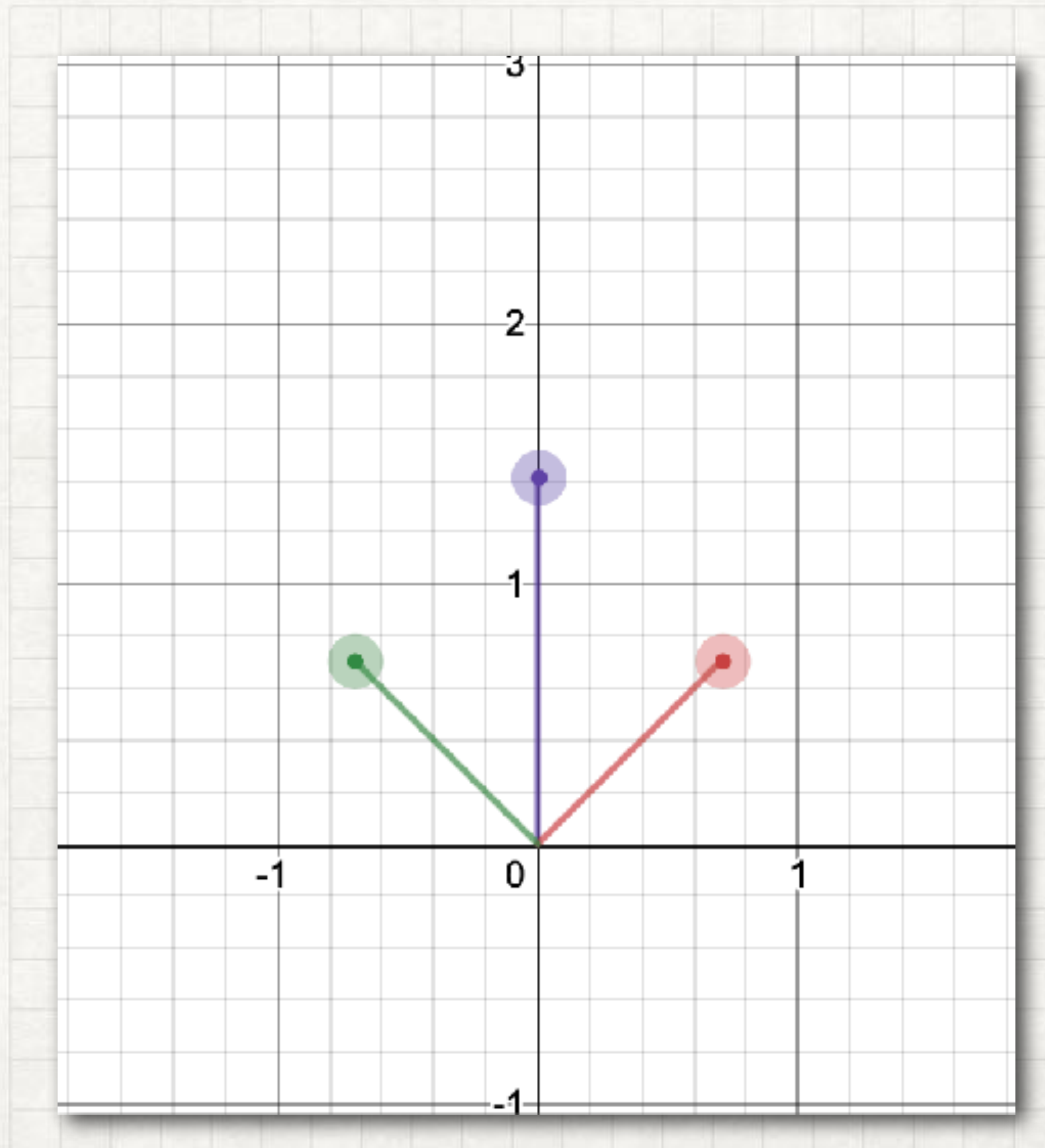
ROTATING AND ADDING

- Here, the purple vector is equal to the red vector plus the green vector.



