NYU Center for Data Science

DS-UA 112
Introduction to Data Science

Week 9: Lecture 2

Models – Working with Random Variables

How can we generalize from a sample to a population?

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Introduction to Data Science

Week 9: Lecture 2
Models – Working with Random Variables

Adapted from Dudoit, Nolan, Gonzalez, Lau

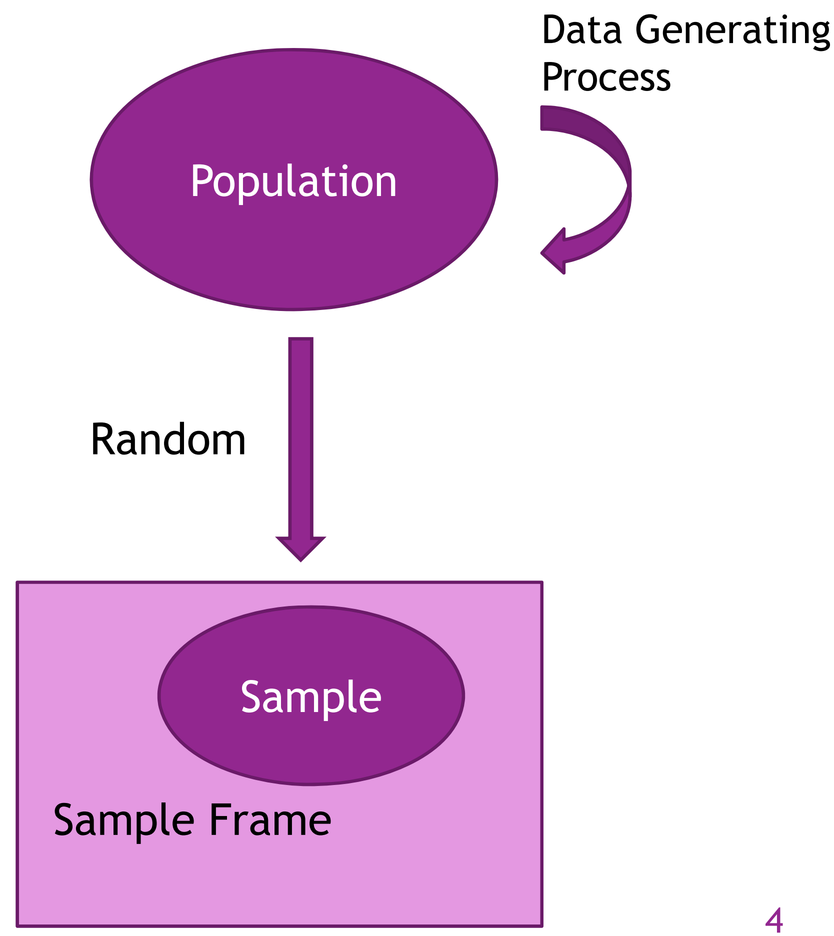
Announcements

- Please check Week 9 agenda on NYU Classes
  - Lab 7
    - Due on Friday March 27 at 12PM
  - Project 1
    - Due on Monday April 6 at 12PM
  - Survey

