# Assignment 1: Partitioned Integration

- The assignment should be handed in at the latest Wednesday 6th March 2024 12:00 in order to receive full credit, otherwise a 2 point penalty is applied to the grade of the assignment.
- The assigment can be made in groups of **at most 3 students**.
- Please be sure to include the names and student numbers in your report.
- Please hand in an electronic version (also a scanned version of a (clearly) hand-written report is accepted) through Brightspace (only one submission per group is required).
- You do not have to make a full report (including summary, list of figures, list of symbols etc), but indicate clearly the question numbers when giving your answer / results / discussion of results.
- When an implementation is requested, please provide (only) the code snippets where you made your implementation.

## 1 One-dimensional piston problem

In the lectures we discussed the Finite Volume approach to discretize the fluid dynamics equations. We are going to use this method to model the one-dimensional piston problem for inviscid, isentropic, linearised flow. The piston problem is sketched in Fig. 1 and consists of a fluid do-

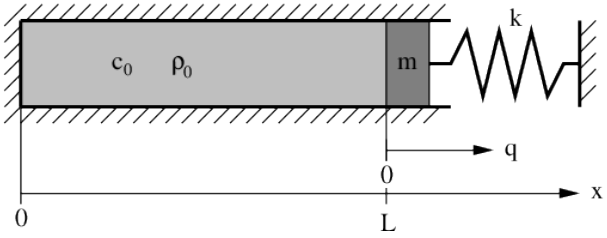


**Figure 1:** One-dimensional piston problem.

main of length $L$ filled with an ideal gas and closed at the right side by a piston with mass $m$ which is attached to a wall by a spring with stiffness $k$.

### Governing equations

Since the flow is invisced and isentropic, the governing equations for the flow are given by conservation of mass and momentum

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)} \begin{pmatrix} \rho \\ \rho\vec{u} \end{pmatrix} \mathrm{d}V + \oint_{\partial\Omega(t)} \begin{pmatrix} \rho\vec{u}\cdot\vec{n} \\ \rho\vec{u}\vec{u}\cdot\vec{n} + p\vec{n} \end{pmatrix} \mathrm{d}S = \vec{0}, \tag{1}$$

wherein $\rho$ the density, $\vec{u}$ the velocity (which only has a contribution in $x$ direction), $\vec{n}$ the outward pointing normal on the boundary $\partial\Omega$ of the control volume $\Omega$. The pressure $p$ satisfies the isentropic relation $p/\rho^\gamma = \text{constant}$, with $\gamma$ the specific heat ratio.

The structure dynamics are governed by linear mass-spring dynamics

$$mS\ddot{q} + kSq = S(p_I - p_0), \tag{2}$$

wherein $q$ the piston displacement, $m$ its mass per unit surface area, $k$ the spring stiffness per unit surface area, $S$ the surface area of the piston (and fluid domain), $p_I$ the pressure at the fluid-structure interface and $p_0$ the ambient pressure. In the following the surface area $S$ of the piston is chosen identical to the "cross section" of the fluid domain and equal to 1.

**Exercise 1.1** Which are the dynamic and kinematic interface conditions to be satisfied at the fluid-structure interface?

### Linearization

First the problem is linearized by assuming only small perturbations of $O(\epsilon)$ compared to the equilibrium solution $\rho_0 = 1, u_0 = 0, p_0 = 1/\gamma$ and a fluid domain length $L$, and for the structure $q_0 = 0, \dot{q}_0 = 0$, where $\dot{q}$ represents the time derivative $\frac{\mathrm{d}q}{\mathrm{d}t}$. For convenience, we drop the formal vector-indication above $u$ and $n$. The variables are now written as

$$\begin{aligned} \rho &= \rho_0 + \rho', \\ u &= u_0 + u', \\ p &= p_0 + p', \\ q &= q_0 + q', \\ \dot{q} &= \dot{q}_0 + \dot{q}'. \end{aligned}$$

