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HW-0

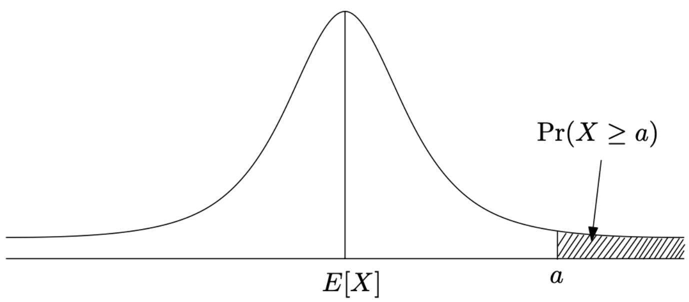
"This homework is designed to provide you with hands-on experience in applying essential probability concepts like Markov's inequality, Chebyshev's inequality, and Chernoff bounds. These concepts form the bedrock for analyzing sublinear and randomized algorithms. If you are already well-versed in these concepts, you may consider skipping this homework."

Prerequisite knowledge(Markov's Inequality):

(Markov's Inequality) *Let X be a non-negative random variable. Then,*

$$\Pr(X \geq a) \leq \frac{E[X]}{a}, \qquad \text{for any } a > 0.$$

Before we discuss the proof of Markov's Inequality, first let's look at a picture that illustrates the event that we are looking at.



Markov's Inequality bounds the probability of the shaded region.

Proof: Suppose X is a discrete random variable, for simplicity.

$$\begin{aligned} E[X] &= \sum_x x \cdot \Pr(X = x) \\ &\geq \sum_{x \geq a} x \cdot \Pr(X = x) \\ &\geq a \cdot \sum_{x \geq a} \cdot \Pr(X = x) \\ &= a \cdot \Pr(X \geq a) \end{aligned}$$

Rearranging, we get

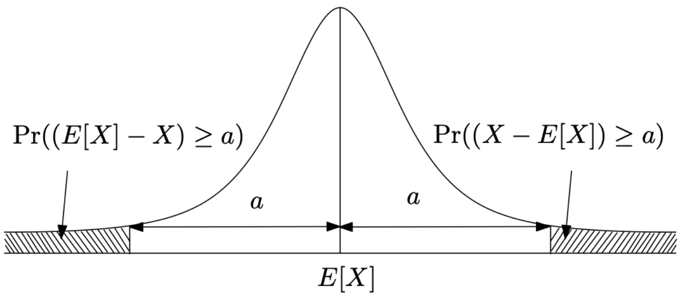
$$\Pr(X \geq a) \leq \frac{E[X]}{a}.$$

Prerequisite knowledge(Chebyshev's Inequality):

(Chebyshev's Inequality) *For any $a > 0$,*

$$\Pr(|X - E[X]| \geq a) \leq \frac{\mathrm{Var}[X]}{a^2}.$$

Again, let us look at a picture that illustrates Chebyshev's Inequality.



Chebyshev's Inequality bounds the probability of the shaded regions.

Proof:

$$\Pr\left(|X - E[X]| \geq a\right) = \Pr\left((X - E[X])^2 \geq a^2\right) = \Pr(Y \geq a^2)$$

Where, $Y = (X - E[X])^2$. Note that Y is a non-negative random variable. Therefore, using Markov's Inequality,

$$\Pr(Y \geq a^2) \leq \frac{E[Y]}{a^2} = \frac{E\left((X - E[X])^2\right)}{a^2} = \frac{\mathrm{Var}[X]}{a^2}.$$

problem 1:

Alice and Bob are running for class president. There are 100 voters. 80 voters prefer Alice and 20 voters prefer Bob. During the election each voter gets confused (independently of other voters) with probability 1/100 and votes for the wrong candidate (i.e. the candidate he likes less).

If A is the number of votes Alice receives and B is the number of votes Bob receives:

1. Calculate E[A] and E[B].

2. Use Markov's inequality to upper bound the probability that Bob wins the election.

3. Use Chebyshev's inequality to upper bound the probability that Bob wins the election.(hint:***Var[X] = E[(X-E[X])2] = E[X2] - (E[X])2***)

Prerequisite knowledge(Chernoff-Hoeffding Bounds):

Hoeffding's inequality

- Let $X_1, \dots X_n$ be independently distributed over $[0,1]$
- Let $p = \mathbf{E}[X_i]$ and $X = \frac{\sum X_i}{n}$

- Hoeffding's inequality: both below hold:
  1. $\Pr[X \geq p + \varepsilon] \leq e^{-2n\varepsilon^2}$
  2. $\Pr[X \leq p - \varepsilon] \leq e^{-2n\varepsilon^2}$

- Combining them, we also have $\Pr[|X - p| \geq \varepsilon] \leq 2e^{-2n\varepsilon^2}$
- The inequalities hold even if $p = \mathbf{E}[X] = \sum_i \frac{p_i}{n}$ where $p_i = \mathbf{E}[X_i]$

Prerequisite knowledge(Chernoff Bounds):

(Chernoff Bound). *Let X be the random variable defined above, and let $\mu = \mathbb{E}[X]$. Then for any $\delta > 0$:*

$$\Pr[X \geq (1+\delta)\mu] \leq e^{-\mu((1+\delta)\ln(1+\delta)-\delta)}$$

problem 2:

You are given an algorithm A for a decision problem (i.e., answer for each input is either 0 or 1), that runs in time T (n) on inputs of size n, with probability of error 1/4. Show how to convert it into a new algorithm B that runs in time O(T (n) log 1/β) with probability of error at most β. (Hint: ***run A O(log 1/β) times and take the "majority", i.e., the most common, answer. Use Chernoff bounds to show that the correct answer is highly likely to be the output.***)

problem 3:

Let f be a function which maps inputs of size n to a number. You are given an approximation scheme $\mathcal{A}$ for f such that $Pr[\frac{f(x)}{1+\epsilon} \leq \mathcal{A}(x) \leq f(x)(1+\epsilon)] \geq 3/4$, and $\mathcal{A}$ runs in time polynomial in $1/\epsilon, |x|$. Construct an approximation scheme $\mathcal{B}$ for f such that $Pr[\frac{f(x)}{1+\epsilon} \leq \mathcal{B}(x) \leq f(x)(1+\epsilon)] \geq 1-\delta$, and $\mathcal{B}$ runs in time polynomial in $\frac{1}{\epsilon}, |x|, \log\frac{1}{\delta}$.

Prerequisite knowledge(Linearity of Expectations):

[Linearity of Expectations]: *For any finite collection of discrete random variables $X_1, X_2, \ldots, X_n$ with finite expectations,*

$$\mathbf{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbf{E}[X_i].$$

problem 4:

(Coupon Collector Problem). Given a die with *n* sides. What is the expected number of times you need to roll the die in order to see each of the *n* sides? (Hint: ***Given that you saw i sides, how many times do you need to roll the die to see the (i + 1)st side? Then use linearity of expectation.***)