Part 4

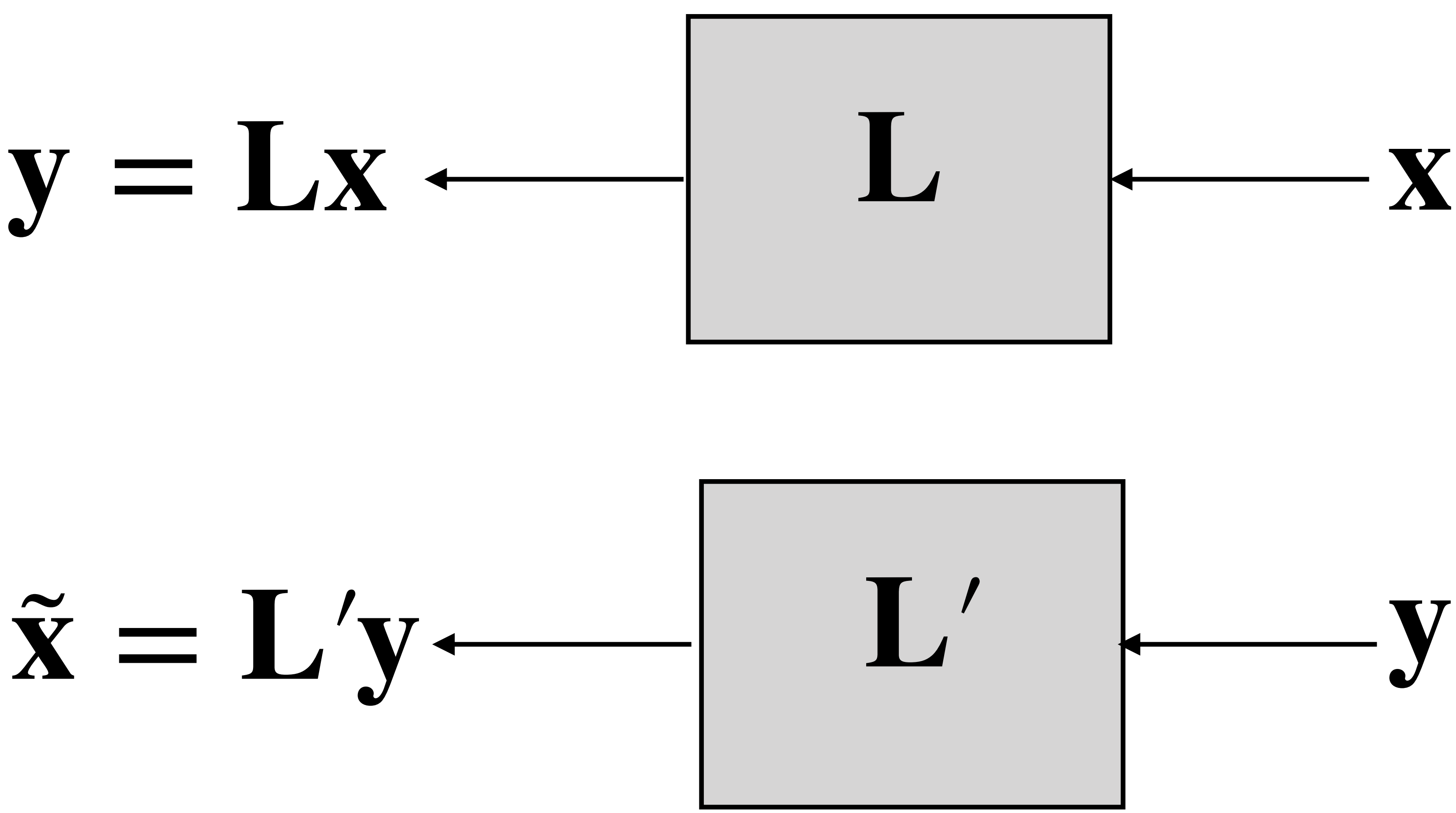
Changing basis to solve sparsity promoting problems: Application to Interpolation

