Topological Stability and Preconditioning of Higher-Order Laplacian Operators on Simplicial Complexes

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Abstract: Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum.

Keywords: NNs, condition number

I. Introduction

II. From Graphs to Simplicial Complex

II.I Simplicial Complexes

Let $V = \{v_1, v_2, \ldots, v_n\}$ be a set of nodes; as discussed above, such set may refer to various interacting entities and agents in the system, e.g. neurons, genes, traffic stops, online actors, publication authors, etc. Then:

Def. 1 **(Simplicial Complex)** The collection of subsets $\mathcal{K}$ of the nodal set V is a (abstract) simplicial complex[1] if for each subset $\sigma \in \mathcal{K}$, referred as a simplex, all its subsets σ', $\sigma' \subseteq \sigma$, referred as face, enter $\mathcal{K}$ as well, $\sigma' \in \mathcal{K}$.

[1] addition of the word "abstract" to the term is more common in the topological setting

A simplex $\sigma \in \mathcal{K}$ on $k+1$ vertices is said to be of the order k, $\operatorname{ord}\sigma = k$. Let $\mathcal{V}_k(\mathcal{K})$ be a set of all k-order simplices in $\mathcal{K}$ and m_k is the cardinality of $\mathcal{V}_k(\mathcal{K})$, $m_k = |\mathcal{V}_k(\mathcal{K})|$; then $\mathcal{V}_0(\mathcal{K})$ is the set of nodes in the simplicial complex $\mathcal{K}$, $\mathcal{V}_1(\mathcal{K})$ — the set of edges, $\mathcal{V}_2(\mathcal{K})$ — the set of triangles, or 3-cliques, and so on, with $\mathcal{K} = \{\mathcal{V}_0(\mathcal{K}), \mathcal{V}_1(\mathcal{K}), \mathcal{V}_2(\mathcal{K}) \ldots\}$. Note that due to the inclusion rule in Definition 1, the number of non-empty $\mathcal{V}_k(\mathcal{K})$ is finite and, moreover, uninterupted in a sense of the order: if $\mathcal{V}_k(\mathcal{K}) = \varnothing$, then $\mathcal{V}_{k+1}(\mathcal{K})$ is also necessarily empty.

Example (Simplicial Complex) 123

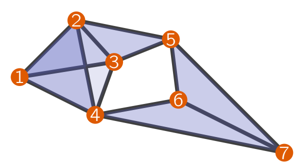

Figure 1: Example of a simplicial complex

III. Matrix nearness problems

IV. Topological Stability of Simplicial Complexes

V. Preconditioning

V.I Preconditioning 101

V.II Cholesky preconditioning for classical graphs

V.III Collapsible simplicial complexes

In this section we borrow the terminology from [Whi39]. The simplex $\sigma \in \mathcal{K}$ is free if it is a face of exactly one simplex $\tau = \tau(\sigma) \in \mathcal{K}$ of the higher order (facet). The collapse $\mathcal{K} \backslash \{\sigma\}$ of $\mathcal{K}$ at a free simplex σ is the transition $\mathcal{K} \to \mathcal{K}' = \mathcal{K} - \sigma - \tau$; namely, one can eliminate a simplex τ if it has an accessible (not included in another simplex) face σ.

The consequent collapses at $\Sigma = \{\sigma_1, \sigma_2, \ldots\}$ are called the collapsing sequence; formally:

Def. 2 **(Collapsing sequence)** Let $\mathcal{K}$ be a simplicial complex. $\Sigma = \{\sigma_1, \sigma_2, \dots\}$ is a collapsing sequence if σ_1 is free in $\mathcal{K}$ and each σ_i, $i > 1$, is free at $\mathcal{K}^{(i)} = \mathcal{K}^{(i-1)} \backslash \{\sigma_i\}$, $\mathcal{K}^{(1)} = \mathcal{K}$. The collapse of $\mathcal{K}$ to a new complex $\mathcal{L}$ at Σ is denoted by $\mathcal{L} = \mathcal{K} \backslash \Sigma$.

By the definition, every collapsing sequence Σ has a corresponding sequence $\mathbb{T} = \{\tau(\sigma_1), \tau(\sigma_2), \dots\}$ of facets being collapsed at every step.

Def. 3 **(Collapsible simplicial complex, [Whi39])** The simplicial complex $\mathcal{K}$ is collapsible if there exists a collapsing sequence Σ such that $\mathcal{K}$ collapses to a single vertex at Σ, $\mathcal{K} \backslash \Sigma = \{v\}$.

Determining whether the complex is collapsible is in general NP-complete, [**?**], but can be almost linear for a set of specific families of $\mathcal{K}$, [CFM$^+$14]. Naturally restricting the collapses to the case of d-collapses (such that $\dim \sigma_i \leq d - 1$), one arrive at the notion of d-collapsibility, [Tan09].

