ISTA (Iterative Shrinkage-Thresholding Algorithm)

M D Sacchi *

GEOPH 531 W-2023

1 Preliminaries

When solving inverse problems and many signal processing problems, such as signal reconstruction, we often encounter the following

$$\begin{aligned}\hat{\mathbf{x}} &= \underset{\mathbf{x}}{\operatorname{argmin}}\ \{f(\mathbf{x}) + \lambda\, g(\mathbf{x})\} \\ &= \underset{\mathbf{x}}{\operatorname{argmin}}\ \{\|\mathbf{A}\mathbf{x}-\mathbf{y}\|_2^2 + \lambda\|\mathbf{x}\|_1\}\,. \end{aligned} \tag{1}$$

Problems that can be written as equation 1 are often called $l_2 - l_1$ problems. The statistical literature often calls the problem given by equation 1 Basis Pursuit Denoising (BPDN). We are trying to find the minimum of J, the sum of two convex functions: the quadratic l_2 loss (misfit) and the l_1 regularization term. The main idea is to recover a sparse signal $\mathbf{x}$ from observations $\mathbf{y}$. These algorithms are used in Compressive Sensing to recover signals that have been compressed via a randomized sampling process [Baraniuk, 2007]. The problem is sometimes formulated as follows

$$\min \|\mathbf{x}\|_1 \quad \text{subject to } \|\mathbf{A}\mathbf{x}-\mathbf{y}\|_2^2 \le \delta \tag{2}$$

Equation 1 is often called the unconstrained form of BPDN [Chen et al., 1998]. Conversely, equation 2 is the constrained form of the problem. In what follows, we will adopt the unconstrained form where the single tradeoff parameter λ could be tuned to yield a sparse solution where $\|\mathbf{A}\mathbf{x}-\mathbf{y}\|_2^2 \le \delta$. In other words, we can find the constrained-form solution from the unconstrained problem. I prefer to use the unconstrained form of BPDN because it reminds me of classical Tikhonov regularization (the damped least-squares method), but

*emal: msacchi@ualberta.ca

with the critical difference that the l_1 regularization replaces the l_2-norm regularization norm. The unconstrained form of the problem also has a simple Bayesian interpretation, whereas the constrained form (to my knowledge) does not.

1.1 ISTA solution

I will start with the general problem where we minimize the function $J = f(\mathbf{x}) + \lambda g(\mathbf{x})$ where f and g are convex functions

$$\hat{\mathbf{x}} = \underset{\mathbf{x}}{\operatorname{argmin}} \; \{ f(\mathbf{x}) + \lambda \, g(\mathbf{x}) \} \tag{3}$$

and then move on with the problem that involves minimizing the l_2 misfit in conjunction with an l_1 regularization term [Daubechies et al., 2004]. The function $f(\mathbf{x})$ can be approximated as follows

$$f(\mathbf{x}) \approx f(\mathbf{x}_k) + \nabla f_k^T (\mathbf{x} - \mathbf{x}_k) + \frac{1}{2\eta} \|\mathbf{x} - \mathbf{x}_k\|_2^2 \tag{4}$$

where ∇f_k^T is the gradient of $f(\mathbf{x})$ at $\mathbf{x}_k$. Notice that if you take the derivative of the last equation and equate it to zero; you will get the classical steepest descent step for updating the variable $\mathbf{x}$. Hence, we can propose an algorithm that updates $\mathbf{x}$ in equation 1 via

$$\mathbf{x}_{k+1} = \underset{\mathbf{x}}{\operatorname{argmin}} \{ \, f(\mathbf{x}_k) + \nabla f_k^T \, (\mathbf{x} - \mathbf{x}_k) + \frac{1}{2\eta} \|\mathbf{x} - \mathbf{x}_k\|_2^2 + \lambda \, g(\mathbf{x}) \, \} \, . \tag{5}$$

I can complete squares in the last expression and obtain the following

$$\mathbf{x}_{k+1} = \underset{\mathbf{x}}{\operatorname{argmin}} \{ \; f(\mathbf{x}_k) + \frac{1}{2\eta} \|(\mathbf{x} - \mathbf{x}_k) + \eta \nabla f_k)\|_2^2 - \frac{\eta}{2} \nabla f_k^T \nabla f_k + \lambda \, g(\mathbf{x}) \} \, . \tag{6}$$

Now, I only keep terms that depend on the variable $\mathbf{x}$ (the others are constants that become zero after differentiation)

$$\mathbf{x}_{k+1} = \underset{\mathbf{x}}{\operatorname{argmin}} \{ \; \frac{1}{2\eta} \|(\mathbf{x} - \mathbf{x}_k) + \eta \nabla f_k)\|_2^2 + \lambda \, g(\mathbf{x}) \} \, . \tag{7}$$

