Doctoral Thesis

# Topological Stability and Preconditioning of Higher-Order Laplacian Operators on Simplicial Complexes

PhD Program in Mathematics: XXXV cycle

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# *Abstract*

✎ Insert here the abstract of the thesis proposal.

# Contents

# List of Figures

# Chapter 1

# Introduction

# Chapter 2

# Simplicial complex as Higher-order Topology Description

## 2.1 From graph to higher-order models

## 2.2 Simplicial Complexes

Let $V = \{v_1, v_2, \ldots, v_n\}$ be a set of nodes; as discussed above, such set may refer to various interacting entities and agents in the system, e.g. neurons, genes, traffic stops, online actors, publication authors, etc. Then:

**Definition 2.1** (Simplicial Complex)**.** The collection of subsets $\mathcal{K}$ of the nodal set $V$ is a (abstract) SC[1] if for each subset $\sigma \in \mathcal{K}$, referred as a simplex, all its subsets $\sigma'$, $\sigma' \subseteq \sigma$, referred as faces, enter $\mathcal{K}$ as well, $\sigma' \in \mathcal{K}$.

A simplex $\sigma \in \mathcal{K}$ on $k+1$ vertices is said to be of the order $k$, $\operatorname{ord}(\sigma) = k$. Let $\mathcal{V}_k(\mathcal{K})$ be a set of all $k$-order simplices in $\mathcal{K}$ and $m_k$ is the cardinality of $\mathcal{V}_k(\mathcal{K})$, $m_k = |\mathcal{V}_k(\mathcal{K})|$; then $\mathcal{V}_0(\mathcal{K})$ is the set of nodes in the simplicial complex $\mathcal{K}$, $\mathcal{V}_1(\mathcal{K})$ — the set of edges, $\mathcal{V}_2(\mathcal{K})$ — the set of triangles, or 3-cliques, and so on, with $\mathcal{K} = \{\mathcal{V}_0(\mathcal{K}), \mathcal{V}_1(\mathcal{K}), \mathcal{V}_2(\mathcal{K}) \ldots\}$. Note that due to the inclusion rule in Theorem 2.1, the number of non-empty $\mathcal{V}_k(\mathcal{K})$ is finite and, moreover, uninterupted in a sense of the order: if $\mathcal{V}_k(\mathcal{K}) = \varnothing$, then $\mathcal{V}_{k+1}(\mathcal{K})$ is also necessarily empty.

**Example 2.1** (Simplicial Complex)**.** *123*

[1]addition of the word "abstract" to the term is more common in the topological setting

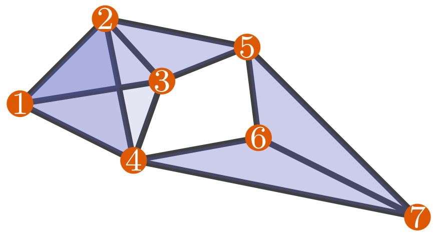


FIGURE 2.1: Example of a simplicial complex

**Example 2.2** (Real Life Simplicial Complex). *123*

Comparing to the general case of the hypergraph described above, it is easy to see that simplicial complex is a special case of a hypergraph where every edge is enclosed with respect to the inclusion. In other words, simplicial complex contains additional structural rigidness which allows to formally describe the topology of $\mathcal{K}$; as a result, one is specifically interested in the formal desctiption of the nested inclusion principle achieved through *boundary operators* defined in the subsections below.

## 2.3 Hodge's Theory

Two linear operators $A$ and $B$ are said to satisfy Hodge's theory if and only if their composition is a null operator,

$$AB = 0 \tag{2.1}$$

which is equivalent to $\operatorname{im} B \subseteq \ker A$.

**Definition 2.2.** For a pair of operators $A$ and $B$ satisfying Hodge's theory, the *quotient space* $\mathcal{H}$ is defined as follows:

$$\mathcal{H} = {}^{\ker A}\!\big/_{\operatorname{im} B} \tag{2.2}$$

where each element of $\mathcal{H}$ is a manifold $\mathbf{x} + \operatorname{im} B = \{\mathbf{x} + \mathbf{y} \mid \forall \mathbf{y} \in \operatorname{im} B\}$ for $\mathbf{x} \in \ker A$. It follows directly from the definition that $\mathcal{H}$ is an abelian group under addition.

By Theorem 2.2, the quotient space $\mathcal{H}$ is a collection✎ in a general sense of *equivalence classes* $\mathbf{x} + \operatorname{im} B$. Then, each class $\mathbf{x} + \operatorname{im} B = \mathbf{x}_H + \operatorname{im} B$ for some $\mathbf{x}_H \perp \operatorname{im} B$ (both $\mathbf{x}, \mathbf{x}_H \in \ker A$); indeed, since the orthogonal component $\mathbf{x}_H$ (referred as *harmonic representative*) of $\mathbf{x}$ with respect to $\operatorname{im} B$ is unique, the map $\mathbf{x}_H \leftrightarrow \mathbf{x} + \operatorname{im} B$ is bijectional.

