Quantitatative Macroeconomics.
Homework 2.

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I. Computing Transitions in a Representative Agent Economy.

Exercise 1

a) Compute the steady-state. Choose z to match an annual capital-output ratio of 4, and an investment-output ratio of .25.

$$\max_{\{c_t\}_{t=0}^{\infty}} \quad E_0\{\textstyle\sum_{t=0}^{\infty} \beta^t u(c_t)\} \tag{1}$$

s.t. $\quad c_t + i_t = y_t$

$y_t = k_t^{1-\theta}(zh_t)^\theta$

$i_t = k_{t+1} - (1-\delta)k_t$

Merging the constraints:

$k_t^{1-\theta}(zh_t)^\theta + (1-\delta)k_t - k_{t+1} - c_t \geq 0$

Set up the Lagrangian:

$\mathscr{L}(c_t, k_{t+1}; \lambda_t) = E_0\{\sum_{t=0}^{\infty} \beta^t \ln c_t + \lambda_t[k_t^{1-\theta}(zh_t)^\theta + (1-\delta)k_t - k_{t+1} - c_t]\}$

FOC-s:

$\frac{\partial \mathscr{L}(c_t,k_{t+1};\lambda_t)}{\partial c_t} : \frac{\beta^t}{c_t} = \lambda_t$

$\frac{\partial \mathscr{L}(c_t,k_{t+1};\lambda_t)}{\partial k_{t+1}} : \lambda_{t+1}(1-\theta)k_{t+1}^{-\theta} + (1-\delta)\lambda_{t+1} - \lambda_t$

Merging the FOC-s we obtain the Euler equation:

$\frac{c_{t+1}}{c_t} = \beta[(zh_{t+1})^\theta(1-\theta)k_{t+1}^{-\theta} + 1 - \delta]$

In the SS $\quad c_{t+1} = c_t \quad$ and $\quad k_t = k_{t+1}$**:**

$\beta[(zh^*)^\theta(1-\theta)k^{*-\theta}]$

Thus the level of capital in the SS:

$k^* = [\frac{\beta^{-1}-1+\delta}{(zh^*)^\theta(1-\theta)}]^{-\frac{1}{\theta}}$

From the feasibility for $\quad c_{t+1} = c_t \quad$ and $\quad k_t = k_{t+1}$**:**

$c^* = (zh^*)^\theta[\frac{\beta^{-1}-1+\delta}{(zh^*)^\theta(1-\theta)}]^{-\frac{1-\theta}{\theta}} - \delta[\frac{\beta^{-1}-1+\delta}{(zh^*)^\theta(1-\theta)}]^{-\frac{1}{\theta}}$

$y^* = k^{*(}1-\theta)(zh^*)^\theta = [\frac{\beta^{-1}-1+\delta}{(zh^*)^\theta(1-\theta)}]^{-\frac{1-\theta}{\theta}}(zh^*)^\theta$

The capital-to-output ratio:

$\frac{k^*}{y^*} = [\frac{1-\theta}{\beta^{-1}-1+\delta}]$

The investment-to-output ratio:

$\frac{i^*}{y^*} = \delta[\frac{1-\theta}{\beta^{-1}-1+\delta}]$

We want $\frac{k^*}{y^*} = 4$ **and**$\frac{i^*}{y^*} = \frac{1}{4}$**which implies:**

$\delta = \frac{i}{k} = \frac{\frac{1}{4}}{4} = \frac{1}{16}$

$\frac{k}{c} = \frac{\frac{k}{y}}{\frac{c}{y}} = \frac{4}{1-\frac{1}{4}} = \frac{16}{3}$

Which together with assumptions on parameters implies:

$\beta = \frac{4}{1-\theta+4(1-\delta)} \approx .98$

In conclusion, any level of the technology parameter z can support $\frac{k^*}{y^*} = 4$ **and**$\frac{i^*}{y^*} = \frac{1}{4}$**as long as the parameters are specified as above.**

The steady state level of capital, consumption and output assuming $z_0 = 1$**:**

$k_{ss} = 2.454475468264121$

$c_{ss} = 0.46021415029952306$

$y_{ss} = 0.6136188670660306$

b) Double permanently the productivity parameter z and solve for the new steady state.

