Design Process of PI, Lag, PD & Lead Compensator

1. Proportional Plus Integral Compensation (PI):

Objective: To make the steady state error zero without affecting the transient response.

Design Process:

- First add a pole at origin
- Now add a zero very close to the origin

Characteristic of PI Compensator:

1. Generalized transfer function of PI compensator, $\boldsymbol{G_C} = \frac{s+a}{s}$
2. Increases system type
3. Error becomes zero
4. Zero at -a is small and negative
5. Active circuits are required to implement

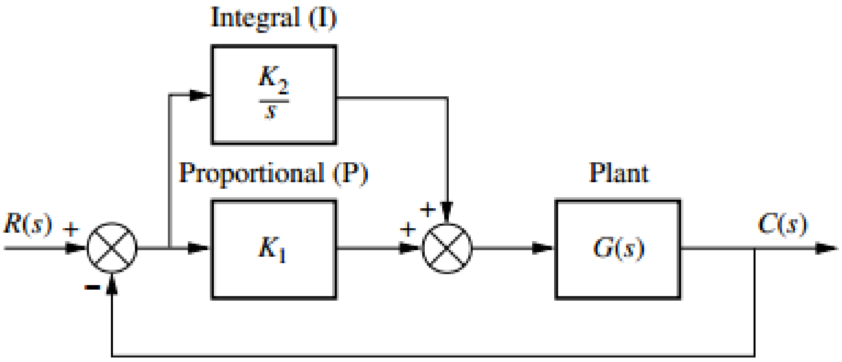

Fig. Close loop feedback with PI controller

$$\boldsymbol{G_C(s) = K_1 + \frac{K_2}{s} = K_1 \frac{s+a}{s}} \quad \textbf{, where } \boldsymbol{a = \frac{K_2}{K_1}}$$

2. Lag Compensator:

Objective: To improve or reduce the steady state error.

Design Process:

$$\frac{K_{P_N}(New\ or\ desired\ K_p)}{K_{P_o}(Old\ or\ previous\ K_p)} = \frac{Z_C}{P_C}$$

Now, arbitrarily selecting a $-P_C$ and then calculate $-Z_C$

Characteristic of lag Compensator:

1. Generalized transfer function of lag compensator, $\boldsymbol{G_C} = \frac{(s+Z_c)}{(s+P_c)}$
2. Error is improved but not driven to zero.
3. Pole at $-P_C$ is small and negative.
4. Zero at $-Z_C$ is close to, and to the left of the pole at $-P_C$ (i.e. $P_C < Z_C$).
5. Active circuits are not required to implement.

N.B: To modify the error, PI or lag compensator is used, but there is very negligible change in desired transient response.

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3. Proportional Plus Derivative Compensation (PD):

Objective: 1. To improve transient response

2. Zero at $-Z_C$ is selected to put design point on the root locus.

Design Process:

1. First evaluate the performance of the uncompensated system:

(a) ξ (zeta) is given, find dominant pole (old) from the root locus of uncompensated system,

$$S_{d_{old}} = -\xi\omega_n + j\omega_n\sqrt{1-\xi^2}$$
$$S_{d_{old}} = -\sigma_d + j\omega_d$$

(b) Now find old settling time, $T_{S_{old}} = \frac{4}{\xi\omega_n} = \frac{4}{\sigma_d}$

2. Evaluate the performance of the compensated system:

(a) From the question, find new $T_{S_{new}}$ and new ω_n [ξ will remain unchanged]

(b) Find dominant pole (new), $S_{d_{new}} = -\xi\omega_n + j\omega_n\sqrt{1-\xi^2}$

$$S_{d_{new}} = -\sigma_d + j\omega_d$$

(c) Design the location of compensating zero, $-Z_C$

- Finding necessary angle (θ_Z) to locate $S_{d_{new}}$ on root locus.
- $\tan\theta_Z = \frac{\omega_d}{Z_c-\sigma_d}$ then find, $-Z_C$

Characteristic of PD Compensator:

i) Generalized transfer function of PD compensator, $\boldsymbol{G_C = s + Z_c}$
ii) Zero at $-Z_C$ is selected to put design point on the root locus.
iii) It increases ω_n.
iv) 'Zeta' for system remains unchanged.
v) 'TYPE' of the system remains unchanged.
vi) It reduces settling time.
vii) Steady state error remains unchanged.
viii) Active circuits are required to implement.

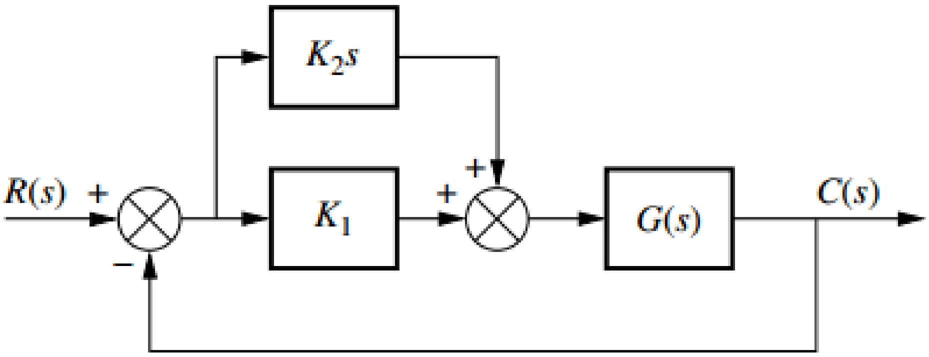

Fig. Close loop feedback with PD controller

Transfer function of PD controller,

$$\boldsymbol{G_C(s) = K_1 + sK_2 = K_2(s+a)} \qquad \textbf{Where, } \boldsymbol{a = \frac{K_1}{K_2}}$$

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4. Lead Compensator:

Objective: 1. To improve transient response
2. Zero at $-Z_C$ and pole at $-P_C$ are selected to put design point on the root locus.

