Lecture 6
Formal derivation of the de-migration operator

M D Sacchi
UNLP Inverse Problems

Acoustic wave equation

$$u_{tt} - c^2 \nabla^2 u = s(t)\delta(x - x_s, z - z_s)$$

$u(t, x, z)$: wavefield (pressure)

$c(x, z)$: subsurface velocity

$s(t)$: source function

x_s, z_s : Source position

Acoustic wave equation

$$u_{tt} - c^2 \nabla^2 u = s(t)\delta(x - x_s, z - z_s)$$

$$u_{tt} = \frac{\partial^2 u}{\partial t^2}$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}$$

Acoustic wave equation: Linearization

$$u_{tt} - c^2 \nabla^2 u = s(t)\delta(x - x_s, z - z_s)$$

Define backgrounds and perturbations

$$u = u_0 + \delta u, \quad c = c_0 + \delta c$$

Replace perturbations in wavefield and parameter

$$(u_0 + \delta u)_{tt} - (c_0 + \delta c)^2 \nabla^2 (u_0 + \delta u) = s(t)\delta(x - x_s, z - z_s)$$

Keep first order terms

$$u_{0_{tt}} + \delta u_{tt} - c_0^2 \nabla^2 u_0 - 2c_0 \delta c \nabla^2 u_0 - c_0 \nabla^2 \delta u = s(t)\delta(x - x_s, z - z_s)$$

Acoustic wave equation: Linearization

$$u_{tt} - c^2 \nabla^2 u = s(t)\delta(x - x_s, z - z_s)$$

Define backgrounds and perturbations

$$u = u_0 + \delta u, \quad c = c_0 + \delta c$$

Replace perturbations in wavefield and parameter

$$(u_0 + \delta u)_{tt} - (c_0 + \delta c)^2 \nabla^2 (u_0 + \delta u) = s(t)\delta(x - x_s, z - z_s)$$

Keep first order terms

$$u_{0_{tt}} + \delta u_{tt} - c_0^2 \nabla^2 u_0 - 2c_0 \delta c \nabla^2 u_0 - c_0^2 \nabla^2 \delta u = s(t)\delta(x - x_s, z - z_s)$$

Terms in blue are the wave eq. in background, terms in orange are a new wave eq. for perturbations

Acoustic wave equation: Linearization

$$u_{0_{tt}} + \delta u_{tt} - c_0^2 \nabla^2 u_0 - 2c_0 \delta c \nabla^2 u_0 - c_0^2 \nabla^2 \delta u = s(t)\delta(x - x_s, z - z_s)$$

$$u_{0_{tt}} - c_0^2 \nabla^2 u_0 = s(t)\delta(x - x_s, z - z_s) \longrightarrow \mathscr{L} u_0 = s(t)\delta(x - x_s, z - z_s)$$

$$\delta u_{tt} - c_0^2 \nabla^2 \delta u = 2c_0 \delta c \nabla^2 u_0 \longrightarrow \mathscr{L} \delta u = 2c_0 \delta c \nabla^2 u_0$$

Both eqs above are of the form:

$$\mathscr{L} v = f \longrightarrow v = \mathscr{L}^{-1} f = \mathscr{F} f \qquad \mathscr{L}^{-1} = \mathscr{F}$$

$\mathscr{F}$ **means solving the wave eq, for instance, via a FD method**

Acoustic wave equation: Linearization

$$u_{0_{tt}} + \delta u_{tt} - c_0^2 \nabla^2 u_0 - 2c_0 \delta c \nabla^2 u_0 - c_0^2 \nabla^2 \delta u = s(t)\delta(x - x_s, z - z_s)$$

$$\mathscr{L} u_0 = s(t)\delta(x - x_s, z - z_s)$$ **Propagation of source**

$$\mathscr{L} \delta u = 2c_0 \delta c \nabla^2 u_0$$ **Propagation of scattered field**

Solutions

$$u_0 = \mathscr{F} s(t)\delta(x - x_s, z - z_s)$$ **(1)**

$$\delta u = \mathscr{F} 2 c_0 \nabla^2 u_0 \delta c$$ **(2)**

Acoustic wave equation: Linearization

$$u_0 = \mathscr{F} s(t) \delta(x - x_s, z - z_s) \tag{1}$$

$$\delta u = \mathscr{F}\, 2\, c_0\, \nabla^2 u_0\, \delta c \tag{2}$$

- Eq. (1) solves for the source wavefield everywhere $u_0(t, x, z)$
- Eq. (2) can be used to estimate the perturbation or “image” $\delta c(x, z)$
- Eq. (2) is often called Born Forward Modelling operator.
- Also notice that the perturbed field is a quality evaluated everywhere in the propagation domain $\delta u(t, x, z)$