For the doubled technology parameter $z = 2z_0 = 2$ **:**

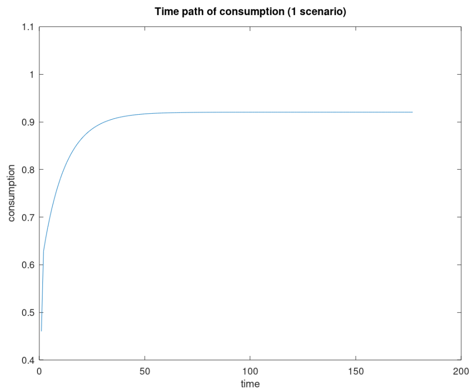
$$k_{2ss} = 4.908950936528243$$

$$c_{2ss} = 0.9204283005990461$$

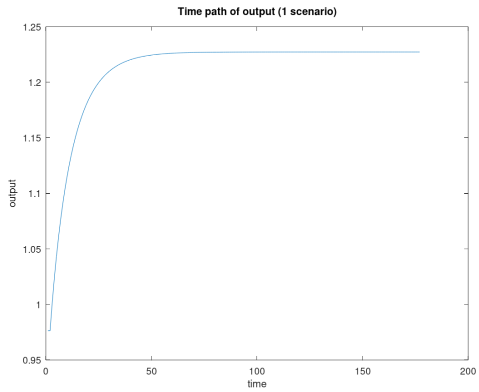
$$y_{2ss} = 1.2272377341320613$$

c) Compute the transition from the first to the second steady state and report the time-path for savings, consumption, labor and output.

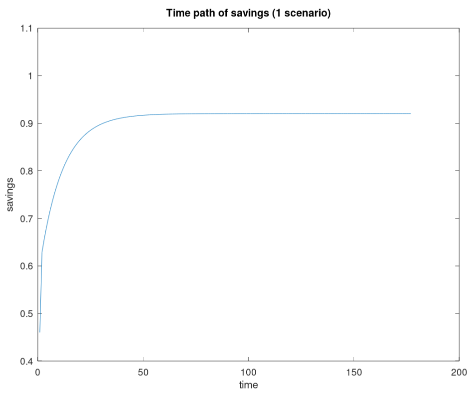
Figure 1: The transition path of aggregate variables after the doubling of productivity



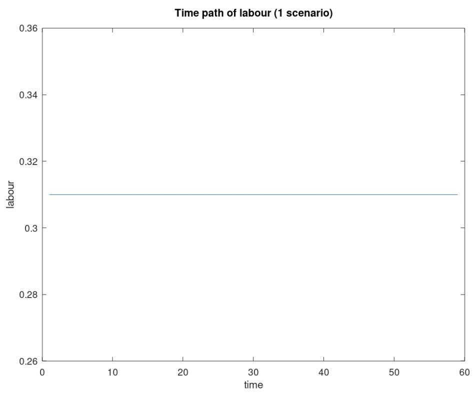
(a) Consumption path to the new SS



(b) Output path to the new SS



(c) Savings path to the new SS

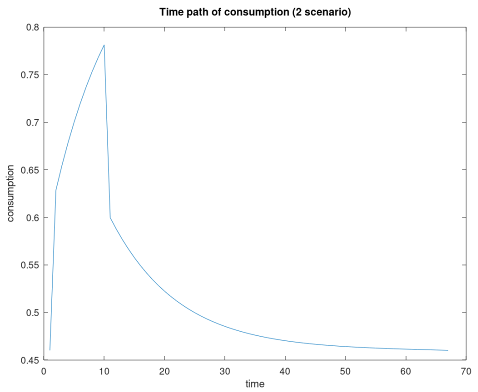


(d) Labour path to the new SS

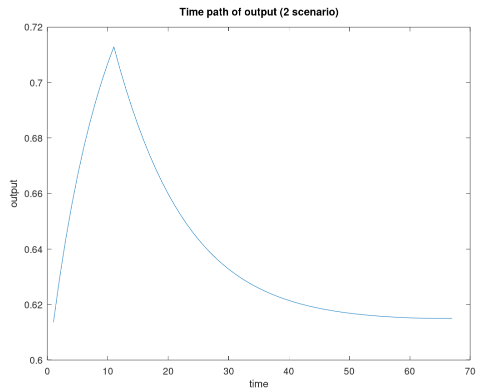
Once the parameter z has doubled permanently, the consumption level jumps initially which is required by the transversality condition (TVC) so that the dynamics of the capital law of motion and Euler equation proceed on the saddle path.The convergence to the new steady state proceeds quickly at the beginning but slows down after the first few periods. It takes the economy about 50 periods to be close to the new steady state.

d) Unexpected shocks. Let the agents believe productivity z_t doubles once and for all periods. However, after 10 periods, surprise the economy by cutting the productivity z_t back to its original value. Compute the transition for savings, consumption, labor and output.)

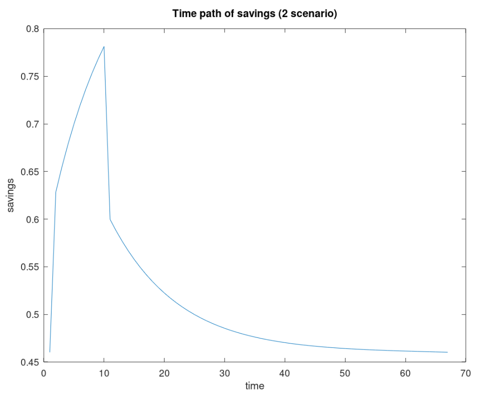
Figure 2: The transition path of aggregate variables after the temporary doubling of productivity



