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Week9-template **Math 468, Spring 2022**

Q 1. (Durrett, Problem 4.1) A salesman flies around between Atlanta, Boston, and Chicago as the following rates (the units are trips per month):

F T	A	B	C
A	-4	2	2
B	3	-4	1
C	5	0	-5

Q 1.1. What is the transition rate matrix $\mathbf{Q}$ for this process?

Solution: THE ANSWER:

$$\mathbf{Q} = \begin{bmatrix} -4 & 2 & 2 \\ 3 & -4 & 1 \\ 5 & 0 & -5 \end{bmatrix}$$

Q 1.2. List the eigenvalues of $\mathbf{Q}$ as a comma-separated list.

Solution: THE ANSWER:

$$-5, -8, 0$$

Q 1.3. Find the (right) diagonalizing matrix S of $\mathbf{Q}$, so that $\mathsf{S}^{-1}\mathbf{Q}\mathsf{S}$ is diagonal. Scale the columns so that the first entry in each column (counting from the top) is 1.

Solution: THE ANSWER:

$$\mathbf{S} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -\frac{1}{3} & 1 \\ -1 & -\frac{5}{3} & 1 \end{bmatrix}$$

Q 1.4. Find the left diagonalizing matrix $\mathbf{L} = \mathbf{S}^{-1}$ of $\mathbf{Q}$, so that $\mathbf{L}\mathbf{Q}\mathbf{L}^{-1}$ is diagonal.

Solution: THE ANSWER:

$$\begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

Q 1.5. What is the stationary distribution π for this Markov process? Hint: Use one of the rows of the matrix found in the previous part. Make sure it is a row vector.

Solution: THE ANSWER:

$$\pi = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

Q 1.6. (Durrett 4.1, part (a)) Find the limiting fraction of time she spends in each city. Only the exact answer yields credit. List the numbers in the order "A, B, C".

Solution: THE ANSWER:

$$1/2, 1/4, 1/4$$

Q 1.7. Find the routing matrix $\mathbf{R}$ for $\mathbf{Q}$.

Solution: THE ANSWER:

$$\mathbf{R} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 3/4 & 0 & 1/4 \\ 1 & 0 & 0 \end{bmatrix}$$

Q 1.8. If she is in Boston now, what is the probability that the first city she will visit next is Chicago?

Solution: THE ANSWER:

$$\mathbb{P}(Y_1 = C | Y_0 = B) = \mathbf{R}_{23} = 1/4$$

Q 1.9. (Durrett 4.1, part (b)) What is her average number of trips each year from Boston to Atlanta?

Solution: Once she is in Boston, she will visit Atlanta next according to the routing matrix $\mathbf{R}$, namely $\mathbf{R}_{21} = 3/4$ of the times. Hence, the answer is $3/4$ of the average number of visits to Boston per year. As the fraction of time spend in Boston is $1/4$ of the time, and the rate of staying in Boston is 4, the everage stay in Boston is $1/4$ (of a month). Therefore, to stay $1/4$ of 1 year, which is 3 months spent in Boston each year, we must visit Boston 12 times a year. Clearly, $3/4$ of those visits is 9 (trips to Atlanta). THE ANSWER:

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Q 1.10. Find the matrix $\mathbf{P}(t) = e^{t\,\mathbf{Q}}$.

Solution: As we know, $\mathbf{Q}$ is diagonalizable and we know its diagonalizing matrix

$$\mathbf{S} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -\frac{1}{3} & 1 \\ -1 & -\frac{5}{3} & 1 \end{bmatrix}$$

and its inverse

$$\mathbf{L} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}.$$

Also, the diagonal form is

$$\mathbf{D} = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore,

$$e^{t\,\mathbf{Q}} = \mathbf{S}e^{t\,\mathbf{D}}\mathbf{S}^{-1} = \mathbf{S}e^{t\,\mathbf{D}}\mathbf{L}.$$

Hence the answer is:

$$\begin{aligned} e^{t\,\mathbf{Q}} &= \begin{bmatrix} 0 & 1 & 1 \\ 1 & -\frac{1}{3} & 1 \\ -1 & -\frac{5}{3} & 1 \end{bmatrix} \begin{bmatrix} e^{-5t} & 0 & 0 \\ 0 & e^{-8t} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 1 \\ 1 & -\frac{1}{3} & 1 \\ -1 & -\frac{5}{3} & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{3}e^{-5t} & \frac{2}{3}e^{-5t} & -\frac{1}{3}e^{-5t} \\ \frac{1}{2}e^{-8t} & -\frac{1}{4}e^{-8t} & -\frac{1}{4}e^{-8t} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2}\,e^{-8t} + \frac{1}{2} & -\frac{1}{4}\,e^{-8t} + \frac{1}{4} & -\frac{1}{4}\,e^{-8t} + \frac{1}{4} \\ -\frac{1}{3}\,e^{-5t} - \frac{1}{6}\,e^{-8t} + \frac{1}{2} & \frac{2}{3}\,e^{-5t} + \frac{1}{12}\,e^{-8t} + \frac{1}{4} & -\frac{1}{3}\,e^{-5t} + \frac{1}{12}\,e^{-8t} + \frac{1}{4} \\ \frac{1}{3}\,e^{-5t} - \frac{5}{6}\,e^{-8t} + \frac{1}{2} & -\frac{2}{3}\,e^{-5t} + \frac{5}{12}\,e^{-8t} + \frac{1}{4} & \frac{1}{3}\,e^{-5t} + \frac{5}{12}\,e^{-8t} + \frac{1}{4} \end{bmatrix} \end{aligned}$$

THE ANSWER:

$$\begin{bmatrix} \frac{e^{-8\,t}}{2} + \frac{1}{2} & \frac{1}{4} - \frac{e^{-8\,t}}{4} & \frac{1}{4} - \frac{e^{-8\,t}}{4} \\ -\frac{e^{-5\,t}}{3} - \frac{e^{-8\,t}}{6} + \frac{1}{2} & \frac{2\,e^{-5\,t}}{3} + \frac{e^{-8\,t}}{12} + \frac{1}{4} & -\frac{e^{-5\,t}}{3} + \frac{e^{-8\,t}}{12} + \frac{1}{4} \\ \frac{e^{-5\,t}}{3} - \frac{5\,e^{-8\,t}}{6} + \frac{1}{2} & -\frac{2\,e^{-5\,t}}{3} + \frac{5\,e^{-8\,t}}{12} + \frac{1}{4} & \frac{e^{-5\,t}}{3} + \frac{5\,e^{-8\,t}}{12} + \frac{1}{4} \end{bmatrix}$$

Q 1.11. If she is in Boston now, what is the probability that she will be in Atlanta two months from now? Your answer must have at least 6 digits of precision.

Solution: The answer is $\mathbb{P}(X(2) = A|X(0) = B) = \mathbf{P}_{21}(2)$ and can be obtained by evaluating the right entry of the matrix from the previous part. THE ANSWER:

$$\begin{aligned}\mathbf{P}_{21}(2) &= -\frac{e^{-5\,t}}{3} - \frac{e^{-8\,t}}{6} + \frac{1}{2}\Big|_{t=2} \\ &= -\frac{e^{-10}}{3} - \frac{e^{-16}}{6} + \frac{1}{2} \approx 0.4999848479342167\end{aligned}$$