Name: ____________________ ____________________ **Student ID:** ____________________

Q 1. (Durrett 3.1.1.) Generalize the proof of Lemma 3.1.1 to conclude that if $\max_{1\le j\le n}|cj,n| \to 0$, $\sum\sum_{j=1}^{n} c_{j,} \to \lambda$ and $\sup_n \sum_{j=1}^{n} |c_{j,n}| < \infty$ then $\prod_{j=1}^{n}(1+c_{j,n}) \to e^{\lambda}$.

The Proof

QED

Q 2. (Durrett 3.1.2.) If the X_i have a Poisson distribution with mean 1, then S_n has a Poisson distribution with mean n, i.e., $\mathbb{P}(S_n = k) = e^{-n}n^k/k!$.

Q 2.1. Use Stirling's formula to show that if $(k-n)/n \to x$ then

$$\sqrt{2\pi\, n}\,\mathbb{P}(S_n = k) \to \exp(-x^2/2).$$

THE PROOF

QED

Q 2.2. Use the asymptotic expansion for $\log n!$ (see *Wikipedia* page for details):

$$\begin{aligned}\ln(k!) - \tfrac{1}{2}\ln k &= \tfrac{1}{2}\ln 1 + \ln 2 + \ln 3 + \cdots + \ln(k-1) + \tfrac{1}{2}\ln k \\ &= k\ln k - k + 1 + \sum_{\ell=2}^{\infty} \frac{(-1)^\ell B_\ell}{\ell(\ell-1)}\left(\frac{1}{k^{\ell-1}} - 1\right)\end{aligned}$$

where B_ℓ is a Bernoulli number, to find the asymptotic expansion of $\mathbb{P}(S_n = k)$ in n. Negative powers of n are to be expected.

THE SCRATCH AREA

THE ANSWER

Q 3. (Durrett 3.1.4.) Suppose $\mathbb{P}(X_i = k) = e^{-1}/k!$ for $k = 0, 1, \ldots$. Show that if $a > 1$

$$\frac{1}{n} \log \mathbb{P}(S_n \geq n\,a) \to a - 1 - a\,\log a.$$

THE PROOF

QED

Q 4. (Custom) Use the method of generating functions to calculate the **the exact** probability that in 6 independent die rolls we get a total score of 15. This problem may seem to involve a lot of computations, but it is not so if the method of generating functions is used efficiently.

Q 4.1. Write the probability generating function for a single die roll score X_1 (uniforml probabilty of $1/6$ for die faces $\Omega = \{1, 2, \ldots, 6\}$

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Q 4.2. Simplify the formula to ratio of polynomials with two terms only. (HINT: Use the formula for the sum of a finite geometric series

$$\sum_{k=0}^{n} x^k = (1 - x^{n+1})/(1 - x) = (1 - x^{n+1})(1 - x)^{-1}.$$

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Q 4.3. Write the probability generating function for the sum $S_n = \sum_{j=1}^{n} X_j$, where each X_j represents the face value of a singl die roll.

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Q 4.4. Use the Cauchy product formula and the binomial expansion for any real α:

$$(1+x)^\alpha = \sum_{k=1}^{\infty} \binom{\alpha}{k} x^k$$

where

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$$

to write down the formula for the probability $P(S_n = k)$. The formula should involve a single summation. Work out the limits. They will involve functions such as $\min$, $\max$, $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$. Also, you may find the following identity useful, which allows rewriting binomial coefficients for negative α with a positive one:

$$\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}$$

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Q 4.5. Evauluate the previously derived formula for $P(X_n = k)$ with $n = 6$ and $k = 15$ "by hand" to get the exact value.

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