Design Process:

1. First evaluate the performance of the uncompensated system:

(a) ξ (zeta) is given, find dominant pole (old) from the root locus of uncompensated system,

$$S_{d_{old}} = -\xi\omega_n + j\omega_n\sqrt{1-\xi^2}$$
$$S_{d_{old}} = -\sigma_d + j\omega_d$$

(b) Now find old settling time, $T_{S_{old}} = \frac{4}{\xi\omega_n} = \frac{4}{\sigma_d}$

2. Evaluate the performance of the compensated system:

(a) From the question, find new $T_{S_{new}}$ and new ω_n [ξ will remain unchanged]

(b) Find dominant pole (new), $S_{d_{new}} = -\xi\omega_n + j\omega_n\sqrt{1-\xi^2}$

$$S_{d_{new}} = -\sigma_d + j\omega_d$$

(c) Now, arbitrarily selecting a $-Z_C$

(d) Design the location of compensating pole, $-P_C$

- Finding necessary angle (Θ_P) to locate $S_{d_{new}}$ on root locus.
- $\tan\Theta_P = \frac{\omega_d}{P_C - \sigma_d}$ then find, $-P_C$

Characteristic of Lead Compensator:

1. Generalized transfer function of lead compensator, $\boldsymbol{G_C} = \frac{(s+Z_c)}{(s+P_c)}$
1. Zero at $-Z_C$ and pole at $-P_C$ are selected to put design point on the root locus.
2. Pole at $-P_C$ is more negative than zero at $-Z_C$ (i.e. $-P_C > -Z_C$).
3. Active circuits are not required to implement.

TABLE 7.2 Relationships between input, system type, static error constants, and steady-state errors

Input	Steady-state error formula	Type 0 Static error constant	Type 0 Error	Type 1 Static error constant	Type 1 Error	Type 2 Static error constant	Type 2 Error
Step, $u(t)$	$\frac{1}{1+K_p}$	$K_p =$ Constant	$\frac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $tu(t)$	$\frac{1}{K_v}$	$K_v = 0$	∞	$K_v =$ Constant	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola, $\frac{1}{2}t^2u(t)$	$\frac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a =$ Constant	$\frac{1}{K_a}$

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Problem Statement 1: A unity negative feedback system which is operating with a closed loop response that has 58% overshoot with a gain, K=164.6 has the following transfer function: $\boldsymbol{G_u(s)} = \frac{\boldsymbol{K}}{(\boldsymbol{s+1})(\boldsymbol{s+2})(\boldsymbol{s+10})}$; Now-

(a) Find the steady state error for a unit step input

(b) Design a PI controller to make the steady state error to zero.

(c) By using both theoretically and MATLAB codes show that steady state error becomes zero?

Design Process:

- First add a pole at origin.
- Now add a zero very close to the origin.
- Generalized transfer function of PI compensator, $\boldsymbol{G_C(s)} = \frac{\boldsymbol{s+a}}{\boldsymbol{s}}$

Theoretical Solution:

It is a type – 0 system, $Old\ or\ previous\ K_{P_o} = \lim_{s\to 0} \boldsymbol{G_u(s)} = \frac{164.6}{20} = 8.23$

Thus, $e_{old}(\infty) = \frac{1}{1+K_{P_o}} = \frac{1}{1+8.23} = 0.1083$

Arbitrarily selecting, $-a = -0.1$

Finally, transfer function of PI compensator, $\boldsymbol{G_C} = \frac{\boldsymbol{s+0.1}}{\boldsymbol{s}}$

