Proving $f(n) = \frac{\phi^n - \psi^n}{\sqrt{5}}$

Proof by induction. The base case is for $f(n) = 0$ and $f(n+1) = 1$. It can be clearly seen that:

$$\frac{\phi^0 - \psi^0}{\sqrt{5}} = 0 \text{ and } \frac{\phi^1 - \psi^1}{\sqrt{5}} = 1. \tag{1}$$

Next, It must be shown that $f(n+2) = f(n+1) + f(n)$ given that $f(n) = \frac{\phi^n - \psi^n}{\sqrt{5}}$ and $f(n+1) = \frac{\phi^{n+1} - \psi^{n+1}}{\sqrt{5}}$. By direct substitution : $f(n+2) = \frac{\psi^n - \psi^n}{\sqrt{5}} + \frac{\phi^{n+1} - \psi^{n+1}}{\sqrt{5}}$. Which yields:

$$f(n+2) = \frac{(\phi^n + \phi^{n+1}) - (\psi^n + \psi^{n+1})}{\sqrt{5}} \tag{2}$$

This can be further reduced by taking $n+2$ as a common factor to get $f(n+2) = \frac{\phi^{n+2}(\phi^{-2} + \phi^{-1}) - \psi^{n+2}(\psi^{-2} + \psi^{-1})}{\sqrt{5}}$. The result can be further simplified as:

$$f(n+2) = \frac{\phi^{n+2}(\frac{\phi+1}{\phi^2}) - \psi^{n+2}(\frac{\psi+1}{\psi^2})}{\sqrt{5}}. \tag{3}$$

Recall that both ϕ and ψ are the solution of the equation $x + 1 = x^2$. Therefore the following is true:

- $\phi + 1 = \phi^2$
- $\psi + 1 = \psi^2$

Substitution with the above produces the result:

$$f(n+2) = \frac{\phi^{n+2} - \psi^{n+2}}{\sqrt{5}} \tag{4}$$

As such, the induction process is complete. Therefore:

$$f(n) = \frac{\phi^n - \psi^n}{\sqrt{5}} \tag{5}$$

Is true for all n in the domain of f. As desired $\square$

$f(n)$ is the closest integer to $\frac{\phi^n}{\sqrt{5}}$

Proof. For $f(n)$ to be the closest integer to $\frac{\phi^n}{\sqrt{5}}$, the following inequality must be true: $|f(n) - \frac{\psi^n}{\sqrt{5}}| \leq \frac{1}{2}$.

However, by equation (5) we obtain $|\frac{\phi^n - \psi^n}{\sqrt{5}} - \frac{\phi^n}{\sqrt{5}}| \leq \frac{1}{2}$, which can be re-written as:

$$|\psi|^n \leq \frac{\sqrt{5}}{2} \tag{6}$$

It is known that $|\psi| < 1$. Therefore, $|\psi|^n < |\psi^{n-1}|$ for all $n \in \mathbb{N}$, and since $|\psi| < \frac{\sqrt{5}}{2}$ it follows that $|\psi|^n < \frac{\sqrt{5}}{2}$, for all n in $\mathbb{N}$

$\square$