Tutorial 3: Bessel and related functions

Mohamed Salaheldeen

November 2023

Under the supervision of: Dr. Abdallah Aboutahoun

1 Important Points

In a similar fashion to Legendre polynomials, there exists a generating function for Bessel functions,

$$\exp\left[\frac{1}{2}x\left(t-\frac{1}{t}\right)\right] = \sum_{n=-\infty}^{\infty} J_n(x)t^n \tag{1}$$

Note here the sum also goes to $-\infty$. From this, we can deduce the most important representation of the Bessel function,

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k+n}}{k!(k+n)!} \tag{2}$$

In this case, n is an integer. Furthermore, you can use this definition to show that $J_{-n}(x) = (-1)^n J_n(x)$. As we have seen in the lectures, we can analytically extend the definition of general case Bessel of order p instead of n by noticing that we can write the factorial in terms of a gamma function.

$$J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k+p}}{k!\Gamma(k+p+1)} \tag{3}$$

You can use this definition to show that,

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin(x) \qquad\qquad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos(x) \tag{4}$$

Furthermore, $J_p(x)$ is a solution to the Bessel differential equation,

$$x^2 y''(x) + xy'(x) + (x^2 - p^2)y = 0$$

Similarly $J_{-p}(x)$ is the second solution to the differential equation. However, for integer p values, J_{-p} is dependent on J_p. Thus, we were required to find a second solution to the DE. Consequently, we introduced $Y_n(x)$,

$$Y_n(x) = \lim_{p\mapsto n} \frac{\cos(p\pi)J_p(x) - J_{-p}(x)}{\sin(p\pi)} \tag{5}$$

Thus, producing a general solution for Bessel DE. You can use equation 3 for Bessel functions to show that it satisfies certain recurrence formulas much like Legendre polynomials.

$$\frac{d}{dx}[x^p J_p(x)] = x^p J_{p-1}(x) \tag{6}$$

$$\frac{d}{dx}[x^{-p} J_p(x)] = -x^{-p} J_{p+1}(x) \tag{7}$$

$$J_p'(x) + \frac{p}{x}J_p(x) = J_{p-1}(x) \tag{8}$$

$$J_p'(x) - \frac{p}{x}J_p(x) = -J_{p+1}(x) \tag{9}$$

$$2J_p'(x) = J_{p-1}(x) - J_{p+1}(x) \tag{10}$$

$$\frac{2p}{x}J_p(x) = J_{p-1}(x) + J_{p+1}(x) \tag{11}$$

Note!! $Y_p(x)$ satisfy the same recurrence formulas since it's just a linear combination of Bessel functions.

Integrations involving Bessel functions: Throughout the rest of the course you will be asked to do several integrations involving Bessel functions.

1. Integrations of the type:
$$I(m,n) = \int x^m J_n(x)dx$$

2. Definite integration

Let us talk about each type separately. Integrations of type 1 usually involve clever integration by parts that use formulas 6 and 7. However, we would like to make a general rule that would help us do any integral in no time. The idea here is to notice a pattern that appears when integrating by parts. Consider integral

$$\int x^m J_n(x)dx = \int x^m x^{-n-1} x^{n+1} J_n(x)dx$$

Using equation 6 by letting $p = n+1$. We can integrate by parts in the following manner,

$$\begin{aligned} u &= x^{m-n-1} & v &= x^{n+1}J_{n+1}(x) \\ du &= (m-n-1)x^{m-n-2}dx & dv &= x^{n+1}J_n(x)dx \end{aligned}$$

Thus, we end up with,

$$\int x^m J_n(x)dx = x^m J_{n+1}(x) - (m-n-1)\int x^{m-1}J_{n+1}dx$$

Notice here the following that we began with an integral $I(m,n)$ and ended up with the need to integrate $(m-1, n+1)$. Thus, using this technique we get,

$$I(m,n) \to I(m-1, n+1)$$

However, we can still use equation 6 in another manner. Instead of integrating the right-hand side, we differentiate,

$$\int x^m J_n(x)dx = \int x^m x^{-n} x^n J_n(x)dx$$

integrate by parts as follows,

$$\begin{aligned} u &= x^n J_n(x) & v &= x^{m-n+1}/(m-n+1) \\ du &= x^n J_{n-1}(x)dx & dv &= x^{m-n}dx \end{aligned}$$