M D Sacchi
University of Alberta

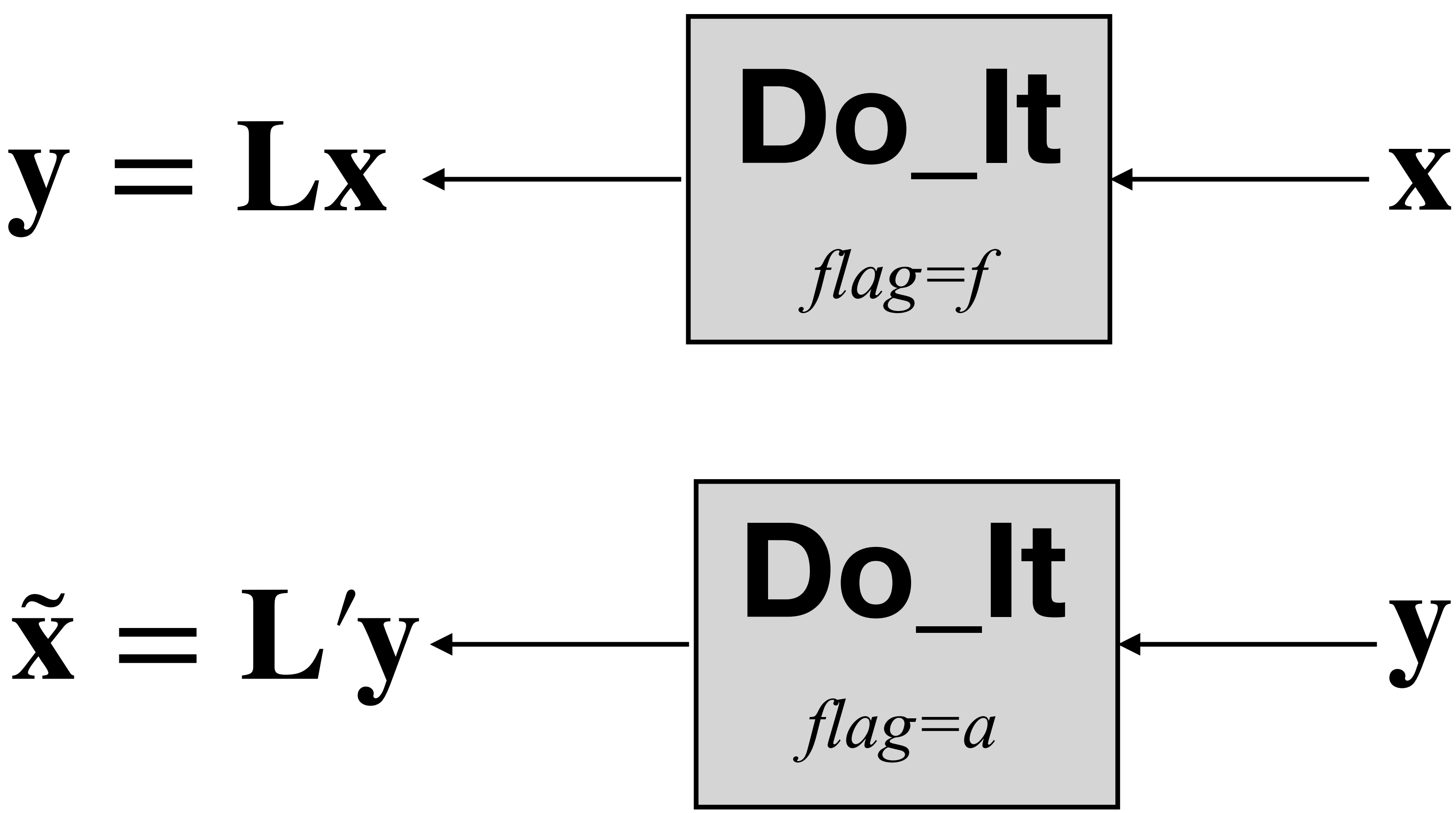
Interpolation problem

- Data to interpolate is not sparse
- Coefficients representing the data are sparse if the basis functions to represent the data are properly selected
- Let's discuss how one can solve the interpolation problem via a sparsity promoting algorithm
    - First we will use explicit operator (Matrices)
    - Then Transforms applied as implicit linear operators

Explicit


$\mathbf{y} = \mathbf{L}\mathbf{x}$
$\mathbf{L}$
$\mathbf{x}$
$\tilde{\mathbf{x}} = \mathbf{L}'\mathbf{y}$
$\mathbf{L}'$
$\mathbf{y}$

Implicit



f: *Forward*
a: *Adjoint or transpose*

Interpolation problem

- Given ideal data $\mathbf{m}$
- Assume the data was sampled to produce observations $\mathbf{d}$
- $\mathbf{d} = \mathbf{S}\mathbf{m}$
- Where $\mathbf{S}$ is the Sampling Operator that extracts N samples of the length N signal $\mathbf{m}$
- $\mathbf{S} \in \mathscr{R}^{N \times M}$

Interpolation Problem: Sampling Operator

- Given ideal complete data $\mathbf{m}$ and a sampling operator that produce observations $\mathbf{d}$ which are a subset of $\mathbf{m}$

Flavour 1 Operator

$$\begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \\ m_7 \\ m_9 \end{bmatrix} \longrightarrow \mathbf{d} = \mathbf{S}\mathbf{m}$$

- Where $\mathbf{S}$ is the Sampling Operator that extracts N samples of the length M signal $\mathbf{m}$

- $\mathbf{S} \in \mathscr{R}^{N \times M}$ **.** The goal is to estimate $\mathbf{m}$ from $\mathbf{d}$ (underdetermined)

Interpolation Problem: Sampling Operator

- The adjoint of sampling replaces empty bins by zeros

$$\begin{bmatrix} d_1 \\ 0 \\ 0 \\ d_2 \\ 0 \\ d_3 \\ 0 \\ 0 \\ d_4 \end{bmatrix} = \mathbf{S}^T \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} \longrightarrow \tilde{\mathbf{m}} = \mathbf{S}^T \mathbf{d}$$

