Name: ____________________ ____________________ **Student ID:** __________________________

Week3-template **Math 563, Fall 2022**

Q 1. (Durrett 1.3.2.) Prove Theorem 1.3.6 when $n = 2$ by checking $\{X_1 + X_2 < x\} \in \mathcal{F}$.

Q 1.1. Let $X = X_1 + X_2$. Prove that we have a partition

$$X^{-1}((-\infty, x]) = \dot{\bigcup_{u \in \mathbb{R}}} X_1^{-1}(u) \cap X_2^{-1}((-\infty, x - u]). \quad (1)$$

THE PROOF

QED

Q 1.2. Why is the fact in the previous part not sufficient to deduce that $X^{-1}((-\infty, x]) \in \mathcal{F}$?

THE SCRATCH AREA

THE ANSWER

Q 1.3. If X_1 is a discrete variable, use the partition in equation (1) to show that $\{X < x\} \in \mathcal{F}$.

THE PROOF

QED

Q 1.4. Write $X^{-1}((-\infty, x])$ as a triple-nested expression (e.g. countable union of countable intersections of countable unions) like the partition above, which shows by inspection that $\{X < x\} \in \mathcal{F}$.

THE SCRATCH AREA

THE ANSWER

Q 2. (Durrett 1.3.5.) A function f is said to be lower semicontinuous or l.s.c. if

$$\liminf_{y \to x} f(y) \geq f(x)$$

and upper semicontinuous (u.s.c.) if $-f$ is l.s.c. Show that f is l.s.c. if and only if $\{x : f(x) \leq a\}$ is closed for each $a \in \mathbb{R}$ and conclude that semicontinuous functions are measurable.

THE PROOF

QED

Q 3. (Durrett 1.4.4.) Prove the Riemann-Lebesgue lemma. If g is integrable (in the sense of Lebesgue) then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} g(x) \cos n\, x\, dx = 0.$$

Hint: If g is a step function, this is easy. Now use the previous exercise (Durrett 1.4.3). NOTE: Use the results of that exercise freely but do not solve it.

THE PROOF

QED

Q 4. (Durrett 1.5.8.) Show that if f is integrable on $[a, b]$, $g(x) = \int_{[a,x]} f(y)\, dy$ is continuous on (a, b).

THE PROOF

QED

Q 5. (Durrett 1.6.14.) Let $X \geq 0$ but do NOT assume $\mathbb{E}(1/X) < \infty$. The book defines this notation

$$\mathbb{E}(X; A) = \int_A X \, d\mathbb{P} = \mathbb{E}(X \cdot \mathbb{1}_A)$$

Show

Q 5.1.

$$\lim_{y \to \infty} y \, \mathbb{E}(1/X; X > y) = 0,$$

THE PROOF

QED

Q 5.2.

$$\lim_{y \downarrow 0} y \, \mathbb{E}(1/X; X > y) = 0.$$

THE PROOF

QED