https://nyu.qualtrics.com/jfe/form/SV_3DCWUa4yc08L0wt

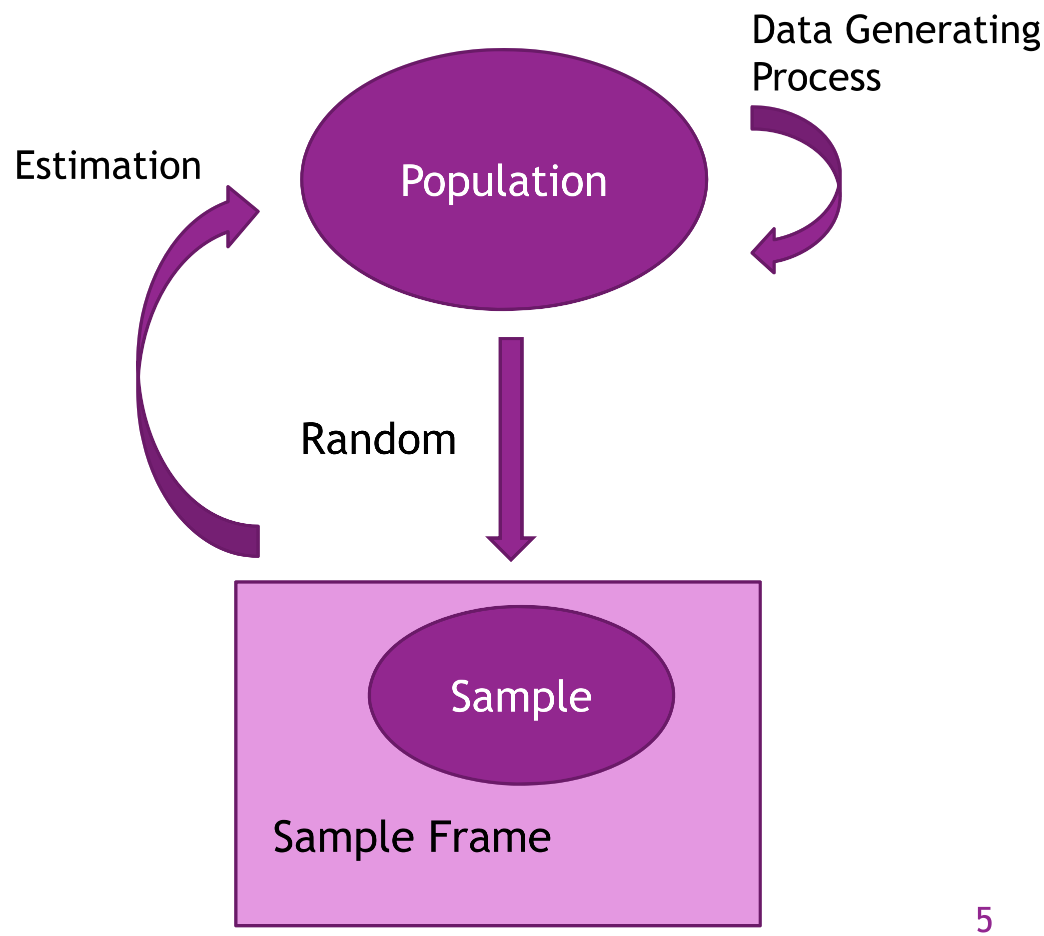
Review

- Model
  - We want to understand a population through random samples
  - If we have guesses about the processes generating the data, then we can propose a model.
  - The data helps us to validate the assumptions behind the model



Review

- Model
  - We can associate numbers called parameters to the population
  - A statistic is an estimate for a parameter obtained from a summary of data
  - How can we determine appropriate estimators of the parameters in the population?