Each of the perturbations are denoted by $(\cdot)'$ and are of $O(\epsilon)$, $\epsilon \ll 1$. Substitution into the governing equation (1), and eliminating all $O(\epsilon^2)$ or higher order terms, gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \begin{pmatrix} \rho' \\ \rho_0 u' \end{pmatrix} \mathrm{d}V + \oint_{\partial\Omega} \begin{pmatrix} \rho_0 u' \cdot n \\ c_0^2 \rho' n \end{pmatrix} \mathrm{d}S = \vec{0}, \tag{3}$$

with $c_0^2 = \frac{\partial p}{\partial \rho} = 1$. The system can also be written in a differential form, and filling in the constant values for $\rho_0$ and $c_0$, and noting that the momentum perturbation $(\rho u)' = \rho u - (\rho u)_0 = \rho_0 u'$, we obtain

$$\frac{\partial \rho'}{\partial t} + \frac{\partial (\rho u)'}{\partial x} = 0, \tag{4}$$

$$\frac{\partial (\rho u)'}{\partial t} + \frac{\partial \rho'}{\partial x} = 0. \tag{5}$$

## Spatial discretization of a single control volume

Since the flow is assumed to be one-dimensional, only variations in the $x$ direction are taken into account ($\vec{u} = u$). The spatial domain is discretized by a number of finite volume cells, which are uniform and contain the solutions of density ($\rho_i$) and momentum ($\rho u_i$) in their centers, see Fig. 2. Since the flow is one-dimensional, the cell spacing $\Delta y = \Delta z = 1$ can be used, and only

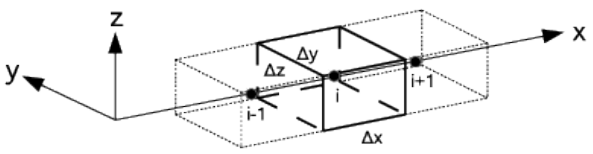


**Figure 2:** One-dimensional piston problem.

the contributions of the cell boundaries that have a normal in the $x$ direction are required.

**Exercise 1.2** For a single control volume $\Omega_i$, $(\Delta x \times \Delta y \times \Delta z)$, discretize the volume and surface integrals of linearized fluid dynamics equations (3) using a linear interpolation to approximate the solution at the cell boundaries.

## Boundary conditions

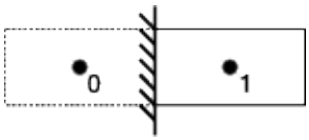


**Figure 3:** Ghost cell at $x = 0$.

At the boundaries of the domain we have to impose boundary conditions. We choose to use a "ghost-cell" approach, see Fig. 3: this means that a virtual cell is modeled behind the boundary for which the solution in that cell is governed by the boundary condition imposed.

E.g. for the solid wall at $x = 0$, the condition is that the velocity (and also its perturbation) $u'_{x=0} = 0$ and when using linear interpolation between two neighboring cells to approximate the solution at the cell boundary we find

$$u'_{x=0} = \frac{u'_0 + u'_1}{2} = 0 \Rightarrow u'_0 = -u'_1.$$

Furthermore, since the velocity at the boundary should be equal to zero, we obtain from the differential form of the momentum equation (5)

$$\frac{\partial(\rho u)'}{\partial t} + \frac{\partial \rho'}{\partial x} = 0,$$

that, since $u'_{x=0} = 0$ and $\frac{\partial u'_{x=0}}{\partial t} = 0$ (solid wall conditions), we obtain that at $x = 0$

$$\frac{\partial \rho'}{\partial x} = 0,$$

so that in a discrete sense

$$\frac{\rho'_1 - \rho'_0}{\Delta x} = 0,$$

which gives the relation of the density perturbation in the ghost cell as a constant extrapolation of the density

$$\rho'_0 = \rho'_1.$$

Substitution of the ghost cell $(\rho'_0, \rho u'_0)$ in the standard discretization derived in Exercise 1.2 incorporates the boundary condition.

