Name: ______________________ ______________________ **Student ID:** ______________________

Week1-template **Math 563, Fall 2022**

Q 1. (Analysis, Set Theory Review)

Q 1.1. Carefully state the **Axiom of Choice**. Use proper notation, consistent with the book.

THE SCRATCH AREA

THE ANSWER

Q 1.2. Let x_i, $i \in I$, be an uncountable family of positive real numbers. Let $|J|$ denote the cardinality of a subset $J \subseteq I$. Prove that

$$\sup_{J \subset I,\ |J| < \infty} \sum_{i \in J} x_i = \infty.$$

Make sure to show the step which relies upon the Axiom of Choice.

THE PROOF

QED

Q 2. (Durrett, Problem 1.1.1) Let $\Omega = \mathbb{R}$, $\mathcal{F}$ = all subsets so that A or A^c is countable, $\mathbb{P}(A) = 0$ in the first case and $= 1$ in the second. Show that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space.

Q 2.1. Which are the **axioms** of a σ-algebra? (When in doubt, follow the book.)

(A) closed under complement

(B) closed under finite intersection

(C) closed under countable union

(D) contains empty set $\varnothing$

(E) contains Ω

(F) contains all subsets of Ω

Q 2.2. Prove that $\mathcal{F}$ is a σ-algebra.

THE PROOF

QED

Q 2.3. Which are **properties** (including the axioms) of $\mathbb{P}$, a general probability measure? (When in doubt, follow the book.)

- (A) The domain of $\mathbb{P}$ is 2^{Ω}.
- (B) The domain of $\mathbb{P}$ is $\mathcal{F}$.
- (C) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$ for $A, B \in \mathcal{F}$.
- (D) $\mathbb{P}(A^c) = \mathbb{P}(A)$
- (E) $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$
- (F) $\mathbb{P}(\varnothing) = 0$.
- (G) If $\mathbb{P}(A) = 0$ then $A = \varnothing$.
- (H) If $A \subseteq B$ then $\mathbb{P}(A) \leq \mathbb{P}(B)$ for $A, B \in \mathcal{F}$.

Q 2.4. Prove that $\mathbb{P}$ defined in 1.1.1 is a probability measure.

THE PROOF

QED

Q 3. (Durret 1.1.2) Let $\mathcal{S}_d$ = the empty set plus all sets of the form

$$(a_1, b_1] \times \ldots (a_d, b_d] \subset \mathbb{R}^d$$

where $-\infty \le a_i < b_i \le \infty$. Show that $\sigma(\mathcal{S}_d) = \mathcal{R}^d$, the Borel subsets of $\mathbb{R}^d$.

THE PROOF

QED

Q 4. (Durret 1.1.3) A σ-field $\mathcal{F}$ is said to be countably generated if there is a countable collection $\mathcal{C} \subseteq \mathcal{F}$ so that $\sigma(\mathcal{C}) = \mathcal{F}$. Show that $\mathcal{R}^d$ is countably generated.

THE PROOF

QED

Q 5. (Durret 1.1.4)

Q 5.1. (i) Show that if $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \dots$ are σ-algebras, then $\bigcup_i \mathcal{F}_i$ is an algebra.

THE PROOF

QED

Q 5.2. (ii) Give an example to show that $\bigcup_i \mathcal{F}_i$ need not be a σ-algebra.

THE SCRATCH AREA

THE ANSWER

Q 6. (Durrett 1.1.5) A set $A \subseteq \{1, 2, \ldots\}$ is said to have asymptotic density θ if

$$\lim_{n\to\infty} |A \cap \{1, 2, \ldots, n\}|/n = \theta$$

Let $\mathcal{A}$ be the collection of sets for which the asymptotic density exists.

Q 6.1. Is $\mathcal{A}$ a σ-algebra?

(A) Yes

(B) No

Q 6.2. an algebra?

(A) Yes

(B) No

Q 6.3. Justify the answers in previous parts.

THE PROOF

QED