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Journal of Ocean Engineering and Science 3 (2018) 316–324

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Iterative algorithm for parabolic and hyperbolic PDEs with nonlocal boundary conditions

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Received 1 July 2018; received in revised form 24 September 2018; accepted 11 October 2018
Available online 5 November 2018

Abstract

In this paper, we are concerned with the numerical solutions for the parabolic and hyperbolic partial differential equations with nonlocal boundary conditions. Thus, we presented a new iterative algorithm based on the Restarted Adomian Decomposition Method (RADM) for solving the two equations of different types involving dissimilar boundary and nonlocal conditions. The algorithm presented transforms the given nonlocal initial boundary value problem to a local Dirichlet one and then employs the RADM for the numerical treatment. Numerical comparisons were made between our proposed method and the Adomian Decomposition Method (ADM) to demonstrate the efficiency and performance of the proposed method.

Keywords: Adomian Decomposition Method; Restarted method; Parabolic and hyperbolic PDEs; Nonlocal boundary conditions.

1. Introduction

Many problems in science and engineering tend to be modelled as nonlocal mathematical problems as a result of viscoelastic behaviour of material or previous data. In particular, many branches of ocean engineering like the marine geology, geophysics, oceanography and acoustical engineering among others make use of nonlocal mathematical problems while modelling and design of various oceanographic devices and instruments. Additionally, it is sometimes better to impose nonlocal conditions since the measurements needed by a nonlocal condition may be more precise than the measurement given by a local condition, see [1–4]. Nonlocal problems are problems with presence of an integral term in a boundary condition which greatly complicate the application of standard numerical techniques like the finite difference, finite elements and spectral methods to mention a few. However, it is therefore important to be able to convert a nonlocal boundary value problem to a more desirable form of practical interest where several methods can be employed. For instance, the spectral collocation [5] with preconditioning and the radial basis functions collocation methods [6] were applied to solve nonlocal parabolic partial differential equations (PDEs) with Neumann and nonlocal boundary conditions. The Adomian's decomposition method for nonlocal heat problem and a class of hyperbolic equations with Dirichlet and nonlocal boundary conditions were treated in [4,7,8] based on the Beilin's transformation [9]. The solution of the one-dimensional nonlocal hyperbolic equation is presented using the method of lines in [10,11], the He's homotopy perturbation method was employed for solving the initial-boundary value problem with Dirichlet and integral condition for the wave equation [12] and lastly problems with the combined Neumann and integral conditions were considered by Dehghan [13] using new finite difference schemes and Bouziani [14], respectively.

However, in this paper, an iterative algorithm based on the RADM [15,16] of ADM for solving parabolic and hyperbolic PDES with nonlocal boundary conditions of different kinds will be proposed. ADM is a very powerful semi-analytical method with rich literature that has undergone several modifications and improvements, see [17–22,24–26] while the

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https://doi.org/10.1016/j.joes.2018.10.001
2468-0133/

RADM is an algorithm that improves the accuracy of ADM and gives better approximations. In the process of computation, all calculations will be performed using the Maple software package. The structure of this paper is as follows: In Section 2, we present the nonlocal systems of interest while Section 3 presents the standard Adomian decomposition method. In Section 4, we give the proposed method based on the Restarted Adomian Decomposition Method. Section 5 presents the application of the proposed method and Section 6 gives a brief conclusion.

2. The nonlocal systems

In this section, we give the general forms of the parabolic and hyperbolic systems subject to initial and certain nonlocal boundary conditions together with some lemmas in each case for transforming the each system to a handable local problem.

2.1. Nonlocal parabolic System

Consider the general parabolic partial differential equation

$$L_t u(x,t) - p(x,t) L_{xx} u(x,t) + q(x,t) u(x,t) \\ = g(x,t) + Nu(x,t), \tag{2.1}$$

with the initial condition

$$u(x,0) = \alpha(x), \quad a \le x \le b, \quad t > 0, \tag{2.2}$$

and the Dirichlet (or Neumann) with nonlocal inhomogeneous boundary conditions of the integral type

$$u(a,t) = \beta_1(t), \quad (\text{or } u_x(a,t) = \beta_1(t)), \\ \int_a^b \varphi(x) u(x,t) dx = \beta_2(t), \tag{2.3}$$

where $L_t u(x,t) = \frac{\partial u}{\partial t}, L_{xx} u(x,t) = \frac{\partial^2 u}{\partial x^2}, g(x,t)$ is a source function; $Nu(x, t)$ is a nonlinear term; $\alpha(x)$, $\varphi(x)$, and $\beta_i(t)$ for $i = 1, 2$ are given continuous functions.

Lemma 1. *The nonlocal parabolic system given in Eqs. (2.1)–(2.3) reduces to a local system of the form*

$$\begin{cases} v_{tx} + r(x,t) v_x + s(x,t) v_{xx} - p(x,t) v_{xxx} = f(x,t) + F(v), \\ v_x(x,0) = h(x), \\ v(a,t) = 0, \text{ and } v(b,t) = \beta(t), \end{cases} \tag{2.4}$$

where,

$$r(x,t) = -p(x,t) \left(\frac{1}{\varphi(x)} \right)'' \varphi(x) + q(x,t),$$

$$s(x,t) = -2p(x,t) \left(\frac{1}{\varphi(x)} \right)' \varphi(x),$$

$$\beta(t) = \beta_1(t) + \beta_2(t), \quad h(x) = \varphi(x) \alpha(x),$$

$$f(x,t) = \varphi(x) g(x,t), \quad \text{and} \quad F(v) = \varphi(x) N\left(\frac{v_x}{\varphi(x)} \right). \tag{2.5}$$