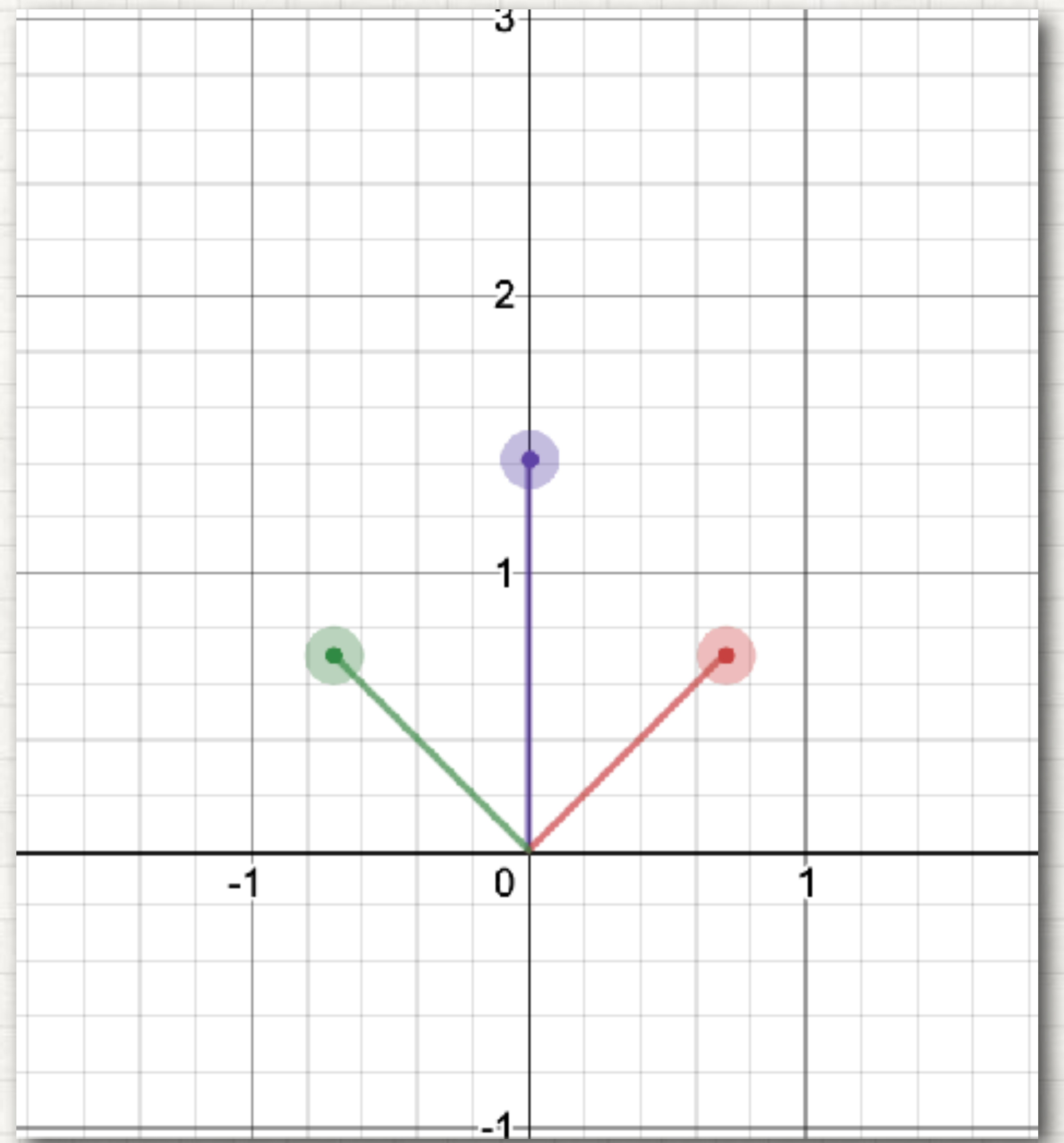
ROTATING AND ADDING

- If you rotate all of the vector by 45 degrees, you will see that the purple vector is still the red vector plus the green vector.



ROTATING AND ADDING

- The fact about rotating and adding was that R(u) + R(v) = R(u + v).
- In the example you just saw, u was the red vector, v was the green vector, and u + v was the purple vector.



ROTATING A VECTOR

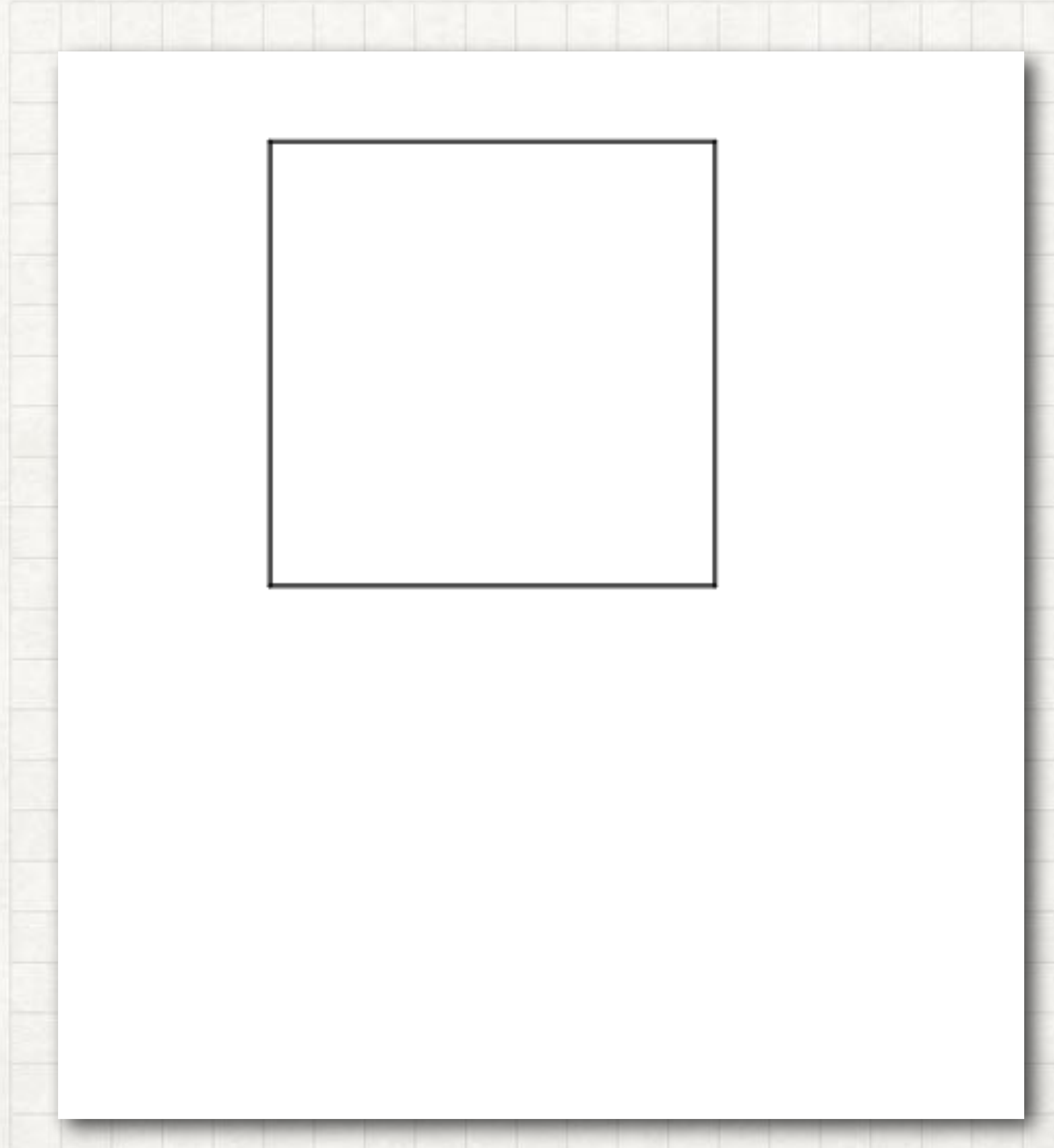
- $R(v) \cdot k = R(k \cdot v)$.
- $R(u) + R(v) = R(u + v)$.
- These two facts let us solve the problem of rotating a vector we considered earlier.
    - Suppose that you are given a vector (x_0, y_0). Let (x_1, y_1) be the vector you get from rotating (x_0, y_0) by 45 degrees counterclockwise.
    - How do you compute (x_1, y_1)?
- How?

ROTATING A VECTOR

- Any vector (x, y) can be written as $x \cdot (1, 0) + y \cdot (0, 1)$.
- $R(x, y)$
  $= R(x \cdot (1, 0) + y \cdot (0, 1))$
  $= R(x \cdot (1, 0)) + R(y \cdot (0, 1))$ because $R(u + v) = R(u) + R(v)$
  $= x \cdot R(1, 0) + y \cdot R(0, 1)$ because $R(k \cdot v) = k \cdot R(v)$
- If you know $R(1, 0)$ and $R(0, 1)$, then you can easily compute $R(x, y)$.