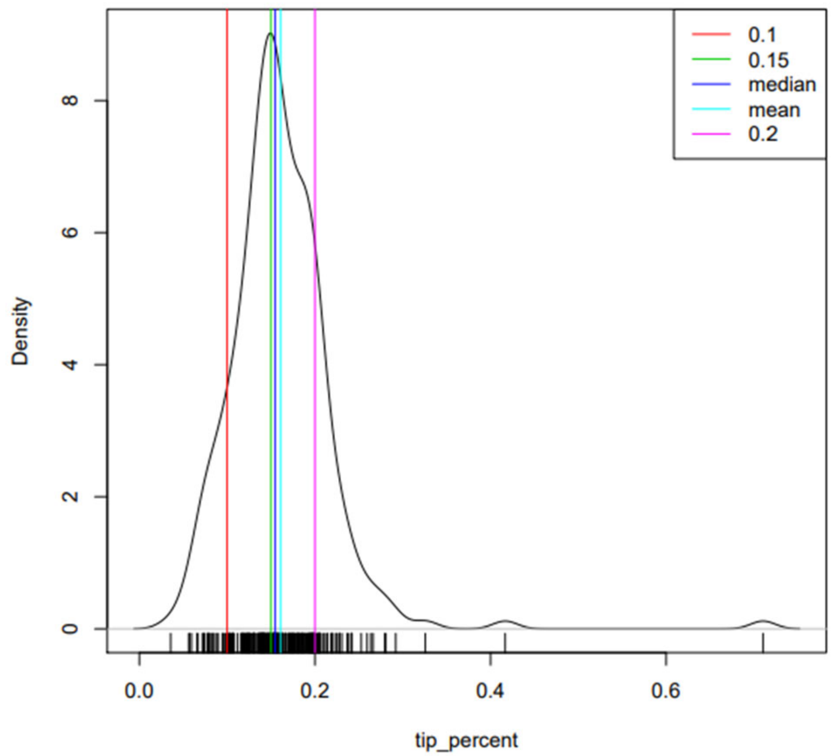


Review

	total_bill	tip	sex	smoker	day	time	size
0	16.99	1.01	Female	No	Sun	Dinner	2
1	10.34	1.66	Male	No	Sun	Dinner	3
2	21.01	3.50	Male	No	Sun	Dinner	3
...	...	...	...	...	...	...	...
241	22.67	2.00	Male	Yes	Sat	Dinner	2
242	17.82	1.75	Male	No	Sat	Dinner	2
243	18.78	3.00	Female	No	Thur	Dinner	2

- For example, the population could be tips at restaurants in the United States
- The sample could consist of 244 observations collected at a restaurant
- The customary amount for a tip is 15%. So our model could assert that tips are 15% on average.
- How could we estimate the average?

Review



- For sample size n = 244 we can label the observations as $x_1,\ldots,x_n$
- We could summarize with the mean

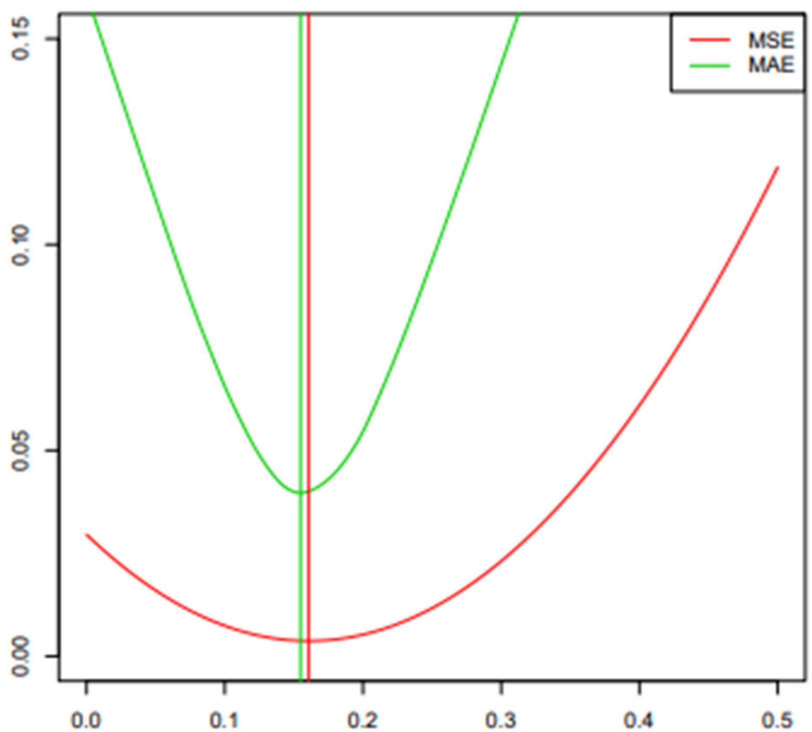
$$x_{mean} = \frac{1}{n}(x_1 + \ldots + x_n)$$

