| Name: | Student ID: | |
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| Week6-template | | Math 563, Fall 2022 |

Q 1. (Durrett 3.1.1.) Generalize the proof of Lemma 3.1.1 to conclude that if $\max_{1 \le j \le n} |cj,n| \to 0$, $\sum_{j=1}^n c_{j,} \to \lambda$ and $\sup_n \sum_{j=1}^n |c_{j,n}| < \infty$ then $\prod_{j=1}^n (1+c_{j,n}) \to e^{\lambda}$.

| THE PROOF | |
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| | QED |

- (Durrett 3.1.2.) If the X_i have a Poisson distribution with mean 1, then S_n has a Poisson distribution Q 2. with mean n, i.e., $\mathbb{P}(S_n = k) = e^{-n}n^k/k!$. Use Stirling's formula to show that if $(k-n)/n \to x$ then
- Q 2.1.

$$\sqrt{2\pi n} \mathbb{P}(S_n = k) \to \exp(-x^2/2).$$

| THE PROOF | |
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Q 2.2. Use the asymptotic expansion for $\log n!$ (see *Wikipedia* page for details):

$$\ln(k!) - \frac{1}{2} \ln k = \frac{1}{2} \ln 1 + \ln 2 + \ln 3 + \dots + \ln(k-1) + \frac{1}{2} \ln k$$
$$= k \ln k - k + 1 + \sum_{\ell=2}^{\infty} \frac{(-1)^{\ell} B_{\ell}}{\ell(\ell-1)} \left(\frac{1}{k^{\ell-1}} - 1\right)$$

where B_{ℓ} is a Bernoulli number, to find the asymptotic expansion of $\mathbb{P}(S_n = k)$ in n. Negative powers of n are to be expected.

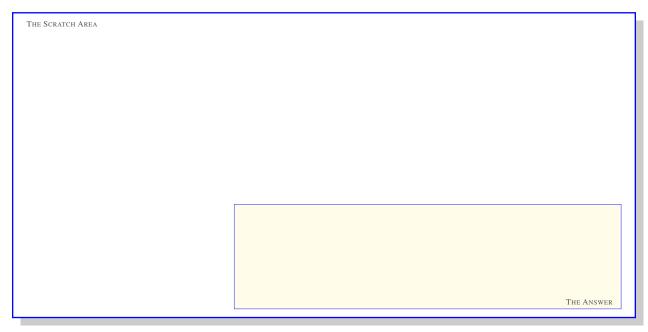


Q 3. (Durrett 3.1.4.) Suppose $\mathbb{P}(X_i = k) = e^{-1}/k!$ for k = 0, 1, ... Show that if a > 1

$$\frac{1}{n}\log \mathbb{P}(S_n \ge n\,a) \to a - 1 - a\,\log a.$$

THE PROOF QED

- **Q 4.** (Custom) Use the method of generating functions to calculate the **the exact** probability that in 6 independent die rolls we get a total score of 15. This problem may seem to involve a lot of computations, but it is not so if the method of generating functions is used efficiently.
- **Q 4.1.** Write the probability generating function for a single die roll score X_1 (uniforml probability of 1/6 for die faces $\Omega = \{1, 2, ..., 6\}$

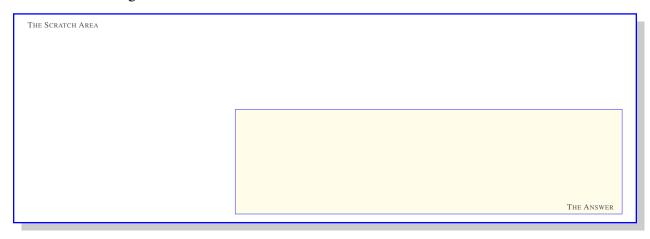


Q 4.2. Simplify the formula to ratio of polynomials with two terms only. (HINT: Use the formula for the sum of a finite geometric series

$$\sum_{k=0}^{n} x^{k} = (1 - x^{n+1})/(1 - x) = (1 - x^{n+1})(1 - x)^{-1}.$$



| Q 4.3. | Write the probability generating function for the sum $S_n = \sum_{j=1}^n X_j$, where each X_j represents the |
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| | face value of a singl die roll. |



Q 4.4. Use the Cauchy product formula and the binomial expansion for any real α :

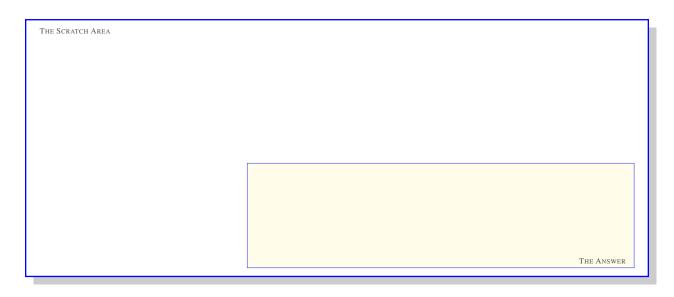
$$(1+x)^{\alpha} = \sum_{k=1}^{\infty} {\alpha \choose k} x^k$$

where

$$\binom{\alpha}{k} = \frac{\alpha(\alpha - 1)\cdots(\alpha - k + 1)}{k!}$$

to write down the formula for the probability $P(S_n = k)$. The formula should involve a single summation. Work out the limits. They will involve functions such as min, max, $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$. Also, you may find the following identity useful, which allows rewriting binomial coefficients for negative α with a positive one:

$$\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}$$



Q 4.5. Evaluate the previously derived formula for $P(X_n = k)$ with n = 6 and k = 15 "by hand" to get the exact value.