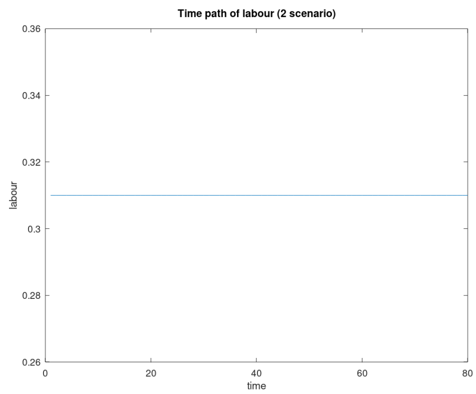
(a) Consumption path to the old SS



(b) Output path to the old SS



(c) Savings path to the old SS



(d) Labour path to the old SS

Once the parameter z has doubled, the time paths of variables of interest remain the same as in the previous point until period 11 when it turns out that the shock was not permanent. The consumption level plunges so that the next period capital sets the agent on the saddle path which is consistent with the TVC condition. The speed of convergence to the old steady state is very fast initially but then it slows down as in the previous point and again it takes rougly 50 periods for the economy to get close to the old steady state.

II. Solve the optimal COVID-19 lockdown model posted in the slides.

Exercise 2

a) Show your results for a continuum of combinations of the $\beta \in [1, 0]$ parameter (vertical axis) and the c(TW)∈[0; 1] parameter (horizontal axis). That is, plot for each pair of β and c(TW) the optimal allocations of H, H_f , H_{nf} , $\frac{H_f}{H}$, output, welfare, amount of infections and deaths. Note that if H = N there is no lockdown, so pay attention to the potential non-binding constraint H < N.
Discuss your results. You may want to use the following parameters:
$A_f = A_nf = 1$; ρ = 1:1, $\kappa_f = \kappa_{nf} = 0.2$, $\omega = 20$, γ= 0.9, $i_o = 0.2$ and N = 1.

The code construction

The maximization problem is solved in the .py code using the FOC-s and 2 cases:
1) $\lambda = 0$ and using the *least squares* function we find the root of the system:

$$\begin{aligned} A_f[\frac{H_f}{Y}]^{-\frac{1}{\rho}} = \kappa_f + \omega(1-\gamma)\beta(HC)2i_0H_f\frac{1}{N} \\ c(TW)A_f[\frac{H_{nf}}{Y}]^{-\frac{1}{\rho}} = \kappa_{nf} \\ \textbf{subject to:} \quad H_f + H_{nf} \leq N \end{aligned}$$

The constraint verification is nested in the root-searching procedure which minimizes the squared distance from zero. I have chosen this method because it is not susceptible to the choice of initial guess and minimizes the distance function over the whole domain of interest. It does not rely on the derivative approximation so it is not susceptible to the curvature of the function.
2) $\lambda \neq 0$ which means that the constraint is bindings. Merging the FOC-s and the labour supply constraint we look for their root of:

$$\frac{\kappa_f + 2\omega(1-\gamma)\beta(HC)\frac{i_0}{N}H_f}{\kappa_{nf}} = \frac{1}{c(TW)}[\frac{H_f}{H_{nf}}]^{(}-\frac{1}{\rho})$$

If we find a candidate solution in both steps, we take stock in the next step of the algorithm. Bounds on H_f, H_{nf} and the labour constraint is verified at all times.

Interpretation

Figures 3 reflect the impact of the parameters β and c_{tw} on the structure of employment in the economy. The higher the probability of infections conditional on human contact β, the lower the labour supply on site (3a), the higher the telework labour supply(3b) and the lower the share of on site worker in aggregate employment (3c). The higher the c_{tw}, that is the lower the cost of telework, the lower the labour supply on site(3a), the higher the telework labour supply(3b) and the lower the share of on site worker in aggregate employment (3c). The results are smooth across the β, c_{tw} matrix and are in line with the intuition.

Figures 4 reflect the impact of the parameters β and c_{tw} on the aggregate outcomes in the economy. For the parametrization $\rho = 1.1$ and $\omega = 20$ the propagation of the disease has negligible impact on aggregate employment(4a). This is due to relatively high substitutability of on-site and teleworking labour force and relatively low utility cost of death. Figures 4b and 4c indicate that the virus infectiousness parameter β has no impact on production and welfare, whereas higher cost of telework (which corresponds to the lower values of c_{tw}) has a very substantial negative impact on both production and welfare.

