Topological Stability and Preconditioning of Higher-Order Laplacian Operators on Simplicial Complexes

Anton Savostianov[1]

[1]Gran Sasso Science Institute, viale F.Crispi 7, L'Aquila, Italy, 67100, email: `anton.savostianov@gssi.it`

Abstract: Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum.

Keywords: NNs, condition number

I. Introduction

II. From Graphs to Simplicial Complex

II.I Simplicial Complexes

Let $V = \{v_1, v_2, \ldots, v_n\}$ be a set of nodes; as discussed above, such set may refer to various interacting entities and agents in the system, e.g. neurons, genes, traffic stops, online actors, publication authors, etc. Then:

Def. 1 **(Simplicial Complex)** The collection of subsets $\mathcal{K}$ of the nodal set V is a (abstract) simplicial complex if for each subset $\sigma \in \mathcal{K}$, referred as a simplex, all its subsets σ', $\sigma' \subseteq \sigma$, referred as faces, enter $\mathcal{K}$ as well, $\sigma' \in \mathcal{K}$.

addition of the word "abstract" to the term is more common in the topological setting

A simplex $\sigma \in \mathcal{K}$ on $k + 1$ vertices is said to be of the order k, $\operatorname{ord}(\sigma) = k$. Let $\mathcal{V}_k(\mathcal{K})$ be a set of all k-order simplices in $\mathcal{K}$ and m_k is the cardinality of $\mathcal{V}_k(\mathcal{K})$, $m_k = |\mathcal{V}_k(\mathcal{K})|$; then $\mathcal{V}_0(\mathcal{K})$ is the set of nodes in the simplicial complex $\mathcal{K}$, $\mathcal{V}_1(\mathcal{K})$ — the set of edges, $\mathcal{V}_2(\mathcal{K})$ — the set of triangles, or 3-cliques, and so on, with $\mathcal{K} = \{\mathcal{V}_0(\mathcal{K}), \mathcal{V}_1(\mathcal{K}), \mathcal{V}_2(\mathcal{K}) \ldots\}$. Note that due to the inclusion rule in Definition 1, the number of non-empty $\mathcal{V}_k(\mathcal{K})$ is finite and, moreover, uninterupted in a sense of the order: if $\mathcal{V}_k(\mathcal{K}) = \varnothing$, then $\mathcal{V}_{k+1}(\mathcal{K})$ is also necessarily empty.

Example (Simplicial Complex) 123

Example (Real Life Simplicial Complex) 123

Comparing to the general case of the hypergraph described above, it is easy to see that simplicial complex is a special case of a hypergraph where every edge is enclosed with respect to the inclusion. In other words, simplicial complex contains additional structural rigidity which allows to formally describe the topology of $\mathcal{K}$; as a result, one is specifically interested in the formal desctiption of the nested inclusion principle achieved through **boundary operators** defined in the subsections below.

Figure II.1:   Example of a simplicial complex

II.II Hodge's Theory

Two linear operators A and B are said to satisfy Hodge's theory if and only if their composition is a null operator,

$$AB = 0 \qquad \text{(Eqn. 1)}$$

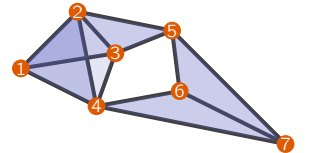which is equivalent to $\operatorname{im} B \subseteq \ker A$.

Def. 2

For a pair of operators A and B satisfying Hodge's theory, the **quotient space** $\mathcal{H}$ is defined as follows:

$$\mathcal{H} = \ker A \big/ \mathrm{im}\, B \qquad \text{(Eqn. 2)}$$

where each element of $\mathcal{H}$ is a manifold $\mathbf{x} + \mathrm{im}\, B = \{\mathbf{x} + \mathbf{y} \mid \forall \mathbf{y} \in \mathrm{im}\, B\}$ for $\mathbf{x} \in \ker A$. It follows directly from the definition that $\mathcal{H}$ is an abelian group under addition.

By Definition 2, the quotient space $\mathcal{H}$ is a collection of **equivalence classes** $\mathbf{x} + \mathrm{im}\, B$. ⎯ in a general sense

Then, each class $\mathbf{x} + \mathrm{im}\, B = \mathbf{x}_H + \mathrm{im}\, B$ for some $\mathbf{x}_H \perp \mathrm{im}\, B$ (both $\mathbf{x}, \mathbf{x}_H \in \ker A$); indeed, since the orthogonal component $\mathbf{x}_H$ (referred as **harmonic representative**) of $\mathbf{x}$ with respect to $\mathrm{im}\, B$ is unique, the map $\mathbf{x}_H \leftrightarrow \mathbf{x} + \mathrm{im}\, B$ is bijectional.

Theorem 1

([**Lim20, Thm 5.3**]) Let A and B be linear operators, $AB = 0$. Then the homology group $\mathcal{H}$ satisfies:

$$\mathcal{H} = \ker A \big/ \mathrm{im}\, B \cong \ker A \cap \ker B^\top, \qquad \text{(Eqn. 3)}$$

where $\cong$ denotes the isomorphism.

