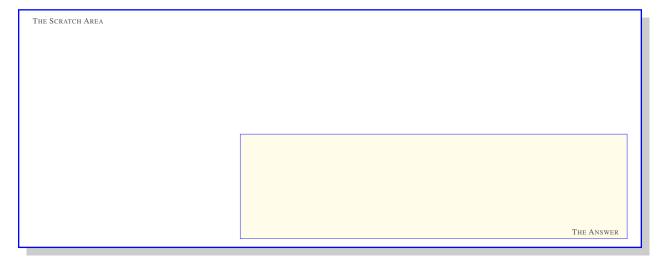
| Name:          | Student ID: |                     |
|----------------|-------------|---------------------|
| Week3-template |             | Math 563, Fall 2022 |

- (Durrett 1.3.2.) Prove Theorem 1.3.6 when n=2 by checking  $\{X_1+X_2 < x\} \in \mathcal{F}$ . Let  $X=X_1+X_2$ . Prove that we have a partition Q 1.
- Q 1.1.

$$X^{-1}((-\infty, x]) = \bigcup_{u \in \mathbb{R}} X_1^{-1}(u) \cap X_2^{-1}((-\infty, x - u]).$$
 (1)

| THE PROOF |     |
|-----------|-----|
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           | QED |

Why is the fact in the previous part not sufficient to deduce that  $X^{-1}((-\infty,x]) \in \mathcal{F}$ ? Q 1.2.



**Q 1.3.** If  $X_1$  is a discrete variable, use the partition in equation (1) to show that  $\{X < x\} \in \mathcal{F}$ .



**Q 1.4.** Write  $X^{-1}((-\infty, x])$  as a triple-nested expression (e.g. countable union of countable intersections of countable unions) like the partition above, which shows by inspection that  $\{X < x\} \in \mathcal{F}$ .



$$\liminf_{y \to x} f(y) \ge f(x)$$

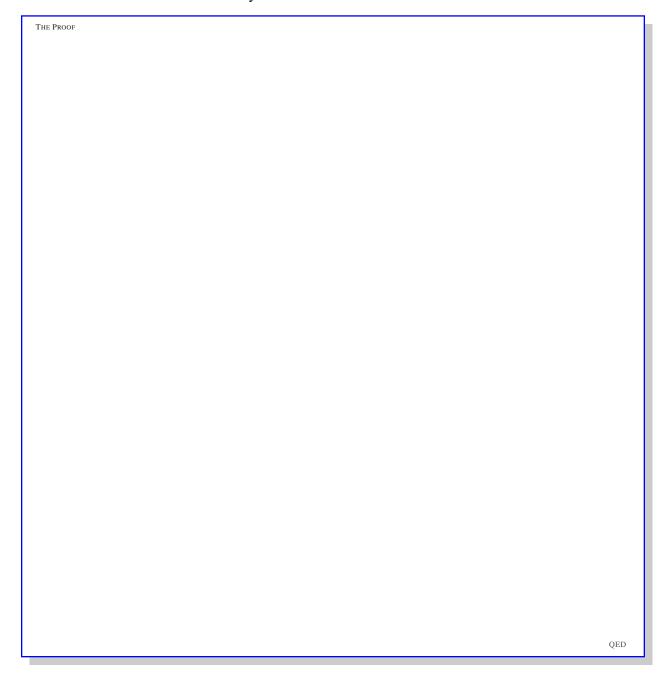
and upper semicontinuous (u.s.c.) if -f is l.s.c. Show that f is l.s.c. if and only if  $\{x: f(x) \le a\}$  is closed for each  $a \in \mathbb{R}$  and conclude that semicontinuous functions are measurable.

| THE PROOF |     |
|-----------|-----|
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           | QED |
|           |     |

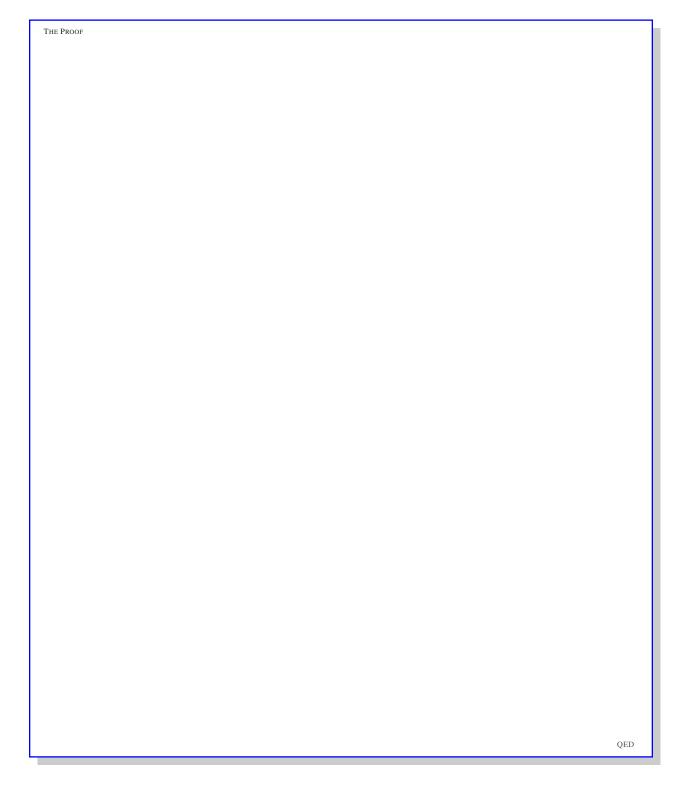
| Q 3. | (Durrett 1.4.4.) Prove the Riemann-Lebesgue lemma. If $g$ is integrable (in the sense of Lebesgue) |
|------|--|
|      | then $C^{\infty}$  |

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}g(x)\cos n\,x\,dx=0.$$

Hint: If g is a step function, this is easy. Now use the previous exercise (Durrett 1.4.3). NOTE: Use the results of that exercise freely but do not solve it.



**Q 4.** (Durrett 1.5.8.) Show that if f is integrable on [a,b],  $g(x) = \int_{[a,x]} f(y) dy$  is continuous on (a,b).



**Q 5.** (Durrett 1.6.14.) Let  $X \ge 0$  but do NOT assume  $\mathbb{E}(1/X) < \infty$ . The book defines this notation

$$\mathbb{E}(X;A) = \int_A X d\mathbb{P} = \mathbb{E}(X \cdot \mathbb{1}_A)$$

Show

Q 5.1.

$$\lim_{y\to\infty}y\,\mathbb{E}\big(1/X;X>y\big)=0,$$

| THE PROOF |     |
|-----------|-----|
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           |     |
|           | QED |
|           |     |

Q 5.2.

$$\lim_{y\downarrow 0}y\,\mathbb{E}(1/X;X>y)=0.$$

THE PROOF