On the behavior of some Hamiltonian impact systems

Lars Becker, Steven Elliott, Benjamin Firester, Segev Gonen Cohen

July 2018

Abstract

Hamiltonian impact systems model diverse real world dynamics of physical systems and have rich applications in many fields. In this investigation we consider a specific Hamiltonian system with two degrees of freedom and elastic collisions wherein one d.o.f.'s collision depends on the state of the other. We derive the explicit interval exchange maps in the case of two linear oscillators and an algorithm to derive it in the nonlinear case and for potentially multiple walls. We also prove the topological equivalence between the most general case to billiards on a rectangle.

1 Introduction

A dynamical system describes how an object's state changes. Hamiltonian systems with 1 degree of freedom[i] are completely integrable[ii] meaning that the dynamics are completely understood in the most general case. We deal with a 2 degrees of freedom system with the addition of elastic impacts—impacts that mirror reflect the trajectory and preserve energy. Our specific system looks at 2 independent oscillators with hard impacts in each system dependent on the position of the other.

We consider the phase space of our total system- the space of conjugate momenta and positions, which for our case is simply (q_1, q_2, p_1, p_2). In phase space one can trace the possible trajectories of our system, and since dynamical systems are deterministic, it is important to note that no trajectories intersect. We construct a corner in configuration space (q_1, q_2), a projection of the total phase space. Colliding with it mirrors the trajectory in configuration space along the line normal to the wall at the impact location.

[i]Since we are dealing with autonomous mechanical Hamiltonians, we have a degree of freedom for every particle in each dimension—the dimension of the phase space is double the number of degrees of freedom.

[ii]Completely integrable Hamiltonian in this context means that the system has n independent, globally defined first integrals of motion (a function of a trajectory that is constant over time) and thus the motion is solvable.

2 Setup and Definitions

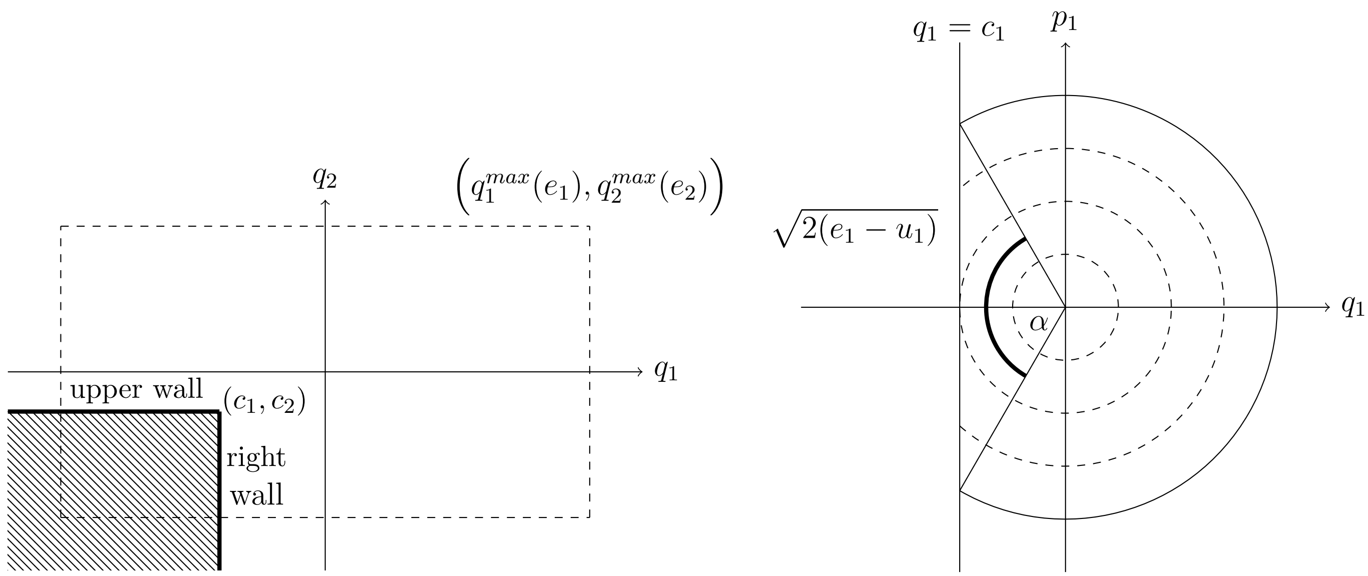
We consider a general, autonomous Hamiltonian H_{tot} which is separable into two independent Hamiltonian systems:

