Notes on the prelimit patterns in synchronized asymmetrically coupled van der Pol oscillators

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Abstract:

Keywords: coupled oscillators

I. Introduction

Well, it would be nice to have one.

You know, somewhere here, maybe...

Related works.

Contributions. we do contribute something, right?

Outline. it's all over the place...

II. Model

We consider the space-time normalized model of two dissipatively coupled van der Pol (vdP) oscillators given by:

$$\begin{aligned} &\ddot{x} - (1 - x^2)\dot{x} + (1 - \Delta\omega)x + \mu_1(\dot{x} - \dot{y}) = 0 \\ &\ddot{y} - (1 - y^2)\dot{y} + (1 + \Delta\omega)y + \mu_2(\dot{y} - \dot{x}) = 0 \end{aligned}, \quad \text{(Eqn. 1)}$$

where $0 < \Delta\omega < 1$ is a **normalized relative frequency difference** and $\mu_i \le 0$ describe couplings. Note that x is set to be a "faster" of two oscillators and one should consider different setups in terms of which oscillator (the "fast" one or the "slow" one) has larger coupling.

in case it might be useful, lets set $\mu_1 + \mu_2 = 2\mu$ and $|\mu_1 - \mu_2| = 2\Delta\mu$

Notable cases:

(i) **symmetric coupling** , $\mu_1 = \mu_2$, previously studied and should be used as reference point;

(ii) **fancy RHS**, e.g. $\mu_2 = 0$, so

$$\begin{aligned} &\ddot{x} - (1 - \mu_1 - x^2)\dot{x} + (1 - \Delta\omega)x = \mu_1\dot{y} \\ &\ddot{y} - (1 - y^2)\dot{y} + (1 + \Delta\omega)y = 0 \end{aligned} \quad \text{(Eqn. 2)}$$

which mimic a case of vdP with a "resonant" RHS.

it is not really a resonance, since the amplitude on the right correspond to some function a frequency on the left, but maybe...

Rem II.1 **(Half-sum/half-diff notation)** On par with the previous papers, we provide the same dynamics as in Equation (1) under the following change of variables:

$$\begin{cases} 2u = x + y \\ 2v = x - y \end{cases} \qquad \begin{cases} x = u + v \\ y = u - v \end{cases}, \quad \text{(Eqn. 3)}$$

so we obtain

$$\ddot{u} - (1 - u^2 - v^2)\dot{u} + u - \Delta\omega v + 2uv\dot{v} + 2\Delta\mu\dot{v} = 0$$
$$\ddot{v} - (1 - u^2 - v^2)\dot{v} + v - \Delta\omega u + 2vu\dot{u} + 2\mu\dot{v} = 0 \qquad \text{(Eqn. 4)}$$

with the highlighted term concentrates the asymmetry of the coupling.

III. Synchronization

All of our consideration and defined entities are well-posed only for synchronized pair of oscillators. Specifically, one normally distinguishes two types of synchronization, in **frequency** and **phase** . Recall that in the case of symmetric coupling $\mu_1 = \mu_2$, the synchronization seemed to be joint, i.e. frequency synchronization necessarily implied phase synchronization and vice versa.

it is worth revisiting it around the threshold, we haven't properly tested it in the "battle" region

Rem III.2 **(Synchronization inequality from simpler times)** Under specific simplifying assumptions, one may reduce symmetrically coupled vdP system (Equation (1)) to a simple Kuramoto model where the synchronization is achieved if and only if $\mu > \Delta\omega$; previously, we inherited that property for the symmetric vdP system which seeemed to hold experimentally.

At the same time, in the nonsymmetric Kuramoto case, assuming initial frequencies Ω_1 and Ω_2, the system is synchronized if

$$\mu_1 > \Omega_1 - \Omega_0 \text{ and } \mu_2 > \Omega_2 - \Omega_0$$

where Ω_0 is the common synchronized frequency, $\Omega_0 = \frac{\mu_1\Omega_1 + \mu_2\Omega_2}{\mu_1 + \mu_2}$. Note that this principle may be expected to largely translate to the vdP case with certain caveats; specifically, it is challenging to redefine Ω_0 since even for $\mu_1 = \mu_2 = \mu$ the common frequency was shown to be dependent on μ, $\Omega_0 = \Omega_0(\mu)$, albeit this dependence is rather small.

probably worth recalling some series expansions here

.