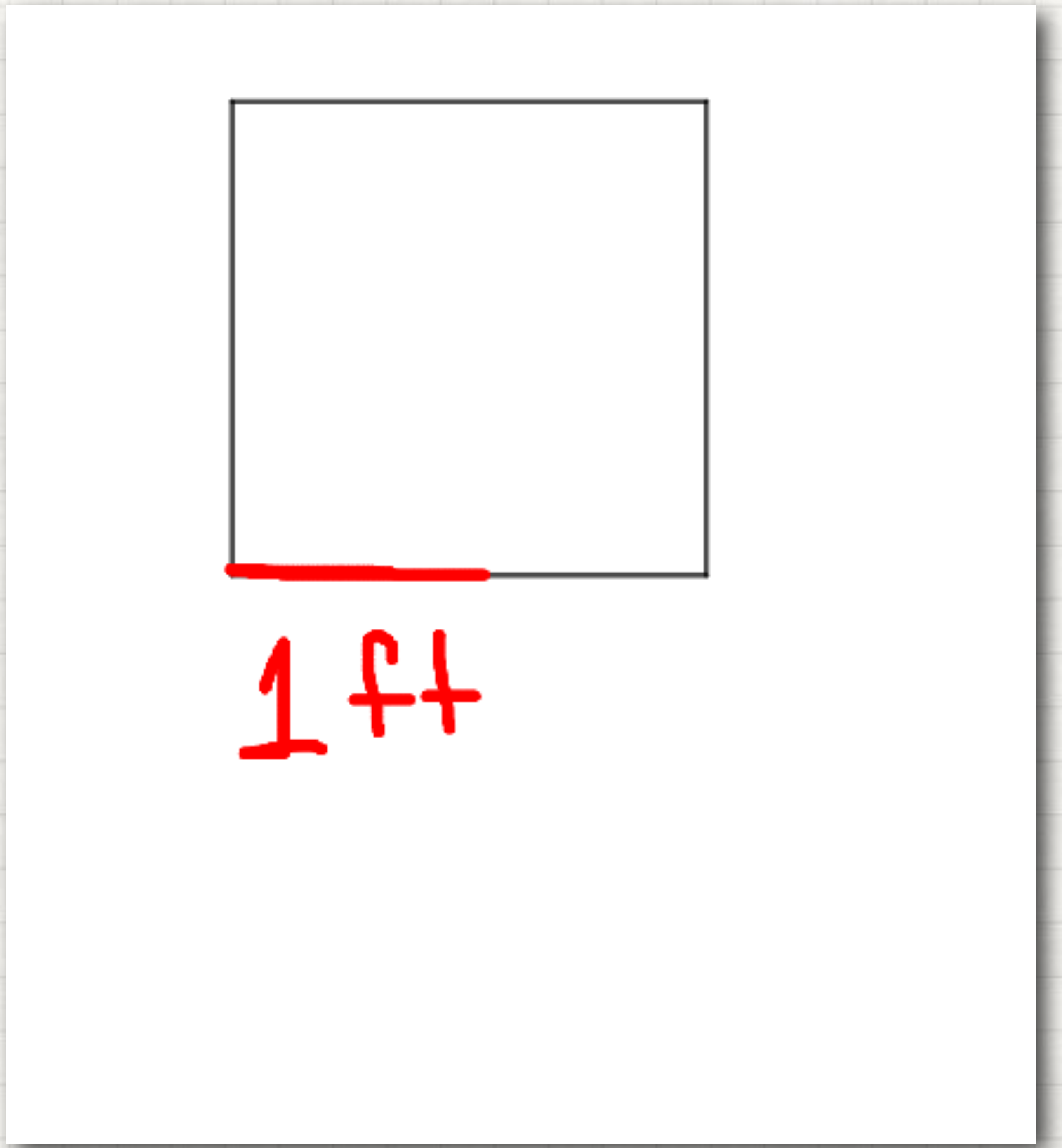
ROTATING A VECTOR

- What is R(1, 0)?
- Here is a quick and dirty way to do this by hand.
- First, you need a surface with a right angle, like a piece of paper or a table.
- The lower left of the surface will act as (0, 0).