Proof. The proof follows immediately after setting

$$v(x,t) = \int_a^x \varphi(x) u(x,t) dx \; \left(\text{or } u(x,t) = \frac{v_x}{\varphi(x)} \right), \tag{2.6}$$

in the Eq. (2.1)–(2.3). □

2.2. Nonlocal hyperbolic System

Consider the general hyperbolic partial differential equation

$$L_{tt} u(x,t) - p(x,t) L_{xx} u(x,t) + q(x,t) u(x,t) \\ = g(x,t) + Nu(x,t), \tag{2.7}$$

subject to the initial conditions

$$u(x,0) = \alpha_1(x), \quad u_t(x,0) = \alpha_2(x), \quad a \le x \le b, t > 0, \tag{2.8}$$

and the Dirichlet (or Neumann) with nonlocal inhomogeneous boundary conditions of the integral type given in (2.3); where $L_{tt} u(x,t) = \frac{\partial^2 u}{\partial t^2}$, and $L_{xx} u(x,t) = \frac{\partial^2 u}{\partial x^2}$, $g(x,t)$ is a source function, $Nu(x, t)$ is a nonlinear term.

Lemma 2. *The nonlocal hyperbolic system given in Eq. (2.7) and (2.8) and (2.3) reduces to a local system of the form*

$$\begin{cases} v_{ttx} + r(x,t) v_x + s(x,t) v_{xx} - p(x,t) v_{xxx} = f(x,t) + F(v), \\ v_x(x,0) = h_1(x), \text{ and } v_{tx}(x,0) = h_2(x), \\ v(a,t) = 0, \text{ and } v(b,t) = \beta(t), \end{cases} \tag{2.9}$$

where,

$$h_i(x) = \varphi(x) \alpha_i(x), \quad i = 1, 2. \tag{2.10}$$

Remark 1. The proof follows as in above.

3. Standard Adomian Decomposition Method

This section presents the method of solution of the above two problems using the standard Adomian Decomposition Method (ADM) as follows:

3.1. ADM for the nonlocal parabolic system

Rewriting the PDE in the system given in Eq. (2.4) in an operator form, we get:

$$L_{tx} v = f + Rv + F(v), \tag{3.1}$$

where $L_{tx} = v_{tx}$ and $Rv = -r(x,t) v_x - s(x,t) v_{xx} + p(x,t) v_{xxx}$. We define two inverse linear operators as

$$L_{a,tx}^{-1}(.) = \int_a^x \int_0^t (.) dt dx, \tag{3.2}$$

and

$$L_{b,tx}^{-1}(.) = \int_x^b \int_0^t (.) dt dx. \tag{3.3}$$

Applying these operators to Eq. (3.1) alongside taking into account the initial and boundary conditions in Eq. (2.4); we obtain

$$v(x,t) = \int_a^x h(x)dx + L_{a,tx}^{-1}f + L_{a,tx}^{-1}Rv + L_{a,tx}^{-1}F(v). \tag{3.4}$$

and

$$v(x,t) = \beta(t) - \int_x^b h(x)dx - L_{b,tx}^{-1}f - L_{b,tx}^{-1}Rv - L_{b,tx}^{-1}F(v). \tag{3.5}$$

respectively. On averaging Eq. (3.4) and (3.5), we obtain

$$\begin{aligned} v(x,t) &= \frac{1}{2}\left(\int_a^x h(x)dx + \beta(t) - \int_x^b h(x)dx + L_{a,tx}^{-1}f - L_{b,tx}^{-1}f\right) \\ &+\frac{1}{2}\left(L_{a,tx}^{-1}Rv + L_{a,tx}^{-1}F(v) - L_{b,tx}^{-1}Rv - L_{b,tx}^{-1}F(v)\right). \end{aligned} \tag{3.6}$$

Therefore, the ADM offers the solution of $v(x, t)$ by an infinite series of components and the nonlinear $F(v)$ by a series of polynomials given by

$$v(x,t) = \sum_{n=0}^{\infty} v_n(x,t), \text{ and } F(v) = \sum_{n=0}^{\infty} A_n, \tag{3.7}$$

respectively; and A_n is calculated from

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{n}\lambda^i u_i\right)\right]_{\lambda=0}, \quad i = 0, 1, 2, \ldots, n. \tag{3.8}$$

Thus, substituting Eq. (3.7) into Eq. (3.6); ADM gives the solution of Eq. (2.4) recursively as follows:

$$\begin{aligned} v_0(x,t) &= \frac{1}{2}\left(\int_a^x h(x)dx + \beta(t) - \int_x^b h(x)dx + L_{a,tx}^{-1}f - L_{b,tx}^{-1}f\right), \\ v_{n+1}(x,t) &= \frac{1}{2}\left(L_{a,tx}^{-1}Rv_n + L_{a,tx}^{-1}A_n - L_{b,tx}^{-1}Rv_n - L_{b,tx}^{-1}A_n\right), n \ge 0, \end{aligned} \tag{3.9}$$

Note that, the original function $u(x, t)$ can be obtained on using Eq. (2.6).

3.2. ADM for the nonlocal hyperbolic system

Considering the PDE in the system given in Eq. (2.9) with two inverse linear operators defined by

$$L_{a,ttx}^{-1}(.) = \int_a^x \int_0^t \int_0^t (.)dtdtdx, \tag{3.10}$$

and

$$L_{b,ttx}^{-1}(.) = \int_x^b \int_0^t \int_0^t (.)dtdtdx; \tag{3.11}$$

we proceed as in the above and taking into account the initial and second boundary conditions in Eq. (2.9); we get the solution in a recursive manner as follows:

$$\begin{aligned} v_0(x,t) &= \frac{1}{2}\left(\int_a^x h_1(x)dx + t\int_a^x h_2(x)dx + \beta(t)\right) \\ &+\frac{1}{2}\left(-\int_x^b h_1(x)dx - t\int_x^b h_2(x)dx + L_{a,ttx}^{-1}f - L_{b,ttx}^{-1}f\right), \\ v_{n+1}(x,t) &= \frac{1}{2}\left(L_{a,ttx}^{-1}Rv_n + L_{a,ttx}^{-1}A_n - L_{b,ttx}^{-1}Rv_n - L_{b,ttx}^{-1}A_n\right), n \ge 0. \end{aligned} \tag{3.12}$$