Proof

One builds the isomorphism through the harmonic representative, as discussed above. It sufficient to note that $\mathbf{x}_H \perp \mathrm{im}\, B \Leftrightarrow \mathbf{x}_H \in \ker B^\top$ in order to complete the proof. ∎

Lemma 1

([**Lim20, Thm 5.2**]) Let A and B be linear operators, $AB = 0$. Then:

$$\ker A \cap \ker B^\top = \ker\left(A^\top A + BB^\top\right) \qquad \text{(Eqn. 4)}$$

Proof

Note that if $\mathbf{x} \in \ker A \cap \ker B^\top$, then $\mathbf{x} \in \ker A$ and $\mathbf{x}\,\ker B^\top$, so $\mathbf{x} \in \ker\left(A^\top A + BB^\top\right)$. As a result, $\ker A \cap \ker B^\top \subset \ker\left(A^\top A + BB^\top\right)$.

On the other hand, let $\mathbf{x} \in \ker\left(A^\top A + BB^\top\right)$, then

$$A^\top A \mathbf{x} + BB^\top \mathbf{x} = 0 \qquad \text{(Eqn. 5)}$$

Exploiting $AB = 0$ and multiplying the equation above by $B^\top$ and A one gets the following:

$$B^\top BB^\top \mathbf{x} = 0$$
$$AA^\top A \mathbf{x} = 0 \qquad \text{(Eqn. 6)}$$

Note that $AA^\top A\mathbf{x} = 0 \Leftrightarrow A^\top A \mathbf{x} \in \ker A$, but $A^\top A \mathbf{x} \in \mathrm{im}\, A^\top$, so by Fredholm alternative, $A^\top A \mathbf{x} = 0$. Finally, for $A^\top A \mathbf{x} = 0$:

$$A^\top A \mathbf{x} = 0 \implies \mathbf{x}^\top A^\top A \mathbf{x} = 0 \iff \|A\mathbf{x}\|^2 = 0 \implies \mathbf{x} \in \ker A \quad \text{(Eqn. 7)}$$

Similarly, for the second equation, $\mathbf{x} \in \ker B^\top$ which completes the proof. ∎

ker A, im B, $\mathbf{x}$, $\mathbf{x}_H$

Figure II.2: Illustration of a harmonic representative for an equivalence class

╌╌ TMP ╌╌

Here we need some words about the transitions.

Since $AB = 0$, $B^\top A^\top = 0$ or $\mathrm{im}\, A^\top \subset \ker B^\top$. Then, exploiting $\mathbb{R}^n = \ker A \oplus \mathrm{im}\, A^\top$:

$$\ker B^\top = \ker B^\top \cap \mathbb{R}^n = \ker B^\top \cap \left(\ker A \oplus \mathrm{im}\, A^\top\right) =$$
$$= \left(\ker A \cap \ker B^\top\right) \oplus \left(\mathrm{im}\, A^\top \cap \ker B^\top\right) \qquad \text{(Eqn. 8)}$$

Given Lemma 1, $\ker A \cap \ker B^\top = \ker\left(A^\top A + BB^\top\right)$ and, since $\mathrm{im}\, A^\top \subset \ker B^\top$, $\mathrm{im}\, A^\top \cap \ker B^\top = \mathrm{im}\, A^\top$, yeilding the decomposition of the whole space:
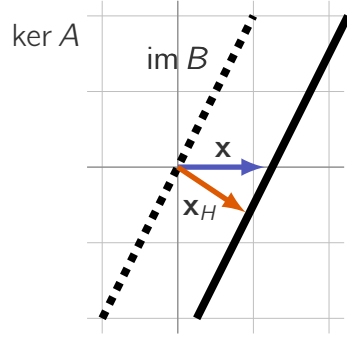
Theorem 2

(Hodge Decomposition) Let A and B be linear operators, $AB = 0$. Then:

$$\mathbb{R}^n = \overbrace{\operatorname{im} A^\top \oplus \underbrace{\ker\left(A^\top A + BB^\top\right)}_{\ker A} \oplus \operatorname{im} B}^{\ker B^\top} \qquad \text{(Eqn. 9)}$$

II.III Boundary and Laplacian Operators

II.III.I Boundary operators

Each simplicial complex $\mathcal{K}$ has a nested structure of simplicies: indeed, if σ is a simplex of order k, $\sigma \in \mathcal{V}_k(\mathcal{K})$, then all of $(k-1)$-th order faces forming the boundary of σ also belong to $\mathcal{K}$: for instance, for the triangle $\{1, 2, 3\}$ all the border edges $\{1, 2\}$, $\{1, 3\}$ and $\{2, 3\}$ are also in the simplicial complex, Figure II.1.

This nested property implies that one can build a formal map from a simplex to its boundary enclosed inside the simplicial complex.

Def. 3

(Chain spaces) Let $\mathcal{K}$ be a simplicial complex; then the space C_k of formal sums of simplices from $\mathcal{V}_k(\mathcal{K})$ over real numbers is called **a k-th chain space**.

Chain spaces on its own are naturally present in the majority of the network models: C_0 is a space of states of vertices (e.g. in the dynamical system $\dot{\mathbf{x}} = A\mathbf{x}$ the evoloving vector $\mathbf{x} \in C_0$), C_1 — is a space of (unrestricted) flows on graphs edges, and so on. ⌐ refs?

Since C_k is a linear space, versor vectors corresponding to the elements of $\mathcal{V}_k(\mathcal{K})$ is a natural basis of C_k; in order to proceed with the boundary maps, one needs to fix an order of the simplices inside $\mathcal{V}_k(\mathcal{K})$ (alas one fixes the order of vertices in the graph to form an adjacency matrix) as a matter of bookkeeping. Addiitionally, since C_k is a space over $\mathbb{R}$, chains σ and $-\sigma$ naturally introduce the notion of **orientation** of the simplex.