$K_{P_N} = \lim_{s\to 0} G_e(s) = G_C(s)\times G_u(s) = \infty$

$e_{new}(\infty) = \frac{1}{1+K_{P_N}} = 0$

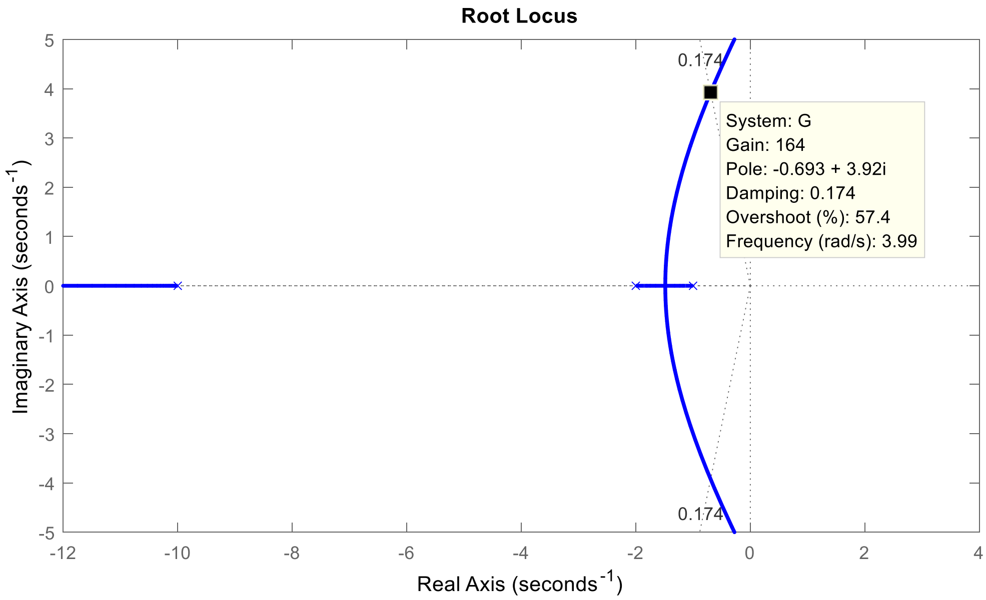

Fig. 1(a). Root locus for uncompensated system

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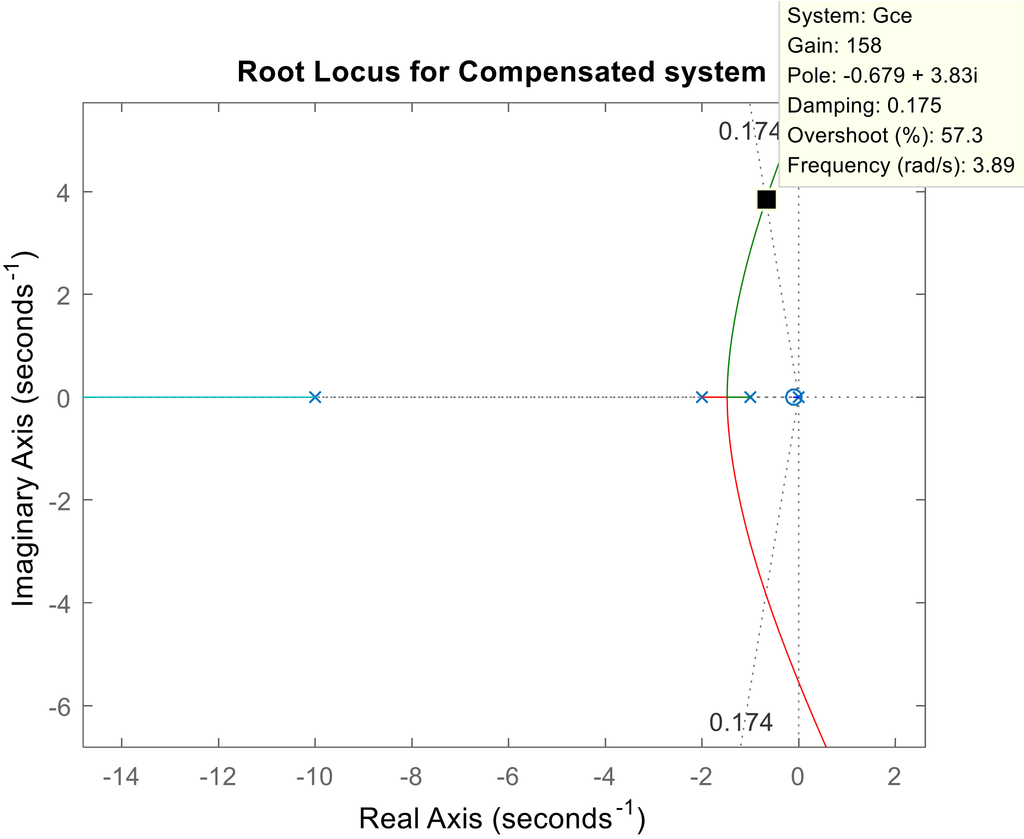

Fig. 1(b). Root locus for PI compensated system

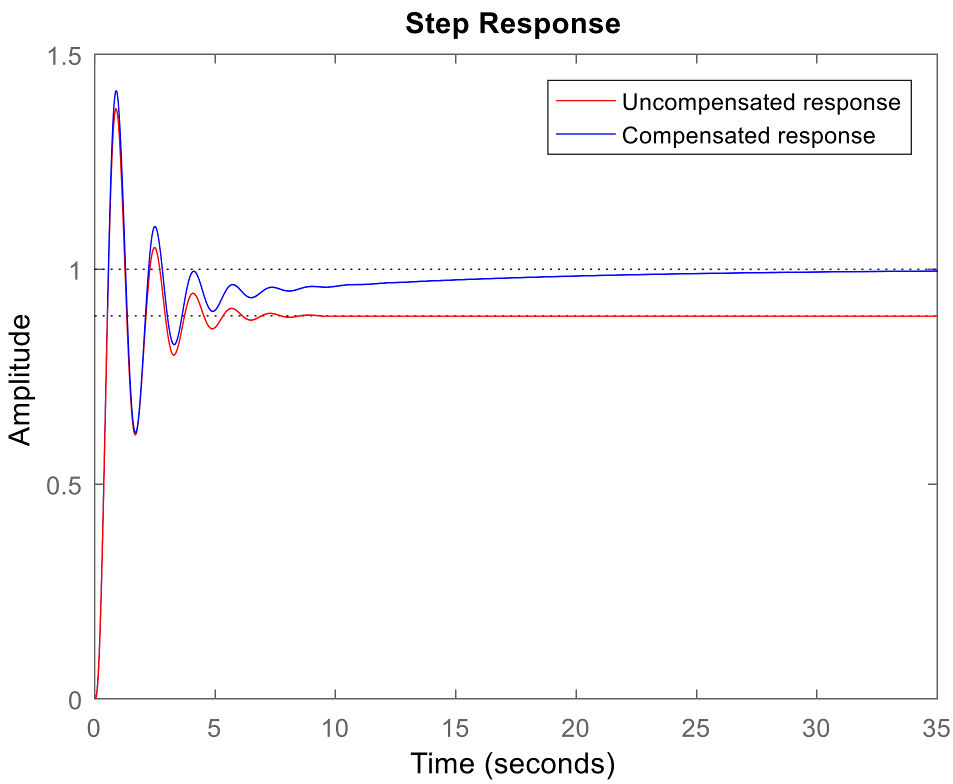

Fig. 1(c). Output Response of Uncompensated and PI Compensated System

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Problem Statement 2: A unity negative feedback system which is operating with a damping ratio of 0.174 has the following transfer function: $\boldsymbol{G(s)} = \frac{\boldsymbol{K}}{(\boldsymbol{s+1})(\boldsymbol{s+2})(\boldsymbol{s+10})}$; Now-

(a) Find the steady state error for a unit step input.
(b) Design a lag compensator to improve the steady state error by a factor of 10.
(c) Evaluate the steady state error for a unit step input to your compensated system.