4. Restarted Adomian Decomposition Method

Restarted Adomian Decomposition Method (RADM) was introduced Babolian and Javadi [15] to facilitate the rate of convergence of ADM and successfully used by Babolian et al. [16] and Vahidi et al. [23] which turned out remarkable numerical solution. Thus, we present this novel idea for our problems under consideration and also give algorithms in each case as follows:

4.1. RADM for the parabolic system

RADM first introduced a fixed term say ω that starts or rather facilitates the rate of convergence. Now, consider Eq. (3.6) and then add a fixed term say ω to both sides and proceed as in ADM. The RADM associates the ω to v_0 and then carefully select v_1 and the remaining follow recursively as given below:

$$\begin{aligned} v_0(x,t) &= \omega, \\ v_1(x,t) &= \frac{1}{2}\left(\int_a^x h(x)dx + \beta(t) - \int_x^b h(x)dx + L_{a,tx}^{-1}f - L_{b,tx}^{-1}f\right) \\ &\quad -\omega + \frac{1}{2}\left(L_{a,tx}^{-1}Rv_0 + L_{a,tx}^{-1}A_0 - L_{b,tx}^{-1}Rv_0 - L_{b,tx}^{-1}A_0\right), \\ v_{n+1}(x,t) &= \frac{1}{2}\left(L_{a,tx}^{-1}Rv_n + L_{a,tx}^{-1}A_n - L_{b,tx}^{-1}Rv_n - L_{b,tx}^{-1}A_n\right), n \ge 1. \end{aligned} \tag{4.1}$$

Algorithm 1 Choose small natural numbers m and n.

Step 1:
Apply the ADM and calculate $v_0, v_1, v_2, \ldots, v_n$.
set $\omega^1 = v_0 + v_1 + v_2 + \cdots + v_n$.
Step 2: For $i = 2 : m$, do
$v_0 = \omega^{i-1}$,
$v_1 = \frac{1}{2}\left(\int_a^x h(x)dx + \beta(t) - \int_x^b h(x)dx + L_{a,tx}^{-1}f - L_{b,tx}^{-1}f\right) - \omega^{i-1} + \frac{1}{2}\left(L_{a,tx}^{-1}Rv_0 + L_{a,tx}^{-1}A_0 - L_{b,tx}^{-1}Rv_0 - L_{b,tx}^{-1}A_0\right)$,
$v_{n+1} = \frac{1}{2}\left(L_{a,tx}^{-1}Rv_n + L_{a,tx}^{-1}A_n - L_{b,tx}^{-1}Rv_n - L_{b,tx}^{-1}A_n\right)$.
set $\omega^i = v_0 + v_1 + v_2 + \cdots + v_n$.
end of algorithm.

4.2. RADM for the hyperbolic system

Here also we present as in above, the RADM scheme for the hyperbolic system in Eq. (2.9) the following:

$$v_0(x,t) = \omega$$

$$v_1(x,t) = \frac{1}{2}\left(\int_a^x h_1(x)dx + t\int_a^x h_2(x)dx + \beta(t)\right)$$
$$+\frac{1}{2}\left(-\int_x^b h_1(x)dx - t\int_x^b h_2(x)dx + L_{a,ttx}^{-1}f - L_{b,ttx}^{-1}f\right)$$
$$-\omega + \frac{1}{2}\left(L_{a,ttx}^{-1}Rv_0 + L_{a,ttx}^{-1}A_0 - L_{b,ttx}^{-1}Rv_0 - L_{b,ttx}^{-1}A_0\right),$$
$$v_{n+1}(x,t) = \frac{1}{2}\left(L_{a,ttx}^{-1}Rv_n + L_{a,ttx}^{-1}A_n - L_{b,ttx}^{-1}Rv_n - L_{b,ttx}^{-1}A_n\right), n \geq 1. \tag{4.2}$$

Also, for the algorithm, one follows similar procedure as in above.

5. Application and results

In this section, we demonstrate the efficiency proposed schemes by studying some test problems of parabolic and hyperbolic systems drawn from literature.

5.1. Example one

Consider the nonlocal inhomogeneous parabolic linear PDE

$$u_t - u_{xx} = \sin(x), \quad 0 \leq x \leq \pi, \quad t \geq 0, \tag{5.1}$$

subject to the initial condition

$$u(x,0) = \cos(x), \tag{5.2}$$

with Neumann and nonlocal boundary conditions

$$u_x(0,t) = 1 - e^{-t},$$
$$\int_0^\pi u(x,t)dx = 2(1 - e^{-t}). \tag{5.3}$$

Thus, on using Lemma 1. Eq. (5.1) and conditions in Eq. (5.2) and (5.3) converts to the following system:

$$\begin{cases} v_{tx} - v_{xxx} = \sin(x), \\ v_x(x,0) = \cos(x), \\ v(0,t) = 0, \quad \text{and} \quad v(\pi,t) = 2(1 - e^{-t}), \end{cases} \tag{5.4}$$

The **ADM** gives the solution of Eq. (5.4) recursively as follows:

$$v_0(x,t) = \frac{1}{2}\left(\int_0^x \cos(x)dx - \int_x^\pi \cos(x)dx + 2(1 - e^{-t})\right.$$
$$\left.+L_{0,ttx}f - L_{\pi,ttx}f\right),$$
$$v_{n+1}(x,t) = \frac{1}{2}\left(L_{0,tx}^{-1}Rv_n - L_{\pi,tx}^{-1}Rv_n\right), n \geq 0, \tag{5.5}$$

with some few terms as

$$v_0 = \sin(x) + (1 - e^{-t}) - t\cos(x),$$
$$v_1 = -t\sin(x) + \frac{t^2}{2}\cos(x),$$
$$v_2 = \frac{t^2}{2}\sin(x) - \frac{t^3}{6}\cos(x),$$
$$\vdots$$

