Interpolation of EigenSpaces in Graph/SC-induced operators

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Abstract: There are the notes for the interpolation of the eigenspaces for graph-induced operators.

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I. Initial definitions

Def. 1 **(Parametric family)** Let $p \in [0; 1]$ be a real parameter for a family of square matrices $A(p) \in \mathbb{R}^{n\times n}$ (such that each element $a_{ij} = a_{ij}(p)$). In order to simplify the discussion, we assume that $\forall p$: $A^\top(p) = A(p)$ and each $A(p)$ has real spectrum.

The problem we want to deal with is the question of eigenspace interpolation: indeed, let $A(p)$ first have only **simple** eigenvalues; then, we aim to find first k eigenvectors of $A(p)$ for every p given several exact estimations at $\{p_1, \dots p_N\}$. We denote the matrix composed of k first unit eigenvalues by $C(p) \in \mathbb{R}^{n\times k}$, so

$$\begin{aligned} C(p)^\top C(p) &= I_k \\ A(p)C(p) &= C(p)\Lambda(p) \end{aligned} \tag{Eqn. 1}$$

such that $\Lambda(p) = \operatorname{diag}\{\lambda_1, \dots \lambda_k\}$ with $\lambda_1 > \lambda_2 > \dots > \lambda_k$.

We also adopt the convention $X(p_i) = X_i$.

II. Two point problem

Let us assume we have only two point estimation (A_0, C_0) and (A_1, C_1). How can we estimate C_α?

- ◇ compute a polynomial estimation: $C_\alpha = \alpha C_0 + (1-\alpha)C_1$;
- ◇ one needs C_α to uphold Equation (1).

What do we make of Equation (1)?

$$\begin{aligned} C_\alpha^\top C_\alpha &= (\alpha C_0 + (1-\alpha)C_1)^\top (\alpha C_0 + (1-\alpha)C_1) = \\ &= \alpha^2 C_0^\top C_0 + 2\alpha(1-\alpha)\operatorname{Sym}(C_0^\top C_1) + (1-\alpha)^2 C_1^\top C_1 = \\ &= (1 - 2\alpha + 2\alpha^2) I_k + 2\alpha(1-\alpha)\operatorname{Sym}(C_0^\top C_1) = \\ &= I_k + 2\alpha(1-\alpha)\operatorname{Sym}(C_0^\top C_1 - I_k) \end{aligned} \tag{Eqn. 2}$$

Additionally,

$$A_\alpha C_\alpha = A_\alpha(\alpha C_0 + (1-\alpha)C_1) = \alpha A_\alpha C_0 + (1-\alpha)A_\alpha C_1 \tag{Eqn. 3}$$

This is clearly not enough: we need $A_\alpha C_\alpha = C_\alpha \Lambda_\alpha$ such that Λ_α is diagonal. Instead, we can write $C_\alpha^\top A_\alpha C_\alpha = \Lambda_\alpha$; so we would need $C_\alpha^\top A_\alpha C_\alpha$ be as close to diagonal as possible:

$$
\begin{aligned}
C_\alpha^\top A_\alpha C_\alpha &= (\alpha C_0 + (1-\alpha) C_1)^\top A_\alpha (\alpha C_0 + (1-\alpha) C_1) = \\
&= \alpha^2 C_0^\top A_\alpha C_0 + (1-\alpha)^2 C_1^\top A_\alpha C_1 + \\
&+ 2\alpha(1-\alpha)\, \mathrm{Sym}(C_0^\top A_\alpha C_1)
\end{aligned}
\tag{Eqn. 4}
$$

Let $\Delta A_0 = A_\alpha - A_0$ and $\Delta A_1 = A_\alpha - A_1$, so:

$$
\begin{aligned}
C_0^\top A_\alpha C_0 &= C_0^\top \Delta A_0 C_0 + \Lambda_0 \\
C_1^\top A_\alpha C_1 &= C_1^\top \Delta A_0 C_1 + \Lambda_1
\end{aligned}
\tag{Eqn. 5}
$$

Can we inject something into $C_0^\top A_\alpha C_1 + C_1^\top A_\alpha C_0$?

$$
\begin{aligned}
\mathrm{Sym}(C_0^\top A_\alpha C_1) &= \mathrm{Sym}(C_0^\top A_0 C_1) + \mathrm{Sym}(C_1^\top \Delta A_0 C_0) \\
\mathrm{Sym}(C_0^\top A_\alpha C_1) &= \mathrm{Sym}(C_0^\top A_1 C_1) + \mathrm{Sym}(C_1^\top \Delta A_1 C_0)
\end{aligned}
\tag{Eqn. 6}
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References

[KS16] Rasmus Kyng and Sushant Sachdeva. Approximate gaussian elimination for laplacians: Fast, sparse, and simple. May 2016.