Properties that can be easily shown: $\mathbf{S}\mathbf{S}^T = \mathbf{I}\,,\ \mathbf{S}^T\mathbf{S} \neq \mathbf{I}$

Interpolation Problem: Naive Solution

Because is an underdetermined problem, we could try the minimum norm solution:

$$\mathbf{d} = \mathbf{S}\mathbf{m} \longrightarrow \mathbf{m}_{mn} = \mathbf{S}^T(\mathbf{S}\mathbf{S}^T)^{-1}\mathbf{d}$$

But $\mathbf{S}\mathbf{S}^T = \mathbf{I} \longrightarrow \mathbf{m}_{mn} = \mathbf{S}^T\mathbf{d}$

(1)

$$\mathbf{m}_{mn} = \mathbf{S}^T \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} d_1 \\ 0 \\ 0 \\ d_2 \\ 0 \\ d_3 \\ 0 \\ 0 \\ d_4 \end{bmatrix} = \begin{bmatrix} m_1 \\ 0 \\ 0 \\ m_4 \\ 0 \\ m_6 \\ 0 \\ 0 \\ m_9 \end{bmatrix}$$

The interpolation via the minimum norm solution did not work. There is nothing better than zeros to make the norm of the solution small!

Interpolation Problem: Naive Solution, second attempt

Because is an underdetermined problem, we could try the minimum norm solution but on an orthogonal basis like Fourier.

$$\mathbf{d} = \mathbf{S}\mathbf{m}$$

$\mathbf{m} = \mathbf{F}\mathbf{a}$ The Fourier inverse transform is $\mathbf{F}$ with $\mathbf{F}\mathbf{F}^H = \mathbf{I}$

$\mathbf{d} = \mathbf{S}\mathbf{F}\mathbf{a}$ Then,

$$\mathbf{a}_{mn} = \mathbf{F}^H\mathbf{S}^T(\mathbf{S}\mathbf{F}\mathbf{F}^H\mathbf{S}^T)^{-1}\mathbf{d} = \mathbf{F}^H\mathbf{S}^T\mathbf{d}$$

$$\mathbf{m}_{mn} = \mathbf{F}\mathbf{a}_{mn} = \mathbf{F}\mathbf{F}^H\mathbf{S}^T\mathbf{d} = \mathbf{S}^T\mathbf{d}$$

See (1) in previous slide

Uppps!! again, we have replaced missing data by zeros like in the previous example

Interpolation: Sparse or Compressive Solution

- Minimum norm solution in data space or transformed domain space did not work!! So we need something else. What about using sparsity?

- Asking for the solution to be sparse does not make any sense. However, we can say that solution can be represented by a transform (Forward or Synthesis transform)

$$\mathbf{m} = \mathbf{F}\mathbf{a}$$

- Where $\mathbf{F}$ is the forward transform and $\mathbf{a}$ are the coefficients that model the ideal data $\mathbf{m}$

- So we have the following two problems:

Interpolation: Sparse or Compressive Solution

- Find the coefficient such that

$$\mathbf{d} = \mathbf{S}\mathbf{m}, \quad \mathbf{m} = \mathbf{F}\mathbf{a},$$

Measurements

- The above is equivalent to $\mathbf{d} = \mathbf{S}\mathbf{F}\mathbf{a},$

- Which can be solved by minimizing the l2-l1 cost function

$$J = \|\mathbf{d} - \mathbf{S}\mathbf{F}\mathbf{a}\|_2^2 + \mu\|\mathbf{a}\|_1$$

Interpolation

- Interpolated data if found by solving

$$\hat{\mathbf{a}} = \mathbf{argmin}_{\mathbf{a}} \|\mathbf{d} - \mathbf{SFa}\|_2^2 + \mu\|\mathbf{a}\|_1$$

$$\hat{\mathbf{m}} = \mathbf{F}\hat{\mathbf{a}},$$

- This is also the main idea behind CS (Compressive Sensing) where Sampling is Random

- Can be applied to ND-problems by using an ND transform (e.g. ND Fourier or Curvelet transforms)

Interpolation: Example (Explicit)

- Interpolated data is found by solving

$$\hat{\mathbf{a}} = \mathbf{argmin_a} \, \|\mathbf{d} - \mathbf{SFa}\|_2^2 + \mu\|\mathbf{a}\|_1$$

$$\hat{\mathbf{m}} = \mathbf{F}\hat{\mathbf{a}},$$

- I can write the transform in matrix form for some simple problems. For instance the Fourier synthesis operator can be written easily as a matrix (DFT Matrix)

- Of course, it is much faster to use the FFT rather than the DFT expressed as a matrix multiplication operation

Interpolation: Example (Explicit)

- F is the inverse DFT, that we will call the Fourier Synthesis Operator

$$x_k = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} X_k e^{i2\pi nk/N}$$

- In matrix form

$$\mathbf{x} = \mathbf{F}\,\mathbf{X} \qquad\qquad \mathbf{X} = \mathbf{F}^H\,\mathbf{x}$$