or the median x_{median}

$$\sum_{x_i < x_{median}} (1) = \sum_{x_i > x_{median}} (1)$$

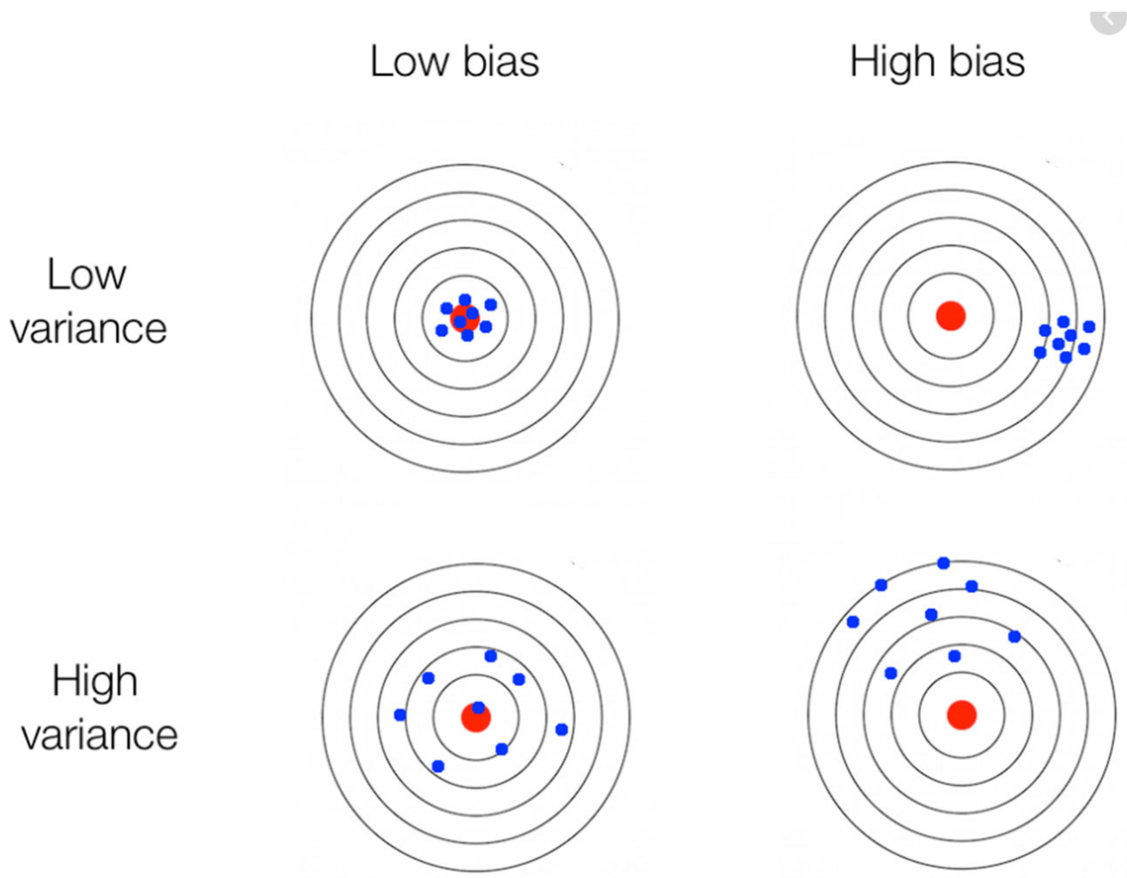
- How would one summary determine a better estimator than another summary?

Review



- Loss Functions
  - Functions depending on $x_1,\ldots,x_n$ along variables. These variables are the unknown quantities corresponding to different choices for summaries.
  - We choose between the different summaries by the value of the loss function.
    - Low value is good
    - High value is bad

