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Week9-template **Math 468, Spring 2022**

Q 1. (Durrett, Problem 4.1) A salesman flies around between Atlanta, Boston, and Chicago as the following rates (the units are trips per month):

F T	A	B	C
A	-4	2	2
B	3	-4	1
C	5	0	-5

Q 1.1. What is the transition rate matrix $\mathbf{Q}$ for this process?

THE SCRATCH AREA

$$\mathbf{Q} = \begin{bmatrix} -4 & 2 & 2 \\ 3 & -4 & 1 \\ 5 & 0 & -5 \end{bmatrix}$$

THE ANSWER

Q 1.2. List the eigenvalues of $\mathbf{Q}$ as a comma-separated list.

THE SCRATCH AREA

$$-5, -8, 0$$

THE ANSWER

Q 1.3. Find the (right) diagonalizing matrix $\mathbf{S}$ of $\mathbf{Q}$, so that $\mathbf{S}^{-1}\mathbf{Q}\mathbf{S}$ is diagonal. Scale the columns so that the first entry in each column (counting from the top) is 1.

THE SCRATCH AREA

$$\mathbf{S} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -\frac{1}{3} & 1 \\ -1 & -\frac{5}{3} & 1 \end{bmatrix}$$

THE ANSWER

Q 1.4. Find the left diagonalizing matrix $\mathbf{L} = \mathbf{S}^{-1}$ of $\mathbf{Q}$, so that $\mathbf{L}\mathbf{Q}\mathbf{L}^{-1}$ is diagonal.

THE SCRATCH AREA

$$\begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

THE ANSWER

Q 1.5. What is the stationary distribution π for this Markov process? Hint: Use one of the rows of the matrix found in the previous part. Make sure it is a row vector.

THE SCRATCH AREA

$$\pi = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

THE ANSWER

Q 1.6. (Durrett 4.1, part (a)) Find the limiting fraction of time she spends in each city. Only the exact answer yields credit. List the numbers in the order "A, B, C".

THE SCRATCH AREA

$$1/2, 1/4, 1/4$$

THE ANSWER

Q 1.7. Find the routing matrix $\mathbf{R}$ for $\mathbf{Q}$.

THE SCRATCH AREA

$$\mathbf{R} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 3/4 & 0 & 1/4 \\ 1 & 0 & 0 \end{bmatrix}$$

THE ANSWER

Q 1.8. If she is in Boston now, what is the probability that the first city she will visit next is Chicago?

THE SCRATCH AREA

$$\mathbb{P}(Y_1 = C | Y_0 = B) = \mathbf{R}_{23} = 1/4$$

THE ANSWER

Q 1.9. (Durrett 4.1, part (b)) What is her average number of trips each year from Boston to Atlanta?

THE SCRATCH AREA

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THE ANSWER

Q 1.10. Find the matrix $\mathbf{P}(t) = e^{t\,\mathbf{Q}}$.

THE SCRATCH AREA

$$\begin{bmatrix} \frac{e^{-8\,t}}{2} + \frac{1}{2} & \frac{1}{4} - \frac{e^{-8\,t}}{4} & \frac{1}{4} - \frac{e^{-8\,t}}{4} \\ -\frac{e^{-5\,t}}{3} - \frac{e^{-8\,t}}{6} + \frac{1}{2} & \frac{2\,e^{-5\,t}}{3} + \frac{e^{-8\,t}}{12} + \frac{1}{4} & -\frac{e^{-5\,t}}{3} + \frac{e^{-8\,t}}{12} + \frac{1}{4} \\ \frac{e^{-5\,t}}{3} - \frac{5\,e^{-8\,t}}{6} + \frac{1}{2} & -\frac{2\,e^{-5\,t}}{3} + \frac{5\,e^{-8\,t}}{12} + \frac{1}{4} & \frac{e^{-5\,t}}{3} + \frac{5\,e^{-8\,t}}{12} + \frac{1}{4} \end{bmatrix}$$

THE ANSWER

Q 1.11. If she is in Boston now, what is the probability that she will be in Atlanta two months from now? Your answer must have at least 6 digits of precision.

THE SCRATCH AREA

$$\begin{aligned}\mathbf{P}_{21}(2) &= -\frac{e^{-5\,t}}{3} - \frac{e^{-8\,t}}{6} + \frac{1}{2}\Big|_{t=2} \\ &= -\frac{e^{-10}}{3} - \frac{e^{-16}}{6} + \frac{1}{2} \approx 0.4999848479342167\end{aligned}$$

THE ANSWER