Notes on the prelimit patterns in synchronized asymmetrically coupled van der Pol oscillators

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Abstract:

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I. Introduction

Well, it would be nice to have one.

You know, somewhere here, maybe...

Related works.

Contributions. we do contribute something, right?

Outline. it's all over the place...

II. Model

We consider the space-time normalized model of two dissipatively coupled van der Pol (vdP) oscillators given by:

$$\begin{aligned} &\ddot{x} - (1 - x^2)\dot{x} + (1 - \Delta\omega)x + \mu_1(\dot{x} - \dot{y}) = 0 \\ &\ddot{y} - (1 - y^2)\dot{y} + (1 + \Delta\omega)y + \mu_2(\dot{y} - \dot{x}) = 0 \end{aligned}, \qquad \text{(Eqn. 1)}$$

where $0 < \Delta\omega < 1$ is a **normalized relative frequency difference** and $\mu_i \leq 0$ describe couplings. Note that x is set to be a "faster" of two oscillators and one should consider different setups in terms of which oscillator (the "fast" one or the "slow" one) has larger coupling.

in case it might be useful, lets set $\mu_1 + \mu_2 = 2\mu$ and $|\mu_1 - \mu_2| = 2\Delta\mu$

Notable cases:

(i) **symmetric coupling** , $\mu_1 = \mu_2$, previously studied and should be used as reference point;

(ii) **fancy RHS**, e.g. $\mu_2 = 0$, so

$$\begin{aligned} &\ddot{x} - (1 - \mu_1 - x^2)\dot{x} + (1 - \Delta\omega)x = \mu_1\dot{y} \\ &\ddot{y} - (1 - y^2)\dot{y} + (1 + \Delta\omega)y = 0 \end{aligned} \qquad \text{(Eqn. 2)}$$

which mimic a case of vdP with a "resonant" RHS.

it is not really a resonance, since the amplitude on the right correspond to some function a frequency on the left, but maybe...

Rem II.1 **(Half-sum/half-diff notation)** On par with the previous papers, we provide the same dynamics as in Equation (1) under the following change of variables:

$$\begin{cases} 2u = x + y \\ 2v = x - y \end{cases} \qquad \begin{cases} x = u + v \\ y = u - v \end{cases}, \qquad \text{(Eqn. 3)}$$

so we obtain

$$\ddot{u} - (1 - u^2 - v^2)\dot{u} + u - \Delta\omega v + 2uv\dot{v} + 2\Delta\mu\dot{v} = 0$$
$$\ddot{v} - (1 - u^2 - v^2)\dot{v} + v - \Delta\omega u + 2vu\dot{u} + 2\mu\dot{v} = 0 \qquad \text{(Eqn. 4)}$$

with the highlighted term concentrates the asymmetry of the coupling.

III. Synchronization

All of our consideration and defined entities are well-posed only for synchronized pair of oscillators. Specifically, one normally distinguishes two types of synchronization, in **frequency** and **phase** . Recall that in the case of symmetric coupling $\mu_1 = \mu_2$, the synchronization seemed to be joint, i.e. frequency synchronization necessarily implied phase synchronization and vice versa.

it is worth revisiting it around the threshold, we haven't properly tested it in the "battle" region

Rem III.2 **(Synchronization inequality from simpler times)** Under specific simplifying assumptions, one may reduce symmetrically coupled vdP system (Equation (1)) to a simple Kuramoto model where the synchronization is achieved if and only if $\mu > \Delta\omega$; previously, we inherited that property for the symmetric vdP system which seeemed to hold experimentally.

At the same time, in the nonsymmetric Kuramoto case, assuming initial frequencies Ω_1 and Ω_2, the system is synchronized if

$$\mu_1 > \Omega_1 - \Omega_0 \text{ and } \mu_2 > \Omega_2 - \Omega_0$$

where Ω_0 is the common synchronized frequency, $\Omega_0 = \frac{\mu_2\Omega_1+\mu_1\Omega_2}{\mu_1+\mu_2}$. Note that this principle may be expected to largely translate to the vdP case with certain caveats; specifically, it is challenging to redefine Ω_0 since even for $\mu_1 = \mu_2 = \mu$ the common frequency was shown to be dependent on μ, $\Omega_0 = \Omega_0(\mu)$, albeit this dependence is rather small.