Review

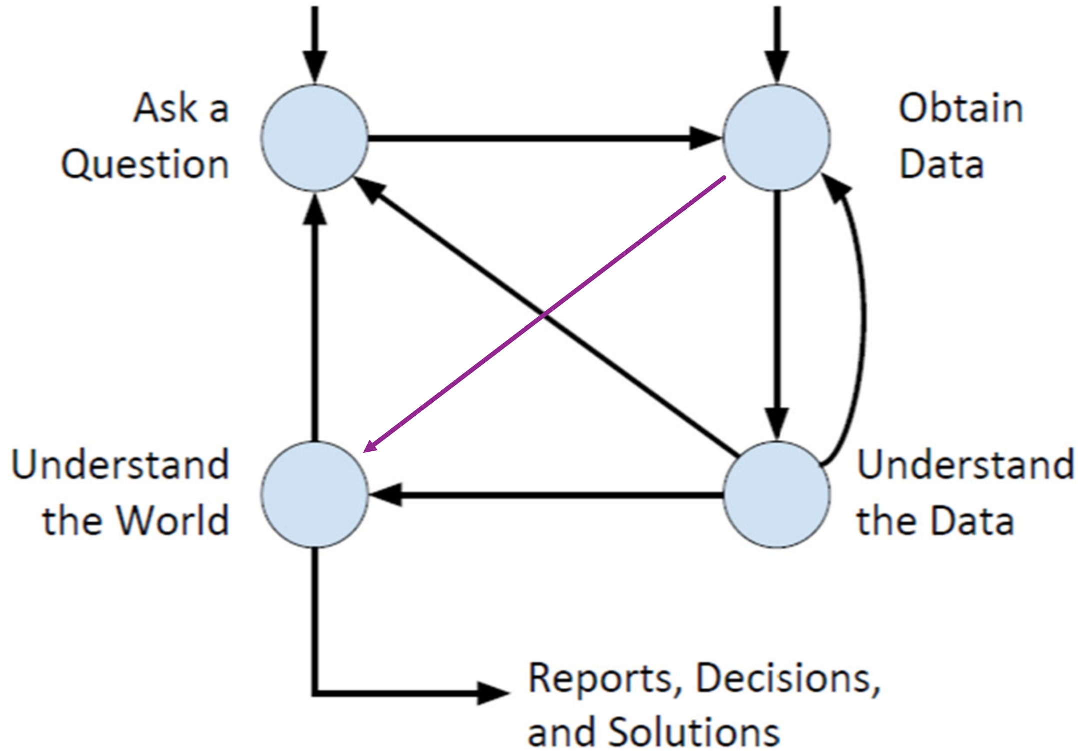


- Loss Functions
  - Find the values of the unknown quantities that minimizes the loss function gives a systematic way to choose between statistics
  - Loss functions should allow us to assess the
    - Bias
    - Variance

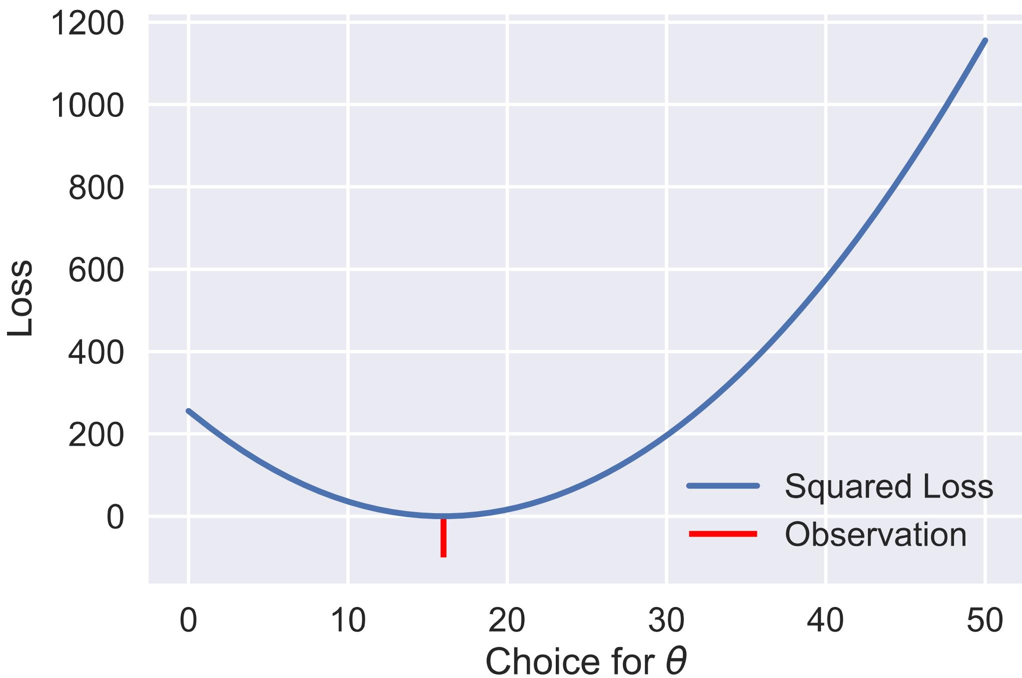
  of the choices

Agenda

- Loss Functions
- Probability Distributions
  - Bernoulli
  - Binomial
- Random Variables
  - Expectation
  - Variance