$$H_{tot} = H_1(q_1, p_1) + H_2(q_2, p_2) = \frac{p_1^2}{2m_1} + V(q_1) + \frac{p_2^2}{2m_2} + V(q_2)$$

Our equations of motion are:

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \text{ and } \dot{q}_i = \frac{\partial H}{\partial p_i}$$

We let e_i be the (constant) energy of each system. Without loss of generality, the coordinates are chosen such that the point of lowest potential of the system is at $(0, 0)$, and we define $q_i^{max}(e_i)$ as the farthest displacement of q_i from the equilibrium position. Within the configuration space, we construct a wall in the region $q_1 \leq c_1$ and $q_2 \leq c_2$. We define u_i to be the potential energy of system i at c_i. When impacted, the trajectory is mirrored in the configuration space so that the colliding momenta is negated and the other is maintained. The flow is discontinuous at these points, where it jumps from (q_1, q_2, p_1, p_2) to $(q_1, q_2, -p_1, p_2)$ or $(q_1, q_2, p_1, -p_2)$, everywhere else it is smooth. No energy is lost in collisions and they are instantaneous. We also define $\{x\} := x - \lfloor x \rfloor$, being the fractional part of x. We let all the masses be 1 for simplicity.



(a) Our configuration space (b) Phase space when impacting

Figure 1: Configuration space and phase space for the linear oscillators

3 Transformation to action-angle variables

In order to simplify a more general case, we can apply the canonical transformation $(q, p) \mapsto (I, \theta)$ where

$$I = \frac{1}{2\pi} \oint p \, dq \text{ and } \theta = \int \frac{\partial H}{\partial I} dt = \int \omega(I) dt = \omega(I)t + \theta_0 \mod 2\pi$$

Since H_i is constant, the magnitude of the momenta are determined. For any e_i, we can therefore represent a point along the trajectory by some 4-tuple, adding in the direction of the momenta:

$$(q_1, q_2, \psi_1, \psi_2) \in [-q_1^{max}(e_1), q_1^{max}(e_1)] \times [-q_2^{max}(e_2), q_2^{max}(e_2)] \times \{-1, 1\}^2 = \mathcal{C}$$

where ψ just represents the signs of the momenta. We can also define some inverse function $\theta_i : [-q_i^{max}, q_i^{max}] \mapsto [0, \pi]$, which assigns every position in q-space the corresponding angle between 0 and π. This is well defined because although there are two possible angles for each q_i value, one will always be 2π minus the other by symmetry, and so only one will be in $[0, \pi]$.

4 Oscillators

A separable, autonomous, smooth Hamiltonian system will oscillate around a fixed point q_i^{fix} in the i direction when $\frac{d^2V_i}{dq_i^2} > 0$ in the neighbourhood of that q_i^{fix}.

Example: Linear oscillators

The system involves 2 harmonic oscillators with each system having a corresponding constant angular velocity ω_i. We have

$$H_{tot} = H_1 + H_2 = \frac{p_1^2}{2} + \frac{\omega_1^2}{2} q_1^2 + \frac{p_2^2}{2} + \frac{\omega_2^2}{2} q_2^2$$

Within the proper range of impact, an angle is removed from the phase space of each system. We define these angles as α and β for systems 1 and 2 respectively (see figure: 1).