Design Process:

It is a type – 0 system, $Old\ or\ previous\ K_{P_o} = \lim_{s\to 0} G(s) = \frac{164.6}{20} = 8.23$

[K=164.6 which was extracted from the Root Locus of Uncompensated System]

Thus, $e_{old}(\infty) = \frac{1}{1+K_{P_o}} = \frac{1}{1+8.23} = 0.1083$

But, $e_{new}(\infty) = \frac{0.1083}{10} = 0.01083 = \frac{1}{1+K_{P_N}}$

Rearranging and solving for the required, $K_{P_N} = 91.59$

The improvement in K_P from the uncompensated system to the compensated system is the required ratio of the compensator zero to the compensator pole,

$$\frac{K_{P_N}(New\ or\ desired\ K_p)}{K_{P_o}(Old\ or\ previous\ K_p)} = \frac{Z_C}{P_C} = \frac{91.59}{8.23} = 11.13$$

Arbitrarily selecting, $-P_C = -0.01$

Thus $-Z_C = 11.13P_C = -0.111$

Finally, transfer function of lag compensator, $\boldsymbol{G_C} = \frac{(\boldsymbol{s+0.111})}{(\boldsymbol{s+0.01})}$

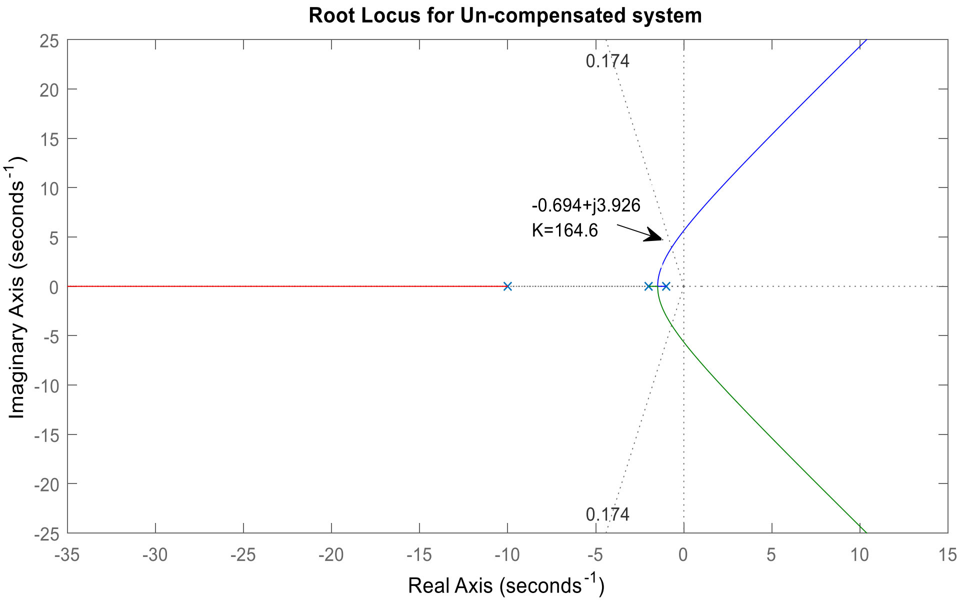

Fig 2(a). Root Locus for Uncompensated System

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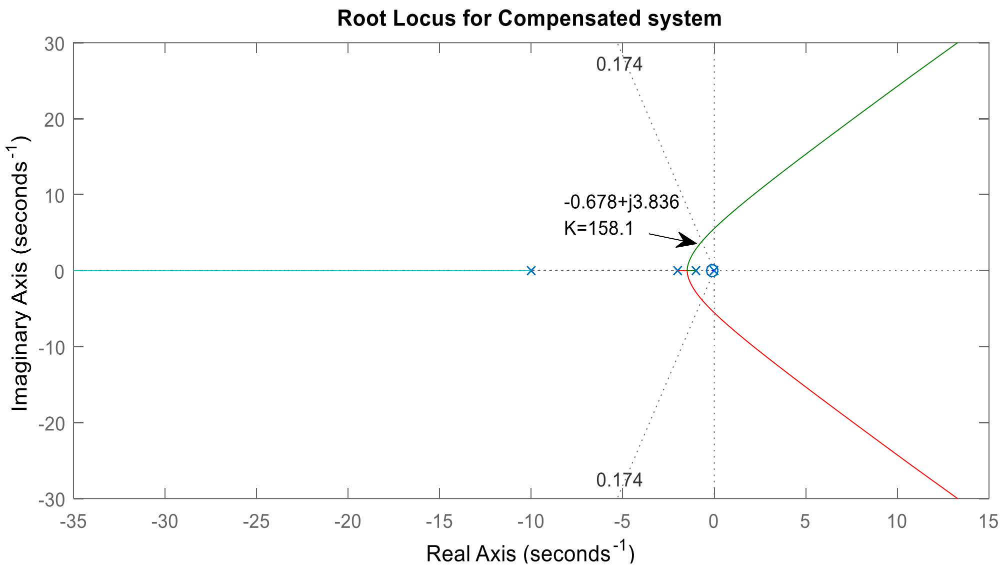