Table 1
Absolute errors comparisons for Example one.

x	RADM	ADM
0.0	1.45830000e−06	2.02427000e−05
0.31	1.41540000e−06	1.97025000e−05
0.63	1.23370000e−06	1.72339000e−05
0.94	9.31300000e−07	1.30783000e−05
1.26	5.37800000e−07	7.64230000e−06
1.57	9.17000000e−08	1.45840000e−06
1.88	3.63400000e−07	4.86830000e−06
2.20	7.83000000e−07	1.07185000e−05
2.51	1.12590000e−06	1.55195000e−05
2.83	1.35850000e−06	1.88013000e−05
3.14	1.45830000e−06	2.02427000e−05

Therefore,

$$v(x,t) = \sin(x) + (1 - e^{-t}) - t\cos(x) - t\sin(x)$$
$$+\frac{t^2}{2}\cos(x) + \frac{t^2}{2}\sin(x) - \frac{t^3}{6}\cos(x) + \ldots$$

Thus, for the original function $u(x, t)$; Eq. (2.6) gives

$$u(x,t) = \frac{v_x}{\varphi(x)},$$
$$= \cos(x)\left(1 - t + \frac{t^2}{2!} + \ldots\right)$$
$$+\sin(x)\left(t - \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots\right). \tag{5.6}$$

The **RADM** gives the solution of Eq. (5.4) by employing the ADM to calculate $v_0, v_1, v_2, \ldots, v_n$. Thus, for $m = 3$, we obtain the following:

$$\omega^1 = v_0 + v_1 + v_2,$$
$$= \sin(x) + (1 - e^{-t}) - t\cos(x) - t\sin(x) + \frac{t^2}{2}\cos(x)$$
$$+\frac{t^2}{2}\sin(x) - \frac{t^3}{6}\cos(x).$$

Set

$$v_0 = \omega^1,$$
$$v_1 = \frac{-1}{6}t^3\sin(x) + \frac{1}{24}t^4\cos(x),$$
$$v_2 = \frac{1}{24}t^4\sin(x) - \frac{1}{120}t^5\cos(x).$$

Set

$$\omega^2 = v_0 + v_1 + v_2. \tag{5.7}$$

Then $v_0 = \omega^2$, and so on. Hence, we obtain $v(x,t) = \sum_{n=0}^{2} v_n$, and $u(x, t)$ from Eq. (2.6). The absolutes errors for the RADM and ADM for $t = 0.5$ and $0 \leq x \leq \pi$ is given in Table 1 and Fig. 1, respectively.

5.2. Example two

Consider the nonlocal homogeneous nonlinear parabolic PDE

$$u_t - u_{xx} = -uu_x, \quad 0 \leq x \leq 1, \quad t \geq 0, \tag{5.8}$$

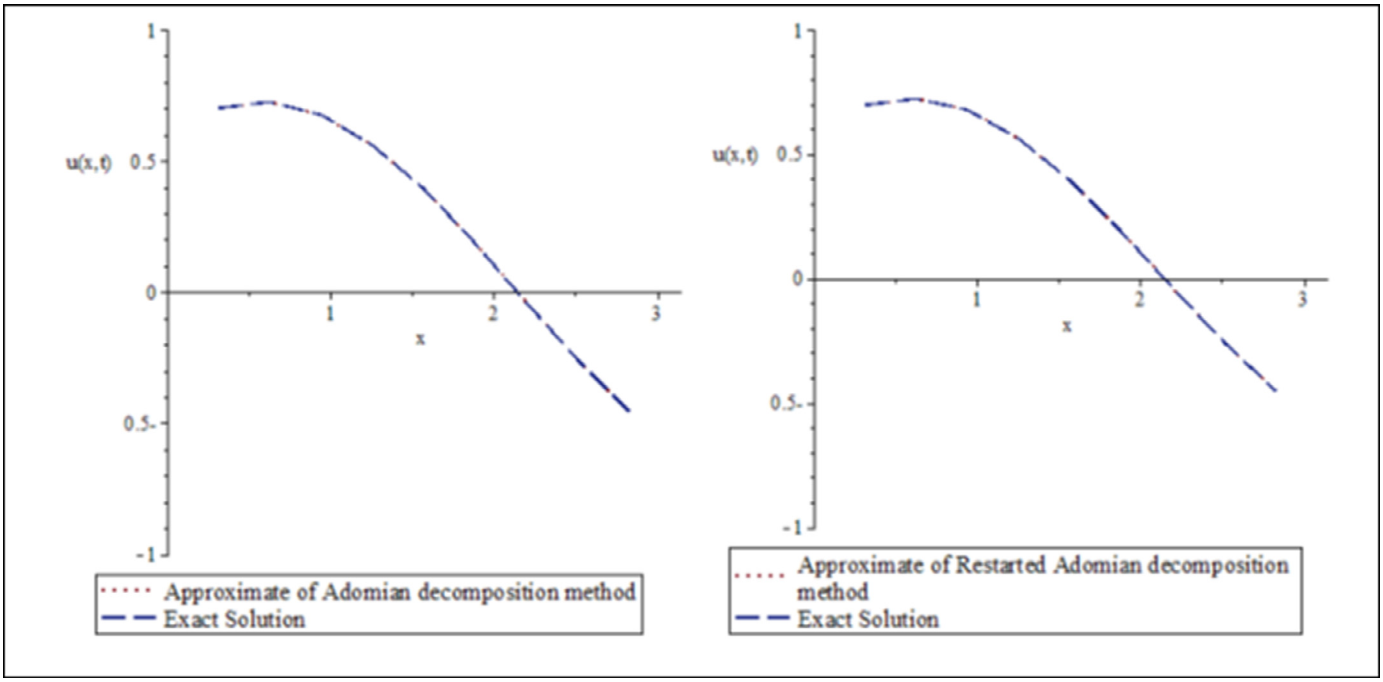

Fig. 1. Exact and approximate solutions comparisons for Example one.

