Name: ____________________ ____________________ **Student ID:** ____________________

Week7-template **Math 563, Fall 2022**

Q 1. (Durrett 4.1.1.) **Bayes's Formula.** Let $G \in \mathcal{G}$.

Q 1.1. Show that

$$\mathbb{P}(G \mid A) = \frac{\int_G \mathbb{P}(A \mid \mathcal{G})\, d\mathbb{P}}{\int_\Omega \mathbb{P}(A \mid \mathcal{G})\, d\mathbb{P}}$$

THE PROOF

QED

Q 1.2. Show that when $\mathcal{G}$ is generated by a partition $\{G_1, G_2, \ldots\}$, this reduces to the usual Bayes' formula:

$$\mathbb{P}(G_i \mid A) = \frac{\mathbb{P}(A \mid G_i)\,\mathbb{P}(G_i)}{\sum_j \mathbb{P}(A \mid G_j)\,\mathbb{P}(G_j)}.$$

THE PROOF

QED

Q 2. (Durrett 4.1.2.) Prove **Chebyshev's inequality.** If $a > 0$ then

$$\mathbb{P}(|X| \geq a \mid \mathcal{F}) \leq a^{-2}\mathbb{E}(X^2 \mid \mathcal{F})$$

THE PROOF

QED

Q 3. (Durrett 4.1.5.) Give an example on $\Omega = \{a, b, c\}$ in which:

$$\mathbb{E}(\mathbb{E}(X \mid \mathcal{F}_1) \mid \mathcal{F}_2) \neq \mathbb{E}(\mathbb{E}(X \mid \mathcal{F}_2) \mid \mathcal{F}_1)$$

THE SCRATCH AREA

THE ANSWER

Q 4. (Durrett 4.1.9.) Show that if X and Y are random variables with $\mathbb{E}(Y \mid \mathcal{G}) = X$ and $\mathbb{E}(Y^2) = \mathbb{E}(X^2) < \infty$, then $X = Y$ a.s.

THE SCRATCH AREA

THE ANSWER

Q 5. (Durrett 4.1.10.) **Bonus problem!** The result of the last exercise implies that if $\mathbb{E}Y^2 < \infty$ and $\mathbb{E}(Y \mid \mathcal{G})$ has the same distribution as Y then $\mathbb{E}(Y \mid \mathcal{G}) = Y$ a.s. Prove that under the assumption $\mathbb{E}|Y| < \infty$. Hint: The trick is to prove that $\operatorname{sgn}(X) = \operatorname{sgn}(\mathbb{E}(X \mid \mathcal{G}))$ a.s., and then take $X = Y - c$ to get the desired result.

THE SCRATCH AREA

THE ANSWER