**Theorem 2.3** ([1, Thm 5.3]). *Let $A$ and $B$ be linear operators, $AB = 0$. Then the homology group $\mathcal{H}$ satisfies:*

$$\mathcal{H} = {}^{\ker A}\!\big/_{\operatorname{im} B} \cong \ker A \cap \ker B^\top, \tag{2.3}$$

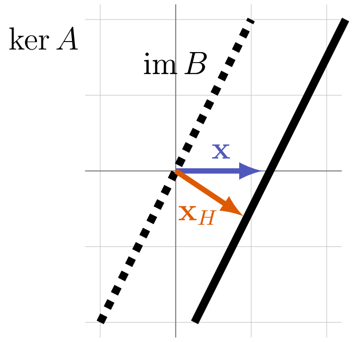


FIGURE 2.2: Illustration of a harmonic representative for an equivalence class

*where* $\cong$ *denotes the isomorphism.*

*Proof.* One builds the isomorphism through the harmonic representative, as discussed above. It sufficient to note that $\mathbf{x}_H \perp \operatorname{im} B \Leftrightarrow \mathbf{x}_H \in \ker B^\top$ in order to complete the proof. $\square$

**Lemma 2.4** ([1, Thm 5.2])**.** *Let $A$ and $B$ be linear operators, $AB = 0$. Then:*

$$\ker A \cap \ker B^\top = \ker\left(A^\top A + BB^\top\right) \tag{2.4}$$

*Proof.* Note that if $\mathbf{x} \in \ker A \cap \ker B^\top$, then $\mathbf{x} \in \ker A$ and $\mathbf{x} \ker B^\top$, so $\mathbf{x} \in \ker\left(A^\top A + BB^\top\right)$. As a result, $\ker A \cap \ker B^\top \subset \ker\left(A^\top A + BB^\top\right)$.

On the other hand, let $\mathbf{x} \in \ker\left(A^\top A + BB^\top\right)$, then

$$A^\top A\mathbf{x} + BB^\top \mathbf{x} = 0 \tag{2.5}$$

Exploiting $AB = 0$ and multiplying the equation above by $B^\top$ and $A$ one gets the following:

$$\begin{aligned} B^\top BB^\top \mathbf{x} &= 0 \\ AA^\top A\mathbf{x} &= 0 \end{aligned} \tag{2.6}$$

Note that $AA^\top A\mathbf{x} = 0 \Leftrightarrow A^\top A\mathbf{x} \in \ker A$, but $A^\top A\mathbf{x} \in \operatorname{im} A^\top$, so by Fredholm alternative, $A^\top A\mathbf{x} = 0$. Finally, for $A^\top A\mathbf{x} = 0$:

$$A^\top A\mathbf{x} = 0 \Longrightarrow \mathbf{x}^\top A^\top A\mathbf{x} = 0 \iff \|A\mathbf{x}\|^2 = 0 \Longrightarrow \mathbf{x} \in \ker A \tag{2.7}$$

Similarly, for the second equation, $\mathbf{x} \in \ker B^\top$ which completes the proof. $\square$

✎ Here we need some words about the transitions.

Since $AB = 0$, $B^\top A^\top = 0$ or $\operatorname{im} A^\top \subset \ker B^\top$. Then, exploiting $\mathbb{R}^n = \ker A \oplus \operatorname{im} A^\top$:

$$\begin{aligned} \ker B^\top = \ker B^\top \cap \mathbb{R}^n = \ker B^\top \cap \left(\ker A \oplus \operatorname{im} A^\top\right) = \\ = \left(\ker A \cap \ker B^\top\right) \oplus \left(\operatorname{im} A^\top \cap \ker B^\top\right) \end{aligned} \tag{2.8}$$

Given Theorem 2.4, $\ker A \cap \ker B^\top = \ker\left(A^\top A + BB^\top\right)$ and, since $\operatorname{im} A^\top \subset \ker B^\top$, $\operatorname{im} A^\top \cap \ker B^\top = \operatorname{im} A^\top$, yeilding the decomposition of the whole space:

**Theorem 2.5** (Hodge Decomposition)**.** *Let $A$ and $B$ be linear operators, $AB = 0$. Then:*

$$\mathbb{R}^n = \overbrace{\operatorname{im} A^\top \oplus \underbrace{\ker\left(A^\top A + BB^\top\right)}_{\ker A}}^{\ker B^\top} \oplus \operatorname{im} B \tag{2.9}$$

## 2.4 Boundary and Laplacian Operators

### 2.4.1 Boundary operators

Each simplicial complex $\mathcal{K}$ has a nested structure of simplicies: indeed, if $\sigma$ is a simplex of order $k$, $\sigma \in \mathcal{V}_k(\mathcal{K})$, then all of $(k-1)$-th order faces forming the boundary of $\sigma$ also belong to $\mathcal{K}$: for instance, for the triangle $\{1, 2, 3\}$ all the border edges $\{1, 2\}$, $\{1, 3\}$ and $\{2, 3\}$ are also in the simplicial complex, Figure 2.1.

This nested property implies that one can build a formal map from a simplex to its boundary enclosed inside the simplicial complex.