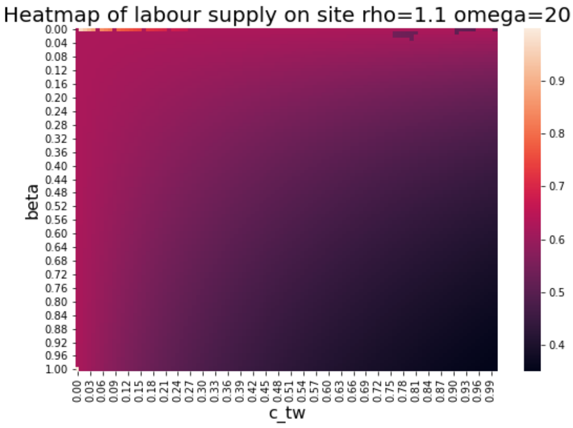
This result may be counterintuitive as one would probably expect that both β and c_{tw} have impact on production and welfare. However, the high value of parameter $\rho = 1.1$ corresponding to the on-site and teleworkers being gross substitutes means that it is relatively easy to mitigate the relatively higher negative impact of human contact by substituting on-site work with telework. Moreover, for any specific level of β the lower values of parameter c_{tw} associated with higher cost of telework and lower productivity have a strong and direct negative impact on output and welfare.

Figures 5 reflect the impact of the parameters β and c_{tw} on the propagation of the virus. The results are completely in line with expectations. Both the higher values of parameter β, which corresponds to the infectiousness of the disease, and the lower values of parameter c_{tw}, which corresponds to high productivity losses caused by moving to telework, are associated with higher numbers of infections and deaths as indicated by the brighter colours in the lower left corners of the figures 5a and 5b.

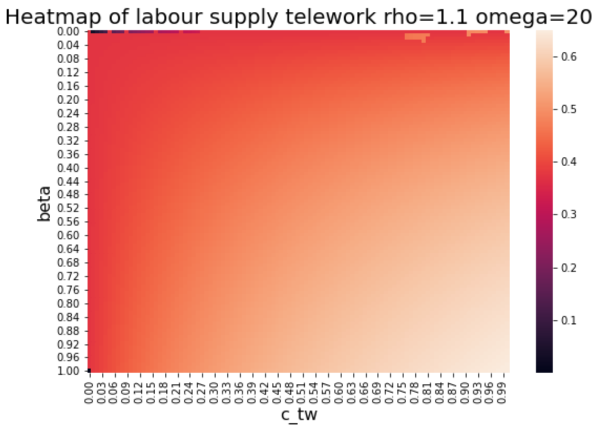
For this parametrization high substitutability of on-site and telework can help mitigate the negative impact of infectiousness in most cases. The disease only becomes particularly difficult to control when it both spreads easily in the workplace and the productivity losses due to telework are very high.

Plots for Exercise 2 a)

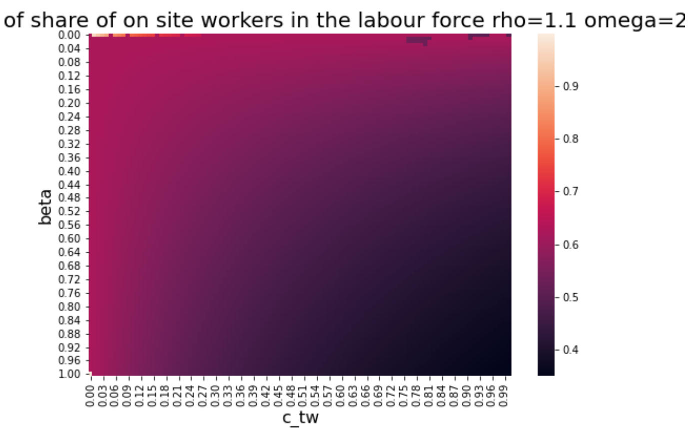
Figure 3: Labour market behaviour for $\rho = 1.1$ and $\omega = 20$



(a) Employment on site for $\rho = 1.1$ and $\omega = 20$

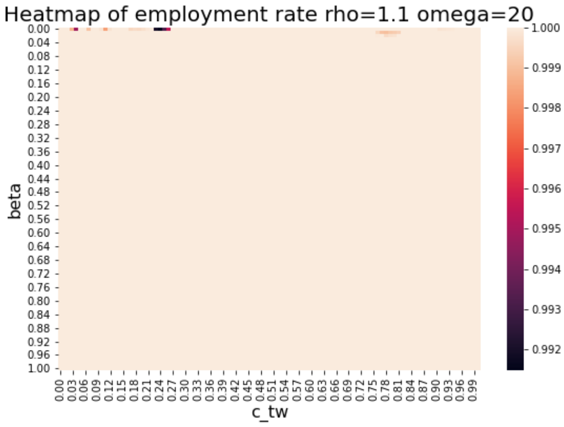


(b) Employment in teleworking sector for for $\rho = 1.1$ and $\omega = 20$

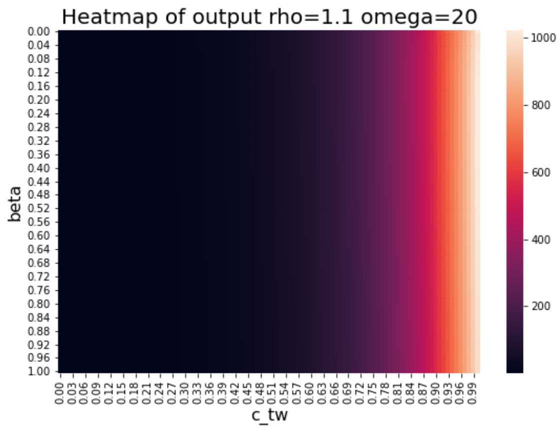


(c) Share of on site workers in the active labour force for $\rho = 1.1$ and $\omega = 20$

