MSA-WARES **(D_S)**

A Simple Guide to Discrete Logic

This guide covers the fundamentals of logic, from basic statements to the rules of building a valid argument.

Part 1: Propositional Logic

Propositional logic deals with simple, declarative statements (propositions) that are either **True (T)** or **False (F)**.

Proposition (P, Q, R...): A statement that is either T or F.

- **Example 1:** "The sky is blue." (This is a proposition. We'll call it P).
- **Example 2:** "2 + 2 = 5." (This is a proposition. We'll call it Q).
- **Example 3:** "It is raining outside." (A proposition, its truth value depends on the situation).
- **Example 4:** "What time is it?" (NOT a proposition, it's a question).
- **Example 5:** "He is tall." (NOT a proposition, "he" is not defined. This leads to Predicate Logic).
- **Example 6:** "This statement is false." (A paradox, not a simple proposition).

We use **Truth Tables** to see how propositions combine.

P	**¬P (NOT P)**
T	F
F	T

P	**Q**	**P∧Q (P AND Q)**	**P∨Q (P OR Q)**
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

1. Conditional Statements (Implication)

This is the "if... then..." statement.

- **Symbol:** $P \to Q$
- **Reads:** "If P, then Q."
- **P is the hypothesis**, **Q is the conclusion**.

Key Tip: The *only* time an implication is **FALSE** is when the hypothesis (P) is **True** and the conclusion (Q) is **False**. Think of it as a "broken promise."

Truth Table for $P \to Q$

P	Q	$P \to Q$	Why?
T	T	T	You promised, and it happened. (Promise kept)
T	F	F	You promised, but it didn't happen. (Broken Promise)
F	T	T	You didn't promise, but it happened anyway. (You got lucky! No promise broken).
F	F	T	You didn't promise, and it didn't happen. (No promise broken).

Examples:

- **Example 1:** "If you get an A (P), I will buy you a car (Q)."
    - T $\to$ T: You get an A, I buy a car. (True)
    - T $\to$ F: You get an A, I don't buy a car. (False - I broke my promise)
    - F $\to$ T: You don't get an A, I buy you a car. (True - I'm just nice!)
    - F $\to$ F: You don't get an A, I don't buy a car. (True - No promise broken)
- **Example 2:** "If 2+2=4 (P), then the sun is hot (Q)." (T $\to$ T, so True)
- **Example 3:** "If pigs can fly (P), then 1+1=3 (Q)." (F $\to$ F, so True. This is called "vacuously true").
- **Example 4:** "If 1+1=2 (P), then the moon is green (Q)." (T $\to$ F, so False).
- **Example 5:** "If it is raining (P), then the ground is wet (Q)." (This statement is True, because T $\to$ F is impossible. You can't have it be raining and the ground *not* be wet).
- **Example 6:** "If the earth is flat (P), then 2 is an odd number (Q)." (F $\to$ F, so True).

2. Biconditional Statements

This is the "if and only if" (iff) statement.

- **Symbol:** $P \leftrightarrow Q$
- **Reads:** "P if and only if Q."
- **Meaning:** It's the same as $(P \to Q) \land (Q \to P)$.

Key Tip: The biconditional is True *only* when P and Q have the **same truth value** (both are T or both are F). It's an "equivalence" operator.

Truth Table for $P \leftrightarrow Q$

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F

| F | T | F |
| F | F | T |

Examples:

- **Example 1:** "You can vote (P) if and only if you are over 18 (Q)."
  - If you vote (T) and are over 18 (T), the statement is True.
  - If you don't vote (F) and are not over 18 (F), the statement is True.
  - If you vote (T) but are not over 18 (F), the statement is False.
- **Example 2:** "A number is even (P) iff it is divisible by 2 (Q)." (This is True, they always match).
- **Example 3:** "2+2=5 (P) if and only if 1+1=3 (Q)." (F $\leftrightarrow$ F, so this statement is True).
- **Example 4:** "The sky is blue (P) iff pigs can fly (Q)." (T $\leftrightarrow$ F, so this statement is False).
- **Example 5:** "x > 0 (P) $\leftrightarrow$ $x^2 > 0$ (Q)." (This is False. Try x = -2. P is False, but Q is True. F $\leftrightarrow$ T is False).
- **Example 6:** "It is snowing (P) $\leftrightarrow$ it is 32°F or below (Q)." (This is False. It could be 20°F (Q=T) but not snowing (P=F)).