Moreover, $\mu_1 > \frac{\mu_2\Omega_1+\mu_1\Omega_2}{\mu_1+\mu_2} - \Omega_1 = \frac{\mu_1(\Omega_2-\Omega_1)}{\mu_1+\mu_2} = \frac{2\mu_1\Delta\omega}{\mu_1+\mu_2} \Leftrightarrow \frac{\mu_1+\mu_2}{2} > \Delta\omega$. Hence, in the case of RHS, for $\mu_2 = 0$ we get $\mu_1 \geq 2\Delta\omega$.

probably worth recalling some series expansions here

.

III.I Barcode diagrams

Here we formulate a straightforward method to test the synchronization of two oscillators, Equation (1).

We base our estimates on the relative placement of local minima of $\{x(t), y(t)\}$; namely,

(i) solve the system (1) for long-enough time (specifically, we solve for $[0; T] = [0; 2\pi N]$, $N = 100$); this length is based on the notion that the first order approximation of the isolated vdP frequency is 1, so the first order period is 2π ;

here a reference for the series decomposition should be

(ii) for each oscillator $\{x(t), y(t)\}$ we extract position of local minima $\{\tau_x^{(i)}\}$ and $\{\tau_y^{(i)}\}$;

one can do this with preexisting tools, but the fact is that ODEsolver computes an approximation of $\dot{x}$ and $\dot{y}$, so one can extract minima from them

(iii) then one can define *instantaneous* frequencies as $\left\{\Omega_x^{(i)}\right\} = \left\{\frac{1}{\tau_x^{(i+1)}-\tau_x^{(i)}}\right\}$ and $\left\{\Omega_y^{(i)}\right\} = \left\{\frac{1}{\tau_y^{(i+1)}-\tau_y^{(i)}}\right\}$ respectively; we measure the standard deviation of the frequency sequences $\left\{\Omega_x^{(i)}\right\}$ over the last 20 minima (relative to the mean value) to ascertain synchronization;

(iv) finally, we posit that the phase difference $\Delta\varphi$ never exceeds half a period, so in order to calculate it, we find for each minimum $\tau_x^{(i)}$ the closest minimum $\tau_y^{(j)}$; then, we say that $\Delta\phi^{(i)} = \left|\tau_x^{(i)} - \tau_y^{(j)}\right|$ and compute the stability of the sequence in the same manner.

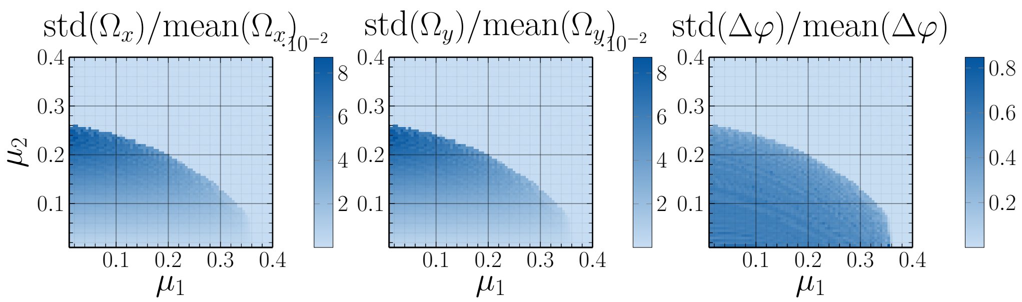


Figure III.1: Synchronization of oscillators (Equation (1)) in frequency (left and center panes) and in phase (right pane). $\Delta\omega = 0.2$, integration time is set to $N = 100$.

We provide the stabilization of frequency and the phase difference in Figure III.1 and synchronization boundary for different values of $\Delta\omega$ in Figure III.2:

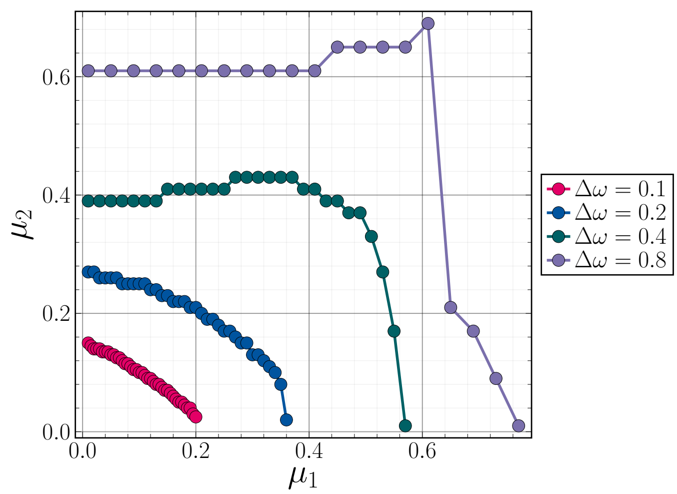


Figure III.2: Synchronization boundary of oscillators (Equation (1)) for different values $\Delta\omega$. Integration time is set to $N = 100$.