Thus, we get,

$$\int x^m J_n(x)dx = \frac{x^{m+1}J_n(x)}{(m-n+1)} - \frac{1}{(m-n+1)}\int x^{m+1}J_{n-1}dx$$

Thus, we ended up with the inverse of the integral case,

$$I(m,n) \to I(m+1, n-1)$$

In a similar fashion, you can do the same with equation 7. Results are shown in table 1. You can easily find the following table.

Notice that we cannot use the second and last formula when $m-n = -1$ and $m+1 = -1$ respectively. If you look again at the right-hand side of equations 6 and 7, we see that we can exactly integrate when the difference $m-n = 1$ or $m+n = 1$. Thus, we make some integrations successfully after one another until we reach the desired result. Furthermore, notice that if $m+n$ is an odd number the integral can evaluated in a closed form while

Transformation	Formula
$I(m,n) \to I(m-1,n+1)$	$x^m J_{n+1}(x) - (m-n-1)I(m-1,n+1)$
$I(m,n) \to I(m+1,n-1)$	$\frac{x^{m+1}J_n(x)}{(m-n+1)} - \frac{I(m+1,n-1)}{(m-n+1)}$
$I(m,n) \to I(m-1,n-1)$	$-x^m J_{n-1}(x) + (m+n-1)I(m-1,n-1)$
$I(m,n) \to I(m+1,n+1)$	$\frac{x^{m+1}J_n(x)}{(m+n+1)} + \frac{I(m+1,n+1)}{(m+n+1)}$

Table 1: Table for integration chains of type 1.

$I_n(x)$	$K_n(x)$
$\frac{d}{dx}[x^p I_p(x)] = x^p I_{p-1}(x)$	$\frac{d}{dx}[x^p K_p(x)] = -x^p K_{p-1}(x)$
$\frac{d}{dx}[x^{-p} I_p(x)] = x^{-p} I_{p+1}(x)$	$\frac{d}{dx}[x^{-p} K_p(x)] = -x^{-p} K_{p+1}(x)$
$I_p'(x) + \frac{p}{x} I_p(x) = I_{p-1}(x)$	$K_p'(x) + \frac{p}{x} K_p(x) = -K_{p-1}(x)$
$I_p'(x) - \frac{p}{x} I_p(x) = I_{p+1}(x)$	$K_p'(x) - \frac{p}{x} K_p(x) = -K_{p+1}(x)$
$2I_p'(x) = I_{p-1}(x) + I_{p+1}(x)$	$-2K_p'(x) = K_{p-1}(x) + K_{p+1}(x)$
$\frac{2p}{x} I_p(x) = I_{p-1}(x) - I_{p+1}(x)$	$-\frac{2p}{x} K_p(x) = K_{p-1}(x) - K_{p+1}(x)$

Table 2: Recursion formula for modified Bessel function

if $m+n$ is even the integral will depend on the residual element $\int J_0(x)dx$ which has no analytic form. Lastly, again, notice that this is also the case for the Bessel function of the second kind.

Second type of integrations is definite integrations of Bessel functions. The most basic approach to solve these problems is by using the series representation of Bessel and simplifying the equation. However, it's a context-dependent case.