Def. 4 **(d-Core)** A d-Core is a subcomplex of $\mathcal{K}$ such that every simplex of dimension $d - 1$ belongs to at least 2 d-simplices. E.g. 2-Core is such a subcomplex of the original 2-skeleton $\mathcal{K}$ that every edge from $\mathcal{V}_1(\mathcal{K})$ belong to at least 2 triangles from $\mathcal{V}_2(\mathcal{K})$.

Lemma 1 **([LN21])** $\mathcal{K}$ is d-collapsible if and only if it does not contain a d-core.

Proof The proof of the lemma above naturally follows from the definition of the core: if the d-collapsing sequence is stuck then the simplex collapsed up to d-Core; conversely if a d-Core exists in the complex, the collapsing sequence necessarily includes its $(d-1)$-faces which are not collapsible. ∎

The d-Core is the generalization of the cycle for the case of 1-collapsibility of the classical graph; additionally, the d-Core is very dense due to its definition. In the case of 2-Core, we provide simple exemplary structures on Figure 2 which imply various possible configurations for a d-Core, $d \geq 2$, hence a search for d-Core inside $\mathcal{K}$ is neither trivial, no computationally cheap.

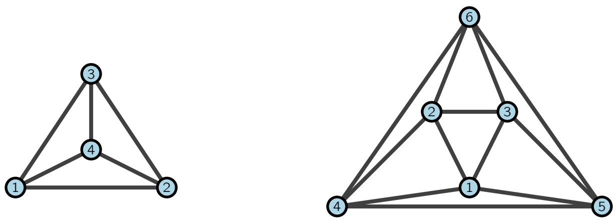

Figure 2: 2-Core, examples.

Additionally, we demonstrate that an arbitrary simplicial complex $\mathcal{K}$ tends to contain 2-Cores as long as $\mathcal{K}$ is denser than a trivially collapsible case. Assume the complex formed by triangulation of m_0 random points on the unit square with a sparsity pattern ν; the triangulation itself with the corresponding ν_Δ is collapsible, but a reasonably small addition of edges already creates a 2-Core (since it is local), Figure 3, left. Similarly, sampled sensor networks, where $\exists \sigma \in \mathcal{V}_1(\mathcal{K}) : \sigma = [v_1, v_2] \iff \|v_1 - v_2\|_2 < \varepsilon$ for a chosen percolation parameter $\varepsilon > 0$, quickly form a 2-Core upon the densifying of the network.

However, in the following, we observe that a weaker condition is enough to efficiently design a preconditioner for any "sparse enough" simplicial complex.

V.IV Weak collapsibility

Let the complex $\mathcal{K}$ be restricted up to its 2-skeleton, $\mathcal{K} = \mathcal{V}_0(\mathcal{K}) \cup \mathcal{V}_1(\mathcal{K}) \cup \mathcal{V}_2(\mathcal{K})$, and $\mathcal{K}$ is collapsible. Then the collapsing sequence Σ necessarily involves collapses at simplices

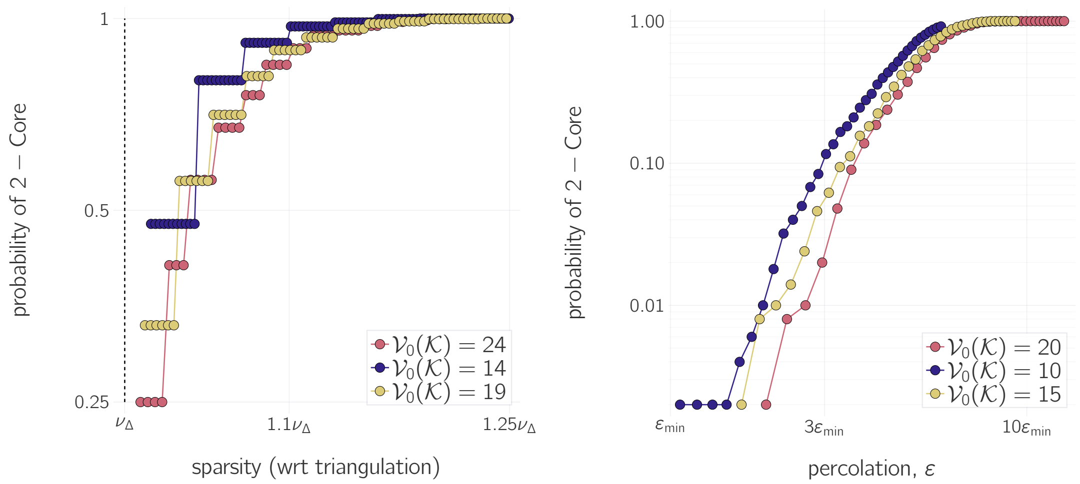

Figure 3: The probability of the 2-Core in richer-than-triangulation simplicial complexes: triangulation of random points modified to have $\left\lceil \nu \frac{|\mathcal{V}_0(\mathcal{K})|\cdot(|\mathcal{V}_0(\mathcal{K})|-1)}{2} \right\rceil$ edges on the left; random sensor networks with ε-percolation on the right. ν_Δ defines the initial sparsity of the triangulated network; $\varepsilon_{\min} = \mathbb{E} \min_{x,y\in[0,1]^2} \|x-y\|_2$ is the minimal possible percolation parameter.