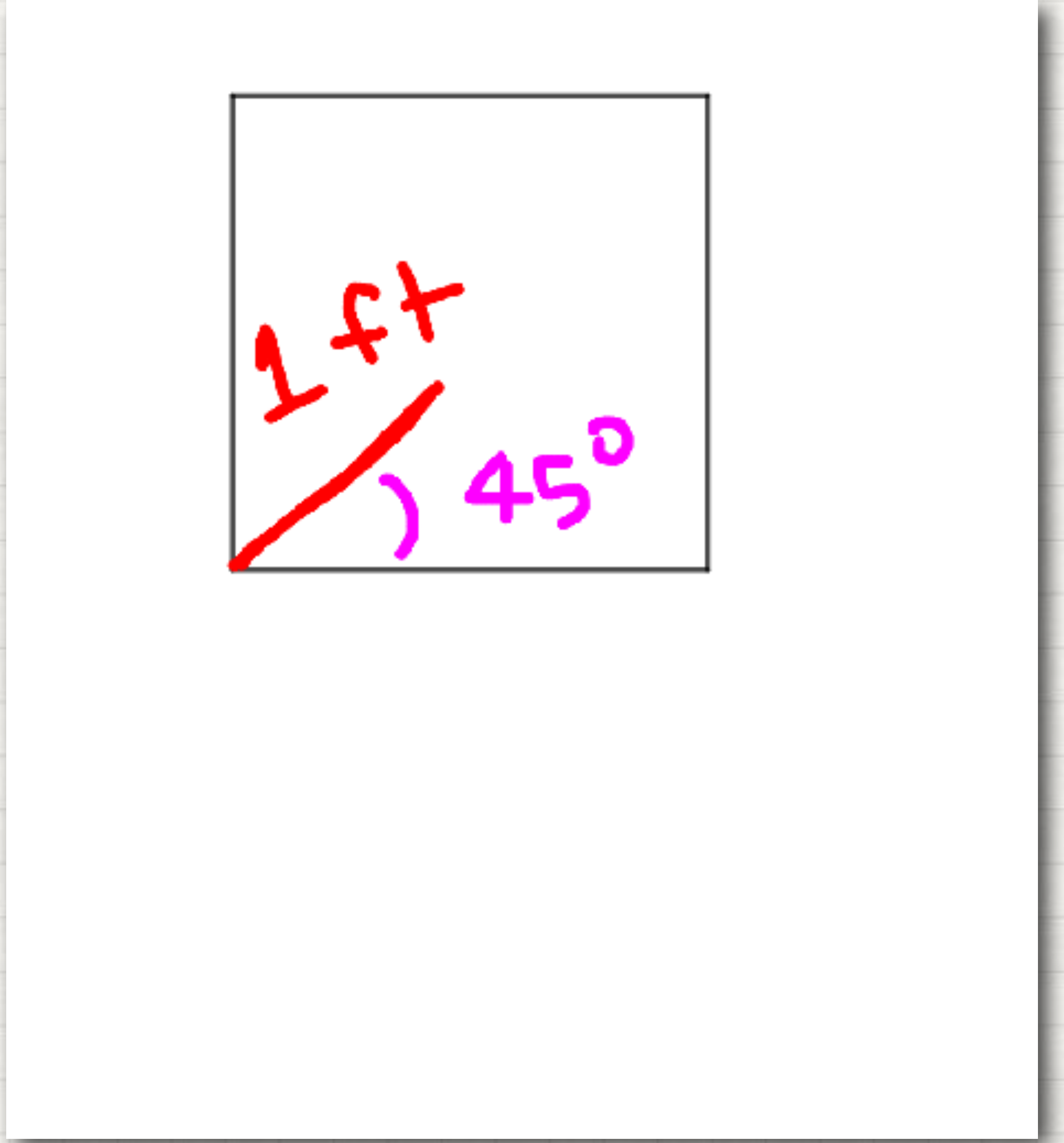
ROTATING A VECTOR

- Second, get something that is one foot long, like a ruler or a piece of string, and lay it on the bottom edge of the surface.
- The end point of the foot long thing is (1, 0).



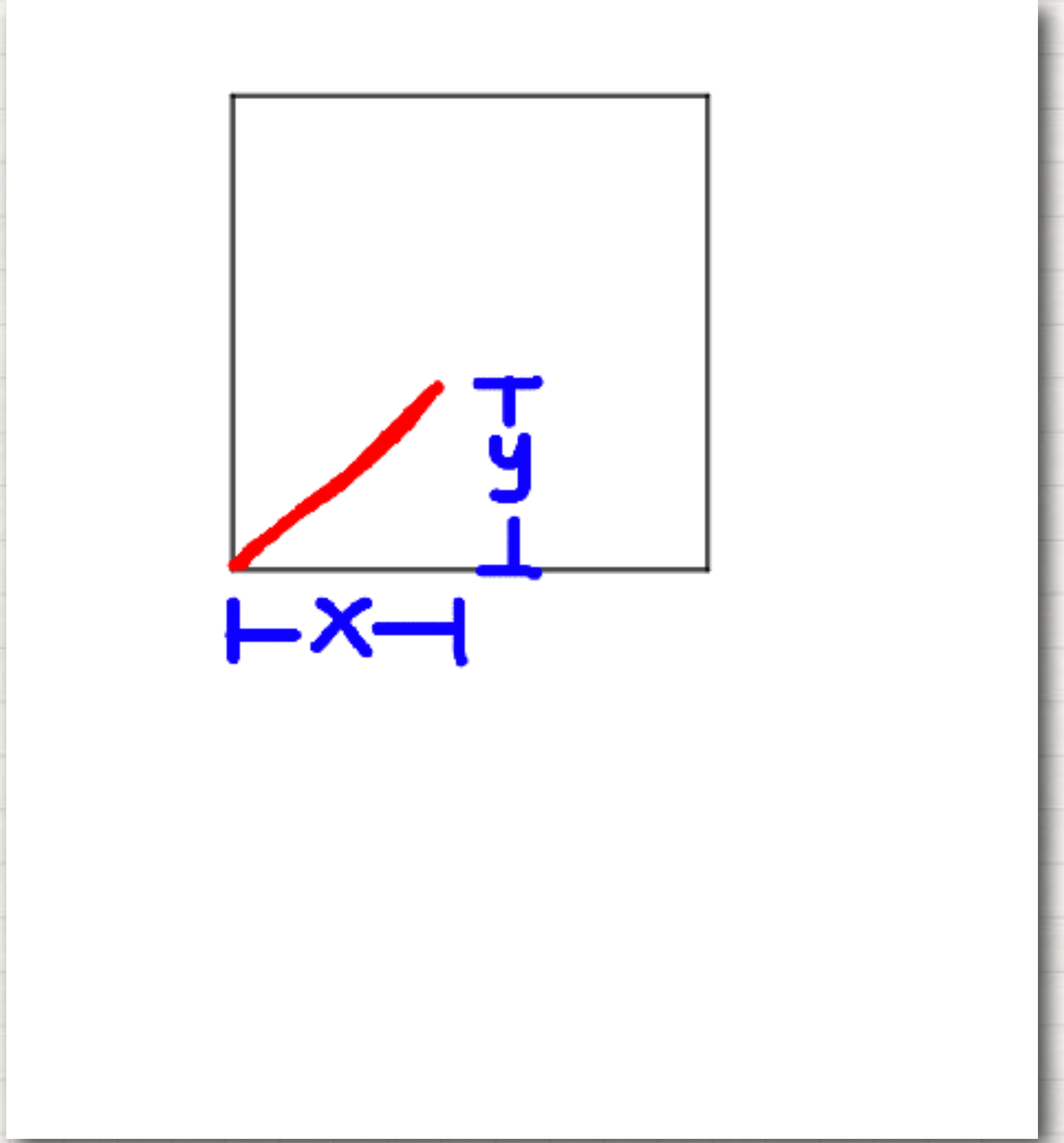
ROTATING A VECTOR

- Third, rotate the foot long thing by 45 degrees counterclockwise.