It is worth pointing out that:

- ◇ point $\mu_1 = \mu_2 = \Delta\omega$ lands on the boundary, upholding observed behaviour in the symmetric case;
- ◇ in the case of $\mu_1 = 0$, Kuramoto synchronization (see Remark III.2) requires $\mu_2 \geq 2\Delta\omega$ (and vice versa), which is evidently too restrictive for the vdP case; moreover, same holds for the neighbouring areas (one small coupling + one medium coupling; insufficient for Kuramoto, but seemingly enough for vdP);
- ◇ Kuramoto oscillators are the same in terms of the synchronization boundary; instead, vdP oscillators are not symmetric. More importantly, this asymmetry creates an area in the coupling space sufficient for the synchronization of Kuramoto but insufficient for vdP;
- ◇ phase and frequency synchronizations remain joint and are achieved simultaneously;

◇ the boundary is impressively pronounced, so one may at least attempt to speculate about the actual function and synchronization inequality.

III.II The smiling whale conundrum

We have established a problem for higher values of $\Delta\omega$, Figure III.3, which we refer as a **smiling whale** problem.

is there an idea about $\frac{1+\Delta\omega}{1-\Delta\omega}$ in comparison with 2? So $3\Delta\omega \geq 1$

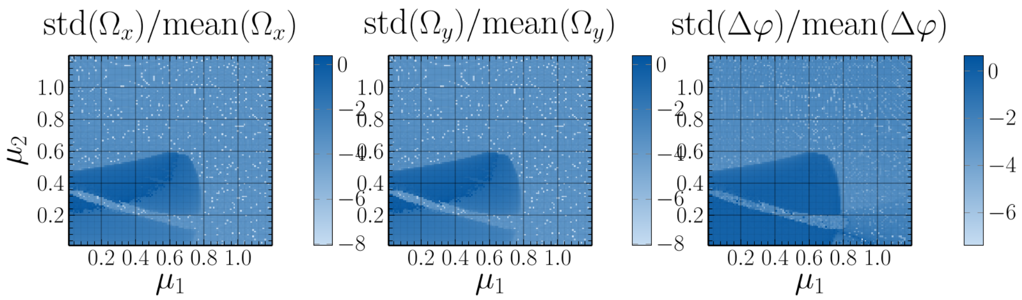
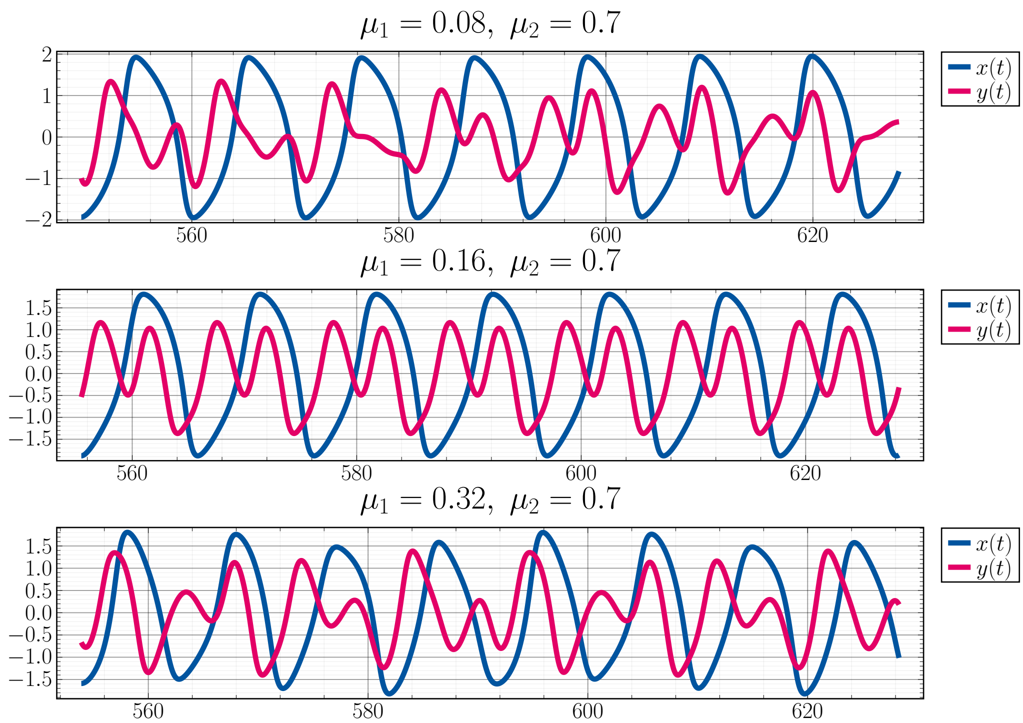