subject to the initial condition

$$u(x,0)=x, \tag{5.9}$$

with Dirichlet and nonlocal boundary conditions

$$u(0,t)=0,$$
$$\int_0^1 u(x,t)dx=\frac{1}{2(1+t)}. \tag{5.10}$$

As in above, Lemma 1 converts Eq. (5.8) with conditions in Eq. (5.9) and (5.10) to a system of the form given in Eq. (2.4) and (2.5). Thus, the **ADM** offers the following recursive terms:

$$v_0=\frac{1}{2}x^2+\frac{1}{4(1+t)}-\frac{1}{4},$$
$$v_1=-\frac{1}{2}x^2t+\frac{1}{4}t,$$
$$v_2=\frac{1}{2}x^2t-\frac{1}{4}t^2,$$
$$\vdots$$

Therefore,

$$v(x,t)=\frac{1}{2}x^2+\frac{1}{4(1+t)}-\frac{1}{4}-\frac{1}{2}x^2t+\frac{1}{4}t+\frac{1}{2}x^2t^2-\frac{1}{4}t^2+\ldots$$

Thus, for the original function $u(x,\ t)$; Eq. (2.6) gives

$$u(x,t)=\frac{v_x}{\varphi(x)}, \\ =x\left(1-t+t^2-t^3+\ldots\right), \\ =\frac{x}{1+t}, \tag{5.11}$$

which is also the exact solution.

The **RADM** starts by using ADM to calculate $v_0, v_1, v_2, \ldots, v_n$. We set $m=3$, to get the following series:

$$\omega^1=v_0+v_1+v_2, \\ =\left(\frac{1}{2}x^2+\frac{1}{4(1+t)}-\frac{1}{4}\right)+\left(-\frac{1}{2}x^2t+\frac{1}{4}t\right)+\left(\frac{1}{2}x^2t^2-\frac{1}{4}t^2\right),$$
$$v_0=\omega^1,$$
$$v_1(x,t)=\frac{-1}{10}x^2t^5+\frac{1}{4}x^2t^4-\frac{1}{2}x^2t^3+\frac{1}{20}t^5-\frac{1}{8}t^4+\frac{1}{4}t^3,$$
$$v_2(x,t)=\frac{1}{40}x^2t^8-\frac{1}{10}x^2t^7+\frac{17}{60}x^2t^6-\frac{3}{10}x^2t^5+\frac{1}{4}x^2t^4 \\ -\frac{1}{80}t^8+\frac{1}{20}t^7-\frac{17}{120}t^6+\frac{3}{20}t^5-\frac{1}{8}t^4. \tag{5.12}$$

Set $\omega^2=v_0+v_1+v_2$, then $v_0=\omega^2$, and proceed as in above to obtain $u(x,\ t)$.

The absolutes errors for the RADM and ADM at $t=0.5$ and $0\le x\le 1$ is given in Table 2 and Fig. 2, respectively.

5.3. Example three

Consider the nonlocal homogeneous hyperbolic linear PDE

$$u_{tt}-u_{xx}+\lambda u=0, \lambda=-(1+4\pi),\ 0\le x\le 1,\quad t\ge 0, \tag{5.13}$$

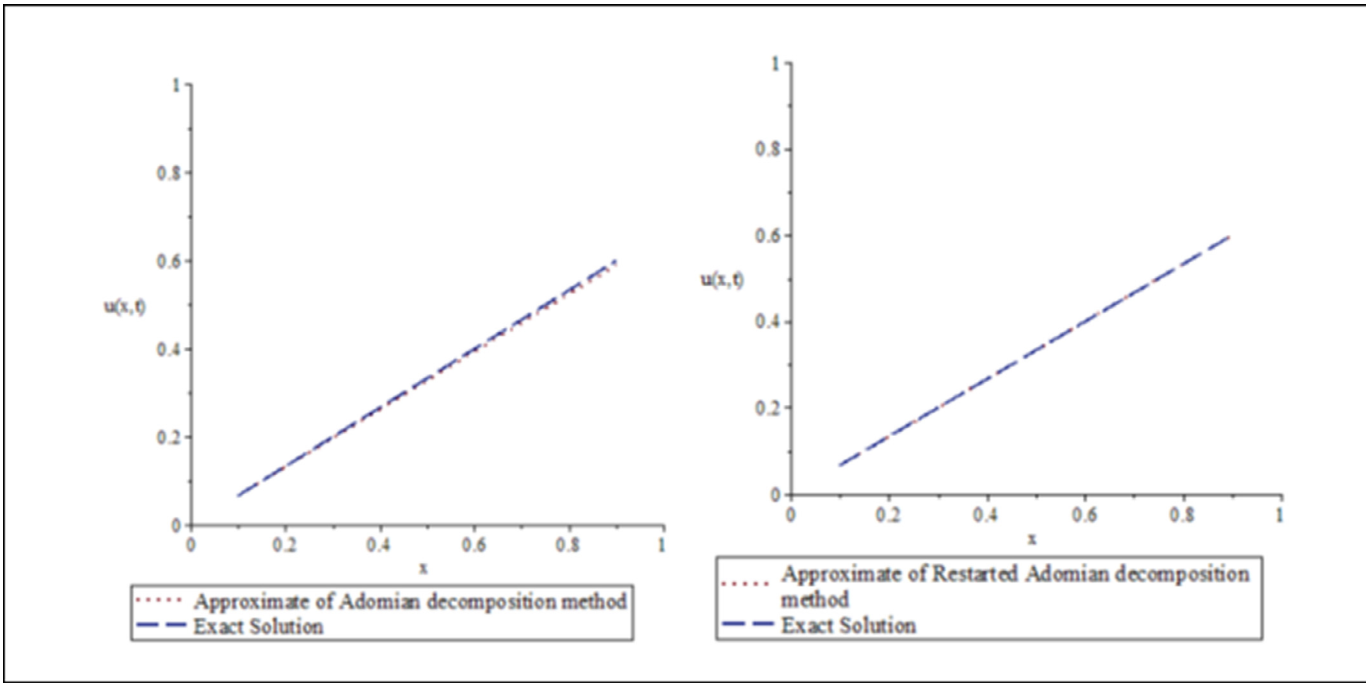

Fig. 2. Exact and approximate solutions comparisons for Example two.