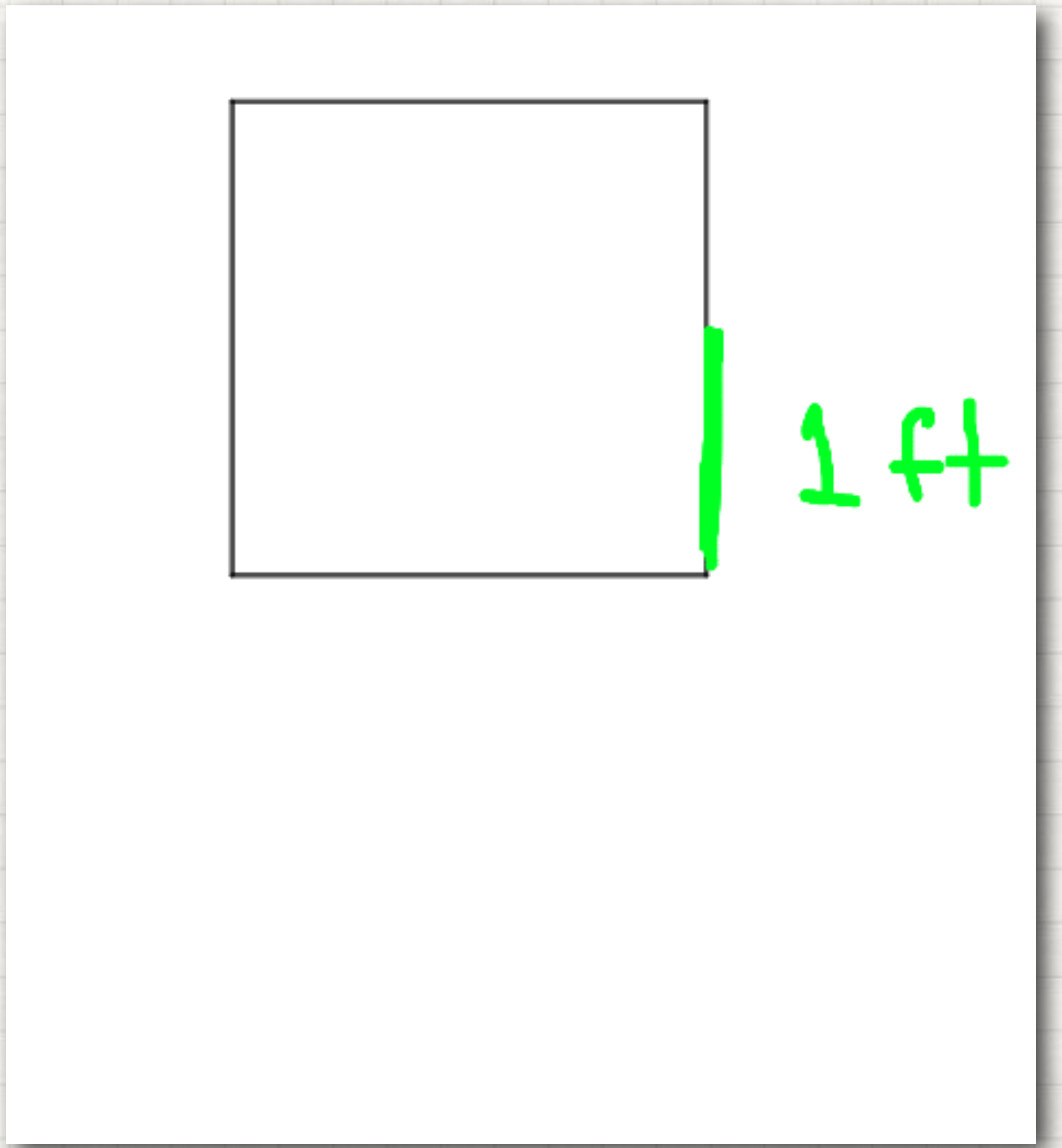
ROTATING A VECTOR

- Fourth, get a ruler and measure x and y.
- x and y are going to be the coordinates of R(1, 0).
- That is, R(1, 0) = (x, y).
- Remember to make this measurement in feet, not inches or centimeters.



