Applied Mathematics Department

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INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS

MATH - 302

FINAL PROJECT

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Soliton Waves

Solving the Korteweg de-Vries (KdV) Equation

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Contents

1 Abstract

This report is intended to present a condensed overview over the theory of soliton waves, in one of its most important and historical equation known as the Korteweg de-Vries Equation. Such wave equation with its dispersive and dissipative version is considered to be a version from the Burger's equations categorization, that was mainly observed by Russell in the unusual behaviour he anticipated experimentally within the water waves, modelled mathematically by Korteweg after about half a century. This non-linear partial differential equation is considered to be the base start for the modern water wave theory, and has been historically extended in rather different fields, like modelling the blood flow in an artery up to modelling the behaviour of the quantum wave packets form the Non-Linear Schrödinger equation developed by the insights of the KdV equation! Tremendous efforts has then been exerted in trails of obtaining accurate and precise analytic solutions for the 1-soliton case followed by the general N-soliton case.

2 Introduction

The Korteweg de-Vries (KdV) equation can be interrupted as follows:

$$u_t + auu_x + u_{xxx} = 0 \tag{1}$$

where the subscripts represent the partial derivatives. The parameter a is usually set as ± 1 or ± 6. This equation models a variety of non-linear phenomena that occurs in nature such as, ion acoustic waves in plasma, and shallow water waves. As it is pointed out the form of the equation, one can easily interpret the behaviours of u, if one know the initial behaviour of the function $u(x, 0)$ through numerical simulation. We will first examine the behaviour of u as time evolves using numerical through implementation of finite difference time domain scheme. We will also point out the effect of the non-linear term and triple derivative term inside the equation. It will be demonstrated numerically that non-linearity of uu_x tends to localize the wave whereas dispersion u_{xxx} spreads the wave out. The delicate balance between the weak non-linearity of uu_x and the linear dispersion of u_{xxx} defines the formulation of solitons that consist of single humped waves. The stability of solitons is a result of the delicate equilibrium between the two effects of non-linearity and dispersion.

3 Derivation of KdV Equation

3.1 Formulation and Geometry

Starting from Euler's standard equations of fluid flow "essentially the *continuity equations* describing the conservation of mass and momentum":

$$\begin{gathered} \partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0 \\ \rho \left(\partial_t + \vec{v} \cdot \nabla \right) \mathbf{v} = -\nabla P + \mathbf{F} \end{gathered} \tag{2}$$

where ρ is the fluid density, $\mathbf{v}$ is the velocity vector of the fluid, whilst the right hand side of the latter equation relates the pressures acting on and from the fluid, namely P as the internal pressure and $\mathbf{f}$ as the external force acting on the fluid. To start off, we will begin the analysis by deriving the linear approximation of the KdV equation, to give as an insight of the liquid's behaviour under the constrictions of **irrotational laminar** smooth flow of an **incompressible** fluid. Whilst the governing KdV equation accounts for the dispersive factors within the waves, we will then be using perturbation ratios "will be derived along the analysis" to present the actual differential equation.

To consider the laminar flow, we say that the fluid is irrotational. Hence:

$$\nabla \times \mathbf{v} = 0 \tag{3}$$

By the aid of vector calculus, we can express the velocity in terms of some scalar field φ, as follows:

$$\mathbf{v} = \nabla \varphi = u\hat{\mathbf{i}} + w\hat{\mathbf{j}} \tag{4}$$

Before proceeding any further, we make a note that this scalar potential has to satisfy the above mentioned equations in Eqn.2. Thus, an additional constraint is applied to φ, where it meets the Laplacian Equation:

$$\nabla^2 \varphi = 0 \tag{5}$$

Given that the fluid is incompressible, we get the following set of constraints:

$$\nabla \rho = 0, \quad \partial_t \rho = 0, \tag{6}$$

Considering that the only external acting force is the force of gravity, we say that $\mathbf{F} = -\rho g \hat{\mathbf{j}}$. Knowing that our velocity vector has zero curl, we make use of a vector identity as follows:

$$\mathbf{v} \times (\nabla \times \mathbf{v}) = -(\mathbf{v} \cdot \nabla)\mathbf{v} + \frac{1}{2}\nabla(\mathbf{v} \cdot \mathbf{v}) = 0 \tag{7}$$

The momentum equation can then be rewritten as follows:

$$\begin{gathered} \nabla \left(\partial_t \varphi + \tfrac{1}{2} v^2 + \tfrac{P}{\rho} + gy \right) = 0 \\ \therefore \nabla \left(\partial_t \varphi + \tfrac{1}{2} \left(u^2 + w^2 \right) + \tfrac{P}{\rho} + gy \right) = 0 \end{gathered} \tag{8}$$