II.IV Weigthed and Normalised Boundary Operators

Note that, from the definition $\overline{B}_k = W_{k-1}^{-1} B_k W_k$ and **??**, we immediately have that $\overline{B}_k \overline{B}_{k+1} = 0$. Thus, the group $\overline{\mathcal{H}}_k = \ker \overline{B}_k / \operatorname{im} \overline{B}_{k+1}$ is well defined for any choice of positive weights w_k and is isomorphic to $\ker \overline{L}_k$. While the homology group may depend on the weights, we observe below that its dimension does not. Precisely, we have

Prop.

The dimension of the homology groups of $\mathcal{K}$ is not affected by the weights of its k-simplices. Precisely, if W_k are positive diagonal matrices, we have

$$\dim \ker \overline{B}_k = \dim \ker B_k, \quad \dim \ker \overline{B}_k^\top = \dim \ker B_k^\top, \quad \dim \overline{\mathcal{H}}_k = \dim \mathcal{H}_k \quad \text{(Eqn. 10)}$$

Moreover, $\ker B_k = W_k \ker \overline{B}_k$ and $\ker B_k^\top = W_{k-1}^{-1} \ker \overline{B}_k^\top$.

Proof

Since W_k is an invertible diagonal matrix,

$$\overline{B}_k \mathbf{x} = 0 \iff W_{k-1}^{-1} B_k W_k \vec{x} = 0 \iff B_k W_k \vec{x} = 0.$$

Hence, if $\vec{x} \in \ker \overline{B}_k$, then $W_k \vec{x} \in \ker B_k$, and, since W_k is bijective, $\dim \ker \overline{B}_k = \dim \ker B_k$. Similarly, one observes that $\dim \ker \overline{B}_k^\top = \dim \ker B_k^\top$. Moreover, since $\overline{B}_k \overline{B}_{k+1} = 0$, then $\operatorname{im} \overline{B}_{k+1} \subseteq \ker \overline{B}_k$ and $\operatorname{im} \overline{B}_k^\top \subseteq \ker \overline{B}_{k+1}^\top$. This yields $\ker \overline{B}_k \cup \ker \overline{B}_{k+1}^\top = \mathbb{R}^{\mathcal{V}_k} = \ker B_k \cup \ker B_{k+1}^\top$. Thus, for the homology group

3

it holds:

$$\dim \overline{\mathcal{H}}_k = \dim \left(\ker \overline{B}_k \cap \ker \overline{B}_{k+1}^\top \right) =$$
$$= \dim \ker \overline{B}_k + \dim \ker \overline{B}_{k+1}^\top - \dim \left(\ker \overline{B}_k \cup \ker \overline{B}_{k+1}^\top \right) =$$
$$= \dim \ker B_k + \dim \ker B_{k+1}^\top - \dim \left(\ker B_k \cup \ker B_{k+1}^\top \right) = \dim \mathcal{H}_k$$

∎

III. Matrix nearness problems

III.I 101 on MNP

Generally speaking, for a given matrix A a **matrix nearness problem** consists of finding the closest possible matrix X among the admissible set with a number of desired properties. For instance, one may search for the closest (in some metric) symmetric positive/negative definite matrix, unitary matrix or the closest graph Laplacian.

Motivated by the topological meaning of the **kernel** of Hodge Laplacians L_k, we assume the specific case of **spectral** MNPs: here one aims for the target matrix X to have a specific spectrum $\sigma(X)$. For instance in the stability study of the dynamical system $\dot{\mathbf{x}} = A\mathbf{x}$ one can search for the closest Hurwitz matrix such that $\mathrm{Re}\,[\lambda_i] < 0$ for all $\lambda_i \in \sigma(X)$; similarly, assuming given matrix A is a graph Laplacian, one can search for the closest disconnected graph (so the algebraic connectivity $\lambda_2 = 0$).

Here we recite the optimization framework developed by REFREFREF for the class of the spectral matrix nearness problems; one should note, however, that this is by far not the only approach to the task, REFREFREF.

> fix it

> also fix it with Nicholas and others, I guess?

Functional and Gradient Flow   Let as assume that $X = A + \Delta$ and intead of searching for X, we search for the perturbation matrix Δ; additionally, we assume that Ω is the admissible set containing all possible perturbations Δ.

III.II Direct approach: failure and discontinuity problems

III.II.I Principal spectral inheritance

Before moving on to the next section, we recall here a relatively direct but important spectral property that connects the spectra of the k-th and $(k+1)$-th order Laplacians.

