Methods 3, Week 2:

Introduction to Probability Theory

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What are probabilities?

- Means to capture uncertainty
- 2 types of uncertainty
  1. Outside (empirical, objective, long run relative frequencies) - Probability is a property of the world
  2. Inside (degree of belief, state of knowledge, subjective) - Probability is a property of the mind
- 2 types of probabilities
  1. Categorical
  2. Conditional

Unified formal machinery

- Statisticians usually talk about events in terms of set theory
- Axiomatization by Kolmogorov (1933)
  - 3 axioms, 2 definitions
    - All probability relations can be derived from these axioms
- Presented here according to probability type
  - Categorical Probability = $Pr(\ )$
  - Conditional Probability = $Pr(\ |\ )$

Categorical Probabilities

Formal conventions:

$$0 \leq Pr(A) \leq 1 \tag{1}$$
$$Pr(\text{certain event}) = 1 \tag{2}$$
$$Pr(\text{impossible event}) = 0 \tag{3}$$
$$Pr(\sim A) = 1 - Pr(A) \tag{4}$$

Mutual exclusivity

- Two events are **mutually exclusive** (disjoint, incompatible) if they can't both occur at once
  - If two events A and B are mutually exclusive, the probability that one or the other happens is the sum of their probabilities
  - When a or B are not mutually exclusive, we have to subtract the probability of their overlap

  If A and B are mutually exclusive, then:
  $Pr(A \cup\ B) = Pr(A) + Pr(B)$ Otherwise:
  $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$

Independence

- Two events are **independent** when the occurrence of one does not influence the probability of occurrence of the other
  - People do not always understand independence very well
  - Everybody knows that it is possible to multiply probabilities
    - *The probabilities of **independent** events can be multiplied*

  If A and B are independent, then:
  $Pr(A \cap B) = Pr(A) \times Pr(B)$

Categorical Probability Quiz!

Conditional Probabilities

If Pr(B) > 0, then
$Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)}$

- Better understood as ratios of areas
  - The probability of A given B is the ratio between everything contained in both B and A to everything contained in B
  - The fraction of B occupied by A
- The probability of an event can vary under different conditions
- Non commutativity
  - Bayes' theorem addresses this

Conditional properties and operations

- Properties and operations for categorical probabilities also prevail in conditional context

Hence:

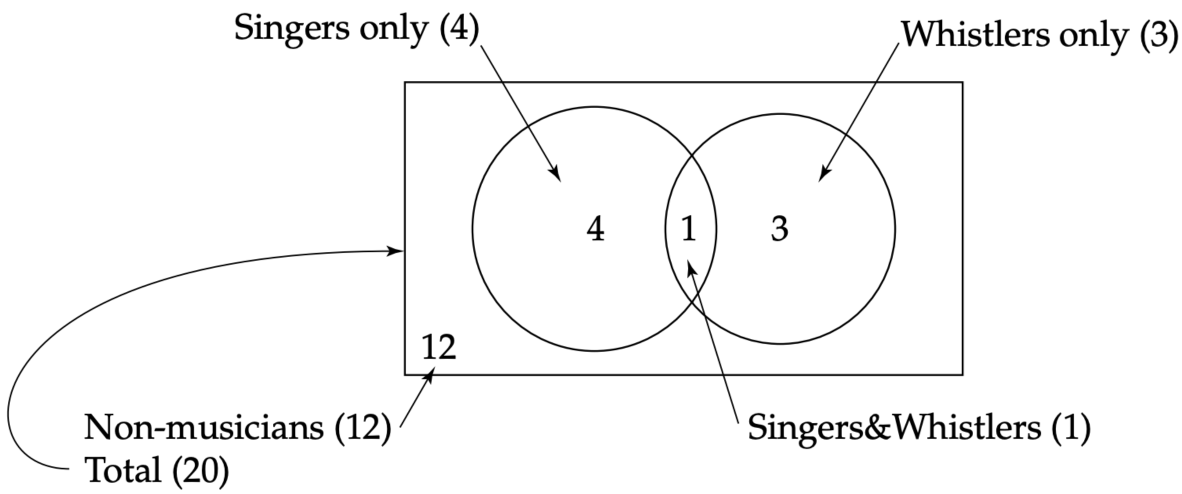
$$0 \leq Pr(A|B) \leq 1 \tag{5}$$
$$Pr(\text{Sure Event}|B) = 1 \tag{6}$$
$$Pr(A \cup B|C) = Pr(A|C) + Pr(B|C) - Pr(A \cap B|C) \tag{7}$$
$$Pr(A|(B \cap C)) = \frac{Pr((A \cap B)|C)}{Pr(B|E)} \tag{8}$$

Statistical Independence

- The probability of A should be no different from the probability of A given B, Pr(A/B)
- A is independent of B only if B is independent of A

When 0 < Pr(A) and 0 < Pr(B), we expect that:
If $Pr(A|B) = Pr(A)$, then $Pr(B|A) = Pr(B)$ (and vice versa)

An example using Venn diagram



Normality? Certainty? Complementation? Additivity? Overlap? Conditional?

Conditional Probability Quiz!

Conditional Probability

Marginal Probabilities

- The marginal probability of a variable is given by the area of the corresponding circle, independently fo the circles

Conditional probability

$P(A|B) \neq P(B|A)$