Fig 2(b). Root Locus for Lag Compensated System

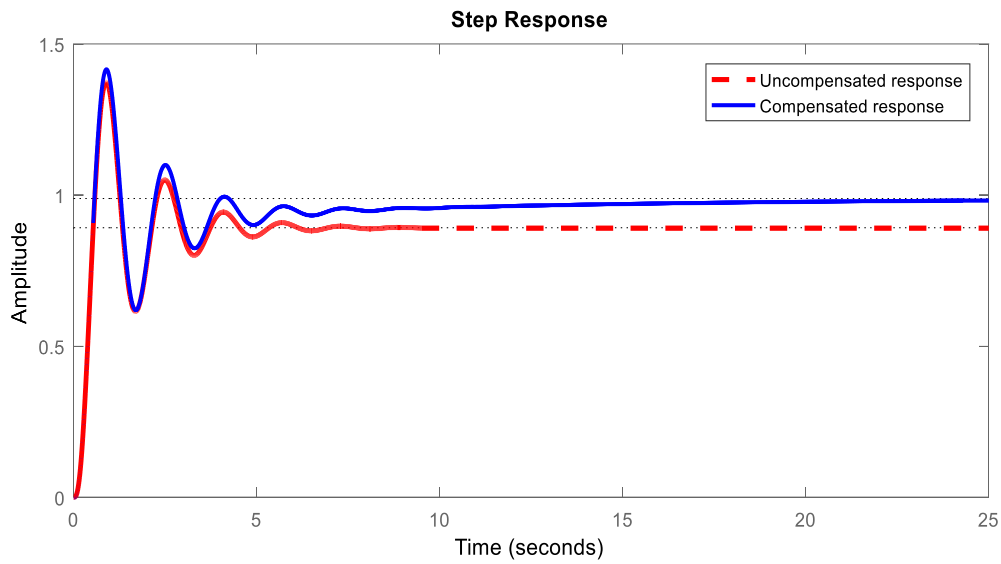

Fig 2(c). Output Response of Uncompensated and Lag Compensated System

N.B: To modify the error, PI or lag compensator is used, but there is very negligible change in desired transient response.

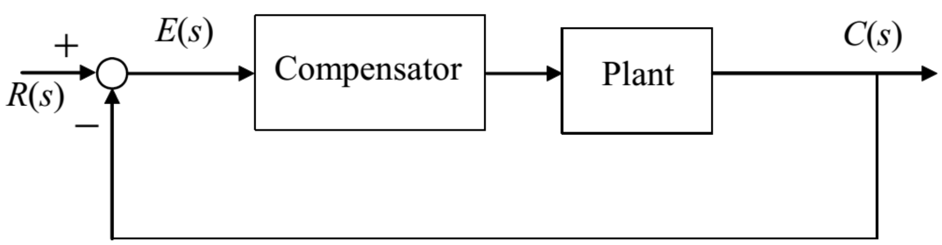

Fig. Cascaded Compensation

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<u>Problem Statement 3:</u> A unity negative feedback system has forward transfer function, $\mathbf{G(s) = K / [s(s+4)(s+6)]}$. Find the location of compensating zero for designing a PD compensator to yield a 16% overshoot, with a threefold reduction in settling time. **[Uncompensated System has ω_n= 2.391 rad/s].**

<u>Design Process:</u>

<u>1. First evaluate the performance of the uncompensated system:</u>

(a) We know, $\%O.S = exp^{\left(-\pi\xi/\sqrt{1-\xi^2}\right)} \times 100$ So, $\xi = 0.504$. Now, find dominant pole (old) from the root locus of uncompensated system,

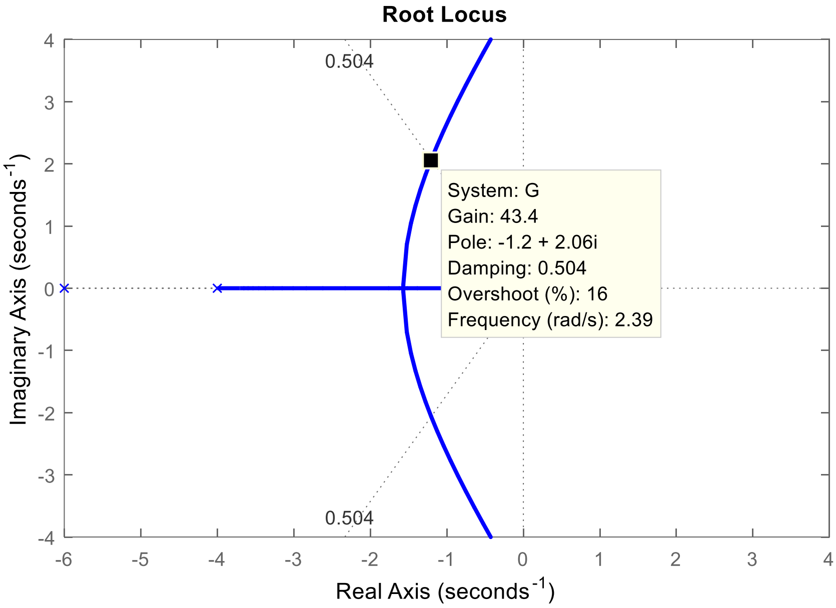

Fig. 3(a). Root locus for the uncompensated system

$Thus, S_{d_{old}} = -1.205 + j2.064$

(b) Now find old settling time, $T_{s_{old}} = \frac{4}{\xi\omega_n} = 3.320$ sec

<u>2. Evaluate the performance of the compensated system:</u>

(a) From the question, find new $T_{s_{new}}$ and new ω_n [ξ will remain unchanged which is 0.504]

$$T_{s_{new}} = \frac{3.320}{3} = 1.107\ sec \text{ and new } \omega_n = 7.17\ rad/s$$

(b) Find dominant pole (new), $S_{d_{new}} = -\xi\omega_n + j\omega_n\sqrt{1-\xi^2}$

$$S_{d_{new}} = -\sigma_d + j\omega_d = -3.613 + j6.193$$

(c) Design the location of compensating zero, $-Z_C$

- Finding necessary angle (θ_Z) to locate $S_{d_{new}}$ on root locus.
- $\tan\theta_Z = \frac{\omega_d}{Z_C - \sigma_d}$ then find, $-Z_C$