Next we define the ghost cell at the fluid-structure interface (Fig. 4) by imposing that the

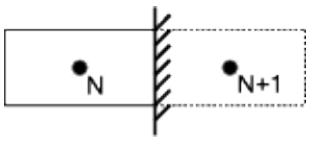


**Figure 4:** Ghost cell at $x = L$.

velocity of the flow at the location of the piston is equal to the piston velocity, hence $u'_{x=L} = \dot{q}'$, and we find a value for the velocity perturbation in the ghost cell $N + 1$

$$u'_{x=L} = \frac{u'_N + u'_{N+1}}{2} = \dot{q}' \Rightarrow u'_{N+1} = 2\dot{q}' - u'_N.$$

Again, using the differential form of the momentum equation (5), we find that

$$\frac{\partial(\rho u_{x=L})'}{\partial t} + \frac{\partial \rho'_{x=L}}{\partial x} = 0,$$

or

$$\frac{\partial(\rho \dot{q}_{x=L})'}{\partial t} + \frac{\partial \rho'_{x=L}}{\partial x} = 0,$$

which gives

$$\rho_0 \ddot{q}_{x=L} + \frac{\partial \rho'_{x=L}}{\partial x} = 0,$$

where $\rho_0 = 1$ is the reference density value (not to be confused with the $\rho'_0$ which is the density perturbation in ghost cell 0. This shows that formally, the density gradient (or pressure gradient) at the wall, depends on the acceleration of the wall. In this assignment we make the simplification that the gradient $\frac{\partial \rho'_{x=L}}{\partial x} = 0$, so we obtain

$$\frac{\rho'_{N+1} - \rho'_N}{\Delta x} = 0,$$

which gives the relation of the density perturbation in the ghost cell as a constant extrapolation of the density

$$\rho'_{N+1} = \rho'_N.$$

### Semi-discrete system

With the boundary and interface conditions discretized, we can now write the problem as a linear system of equations

$$\frac{\mathrm{d}\mathbf{W}}{\mathrm{d}t} = A\mathbf{W}, \tag{6}$$

wherein $A$ the spatial discretization matrix and $\mathbf{W}$ the state vector, containing both the discrete solutions of the flow at the cell centers and the structural displacements and velocity. When using $N$ finite volume cells to discretize the fluid domain, the ordering of variables in the state vector $\mathbf{W}$ is

$$\mathbf{W} = \begin{pmatrix} \mathbf{W}_s \\ \mathbf{W}_f \end{pmatrix} = \begin{pmatrix} \dot{q}' \\ q' \\ \rho'_1 \\ \vdots \\ \rho'_N \\ \rho u'_1 \\ \vdots \\ \rho u'_N \end{pmatrix}, \tag{7}$$

and the matrix $A$ consists of four blocks

$$A = \begin{pmatrix} A_s & A_{sf} \\ A_{fs} & A_f \end{pmatrix}. \tag{8}$$

**Exercise 1.3** Write down the $2 \times 2$ matrix $A_s$, the $2 \times 2N$ matrix $A_{sf}$, the $2N \times 2$ matrix $A_{fs}$ and the $2N \times 2N$ matrix $A_f$. Discuss how / where the coupling between the flow and structure is present.

## 2 Time integration

In the previous section the governing equations for the fluid have been discretized in space using a finite volume approximation, resulting in a semi discrete system. The next step is to apply a time integration scheme to obtain the fully discrete system. In this exercise we use the $\theta$-scheme:

$$\left.\frac{\mathrm{d}\mathbf{W}}{\mathrm{d}t}\right|_{t_{n+\theta}} \approx \frac{\mathbf{W}^{n+1}-\mathbf{W}^{n}}{\Delta t} = \theta\mathbf{F}^{n+1} + (1-\theta)\mathbf{F}^{n}, \tag{9}$$