3. Tautology, Contradiction, and Contingency

- **Tautology:** A statement that is **always True**, no matter the truth values of its propositions.
- **Contradiction:** A statement that is **always False**, no matter the truth values.
- **Contingency:** A statement that can be either True or False.

Truth Table Example:

P	$\neg P$	$P \lor \neg P$ (Tautology)	$P \land \neg P$ (Contradiction)
T	F	T	F
F	T	T	F

Examples of Tautologies:

1. $P \lor \neg P$ ("It is raining or it is not raining.")
2. $P \to P$ ("If it's raining, then it's raining.")
3. $(P \land Q) \to P$ ("If it's raining and it's cold, then it's raining.")
4. $\neg (P \land \neg P)$ (The negation of a contradiction).
5. $(P \to Q) \leftrightarrow (\neg P \lor Q)$ (This is a very important equivalence).

Examples of Contradictions:

1. $P \land \neg P$ ("It is raining and it is not raining.")
2. $\neg (P \lor \neg P)$ (The negation of a tautology).
3. $(P \to Q) \land (P \land \neg Q)$ (This says "a promise is kept" AND "the promise was broken," which is impossible).
4. $P \leftrightarrow \neg P$ ("This statement is its own opposite.")

5. F (The simple proposition "False").

4. Equivalent Statements

Two statements are **logically equivalent** if they have the exact same truth table.

- **Symbol:** $\equiv$

The Most Important Equivalences: The Conditional
Given the statement: $P \to Q$ ("If it is raining, the ground is wet.")

1. **Converse:** $Q \to P$ ("If the ground is wet, it is raining.")
   - **NOT Equivalent.** (The ground could be wet from a sprinkler).
2. **Inverse:** $\neg P \to \neg Q$ ("If it is not raining, the ground is not wet.")
   - **NOT Equivalent.** (Again, the sprinkler).
3. **Contrapositive:** $\neg Q \to \neg P$ ("If the ground is not wet, it is not raining.")
   - **IS Equivalent!** This is a very powerful tool.

Truth Table Confirmation:

P	Q	$\neg P$	$\neg Q$	$P \to Q$	$\neg Q \to \neg P$ (Contrapositive)
T	T	F	F	T	F $\to$ F = T
T	F	F	T	F	T $\to$ F = F
F	T	T	F	T	F $\to$ T = T
F	F	T	T	T	T $\to$ T = T

(See! The columns are identical!)
Other Key Equivalences:

- **De Morgan's Laws (Key Tip: "Break the line, flip the sign")**
  - **Example 1:** $\neg(P \land Q) \equiv \neg P \lor \neg Q$
    - "It is NOT (raining AND cold)" $\equiv$ "It is (NOT raining) OR (NOT cold)."
  - **Example 2:** $\neg(P \lor Q) \equiv \neg P \land \neg Q$
    - "I am NOT (happy OR rich)" $\equiv$ "I am (NOT happy) AND (NOT rich)."
- **Implication Equivalence (Very useful!)**
  - **Example 3:** $P \to Q \equiv \neg P \lor Q$
    - "If it's raining (P), I have an umbrella (Q)" $\equiv$ "It is NOT raining ($\neg P$) OR I have an umbrella (Q)."

Part 2: Predicate Logic

Predicate logic is "smarter" than propositional logic. It uses **predicates** (properties) and **quantifiers** (how many).

- **Predicate:** A statement with variables, e.g., $P(x) = \text{"x > 3"}$. Its truth value depends on x.
- **Domain:** The set of values the variables can be (e.g., all integers, all people).

1. Universal Quantifier ($\forall$)

- **Symbol:** $\forall$
- **Reads:** "For all..." or "For every..."
- $\forall x P(x)$ means "For every x in the domain, P(x) is true."

Key Tip: To prove $\forall x P(x)$ is **True**, you must show it for *all* cases. To prove it's **False**, you only need *one* **counterexample**.

