Complex Tutorial

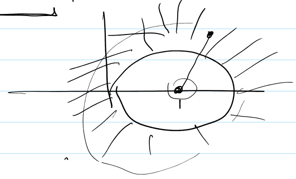
18 January 2024 22:04

Q1 $f(z) = \dfrac{e^z}{z-1}$ $\qquad 0 < |z-1|$

$$\frac{e^z}{z-1} = \frac{e^{z-1+1}}{z-1} = \frac{e^1 e^{z-1}}{z-1} = \frac{e}{z-1} \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}$$

$$= e \sum_{n=0}^{\infty} \frac{(z-1)^{n-1}}{n!}.$$

Q2 $f(z) = \dfrac{z^2-2z+2}{z-2}$ $\qquad \underline{1 < |z-1|}$



$$\frac{1}{z-2} = \frac{1}{z-1-1} = \frac{1}{z-1} \cdot \frac{1}{1-\frac{1}{z-1}}$$

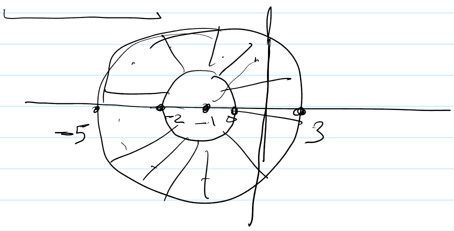
$$= \frac{1}{z-1} \sum_{n=0}^{\infty} \left(\frac{1}{z-1}\right)^n$$

$$z^2-2z+2 = z^2-2z+2 = z^2-2z+1+1 = (z-1)^2+1$$

$$f(z) = \left[(z-1)^2+1\right] \frac{1}{z-1} \sum_{n=0}^{\infty} (z-1)^{-n}$$

$$= \left[(z-1)+(z-1)^{-1}\right] \sum (z-1)^{-n} = \sum_{n=0}^{\infty} (z-1)^{-n+1} + \sum_{n=0}^{\infty} (z-1)^{-n-1}$$

Q3 $f(z) = \dfrac{1}{z(z-3)}$ $\qquad \underline{1 < |z+1|} < 4$



$$= \frac{A}{z} + \frac{B}{z-3}$$

$$A(z-3) + Bz = 1$$

$A+B=0$
$-3A=1$
$A=-1/3$
$B=1/3$

$$f(z) = -\frac{1}{3z} + \frac{1}{3(z-3)} = \frac{1}{3}\left[-\frac{1}{z} + \frac{1}{z+3}\right]$$

$$\frac{1}{z} = \frac{1}{z+1-1} = \frac{1}{z+1} \cdot \frac{1}{1-\frac{1}{z+1}} = \frac{1}{z+1} \sum_{n=0}^{\infty} (z+1)^{-n}$$

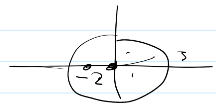
$$\frac{1}{z-3} = \frac{1}{z+1-1-3} = \frac{1}{(z+1)-4} = -\frac{1}{4} \cdot \frac{1}{1-\left(\frac{z+1}{4}\right)}$$

$$= -\frac{1}{4}\left[\sum_{n=0}^{\infty} \left(\frac{z+1}{4}\right)^n\right.$$

$$f(z) = \frac{1}{3}\left[-\sum_{n=0}^{\infty} (z+1)^{-n-1} - \frac{1}{4} \sum_{n=0}^{\infty} \left(\frac{z+1}{4}\right)^n\right]$$

$$f(z) = \frac{1}{3}\left[- \sum_{n=0}^{\infty} (z+1)^{-n-1} - \frac{1}{4}\sum_{n=0}^{\infty} \left(\frac{z+1}{4}\right)^n \right]$$

$$\oint_C \frac{e^z}{z^2(z+2)}\,dz \quad , \quad C: |z| = 3$$



$z = 0, -2$

$$= 2\pi i \sum \text{Res} = 2\pi i \left(\text{Res}[f, 0] , \text{Res}[f, -2] \right)$$

$$\text{Res}[f, -2] = \lim_{z \to -2} \frac{e^z}{z^2} = \frac{e^{-2}}{4} = \frac{1}{4e^2}$$