**Definition 2.6** (Chain spaces)**.** Let $\mathcal{K}$ be a simplicial complex; then the space $C_k$ of formal sums of simplices from $\mathcal{V}_k(\mathcal{K})$ over real numbers is called *a $k$-th chain space.*

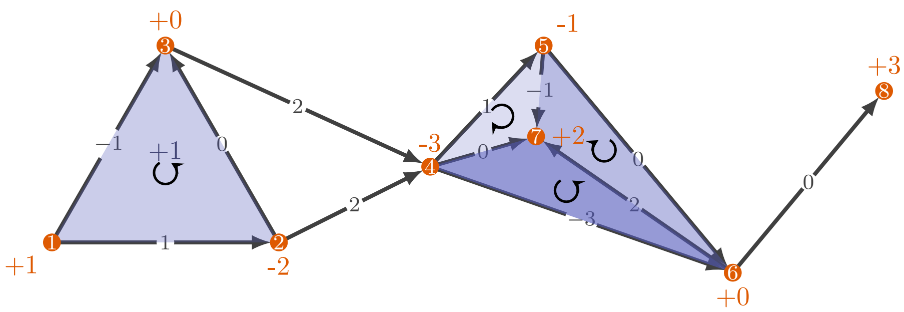


FIGURE 2.3: Example of chains on the simplicial complex

**Example 2.3.**

Chain spaces on its own are naturally present in the majority of the network models: $C_0$ is a space of states of vertices (e.g. in the dynamical system $\dot{\mathbf{x}} = A\mathbf{x}$ the evoloving vector $\mathbf{x} \in C_0$), $C_1$ — is a space of (unrestricted) flows on graphs edges, and so on✎ refs?.

Since $C_k$ is a linear space, versor vectors corresponding to the elements of $\mathcal{V}_k(\mathcal{K})$ is a natural basis of $C_k$; in order to proceed with the boundary maps, one needs to fix an order of the simplices inside $\mathcal{V}_k(\mathcal{K})$ (alas one fixes the order of vertices in the graph to form an adjacency matrix) as a matter of bookkeeping. Addiitionally, since $C_k$ is a space over $\mathbb{R}$, chains $\sigma$ and $-\sigma$ naturally introduce the notion of *orientation* of the simplex.

## 2.5 Weigthed and Normalised Boundary Operators

Note that, from the definition $\overline{B}_k = W_{k-1}^{-1} B_k W_k$ and (**??**), we immediately have that $\overline{B}_k \overline{B}_{k+1} = 0$. Thus, the group $\overline{\mathcal{H}}_k = \ker \overline{B}_k / \operatorname{im} \overline{B}_{k+1}$ is well defined for any choice of positive weights $w_k$ and is isomorphic to $\ker \overline{L}_k$. While the homology group may depend on the weights, we observe below that its dimension does not. Precisely, we have

**Proposition 2.7.** *The dimension of the homology groups of $\mathcal{K}$ is not affected by the weights of its $k$-simplicies. Precisely, if $W_k$ are positive diagonal matrices, we have*

$$\dim \ker \overline{B}_k = \dim \ker B_k, \quad \dim \ker \overline{B}_k^\top = \dim \ker B_k^\top, \quad \dim \overline{\mathcal{H}}_k = \dim \mathcal{H}_k \,. \tag{2.10}$$

*Moreover,* $\ker B_k = W_k \ker \overline{B}_k$ *and* $\ker B_k^\top = W_{k-1}^{-1} \ker \overline{B}_k^\top$.

*Proof.* Since $W_k$ is an invertible diagonal matrix,

$$\overline{B}_k \mathbf{x} = 0 \iff W_{k-1}^{-1} B_k W_k \vec{x} = 0 \iff B_k W_k \vec{x} = 0.$$

Hence, if $\vec{x} \in \ker \overline{B}_k$, then $W_k \vec{x} \in \ker B_k$, and, since $W_k$ is bijective, $\dim \ker \overline{B}_k = \dim \ker B_k$. Similarly, one observes that $\dim \ker \overline{B}_k^\top = \dim \ker B_k^\top$.

Moreover, since $\overline{B}_k \overline{B}_{k+1} = 0$, then $\operatorname{im} \overline{B}_{k+1} \subseteq \ker \overline{B}_k$ and $\operatorname{im} \overline{B}_k^\top \subseteq \ker \overline{B}_{k+1}^\top$. This yields $\ker \overline{B}_k \cup \ker \overline{B}_{k+1}^\top = \mathbb{R}^{\mathcal{V}_k} = \ker B_k \cup \ker B_{k+1}^\top$. Thus, for the homology group it holds:

$$\begin{aligned}
\dim \overline{\mathcal{H}}_k &= \dim \left( \ker \overline{B}_k \cap \ker \overline{B}_{k+1}^\top \right) = \\
&= \dim \ker \overline{B}_k + \dim \ker \overline{B}_{k+1}^\top - \dim \left( \ker \overline{B}_k \cup \ker \overline{B}_{k+1}^\top \right) = \\
&= \dim \ker B_k + \dim \ker B_{k+1}^\top - \dim \left( \ker B_k \cup \ker B_{k+1}^\top \right) = \dim \mathcal{H}_k
\end{aligned}$$

□

# Chapter 3

# Matrix nearness problems

## 3.1 101 on MNP

Generally speaking, for a given matrix $A$ a *matrix nearness problem* consists of finding the closest possible matrix $X$ among the admissible set with a number of desired properties. For instance, one may search for the closest (in some metric) symmetric positive/negative definite matrix, unitary matrix or the closest graph Laplacian.

Motivated by the topological meaning of the *kernel* of Hodge Laplacians $L_k$, we assume the specific case of *spectral* MNPs: here one aims for the target matrix $X$ to have a specific spectrum $\sigma(X)$. For instance in the stability study of the dynamical system $\dot{\mathbf{x}} = A\mathbf{x}$ one can search for the closest Hurwitz matrix such that $\mathrm{Re}\,[\lambda_i] < 0$ for all $\lambda_i \in \sigma(X)$; similarly, assuming given matrix $A$ is a graph Laplacian, one can search for the closest disconnected graph (so the algebraic connectivity $\lambda_2 = 0$).