We note that the ∇ operator corresponds to the *spatial* partial derivatives, implemented in a problem where time is an accountable variable. Since the three spatial partial derivatives equate to zero, then the expression denoted under the nabla sign equates to a function in time, that could be absorbed by our scalar potential φ, as it rather finds the velocities, and the addition of a time component doesn't affect its role. Hence, in addition to the Laplacian equation, we have:

$$\partial_t \phi + \frac{1}{2}(\nabla\phi)^2 + P/\rho + gy = \partial_t \phi + \frac{1}{2}\left(u^2 + w^2\right) + P/\rho + gy = 0 \tag{9}$$

For now, the geometry of the problem has to be set, in order to identify the boundary conditions to be able to complete the solution. Consider a narrow channel whose length is infinitely larger than its depth, this makes our channel one-dimensional. Moreover, it's depth does not vary along different positions, i.e. not a function of x. Let the depth be denoted by h, which also corresponds to the height of the water at rest, given that we adjust our coordinate system as follows: We impose that the $\hat{x}$-direction is along the length of the channel at the surface of the water and constructing the $\hat{y}$-direction vertically. The bottom of the water corresponds to $y = -h$.

We now consider the terminology "*shallow*", another fancy way of imposing a *long wavelength approximation*. Defining that λ is the charachterstic wavelength of a given wave, we then require that $\lambda \gg h$. The first perturbation ratio is then defined as follows for the relevance of the upcoming analysis:

$$\delta \equiv \left(\frac{h}{\lambda}\right)^2 \tag{10}$$

Irrotationality of the fluid imposes the absence of any turbulence. What this actually means is that given some travelling wave over the surface of the fluid with an amplitude a, then we require that $a \ll h$ to avoid creating disturbance at the surface of the channel. We now introduce the second perturbation ratio as follows:

$$\varepsilon \equiv \left(\frac{a}{h}\right) \tag{11}$$

Naively, it could be argued that both ratios are not of the same order, given the second power definition in δ, but as we proceed further, we will show that the modulo overall factors,the first power of (h/λ) does not appear, while the quadratic power does.

3.2 Boundary Conditions

1. The pressure, denoted P, vanishes *near and at* the surface of the given fluid, which in turn means that the y-th component of the velocity vector vanishes. This imposes the following BC:

$$\mathbf{v}_y|_{\text{bottom}} = v|_{\text{bottom}} = \frac{\partial\varphi}{\partial y}(x, 0) = 0 \tag{12}$$

2. In general, the amplitude of a given travelling wave is given as a function of space and time. So, in our case, we define a function called $\xi = \xi(x, t)$. Then:

$$y|_{\text{water surface}} = h + \xi(x, t) \tag{13}$$

$$\implies v\Big|_{\text{water surface}} = \left.\frac{dy}{dt}\right|_{\text{water surface}} = \partial_t \xi + \partial_x \xi \left.\frac{dx}{dt}\right|_{\text{water surface}} \tag{14}$$

$$\implies \varphi_y|_{\text{water surface}} = \partial_t \xi + \varphi_x \xi \tag{15}$$

3. We can re-state Eqn.9 using the above formulations as follows:

$$\varphi_t|_{\text{water surface}} + \frac{1}{2}\left(u^2 + w^2\right)|_{\text{water surface}} + g\xi = 0 \tag{16}$$

We now start our linear approximations under the aforementioned restrictions, we get:

$$\nabla^2\varphi = 0, \quad \text{where } \varphi_y\left(x, y = -h\right) = 0, \tag{17}$$

and at the water surface, we get:

$$\partial_t\xi - \partial_y\varphi = 0 = \partial_t\varphi + g\xi \tag{18}$$

3.3 Solution to the Linearized Version of KdV

We are now ready for developing the solution to the linear equation. We first start by eliminating the amplitude function ξ, achieved by differentiating the R.H.S of eqn.18 with respect to time, we then get:

$$\partial_{tt}\varphi = -g\partial_t\xi \tag{19}$$

Hence, substituting in L.H.S of eqn.18, we get the following differential equation:

$$\partial_{tt}\varphi + g\partial_y\varphi = 0 \tag{20}$$

The above relations will help us restore back the amplitude function ξ after gaining the solution of the scalar potential φ. We already note our acquired knowledge of expecting wavelike solutions, then, the proposed solution is presented as follows:

$$\varphi = Y\left(y\right)\sin\left(kx - \omega t\right) \tag{21}$$

where k denotes the wave-number = $2\pi/\lambda$ and ω denotes the angular frequency of the travelling wave.