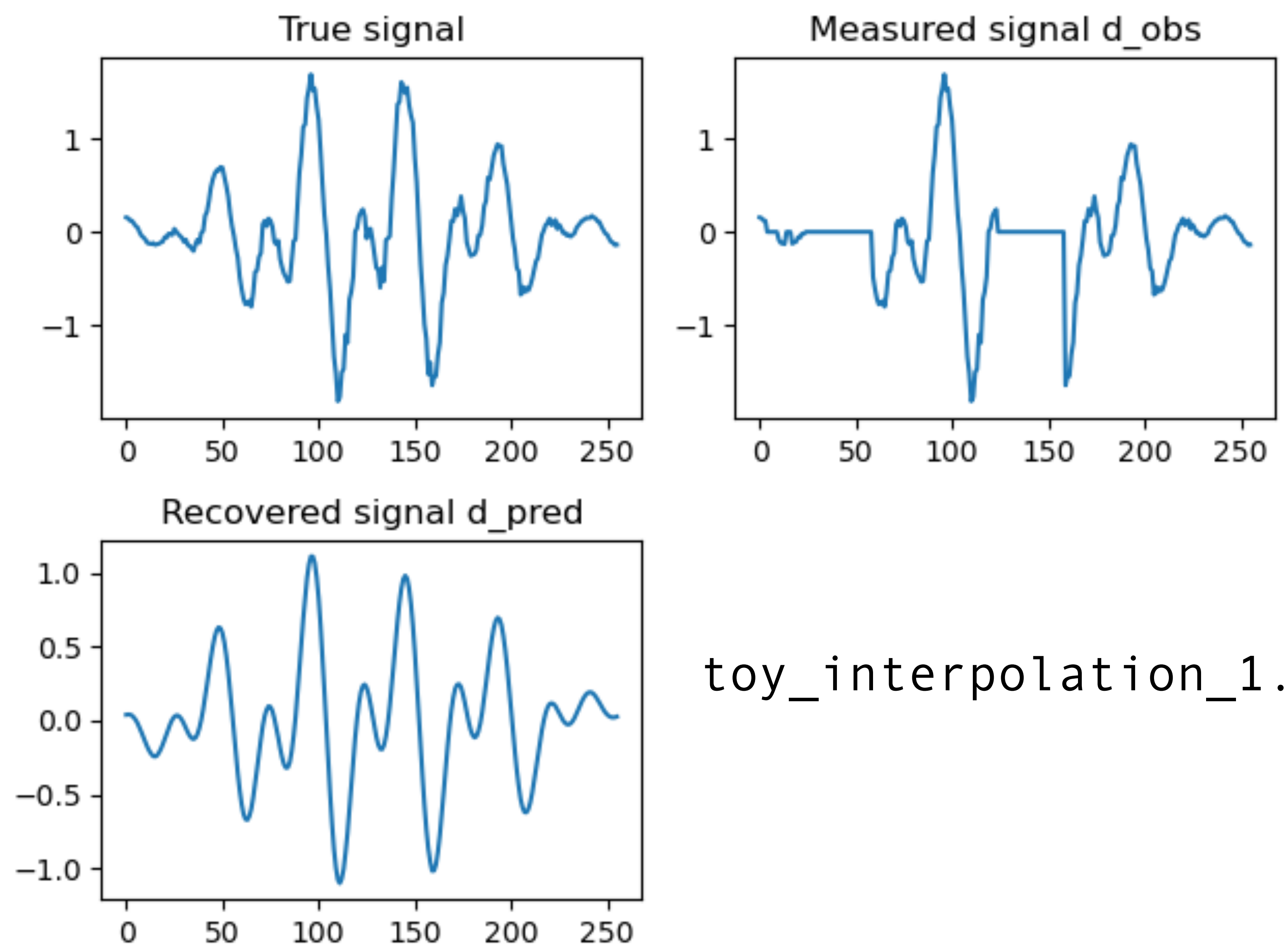
- Where the elements of the Fourier operator (Matrix) are