Here we recite the optimization framework developed by REFREFREF✎ fix it for the class of the spectral matrix nearness problems; one should note, however, that this is by far not the only approach to the task, REFREFREF✎ also fix it with Nicholas and others, I guess?.

**Functional and Gradient Flow** Let as assume that $X = A + \Delta$ and intead of searching for $X$, we search for the perturbation matrix $\Delta$; additionally, we assume that $\Omega$ is the admissible set containing all possible perturbations $\Delta$.

## 3.2 Direct approach: failure and discontinuity problems

### 3.2.1 Principal spectral inheritance

Before moving on to the next section, we recall here a relatively direct but important spectral property that connects the spectra of the $k$-th and $(k+1)$-th order Laplacians.

**Theorem 3.1** (HOL's spectral inheritance). *Let $L_k$ and $L_{k+1}$ be higher-order Laplacians for the same simplicial complex $\mathcal{K}$. Let $\overline{L}_k = \overline{L}_k^{down} + \overline{L}_k^{up}$, where $\overline{L}_k^{down} = \overline{B}_k^\top \overline{B}_k$ and $\overline{L}_k^{up} = \overline{B}_{k+1}\overline{B}_{k+1}^\top$. Then:*

1. *$\sigma_+(\overline{L}_k^{up}) = \sigma_+(\overline{L}_{k+1}^{down})$, where $\sigma_+(\cdot)$ denotes the positive part of the spectrum;*
2. *if $0 \neq \mu \in \sigma_+(\overline{L}_k^{up}) = \sigma_+(\overline{L}_{k+1}^{down})$, then the eigenvectors are related as follows:*
   *(a) if $\mathbf{x}$ is and eigenvector for $\overline{L}_k^{up}$ with the eigenvalue $\mu$, then $\mathbf{y} = \frac{1}{\sqrt{\mu}}\overline{B}_{k+1}^\top \mathbf{x}$ is an eigenvector for $\overline{L}_{k+1}^{down}$ with the same eigenvalue;*
   *(b) if $\mathbf{u}$ is and eigenvector for $\overline{L}_{k+1}^{down}$ with the eigenvalue $\mu$ and $\mathbf{u} \notin \ker \overline{B}_{k+1}$, then $\mathbf{v} = \frac{1}{\sqrt{\mu}}\overline{B}_{k+1}\mathbf{u}$ is an eigenvector for $\overline{L}_k^{up}$ with the same eigenvalue;*
3. *for each Laplacian $\overline{L}_k$: if $\mathbf{v} \notin \ker \overline{L}_k^{down}$ is the eigenvector for $\overline{L}_k^{down}$, then $\mathbf{v} \in \ker \overline{L}_k^{up}$; vice versa, if $\mathbf{u} \notin \ker \overline{L}_k^{up}$ is the eigenvector for $\overline{L}_k^{up}$, then $\mathbf{v} \in \ker \overline{L}_k^{down}$;*
4. *consequently, there exist $\mu \in \sigma_+(\overline{L}_k)$ with an eigenvector $\mathbf{u} \in \ker \overline{L}_k^{up}$, and $\nu \in \sigma_+(\overline{L}_{k+1})$ with an eigenvector $\mathbf{u} \in \ker \overline{L}_{k+1}^{down}$, such that:*

$$\overline{B}_k^\top \overline{B}_k \mathbf{v} = \mu \mathbf{v}, \qquad \overline{B}_{k+2}\overline{B}_{k+2}^\top \mathbf{u} = \nu \mathbf{u}\,.$$

*Proof.* For $(2a)$ it is sufficient to note that $\overline{L}_{k+1}^{down}\mathbf{y} = \overline{B}_{k+1}^\top \overline{B}_{k+1}\frac{1}{\sqrt{\mu}}\overline{B}_{k+1}^\top \mathbf{x} = \frac{1}{\sqrt{\mu}}\overline{B}_{k+1}^\top \overline{L}_k^{up}\mathbf{x} = \sqrt{\mu}\overline{B}_{k+1}^\top \mathbf{x} = \mu \mathbf{y}$. Similarly, for $(2b)$: $\overline{L}_k^{up}\mathbf{v} = \overline{B}_{k+1}\overline{B}_{k+1}^\top \frac{1}{\sqrt{\mu}}\overline{B}_{k+1}\mathbf{u} = \frac{1}{\sqrt{\mu}}\overline{B}_{k+1}\overline{L}_{k+1}^{down}\mathbf{u} = \mu\mathbf{v}$; joint $2(a)$ and $2(b)$ yield (1). *Hodge decomposition* immediately yields the strict separation of eigenvectors between $\overline{L}_k^{up}$ and $\overline{L}_k^{down}$, (3); given (3), all the inherited eigenvectors from $(2a)$ fall into the $\ker \overline{L}_{k+1}^{down}$, thus resulting into (4). □

In other words, the variation of the spectrum of the $k$-th Laplacian when moving from one order to the next one works as follows: the down-term $\overline{L}_{k+1}^{down}$ inherits the positive part of the spectrum from the up-term of $\overline{L}_k^{up}$; the eigenvectors corresponding to the inherited positive part of the spectrum lie in the kernel of $\overline{L}_{k+1}^{up}$; at the same time, the "new" up-term $\overline{L}_{k+1}^{up}$ has a new, non-inherited, part of the positive spectrum (which, in turn, lies in the kernel of the $(k+2)$-th down-term).