There are 2 possible periods to consider for each system, defined as follows: T_i is the period of an unimpeded orbit, whilst t_i is the period of an orbit with a collision. By considering the phase space of our systems it is shown in figure 1 that

$$\alpha = 2 \ tan^{-1}(\frac{\sqrt{2(e_1 - u_1)}}{-c_1}); \ \beta = 2 \ tan^{-1}(\frac{\sqrt{2(e_2 - u_2)}}{-c_2})$$

We also know that

$$T_1 = \frac{2\pi}{\omega_1}; \ T_2 = \frac{2\pi}{\omega_2}; \ t_1 = \frac{2\pi - \alpha}{\omega_1}; \ t_2 = \frac{2\pi - \beta}{\omega_2}$$

We define our Poincaré section to be the section where $\theta_1 = 0$. This will tell us about the dynamics of θ_2 upon every return to $\theta_1 = 0$. We consider solely the the dynamics of θ_2, which completely describes the dynamics of the entire system, as e_1, e_2 and θ_1 are all constrained, hence our problem is greatly simplified. By considering the mapping $(\theta_1, \theta_2) \mapsto (q_1, q_2)$ given by $q_1 = q_1^{max} cos(\theta_1)$ and $q_2 = q_2^{max} cos(\theta_2)$ we see that this section corresponds to $q_1 = q_1^{max}$, which is the right hand boundary of a projection of a trajectory to configuration space. We investigate the return map of our system and we see that there are three cases of the map, determined uniquely by the initial value of θ_2. The first case[iii] is when the particle hits the right hand wall and rebounds. Because $\dot{q}_1$ is negative after the collision, the particle must return to our section without any further collisions, and so it is trivial to find the return map in this possibility:

$$\theta_2^{n+1} = \theta_2^n + t_1\omega_2$$

Furthermore, we note that the particle collides with the right wall when $\theta_1 = \pi - \dfrac{\alpha}{2}$ and $\theta_2 \in (\dfrac{2\pi - \beta}{2}, \dfrac{2\pi + \beta}{2})$.[iv] Reversing their trajectory, we can explicitly specify the set of all θ_2^n on the section whose orbits will collide with the right wall before returning to the section.

Cases II and III are scenarios in which the particle does not impact the right wall and bounces potentially one or more times on the upper wall before exiting the region above the wall and returning to the Poincaré section. The particle is above the wall if $\pi - \dfrac{\alpha}{2} < \theta_1 < \pi + \dfrac{\alpha}{2}$. Considering only the locations of impact with the wall (for which θ_1 does not leave this segment) one gets the subsequent auxiliary angle of the form

$$\theta_1^{n+1} = \theta_1^n + t_2\omega_1$$

The return map can only be applied at most $\left\lfloor \dfrac{\alpha}{t_2\omega_2} \right\rfloor$ times because α is the distance to travel and $t_2\omega_2$ is the distance between impacts, and so we have two possibilities—the map is applied $\left\lfloor \dfrac{\alpha}{t_2\omega_2} \right\rfloor$ times and the number of collisions is $\left\lfloor \dfrac{\alpha}{t_2\omega_2} \right\rfloor + 1$ (case II) or the particle could exit the region above the wall before it collides with it again, hence missing the last collision (case III).[v]

In summary, we present the general intervals and return maps, both of which are mod 2π.

Case I:

$$\theta_2^n \in (\frac{2\pi - \beta}{2} - \frac{\omega_2}{2}t_1, \frac{2\pi + \beta}{2} - \frac{\omega_2}{2}t_1)$$

Or equivalently:

$$\theta_2^n \in (\frac{\omega_2}{2}(t_2 - t_1), \frac{\omega_2}{2}(t_2 - t_1) + \beta)$$

[iii]We consider cases where the parameters are chosen such that collisions with both walls are possible

[iv]Note that our intervals are open as the behaviour of corner collisions is undefined and the set of starting positions leading to this situation has 0 measure.

