Part 1
Preliminaries, Notation and Linear Algebra

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Problemas Inversos - UNLP

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Preliminaries: Notation

$f(\mathbf{x})$: **scalar function of a vector**

$\mathbf{x}$: **a vector**

x : **a scalar**

α : **a scalar**

$\mathbf{M}$: **a matrix**

$M_{i,j}$: **element of the matrix**

J : **scalar cost function**

In general, we will upper case bold fonts for matrices, and lower case bold fonts for vectors.
Lower case non-bold are scalars. Operators are denoted by calligraphic fonts. All the above is true unless we indicate there is an exception.

Preliminaries: Notation

- $J = f(\mathbf{x})$
- $\mathbf{u} = f(\mathbf{x})$
- $\alpha = \mathbf{M}$
- $\mathbf{AB} = \mathbf{C}$
- $\alpha + \mathbf{M} = \mathbf{v}$
- $\|\mathbf{v}\|_2^2 = \alpha$
- $\|\mathbf{v}\|_2^2 = \mathbf{u}$

Preliminaries: Notation

- $J = f(\mathbf{x})$ ok
- $\mathbf{u} = f(\mathbf{x})$ X
- $\alpha = \mathbf{M}$ X
- $\mathbf{AB} = \mathbf{C}$ ok
- $\alpha + \mathbf{M} = \mathbf{v}$ X
- $\|\mathbf{v}\|_2^2 = \alpha$ ok
- $\|\mathbf{v}\|_2^2 = \mathbf{u}$ X

Preliminaries: ell-2 norm

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_N \end{pmatrix}$$

- Real vector

$$\mathbf{u} : N \times 1 \longrightarrow \|\mathbf{u}\|_2^2 = \mathbf{u}^T\mathbf{u} = \sum_{i=1}^{N} u_i^2$$

- Complex vector

$$\mathbf{u} : N \times 1 \longrightarrow \|\mathbf{u}\|_2^2 = \mathbf{u}^H\mathbf{u} = \sum_{i=1}^{N} u_i u_i^* = \sum_{i=1}^{N} |u_i|^2$$

- The ell-2 norm

$$l_2 = \|\mathbf{u}\|_2 = \sqrt{\|\mathbf{u}\|_2^2}$$

Preliminaries: ell-2 norm

$$\|\mathbf{u}\|_2^2 = \mathbf{u}^T\mathbf{u} = \begin{pmatrix} u_1 & u_2 & u_3 & u_4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \sum_{k=1}^{4} u_k^2$$

$$\|\mathbf{u}\|_2^2 = \mathbf{u}^H\mathbf{u} = \begin{pmatrix} u_1^* & u_2^* & u_3^* & u_4^* \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \sum_{k=1}^{4} |u_k|^2$$

Preliminaries: Matrix-times-vector

$\mathbf{A}$: *Matrix*
$\mathbf{x}$: *Vector*
$\mathbf{y}$: *Vector*
α : *scalar*

A_{ij} : *Element of the Matrix* $\mathbf{A}$
x_i : *Element of Vector* $\mathbf{x}$
y_i : *Element of Vector* $\mathbf{y}$

Matrix Multiplication

$$\underbrace{\mathbf{y}}_{N \times 1} = \underbrace{\mathbf{A}}_{N \times M} \; \underbrace{\mathbf{x}}_{M \times 1}$$

Preliminaries: Matrix-times-vector

$$\underset{N \times 1}{\mathbf{y}} = \underset{N \times M}{\mathbf{A}} \quad \underset{M \times 1}{\mathbf{x}}$$

Inner Loop

Outer Loop

$$y_i = \sum_{j=1}^{M} A_{ij} \, x_j \qquad i = 1 \ldots N$$

Input index

Output index

Notation and Review of Linear Algebra

$$\mathbf{y} = \mathbf{A} \quad \mathbf{x}$$

Matrix-times-vector multiplication

In [1]:
```
N = 4
M = 5
A = randn(Float32,N,M)
x = randn(Float32,M)

y = A*x
```

Out[1]:
```
4-element Vector{Float32}:
 -0.48016113
  3.7603946
  4.958347
  0.91909456
```

My own code...

In [2]:
```
y= zeros(Float32,N)
for i = 1:N                               # Outer loop over data parameters
    for j=1:M                             # Inner loop over model parameters
        y[i]+=A[i,j]*x[j]
    end
end
y
```

Out[2]:
```
4-element Vector{Float32}:
 -0.48016113
  3.7603943
  4.9583473
  0.91909456
```

Preliminaries: Matrix-times-vector

$$\underset{N \times 1}{\mathbf{y}} = \underset{N \times M}{\mathbf{A}} \underset{M \times 1}{\mathbf{x}}$$

$$y_i = \sum_{j=1}^{M} A_{ij} \, x_j \quad i = 1 \ldots N$$

Forward mode

$$\underset{M \times 1}{\mathbf{x}'} = \underset{M \times N}{\mathbf{A}^T} \underset{N \times 1}{\mathbf{y}}$$

$$x'_j = \sum_{i=1}^{N} A_{ij} \, y_i \quad j = 1 \ldots M$$

Transpose mode

Preliminaries: Matrix-times-vector

$$\underset{N\times 1}{\mathbf{y}} = \underset{N\times M}{\mathbf{A}} \ \underset{M\times 1}{\mathbf{x}}$$

$$y_i = \sum_{j=1}^{M} A_{ij}\, x_j \quad i = 1\ldots N$$

Forward mode

$$\underset{M\times 1}{\mathbf{x}'} = \underset{M\times N}{\mathbf{A}^H} \ \underset{N\times 1}{\mathbf{y}}$$

$$x'_j = \sum_{i=1}^{N} A^*_{ij}\, y_i \quad j = 1\ldots M$$

Conjugate mode