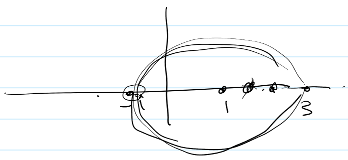
$$\text{Res}[f, 0] = \lim_{z \to 0} \frac{d}{dz}\left(\frac{e^z}{(z+2)} \right) = \lim_{z \to 0} \left[\frac{e^z}{(z+2)} - \frac{e^z}{(z+2)^2} \right]$$

$$g(z) = \frac{f(z)}{(z-z_0)^n} \quad \rightarrow \text{Res}[g(z), z_0] = \underbrace{\frac{1}{(n-1)!} \lim_{z \to z_0}}\frac{d^{n-1}}{dz^{n-1}}\left((z-z_0)^n g(z) \right)$$

$$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$\oint_C = 2\pi i \left[\frac{1}{4e^2} + \frac{1}{4} \right] = \frac{\pi i}{2}\left[1 + e^{-2} \right].$$

$$\oint_C \frac{\tan(z)}{z}\,dz \qquad |z-1| = 2$$



$z = 0$, $\quad \tan z = \frac{\sin z}{\cos z}$, $\quad \cos(z) = 0$

$z = (n + \frac{1}{2})\pi \qquad n = 1 \qquad \pi/2 \sim 1.21$

$\Rightarrow \quad f(z)$ pole of order n (z_0)

$$\frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{n-1}} \cdots \; a_0$$

$(n-1)$

$\lim_{z \to z_0} \underbrace{f(z)(z-z_0)^n} \longrightarrow$ finite

$$f(z) \quad \lim_{z \to \pi/2} (z - \pi/2)^1 \tan(z) = \lim_{z \to \pi/2} \frac{(z-\pi/2)\sin z}{\cos z}$$

$$= \lim_{z \to \pi/2} \frac{\sin z + \cos z\,(z - \pi/2)}{-\sin z} = \frac{1}{-1} = -1$$

0, pole of order 1

$\pi/2$, pole of order 1

$$\oint_C = 2\pi i \sum \text{Res}$$

$$= 2\pi i \left[\text{Res}(f, 0) + \text{Res}(f, \pi/2) \right]$$

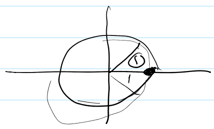
$$\operatorname{Res}(f,0) = \lim_{z\to 0} \cancel{z}\,\frac{\tan(z)}{\cancel{z}} = \lim_{z\to 0} \tan z = 0$$

$$\operatorname{Res}(f,\pi/2) = \lim_{z\to \pi/2} (z-\pi/4)\,\frac{\tan z}{z} = \lim_{z\to \pi/2} \frac{(z-\pi/4)\sin z}{z\cos(z)}$$

$$= \lim_{z\to\pi/2} \frac{\sin z + (z-\pi/4)\cos z}{\cos(z) \mp z\sin z} = \frac{1}{-\pi/2} = -\frac{2}{\pi}$$

$$\oint_c = 2\pi i\left(-\frac{2}{\pi}\right) = -4i$$

Q $$\int_0^{2\pi} \frac{\cos 2\theta\, d\theta}{5-4\cos\theta}$$



$$z = e^{i\theta} = \cos\theta + i\sin\theta$$
$$z^{-1} = e^{-i\theta} = \cos\theta - i\sin\theta$$
$$\longrightarrow \cos\theta = \frac{z+z^{-1}}{2}, \quad \sin\theta = \frac{z-z^{-1}}{2i}$$

$$\oint_{|z|=0} \frac{\left(\frac{z^2+z^{-2}}{2}\right)}{\left[5-4\left(\frac{z+z^{-1}}{2}\right)\right]}\,\frac{dz}{iz}$$

$$z^2 = e^{2i\theta} = \cos(2\theta) + i\sin 2\theta$$
$$\Rightarrow \cos(2\theta) = \frac{z^2+z^{-2}}{2}$$
$$z = e^{i\theta} \Rightarrow dz = i e^{i\theta} d\theta \Rightarrow d\theta = \frac{dz}{iz}$$

$$= \frac{1}{2i}\oint_c \frac{z^2\,(z^2+z^{-2})\,dz}{z^2\,5z - 2z^2 - 2} = -\frac{1}{2i}\oint \frac{(z^4+1)\,dz}{z^2(2z^2-5z+2)}$$

$$= -\frac{1}{2i}\oint \frac{(z^4+1)\,dz}{z^2(2z-1)(z-2)}$$

pole $\{0, \frac{1}{2}\}$ (2, 1)