[v]These cases cover the entire range except for 3 distinct cases in which the corner is struck. These can be avoided as there is no theoretical dynamics that occur here and this set has measure 0.

$$\theta_2^{n+1} = \theta_2^n + t_1\omega_2$$

Case II:

$$\theta_2^n \in (\frac{2\pi - \beta}{2} - \frac{\omega_2}{2}t_1 - \{\frac{\alpha}{t_2\omega_1}\}t_2\omega_2, \frac{2\pi - \beta}{2} - \frac{\omega_2}{2}t_1)$$

Or equivalently:

$$\theta_2^n \in (\ \frac{\omega_2}{2}(t_2 - t_1) - \{\frac{\alpha}{t_2\omega_1}\}t_2\omega_2,\ \frac{\omega_2}{2}(t_2 - t_1))$$

$$\theta_2^{n+1} = \theta_2^n + t_1\omega_2 + \beta + \{\frac{\alpha}{t_2\omega_1}\}t_2\omega_2 + 2\pi\left\lfloor\frac{\alpha}{t_2\omega_2}\right\rfloor$$

Case III:

$$\theta_2^n \in (\frac{2\pi + \beta}{2} - \omega_2\frac{t_1}{2}, \frac{2\pi - \beta}{2} - \omega_2\frac{t_1}{2} - \{\frac{\alpha}{t_2\omega_1}\}t_2\omega_2)$$

Or equivalently:

$$\theta_2^n \in (\ \frac{\omega_2}{2}(t_2 - t_1) + \beta, \frac{\omega_2}{2}(t_2 - t_1) - \{\frac{\alpha}{t_2\omega_1}\}t_2\omega_2)$$

$$\theta_2^{n+1} = \theta_2^n + t_1\omega_2 + \{\frac{\alpha}{t_2\omega_1}\}t_2\omega_2 + 2\pi\left\lfloor\frac{\alpha}{t_2\omega_2}\right\rfloor$$

5 Equivalence to billiards

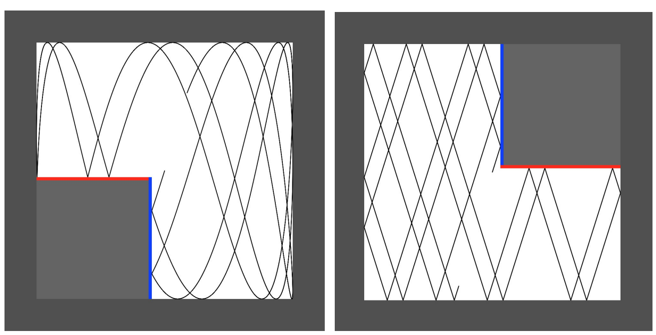


Figure 2: A simulation of the configuration space of the system with its corresponding matching billiard setup in a rectangle. There is a one-to-one correspondence between impacts on each wall in both systems and a bijection between them.

We now apply the transformation from phase space to angle space to our problem (See section 3). The flow on the torus is then given by $\theta_i(t) = \theta_i(0) + t\omega_i$ unless the trajectory hits a wall. Because of the symmetry of the general solution of a one degree of freedom Hamiltonian, the $q_i(\theta_i) = q_i(2\pi - \theta_i)$ and $p_i(\theta_i) = -p_i(2\pi - \theta_i)$. Therefore the singularities become jumps in one of the angles, explicitly (θ_1, θ_2) to $(\theta_1, 2\pi - \theta_2)$ or $(2\pi - \theta_1, \theta_2)$. They still only occur at the borders of rectangles (by definition).
Lines parallel to the coordinate axes in configuration space have constant q_i values, and hence constant θ_i values (since e_1 and e_2 are fixed), which correspond to straight lines parallel to the axes of the torus. Since rectangles are simply circumscribed by four straight lines,

rectangles parallel to the axes in configuration space will map to rectangles on the torus.