Loss Functions



- Mean Square Error has minimum value at the mean of the data
- The derivative tells us the rate of change of a function. We calculate the change in output divided by the change in input.
- If the derivative is zero, then mean square error has obtained it minimum value

Loss Functions

$$L(\theta, x_1, \ldots, x_n) = \frac{1}{n} \sum (x_i - \theta)^2$$

$$\frac{\partial}{\partial \theta} L(\theta, x_1, \ldots, x_n) = \frac{1}{n} \sum (2)(x_i - \theta)(-1)$$

$$= -\frac{2}{n} \left(\sum (x_i) - n\theta \right)$$

$$\sum (x_i) - n\hat{\theta} = 0$$

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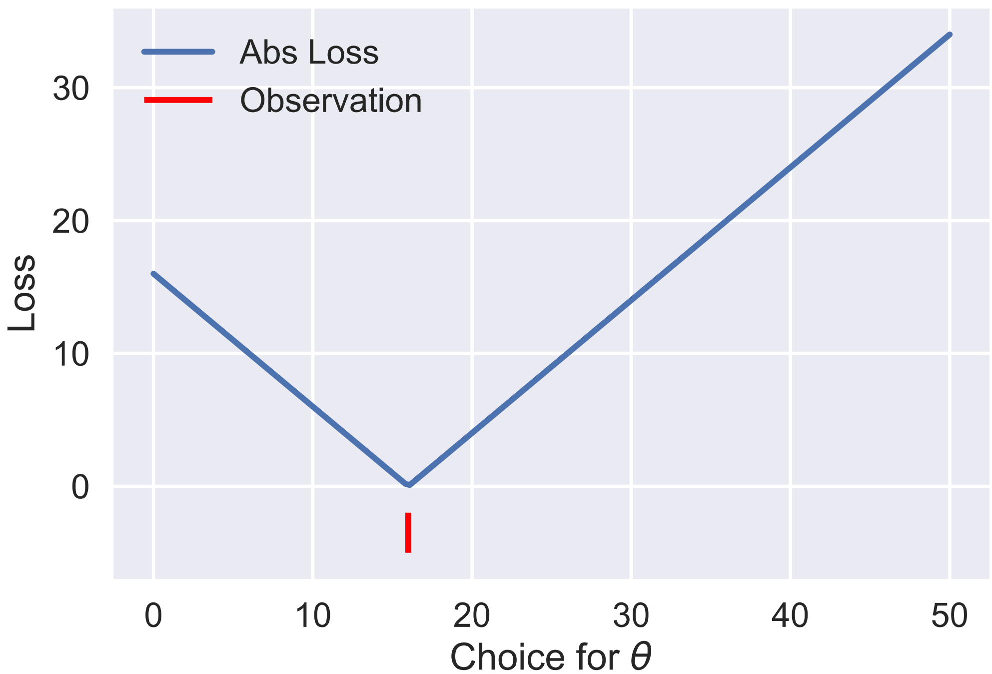
Loss Functions



- Mean Absolute Error has minimum value at the median of the data
- The derivative of absolute value function gets tricky around the value 0 where it jumps from -1 to 1
- However we can split up the summation in the loss function to make the calculation easier