Acoustic wave equation: Linearization

- Consider that we have solved the forward problem (1) to compute $u_0(t, x, z)$
- Let's now further explore Eq. (2)

$$\delta u = \mathscr{F}\, 2\, c_0\, \nabla^2 u_0\, \delta c$$

- The data is only part of the scattered field. So we introduce a sampling operator:

$$d^{obs} = S\,\delta u = S\,\mathscr{F}\, 2\, c_0\, \nabla^2 u_0\, \delta c$$

$$d^{obs}(t, x_r, 0) = \underset{\substack{\uparrow \\ \text{Sampling operator}}}{S}\,\delta\, u(t, x, z)$$

Born forward operator

- Forward problem

Finite diference modelling operator

$$d^{obs} = S\,\delta u = S\,\mathscr{F}\,2\,c_0\,\nabla^2 u_0\,\delta c$$

Data: shot gather Sampling operator Unknown

- Remember that if you consider a point away from the source position

$$u_{0_{tt}} - c_0^2\,\nabla^2 u_0 = 0 \longrightarrow \nabla^2 u_0 = \frac{1}{c_0^2} u_{0_{tt}}$$

$$d^{obs} = S\,\delta u = S\,\mathscr{F}\,u_{0_{tt}}\,2\frac{\delta c}{c_0}$$

$$d^{obs} = S\,\delta u = S\,\mathscr{F}\,u_{0_{tt}}\,m \qquad m = 2\frac{\delta c}{c_0}$$

"Reflectivity"

Born forward operator

Finite diference modelling operator

- Forward problem $\quad d^{obs} = S\,\delta u = S\,\mathscr{F} u_{0_{tt}}\, m.\quad$ Let $W_s(t,x,z) = u_{0_{tt}}$

Data: shot gather   Sampling operator   Unknown

- Introducing the copy operator

Copy m to all times

$$d^{obs}(t, x_r, 0) = S\,\delta u(t,x,z) = S\,\mathscr{F} W_s(t,x,z) \circ C\, m(x,z)$$

Element-wise multiplication

- Adjoint

$$< m(x,z) > = C' W_s(t,x,z) \circ \mathscr{F}'\, S'\, d^{obs}(t, x_r, 0)$$

$$= C' W_s(t,x,z) \circ W_r(t,x,z)$$

$$= \sum_t W_s(t,x,z) \circ W_r(t,x,z)$$

Copying and sum

Copying $$\begin{pmatrix}1\\1\\1\end{pmatrix} a = \begin{pmatrix}a\\a\\a\end{pmatrix} \qquad \equiv \qquad Cu = v$$

Sum $$\begin{pmatrix}1 & 1 & 1\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = a + b + c \qquad \equiv \qquad C'v = u$$

The adjoint of copying is the sum operator

RTM image (Reverse Time-Migration)

Source side wavefield

$$u_0 = \mathscr{F} s(t)\delta(x - x_s, z - z_s)$$

$$W_s(t, x, z) = u_{0_{tt}}$$

Receiver side wavefield

$$W_r(t, x, z) = \mathscr{F}' S' d^{obs}(t, x_r, 0)$$

Image

$$< m(x, z) > = \sum_t W_s(t, x, z) \circ W_r(t, x, z)$$ **RTM for 1 source**

RTM image (Reverse Time-Migration)

Source side wavefields

$$u_0^k = \mathscr{F} s(t)\delta(x - x_s^k, z - z_s^k), \quad k = 1\ldots N_s$$

$$W_s^k(t, x, z) = u_{0_{tt}}^k$$

Receiver side wavefields

Shot gather k

$$W_r^k(t, x, z) = \mathscr{F}'\, S'\, d^k(t, x_r, 0)$$

Image

$$< m(x, z) > = \sum_k \sum_t W_s^k(t, x, z) \circ W_r^k(t, x, z)$$

RTM for many sources

RTM in a nutshell

- For Source k
- Compute Synthetic Source Wavefield for a give shot (Forward Modelling)
- Read Shot k and Compute Receiver Wavefield using Adjoint operator
- Evaluate image contribution for source k, <m(k)>
- <m> = <m> + <m(k)>
- Next source