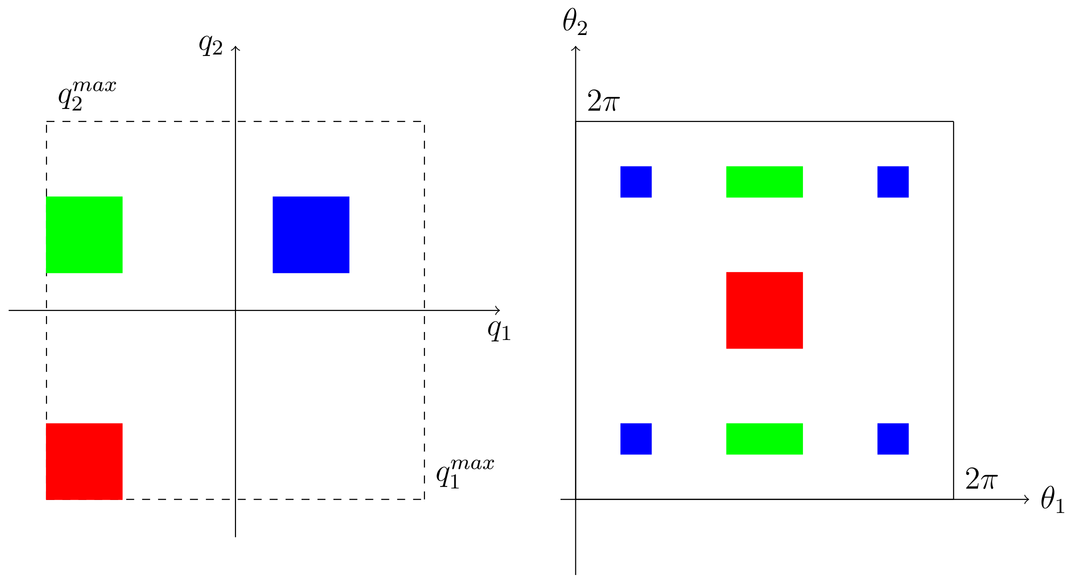


Figure 3: Configuration space showing 3 example walls and their angle-angle torus representation

The rectangles on the torus will be symmetric over the lines $\theta_1 = \pi$ and $\theta_2 = \pi$, because if $\pi - \delta$ is inside a wall, then $\pi + \delta$ is also inside it. Therefore, a rectangle in configuration space maps to either 1, 2 or 4 rectangles on the torus. If the rectangle touches the $q_1 = -q_1^{max}$ border of the configuration space, rectangles will merge on the torus and be centred on $q_1 = \pi$, and similarly for q_2 (see fig. 3).

Now we map the torus $[0, 2\pi) \times [0, 2\pi)$ back to a square a quarter its size by applying the function $\phi = f(\theta) = \pi - |\pi - \theta|$ to each coordinate separately. This just folds the 3 squares $[0, \pi] \times (\pi, 2\pi)$, $(\pi, 2\pi) \times [0, \pi]$ and $(\pi, 2\pi) \times (\pi, 2\pi)$ to $[0, \pi] \times [0, \pi]$. Because of the symmetry of the rectangles, this transformation maps rectangles onto their symmetric copies. Since the map is just a reflection for each of the squares, the flow ϕ will still be linear, now with velocities $(\pm\omega_1, \pm\omega_2)$, where the signs depend on the square on the torus in which the flow takes place. If the flow ϕ hits the border of the square, the corresponding flow θ moves to another square, so ϕ changes direction, specifically the velocity perpendicular to the border gets negated. This again is just a elastic reflection at the border of the square. The singularities that occurred in the flow θ on the torus now become jumps from $(\phi_1, \phi_2, \omega_1, \omega_2)$ to $(\phi_1, \phi_2, \omega_1, -\omega_2)$ or $(\phi_1, \phi_2, -\omega_1, \omega_2)$, which is again just an elastic reflection on the rectangles. So ϕ evolves the same way as billiards. Now the position on the torus uniquely determines the state on the square, and the position on the square together with the direction in which the particle is moving, uniquely determines its position on the torus.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "f_t"] \arrow[d, leftrightarrow, "\mathcal{T}"] & \mathcal{C} \arrow[d, leftrightarrow, "\mathcal{T}"] \\
\mathcal{B} \arrow[r, "g_t"] & \mathcal{B}
\end{tikzcd}$$