Applying the Laplacian equation restriction, we find ourselves that we require $Y = \alpha e^{+ky} + \beta e^{-ky}$. Inserting this ansatz in the boundary condition evaluated at the bottom, i.e. eqn (12), and evaluating, we get $\beta/\alpha = e^{-2kh}$. The solution φ can then be re-stated as follows:

$$\varphi = 2\alpha e^{-kh}\cosh\left(k\left(y+h\right)\right)\sin\left(kx - \omega t\right) \tag{22}$$

We would now want to restore the amplitude function ξ, so by returning back and substituting in eqn(19), we tackle a formula for the angular frequency ω, which is given as follows:

$$\omega^2 = gk\tanh\left(k\left(y+h\right)\right)|_{\text{water surface}} = gk\tanh\left(kh\right) \tag{23}$$

And at last, we find an explicit formulation for the wave-amplitude function ξ, given as follows:

$$\xi\left(x,t\right) = -\frac{1}{g}\varphi_t = \alpha\sqrt{\left(2k/g\right)\sinh\left(2kh\right)}\sin\left(kx - \omega t\right) \tag{24}$$

We also note down an interesting phenomenon here, in essence the existence of "wave-length *dependent* amplitude", in addition to a dispersion correlation relating ω to k, by a direct proportionality relation, with the proportionality constant resembling the speed given as $v_0 = 1/\sqrt{gh}$, under the constraint of the long wavelengths approximation, i.e. as $k \to 0$. Noticing that the speed goes as $\sqrt{g/k} = v_0\sqrt{kh} = 2\pi v_0\sqrt{\lambda/h}$ as the characteristic wavelength tends to be shorter and shorter, we can then note down that long wavelengths travel faster with lesser dispersion in space.

3.4 General KdV Derivation - Perturbation and Dispersive Ratios

By harnessing the aforementioned insight from the linearized model, we can now take a step in relaxing the perturbation constraints and look into the higher levels of perturbation. Referring to Eq.(9), and differentiating it with respect to the length of the channel, accompanied with its evaluation at the surface of the water, we get:

$$\varphi_{tx} + \varphi_x \varphi_{xx} + \varphi_y \varphi_{xy} + g\xi_x = u_t + uu_x + ww_x + g\xi_x = 0 \tag{25}$$

Creating an ansatz solution (standard approach is to expand the solution in the form of a power series) for small amplitudes as required earlier, we get the following:

$$\varphi = \sum_{n=0}^{\infty} y^n \varphi(x,t) \tag{26}$$

which in turn implies when differentiating with respect to y:

$$\varphi_y = \sum_{n=1}^{\infty} n y^{n-1} \varphi_n \tag{27}$$

Before proceeding forward, one important requirement is obtained when evaluating the above relation at the bottom of the channel, i.e. solving it with eqn.(12):

$$\boxed{\varphi_1 = 0} \tag{28}$$

And since the function φ satisfies the Laplace's equation, we can insert the summation into it, to get the following:

$$\sum_{n=0}^{\infty} y^n \left\{\varphi_{n,xx} + (n+2)(n+1)\varphi_{n+2}\right\} = 0 \quad \Longrightarrow \quad \varphi_{n,xx} + (n+2)(n+1)\varphi_{n+2} = 0 \tag{29}$$

Substituting eqn.(28), we note down that all the *odd terms vanishes out*, and the even terms are given as follows:

$$\varphi(x,t) = \sum_{m=0}^{\infty} \frac{(-1)^m y^{2m}}{(2m)!} \varphi_0^{(2m)}, \qquad \varphi_0 \equiv \varphi^0$$
$$\Longrightarrow \begin{cases} u = \varphi_x = \varphi_x^0 - \frac{1}{2} y^2 \varphi_{xxx}^0 + \dots \\ \\ w = \varphi_y = -y\varphi_{xx}^0 + \frac{1}{6} y^3 \varphi_{xxxx}^0 + \dots \end{cases} \tag{30}$$

For the sake of easiness in treatment with the equations, we ought to deal with dimensionless variables. For this sake, we define the following:

The linear speed is defined as $c_0 \equiv \sqrt{gh}$

$$\tilde{x} \equiv \frac{x}{\lambda}, \quad \tilde{y} \equiv \frac{y}{h}, \quad \tilde{t} \equiv \frac{t}{\lambda/c_0}$$
$$\tilde{\xi} \equiv \frac{\xi}{a}, \quad \tilde{\varphi} \equiv \frac{h\varphi}{a\lambda c_0} = \frac{\varphi}{\varepsilon\lambda c_0}, \quad \Longrightarrow \quad \text{the surface's height is: } \tilde{y} = 1 + \varepsilon\tilde{\xi}$$
$$\Longrightarrow \begin{cases} \tilde{u} = u/\varepsilon c_0 \,, \\ \tilde{w} = \left(\delta/\varepsilon c_0\right) w \,, \\ \varphi_0 \equiv \varphi^0 \equiv H(x,t) \,, \\ \tilde{H} = H/\varepsilon\lambda c_0 \,. \end{cases} \tag{31}$$