Table 2
Absolute errors comparisons for Example two.

x	RADM	ADM
0.0	0.00000000e+00	0.00000000e+00
0.1	4.47024000e−06	1.04166667e−03
0.2	8.94050000e−06	2.08333330e−03
0.3	1.34107000e−05	3.12500000e−03
0.4	1.78809000e−05	4.16666670e−03
0.5	2.23513000e−05	5.20833330e−03
0.6	2.68215000e−05	6.25000000e−03
0.7	3.12917000e−05	7.29166670e−03
0.8	3.57620000e−05	8.33333330e−03
0.9	4.02322000e−05	9.37500000e−03
1.0	4.47024000e−05	1.04166667e−02

subject to the initial conditions

$$u(x,0)=\cos(2\pi x),\quad u_t(x,0)=-\cos(2\pi x), \tag{5.14}$$

with Dirichlet and nonlocal boundary conditions

$$u(0,t)=e^{-t},\quad \int_0^1 u(x,t)dx=0. \tag{5.15}$$

Thus, Lemma 2 converts Eq. (5.13) with condition in Eq. (5.14) and (5.15) to the following system:

$$\begin{cases} v_{ttx}+\lambda v_x - v_{xxx}=0, \\ v_x(x,0)=\cos(2\pi x),\quad v_{tx}(x,0)=-\cos(2\pi x), \\ v(0,t)=0,\ \text{ and } v(1,t)=0. \end{cases} \tag{5.16}$$

Then **ADM** gives the following terms recursively:

$$\begin{aligned} v_0 &= \frac{1}{2\pi}\sin(2\pi x)-\frac{1}{2\pi}t\sin(2\pi x), \\ v_1 &= -\frac{1}{12\pi}t^2\sin(2\pi x)(t-3), \\ v_2 &= -\frac{1}{240\pi}t^4\sin(2\pi x)(t-5), \\ &\vdots \end{aligned} \tag{5.17}$$

Therefore,

$$\begin{aligned} v(x,t) = {} & \frac{1}{2\pi}\sin(2\pi x)-\frac{1}{2\pi}t\sin(2\pi x) \\ & -\frac{1}{12\pi}t^2\sin(2\pi x)(t-3) \\ & -\frac{1}{240\pi}t^4\sin(2\pi x)(t-5)+\ldots \end{aligned}$$

Thus, for the original function $u(x,t)$; Eq. (2.6) gives

$$\begin{aligned} u(x,t) &= \frac{v_x}{\varphi(x)}, \\ &= 2\pi\cos(2\pi x)\left(\frac{1}{2\pi}-\frac{1}{2\pi}t-\frac{1}{12\pi}t^2(t-3)\right. \\ &\quad \left.-\frac{1}{240\pi}t^4(t-5)+\ldots\right). \end{aligned}$$

For the **RADM** solution, we use ADM to calculate $v_0, v_1, v_2, \ldots, v_n$. set $m=3$, to get the following series:

$$\begin{aligned} \omega^1 &= v_0+v_1+v_2, \\ &= \left(\frac{1}{2\pi}\sin(2\pi x)-\frac{1}{2\pi}t\sin(2\pi x)\right) \\ &\quad +\left(-\frac{1}{12\pi}t^2\sin(2\pi x)(t-3)\right) \\ &\quad +\left(-\frac{1}{240\pi}t^4\sin(2\pi x)(t-5)\right), \\ v_0 &= \omega^1, \end{aligned}$$