Figure 4: Commutative diagram showing the topological equivalence of our system restricted to a particular level set of e_1 and e_2 and billiards in a rectangle. $\mathcal{C}(e_1, e_2)$ is the signed configurations space (See section 3) with f_t being the flow induced by the Hamiltonian on it. $\mathcal{B}$ represents the particular corresponding billiard space with punctured rectangles with g_t being the billiard flow. $\mathcal{T}(e_1, e_2)$ is the bijection between these two spaces.

Since the movement on the torus and in phase space can be transformed into each other (See section 3) it follows that the above diagram commutes (See fig. 4). This means that our problem is reduced to billiards on a square punctured by rectangles, which has known dynamics.[vi]

6 Derivation of return maps

We will now derive the interval exchange maps of these billiards for different configurations of rectangles by looking at the torus representation.

Our goal is to derive a return map for a section $\theta_1 = const.$ This maps a θ_2^n to a θ_2^{n+1} value of the next intersection of the trajectory starting at θ_2^n with the section. For now, we will only look at configurations of rectangles that don't intersect $\theta_1 = 0$. To find the return map we split the interval $[0, 2\pi)$, which θ_1 has to cross before returning to our section, into smaller intervals. We use auxiliary sections $\theta_1 = const.$ placed at all the right or left borders of walls on the torus. We will denote them by A, B, ..., corresponding to $\theta_1 = a, b, ...$ with $a < b < ...$

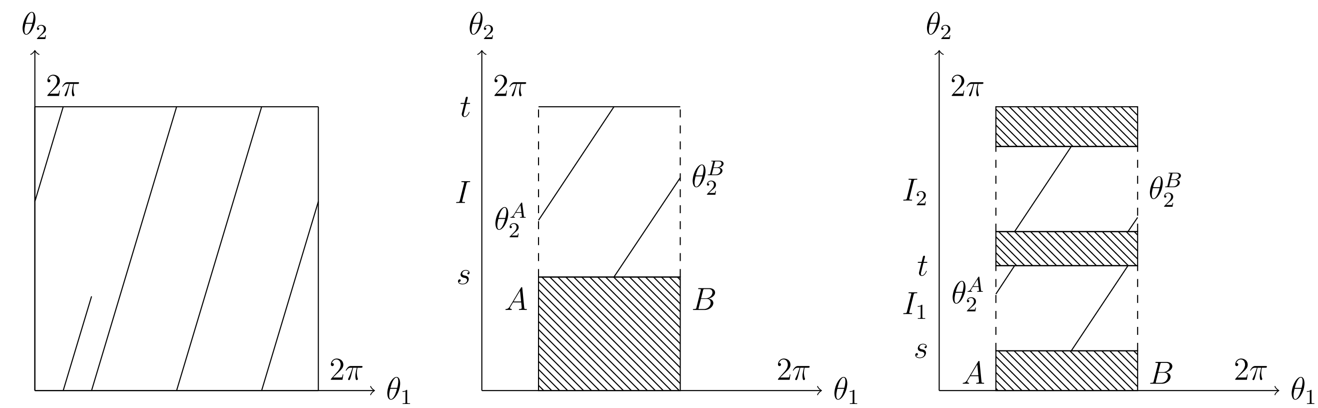
In the following we consider maps $f^I_{A,B} : [0, 2\pi) \mapsto [0, 2\pi)$, which represent the evolution of an angle $\theta_2 \in I = (s, t)$ between the sections A and B on the torus $[a, b) \times [0, 2\pi)$ (see fig: 5), if θ_2 has to stay in I (meaning it jumps if it hits one of the borders and continues on the other side). This is, restricted to I, just a shift by $m(t-s) \bmod |t-s|$, where $m = \frac{\omega_2}{\omega_1}$ is the gradient of the movement on the torus (when mapped to the plane as shown). Taking into