The last expression can be written as a denoising problem

$$\mathbf{x}_{k+1} = \underset{\mathbf{x}}{\operatorname{argmin}} \{ \; \frac{1}{2} \|(\mathbf{x} - \mathbf{u})\|_2^2 + \eta \lambda \, g(\mathbf{x}) \} \, , \tag{8}$$

where $\mathbf{u} = \mathbf{x}_k - \eta \nabla f_k$. The above is a denoising problem where one tries to approximate the vector $\mathbf{u}$ by $\mathbf{x}$ with an additional regularization $g(\mathbf{x})$. The proximal operator gives the solution to equation 8. We generally choose g such that the solution reduces to a univariate minimization problem with an analytical answer. Equation 8 is written as follows

$$\begin{aligned} \mathbf{x}_{k+1} =& \mathrm{Prox}_{g,\lambda\eta}[\mathbf{u}] & (9) \\ =& \mathrm{Prox}_{g,\lambda\eta}[\mathbf{x}_k - \eta \nabla f_k] \,. & (10) \end{aligned}$$

1.2 Proximal operator for $g(\mathbf{x}) = \|\mathbf{x}\|_1$

The proximal operator for $g(\mathbf{x}) = \|\mathbf{x}\|_1$ is named the soft-thresholding operator $\mathcal{S}_{\lambda\eta}$ and is the solution that minimizes

$$\mathcal{L} = \frac{1}{2}\|\mathbf{x} - \mathbf{v}\|_2^2 + a\,\|\mathbf{x}\|_1 = \frac{1}{2}\sum_i |x_i - v_i|^2 + a\,\sum_i |x_i|\,, \tag{11}$$

where $a > 0$. Setting $\frac{\partial \mathcal{L}}{\partial x_k} = 0$ leads to

$$x_k - v_k + a\,\mathrm{sign}(x_k) = 0\,. \tag{12}$$

The latter can be split into

$$\begin{aligned} v_k =& x_k - a \quad \text{if} \quad x_k < 0 & (13) \\ v_k =& x_k + a \quad \text{if} \quad x_k > 0 & (14) \end{aligned}$$

The last expression needs to be inverted because we need x_k as a function of v_k, an operation that can be carried out graphically by first plotting the last expression $v_k = h_a(x_k)$ and then graphically (Figure 1) finding $x_k = h_a^{-1}(v_k) = \mathcal{S}_a(v_k)$ which is the soft-thresholding operator

$$\mathcal{S}_a(v_k) = \begin{cases} v_k - a & v_k > a \\ 0 & |v_k| \le a \\ v_k + a & v_k < -a \end{cases} \tag{15}$$

The operator can be written in a more compact form as follows

$$\mathcal{S}_a(v_k) = \mathrm{sign}(v_k)\max(|v_k| - a, 0)\,. \tag{16}$$

1.3 Recap ISTA

Let us go back to our original problem $f(\mathbf{x}) = \|\mathbf{Ax}-\mathbf{y}\|_2^2$ and $g(\mathbf{x}) = \|\mathbf{x}\|_1$. Then, according to equations 10 and 15

$$\mathbf{x}_{k+1} = \mathcal{S}_{\lambda\eta}[\mathbf{u}] \tag{17}$$

$$= \mathcal{S}_{\lambda\eta}[\mathbf{x}_k - \eta\mathbf{A}^T(\mathbf{Ax}_k - \mathbf{y})] \, . \tag{18}$$

where the proximal operator (Soft thresholding, equation 15) is applied element-wise. The step length η must satisfy $\eta < 1/\lambda_{max}$ where λ_{max} is the maximum eigenvalue of $\mathbf{A}^T\mathbf{A}$. The maximum eigenvalue of $\mathbf{A}^T\mathbf{A}$ can be iteratively found via the Power Method [Golub and Van Loan, 1996].