Calculation:

<u>Angle Criterion:</u> $\sum \boldsymbol{Open\ loop\ poles\ angle} - \sum \boldsymbol{Open\ loop\ zeros\ angle} = \mathbf{180°}$

$$-\boldsymbol{\theta_Z} + \boldsymbol{\theta_{P_0}} + \boldsymbol{\theta_{P_{-4}}} + \boldsymbol{\theta_{P_{-6}}} = \mathbf{180°}$$

$$-\theta_Z + \{180° - \cos^{-1}(0.504)\} + \tan^{-1}\left(\frac{6.193}{4-3.613}\right) + \tan^{-1}\left(\frac{6.193}{6-3.613}\right) = 180°$$

Thus, $\theta_Z = 95.6°$

Now, $\tan\theta_Z = \frac{\omega_d}{Z_C - \sigma_d}$ So, $-Z_C = -3.006$ Finally, $\boldsymbol{G_C = s + 3.006}$

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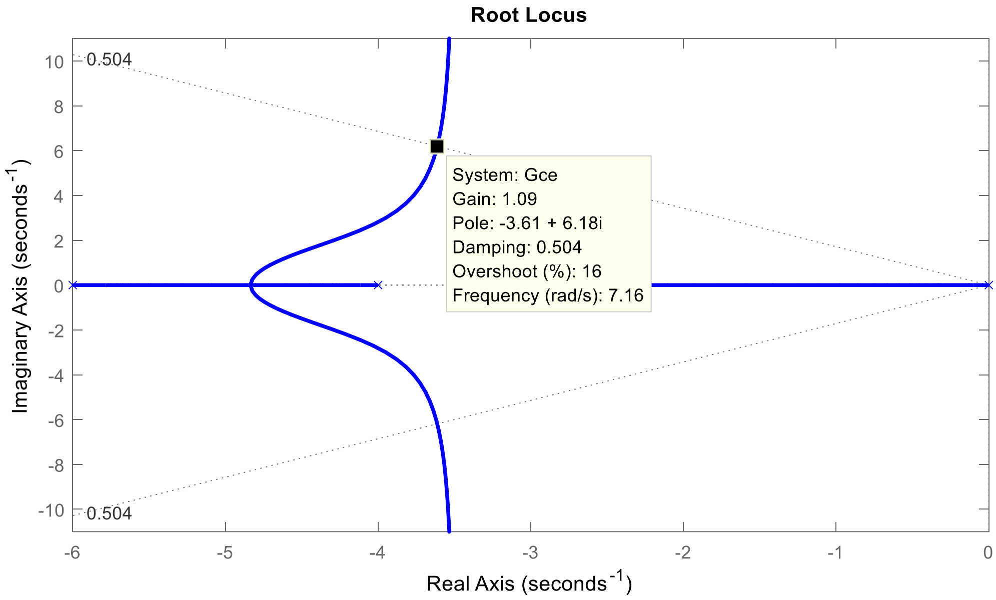

Fig. 3(b). Root locus for the compensated system

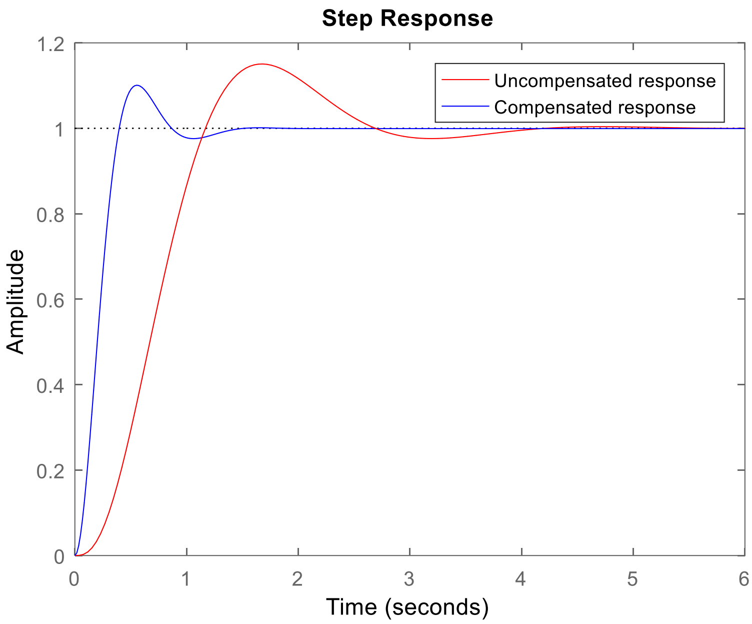

Fig. 3(c): Output Response of Uncompensated and PD Compensated System

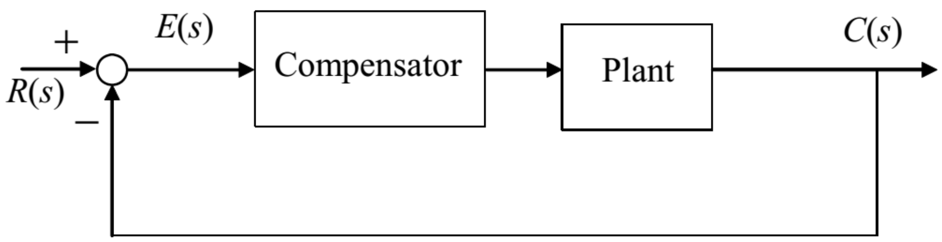

Fig. Cascaded Compensation

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Problem Statement 4: A unity negative feedback system has forward transfer function, $\mathbf{G(s) = K / [s(s+4)(s+6)]}$. Find the location of compensating pole for designing three lead compensator [Considering individually, $-Z_C = -5, -4, -2$] that will reduce the settling time by a factor of 2 while maintaining 30% overshoot.