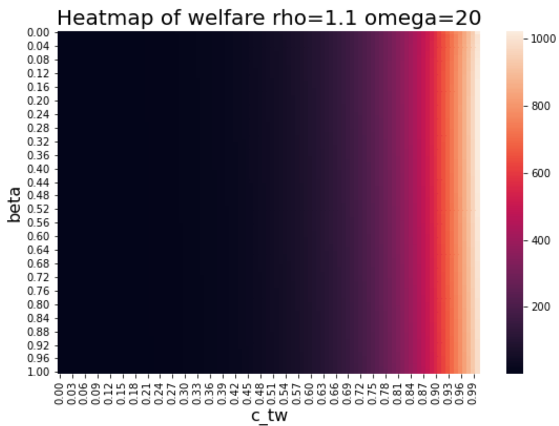
Figure 4: The aggregate measures for $\rho = 1.1$ and $\omega = 20$



(a) Total employment rate for $\rho = 1.1$ and $\omega = 20$

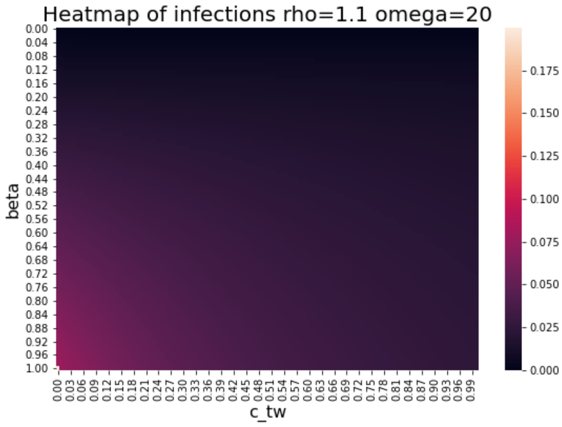


(b) Output for $\rho = 1.1$ and $\omega = 20$

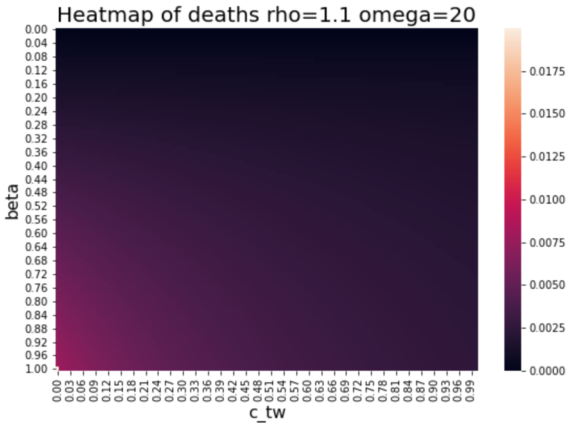


(c) Social planner's welfare function for $\rho = 1.1$ and $\omega = 20$

Figure 5: The COVID stats for $\rho = 1.1$ and $\omega = 20$



(a) Infections for $\rho = 1.1$ and $\omega = 20$



(b) Deaths for $\rho = 1.1$ and $\omega = 20$

b) What happens to your results when you increase (decrease) ρ or ω?

In the first modification of the parameters I have decreased the parameter $\rho = .8$ so that now on-site work and telework are gross complements and are more difficult to substitute each other and increased the parameter $\omega = 40$ simply increasing the utility cost of death.

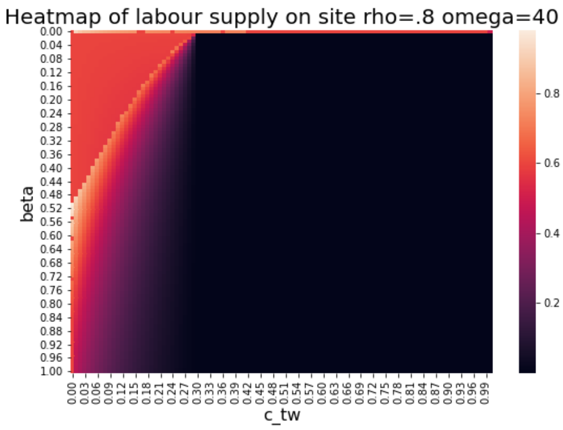
The impact of the change in parameters β and c_{tw} on employment (6) is much stronger than in the previous point of the exercise. Since both types of labour are no longer substitutable and the disutility of death is high, the labour supply of both types of workers comoves and reacts much stronger to changes in parameters β and c_{tw} than in the previous setup.