We have seen that the Bessel functions are even analytic solutions to a more general class of differential equations.

$$x^2y'' + (1-2a)xy' + [b^2c^2x^{2c} + (a^2 - c^2p^2)]y = 0 \qquad p \geq 0, b > 0 \tag{12}$$

which has a solution,

$$y = x^a[C_1 J_p(bx^c) + C_2 Y_p(bx^c)]$$

If we allow b^2 to be negative thus introducing an i factor in the arguments, we get another type of solution with non-oscillatory behavior called the modified Bessel functions. $I_p(x)$ and $K_p(x)$ where,

$$I_p(x) = i^{-p} J_p(ix) = \sum_{k=0}^{\infty} \frac{(x/2)^{3k+p}}{k!\Gamma(k+p+1)}$$

$$K_p(x) = \frac{\pi}{2} \frac{I_{-p} - I_p}{\sin(p\pi)}$$

Using these definitions, we can in a similar manner to the usual Bessel function derive some recurrence formulas as in Table 2. Notice here the difference between Bessel of the first and second kind mainly the position of the minus signs. Lastly, you can deal with the integration of modified Bessel in a similar manner we did with usual Bessel functions. However, the integration rules change due to the minus signs (Can you derive integration rules for type 1 integrations like table 1 ?!).

2 Selected Exercises

Problem 1
Show that,

$$J_1'(0) = \frac{1}{2} \qquad\qquad J_n'(0) = 0$$

with $n \in \mathbb{Z} - 1, -1$

Solution: The basic method to solve such an equation is through the generating function. Clearly, we want to find the derivative. Thus, we take the derivative of equation 1 and then substitute $x = 0$. We, thus, get,

$$\sum_{n=-\infty}^{\infty} J_n'(0)t^n = \frac{1}{2}\left(t - \frac{1}{t}\right) \exp\left\{\frac{1}{2}x\left(t - \frac{1}{t}\right)\right\}\bigg|_{x=0} = \frac{1}{2}\left(t - \frac{1}{t}\right)$$

Thus, by comparing coefficients, we clearly get,

$$J_1'(0) = -J_{-1}'(0) = \frac{1}{2} \qquad\qquad J_n(0) = 0$$

Problem 2
Show that,

$$\frac{d}{dx}J_p(kx) = -kJ_{p+1}(kx) + \frac{p}{x}J_p(kx)$$

with $k > 0$.

Solution: We first notice that this equation has a great resemblance with recurrence formulae 9. However, the argument of Bessel is different. Thus, a simple solution would just be replacing every x with kx (note this also includes the x in the derivative sign). Making, the above substitution, we get,

$$\frac{d}{d(kx)}J_p(kx) - \frac{p}{kx}J_p(kx) = -J_{p+1}(kx)$$

Since k is a constant $\frac{d}{d(kx)} = \frac{1}{k}\frac{d}{d(x)}$. Thus, by multiplying both sides by k, we reach out result.

Problem 3
Show that,

$$J_p(x) = \frac{(x/2)^p}{\sqrt{\pi}\Gamma(p+1/2)}\int_{-1}^{1}(1-t^2)^{p-1/2}e^{ixt}dt$$

Solution: This is just a brute-force evaluation of the right-hand integral.

$$
\begin{aligned}
\int_{-1}^{1}(1-t^2)^{p-1/2}e^{ixt}dt &= \int_{-1}^{1}(1-t^2)^{p-1/2}[\cos(xt)+i\sin(xt)]dt && \text{(using euler formula)}\\
&= 2\int_{0}^{1}(1-t^2)^{p-1/2}\cos(xt)dt && \text{(Odd/even function on sym interval)}\\
&= 2\int_{0}^{1}(1-t^2)^{p-1/2}\left[\sum_{n=0}^{\infty}\frac{(-1)^n(xt)^{2n}}{(2n)!}\right]dt && \text{(Expand cos(xt))}\\
&= 2\sum_{n=0}^{\infty}\frac{(-1)^n(x)^{2n}}{(2n)!}\int_{0}^{1}(1-t^2)^{p-1/2}t^{2n}dt\\
&= \sum_{n=0}^{\infty}\frac{(-1)^n(x)^{2n}}{(2n)!}\frac{\Gamma(p+1/2)\Gamma(n+1/2)}{\Gamma(p+n+1)} && \text{(beta function definition)}\\
&= \Gamma(p+1/2)\sqrt{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n(x/2)^{2n}}{n!\Gamma(p+n+1)} && (\Gamma(n+1/2)=\frac{\sqrt{\pi}(2n)!}{2^{2n}n!})\\
&= \frac{\Gamma(p+1/2)\sqrt{\pi}}{(x/2)^p}\sum_{n=0}^{\infty}\frac{(-1)^n(x/2)^{2n+p}}{n!\Gamma(p+n+1)} && \text{multiplying by } \frac{(x/2)^p}{(x/2)^p}\\
&= \frac{\Gamma(p+1/2)\sqrt{\pi}}{(x/2)^p}J_p(x) && \text{(using eqn 1)}
\end{aligned}
$$