σ_i of different orders: at edges (eliminating edges and triangles) and at vertices (eliminating vertices and edges). One can show that for a given collapsing sequence Σ there is a reordering $\tilde{\Sigma}$ such that $\dim \tilde{\sigma}_i$ are non-increasing, [CFM+14, Lemma 2.5]. Namely, if such a complex is collapsible, then there is a collapsible sequence $\Sigma = \{\Sigma_1, \Sigma_0\}$ where Σ_1 contains all the collapses at edges first and Σ_0 is composed of collapses at vertices. Note that the partial collapse $\mathcal{K}\backslash\Sigma_1 = \mathcal{L}$ eliminates all the triangles in the complex, $\mathcal{V}_2(\mathcal{L}) = \varnothing$; otherwise, the whole sequence Σ is not collapsing $\mathcal{K}$ to a single vertex. Since $\mathcal{V}_2(\mathcal{L}) = \varnothing$, the associated up-Laplacian $L_1^\uparrow(\mathcal{L}) = 0$.

Def. 5 **(Weakly collapsible complex)** Simplicial complex $\mathcal{K}$ restricted to its 2-skeleton is called weakly collapsible, if there exists a collapsing sequence Σ_1 such that the simplicial complex $\mathcal{L} = \mathcal{K}\backslash\Sigma_1$ has no simplices of order 2, $\mathcal{V}_2(\mathcal{L}) = \varnothing$ and $L_1^\uparrow(\mathcal{L}) = 0$.

Example Note that a collapsible complex is necessarily weakly collapsible; the opposite does not hold. Consider the following example in Figure 4: the initial complex is weakly collapsible either by a collapse at $[3,4]$ or at $[2,4]$. After this, the only available collapse is at the vertex $[4]$ leaving the uncollapsible 3-vertex structure.

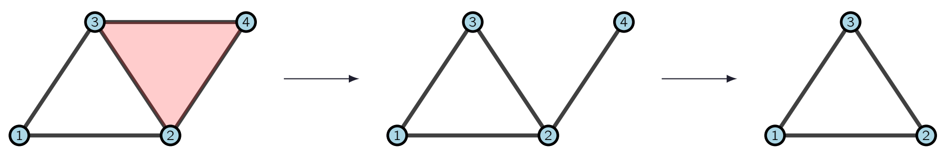

Figure 4: Example of weakly collapsible but not collapsible simplicial complex

Theorem 1 Weak collapsibility of 2-skeleton $\mathcal{K}$ is polynomially solvable.

Proof The greedy algorithm for the collapsing sequence intuitively operates as follows: at each iteration perform any of possible collapses; in the absence of free edges, the complex should be considered not collapsible, Algorithm 1. Clearly, such an algorithm runs polynomially with respect to the number of simplexes in $\mathcal{K}$.

The failure of the greedy algorithm indicates the existence of a weakly collapsible complex $\mathcal{K}$ such that the greedy algorithm gets stuck at a 2-Core, which is avoidable for another possible order of collapses. Among all the counter exemplary complexes, let $\mathcal{K}$ be a minimal one with respect to the number of triangles m_2. Then there exist a free edge $\sigma \in \mathcal{V}_1(\mathcal{K})$ such that $\mathcal{K}\backslash\{\sigma\}$ is collapsible and another $\sigma' \in \mathcal{V}_2(\mathcal{K})$ such that $\mathcal{K}\backslash\{\sigma'\}$ is not collapsible.