Design Process:

1. First evaluate the performance of the uncompensated system:

(a) We know, $\%O.S = exp^{\left(-\pi\xi / \sqrt{1-\xi^2}\right)} \times 100$ So, $\xi = 0.358$. Now, find dominant pole (old) from the root locus of uncompensated system,

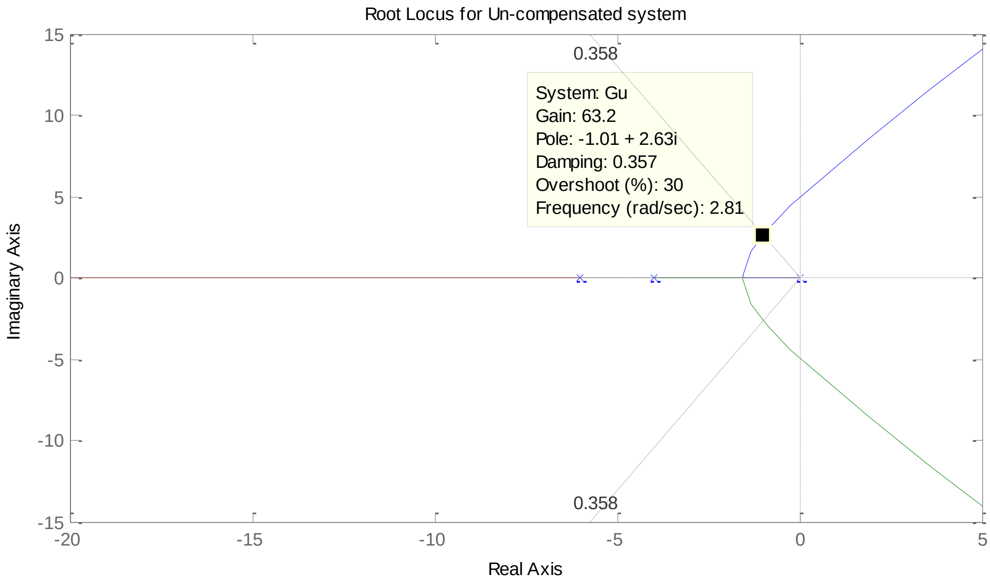

Fig. 4(a) Root Locus for Uncompensated System

$Thus, S_{d_{old}} = -1.01 + j2.63$

(b) Now find old settling time, $T_{S_{old}} = \frac{4}{\xi\omega_n} = \frac{4}{\sigma_d} = \frac{4}{1.01} = 3.96\ sec$

2. Evaluate the performance of the compensated system:

(a) From the question, find new $T_{S_{new}}$ and new ω_n [ξ will remain unchanged which is 0.358]

$$T_{S_{new}} = \frac{3.96}{2} = 1.98\ sec \text{ and new } \omega_n = 5.62\ rad/s$$

(b) Find dominant pole (new), $S_{d_{new}} = -\xi\omega_n + j\omega_n\sqrt{1-\xi^2}$

$$S_{d_{new}} = -\sigma_d + j\omega_d = -2.012 + j5.25$$

(c) Now, arbitrarily selecting a $-Z_C = -5$

(d) Design the location of compensating pole, $-P_C$

- Finding necessary angle (θ_P) to locate $S_{d_{new}}$ on root locus.
- $\tan\theta_P = \frac{\omega_d}{P_C - \sigma_d}$ then find, $-P_C$

Calculation:

Angle Criterion: $\sum \boldsymbol{Open\ loop\ poles\ angle} - \sum \boldsymbol{Open\ loop\ zeros\ angle} = \mathbf{180°}$

$\boldsymbol{\theta_P + \theta_{P_0} + \theta_{P_{-4}} + \theta_{P_{-6}} - \theta_{Z_{-5}} = 180°}$ **[From Root Locus of Lead Compensated System]**

$$\theta_P + \{180° - \cos^{-1}(0.358)\} + \tan^{-1}\left(\frac{5.252}{4-2.012}\right) + \tan^{-1}\left(\frac{5.252}{6-2.012}\right) - \tan^{-1}\left(\frac{5.252}{5-2.012}\right) = 180°$$

Thus, $\theta_P = 7.31°$

Now, $\tan\theta_P = \frac{\omega_d}{P_C - \sigma_d}$ So, $-P_C = -42.95$ Finally, $\boldsymbol{G_C = \frac{(s+5)}{(s+42.95)}}$

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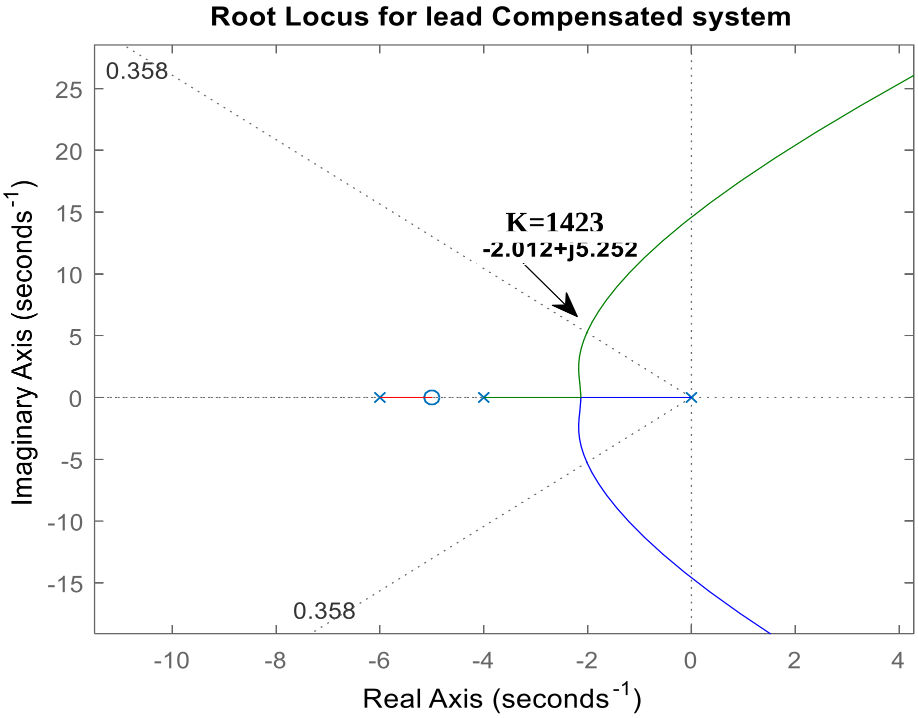