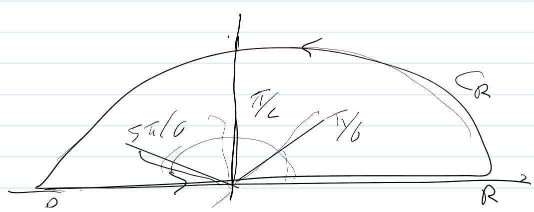
$$= -\frac{1}{2i}\,2\pi i\,\left(\operatorname{Res}[0], \operatorname{Res}[\tfrac{1}{2}]\right)$$
$$= -\pi\left(\sum \operatorname{Res}\right)$$

$$\operatorname{Res}[0] = \lim_{z\to 0}\frac{d}{dz}\left(\frac{z^4+1}{(2z-1)(z-2)}\right) = \lim_{z\to 0}\left[\frac{4z^3}{(2z-1)(z-2)} - \frac{(z^4-1)}{(2z-1)^2(z-2)^2}(2z-1+2z-4)\right]$$

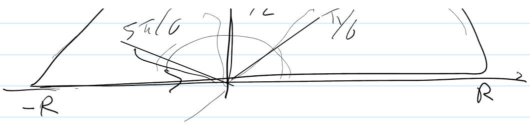
$$= \frac{1}{4}\left[-1-4\right] = \frac{5}{4}$$

$$\operatorname{Res}[\tfrac{1}{2}] = \lim_{z\to 1/2} \frac{z^4+1}{z^2(z-2)} = \frac{1/2^4+1}{(1/2)^2(-3/2)} = \alpha$$

Q $$\int_0^\infty \frac{dx}{x^6+1}$$



$\int_0 \frac{}{x^6+1}$



$$\oint f(z)dz = 2\pi i \sum \text{Res}$$
$$= \int_{-R}^{R} f(z)\,dz + \cancel{\int_{C_R} f(z)\,dz}^{\,0}$$

$\frac{P_n(z)}{Q_m(z)}\,dz$

$m \geq n+2$

$$\oint f(z)dz = 2\pi i \sum \text{Res}[\uparrow]$$
$$= \lim_{R\to\infty} \int_{-R}^{R} f(z)\,dz = \int_{-\infty}^{\infty} f(z)\,dz$$

$$\int_0^\infty \frac{dx}{x^6+1} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{dx}{x^6+1} \longrightarrow \frac{1}{2}\oint \frac{dz}{z^6+1} = \frac{1}{2}(2\pi i)\sum \text{Res}[f,\uparrow]$$

$z^6+1=0$, $z^6 = -1 = e^{\pi i}$, $e^{\left(\frac{\pi+2\pi k}{6}\right)i}$

Roots $\left\{ e^{i\pi/6},\ e^{i(\pi/2)},\ e^{i(5\pi/6)},\ \cancel{e^{7\pi/6\, i}} \right\}$

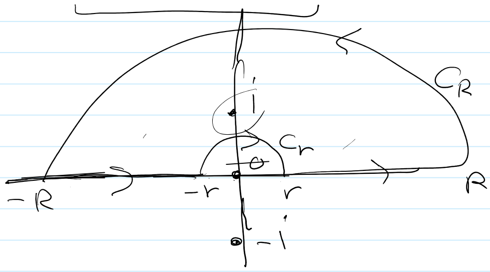
$$\pi i\left(\text{Res}(f, e^{i\pi/6}) + \text{Res}(f, e^{i\pi/2}) + \text{Res}(f, e^{i5\pi/6}) \right).$$

complete $\longrightarrow$

$$\int_{-\infty}^{\infty} \frac{\sin x\, dx}{x(x^2+1)} = \int_{-\infty}^{\infty} \frac{\text{Im}\{e^{ix}\}\,dx}{x(x^2+1)} = \text{Im}\left\{ \underbrace{\int_{-\infty}^{\infty} \frac{e^{ix}dx}{x(x^2+1)}} \right\}$$

$$\int_{-\infty}^{\infty} \frac{e^{ix}\,dx}{x(x^2+1)}$$

Jordan's lemma



$$\oint f(z)dz = \int_{-R}^{-r} - \int_{C_r} + \int_r^R + \cancel{\int_{C_R}}^{\,0}$$