Theorem 3

(HOL's spectral inheritance) Let L_k and L_{k+1} be higher-order Laplacians for the same simplicial complex $\mathcal{K}$. Let $\overline{L}_k = \overline{L}_k^{down} + \overline{L}_k^{up}$, where $\overline{L}_k^{down} = \overline{B}_k^\top \overline{B}_k$ and $\overline{L}_k^{up} = \overline{B}_{k+1}\overline{B}_{k+1}^\top$. Then:

1. $\sigma_+(\overline{L}_k^{up}) = \sigma_+(\overline{L}_{k+1}^{down})$, where $\sigma_+(\cdot)$ denotes the positive part of the spectrum;

2. if $0 \neq \mu \in \sigma_+(\overline{L}_k^{up}) = \sigma_+(\overline{L}_{k+1}^{down})$, then the eigenvectors are related as follows:

    (a) if $\vec{x}$ is and eigenvector for $\overline{L}_k^{up}$ with the eigenvalue μ, then $\vec{y} = \frac{1}{\sqrt{\mu}}\overline{B}_{k+1}^\top \vec{x}$ is an eigenvector for $\overline{L}_{k+1}^{down}$ with the same eigenvalue;

    (b) if $\vec{u}$ is and eigenvector for $\overline{L}_{k+1}^{down}$ with the eigenvalue μ and $\vec{u} \notin \ker \overline{B}_{k+1}$, then $\vec{v} = \frac{1}{\sqrt{\mu}}\overline{B}_{k+1}\vec{u}$ is an eigenvector for $\overline{L}_k^{up}$ with the same eigenvalue;

3. for each Laplacian $\overline{L}_k$: if $\vec{v} \notin \ker \overline{L}_k^{down}$ is the eigenvector for $\overline{L}_k^{down}$, then $\vec{v} \in \ker \overline{L}_k^{up}$; vice versa, if $\vec{u} \notin \ker \overline{L}_k^{up}$ is the eigenvector for $\overline{L}_k^{up}$, then $\vec{v} \in \ker \overline{L}_k^{down}$;

4. consequently, there exist $\mu \in \sigma_+(\overline{L}_k)$ with an eigenvector $\vec{u} \in \ker \overline{L}_k^{up}$, and

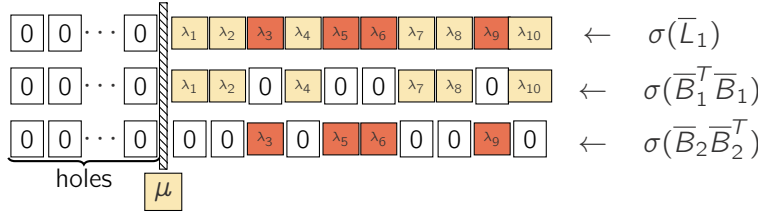**Figure III.1:** Illustration for the principal spectrum inheritance (Theorem 3) in case $k = 0$: spectra of $\overline{L}_1$, $\overline{L}_1^{down}$ and $\overline{L}_1^{up}$ are shown. Colors signify the splitting of the spectrum, $\lambda_i > 0 \in \sigma(\overline{L}_1)$; all yellow eigenvalues are inherited from $\sigma_+(\overline{L}_0)$; red eigenvalues belong to the non-inherited part. Dashed barrier μ signifies the penalization threshold (see the target functional in **??**) preventing homological pollution (see **??**).

> $\nu \in \sigma_+(\overline{L}_{k+1})$ with an eigenvector $\vec{u} \in \ker \overline{L}_{k+1}^{down}$, such that:
>
> $$\overline{B}_k^\top \overline{B}_k \vec{v} = \mu \vec{v}, \qquad \overline{B}_{k+2} \overline{B}_{k+2}^\top \vec{u} = \nu \vec{u}.$$

Proof | For $(2a)$ it is sufficient to note that $\overline{L}_{k+1}^{down} \vec{y} = \overline{B}_{k+1}^\top \overline{B}_{k+1} \frac{1}{\sqrt{\mu}} \overline{B}_{k+1}^\top \vec{x} = \frac{1}{\sqrt{\mu}} \overline{B}_{k+1}^\top \overline{L}_k^{up} \vec{x} = \sqrt{\mu} \overline{B}_{k+1}^\top \vec{x} = \mu \vec{y}$. Similarly, for $(2b)$: $\overline{L}_k^{up} \vec{v} = \overline{B}_{k+1} \overline{B}_{k+1}^\top \frac{1}{\sqrt{\mu}} \overline{B}_{k+1} \vec{u} = \frac{1}{\sqrt{\mu}} \overline{B}_{k+1} \overline{L}_{k+1}^{down} \vec{u} = \mu \vec{v}$; joint $2(a)$ and $2(b)$ yield (1). **Hodge decomposition** immediately yields the strict separation of eigenvectors between $\overline{L}_k^{up}$ and $\overline{L}_k^{down}$, (3); given (3), all the inherited eigenvectors from $(2a)$ fall into the $\ker \overline{L}_{k+1}^{down}$, thus resulting into (4). ∎

In other words, the variation of the spectrum of the k-th Laplacian when moving from one order to the next one works as follows: the down-term $\overline{L}_{k+1}^{down}$ inherits the positive part of the spectrum from the up-term of $\overline{L}_k^{up}$; the eigenvectors corresponding to the inherited positive part of the spectrum lie in the kernel of $\overline{L}_{k+1}^{up}$; at the same time, the "new" up-term $\overline{L}_{k+1}^{up}$ has a new, non-inherited, part of the positive spectrum (which, in turn, lies in the kernel of the $(k+2)$-th down-term).

In particular, we notice that for $k = 0$, since $B_0 = 0$ and $\overline{L}_0 = \overline{L}_0^{up}$, the theorem yields $\sigma_+(\overline{L}_0) = \sigma_+(\overline{L}_1^{down}) \subseteq \sigma_+(\overline{L}_1)$. In other terms, the positive spectrum of the $\overline{L}_0$ is inherited by the spectrum of $\overline{L}_1$ and the remaining (non-inherited) part of $\sigma_+(\overline{L}_1)$ coincides with $\sigma_+(\overline{L}_1^{up})$. Figure III.1 provides an illustration of the statement of Theorem 3 for $k = 0$.