Figure III.3: Synchronization of oscillators (Equation (1)) in frequency (left and center panes) and in phase (right pane). $\Delta\omega = 0.6$, integration time is set to $N = 100$. Logarithmic scale in the colorbar.

Let us consider the three principal points:

$$(\mu_1, \mu_2) = (0.7, 0.08) \qquad (\mu_1, \mu_2) = (0.7, 0.16) \qquad (\mu_1, \mu_2) = (0.7, 0.32) \qquad \text{(Eqn. 5)}$$

you may have to change μ_1 and μ_2 order here



The point of the plot above is more or less simple: we get at least twice as much minima per period for the faster oscillator $y(t)$, which makes our minima stabilization procedure to go wild. Instead one should probably think about a more nuanced approach.

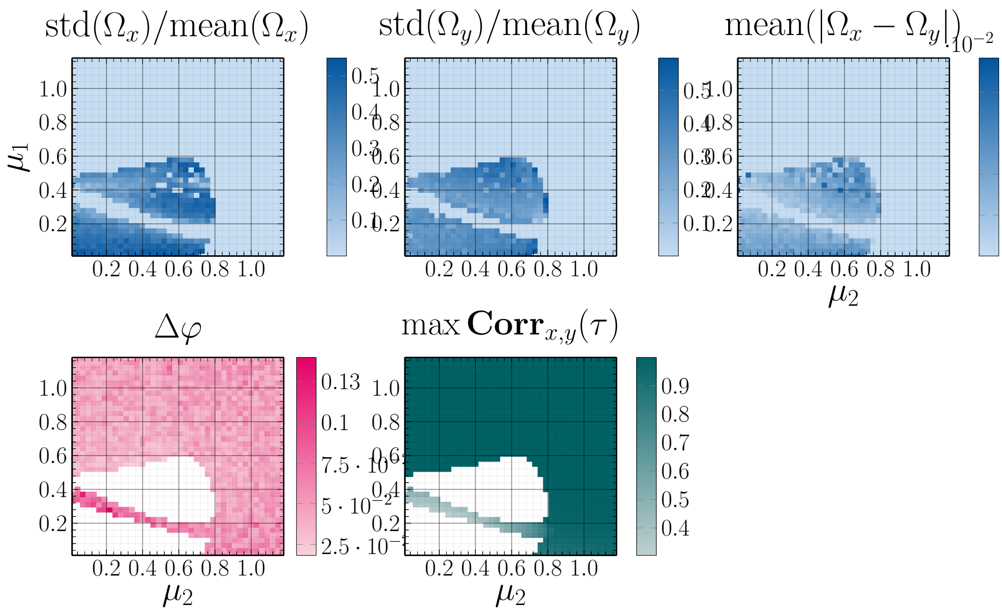
IV. Proper way to define synchronization

Let us redefine synchronization in a more rigorous way that requires both frequency synchronization and deal with the shape of the solutions.

(i) as earlier, let us consider *instantaneous* frequencies as $\left\{\Omega_x^{(i)}\right\} = \left\{\frac{1}{\tau_x^{(i+1)} - \tau_x^{(i)}}\right\}$ and $\left\{\Omega_y^{(i)}\right\} = \left\{\frac{1}{\tau_y^{(i+1)} - \tau_y^{(i)}}\right\}$ respectively; we require them to be both *stable* and the *same*. Let us refer to such configurations as **weakly synchronized**: here we get frequency synchronization, but allow fundamentally different shapes of two solutions (e.g. like in the center pane of Subsection III.II);

(i) let $\Omega_x^{(i)} \approx \Omega_y^{(i)} = \Omega$ and $T = \frac{2\pi}{\Omega}$ are "common" frequency and period respectively; then for any $\tau \in [0; T]$ we can compute the sliding correlation $\mathbf{Corr}_{x,y}(\tau)$ between $x(t)$ and $y(t+\tau)$ for $t > T^*$.

