QUANTUM WINTER 2020

Problem Statement 1:

Using the finite difference approach, discretize the 1D wave equation with 2nd order spatial spatial accuracy and 1st order time accuracy with the implicit time integration scheme.

Solution:

The 1D wave equation is the best description we have for understanding a wide variety of topics from Modelling the longitudinal and torsional vibration of a rod to heat transfer among bodies to name a few.

We begin with the 1D wave equation as specified in the question:

$$\frac{du}{dt} + C\frac{du}{dx} = 0 \text{ - - - - - - - - - eq(1)}$$

We are required to discretize this equation discretize this equation in 2nd order spatial accuracy and 1st order time accuracy with implicit time integration scheme. Applying finite difference approximations to partial derivatives w.r.t time and space as given:

Assume $U_x = \frac{\partial U}{\partial x}$ and $U_t = \frac{\partial U}{\partial t}$.

$$U_i^n = U(t_n, x_i)$$

Space Discretization:

The computational domain contains an infinite number of x values due to its continuous nature. Let the smallest step be Δx be:

$$\Delta x = \frac{(v-u)}{N-1}$$

Where v and u are the boundary points, N is the number of points between (or the domain): N ϵ $[\boldsymbol{u}, \boldsymbol{v}]$

We now apply the central difference formula with 2nd order space accuracy:

$$U_x = \frac{\partial U}{\partial x} = \frac{(U_{i+1}^n - U_{i-1}^n)}{2\Delta x} \qquad \text{- - - - - - - - eq(2)}$$

At the instant t_n

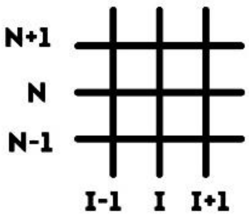
Time Discretization:

Let the time step be $t_n = \Delta t + t$:

We use the forward difference formula for time discretization with 1st order accuracy which can be applied to U_t as:

$$U_t = \frac{\partial u}{\partial t} = \frac{U_i^{n+1} - U_i^n}{\Delta t} \text{ - - - - - - - - eq3}$$

Discretized variables visualised on the grid as follows:



$$U_i^n = U(x, t)$$

$$U_{i+1}^n = U(x + \Delta x, t)$$

$$U_i^{n+1} = U(x, t + \Delta t)$$

Hence substituting eq(2) and eq(3) in eq(1), we get:

$$\frac{(U_i^{n+1} - U_i^n)}{\Delta t} + C\frac{(U_{i+1}^n - U_{i-1}^n)}{2\Delta x} = 0$$

Rearranging the above equation implicitly, we get for each timestep we get:

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} = -\frac{C}{2\Delta x}(U_{i+1}^n - U_{i-1}^n)$$