III.I Barcode diagrams

Here we formulate a straightforward method to test the synchronization of two oscillators, Equation (1).

We base our estimates on the relative placement of local minima of $\{x(t), y(t)\}$; namely,

(i) solve the system (1) for long-enough time (specifically, we solve for $[0; T] = [0; 2\pi N]$, $N = 100$); this length is based on the notion that the first order approximation of the isolated vdP frequency is 1, so the first order period is 2π ;

here a reference for the series decomposition should be

(ii) for each oscillator $\{x(t), y(t)\}$ we extract position of local minima $\{\tau_x^{(i)}\}$ and $\{\tau_y^{(i)}\}$;

one can do this with preexisting tools, but the fact is that ODEsolver computes an approximation of $\dot{x}$ and $\dot{y}$, so one can extract minima from them

(iii) then one can define *empirical* frequencies as $\left\{\Omega_x^{(i)}\right\} = \left\{\frac{1}{\tau_x^{(i+1)} - \tau_x^{(i)}}\right\}$ and $\left\{\Omega_y^{(i)}\right\} = \left\{\frac{1}{\tau_y^{(i+1)} - \tau_y^{(i)}}\right\}$ respectively; we measure the standard deviation of the frequency sequences $\left\{\Omega_x^{(i)}\right\}$ over the last 20 minima (relative to the mean value) to ascertain synchronization;

(iv) finally, we posit that the phase difference $\Delta\varphi$ never exceeds half a period, so in order to calculate it, we find for each minimum $\tau_x^{(i)}$ the closest minimum $\tau_y^{(j)}$; then, we say that $\Delta\phi^{(i)} = \left|\tau_x^{(i)} - \tau_y^{(j)}\right|$ and compute the stability of the sequence in the same manner.

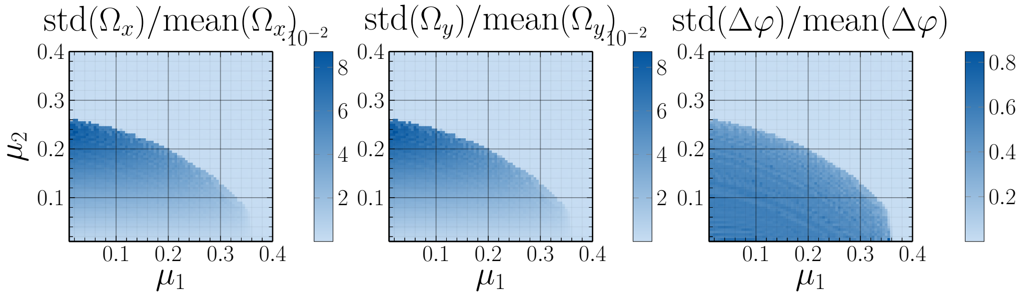

Figure III.1: Synchronization of oscillators (Equation (1)) in frequency (left and center panes) and in phase (right pane). $\Delta\omega = 0.2$, integration time is set to $N = 100$.

We provide the stabilization of frequency and the phase difference in Figure III.1. It is worth pointing out that:

- point $\mu_1 = \mu_2 = \Delta\omega$ lends on the boundary, upholding observed behaviour in the symmetric case;
- in the case of $\mu_1 = 0$, Kuramoto synchronization (see Remark III.2) requires $\mu_2 \geq \Delta\omega$ (and vice versa), which is evidently too restrictive for the vdP case;
- phase and frequency synchronizations remain joint and are achieved simultaneously;
- the boundary is impressively pronounced, so one may at least attempt to speculate about the actual function and synchronization inequality.

References