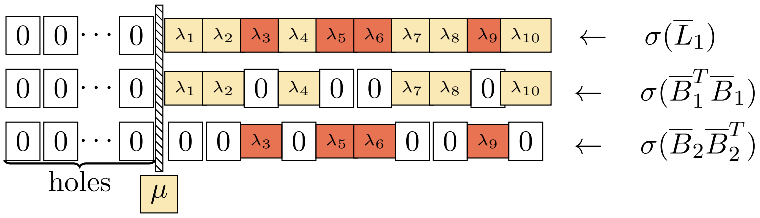


FIGURE 3.1: Illustration for the principal spectrum inheritance (Theorem 3.1) in case $k = 0$: spectra of $\overline{L}_1$, $\overline{L_1^{\downarrow}}$ and $\overline{L_1^{\downarrow}}$ are shown. Colors signify the splitting of the spectrum, $\lambda_i > 0 \in \sigma(\overline{L}_1)$ ; all yellow eigenvalues are inherited from $\sigma_+(\overline{L}_0)$; red eigenvalues belong to the non-inherited part. Dashed barrier $\mu$ signifies the penalization threshold (see the target functional in **??**) preventing homological pollution (see **??**).

In particular, we notice that for $k = 0$, since $B_0 = 0$ and $\overline{L}_0 = \overline{L}_0^{up}$, the theorem yields $\sigma_+(\overline{L}_0) = \sigma_+(\overline{L_1}^{down}) \subseteq \sigma_+(\overline{L}_1)$. In other terms, the positive spectrum of the $\overline{L}_0$ is inherited by the spectrum of $\overline{L}_1$ and the remaining (non-inherited) part of $\sigma_+(\overline{L}_1)$ coincides with $\sigma_+(\overline{L}_1^{up})$. Figure 3.1 provides an illustration of the statement of Theorem 3.1 for $k = 0$.

# Chapter 4

# Topological Stability of Simplicial Complexes

# Chapter 5

# Preconditioning

✎ Here we need to say general words about how we need an efficient preconditioning scheme.

## 5.1 Iterative methods for Positive Definite Systems

## 5.2 Preconditioning 101

## 5.3 Cholesky preconditioning for classical graphs

## 5.4 Classical collapsibility

In this section we borrow the terminology from [2]; additionally, let us assume that considered simplicial complex $\mathcal{K}$ is restricted to its 2-skeleton, so $\mathcal{K}$ consists only of nodes, edges, and triangles, $\mathcal{K} = \mathcal{V}_0(\mathcal{K}) \cup \mathcal{V}_1(\mathcal{K}) \cup \mathcal{V}_2(\mathcal{K})$.

Simplex $\tau \in \mathcal{K}$ is called an (inlusion-wise) maximal face of simplex $\sigma \in \mathcal{K}$ if $\tau$ is maximal by inlusion simplex such that $\sigma \subseteq \tau$ and $\operatorname{ord}(\sigma) < \operatorname{ord}(\tau)$. For instance, in Figure 5.1 the edge $\{1,2\}$ and nodes $\{1\}$ and $\{2\}$ have two maximal faces, $\{1,2,3\}$ and $\{1,2,4\}$, while all the other edges and nodes have unique maximal faces — their corresponding triangles. Note that in the case of the node $\{1\}$, there are bigger simplices containing it besides the triangles (e.g. the edge $\{1,2\}$), but they are not maximal by inclusion.

**Definition 5.1** (Free simplex)**.** The simplex $\sigma \in \mathcal{K}$ is free if it has exactly one maximal face $\tau$, $\tau = \tau(\sigma)$. F.i. edges $\{1,3\}$, $\{1,4\}$, $\{2,3\}$ and $\{2,4\}$ are all free in Figure 5.1.

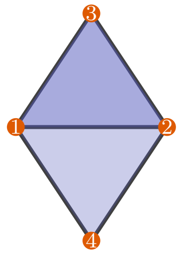


FIGURE 5.1: Example of a simplicial complex: free simplices and maximal faces.

The collapse $\mathcal{K}\backslash\{\sigma\}$ of $\mathcal{K}$ at a free simplex $\sigma$ is the transition from the original simplicial complex $\mathcal{K}$ to a smaller simplicial complex $\mathcal{L}$ without the free simplex $\sigma$ and the corresponding maximal face $\tau$, $\mathcal{K} \to \mathcal{K}' = \mathcal{K} - \sigma - \tau$; namely, one can eliminate a simplex $\tau$ if it has an accessible (not included in another simplex) face $\sigma$.

Naturally, one can perform several consequent collapses at $\Sigma = \{\sigma_1, \sigma_2, \ldots\}$ assuming $\sigma_i$ is free in collapse simplicial complex from the previous stage; $\Sigma$ is called the collapsing sequence. Formally:

**Definition 5.2** (Collapsing sequence)**.** Let $\mathcal{K}$ be a simplicial complex. $\Sigma = \{\sigma_1, \sigma_2, \ldots\}$ is a collapsing sequence if $\sigma_1$ is free in $\mathcal{K}$ and each $\sigma_i$, $i > 1$, is free at $\mathcal{K}^{(i)} = \mathcal{K}^{(i-1)}\backslash\{\sigma_i\}$, $\mathcal{K}^{(1)} = \mathcal{K}$. The collapse of $\mathcal{K}$ to a new complex $\mathcal{L}$ at $\Sigma$ is denoted by $\mathcal{L} = \mathcal{K}\backslash\Sigma$.