Problem 4
Show that,

1. $\int x^3J_0(x)dx = (x^3-4x)J_1(x)+2x^2J_0(x)$
2. $\int x^{-2}J_2(x)dx = -\frac{2}{3x^2}J_1(x)-\frac{1}{3}J_1(x)+\frac{1}{3x}J_0(x)+\frac{1}{3}\int J_0(x)dx$

Solution: To solve this problem, we will be using the integration chains we discussed above. Notice we can either reduce/increase power of x with increasing/decreasing order of J or both decrease/increase.

1. Since the sum $3+0$ is odd the integral can be evaluated in a closed form. Consider the following integration chain,

$$I(3,0)\to I(2,1)\to I(1,0)$$

Thus, we are using the first and third equations in Table 1.

$$
\begin{aligned}
I(3,0) &= x^3J_1(x)-(3-0-1)I(2,1) && \text{(first equation)}\\
&= x^3J_1(x)-2\left[-x^2J_0(x)+(2+1-1)I(1,0)\right] && \text{(third equation)}\\
&= x^3J_1(x)-2\left[-x^2J_0(x)+2xJ_1(x)\right] && \text{(eqn 6)}\\
&= (x^3-4x)J_1(x)+2x^2J_0(x)
\end{aligned}
$$

2. Similarly, since -2+2=0 is even the integral must have a residual J_0 integral, Consider the following integration chain,

$$I(-2,2)\to I(-1,1)\to I(0,0)$$

Thus,

$$\begin{aligned}
I(-2,2) &= \frac{1}{-2-2+1}x^{-1}J_2(x) - \frac{1}{-2-2+1}I(-1,1) && \text{(second equation)}\\
&= -\frac{1}{3}x^{-1}J_2(x) + \frac{1}{3}\left[-J_1(x) + I(0,0)\right] && \text{(second equation)}\\
&= -\frac{1}{3}x^{-1}\left[\frac{2}{x}J_1(x) - J_0(x)\right] + \frac{1}{3}\left[-J_1(x) + I(0,0)\right] && \text{(eqn 11)}\\
&= -\frac{2}{3x^2}J_1(x) - \frac{1}{3}J_1(x) + \frac{1}{3x}J_0(x) + \frac{1}{3}\int J_0(x)dx
\end{aligned}$$

Problem 5
Show that,

$$\int_0^\infty e^{-ax^2}x^{p+1}J_p(bx)dx = \frac{b^p}{(2a)^{p+1}}e^{-b^2/4a}$$

Solution: This is a brute force computation as before. Substitute the series representation of J_p and evaluate the integral.