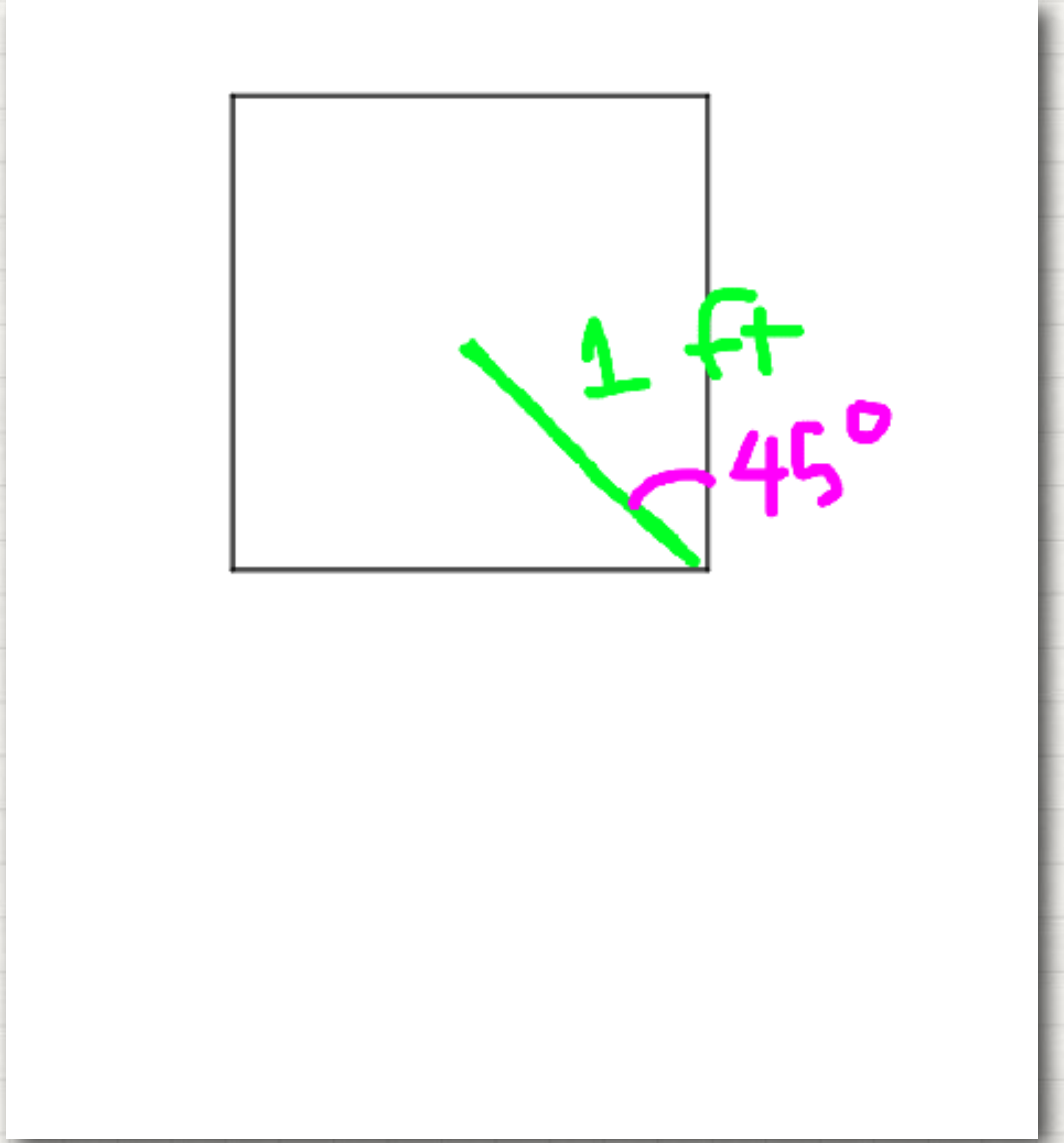
ROTATING A VECTOR

- You can do the same thing for (0, 1).
- This time, let the lower right act as (0, 0).



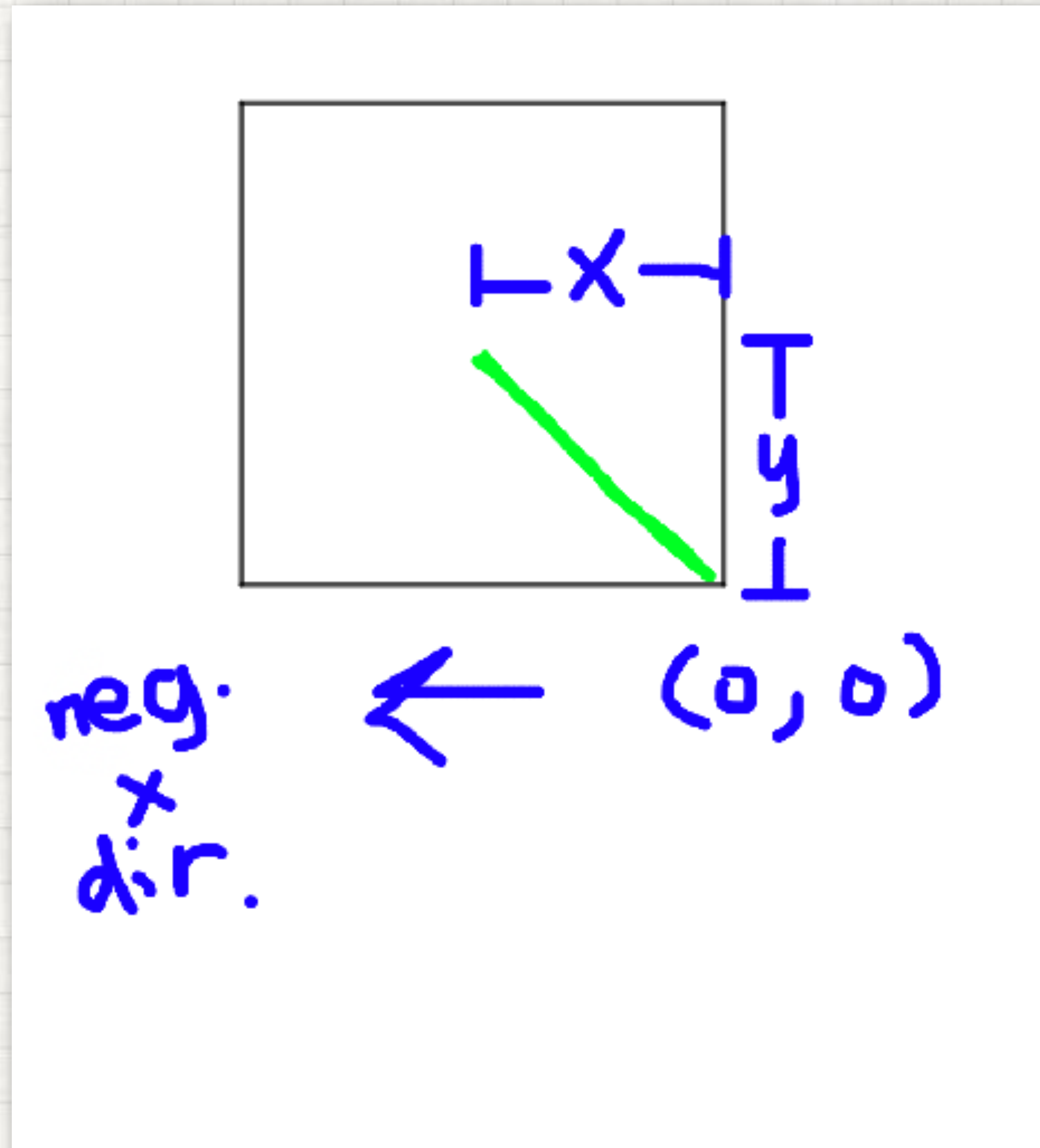
ROTATING A VECTOR

- Then, rotate the foot long thing by 45 degrees counterclockwise.



ROTATING A VECTOR

- Then, measure x and y. Keep in mind that the x-coordinate is negative because it is to the left of (0, 0).



ROTATING A VECTOR

- Here is the problem again.
    - Suppose that you are given a vector (x_0, y_0). Let (x_1, y_1) be the vector you get from rotating (x_0, y_0) by 45 degrees counterclockwise.
    - How do you compute (x_1, y_1)?
- $R(1, 0)$ is roughly $(0.707, 0.707)$, and $R(0, 1)$ is roughly $(-0.707, 0.707)$.
- This means $(x_1, y_1) = R(x_0, y_0) = x_0 \cdot (0.707, 0.707) + y_0 \cdot (-0.707, 0.707)$ by the reasoning on slide 23.

ROTATING A VECTOR

- (x_1, y_1)
  $= R(x_0, y_0)$
  $= x_0 \cdot (0.707, 0.707) + y_0 \cdot (-0.707, 0.707)$
  $= (x_0 \cdot 0.707, x_0 \cdot 0.707) + (y_0 \cdot -0.707, y_0 \cdot 0.707)$
  $= (0.707 \cdot x_0 - 0.707 \cdot y_0, 0.707 \cdot x_0 + 0.707 \cdot y_0)$

- This means that…
  $x_1 = 0.707 \cdot x_0 - 0.707 \cdot y_0$
  $y_1 = 0.707 \cdot x_0 + 0.707 \cdot y_0$

- We usually write this using matrix notation.