A curious characteristic of this setup is that the higher productivity losses in the telework sector (lower values of c_{tw}) are associated with higher levels of employment (6c). If the safer telework is less productive but no substitution can take place the employment in both sectors go up. If the telework is sufficiently productive higher output can be achieved with lower gross employment. In other words, if the teleworking sector is relatively more productive (c_{tw} is higher) the relatively low net utility of employing on-site workers constricts employment in the telework sector through the gross complementarity effect. The final results is lower employment in all sectors(as in 7a), lower output(as in 7b) and welfare (as in 7c).

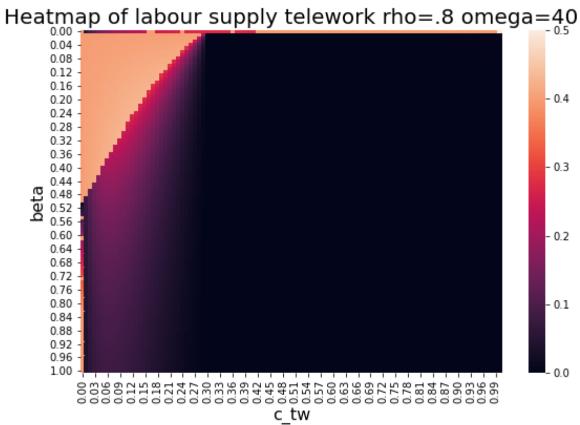
An important additional remark is that the gross complementary renders the social planner unable to escape the trade-off between work and death, hence the white triangle in the upper left corner of 7a.

Plots for exercise 2 b) $\rho = .8$ and $\omega = 40$

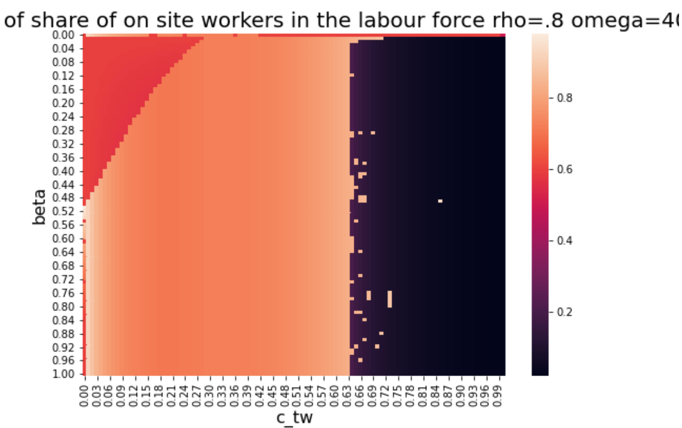
Figure 6: Labour market behaviour for $\rho = .8$ and $\omega = 40$



(a) Employment on site for $\rho = .8$ and $\omega = 40$

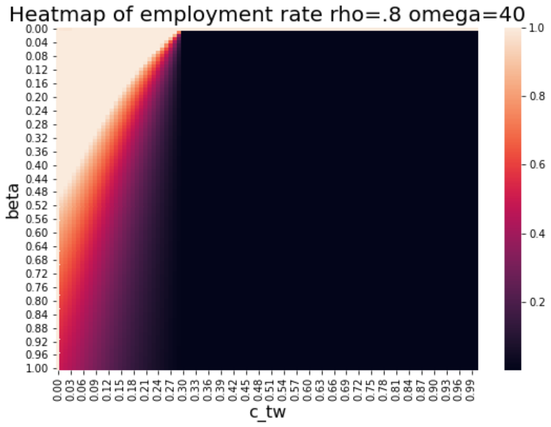


(b) Employment in teleworking sector for $\rho = .8$ and $\omega = 40$

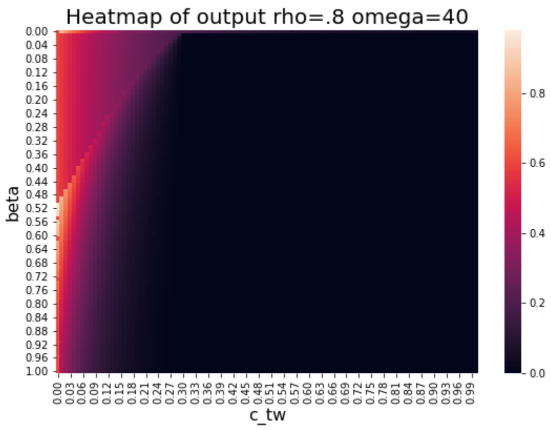


(c) Share of on site workers in the active labour force for $\rho = .8$ and $\omega = 40$

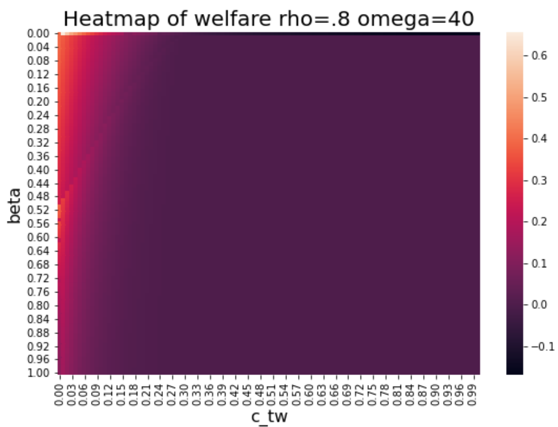
Figure 7: The aggregate measures for $\rho = .8$ and $\omega = 40$