$$F_{n,k} = \frac{1}{\sqrt{N}} e^{i2\pi nk/N}$$

Explicit DFT-matrix IRLS code

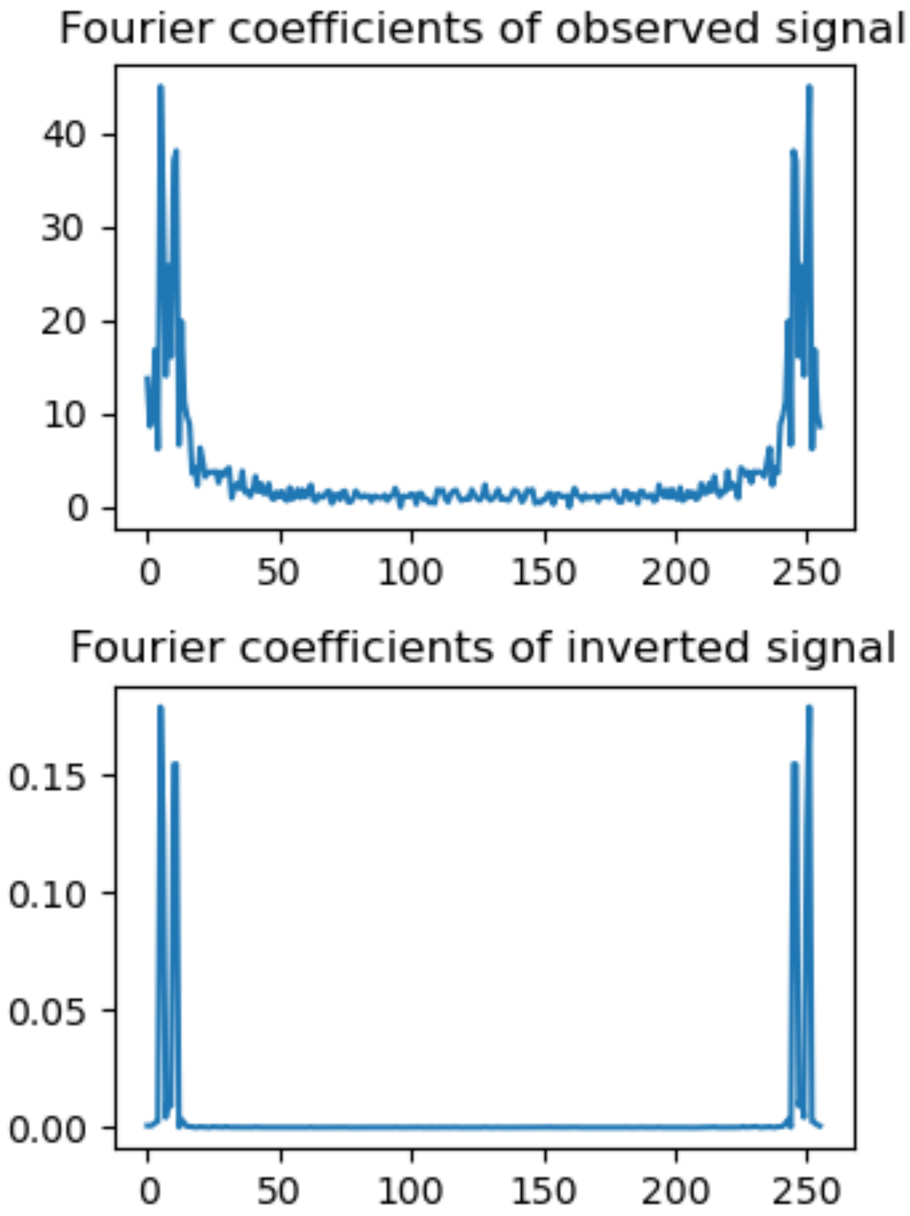
```
L = S*F          # This is the linear operator to invert

mu = 10.1

# Start IRLS

a = L'*d_obs

for k=1:10
    q = 1.0./(abs.(a).+0.001)
    Q = Diagonal(q)
a = (L'*L+mu*Q)\(L'*d_obs)

end
d_interp = F*a
```

Interpolation: Example (Explicit)



toy_interpolation_1.ipynb

Interpolation: Example (Explicit)



toy_interpolation_1.ipynb

This examples shows the initial and inverted (Sparse) Fourier Coefficients

Interpolation: Example (Implicit)

- Interpolated data if found by solving

$$\hat{\mathbf{a}} = \mathbf{argmin}_{\mathbf{a}} \|\mathbf{d} - \mathbf{SFa}\|_2^2 + \mu \|\mathbf{a}\|^2$$

$$\hat{\mathbf{m}} = \mathbf{F}\hat{\mathbf{a}},$$

- Now in the implicit case, $\mathbf{F}$ is not a matrix, $\mathbf{F}$ = `FFT`

- Therefore, I can't treat the problem as before and construct operators such $((\mathbf{SF})^H \mathbf{SF} + \mu \mathbf{Q})^{-1}$ as needed by the sparse inversion solver (IRLS).

Interpolation: Example (Implicit)

- Let's review IRLS

$$\hat{\mathbf{a}} = \mathbf{argmin}_{\mathbf{a}} \|\mathbf{d} - \mathbf{SFa}\|_2^2 + \mu\|\mathbf{a}\|_1$$

- Gradient of cost $\mathbf{g} = \mathbf{L}^H(\mathbf{La} - \mathbf{d}) + \mu\mathbf{Qa}$

- With $\mathbf{L} = \mathbf{SF}$ $\mathbf{Q}_{ii} = 1/(|a|_i + \epsilon)$

- Steepest descent iteration

$$\mathbf{a}^{new} = \mathbf{a}^{old} - \lambda[\mathbf{L}^H(\mathbf{La}^{old} - \mathbf{d}) + \mu\mathbf{Qa}^{old}]$$

Interpolation: Example (Implicit)