Examples (Domain: All Integers)

1. $\forall x (x^2 \ge 0)$
   - **True.** Every integer squared is non-negative.
2. $\forall x (x > 0)$
   - **False.** Counterexample: $x = -1$.
3. $\forall x (x = x+1)$
   - **False.** Counterexample: $x = 5$ (5 != 6).
4. $\forall x ((x > 4) \to (x^2 > 16))$
   - **True.**
5. **Domain: All people:** $\forall p$ ("p has a heart")
   - **True.**
6. **Domain: All students in this class:** $\forall s$ ("s passed the exam")
   - **False.** (If even one student failed, this is false).

2. Existential Quantifier ($\exists$)

- **Symbol:** $\exists$
- **Reads:** "There exists..." or "For some..."
- $\exists x P(x)$ means "There is at least one x in the domain for which P(x) is true."

Key Tip: To prove $\exists x P(x)$ is **True**, you only need *one* **example**. To prove it's **False**, you must show it's false for *all* cases.

Examples (Domain: All Integers)

1. $\exists x (x > 5)$
   - **True.** Example: $x = 6$.
2. $\exists x (x^2 = -1)$
   - **False.** (Assuming domain is integers or reals. It's True if domain is complex numbers!)
3. $\exists x (x = x+1)$
   - **False.** There is no number that equals itself plus one.
4. $\exists x (x \text{ is even})$
   - **True.** Example: $x = 2$.
5. **Domain: All people:** $\exists p$ ("p is over 10 feet tall")
   - **False.**
6. **Domain: All students in this class:** $\exists s$ ("s got an A")
   - **True.** (If at least one student got an A).

3. Nested Quantifiers

This is when you combine quantifiers. ORDER MATTERS!
Let the Domain be all people and $L(x, y)$ be the predicate "x loves y".

1. $\forall x \forall y L(x, y)$
   - "For all people x, and for all people y, x loves y."
   - "Everybody loves everybody." (False)
2. $\exists x \exists y L(x, y)$
   - "There exists a person x, and there exists a person y, such that x loves y."
   - "Somebody loves somebody." (True)
3. $\forall x \exists y L(x, y)$
   - "For every person x, there exists a person y such that x loves y."
   - "Everybody loves someone." (True)
4. $\exists y \forall x L(x, y)$
   - "There exists a person y such that for all people x, x loves y."
   - "There is a person who is loved by everyone." (The "celebrity" example. Probably False).
5. $\forall y \exists x L(x, y)$
   - "For every person y, there exists a person x such that x loves y."
   - "Everybody is loved by someone." (True)
6. $\exists x \forall y L(x, y)$
   - "There exists a person x such that for all people y, x loves y."
   - "There is someone who loves everyone." (The "saint" example. Probably False).

Part 3: Rules of Inference

These are simple, valid argument forms that let us "prove" a conclusion from a set of premises (hypotheses).

1. Modus Ponens (The Law of Detachment)

This is the most common and intuitive rule.

- **Form:**
  1. $P \to Q$ (Premise 1: "If P, then Q")
  2. P (Premise 2: "P is true")
  3. $\therefore Q$ (Conclusion: "Therefore, Q is true")
- **In English:** If you have a conditional, and you have the "if" part, you can conclude the "then" part.

Examples:

1. **Premise 1:** "If it is raining, the street is wet."
2. **Premise 2:** "It is raining."
3. **Conclusion:** "Therefore, the street is wet."
4. **Premise 1:** "If you have a password, you can log in."

5. **Premise 2:** "You have a password."
6. **Conclusion:** "Therefore, you can log in."
7. **Premise 1:** "All men are mortal." (This is $\forall x (Man(x) \to Mortal(x))$)
8. **Premise 2:** "Socrates is a man."
9. **Conclusion:** "Therefore, Socrates is mortal."
10. **Premise 1:** "If $x > 4$, then $x^2 > 16$."
11. **Premise 2:** "$x = 5$." (This satisfies P)
12. **Conclusion:** "Therefore, $x^2 > 16$."
13. **Premise 1:** "If the code compiles, the program will run."
14. **Premise 2:** "The code compiles."
15. **Conclusion:** "Therefore, the program will run."

2. Modus Tollens (The Law of Contraposition)

This uses the contrapositive.