$$\begin{aligned}
\int_0^\infty e^{-ax^2}x^{p+1}J_p(bx)dx &= \int_0^\infty e^{-ax^2}x^{p+1}\left[\sum_{k=0}^\infty \frac{(-1)^k(bx/2)^{2k+p}}{k!\Gamma(k+p+1)}\right]dx\\
&= \sum_{k=0}^\infty \frac{(-1)^k(b/2)^{2k+p}}{k!\Gamma(k+p+1)}\int_0^\infty e^{-ax^2}x^{2k+2p+1}dx\\
&= \sum_{k=0}^\infty \frac{(-1)^k(b/2)^{2k+p}}{k!\Gamma(k+p+1)}\frac{1}{2a^{k+p+1}}\int_0^\infty e^{-u}u^{k+p}du && \text{(substitute } u = ax^2\text{)}\\
&= \sum_{k=0}^\infty \frac{(-1)^k(b/2)^{2k+p}}{k!\Gamma(k+p+1)}\frac{\Gamma(k+p+1)}{2a^{k+p+1}} && \text{(definition of } \Gamma(x)\text{)}\\
&= \sum_{k=0}^\infty \frac{(-1)^k(b/2)^{2k+p}}{k!2a^{k+p+1}} = \frac{b^p}{(2a)^{p+1}}\sum_{k=0}^\infty \frac{(-b^2/4a)^k}{k!}\\
&= \frac{b^p}{(2a)^{p+1}}e^{-b^2/4a}
\end{aligned}$$

Problem 6
Using,

$$\int_o^\infty e^{-ax}J_0(bx)dx = \frac{1}{\sqrt{a^2+b^2}}$$

Prove that,

$$\int_0^\infty e^{-ax}x^2J_0(bx)dx = \frac{2a^2-b^2}{(a^2+b^2)^{5/2}}$$

Solution: Notice that $\frac{d^2}{da^2}e^{-ax} = x^2e^{-ax}$. Thus,

$$\begin{aligned}
\int_0^\infty e^{-ax}x^2J_0(bx)dx &= \int_0^\infty \frac{d^2}{da^2}[e^{-ax}]J_0(bx)dx\\
&= \frac{d^2}{da^2}\left[\int_0^\infty e^{-ax}J_0(bx)dx\right]\\
&= \frac{d^2}{da^2}\left[\frac{1}{\sqrt{a^2+b^2}}\right] = \frac{2a^2-b^2}{(a^2+b^2)^{5/2}}
\end{aligned}$$

Problem 7
Prove that,

$$\int_0^\infty \frac{\sin(x)}{x} J_0(bx)dx = \sin^{-1}\left(\frac{1}{b}\right)$$

for $b > 1$.

Solution: This exercise is a little bit nontrivial. As you can see we have somehow to insert a sin function and a factor of x inside the integral of J_0. Let us begin with the former. We are going to analytically continue the integral,

$$\int_o^\infty e^{-ax} J_0(bx)dx = \frac{1}{\sqrt{a^2+b^2}}$$

on $a \to ia$. Thus, we can use the Euler formula. Thus, we get,

$$\int_o^\infty e^{-iax} J_0(bx)dx = \int_o^\infty \cos(ax)J_0(bx)dx - i\int_o^\infty \sin(ax)J_0(bx)dx = \frac{1}{\sqrt{b^2-a^2}}$$

Now, we notice that the term on the right-hand side is real if $b > a$ and pure imaginary if $b < a$. Thus, we can deduce the following relation,

$$\int_o^\infty \cos(ax)J_0(bx)dx = \begin{cases} \frac{1}{\sqrt{b^2-a^2}} & b > a \\ 0 & b < a \end{cases}$$

$$\int_o^\infty \sin(ax)J_0(bx)dx = \begin{cases} 0 & b > a \\ \frac{1}{\sqrt{a^2-b^2}} & b < a \end{cases}$$

In our case $a = 1$. However, we still need to introduce the factor of x^{-1}. This is very nontrivial, but to do so, consider the integration,

$$\int_0^{a'} \cos(ax)da = \frac{\sin(a'x)}{x}$$

Thus, all you need to do is to integrate the equation of the cos with respect to a. Doing so, one gets,

$$\int_o^\infty \frac{\sin(a'x)}{x} J_0(bx)dx = \begin{cases} \sin^{-1}\left(\frac{a'}{b}\right) & b > a' \\ 0 & b < a' \end{cases}$$

Thus, $a' \mapsto 1$ to get the desired result.