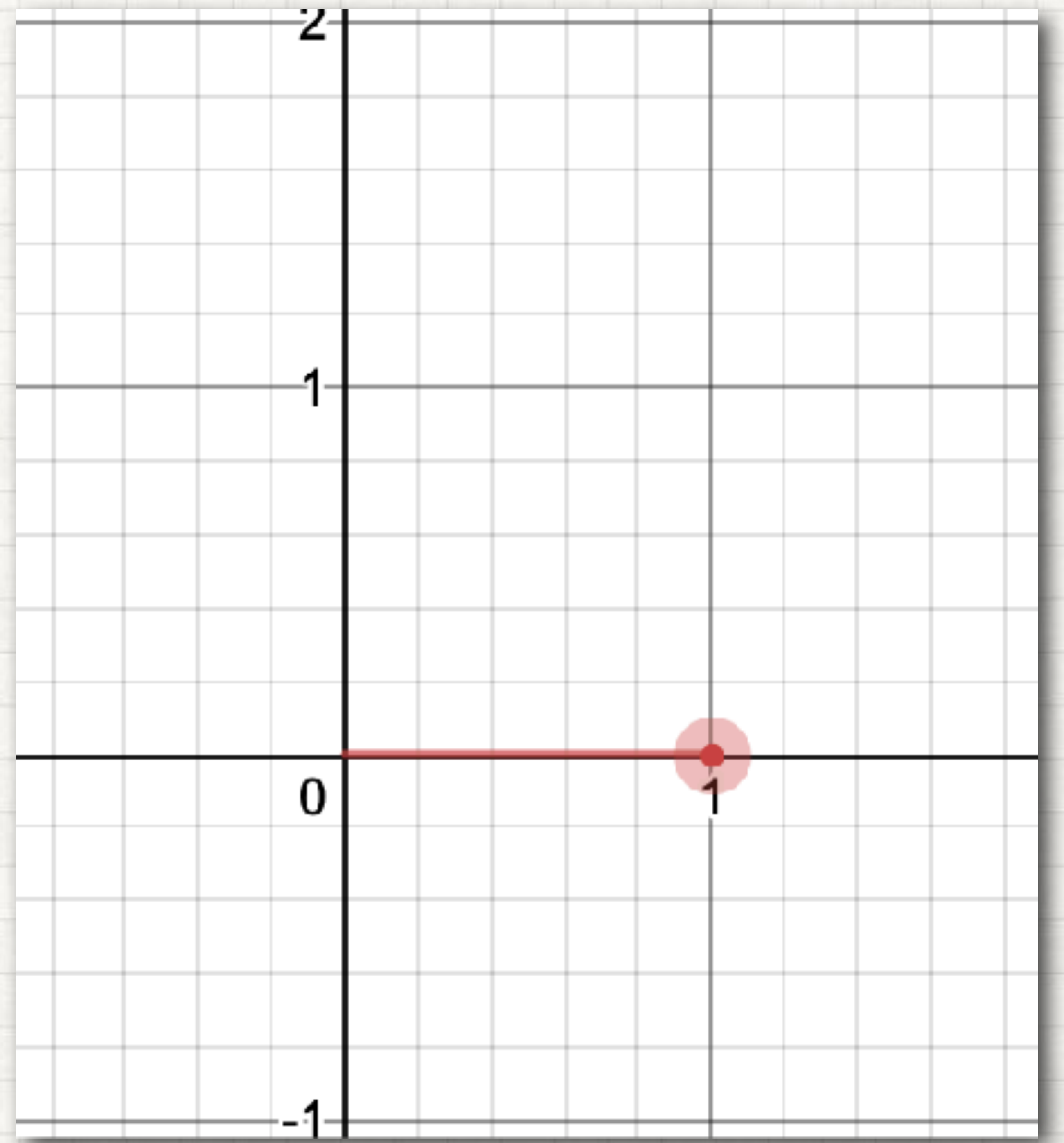
$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0.707 & -0.707 \\ 0.707 & 0.707 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

ROTATING A VECTOR

- You can repeat this procedure for any angle and get the equations/matrix for rotating by that angle counterclockwise.