By the definition, every collapsing sequence $\Sigma$ has a corresponding sequence $\mathbb{T} = \{\tau(\sigma_1), \tau(\sigma_2), \ldots\}$ of maximal faces being collapsed at every step.

**Definition 5.3** (Collapsible simplicial complex, [2])**.** The simplicial complex $\mathcal{K}$ is collapsible if there exists a collapsing sequence $\Sigma$ such that $\mathcal{K}$ collapses to a single vertex at $\Sigma$, $\mathcal{K}\backslash\Sigma = \{v\}$.

Determining whether the complex is collapsible is in general *NP-complete*, [3], but can be almost linear for a set of specific families of $\mathcal{K}$, e.g. if the simplex can be embeded into the triangulation of the $d$-dimensional unit sphere, [4]. Naturally restricting the collapses to the case of $d$-collapses (such that $\operatorname{ord}(\sigma)_i \leq d-1$), one arrive at the notion of $d$-collapsibility, [5].

**Definition 5.4** ($d$-Core)**.** A $d$-Core is a subcomplex of $\mathcal{K}$ such that every simplex of order $d-1$ belongs to at least 2 simplices of order $d$. E.g. 2-Core is such a subcomplex of the original 2-skeleton $\mathcal{K}$ that every edge from $\mathcal{V}_1(\mathcal{K})$ belong to at least 2 triangles from $\mathcal{V}_2(\mathcal{K})$.

**Lemma 5.5** ([6])**.** *$\mathcal{K}$ is $d$-collapsible if and only if it does not contain a $d$-core.*

*Proof.* The proof of the lemma above naturally follows from the definition of the core. Assume $\Sigma$ is a $d$-collapsing sequence, and $\mathcal{K}\backslash\Sigma$ consists of more than a single vertex and has no free simplices of order $\leq d-1$ ("collapsing sequence gets stuck"). Then, each simplex of order $d-1$ is no free but belongs to at least 2 simplices of order $d$, so $\mathcal{K}\backslash\Sigma$ is a $d$-Core.

Conversely if a $d$-Core exists in the complex, the collapsing sequence should necessarily include its simplices of order $d-1$ which can not become free during as a result of a sequence of collapses. Indeed, for $\sigma$ from $d$-Core, $\operatorname{ord}(\sigma) = d-1$, to become free, one needs to collapse at least one of $\sigma$'s maximal faces for $d$-Core , all of whose faces are, in turn, contained in the $d$-Core (since $d$-Core is a simplicial complex). As a result one necessarily needs a prior collapse inside the $d$-Core to perform the first collapse in the $d$-Core, which is impossible. $\square$

In the case of the classical graph model, the 1-Core is a subgraph where each vertex has a degree at least 2; in other words, 1-Core cannot be a tree and necessarily contains a simple cycle. Hence, the collapsibility of a classical graph coincides with the acyclicity. The $d$-Core is the generalization of the cycle for the case of 1-collapsibility of the classical graph; additionally, the $d$-Core is very dense due to its definition. In the case of 2-Core, we provide simple exemplary structures on Figure 5.2 which imply various possible configurations for a $d$-Core, $d \geq 2$, hence a search for $d$-Core inside $\mathcal{K}$ is neither trivial, no computationally cheap.

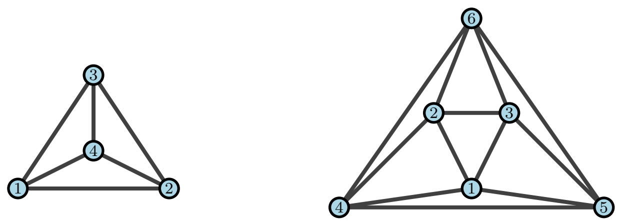


FIGURE 5.2: 2-Core, examples.

Additionally, we demonstrate that an arbitrary simplicial complex $\mathcal{K}$ tends to contain 2-Cores as long as $\mathcal{K}$ is denser than a trivially collapsible case. Assume the complex formed by triangulation of $m_0$ random points on the unit square with a sparsity pattern $\nu$; the triangulation itself with the corresponding $\nu_\Delta$ is collapsible, but a reasonably small addition of edges already creates a 2-Core (since it is local), Figure 5.3, left. Similarly, sampled sensor networks, where $\exists \sigma \in \mathcal{V}_1(\mathcal{K}) : \ \sigma = [v_1, v_2] \iff \|v_1 - v_2\|_2 < \varepsilon$ for a chosen percolation parameter $\varepsilon > 0$, quickly form a 2-Core upon the densifying of the network.

However, in the following, we observe that a weaker condition is enough to efficiently design a preconditioner for any "sparse enough" simplicial complex.