Loss Functions

$$L(\theta, \mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n}|x_i - \theta|$$

$$= \frac{1}{n}\left(\sum_{x_i<\theta}|x_i - \theta| + \sum_{x_i=\theta}|x_i - \theta| + \sum_{x_i>\theta}|x_i - \theta|\right)$$

$$\frac{1}{n}\left(\sum_{x_i<\theta}(-1) + \sum_{x_i=\theta}(0) + \sum_{x_i>\theta}(1)\right) = 0$$

$$\sum_{x_i<\theta}(1) = \sum_{x_i>\theta}(1)$$

- Mean Absolute Error has minimum value at the median of the data
- The derivative of absolute value function gets tricky around the value 0 where it jumps from -1 to 1
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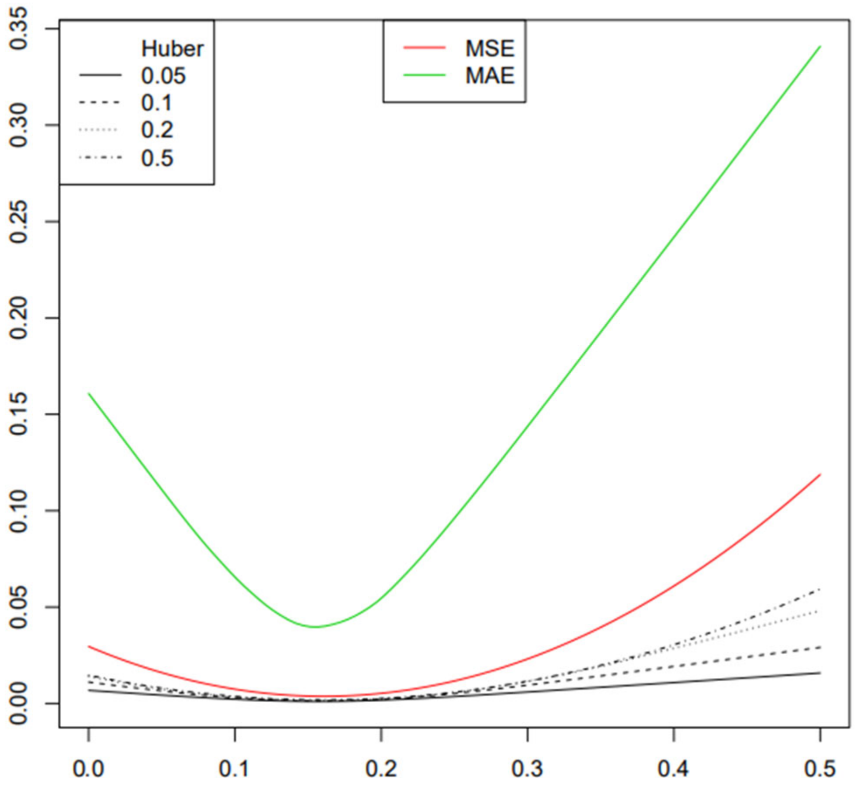
Loss Functions



- The mean square error has derivatives at all values of the function. Derivatives are helpful for finding minimum values.
- However, the mean square error has large output for large input. The function is not robust to outliers
- While the mean absolute error has a tricky derivative, the function does not have the same problem with outlier
- Huber Loss combines benefits of both loss functions

Random Variables

- Represents a numeric value related to a random event.
- Random variables can be discrete (integer values) or continuous (any floating point number).
- We tend to use capital letters for the variables with P denoting probability.
  - For example, X could be the tip as a percentage of the bill
  - $P(X = 2)$ is the probability that X has the value 2.
  - $P(X < 10)$ is the probability that X is less than 10

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Distributions

- The probability distribution of a random variable consists of possible values along with their frequency of occurrence
- We use stem-plots to visualize distributions

$$P(X = 1) = \frac{1}{6}$$
$$P(X = 2) = \frac{1}{6}$$
$$\cdots$$
$$P(X = 6) = \frac{1}{6}$$

X	P(X)
1	1/6
2	1/6
...	...
6	1/6

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X	**P(X)**
1	1/6
2	1/6
...	...
6	1/6

Bernoulli Distribution

- Random variable that takes the value 0 or 1 is called a Bernoulli random variable
- We need to specify the probability that the value is 1

$$X \sim \mathrm{Bern}(p)$$

$$P(X = 1) = p \quad P(X = 0) = 1 - p$$

- Remember by the complement rule that the probability of 0 is 1-0

Binomial Distribution

- Sum of Bernoulli random variables is a Binomial random variable
- Here each random variable is independent
- We need to specify the number of Bernoulli random variables and the probability that each is p

$$X \sim \mathrm{Binom}(n, p)$$

$$P(X = k) = P(k) = \binom{n}{k} p^k (1-p)^{n-k} = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$$

