Name: ____________________ ____________________ **Student ID:** __________________________

Week2-template **Math 563, Fall 2022**

Q 1. (Durrett 1.2.1) Suppose X and Y are random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ and let $A \in \mathcal{F}$. Show that if we let $Z(\omega) = X(\omega)$ for $\omega \in A$ and $Z(\omega) = Y(\omega)$ for $\omega \in A^c$, then Z is a random variable.

Q 1.1. Complete the equation by replacing $\square$ with the correct formula:

$$Z^{-1}(B) = (A \cap (\square)) \cup (A^c \cap (\square))$$

THE SCRATCH AREA

THE ANSWER

Q 1.2. Complete the proof.

THE PROOF

QED

Q 2. (Durrett 1.2.2) Let χ have the standard normal distribution. Use Theorem 1.2.6 to get upper and lower bounds on $\mathbb{P}(\chi \geq 4)$.

Q 2.1. What is the upper bound?

THE SCRATCH AREA

THE ANSWER

Q 2.2. What is the lower bound?

THE SCRATCH AREA

THE ANSWER

Q 3. (Durrett 1.2.3) Show that a distribution function has at most countably many discontinuities.

THE PROOF

QED

Q 4. (Durrett 1.2.4) Show that if $F(x) = \mathbb{P}(X \leq x)$ is continuous then $Y = F(X)$ has a uniform distribution on $(0, 1)$, that is, if $y \in [0, 1]$, $P(Y \leq y) = y$.

Q 4.1. Prove the conclusion with an additional assumption that F is strictly increasing.

THE PROOF

QED

Q 4.2. Carry out the proof (without this additional assumption).

THE PROOF

QED

Q 5. (Durrett 1.2.5) Suppose X has continuous density f , $\mathbb{P}(\alpha \leq X \leq \beta) = 1$ and g is a function that is strictly increasing and differentiable on (α, β). Then

Q 5.1. $g(X)$ has density $f(g^{-1}(y))/g'(g^{-1}(y))$ for $y \in (g(\alpha), g(\beta))$ and 0 otherwise.

THE PROOF

QED

Q 5.2. When $g(x) = ax + b$ with $a > 0$, $g^{-1}(y) = (y - b)/a$ so the answer is $(1/a)f((y - b)/a)$.

THE PROOF

QED

Q 6. (Durrett 1.2.6) Suppose X has a normal distribution. Use the previous exercise to compute the density of $\exp(X)$. (The answer is called the lognormal distribution.)

THE PROOF

QED

Q 7. (Durrett 1.2.7)

Q 7.1. (i) Suppose X has density function f. Compute the distribution function of X^2 and then differentiate to find its density function.

THE PROOF

QED

Q 7.2. (ii) Work out the answer when X has a standard normal distribution to find the density of the chi-square distribution.

THE PROOF

QED

Q 8. (Custom Problem, generalizes Durrett 1.2.5) Suppose X has continuous density f, and g is a continuously differentiable function (not necessarily increasing) such that that the set

$$\{y \in \mathbb{R} : g'(y) = 0\}$$

has Lebesgue measure zero.

Q 8.1. Let $Y = g(X)$. Prove that Y has density h given by the equation

$$h(x) = \sum_{y \in g^{-1}(x)} \frac{f(y)}{|g'(y)|}.$$

THE PROOF

QED

Q 8.2. Let $g : (0,1] \to (0,1]$ be the *fractional part* of $1/x$, i.e. function defined by

$$g(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor$$

Let X be a continuous random variable whose distribution has probability density function (pdf):

$$f(x) = \begin{cases} \frac{1}{\pi}\frac{1}{1+x}, & 0 \le x \le 1, \\ 0 \quad \text{otherwise.} \end{cases}$$

Prove that $Y = g(X)$ is also a continuous variable with the same pdf f.

THE PROOF

QED