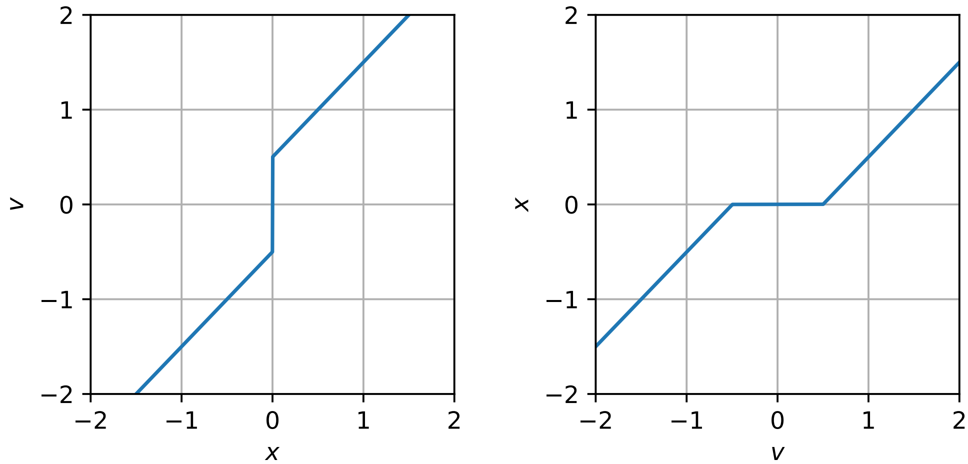

Figure 1: Left is equation 12, $v = h_a(x)$. Right is the soft thresholding operator $x = \mathcal{S}_a(v)$ (equation 15), $a = 0.5$.

2 Example

Figure 2 shows the inversion of a sparse sequence $\mathbf{x}$ that has been compressed via a random matrix $\mathbf{A}$. The compressed data is given by $\mathbf{y} = \mathbf{Ax}+\mathbf{e}$ where $\mathbf{y}$ has 40 points. The original signal $\mathbf{x}$ has 150 points. This is an underdetermined problem, and we are exploiting the fact that $\mathbf{x}$ is sparse to recover it from the measurement vector $\mathbf{y}$. I am comparing ISTA, FISTA (Fast-ISTA) [Beck and Teboulle, 2009] and IRLS [Sacchi et al., 1998]. Figure 2 provides the convergence curves of these three algorithms.

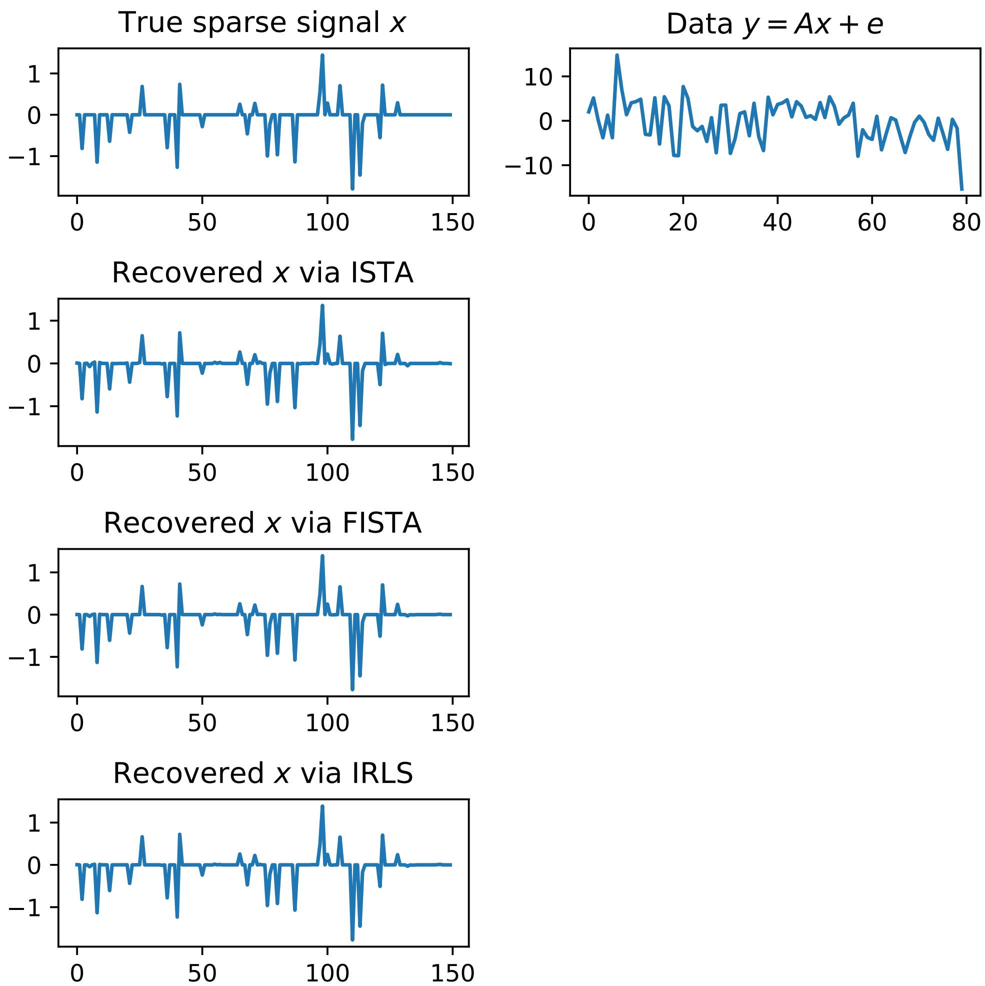

Figure 2: Inversions via ISTA, FISTA and IRLS.

