Density of States of Hodge Laplacians: Decomposition effects and Sparsification of SC

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Abstract:

Keywords: density of states, simplicial complexes, sparsification, generalized effective resistance

I Introduction

Well, it would be nice to have one.
You know, somewhere here, maybe...

Contributions. we do contribute something, right?

Outline. it's all over the place, man...

II Preliminaries

I. Simplicial Complexes, [Lim20]

II. Networks' Density of States, [DBB19]

Def. 1 **(Density of States)** Fro a given symmetric matrix $A = Q\Lambda Q^\top$ with $Q^\top Q = I$ and diagonal $\Lambda = \operatorname{diag}(\lambda_1, \dots \lambda_n)$, the spectral density or density of states (DoS)

$$\mu(\lambda \mid A) = \frac{1}{n}\sum_{i=1}^{n} \delta(\lambda - \lambda_i) \tag{Eqn. 1}$$

Additionally, let $\mathbf{q}_i$ be a corresponding unit eigenvector of A (such that $A\mathbf{q}_i = \lambda_i \mathbf{q}_i$ and $Q = (\mathbf{q}_1 \mid \mathbf{q}_2 \mid \cdots \mid \mathbf{q}_n)$); then one can define a set of local (entry-wise) densities of states (LDoS):

$$\mu_k(\lambda \mid A) = \sum_{i=1}^{n} \left| \mathbf{e}_k^\top \mathbf{q}_i \right|^2 \delta(\lambda - \lambda_i) \tag{Eqn. 2}$$

with $\mathbf{e}_k$ being the corresponding versor.

III. Kernel Polynomial Method (KPM)

Th III.III.1 **(Simplicial Sparsification, [OPW22])** Let $\mathcal{K}$ be a simplicial complex restricted to its p-skeleton, $\mathcal{K} = \bigcup_{i=0}^{p} \mathcal{V}_i(\mathcal{K})$. Let $L_k^{\uparrow}(\mathcal{K})$ be its k-th up-Laplacian and let $m_k = |\mathcal{V}_k(\mathcal{K})|$. For any $\varepsilon > 0$, a sparse simplicial complex $\mathcal{L}$ can be sampled as follows:

(1) compute the probability measure $\mathbf{p}$ on $\mathcal{V}_{k+1}(\mathcal{K})$ proportional to the generalized resistance vector $\mathbf{r} = \operatorname{diag}\left(B_{k+1}^\top (L_k^{\uparrow})^{\dagger} B_{k+1}\right)$, where $\operatorname{diag}(A)$ denotes the vector of the diagonal entries of A;

(2) sample q simplices τ_i from $\mathcal{V}_{k+1}(\mathcal{K})$ according to the probability measure $\mathbf{p}$, where q is chosen so that $q(m_k) \geq 9C^2 m_k \log(m_k/\varepsilon)$, for some absolute constant $C > 0$;

(3) form a sparse simplicial complex $\mathcal{L}$ with all the sampled simplexes of order k and all its faces with the weight $\frac{w_{k+1}(\tau_i)}{q(m_k)\mathbf{p}(\tau_i)}$; weights of repeated simplices are accumulated.

Then, with probability at least $1/2$, the up-Laplacian of the sparsifier $\mathcal{L}$ is ε-close to the original one, i.e. it holds $L_k^{\uparrow}(\mathcal{L}) \underset{\varepsilon}{\approx} L_k^{\uparrow}(\mathcal{K})$.

The bottleneck of the subsampling above is the construction of the appropriate measure $\mathbf{p}$ or, more precisely, generalized effective resistance $\mathbf{r}$. Indeed, one needs a fast pseudo-inverse operator $\left(L_k^{\uparrow}\right)^{\dagger}$ in order to compute $\mathbf{r}$.

we need to assume somewhere $W_k = I$ for simplicity and do not forget about it

Th III.III.2 **(GER through DoS)** For a given simplicial complex mcK with the k-th order up-Laplacian $L_k^{\uparrow} = B_{k+1} W_{k+1}^2 B_{k+1}^\top$, a generalized effective resistance r can be computed through family of local densities of states $\{\mu_i(\lambda \mid L_{k+1}^{\downarrow})\}$:

$$\mathbf{r}_i = \int_{\mathbb{R}} (1 - \mathbb{1}_0(\lambda)) \mu_i(\lambda \mid L_{k+1}^{\downarrow}) d\lambda \quad \text{(Eqn. 3)}$$

weighted egdges === BAD ?