In this setup, the maximum of $\mathbf{Corr}_{x,y}(\tau)$ corresponds to the actual phase difference, $\Delta\varphi = \arg\max_{\tau \in [0;T]} \mathbf{Corr}_{x,y}(\tau)$, and the value $\mathbf{Corr}_{x,y}(\Delta\varphi)$ describes the closeness of the shapes of $x(t)$ and $y(t)$. The case of $\mathbf{Corr}_{x,y}(\Delta\varphi) \approx 1$ could be considered **strongly synchronized**.



Rem IV.3 ▌Strong synchronization removes the whale's smile.

In a sense, we get what we expected: there is some intermediate regime (center pane on Subsection III.II) which corresponds to synchronized in frequency, but not in shape oscillators. If we check the shape of the oscillators via the sliding correlation score $\mathbf{Corr}_{x,y}(\Delta\varphi)$, we can avoid this intermediate regime altogether. However, the synchronization remain non-monotonic (in a sense that one of couplings can increase and destroy the synchronization); one should properly inspect it, or just assume a "monotonic area" of couplings instead (such that if for (μ_1, μ_2) the system is strongly synchronized, then it is synchronized for every (μ_1', μ_2') where $\mu_1 \leq \mu_1'$ and $\mu_2 \leq \mu_2'$).

V. Vertical shift and phase difference

Following the previously established results for the symmetrically coupled vdP oscillators, we revisit previously unseen appearance of pre-limit vertical shift between oscillators.

alrt 1 **(Inconsistency!)** We have a problem: phase difference is defined differently two times (for the synchronization and vertical shift). They *should* be consistent between each other in the synchronized case, but it seems like minima-based definition is somewhat less numerically stable.

Requires further investigation!

Namely, let $x(t)$ and $y(t)$ be solution of Equation (1). Then, for a chosen T^* we define the **phase difference** φ as a horizontal shift minimizing maximal mismatch between solutions for $t > T^*$ and the **vertical shift** C, or, more explicitly:

it seems to be stable wrt T^*...; before we did $T^* = \tau_{LC}$, but it is a more complicated thing to compute, so we provide results for a fixed $T^* = 100T$ for now

$$\varphi = \arg\min_{\varphi\in[0;T]} \left[\max_{t\in[T^*;T^*+T]} (x(t+\varphi) - y(t)) \right]$$
$$\hat{C} = \min_{\varphi\in[0;T]} \left[\max_{t\in[T^*;T^*+T]} (x(t+\varphi) - y(t)) \right] \quad \text{(Eqn. 6)}$$

where T is the common synchronized period of oscillators. Note that such definition of the vertical shift is biased by the amplitude difference ΔA ; in order to track it we introduce an instantaneous amplitude estimator:

it is insignicant and boring, but it is a fair thing to account for

$$A_x = \frac{1}{2}\left[\max_{\tau\in[T^*;T^*+T]} x(\tau) - \min_{\tau\in[T^*;T^*+T]} x(\tau) \right]$$
$$A_y = \frac{1}{2}\left[\max_{\tau\in[T^*;T^*+T]} y(\tau) - \min_{\tau\in[T^*;T^*+T]} y(\tau) \right] \quad \text{(Eqn. 7)}$$
$$C = \hat{C} - \Delta A = \hat{C} - (A_x - A_y)$$