- Steepest descent iteration

$$\mathbf{a}^{new} = \mathbf{a}^{old} - \lambda[\mathbf{L}^H(\mathbf{L}\mathbf{a}^{old} - \mathbf{d}) + \mu \mathbf{Q}\mathbf{a}^{old}]$$

- Replace explicit operators (Matrices) by implicit FFTs

$$\mathbf{L}[\,.\,] = \mathbf{S}[\,\mathbf{ifft}\,[\,.\,]]$$

$$\mathbf{L}^H[\,.\,] = \;\mathbf{fft}\;[\mathbf{S}^T[\,.\,]]$$

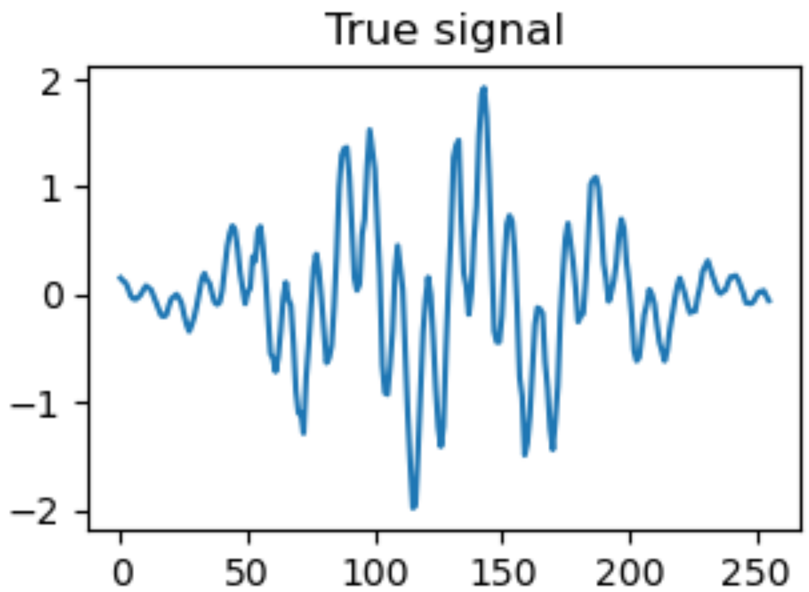
- Program does not require Fourier DFT matrix and hence is faster!!

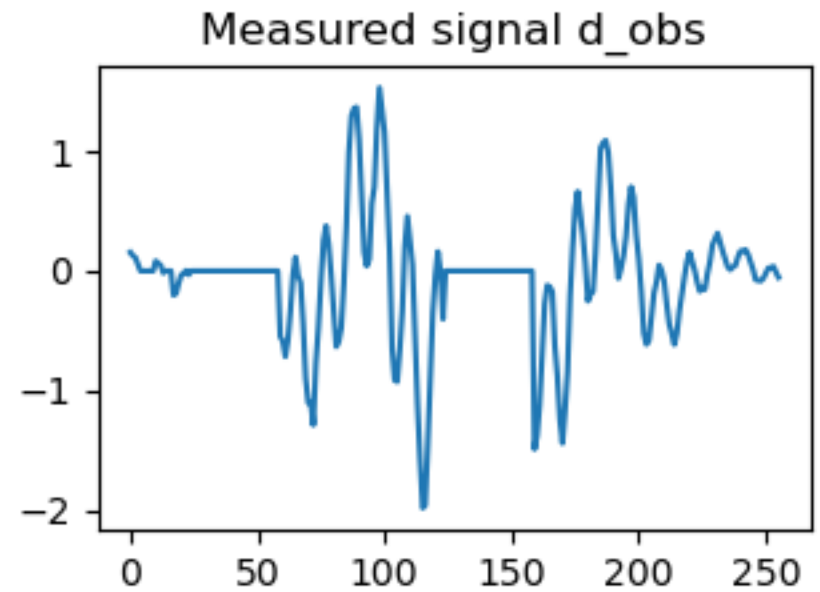
Implicit FFT code via steepest descent

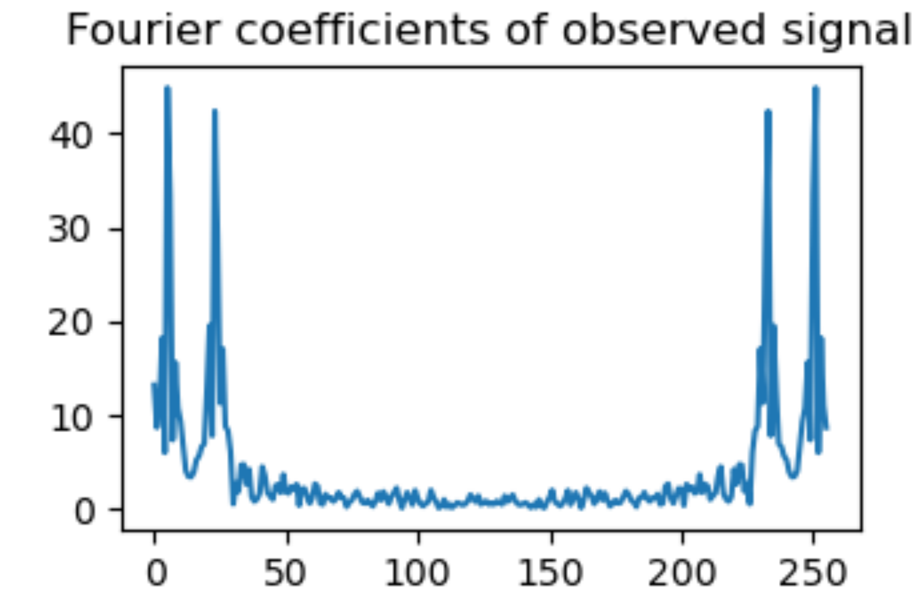
```
for k=1:400
    q = 1.0./(abs.(a).+0.01)
    Q = Diagonal(q)
    r = S*(bfft(a))-d_obs                       # r = SFa-d_obs
    a = a .-0.005*(fft(S'*r).+ mu*Q*a)   # a = a -0.005*(F^H S^T(r))
    Save_P[k,:]=abs.(a)
    Save_Q[k,:]=abs.(q)
end
```