IV. Topological Stability of Simplicial Complexes

V. Preconditioning

> TMP
>
> Here we need to say general words about how we need an efficient preconditioning scheme.

V.I   Iterative methods for Positive Definite Systems
V.II Preconditioning 101
V.III Cholesky preconditioning for classical graphs
V.IV Classical collapsibility

In this section we borrow the terminology from [Whi39]; additionally, let us assume that considered simplicial complex $\mathcal{K}$ is restricted to its 2-skeleton, so $\mathcal{K}$ consists only of nodes, edges, and triangles, $\mathcal{K} = \mathcal{V}_0(\mathcal{K}) \cup \mathcal{V}_1(\mathcal{K}) \cup \mathcal{V}_2(\mathcal{K})$.

Simplex $\tau \in \mathcal{K}$ is called an (inlusion-wise) maximal face of simplex $\sigma \in \mathcal{K}$ if τ is maximal by inlusion simplex such that $\sigma \subseteq \tau$ and $\text{ord}(\sigma) < \text{ord}(\tau)$. For instance, in Figure V.1 the edge $\{1, 2\}$ and nodes $\{1\}$ and $\{2\}$ have two maximal faces, $\{1, 2, 3\}$ and $\{1, 2, 4\}$, while all the other edges and nodes have unique maximal faces — their corresponding triangles. Note that in the case of the node $\{1\}$, there are bigger simplices containing it besides the triangles (e.g. the edge $\{1, 2\}$), but they are not maximal by inclusion.
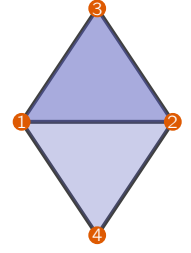
Figure V.1: Example of a simplicial complex: free simplices and maximal faces.

Def. 4 | **(Free simplex)** The simplex $\sigma \in \mathcal{K}$ is free if it has exactly one maximal face τ, $\tau = \tau(\sigma)$. F.i. edges $\{1, 3\}$, $\{1, 4\}$, $\{2, 3\}$ and $\{2, 4\}$ are all free in Figure V.1.

The collapse $\mathcal{K} \backslash \{\sigma\}$ of $\mathcal{K}$ at a free simplex σ is the transition from the original simplicial complex $\mathcal{K}$ to a smaller simplicial complex $\mathcal{L}$ without the free simplex σ and the corresponding maximal face τ, $\mathcal{K} \to \mathcal{K}' = \mathcal{K} - \sigma - \tau$; namely, one can eliminate a simplex τ if it has an accessible (not included in another simplex) face σ.

Naturally, one can perform several consequent collapses at $\Sigma = \{\sigma_1, \sigma_2, \ldots\}$ assuming σ_i is free in collapse simplicial complex from the previous stage; Σ is called the collapsing sequence. Formally:

Def. 5 | **(Collapsing sequence)** Let $\mathcal{K}$ be a simplicial complex. $\Sigma = \{\sigma_1, \sigma_2, \ldots\}$ is a collapsing sequence if σ_1 is free in $\mathcal{K}$ and each σ_i, $i > 1$, is free at $\mathcal{K}^{(i)} = \mathcal{K}^{(i-1)} \backslash \{\sigma_i\}$, $\mathcal{K}^{(1)} = \mathcal{K}$. The collapse of $\mathcal{K}$ to a new complex $\mathcal{L}$ at Σ is denoted by $\mathcal{L} = \mathcal{K} \backslash \Sigma$.

By the definition, every collapsing sequence Σ has a corresponding sequence $\mathbb{T} = \{\tau(\sigma_1), \tau(\sigma_2), \ldots\}$ of maximal faces being collapsed at every step.

Def. 6 | **(Collapsible simplicial complex, [Whi39])** The simplicial complex $\mathcal{K}$ is collapsible if there exists a collapsing sequence Σ such that $\mathcal{K}$ collapses to a single vertex at Σ, $\mathcal{K} \backslash \Sigma = \{v\}$.

Determining whether the complex is collapsible is in general **NP-complete**, [Tan16], but can be almost linear for a set of specific families of $\mathcal{K}$, e.g. if the simplex can be embeded into the triangulation of the d-dimensional unit sphere, [CFM$^+$14]. Naturally restricting the collapses to the case of d-collapses (such that $\text{ord}(\sigma)_i \leq d - 1$), one arrive at the notion of d-collapsibility, [Tan09].

Def. 7 | **(d-Core)** A d-Core is a subcomplex of $\mathcal{K}$ such that every simplex of order $d - 1$ belongs to at least 2 simplices of order d. E.g. 2-Core is such a subcomplex of the original 2-skeleton $\mathcal{K}$ that every edge from $\mathcal{V}_1(\mathcal{K})$ belong to at least 2 triangles from $\mathcal{V}_2(\mathcal{K})$.

Lemma 2 | **([LN21])** $\mathcal{K}$ is d-collapsible if and only if it does not contain a d-core.

Proof | The proof of the lemma above naturally follows from the definition of the core. Assume Σ is a d-collapsing sequence, and $\mathcal{K} \backslash \Sigma$ consists of more than a single vertex and has no free simplices of order $\leq d - 1$ ("collapsing sequence gets stuck"). Then, each simplex of order $d - 1$ is no free but belongs to at least 2 simplices of order d, so $\mathcal{K} \backslash \Sigma$ is a d-Core.

