Name: ______________________ ______________________ **Student ID:** ____________________________

Week5-template **Math 563, Fall 2022**

Q 1. (Durrett 2.3.6.) Metric for convergence in probability. Show

Q 1.1. (a) that $d(X,Y) = \mathbb{E}\left(\frac{|X-Y|}{1+|X-Y|}\right)$ defines a metric on the set of random variables, i.e.,

1. $d(X,Y) = 0$ if and only if $X = Y$ a.s.,
2. $d(X,Y) = d(Y,X)$,
3. $d(X,Z) \leq d(X,Y) + d(Y,Z)$

THE PROOF

QED

Q 1.2. and (b) that $d(X_n, X) \to 0$ as $n \to \infty$ if and only if $X_n \to X$ in probability.

THE PROOF

QED

Q 2. (Durrett 2.3.10.) Kochen-Stone lemma. Suppose $\sum_k \mathbb{P}(A_k) = \infty$. Use Exercises 1.6.6 and 2.3.1 to show that if

$$\limsup_{n\to\infty} \frac{\left(\sum_{k=1}^{n} \mathbb{P}(A_k)\right)^2}{\sum_{1\le j,k\le n} \mathbb{P}(A_j \cap A_k)} = \alpha > 0$$

then $\mathbb{P}(A_n \text{i.o.}) \ge \alpha$. The case $\alpha = 1$ contains Theorem 2.3.7.

THE PROOF

QED

Q 3. (Durrett 2.3.14.) Let $X_1, X_2, \dots$ be independent. Show that $\sup X_n < \infty$ a.s. if and only if $\sum_n \mathbb{P}(X_n > A) < \infty$ for some A.

THE PROOF

QED

Q 4. (Durrett 2.3.15.) Let $X_1, X_2, \ldots$ be i.i.d. with $\mathbb{P}(X_i > x) = e^{-x}$, let $M_n = \max_{1 \le m \le n} X_m$. Show that

Q 4.1. (i) $\limsup_{n\to\infty} X_n / \log n = 1$ a.s.

THE PROOF

QED

Q 4.2. and (ii) $M_n / \log n \to 1$ a.s.

THE PROOF

QED

Q 5. (Durrett 2.3.17.) Let $Y_1, Y_2, \dots$ be i.i.d. Find necessary and sufficient conditions for

Q 5.1. (i) $Y_n/n \to 0$ almost surely,

THE SCRATCH AREA

THE ANSWER

Q 5.2. (ii) $\max_{m \le n} Y_m/n \to 0$ almost surely,

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Q 5.3. (iii) $\max_{m \le n} Y_m/n \to 0$ in probability,

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Q 5.4. and (iv) $Y_n/n \to 0$ in probability.

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THE ANSWER