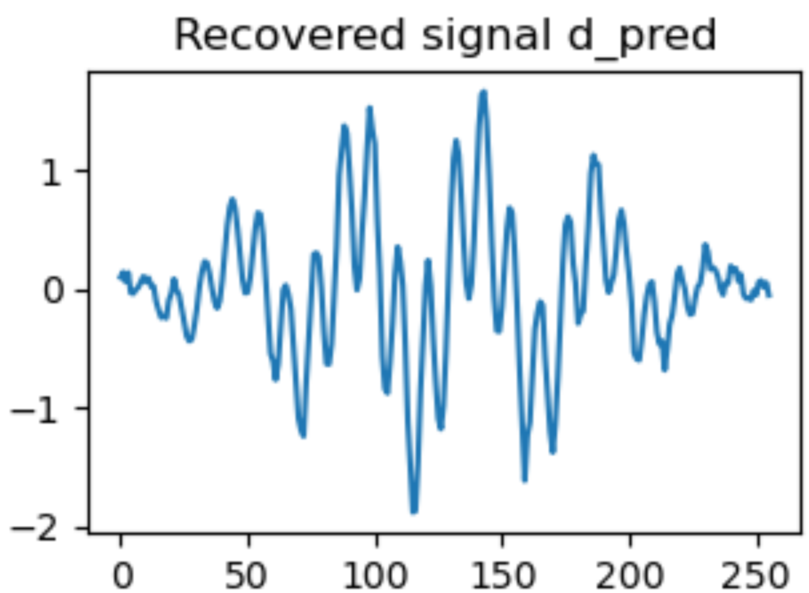
Be careful with the normalization of the FFT and IFFT because I really need the adjoint not the inverse transform (above I use bfft)

Interpolation: Example (Implicit)