(a) Total employment rate for $\rho = .8$ and $\omega = 40$

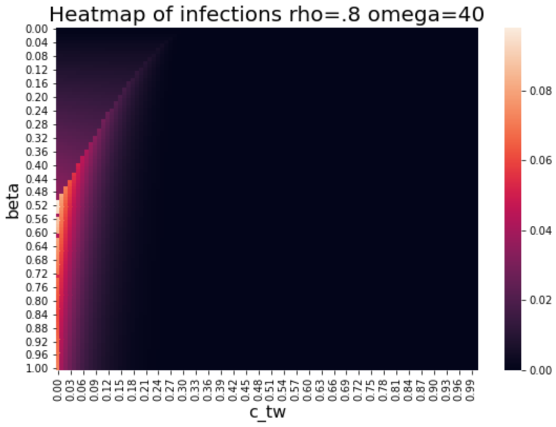


(b) Output for $\rho = .8$ and $\omega = 40$

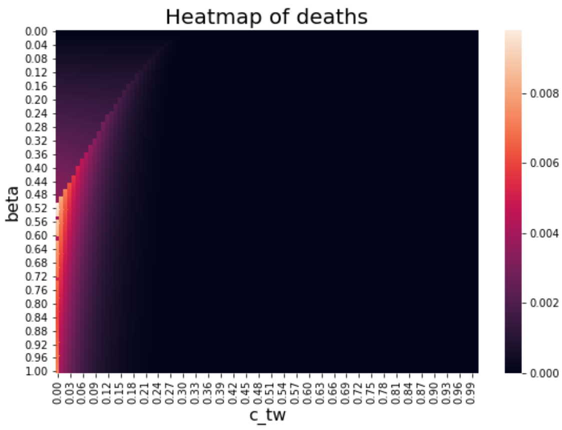


(c) Social planner's welfare function for $\rho = .8$ and $\omega = 40$

Figure 8: The COVID stats for $\rho = .8$ and $\omega = 40$



(a) Infections for $\rho = .8$ and $\omega = 40$



(b) Deaths for $\rho = .8$ and $\omega = 40$

The analysis of the propagation of the virus is best supported by the comparison of 6a and 8. The on-site employment is highest for low values of β and c_{tw} in the upper left corner of the figure 6a due to reasons mentioned before. The number of infections and deaths is the highest when employment on-site is high and the infectiousness β is high at the same time. This is the case for the lower part of the high-employment triangle in the upper left corner of the 8a and 8b.

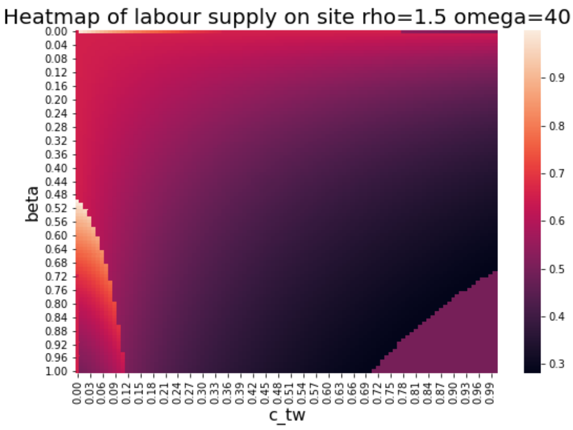
In this section I will briefly discuss the third case, where the utility cost of death is still higher $\omega = 40$ but both labour types are even stronger substitutes than in the first parametrization $\rho = 1.5$. The aim is to verify if higher substitutability will mitigate the negative effects of higher disutility from death.

The conclusions for the labour market (9) are similar to the first setup, in that both types of labour are gross substitutes. Lower levels of parameter β and c_{tw} are associated with higher employment on-site and lower telework employment (9a and 9b similar to 3a and 3b). If the values of β are sufficiently high and c_{tw} sufficiently low the labour supply constraint becomes non-binding (10a) and the demand for telework plummets. Inefficient telework is partially substituted by the on-site work despite the acute virus conditions. Similarly to the first setting (compare 4b, 4c and 10b, 10c), when both types of work are gross substitutes, the production and welfare levels depend only on the values of the productivity parameter c_{tw}.