Conversely if a d-Core exists in the complex, the collapsing sequence should necessarily include its simplices of order $d - 1$ which can not become free during as a result of a sequence of collapses. Indeed, for σ from d-Core, $\text{ord}(\sigma) = d - 1$, to become free, one needs to collapse at least one of σ's maximal faces for d-Core, all of whose faces are, in turn, contained in the d-Core (since d-Core is a simplicial complex). As a result one necessarily needs a prior collapse inside the d-Core to perform the first collapse in the d-Core, which is impossible. ∎

In the case of the classical graph model, the 1-Core is a subgraph where each vertex has a degree at least 2; in other words, 1-Core cannot be a tree and necessarily contains a simple cycle. Hence, the collapsibility of a classical graph coincides with the acyclicity. The d-Core is the generalization of the cycle for the case of 1-collapsibility of the classical graph; additionally, the d-Core is very dense due to its definition. In the case of 2-Core, we provide simple exemplary structures on Figure V.2 which imply various possible configurations for a d-Core, $d \geq 2$, hence a search for d-Core inside $\mathcal{K}$ is neither trivial, no computationally cheap.
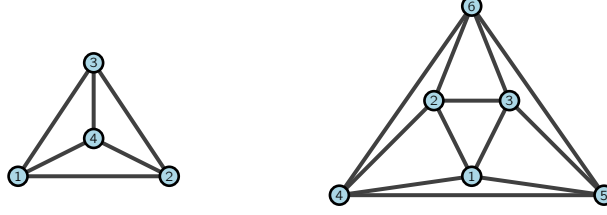
Figure V.2: 2-Core, examples.

Additionally, we demonstrate that an arbitrary simplicial complex $\mathcal{K}$ tends to contain 2-Cores as long as $\mathcal{K}$ is denser than a trivially collapsible case. Assume the complex formed by triangulation of m_0 random points on the unit square with a sparsity pattern ν; the triangulation itself with the corresponding ν_Δ is collapsible, but a reasonably small addition of edges already creates a 2-Core (since it is local), Figure V.3, left. Similarly, sampled sensor networks, where $\exists \sigma \in \mathcal{V}_1(\mathcal{K}): \sigma = [v_1, v_2] \iff \|v_1 - v_2\|_2 < \varepsilon$ for a chosen percolation parameter $\varepsilon > 0$, quickly form a 2-Core upon the densifying of the network.
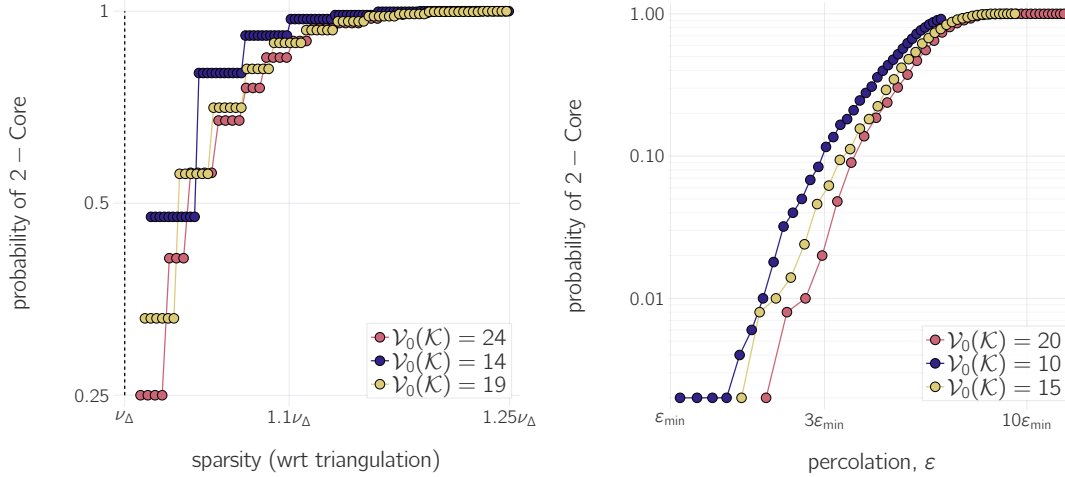

Figure V.3: The probability of the 2-Core in richer-than-triangulation simplicial complexes: triangulation of random points modified to have $\left\lceil \nu \frac{|\mathcal{V}_0(\mathcal{K})| \cdot (|\mathcal{V}_0(\mathcal{K})|-1)}{2} \right\rceil$ edges on the left; random sensor networks with ε-percolation on the right. ν_Δ defines the initial sparsity of the triangulated network; $\varepsilon_{\min} = \mathbb{E} \min_{x,y \in [0,1]^2} \|x - y\|_2$ is the minimal possible percolation parameter.

However, in the following, we observe that a weaker condition is enough to efficiently design a preconditioner for any "sparse enough" simplicial complex.