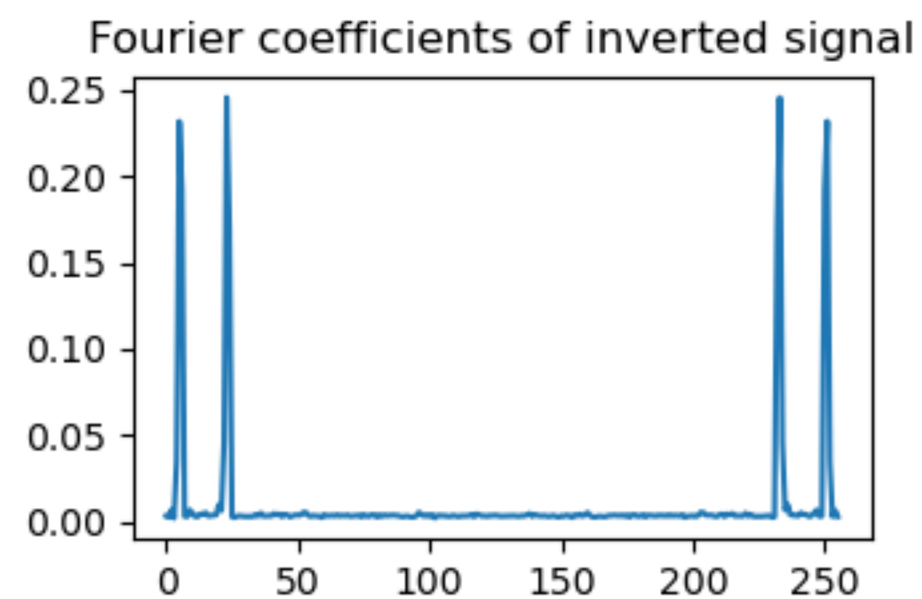






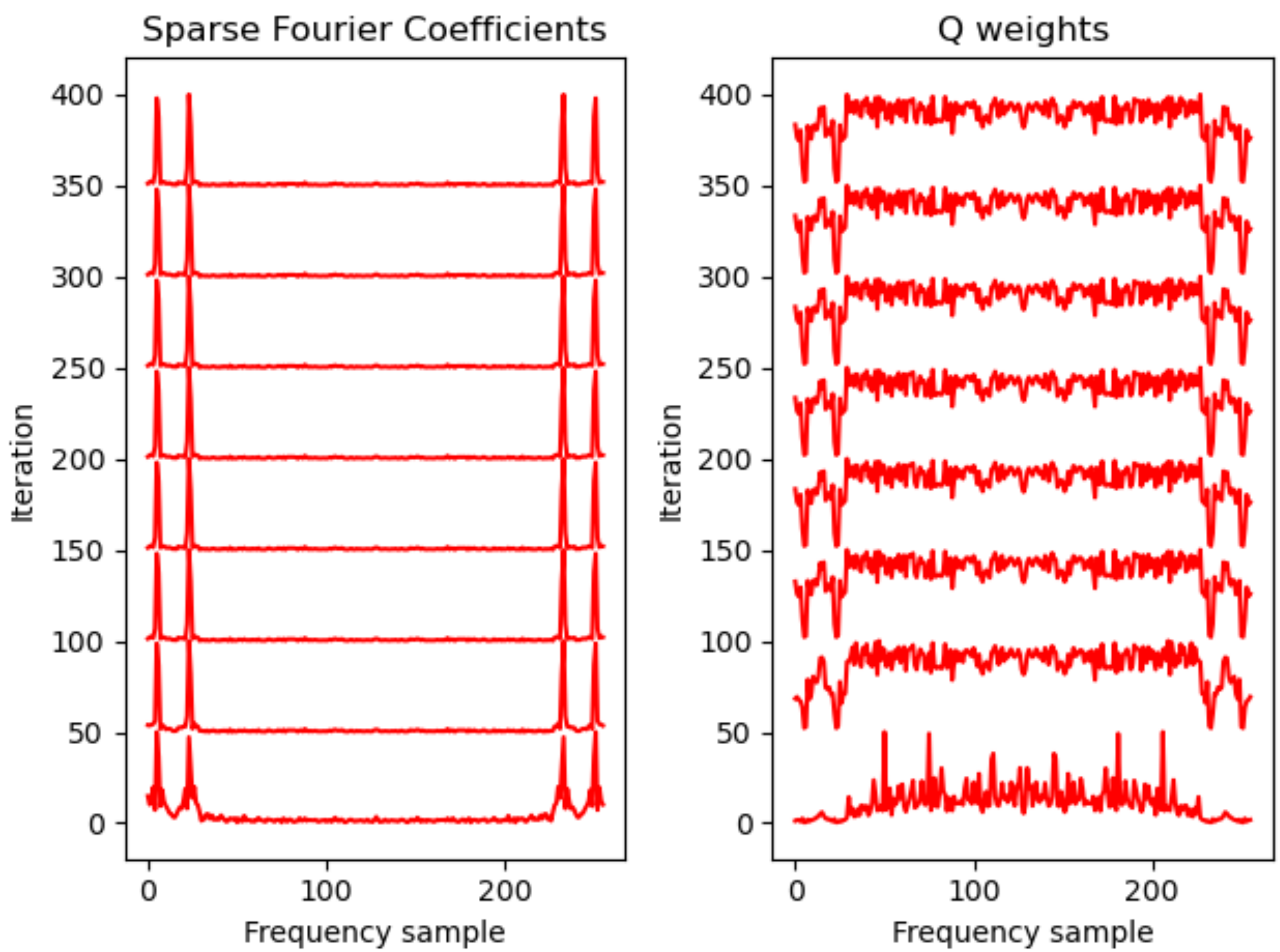


`toy_interpolation_2.ipynb`



Here I used the FFT to synthesize forward and adjoint operators

Interpolation: Example (Implicit)



`toy_interpolation_2.ipynb`

Evolution of Fourier Coefficients and Q weights vs iteration

Two flavour of sampling operator

Flavour 1 (matrix mult)

$$\begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \\ m_7 \\ m_9 \end{bmatrix}$$

$$\mathbf{d} = \mathbf{Sm}$$

$$J = \|\mathbf{d} - \mathbf{SFa}\|_2^2 + \mu\|\mathbf{a}\|_1$$

Flavour 2 (element-to-element mult)

$$\begin{bmatrix} d_1 \\ 0 \\ 0 \\ d_2 \\ 0 \\ d_3 \\ 0 \\ 0 \\ d_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \\ m_7 \\ m_8 \\ m_9 \end{bmatrix}$$

$$\mathbf{d} = \mathbf{Sm} = \mathbf{s} \circ \mathbf{m}$$

$$\mathbf{d} = \mathbf{s} \circ \mathbf{d}$$

Element-to-element multiplication

$$\mathbf{s} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$J = \|\mathbf{s} \circ [\mathbf{d} - \mathbf{Fa}]\|_2^2 + \mu\|\mathbf{a}\|_1$$

$$J = \|\mathbf{d} - \mathbf{s} \circ \mathbf{Fa}]\|_2^2 + \mu\|\mathbf{a}\|_1$$