Proof Let $B_{k+1} W_{k+1} = USV^\top$ where S is diagonal and invertible and both U and V are orthogonal (so it is a truncated SVD decomposition of $B_{k+1} W_{k+1}$ matrix with eliminated obsolete kernel). Then:

$$(L_k^{\uparrow})^{\dagger} = \left(B_{k+1} W_{k+1}^2 B_{k+1}^\top\right)^{\dagger} = \left(US^2U^\top\right)^{\dagger} = US^{-2}U^\top \quad \text{(Eqn. 4)}$$

$$\begin{aligned} \mathbf{r} = \operatorname{diag}\left(W_{k+1} B_{k+1}^\top (L_k^{\uparrow})^{\dagger} B_{k+1} W_{k+1}\right) = \\ = \operatorname{diag}\left(VSU^\top US^{-2}U^\top USV^\top\right) = \operatorname{diag}\left(VV^\top\right) \end{aligned} \quad \text{(Eqn. 5)}$$

As a result, $\mathbf{r}_i = \|V_{i\cdot}\|^2 = \sum_j |v_{ij}|^2$, so the i-th entry of the resistance is defined by the sum of square of i-th components of eigenvectors $\mathbf{v}_j$ of $L_{k+1}^{\downarrow} = W_{k+1} B_{k+1}^\top B_{k+1} W_{k+1}$ operator where $\mathbf{v}_j \perp \ker L_{k+1}^{\downarrow}$.
Note that

$$\mu_i(\lambda \mid L_{k+1}^{\downarrow}) = \sum_{j=1}^{m_{k+1}} \left|\mathbf{e}_i^\top \mathbf{q}_j\right|^2 \delta\left(\lambda - \lambda_j\right) = \sum_{j=1}^{m_{k+1}} \left|q_{ij}\right|^2 \delta\left(\lambda - \lambda_j\right) \quad \text{(Eqn. 6)}$$

so

$$\mathbf{r}_i = \|V_{i\cdot}\|^2 = \sum_j |v_{ij}|^2 = \int_{\mathbb{R}\setminus\{0\}} \sum_{j=1}^{m_{k+1}} |q_{ij}|^2 \, \delta\left(\lambda - \lambda_j\right) d\lambda = \\ = \int_{\mathbb{R}\setminus\{0\}} \mu_i(\lambda \mid L^{\downarrow}_{k+1}) d\lambda = \int_{\mathbb{R}} (1 - \mathbb{1}_0(\lambda)) \mu_i(\lambda \mid L^{\downarrow}_{k+1}) d\lambda \qquad \text{(Eqn. 7)}$$

■

Rem III.1 **(Sensitivity of the Sparsification vis-a-vis sampling measure p)** Compare ε with $\frac{1}{m_2}$: if below, you are fiiiiiiine
INSERT FIGURE HERE

IV KID for LDoS

Note that up- and down-Laplacians $L^{\uparrow}_k$ and $L^{\downarrow}_k$ by their definition have non-trivial kernels of high dimensionality. Indeed, recalling the Hodge decomposition :

ref up

$$\mathbb{R}^{m_k} = \overbrace{\text{im}\, B_k^\top \oplus \underbrace{\ker\left(B_k^\top B_k + B_{k+1} B_{k+1}^\top\right)}_{}}^{\ker B_{k+1}^\top} \oplus\, \text{im}\, B_{k+1}$$

$$\underbrace{\ker\left(B_k^\top B_k + B_{k+1} B_{k+1}^\top\right) \oplus \text{im}\, B_{k+1}}_{\ker B_k} \qquad \text{(Eqn. 8)}$$

the subspaces $\ker B_k = \ker L^{\downarrow}_k$ and $\ker B_{k+1}^\top = \ker L^{\uparrow}_k$ include at least $\text{im}\, B_{k+1}$ and $\text{im}\, B_{k+1}^\top$. As a result, the zero eigenvalue in the corresponding DoS and LDoS for both operators exhibits a dominating pick severely affecting the quality of KPM-approximation.

V Conclusion

Motive filtration is still a thing

V References

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