Problem 8
Find a general solution to the following DE,

1. $y'' + xy = 0$
2. $y'' + k^2x^4y = 0$
3. $x^2y'' + 5xy' + (9x^2 - 12)y = 0$
4. $x^2y'' + xy' - (4x^2 + 1)y = 0$
5. $y'' - y = 0$

*Solution:*All we need to do is to put the above differential equations in a form similar to 12. Then, directly find the equation.

1. To put this into form 12, we need to multiply by x^2. Thus, we get,

$$x^2y'' + x^3y = 0$$

Thus, you can find that

$$(1-2a) = 0$$
$$a^2 - c^2p^2 = 0$$
$$b^2c^2 = 1$$
$$2c = 3$$

Solving these equations together, one can easily find $a = 1/2, p = 1/3, b = 2/3, c = 3/2$. Thus, the general solution is,

$$y = x^{1/2}[C_1J_{1/3}(2x^{3/2}/3) + C_2Y_{1/3}(2x^{3/2}/3)]$$

2. $a = 1/2, p = 1/3, c = 3$. However, note that the value of b will depend on the sign of k. If $k > 0$ then $b = k/3$ and If $k < 0$ then $b = -k/3$ if we require $b > 0$.

3. $a = -2, b = 3, p = \pm 4, c = 1$. For p, we can choose either one since they are dependent on each other.

4. $a = 0, b = 4i, p = 1, c = 1$. Thus, we can write a general solution as,

$$y = C_1J_1(4ix) + C_2Y_1(4ix)$$

You can also show a general solution,

$$y = M_1I_1(4x) + M_2K_1(4x)$$

Problem 9
Show that,

$$K_{1/2}(x) = \sqrt{\frac{\pi}{2x}}e^{-x}$$

Solution: This is a direct substitution in the definition. Thus,

$$K_{1/2} = \frac{\pi}{2\sin(\pi/2)}[I_{-1/2} - I_{1/2}]$$

Now, we need to find $I_{-1/2}$ and $I_{1/2}$. It is a trivial exercise using series representation of modified Bessel function to show that,

$$I_{1/2} = \sqrt{\frac{2}{\pi x}}\sinh(x)$$

$$I_{-1/2} = \sqrt{\frac{2}{\pi x}}\cosh(x)$$

since $\cosh x - \sinh x = (e^x + e^{-x})/2 - (e^x - e^{-x})/2 = e^{-x}$. This directly shows that,

$$K_{1/2}(x) = \sqrt{\frac{\pi}{2x}}e^{-x}$$

Problem 10
Show that,

$$xI_1(x) = 4\sum_{n=1}^{\infty} nI_{2n}(x)$$

Solution: As, it's clear from the form, we need to use a recurrence formula that has no derivatives. The only one with such property is the last one.

$$\frac{2p}{x}I_p(x) = I_{p-1}(x) - I_{p+1}(x)$$

Since the sum is over Is of even order. We make a substitution, $p \to 2p$. Thus, we obtain,

$$\frac{4p}{x}I_{2p}(x) = I_{2p-1}(x) - I_{2p+1}(x)$$

multiplying both sides by x,

$$4pI_{2p}(x) = x\left[I_{2p-1}(x) - I_{2p+1}(x)\right]$$

All we need to do now is to make a sum over integer p from 1 to ∞.

$$\begin{aligned}\sum_{p=1}^{\infty} 4pI_{2p}(x) &= x\sum_{p=1}^{\infty}\left[I_{2p-1}(x) - I_{2p+1}(x)\right] \\ &= x\left[I_1(x) + I_3(x) + I_5(x) + \cdots \right. \\ &\qquad\qquad \left. -I_3(x) - I_5(x) - \cdots\right] \\ &= xI_1(x)\end{aligned}$$

However, this proof is not rigorous enough. Another proof can be made by making a finite sum and then taking the order to ∞, You need then to show that $\lim_{n\to\infty} I_n(x) = 0$.