$\pi i\,\text{Res}[f,0]$

$$= \int_{-\infty}^{\infty} f(z)\,dz - \pi i\,\text{Res}[f,0] = 2\pi i \sum \text{Res}[\text{inside } C]$$

$$\Rightarrow \int_{-\infty}^{\infty} f(z)\,dz = 2\pi i\,\text{Res}(f,i) + \pi i\,\text{Res}(f,0)$$

$$\frac{e^{iz}}{z(z^2+1)} = \frac{e^{iz}}{z(z+i)(z-i)} \Rightarrow \text{Res}[i] = \frac{e^{-1}}{i(2i)} = -\frac{1}{2}e^{-1}$$

$$\text{Res}[0] = \frac{e^0}{(i)(-i)} = 1$$

$\int^\infty$

$$\mathrm{Res}[a] = \frac{e}{(i)(-i)} = 1$$

$$\int_{-\infty}^{\infty} f(z)\,dz = \cancel{2}\pi i\left(-\tfrac{1}{2}e^{-1}\right) + \pi i(1)$$

$$= -\pi i e^{-1} + \pi i = \underline{i\pi(1-e^{-1})}$$

$$\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+1)}\,dx = \mathrm{Im}\left(i\pi(1-e^{-1})\right) = \pi(1-e^{-1})$$

Q $\quad \dfrac{(1-i)^{10}}{(1+i)^3}$, $\quad (1-i)^{10} = e^{\ln(1-i)^{10}} = e^{10\ln(1-i)}$

$$\ln(1-i) = \ln|1-i| + i[\arg(1-i) + 2\pi n]$$

$$|1-i| = \sqrt{(1^2+1^2)} = \sqrt{2}$$

$$\arg(1-i) = \tan^{-1}\left(\frac{-1}{1}\right) = \tan^{-1}(-1) = -\frac{\pi}{4}$$

$$\ln(1-i) = \ln\sqrt{2} + i\left(-\pi/4 + 2\pi n\right)$$

$$e^{10\ln\sqrt{2} + 10i(-\pi/4+2\pi n)} = e^{10\ln\sqrt{2}}\, e^{-10\pi i/4} = e^{\ln(2^5)}\, e^{-5\pi i/2}$$

$$= 2^5 e^{-5\pi i/2} = 2^5\left[\underbrace{\cos(-5\pi/2)}_{0} + i\underbrace{\sin(-5\pi/2)}_{-1}\right]$$

$$= -2^5 i$$

$$(1+i)^3 = e^{3\ln(1+i)} = e^{3[\ln\sqrt{2} + i\pi/4]} = e^{\ln 2^{3/2}}\, e^{3i\pi/4}$$

$$= 2^{3/2}\left[\underbrace{\cos(3\pi/4)}_{-\frac{1}{\sqrt{2}}} + i\underbrace{\sin(3\pi/4)}_{\frac{1}{\sqrt{2}}}\right]$$

$$= -2 + 2i$$

$$\frac{(1-i)^{10}}{(1+i)^3} = -\frac{2^5 i}{-2+2i}\cdot\frac{-2-2i}{-2-2i} = \frac{2^6(i-1)}{2^3} = 2^3(i-1)$$

$$= -8 + 8i$$

Q $\quad \sinh^{-1}(4/3) = \theta$

$$\sinh\theta = z \;\Rightarrow\; \frac{e^{\theta} - e^{-\theta}}{2} = z$$

$$e^{\theta} - e^{-\theta} = 2z \;\Rightarrow\; e^{2\theta} - 1 = 2e^{\theta}z$$

$$e^{\theta} - e^{-\theta} = 2z \Rightarrow e^{2\theta} - 1 = 2e^{\theta}z$$

$$\Rightarrow e^{2\theta} - 2ze^{\theta} - 1 = 0 \qquad u = e^{\theta}$$

$$u^2 - 2zu - 1 = 0$$

$$u = e^{\theta} = \frac{2z \pm \sqrt{4z^2+4}}{2} = z \pm \sqrt{1+z^2}$$

$$\theta = \ln\left[z \pm \sqrt{1+z^2}\right]$$

$$\sinh^{-1}(4/3) = \ln\left[\frac{4}{3} \pm \sqrt{\underbrace{1+\frac{16}{9}}_{25/9}}\right] = \ln\left[\underbrace{\frac{4}{3} \pm 5/3}\right]$$

$$\ln(3) = \ln(3) + i2\pi n$$

$$\ln(-3) = \ln|-3| + i\,\mathrm{avg}(-3) + 2\pi$$

$$= \ln(3) + i(\pi + 2\pi)$$