- **Form:**
  1. $P \to Q$ (Premise 1: "If P, then Q")
  2. $\neg Q$ (Premise 2: "Q is false")
  3. $\therefore \neg P$ (Conclusion: "Therefore, P is false")
- **In English:** If you have a conditional, and the "then" part is false, you can conclude the "if" part is also false.

Examples:

1. **Premise 1:** "If it is raining, the street is wet."
2. **Premise 2:** "The street is NOT wet."
3. **Conclusion:** "Therefore, it is NOT raining."
4. **Premise 1:** "If you have a password, you can log in."
5. **Premise 2:** "You cannot log in."
6. **Conclusion:** "Therefore, you do not have a password." (Assuming the premise is the only way).
7. **Premise 1:** "If a bird is a crow, it is black."
8. **Premise 2:** "This bird is not black."
9. **Conclusion:** "Therefore, this bird is not a crow."
10. **Premise 1:** "If the server is down, the website is inaccessible."
11. **Premise 2:** "The website is accessible."
12. **Conclusion:** "Therefore, the server is not down."
13. **Premise 1:** "If x is an even number, then x is divisible by 2."
14. **Premise 2:** "$x=7$." (This means x is not divisible by 2).
15. **Conclusion:** "Therefore, x is not an even number."

3. Hypothetical Syllogism (The Chain Rule)

This lets you chain implications together.

- **Form:**

    1. $P \to Q$ (Premise 1: "If P, then Q")
    2. $Q \to R$ (Premise 2: "If Q, then R")
    3. $\therefore P \to R$ (Conclusion: "Therefore, If P, then R")
- **In English:** If P leads to Q, and Q leads to R, then P leads to R.

Examples:

1. **Premise 1:** "If you study hard (P), you will get an A (Q)."
2. **Premise 2:** "If you get an A (Q), you will get a scholarship (R)."
3. **Conclusion:** "Therefore, if you study hard (P), you will get a scholarship (R)."
4. **Premise 1:** "If it rains (P), I will stay inside (Q)."
5. **Premise 2:** "If I stay inside (Q), I will play video games (R)."
6. **Conclusion:** "Therefore, if it rains (P), I will play video games (R)."
7. **Premise 1:** "If the power goes out, the server goes down."
8. **Premise 2:** "If the server goes down, the website is offline."
9. **Conclusion:** "Therefore, if the power goes out, the website is offline."
10. **Premise 1:** "If $a=b$, then $a^2 = b^2$."
11. **Premise 2:** "If $a^2=b^2$, then $a^2-b^2=0$."
12. **Conclusion:** "Therefore, if $a=b$, then $a^2-b^2=0$."
13. **Premise 1:** "If I wake up late, I miss the bus."
14. **Premise 2:** "If I miss the bus, I am late for work."
15. **Conclusion:** "Therefore, if I wake up late, I am late for work."

Part 4: Best Tips and Tricks to Remember

1. **The "Broken Promise" ($P \to Q$):** The *only* time a conditional is **FALSE** is $T \to F$. All other times it is True.
2. **The "Same Value" ($P \leftrightarrow Q$):** The biconditional is **TRUE** only if P and Q are *both T* or *both F*.
3. **The Contrapositive (Your Best Friend):** $P \to Q$ is *always* equivalent to $\neg Q \to \neg P$. If an implication looks confusing, try to state its contrapositive.
4. **De Morgan's Law ("Break the Line, Flip the Sign"):**
    - $\neg(P \land Q)$ becomes $\neg P \lor \neg Q$.
    - $\neg(P \lor Q)$ becomes $\neg P \land \neg Q$.
5. **Implication into an OR:** $P \to Q$ is *always* equivalent to $\neg P \lor Q$. This is the "escape clause." ("If you do this, I'll be mad" $\equiv$ "DON'T do this, OR I'll be mad").
6. **Quantifier Counterexamples:**
    - To disprove $\forall x$ ("for all"), you only need **one** counterexample.
    - To prove $\exists x$ ("exists"), you only need **one** example.
7. **Practice, Practice, Practice:** The #1 way to learn this is to translate English sentences *into* logic symbols, and logic symbols *back into* English.

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