Fig. 4(b) Root Locus for Lead Compensated System

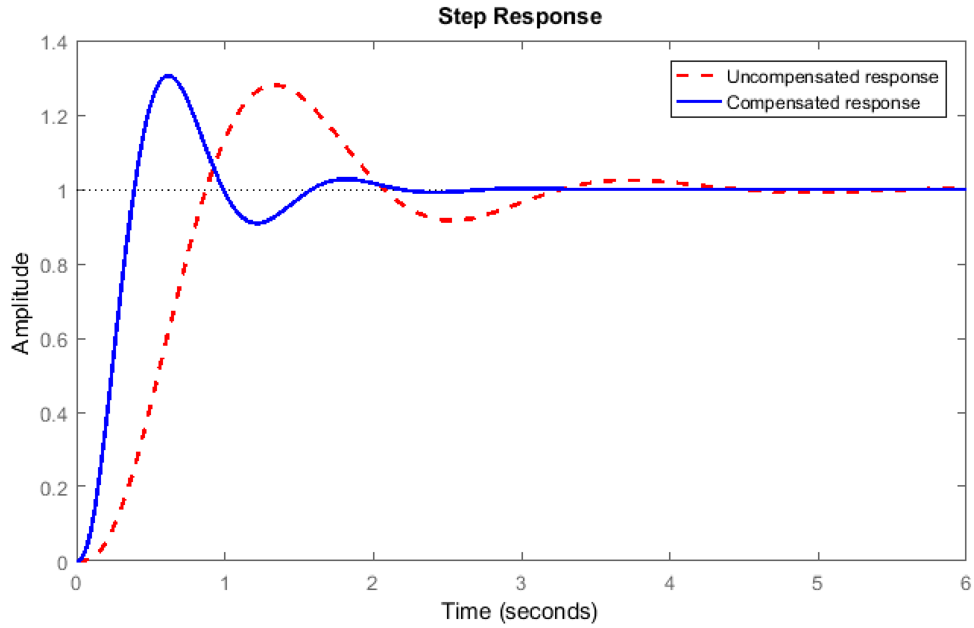


Fig. 4(c): Output Response of Uncompensated and Lead Compensated System

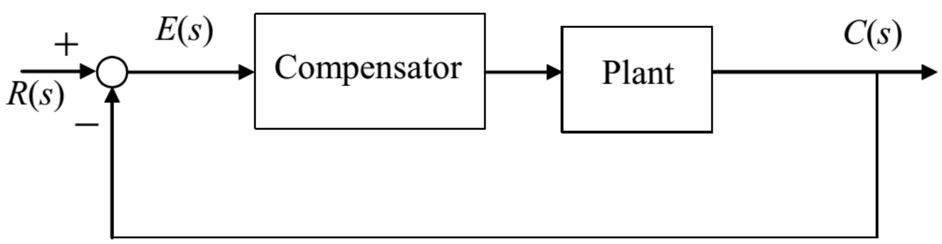
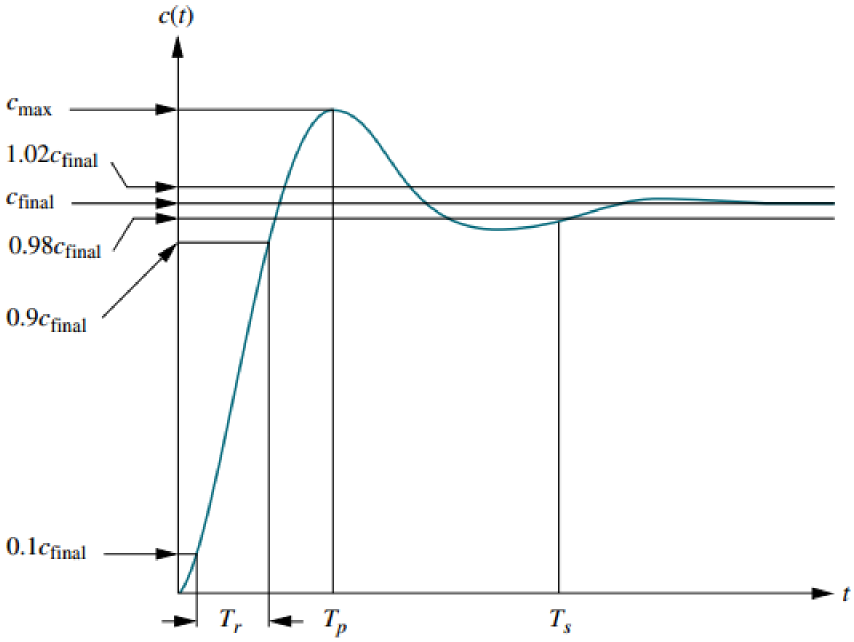


Fig. Cascaded Compensation

Second-order underdamped response specifications:



Reference(s):

[1] Norman S. Nise, "Control Systems Engineering", available Edition, John Wiley & Sons Inc.