wherein $\mathbf{F} = A\mathbf{W}$. A straight forward application of this integration scheme to the semi-discrete system derived above, results in a monolithic discretization in which all equations are solved simultaneously

$$\frac{\mathbf{W}^{n+1}-\mathbf{W}^{n}}{\Delta t} = \theta \begin{pmatrix} A_s & A_{sf} \\ A_{fs} & A_f \end{pmatrix} \mathbf{W}^{n+1} + (1-\theta) \begin{pmatrix} A_s & A_{sf} \\ A_{fs} & A_f \end{pmatrix} \mathbf{W}^{n}, \tag{10}$$

which can be rewritten to $L\mathbf{W}^{n+1} = R\mathbf{W}^{n}$, wherein the left matrix $L$ and the right matrix $R$ are defined by

$$L = \left[ I - \theta\Delta t \begin{pmatrix} A_s & A_{sf} \\ A_{fs} & A_f \end{pmatrix} \right], \tag{11}$$

$$R = \left[ I + (1-\theta)\Delta t \begin{pmatrix} A_s & A_{sf} \\ A_{fs} & A_f \end{pmatrix} \right], \tag{12}$$

so that the iteration matrix $M$ (the fully discrete matrix that defines one time step) is $M = L^{-1}R$. For this academic test problem it is easy to obtain the monolithic solution, but for general cases, when a partitioned approach is used, $L^{-1}$ cannot be solved directly. In `matlab\piston\` you will find the file `linpiston.m`. This file can be used to implement several partitioned methods.

**Exercise 2.1** Write down the $L$ and $R$ matrices for the basic partitioning methods: parallel, serial ($F \to S$) and serial ($S \to F$). Implement these methods into `linpiston.m`.
*Note: you can verify your implementation by setting $\theta > 0.5$ and decreasing the time step $\Rightarrow$ all partitioned methods should converge to the same results as the monolithic scheme.*

## 3 Numerical simulations

For the linear piston problem that we are considering, the energy of the discrete system can be expressed by $e = \frac{1}{2}\mathbf{W}^{T}E\mathbf{W}$, with $E$ the energy coefficient matrix, which only contains elements on the diagonal:

$$E = \begin{pmatrix} m & & & & & \\ & k & & & & \\ & & \Delta x & & & \\ & & & \ddots & & \\ & & & & \Delta x & \end{pmatrix}. \tag{13}$$

We will now investigate the performance of the partitioned integration with respect to the energy in the system. In the `linpiston.m` script, the following plots of the solution at the fluid-structure interface are shown: $q' - t$, $\dot{q}' - t$, $p' - t$ and a plot of the change in system energy $\frac{e(t)-e(0)}{e(0)} - t$. For these investigations we use the settings: `k=1`, `m=2`, `N=64`, `dt=0.1` and `Ndt=100` (number of time steps). An exact solution for this linear piston problem exists and is supplied with the functions `exact_sol` and `exact_omega`.

**Exercise 3.1** $\theta = 0$: Forward (explicit) Euler. First we set `theta=0` and run the code. Discuss the results. Can you explain the results?

**Exercise 3.2** $\theta = 1$: backward (implicit) Euler. Next we set `theta=1` and run the code. Discuss the results.
How large is the error in the system energy at the end of the simulation compared to the exact solution - which scheme performs best?
How large is the error in the system energy for the partitioned schemes at the end of the simulation compared to the monolithic scheme - which scheme performs best?

**Exercise 3.3** $\theta = 0.75$. We set `theta=0.75` and run the code. Discuss the results - which scheme performs best?
How do you expect each of the schemes to perform when the simulation time is increased (larger `Ndt`)?

**Exercise 3.4** $\theta = 0.5$. We set `theta=0.5` and run the code with `Ndt=100` and `Ndt=1000`. Discuss the results.

**Exercise 3.5** Discuss which method you would prefer to determine the performance of the partitioning scheme: by comparing to the exact solution or by comparing to the monolithic (numerical) solution?