V.V Weak collapsibility

Let the complex $\mathcal{K}$ be restricted up to its 2-skeleton, $\mathcal{K} = \mathcal{V}_0(\mathcal{K}) \cup \mathcal{V}_1(\mathcal{K}) \cup \mathcal{V}_2(\mathcal{K})$, and $\mathcal{K}$ is collapsible. Then the collapsing sequence Σ necessarily involves collapses at simplices σ_i of different orders: at edges (eliminating **edges** and **triangles**) and at vertices (eliminating **vertices** and **edges**). One can show that for a given collapsing sequence Σ there is a reordering $\tilde{\Sigma}$ such that $\dim \tilde{\sigma}_i$ are non-increasing, [CFM$^+$14, Lemma 2.5]. Namely, if such

a complex is collapsible, then there is a collapsible sequence $\Sigma = \{\Sigma_1, \Sigma_0\}$ where Σ_1 contains all the collapses at edges first and Σ_0 is composed of collapses at vertices. Note that the partial collapse $\mathcal{K}\backslash\Sigma_1 = \mathcal{L}$ eliminates all the triangles in the complex, $\mathcal{V}_2(\mathcal{L}) = \varnothing$; otherwise, the whole sequence Σ is not collapsing $\mathcal{K}$ to a single vertex. Since $\mathcal{V}_2(\mathcal{L}) = \varnothing$, the associated up-Laplacian $L_1^\uparrow(\mathcal{L}) = 0$.

Def. 8 | **(Weakly collapsible complex)** Simplicial complex $\mathcal{K}$ restricted to its 2-skeleton is called **weakly collapsible**, if there exists a collapsing sequence Σ_1 such that the simplicial complex $\mathcal{L} = \mathcal{K}\backslash\Sigma_1$ has no simplices of order 2, $\mathcal{V}_2(\mathcal{L}) = \varnothing$ and $L_1^\uparrow(\mathcal{L}) = 0$.

Example | Note that a collapsible complex is necessarily weakly collapsible; the opposite does not hold. Consider the following example in Figure V.4: the initial complex is weakly collapsible either by a collapse at $[3, 4]$ or at $[2, 4]$. After this, the only available collapse is at the vertex $[4]$ leaving the uncollapsible 3-vertex structure.
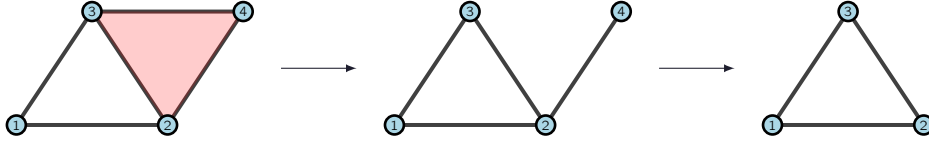

Figure V.4: Example of weakly collapsible but not collapsible simplicial complex

Theorem 4 | Weak collapsibility of 2-skeleton $\mathcal{K}$ is polynomially solvable.

Proof | The **greedy algorithm** for the collapsing sequence intuitively operates as follows: at each iteration perform any of possible collapses; in the absence of free edges, the complex should be considered not collapsible, Algorithm 1. Clearly, such an algorithm runs polynomially with respect to the number of simplexes in $\mathcal{K}$.

The failure of the greedy algorithm indicates the existence of a weakly collapsible complex $\mathcal{K}$ such that the greedy algorithm gets stuck at a 2-Core, which is avoidable for another possible order of collapses. Among all the counter exemplary complexes, let $\mathcal{K}$ be a minimal one with respect to the number of triangles m_2. Then there exist a free edge $\sigma \in \mathcal{V}_1(\mathcal{K})$ such that $\mathcal{K}\backslash\{\sigma\}$ is **collapsible** and another $\sigma' \in \mathcal{V}_2(\mathcal{K})$ such that $\mathcal{K}\backslash\{\sigma'\}$ is **not collapsible**.

Note that if $\mathcal{K}$ is minimal then for any pair of free edges σ_1 and σ_2 belong to the same triangle: $\tau(\sigma_1) = \tau(\sigma_2)$. Indeed, for any $\tau(\sigma_1) \neq \tau(\sigma_2)$, $\mathcal{K}\backslash\{\sigma_1, \sigma_2\} = \mathcal{K}\backslash\{\sigma_2, \sigma_1\}$. Let $\tau(\sigma_1) \neq \tau(\sigma_2)$ for at least one pair of σ_1 and σ_2; in our assumption, either both $\mathcal{K}\backslash\{\sigma_1\}$ and $\mathcal{K}\backslash\{\sigma_2\}$, only $\mathcal{K}\backslash\{\sigma_1\}$ or none are collapsible. In the former case either $\mathcal{K}\backslash\{\sigma_1\}$ or $\mathcal{K}\backslash\{\sigma_2\}$ is a smaller example of the complex satisfying the assumption, hence, violating the minimality. If only $\mathcal{K}\backslash\{\sigma_1\}$ is collapsible, then $\mathcal{K}\backslash\{\sigma_2, \sigma_1\}$ is not collapsible; hence, $\mathcal{K}\backslash\{\sigma_1, \sigma_2\}$ is not collapsible, so $\mathcal{K}\backslash\{\sigma_1\}$ is a smaller example of a complex satisfying the assumption. Finally, if both $\mathcal{K}\backslash\{\sigma_1\}$ and $\mathcal{K}\backslash\{\sigma_2\}$ are collapsible, then for known σ' such that $\mathcal{K}\backslash\{\sigma'\}$ is not collapsible, $\tau(\sigma') \neq \tau(\sigma_1)$ or $\tau(\sigma') \neq \tau(\sigma_2)$, which revisits the previous point.

