COM S 476/576 Homework 2 Solution

Problem 3.1 Define a semi-algebraic model that removes a triangular "nose" from the region shown in Figure 3.4.

Solution Suppose that the vertices of the triangular nose in the **counterclockwise** order is are given by $v_5 = (x_5, y_5)$, $v_6 = (x_6, y_6)$, and $v_7 = (x_7, y_7)$. Define

$$\begin{aligned} f_5(x,y) &= a_5x + b_5y + c_5 \\ f_6(x,y) &= a_6x + b_6y + c_6 \\ f_7(x,y) &= a_7x + b_7y + c_7 \end{aligned}$$

such that $f_i(x,y) < 0$ for all points to the left of the edge from v_i to v_{i+1}, $i \in \{5,6,7\}$, with $v_8 = v_5$. It can be shown that the constants are given by

$$\begin{aligned} a_i &= y_{i+1} - y_i \\ b_i &= x_i - x_{i+1} \\ c_i &= x_{i+1}y_i - x_iy_{i+1} \end{aligned}$$

For each $i \in \{5,6,7\}$, define the algebraic primitives

$$H_i = \{(x,y) \in \mathcal{W} \mid f_i(x,y) \leq 0\}. \tag{1}$$

Note that we define the vertices v_5, v_6, v_7 in the counterclockwise order because we want to get algebraic primitives for all points **inside** of the triangular nose. The triangular nose is then given by $\mathcal{O}_{nose} = H_5 \cap H_6 \cap H_7$.

Therefore, the model that removes a triangular nose from the region shown in Figure 3.4 is given by $\mathcal{O} = (H_1 \cap H_2 \cap H_3 \cap H_4) \setminus (H_5 \cap H_6 \cap H_7)$, where $H_1, \ldots, H_4$ are defined as in Example 3.1.

Problem 3.7 Consider the articulated chain of bodies shown in Figure 3.29. There are three identical rectangular bars in the plane, called $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$. Each bar has width 2 and length 12. The distance between the two points of attachment is 10. The first bar, $\mathcal{A}_1$, is attached to the origin. The second bar, $\mathcal{A}_2$, is attached to $\mathcal{A}_1$, and $\mathcal{A}_3$ is attached to $\mathcal{A}_2$. Each bar is allowed to rotate about its point of attachment. The configuration of the chain can be expressed with three angles, $(\theta_1, \theta_2, \theta_3)$. The first angle, θ_1, represents the angle between the segment drawn between the two points of attachment of $\mathcal{A}_1$ and the x-axis. The second angle, θ_2, represents the angle between $\mathcal{A}_2$ and $\mathcal{A}_1$ ($\theta_2 = 0$ when they are parallel). The third angle, θ_3, represents the angle between $\mathcal{A}_3$ and $\mathcal{A}_2$. Suppose the configuration is $(\pi/4, \pi/2, -\pi/4)$.

(a) Use the homogeneous transformation matrices to determine the locations of points a, b, and c.

(b) Characterize the set of all configurations for which the final point of attachment (near the end of $\mathcal{A}_3$) is at $(0,0)$ (you should be able to figure this out without using the matrices).

Solution

(a) The 2D homogeneous transformation matrix is given by

$$T_i = \begin{bmatrix} \cos\theta_i & -\sin\theta_i & a_{i-1} \\ \sin\theta_i & \cos\theta_i & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

With $\theta_1 = \pi/4$, $\theta_2 = \pi/2$, $\theta_3 = -\pi/4$, $a_0 = 0$, $a_1 = a_2 = 10$, we get

$$\begin{aligned}
T_1 &= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
T_2 &= \begin{bmatrix} 0 & -1 & 10 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
T_3 &= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 10 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
T_1T_2 &= \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} & 10/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} & 10/\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix} \\
T_1T_2T_3 &= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 20/\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

Finally, we obtain the locations of a, b, and c as follows.

$$\begin{aligned}
a &= T_1 \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/\sqrt{2} \\ 5/\sqrt{2} \\ 1 \end{bmatrix} \\
b &= T_1T_2 \begin{bmatrix} 11 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 21/\sqrt{2} \\ 1 \end{bmatrix} \\
c &= T_1T_2T_3 \begin{bmatrix} 11 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 11 + 20/\sqrt{2} \\ 1 \end{bmatrix}
\end{aligned}$$

(b) The configurations such that the final point of attachment is at $(0,0)$ need to be such that $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$ form an equilateral triangle. (The triangle is necessarily equilateral because all the sides have the same length.) This happens when $\theta_2 = \theta_3 = 2\pi/3$ or $\theta_2 = \theta_3 = -2\pi/3$. In other words, the set of all the configurations such that the final point of attachment is at $(0,0)$ is given by

$$\left\{(\theta_1, \theta_2, \theta_3) \in [0, 2\pi) \times [0, 2\pi) \times [0, 2\pi) \mid \theta_2 = \theta_3 = 2\pi/3 \text{ or } \theta_2 = \theta_3 = 4\pi/3\right\}$$