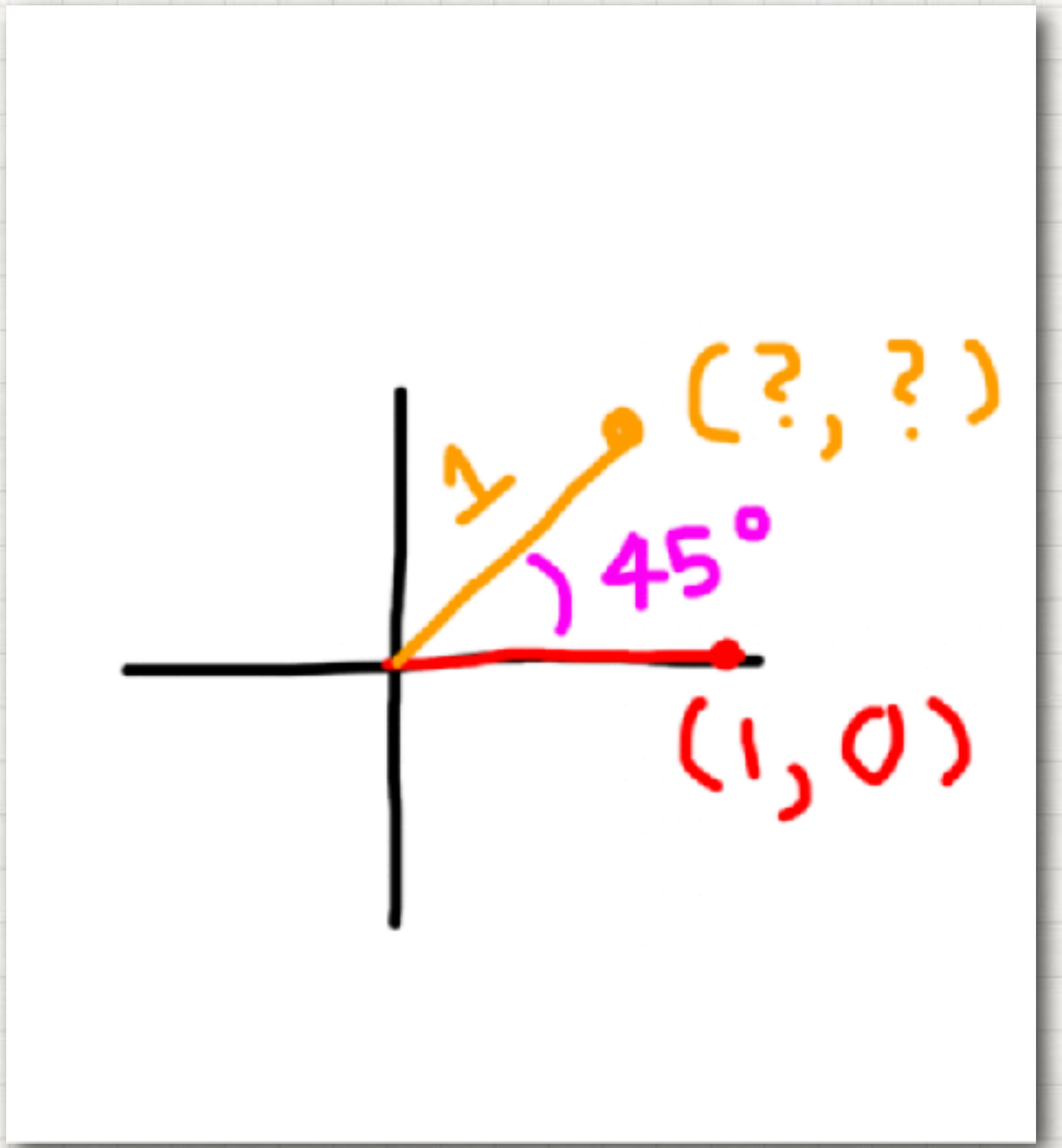
ROTATING A VECTOR

- You can compute the exact values of R(1, 0) and R(0, 1) in the following way.
- Here, we'll do (1, 0).
- The length of (1, 0) is 1.
- If we rotate it by 45 degrees counterclockwise, then the result must also have a length of 1.



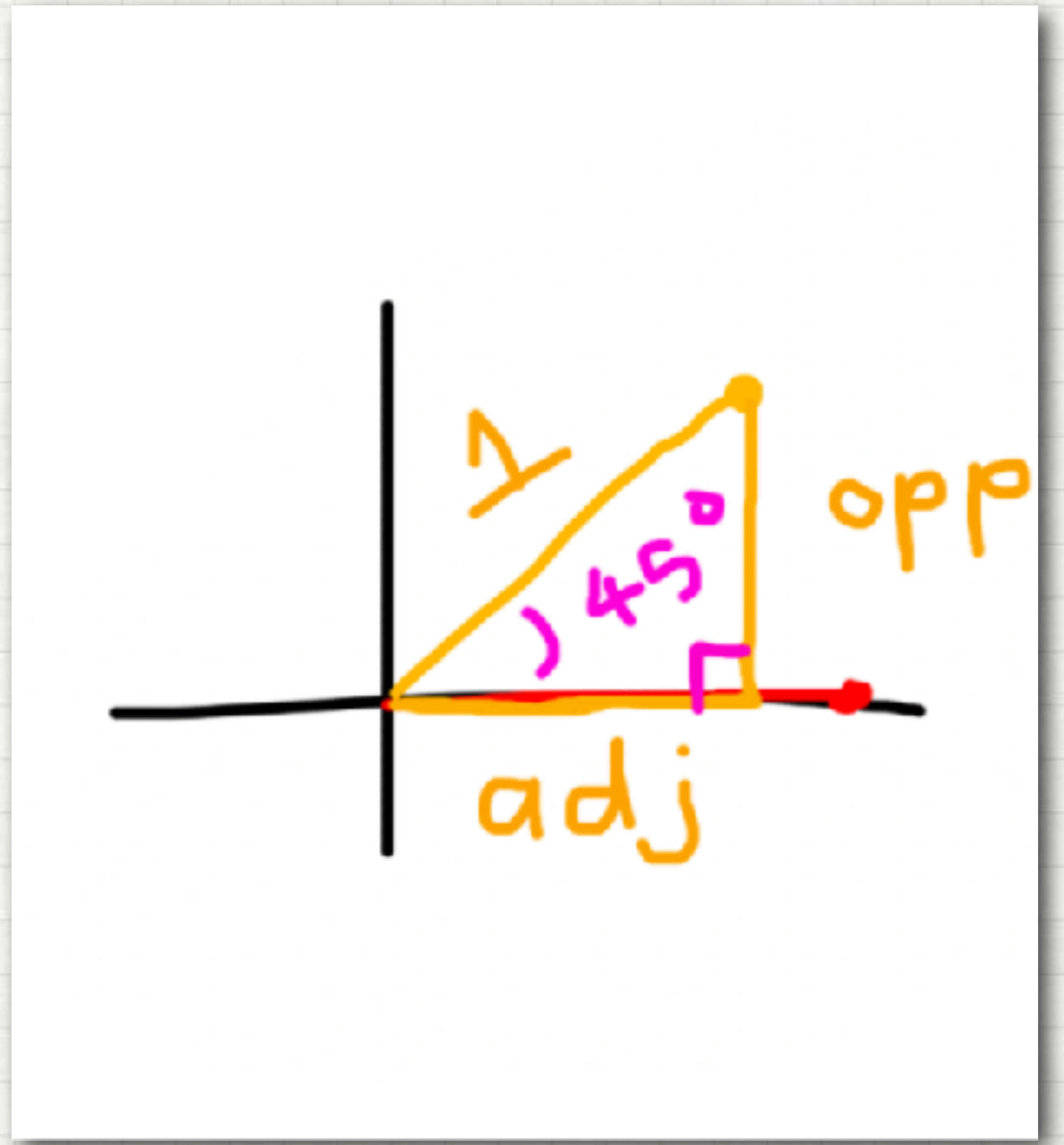
ROTATING A VECTOR

- Here is (1, 0).
- The length of (1, 0) is 1.
- If we rotate it by 45 degrees counterclockwise, then the result must also have a length of 1.



ROTATING A VECTOR

- To find the end point of the orange vector, you can use trigonometry.
- Let A be an angle in a right triangle.
- The hypotenuse of the triangle is the longest side.
- The side that is adjacent to A is the side, other than the hypotenuse, that is part of the angle.
- The side that is opposite to A is the side that is not part of the angle.



ROTATING A VECTOR

- sin(A) = opp / hyp.
- cos(A) = adj / hyp.
- In the diagram, sin(45°) = opp / 1 = opp, and cos(45°) = adj / 1 = adj.
- So, adj = cos(45°) and opp = sin(45°).
- This means that the end point of the orange vector is (adj, opp) = (cos(45°), sin(45°)).