[vi]Athreya, J. S., Eskin, A. & Zorich, A. Right-angled billiards and volumes of moduli spaces of quadratic differentials on $\mathbb{C}P^1$. ArXiv12121660 Math (2012).

account the borders of $I = (s,t)$ this becomes

$$f^I_{A,B}(\theta_2^B) = s + (\theta_2^A - s + (b-a)m) \bmod |t-s|$$

This is just an interval exchange map swapping the intervals $(s, t - \{(b-a)\frac{m}{|t-s|}\}|t-s|)$ and $(t - \{(b-a)\frac{m}{|t-s|}\}|t-s|, t)$.



(a) motion on torus

(b) motion restricted to interval

(c) motion restricted to two symmetrical intervals

Figure 5: exchange maps

This map is applicable whenever θ_2 is restricted to some connected interval, which means that it is only hitting one wall in q-space. In the general case a gap between two rectangles in q-space does not correspond to a connected interval on the torus, but to two intervals lying symmetric to the line $\theta_2 = \pi$. We can however apply the function to the interval we get when we glue those two together. If we call the intervals $I_1 = (s,t)$ and $I_2 = (2\pi - t, 2\pi - s)$, this means explicitly that we shift I_2 down by $2\pi - 2t$, then apply $f^{I_1 \cup I_2}_{A,B}$ restricted to $I_1 \cup I_2$ and then move I_2 back upwards by $2\pi - 2t$. It is easy to see that this will result in an interval exchange map which cyclically permutes 4 intervals. The explicit intervals are as follows where if $m(b-a) \bmod 2(t-s)$ is bigger than $(t-s)$, then M_i goes to N_{i+3}, and otherwise, M_i goes to N_{i+1}, where indices are mod 4:

$$M_0 = (s, t - \{\frac{b-a}{t-s}m\}(t-s))$$

$$M_1 = (t - \{\frac{b-a}{t-s}m\}(t-s), t)$$

$$M_2 = (2\pi - t, 2\pi - s - \{\frac{b-a}{t-s}m\}(t-s))$$

$$M_3 = (2\pi - s - \{\frac{b-a}{t-s}m\}(t-s), 2\pi - s)$$

They map to

$$N_0 = (s, s + \{\frac{b-a}{t-s}m\}(t-s))$$

$$N_1 = (s + \{\frac{b-a}{t-s}m\}(t-s), t)$$

$$N_2 = (2\pi - t, 2\pi - t + \{\frac{b-a}{t-s}m\}(t-s))$$

$$N_3 = (2\pi - t + \{\frac{b-a}{t-s}m\}(t-s), 2\pi - s)$$

We will call that function $g^{I_1}_{A,B} = g^{I_2}_{A,B}$. Furthermore we can define $g^{I}_{A,B} = f^{I}_{A,B}$ for connected intervals I that represent a whole gap. Then

$$\theta_2^B = g^I_{A,B}(\theta_2^A)$$

for $\theta_2^A \in I$ if I is one of the intervals representing a gap and

$$\theta_2^B = \theta_2^A$$

otherwise. Now it is possible to derive the return map by composition of these maps. In the simplest case of one corner we will get three maps: A shift of the $[0, 2\pi)$ by some value mod 2π (a rotation), a rotation of some sub interval, and last of all the same rotation as in the first step. This yields a interval exchange map as shown in fig. 6.