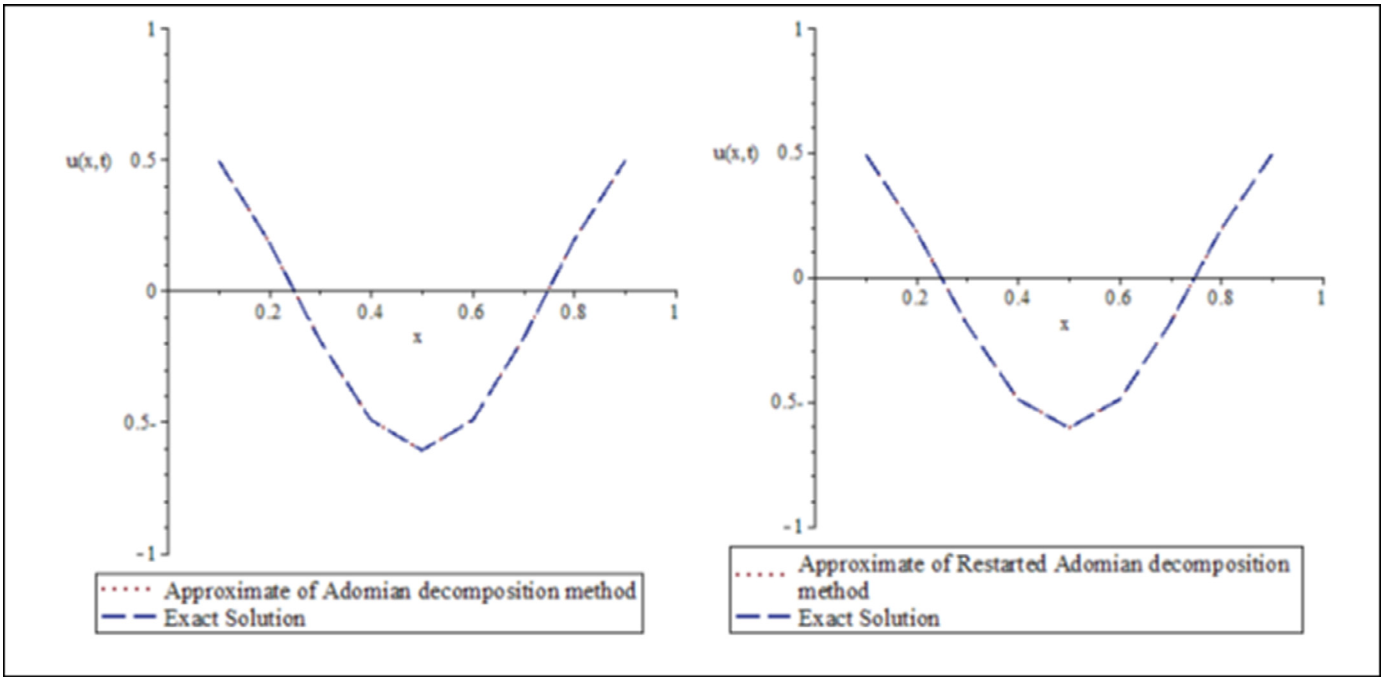

Fig. 3. Exact and approximate solutions comparisons for Example three.

Table 3
Absolute errors comparisons for Example three.

x	RADM	ADM
0.0	6.77489879e−16	4.90760615e−13
0.1	5.48100826e−16	3.97033678e−13
0.2	2.09355886e−16	1.51653370e−13
0.3	2.09355886e−16	1.51653370e−13
0.4	5.48100826e−16	3.97033678e−13
0.5	6.77489879e−16	4.90760615e−13
0.6	5.48100826e−16	3.97033678e−13
0.7	2.09355886e−16	1.51653370e−13
0.8	2.09355886e−16	1.51653370e−13
0.9	5.48100826e−16	3.97033678e−13
1.0	6.77489879e−16	4.90760615e−13

$$v_1(x,t) = -\frac{1}{10080\pi}t^6\sin(2\pi x)(t-7),$$
$$v_2(x,t) = -\frac{1}{725760\pi}t^8\sin(2\pi x)(t-9). \quad (5.18)$$

Set $\omega^2 = v_0 + v_1 + v_2$, then $v_0 = \omega^2$, and proceed as in above to obtain $u(x, t)$.

The absolutes errors for the RADM and ADM at $t = 0.5$ and $0 \le x \le 1$ is given in Table 3 and Fig. 3, respectively.

5.4. Example four

Consider the nonlocal homogeneous hyperbolic linear PDE

$$u_{tt} - u_{xx} = 0, \quad 0 \le x \le 1, \quad t \ge 0, \quad (5.19)$$

subject to the initial conditions

$$u(x,0) = 0, \quad u_t(x,0) = \pi\cos(\pi x), \quad (5.20)$$

with Neumann and nonlocal boundary conditions

$$u_x(0,t) = 0, \quad \int_0^1 u(x,t)dx = 0. \quad (5.21)$$

As in above, we obtain the below new system:

$$\begin{cases} v_{ttx} - v_{xxx} = 0, \\ v_x(x,0) = 0, \quad v_{tx}(x,0) = \pi\cos(\pi x), \\ v(0,t) = 0, \text{ and } v(1,t) = 0. \end{cases} \quad (5.22)$$

Then **ADM** gives the following terms:

$$\begin{aligned} v_0 &= t\sin(\pi x), \\ v_1 &= -\frac{1}{6}\pi^2 t^3 \sin(\pi x), \\ v_2 &= \frac{1}{120}\pi^4 t^5 \sin(\pi x), \\ &\vdots \end{aligned} \quad (5.23)$$

Therefore,

$$v(x,t) = \sin(\pi x)\left(t - \frac{1}{3!}\pi^2 t^3 + \frac{1}{5!}\pi^4 t^5 + ...\right).$$

Thus, for the original function $u(x, t)$; Eq. (2.6) gives

$$\begin{aligned} u(x,t) &= \frac{v_x}{\varphi(x)}, \\ &= \pi\cos(\pi x)\left(t - \frac{1}{3!}\pi^2 t^3 + \frac{1}{5!}\pi^4 t^5 + ...\right). \end{aligned}$$

Also for the **RADM** solution, we use ADM to calculate $v_0, v_1, v_2, \ldots, v_n$. If we set $m = 3$, we get the following series:

$$\omega^1 = v_0 + v_1 + v_2,$$

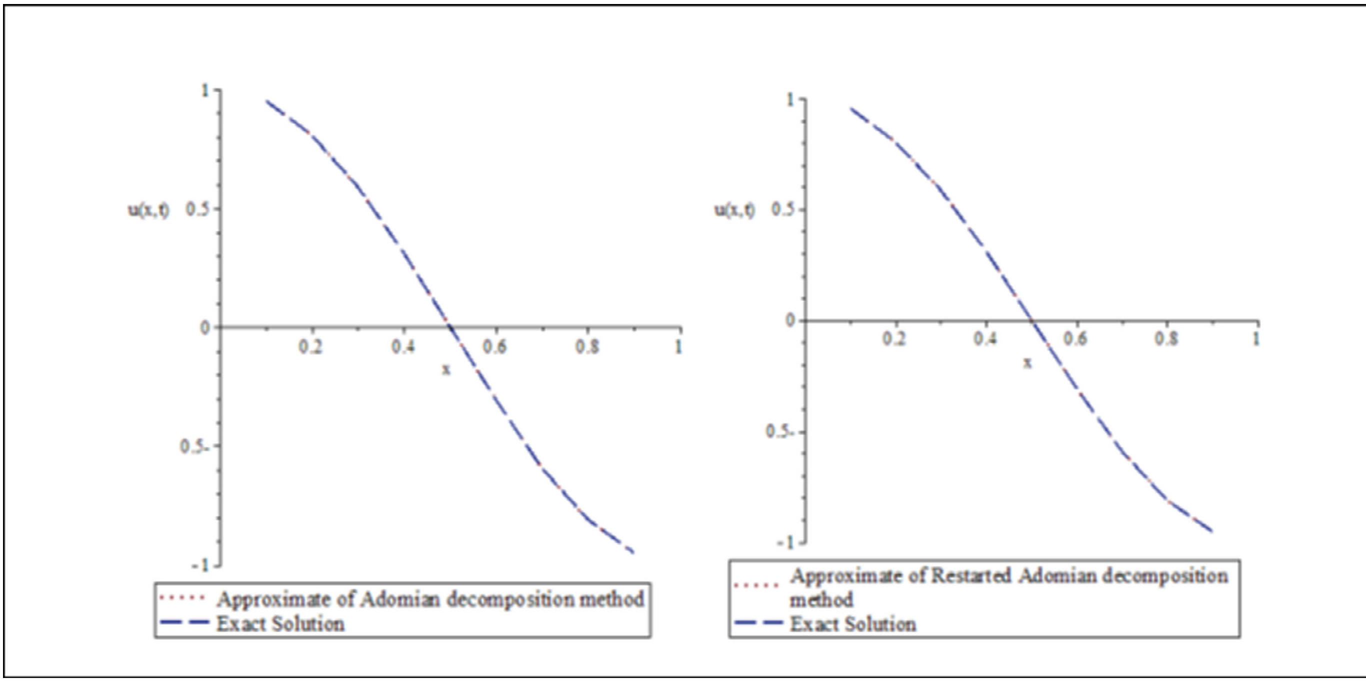

Fig. 4. Exact and approximate solutions comparisons for Example four.

Table 4
Absolute errors comparisons for Example four.

x	RADM	ADM
0.0	1.00000000e−09	5.63000000e−08
0.1	1.00000000e−10	5.37000000e−08
0.2	6.00000000e−10	4.57000000e−08
0.3	1.00000000e−10	3.32000000e−08
0.4	2.00000000e−10	1.74000000e−08
0.5	2.05103381e−10	2.05103369e−10
0.6	2.00000000e−10	1.75000000e−08
0.7	3.00000000e−10	3.31000000e−08
0.8	4.00000000e−10	4.55000000e−08
0.9	7.00000000e−10	5.35000000e−08
1.0	1.00000000e−09	5.63000000e−08

$$= t\sin(\pi x) - \frac{1}{6}\pi^2 t^3 \sin(\pi x) + \frac{1}{120}\pi^4 t^5 \sin(\pi x),$$

$$v_0 = \omega^1,$$

$$v_1(x,t) = -\frac{1}{5040}\pi^6 t^7 \sin(\pi x),$$

$$v_2(x,t) = \frac{1}{362880}\pi^8 t^9 \sin(\pi x). \tag{5.24}$$

Set $\omega^2 = v_0 + v_1 + v_2$, then $v_0 = \omega^2$, and proceed as in above to obtain $u(x, t)$.

The absolutes errors for the RADM and ADM at $t = 0.5$ and $0 \le x \le 1$ is given in Table 4 and Fig. 4, respectively.

5.5. Example five

Consider the nonlocal inhomogeneous hyperbolic nonlinear PDE

$$u_{tt} - u_{xx} = 1 - u^2, \quad 0 \le x \le 1, \quad t \ge 0, \tag{5.25}$$

subject to the initial conditions

$$u(x,0) = 1, \quad u_t(x,0) = 0, \tag{5.26}$$

with Dirichlet and nonlocal boundary conditions

$$u(0,t) = 1, \quad \int_0^1 u(x,t)dx = 1. \tag{5.27}$$

So, the **ADM** gives the following terms:

$$\begin{aligned}
v_0 &= x + \frac{1}{2}xt^2 - \frac{1}{4}t^2,\\
v_1 &= -\frac{1}{2}xt^2 - \frac{1}{12}xt^4 - \frac{1}{120}xt^6 + \frac{1}{4}t^2 + \frac{1}{24}t^4 + \frac{1}{240}t^6,\\
v_2 &= \frac{1}{12}xt^4 - \frac{1}{45}xt^6 + \frac{1}{560}xt^8 + \frac{1}{10800}xt^{10} - \frac{1}{4}t^2 + \frac{1}{24}t^4\\
&\quad + \frac{1}{90}t^6 + \frac{1}{1120}t^8 + \frac{1}{21600}t^{10},\\
&\ \ \vdots
\end{aligned} \tag{5.28}$$

It is obvious that the self-canceling "noise" terms appear between various components. Keeping the remaining non-canceled terms leads immediately to the solution $v(x,t) = x$. Therefore, the original function $u(x,t) = 1$.

The **RADM** solution also gives $v(x,t) = x$ which returns $u(x,t) = 1$.

6. Conclusion

In conclusion, we present a new iterative algorithm based on the Restarted Adomian Decomposition Method for solving nonlocal parabolic and hyperbolic partial differential equations involving dissimilar (Dirichlet or Neumann) boundary conditions including the nonlocal ones. The presented algorithm is then applied to solve effectively a large class of linear and nonlinear parabolic and hyperbolic equations and yields better solutions of high accuracy. Numerical comparisons are also established between the proposed method and the standard Adomian Decomposition Method to assess the efficiency and performance of the proposed method with the help of Maple software. Thus, the presented method is reliable and can be applied to many complicated linear and nonlinear PDEs.

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