Note that if $\mathcal{K}$ is minimal then for any pair of free edges σ_1 and σ_2 belong to the same triangle: $\tau(\sigma_1) = \tau(\sigma_2)$. Indeed, for any $\tau(\sigma_1) \neq \tau(\sigma_2)$, $\mathcal{K}\backslash\{\sigma_1, \sigma_2\} = \mathcal{K}\backslash\{\sigma_2, \sigma_1\}$. Let $\tau(\sigma_1) \neq \tau(\sigma_2)$ for at least one pair of σ_1 and σ_2; in our assumption, either both $\mathcal{K}\backslash\{\sigma_1\}$ and $\mathcal{K}\backslash\{\sigma_2\}$, only $\mathcal{K}\backslash\{\sigma_1\}$ or none are collapsible. In the former case either $\mathcal{K}\backslash\{\sigma_1\}$ or $\mathcal{K}\backslash\{\sigma_2\}$ is a smaller example of the complex satisfying the assumption, hence, violating the minimality. If only $\mathcal{K}\backslash\{\sigma_1\}$ is collapsible, then $\mathcal{K}\backslash\{\sigma_2, \sigma_1\}$ is not collapsible; hence, $\mathcal{K}\backslash\{\sigma_1, \sigma_2\}$ is not collapsible, so $\mathcal{K}\backslash\{\sigma_1\}$ is a smaller example of a complex satisfying the assumption. Finally, if both $\mathcal{K}\backslash\{\sigma_1\}$ and $\mathcal{K}\backslash\{\sigma_2\}$ are collapsible, then for known σ' such that $\mathcal{K}\backslash\{\sigma'\}$ is not collapsible, $\tau(\sigma') \neq \tau(\sigma_1)$ or $\tau(\sigma') \neq \tau(\sigma_2)$, which revisits the previous point.

As a result, for σ ($\mathcal{K}\backslash\{\sigma\}$ is collapsible) and for σ' ($\mathcal{K}\backslash\{\sigma'\}$ is not collapsible) it holds that $\tau(\sigma) = \tau(\sigma') \Rightarrow \sigma \cap \sigma' = \{v\}$, so after collapses $\mathcal{K}\{\sigma\}$ and $\mathcal{K}\backslash\{\sigma'\}$ we arrive at two identical simplicial complexes modulo the hanging vertex irrelevant for the weak collapsibility. A simplicial complex can not be simultaneously collapsible and not collapsible, so the question of weak collapsibility can always be resolved by the greedy algorithm which has polynomial complexity. ■

V.V Computational cost of the greedy algorithm

Let $\mathcal{K}$ be a 2-skeleton; let Δ_σ be a set of triangles of $\mathcal{K}$ containing the edge σ, $\Delta_\sigma = \{t \mid t \in \mathcal{V}_2(\mathcal{K}) \text{ and } \sigma \in t\}$. Then the edge σ is free iff $|\Delta_\sigma| = 1$ and $F = \{\sigma \mid |\Delta_e| = 1\}$ is a set of all free edges. Note that $|\Delta_e| \leq m_0 - 2 = \mathcal{O}(m_0)$.

Algorithm 1 `GREEDY_COLLAPSE`($\mathcal{K}$): greedy algorithm for the weak collapsibility

Require: initial set of free edges F, adjacency sets $\{\Delta_{\sigma_i}\}_{i=1}^{m_1}$

1: $\Sigma = [\,],\ \mathbb{T} = [\,]$ ▷ initialize the collapsing sequence
2: **while** $F \neq \varnothing$ **and** $\mathcal{V}_2(\mathcal{K}) \neq \varnothing$ **do**
3: $\quad \sigma \leftarrow \mathrm{pop}(F),\ \ \tau \leftarrow \tau(\sigma)$ ▷ pick a free edge σ
4: $\quad \mathcal{K} \leftarrow \mathcal{K}\backslash\{\sigma\},\ \ \Sigma \leftarrow [\Sigma\ \ \sigma],\ \ \mathbb{T} \leftarrow [\mathbb{T}\ \tau]$ ▷ τ is a triangle being collapsed; $\tau = [\sigma, \sigma_1, \sigma_2]$
5: $\quad \Delta_{\sigma_1} \leftarrow \Delta_{\sigma_1}\backslash\tau,\ \ \Delta_{\sigma_2} \leftarrow \Delta_{\sigma_2}\backslash\tau$ ▷ remove τ from adjacency lists
6: $\quad F \leftarrow F \cup \{\sigma_i \,|\, i = 1, 2 \text{ and } |\Delta_{\sigma_i}| = 1\}$ ▷ update F if any of σ_1 or σ_2 has become free
7: **end while**
8: **return** $\mathcal{K}$, Σ, $\mathbb{T}$

The complexity of Algorithm 1 rests upon the precomputed $\sigma \mapsto \Delta_\sigma$ structure that de-facto coincides with the boundary operator B_2 (assuming B_2 is stored as a sparse matrix, the adjacency structure describes its non-zero entries). Similarly, the initial F set can be computed alongside the construction of B_2 matrix. Another concession is needed for the complexity of the removal of elements from Δ_{σ_i} and F, which may vary from $\mathcal{O}(1)$ on average up to guaranteed $\log(|\Delta_{\sigma_i}|)$. As a result, given a pre-existing B_2 operator, Algorithm 1 runs linearly, $\mathcal{O}(m_1)$, or almost linearly depending on the realisation, $\mathcal{O}(m_1 \log m_1)$.

V. References

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V. Glossary

face subsimplex of a given simplex . 1

simplex elements of the simplicial complex; subset of vertex set. 1

simplicial complex higher-order network model; hypergraph closed for edge inclusion. 1