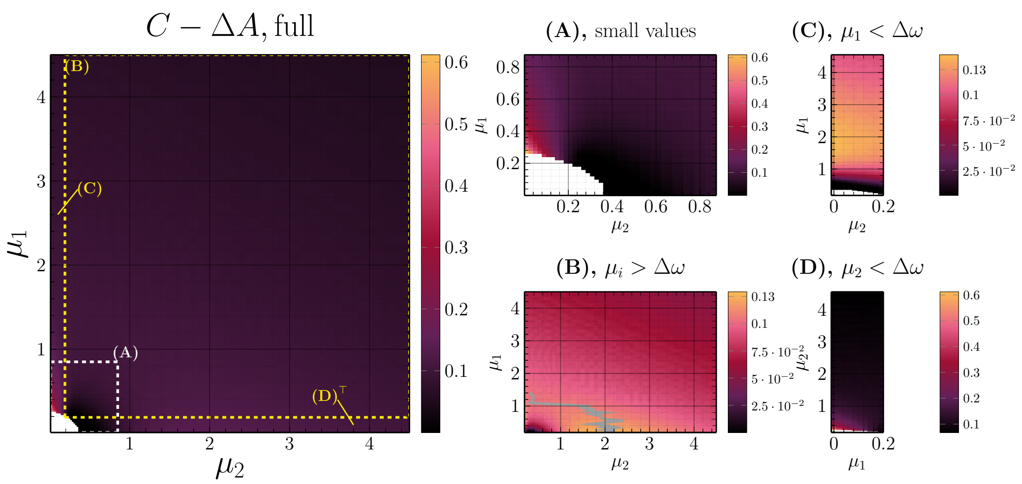


Figure V.1: Vertical shift $C(\mu_1, \mu_2)$ for various values of couplings (enlarged regions of the overall curve are shown on the right). Grey line on **(B)** corresponds to the approximate point of monotonicity change. $\Delta\omega = 0.2$, $T^* = 100T$

Since each pair of solutions is defined by the pair couplings (μ_1, μ_2), each of the introduced values (φ, C, ΔA) are functions of (μ_1, μ_2).

We demonstrate the behaviour of the the unbiased $C(mu_1, mu_2)$ in Figure V.1. The value is not computed for the system lacking synchronization, resulting in the missing area for small values of (μ_1, μ_2) (see Figure V.1A). The most interesting pattern for us is contained in the area $\mu_1 > \Delta\omega, \mu_2 > \Delta\omega$, Figure V.1B: the value of the unbiased vertical shift still exhibits a pronounced "switch" or monotonicity change (grey line on Figure V.1B) ; the switch is con-

C lacks $\hat{C}$ in the figure; will fix later

very shake-y and noisy, will probably be smoothed later

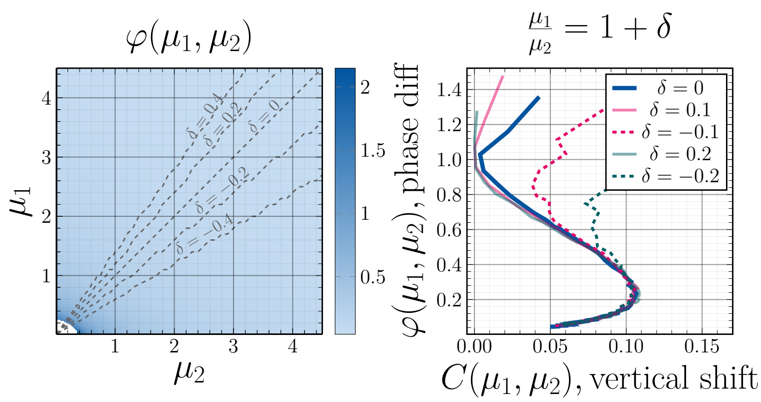


Figure V.2: Phase difference $\varphi(\mu_1, \mu_2)$ for various values of couplings (on the left). (C, φ)-diagrams (on the right) for fixed ratios of couplings ($\mu_2/\mu_1 = 1 + \delta$, corresponding lines are shown on the left). $\Delta\omega = 0.2$, $T^* = 100T$

sistent with the previous results for $\mu_1 = \mu_2$.

Note that the values $\varphi(\mu_1, \mu_2)$ consistently decrease upon any increase in coupling, Figure V.2, as expected. We additionally show (Figure V.2, right) specific slices of $\varphi(\mu_1, \mu_2)$ and $C(\mu_1, \mu_2)$ in the following manner: we fix the ratio between two couplings, $\frac{\mu_2}{\mu_1} = 1 + \delta$ and plot corresponding curve $\gamma_\delta = (C(\mu_1, (1 + \delta)\mu_1), \varphi(\mu_1, (1 + \delta)\mu_1))$ in the (C, φ) phase plane; couplings increase from top to bottom. Besides increasingly erratic behaviour in the case of highly disbalanced and small couplings, one may note that lines overall follow each other very similarly.

References