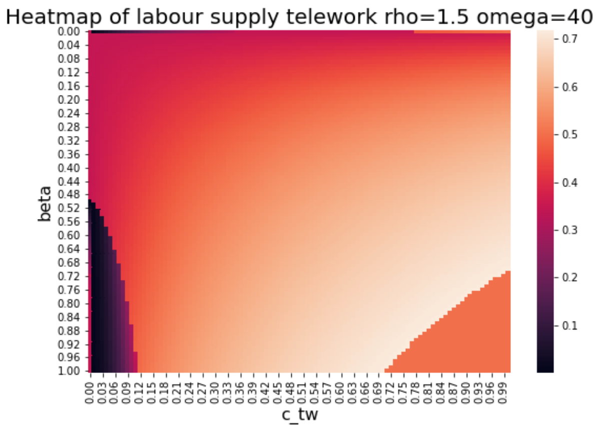
The number of deaths and infections is the highest for the high values of β and low values of c_{tw}, that is when the virus is the most infectious and employment in the on-site sector is the highest (9). The results are similar to 5 but the impact of changes in parameters is stronger due to higher ω.

Plots for exercise 2 b) $\rho = 1.5$ and $\omega = 40$

Figure 9: Labour market behaviour for $\rho = 1.5$ and $\omega = 40$



(a) Employment on site for $\rho = 1.5$ and $\omega = 40$

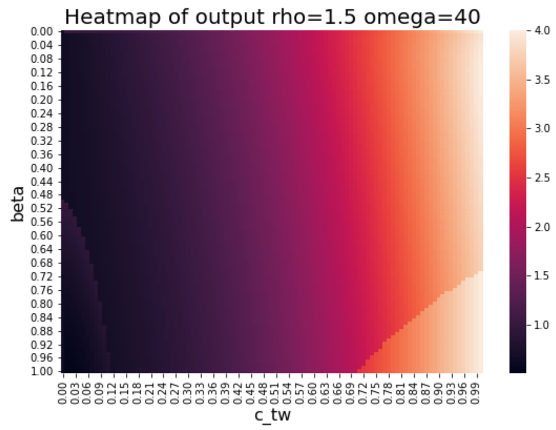


(b) Employment in teleworking sector for $\rho = 1.5$ and $\omega = 40$

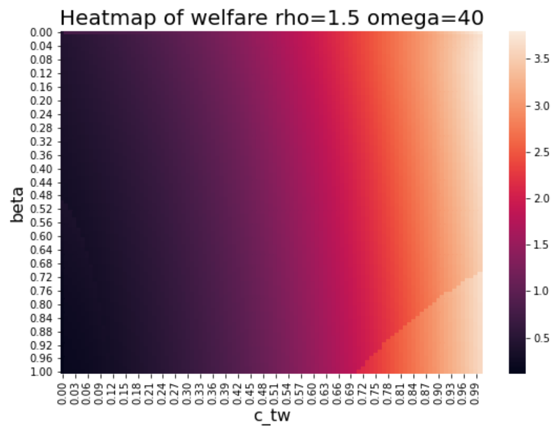
(c) Share of on site workers in the active labour force for $\rho = 1.5$ and $\omega = 40$

Figure 10: The aggregate measures for $\rho = 1.5$ and $\omega = 40$

(a) Total employment rate for $\rho = 1.5$ and $\omega = 40$

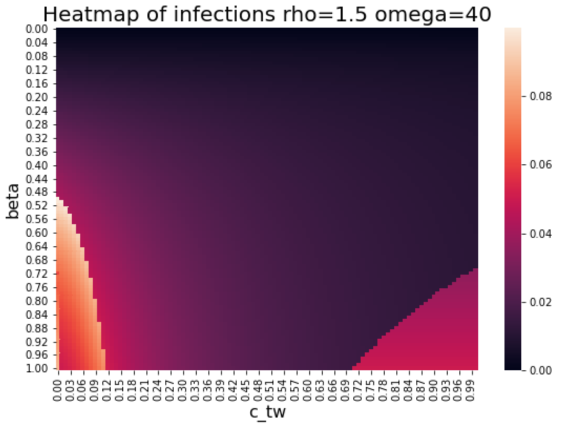


(b) Output for $\rho = 1.5$ and $\omega = 40$

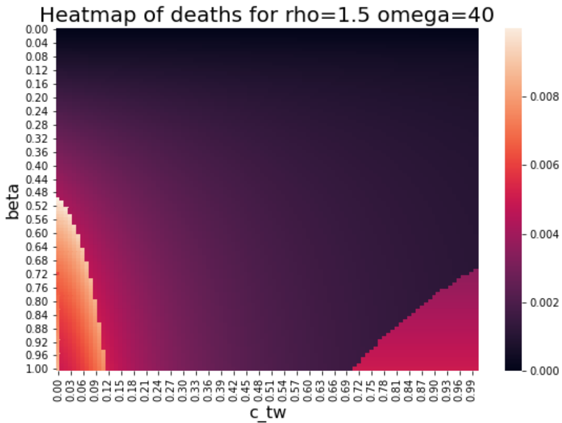


(c) Social planner's welfare function for $\rho = 1.5$ and $\omega = 40$

Figure 11: The COVID stats for $\rho = 1.5$ and $\omega = 40$



(a) Infections for $\rho = 1.5$ and $\omega = 40$



(b) Deaths for $\rho = 1.5$ and $\omega = 40$