Binomial Distribution

$$X \sim \text{Binom}(5, 1/3)$$

$$P(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$P(3) = \binom{5}{3} \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^2 = 0.165$$

- We need to specify the number of Bernoulli random variables and the probability that each is p

$$X \sim \text{Binom}(n, p)$$

$$P(X = k) = P(k) = \binom{n}{k} p^k (1-p)^{n-k} = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$$

Expectation

- Expectation is weighted of the values in the distribution.
- We can think of expectation like the mean expect different values have different weights depending on their frequency.

$$E(X) = \sum_{x \in \mathbb{X}} x \cdot P(X = x)$$

$$X \sim \mathrm{Bern}(p) \qquad \begin{aligned} E(X) &= \sum_{x \in \mathbb{X}} x \cdot P(X = x) \\ &= (1)(p) + (0)(1 - p) \\ &= p \end{aligned}$$

- For example, with Bernoulli random variables we obtain the probability of value 1.

Variance

- Variance is expectation of square difference between random variable and expectation
- We can think of variance like the mean square error. It measures the spread of values of the random variable away from the expectation

$$Var(X) = E\left((X - E(X))^2\right)$$

$$X \sim \mathrm{Bern}(p)$$

$$E(X) = p$$

$$E(X^2) = (1^2)(p) - (0^2)(1-p) = p$$

$$Var(X) = p - p^2 = p(1-p)$$

- For example, with Bernoulli random variables we obtain the probability of value 1 times the probability of value 0

Variance

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- We can think of variance like the mean square error. It measures the spread of values of the random variable away from the expectation

$$Var(X) = E(X^2) - E(X)^2$$

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$$E(X^2) = (1^2)(p) - (0^2)(1-p) = p$$

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- For example, with Bernoulli random variables we obtain the probability of value 1 times the probability of value 0

Summary

- Loss Functions
- Probability Distributions
  - Bernoulli
  - Binomial
- Random Variables
  - Expectation
  - Variance

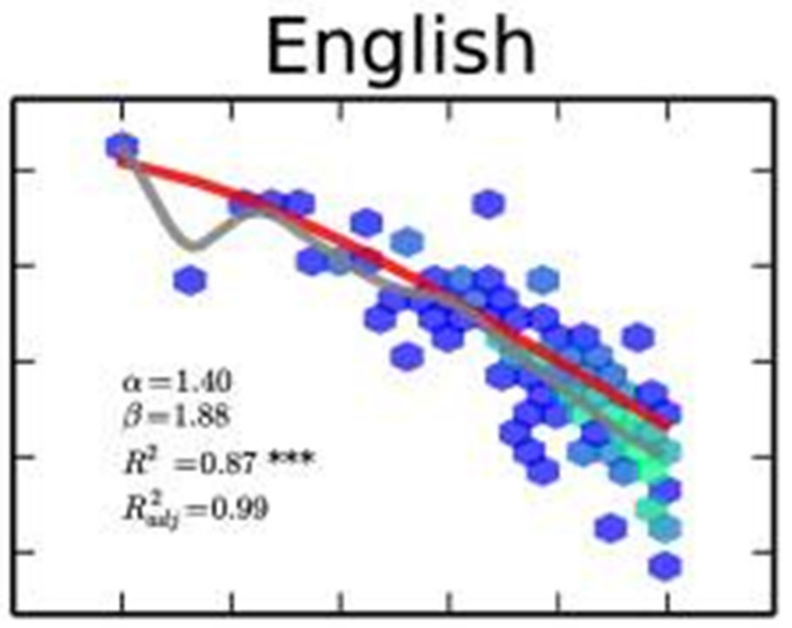
Goals

- How are MSE and MAE different?
- Why would bias and variance help us with estimation of parameters?

Questions

- Questions on Piazza?
  - Please provide your feedback along with questions
- Question for You!

Where else can you find the Zipf Rule?



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