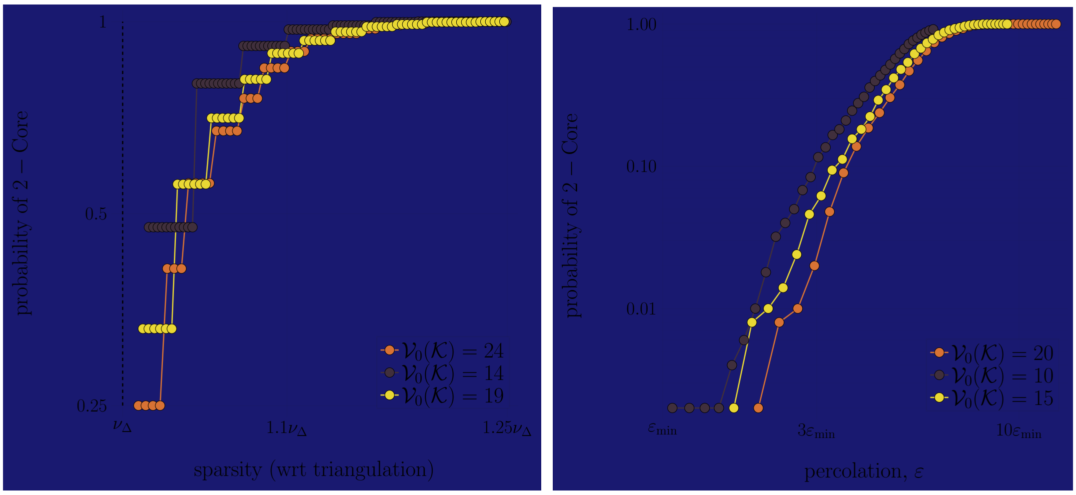


FIGURE 5.3: The probability of the 2-Core in richer-than-triangulation simplicial complexes: triangulation of random points modified to have $\left\lceil \nu \frac{|\mathcal{V}_0(\mathcal{K})|\cdot(|\mathcal{V}_0(\mathcal{K})|-1)}{2} \right\rceil$ edges on the left; random sensor networks with $\varepsilon$-percolation on the right. $\nu_\Delta$ defines the initial sparsity of the triangulated network; $\varepsilon_{\min} = \mathbb{E} \min_{x,y \in [0,1]^2} \|x-y\|_2$ is the minimal possible percolation parameter.

### 5.4.1 Weak collapsibility

Let the complex $\mathcal{K}$ be restricted up to its 2-skeleton, $\mathcal{K} = \mathcal{V}_0(\mathcal{K}) \cup \mathcal{V}_1(\mathcal{K}) \cup \mathcal{V}_2(\mathcal{K})$, and $\mathcal{K}$ is collapsible. Then the collapsing sequence $\Sigma$ necessarily involves collapses at simplices $\sigma_i$ of different orders: at edges (eliminating *edges* and *triangles*) and at vertices (eliminating *vertices* and *edges*). One can show that for a given collapsing sequence $\Sigma$ there is a reordering $\tilde{\Sigma}$ such that $\dim \tilde{\sigma_i}$ are non-increasing, [4, Lemma 2.5]. Namely, if such a complex is collapsible, then there is a collapsible sequence $\Sigma = \{\Sigma_1, \Sigma_0\}$ where $\Sigma_1$ contains all the collapses at edges first and $\Sigma_0$ is composed of collapses at vertices. Note that the partial collapse $\mathcal{K} \backslash \Sigma_1 = \mathcal{L}$ eliminates all the triangles in the complex, $\mathcal{V}_2(\mathcal{L}) = \varnothing$; otherwise, the whole sequence $\Sigma$ is not collapsing $\mathcal{K}$ to a single vertex. Since $\mathcal{V}_2(\mathcal{L}) = \varnothing$, the associated up-Laplacian $L_1^{\uparrow}(\mathcal{L}) = 0$.

**Definition 5.6** (Weakly collapsible complex)**.** Simplicial complex $\mathcal{K}$ restricted to its 2-skeleton is called *weakly collapsible*, if there exists a collapsing sequence $\Sigma_1$ such that the simplicial complex $\mathcal{L} = \mathcal{K} \backslash \Sigma_1$ has no simplices of order 2, $\mathcal{V}_2(\mathcal{L}) = \varnothing$ and $L_1^{\uparrow}(\mathcal{L}) = 0$.

**Example 5.1.** *Note that a collapsible complex is necessarily weakly collapsible; the opposite does not hold. Consider the following example in Figure 5.4: the initial complex is weakly collapsible either by a collapse at* $[3,4]$ *or at* $[2,4]$*. After this, the only available collapse is at the vertex* $[4]$ *leaving the uncollapsible* 3*-vertex structure.*

**Theorem 5.7.** *Weak collapsibility of* 2*-skeleton* $\mathcal{K}$ *is polynomially solvable.*

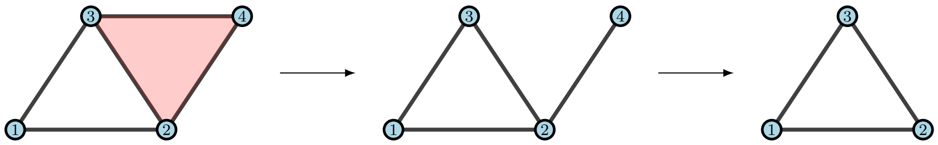


FIGURE 5.4: Example of weakly collapsible but not collapsible simplicial complex

*Proof.* The *greedy algorithm* for the collapsing sequence intuitively operates as follows: at each iteration perform any of possible collapses; in the absence of free edges, the complex should be considered not collapsible, Algorithm 1. Clearly, such an algorithm runs polynomially with respect to the number of simplexes in $\mathcal{K}$.

The failure of the greedy algorithm indicates the existence of a weakly collapsible complex $\mathcal{K}$ such that the greedy algorithm gets stuck at a 2-Core, which is avoidable for another possible order of collapses. Among all the counter exemplary complexes, let $\mathcal{K}$ be a minimal one with respect to the number of triangles $m_2$. Then there exist a free edge $\sigma \in \mathcal{V}_1(\mathcal{K})$ such that $\mathcal{K}\backslash\{\sigma\}$ is *collapsible* and another $\sigma' \in \mathcal{V}_2(\mathcal{K})$ such that $\mathcal{K}\backslash\{\sigma'\}$ is *not collapsible*.