As a result, for σ ($\mathcal{K}\backslash\{\sigma\}$ is collapsible) and for σ' ($\mathcal{K}\backslash\{\sigma'\}$ is not collapsible) it holds that $\tau(\sigma) = \tau(\sigma') \Rightarrow \sigma \cap \sigma' = \{v\}$, so after collapses $\mathcal{K}\{\sigma\}$ and $\mathcal{K}\backslash\{\sigma'\}$ we arrive at two identical simplicial complexes modulo the hanging vertex irrelevant for the weak collapsibility. A simplicial complex can not be simultaneously collapsible and not collapsible, so the question of weak collapsibility can always be resolved by the greedy algorithm which has polynomial complexity. ∎

V.VI Computational cost of the greedy algorithm

Let $\mathcal{K}$ be a 2-skeleton; let Δ_σ be a set of triangles of $\mathcal{K}$ containing the edge σ, $\Delta_\sigma = \{t \mid t \in \mathcal{V}_2(\mathcal{K})$ and $\sigma \in t\}$. Then the edge σ is free iff $|\Delta_\sigma| = 1$ and $F = \{\sigma \mid |\Delta_e| = 1\}$ is a set of all free edges. Note that $|\Delta_e| \leq m_0 - 2 = \mathcal{O}(m_0)$.

Algorithm 1 GREEDY_COLLAPSE($\mathcal{K}$): greedy algorithm for the weak collapsibility

---

Require: initial set of free edges F, adjacency sets $\{\Delta_{\sigma_i}\}_{i=1}^{m_1}$
1:   $\Sigma = [\,]$, $\mathbb{T} = [\,]$ ▷ initialize the collapsing sequence
2:   **while** $F \neq \varnothing$ **and** $\mathcal{V}_2(\mathcal{K}) \neq \varnothing$ **do**
3:       $\sigma \leftarrow \text{pop}(F)$, $\tau \leftarrow \tau(\sigma)$ ▷ pick a free edge σ
4:       $\mathcal{K} \leftarrow \mathcal{K}\backslash\{\sigma\}$, $\Sigma \leftarrow [\Sigma\ \sigma]$, $\mathbb{T} \leftarrow [\mathbb{T}\ \tau]$ ▷ τ is a triangle being collapsed;
        $\tau = [\sigma, \sigma_1, \sigma_2]$
5:       $\Delta_{\sigma_1} \leftarrow \Delta_{\sigma_1}\backslash\tau$, $\Delta_{\sigma_2} \leftarrow \Delta_{\sigma_2}\backslash\tau$ ▷ remove τ from adjacency lists
6:       $F \leftarrow F \cup \{\sigma_i \mid i = 1, 2$ and $|\Delta_{\sigma_i}| = 1\}$ ▷ update F if any of σ_1 or σ_2 has become free
7:   **end while**
8:   **return** $\mathcal{K}$, Σ, $\mathbb{T}$

The complexity of Algorithm 1 rests upon the precomputed $\sigma \mapsto \Delta_\sigma$ structure that de-facto coincides with the boundary operator B_2 (assuming B_2 is stored as a sparse matrix, the adjacency structure describes its non-zero entries). Similarly, the initial F set can be computed alongside the construction of B_2 matrix. Another concession is needed for the complexity of the removal of elements from Δ_{σ_i} and F, which may vary from $\mathcal{O}(1)$ on average up to guaranteed $\log(|\Delta_{\sigma_i}|)$. As a result, given a pre-existing B_2 operator, Algorithm 1 runs linearly, $\mathcal{O}(m_1)$, or almost linearly depending on the realisation, $\mathcal{O}(m_1 \log m_1)$.

V. References

[CFM+14] Michael B. Cohen, Brittany Terese Fasy, Gary L. Miller, Amir Nayyeri, Richard Peng, and Noel Walkington. Solving 1-laplacians in nearly linear time: Collapsing and expanding a topological ball. In Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms, pages 204–216. SIAM, 2014.

[Lim20] Lek-Heng Lim. Hodge Laplacians on Graphs. SIAM Review, 62(3):685–715, January 2020.

[LN21] Davide Lofano and Andrew Newman. The worst way to collapse a simplex. Israel Journal of Mathematics, 244(2):625–647, 2021.

[SBH+20] Michael T. Schaub, Austin R. Benson, Paul Horn, Gabor Lippner, and Ali Jadbabaie. Random Walks on Simplicial Complexes and the Normalized Hodge 1-Laplacian. SIAM Review, 62(2):353–391, January 2020.

[Tan09] Martin Tancer. D-collapsibility is NP-complete for d$\geq$ 4. Electronic Notes in Discrete Mathematics, 34:53–57, 2009.

[Tan16] Martin Tancer. Recognition of collapsible complexes is NP-complete. Discrete & Computational Geometry, 55:21–38, 2016.

[Whi39] John Henry Constantine Whitehead. Simplicial spaces, nuclei and m-groups. Proceedings of the London mathematical society, 2(1):243–327, 1939.

V. List of Figures

V. List of Tables

V. Glossary

collapse transition to a smaller simplicial complex without a free simplex and its maximal face .

collapsible simplex that can be collapsed to a single vertex .

collapsing sequence several consequent collapses .

face subsimplex of a given simplex .

free wrt a simplex: having exactly one facet .

maximal face a maximal simplex of higher-order containing given simplex .

simplex elements of the simplicial complex; subset of vertex set.

simplicial complex higher-order network model; hypergraph closed for edge inclusion.