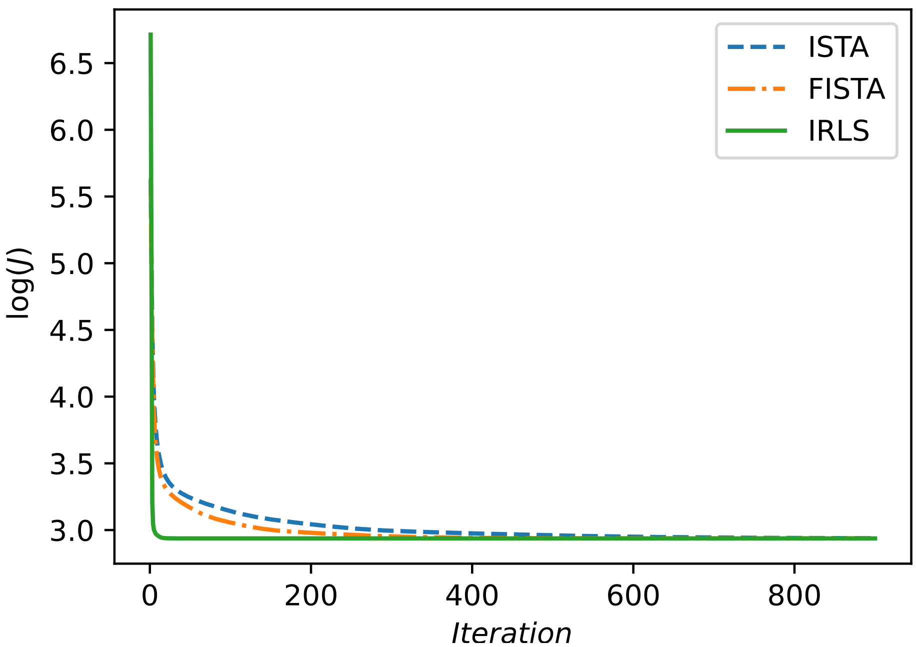

Figure 3: Convergence curves comparing ISTA, FISTA and IRLS.

Notice that in the paper by Sacchi et al. [1998], IRLS is used to solve the Fourier sparse reconstruction problem via a Cauchy sparsity norm. A similar approach is used for multidimensional seismic signal reconstruction by Zwartjes and Gisolf [2007]. The last two references are a good starting point for understanding ND seismic data reconstruction as it is used today by seismic data processing contractors.

	ISTA	FISTA	IRLS
$RMSE \times 100$	0.462	0.178	0.198

Table 1: Recovery error for the example in Figure 2 where $RMSE = \|\mathbf{x} - \mathbf{x}_{true}\|_2^2 / \|\mathbf{x}_{true}\|_2^2$.

3 ISTA Code

```
function ISTA(A,y,Niter,λ)
# ISTA solver. Finds x that minimizes
# J = 1/2||A x - y ||_2^2 + λ ||x||_1

  Soft(x,alpha) = sign(x)*max(abs(x)-alpha, 0)

  N,M = size(A)
  e = Power_Iteration(A)  #
  η = 0.95/e

  x = zeros(Float64,M)

  J = zeros(Niter)
  for k = 1:Niter
    u = x .- η*A'*(A*x.-y)
    x = Soft.(u, η*λ)
    J[k] = 0.5*sum((A*x-y).^2) + λ*sum(abs.(x))
  end
    return x, J
end
```

References

R. G. Baraniuk. Compressive sensing [lecture notes]. *IEEE Signal Processing Magazine*, 24 (4):118–121, 2007.

A. Beck and M. Teboulle. A fast iterative shrinkage-thresholding algorithm for linear inverse problems. *SIAM Journal on Imaging Sciences*, 2(1):183–202, 2009.

S. S. Chen, D. L. Donoho, and M. A. Saunders. Atomic decomposition by basis pursuit. *SIAM Journal on Scientific Computing*, 20(1):33–61, 1998.

I. Daubechies, M. Defrise, and C. De Mol. An iterative thresholding algorithm for linear inverse problems with a sparsity constraint. *Communications on Pure and Applied Mathematics*, 57(11):1413–1457, 2004.

G. H. Golub and C. F. Van Loan. *Matrix Computations*. The Johns Hopkins University Press, third edition, 1996.

M. Sacchi, T. Ulrych, and C. Walker. Interpolation and extrapolation using a high-resolution discrete fourier transform. *IEEE Transactions on Signal Processing*, 46(1): 31–38, 1998.

P. Zwartjes and A. Gisolf. Fourier reconstruction with sparse inversion. *Geophysical Prospecting*, 55(2):199–221, 2007.