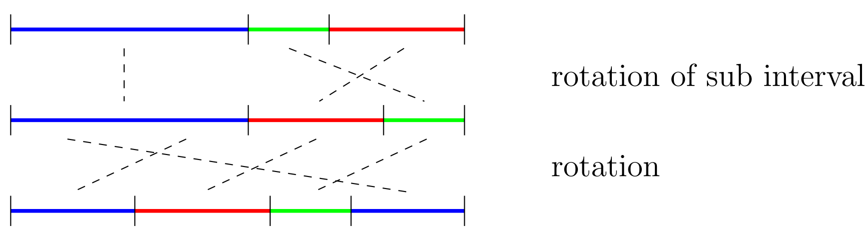
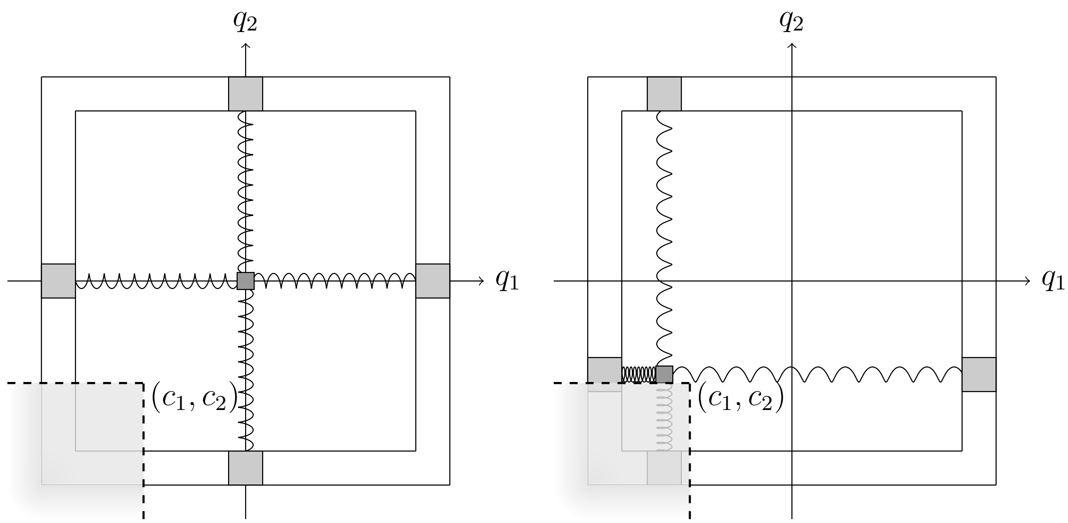


Figure 6: Interval exchange map and rotation in a simple corner

In the simplest case of just one corner, we get two auxiliary sections A and B, placed at $\theta_1 = \pi - \frac{\alpha}{2}$ and $\theta_1 = \pi + \frac{\alpha}{2}$. The map from our basic section to A is then a rotation of $[0, 2\pi)$, so is the map from B back to our basic section. Between A and B, a certain interval stays fixed, while its complement performs a rotation. This and the second rotation are shown in (fig. 6), which does not show the first step. Including the first step and looking at the unit circle just as the interval $[0, 2\pi)$ we would get a interval exchange map with five intervals. (Notice that when performing a rotation one interval gets split at the preimage of 0). However, the actual motion viewed on the unit circle is different just for three intervals, keeping with the results of section 3. The permutation in the one corner case depends on which interval is split up by the rotation, but is in general just the composition of a 5-cycle $(2, 3, 4, 5, 1)^n$and a 3- or 2-cycle containing successive intervals. Whether it is a three or a two cycle depends on if the interval which is invariant between A and B or some other interval is split.

7 Applications

Because oscillators and impact systems are widespread, this work has many potential applications; for example, modelling coupled pendulums or coupled springs. A physical model of the system is shown below. In the model, a block is connected to coupled, bounded springs in the q_1 and q_2 directions, which in turn are connected to blocks that can move along chambers; the center block extends into the plane above the chambers. The wall, marked by a dotted line and set at (c_1, c_2), is also on the plane above the chambers. The side blocks can move under the wall and do not collide with it, while the center block hits the wall and bounces off. As the block bounces off the wall, it moves as predicted earlier in the paper.



(a) Basic state (b) Impact with wall

Figure 7: Real life representation of the system