Note that if $\mathcal{K}$ is minimal then for any pair of free edges $\sigma_1$ and $\sigma_2$ belong to the same triangle: $\tau(\sigma_1) = \tau(\sigma_2)$. Indeed, for any $\tau(\sigma_1) \neq \tau(\sigma_2)$, $\mathcal{K}\backslash\{\sigma_1, \sigma_2\} = \mathcal{K}\backslash\{\sigma_2, \sigma_1\}$. Let $\tau(\sigma_1) \neq \tau(\sigma_2)$ for at least one pair of $\sigma_1$ and $\sigma_2$; in our assumption, either both $\mathcal{K}\backslash\{\sigma_1\}$ and $\mathcal{K}\backslash\{\sigma_2\}$, only $\mathcal{K}\backslash\{\sigma_1\}$ or none are collapsible. In the former case either $\mathcal{K}\backslash\{\sigma_1\}$ or $\mathcal{K}\backslash\{\sigma_2\}$ is a smaller example of the complex satisfying the assumption, hence, violating the minimality. If only $\mathcal{K}\backslash\{\sigma_1\}$ is collapsible, then $\mathcal{K}\backslash\{\sigma_2, \sigma_1\}$ is not collapsible; hence, $\mathcal{K}\backslash\{\sigma_1, \sigma_2\}$ is not collapsible, so $\mathcal{K}\backslash\{\sigma_1\}$ is a smaller example of a complex satisfying the assumption. Finally, if both $\mathcal{K}\backslash\{\sigma_1\}$ and $\mathcal{K}\backslash\{\sigma_2\}$ are collapsible, then for known $\sigma'$ such that $\mathcal{K}\backslash\{\sigma'\}$ is not collapsible, $\tau(\sigma') \neq \tau(\sigma_1)$ or $\tau(\sigma') \neq \tau(\sigma_2)$, which revisits the previous point.

As a result, for $\sigma$ ( $\mathcal{K}\backslash\{\sigma\}$ is collapsible) and for $\sigma'$ ( $\mathcal{K}\backslash\{\sigma'\}$ is not collapsible ) it holds that $\tau(\sigma) = \tau(\sigma') \Rightarrow \sigma \cap \sigma' = \{v\}$, so after collapses $\mathcal{K}\{\sigma\}$ and $\mathcal{K}\backslash\{\sigma'\}$ we arrive at two identical simplicial complexes modulo the hanging vertex irrelevant for the weak collapsibility. A simplicial complex can not be simultaneously collapsible and not collapsible, so the question of weak collapsibility can always be resolved by the greedy algorithm which has polynomial complexity. $\square$

### 5.4.2 Computational cost of the greedy algorithm

Let $\mathcal{K}$ be a 2-skeleton; let $\Delta_\sigma$ be a set of triangles of $\mathcal{K}$ containing the edge $\sigma$, $\Delta_\sigma = \{t \mid t \in \mathcal{V}_2(\mathcal{K}) \text{ and } \sigma \in t\}$. Then the edge $\sigma$ is free iff $|\Delta_\sigma| = 1$ and $F = \{\sigma \mid |\Delta_e| = 1\}$ is a set of all free edges. Note that $|\Delta_e| \leq m_0 - 2 = \mathcal{O}(m_0)$.

**Algorithm 1** GREEDY_COLLAPSE($\mathcal{K}$): greedy algorithm for the weak collapsibility

**Require:** initial set of free edges $F$, adjacency sets $\{\Delta_{\sigma_i}\}_{i=1}^{m_1}$

```
Σ = [ ], 𝕋 = [ ]                                  ▷ initialize the collapsing sequence
while F ≠ ∅ and 𝒱₂(𝒦) ≠ ∅ do
    σ ← pop(F), τ ← τ(σ)                          ▷ pick a free edge σ
    𝒦 ← 𝒦\{σ}, Σ ← [Σ σ], 𝕋 ← [𝕋 τ]               ▷ τ is a triangle being collapsed; τ = [σ, σ₁, σ₂]
    Δ_σ₁ ← Δ_σ₁\τ, Δ_σ₂ ← Δ_σ₂\τ                   ▷ remove τ from adjacency lists
    F ← F ∪ {σ_i | i = 1, 2 and |Δ_σi| = 1}        ▷ update F if any of σ₁ or σ₂ has become free
end while
return 𝒦, Σ, 𝕋
```

The complexity of Algorithm 1 rests upon the precomputed $\sigma \mapsto \Delta_\sigma$ structure that de-facto coincides with the boundary operator $B_2$ (assuming $B_2$ is stored as a sparse matrix, the adjacency structure describes its non-zero entries). Similarly, the initial $F$ set can be computed alongside the construction of $B_2$ matrix. Another concession is needed for the complexity of the removal of elements from $\Delta_{\sigma_i}$ and $F$, which may vary from $\mathcal{O}(1)$ on average up to guaranteed $\log(|\Delta_{\sigma_i}|)$. As a result, given a pre-existing $B_2$ operator, Algorithm 1 runs linearly, $\mathcal{O}(m_1)$, or almost linearly depending on the realisation, $\mathcal{O}(m_1 \log m_1)$.

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