Our equations of interest can now be re-stated to the lowest order in small quantities δ & ε, defined earlier. Order of magnitude 2 (symbolized as O^2) represents the second order of these small quantities like $\varepsilon\delta,\ \delta^2,\ \varepsilon^2$:

$$\begin{gathered}\tilde{\varphi} = \tilde{H} - \tfrac{1}{2}(1+\varepsilon\xi)^2\delta\tilde{H}_{\tilde{x}\tilde{x}} + O^2 \\ \tilde{u} = \tilde{H}_{\tilde{x}} - \tfrac{1}{2}\delta\tilde{H}_{\tilde{x}\tilde{x}\tilde{x}} + O^2, \quad \tilde{v} = -\delta\left[(1+\varepsilon\tilde{\xi})\tilde{H}_{\tilde{x}\tilde{x}} - \tfrac{1}{6}\delta\tilde{H}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}} + O^2\right].\end{gathered} \tag{32}$$

In addition, our boundary condition for the water surface could also be re-written as follows "after factoring out the term $\varepsilon c_0/\sqrt{\delta}$ ":

$$\tilde{\varphi}_{\tilde{y}} = \delta\left(\tilde{\xi}_{\tilde{t}} + \varepsilon\tilde{\xi}_{\tilde{x}}\tilde{\varphi}_{\tilde{x}}\right) \tag{33}$$

We insert the obtained expressions from eqn.(31) into the above equation, and factoring out δ, we obtain:

$$\tilde{\xi}_{\tilde{t}} + \varepsilon\tilde{\xi}_{\tilde{x}}\tilde{H}_{\tilde{x}} + (1+\varepsilon\tilde{\xi})\tilde{H}_{\tilde{x}\tilde{x}} - \frac{1}{6}\delta\tilde{H}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}} + O^2 = 0 \tag{34}$$

Inserting the re-definitions of eqn.(31) into the boundary condition given by eqn.(16), and factoring out the term ag we obtain:

$$\tilde{\varphi}_{\tilde{t}} + \frac{1}{2}\varepsilon\left[\left(\tilde{\varphi}_{\tilde{x}}\right)^2 + \left(\tilde{\varphi}_{\tilde{y}}\right)^2/\delta\right] + \tilde{\xi} = 0 \tag{35}$$

Substituting with the power series expansions, we get:

$$\tilde{H}_{\tilde{t}} - \frac{1}{2}\delta\tilde{H}_{\tilde{x}\tilde{x}\tilde{t}} + \frac{1}{2}\varepsilon\left(\tilde{H}_{\tilde{x}}\right)^2 + \tilde{\xi} + O^2 = 0 \tag{36}$$

Following the same way in analyzing the linear problem, we will proceed to differentiate the above equation with respect to x for as to study its change along the given length. For the easiness of writing, we define $r \equiv \tilde{H}_{\tilde{x}}$[1].
The equations that ought to be solved are now stated as follows:[2]

$$\begin{gathered} r_t - \tfrac{1}{2}\delta r_{xxt} + \varepsilon r r_x + \xi_x = 0 \\ \xi_t + \varepsilon r\xi_x + (1+\varepsilon\xi)r_x - \tfrac{1}{6}\delta r_{xxx} = 0 \end{gathered} \tag{37}$$

A smart beginning that should be apparent to the reader is that we need to check that the equations are satisfied to the zero-th order[3], i.e.:

$$r_t + \xi_x = 0 = r_x + \xi_t \tag{38}$$

Hencem we are now able to separate the given pair of equations as follows:

$$r_{tt} = r_{xx}, \quad \text{and} \quad \xi_{xx} = \xi_{tt} \tag{39}$$

In other words, we should be expecting that both functions have to satisfy the wave equations as previously shown in the linear model, with the previously defined dimensionless linear speed $c_0 = 1$. We also note down that both functions r and ξ share high symmetry in their zero-th order differential structure, which rises a thought ansatz as follows:

$$r \equiv \xi + \varepsilon F + \delta G + O^2 \tag{40}$$

1. Also, given now all our equations of interest are dimensionless, we drop out the over-tildes, and it should be from the context what variable we mean, unless conflicts arise, we will clearly mention the tildes again.
2. Correct up to the second order in both ε & δ, dropping out the O^2 term.
3. The perturbation and dispersion ratios are ignored for now

Two separate terms are anticipated for ε and δ as they are tow totally different perturbation terms, and the differential equations hold only for the second order magnitude and hence, the induced ansatz is consistent. Note down that both F and G functions satisfy the following character:

$$\begin{aligned} F_x + F_t &= O^1 \\ G_x + G_t &= O^1 \end{aligned} \tag{41}$$

We now insert the ansatz into the original pair of equations eqn.(37), bearing ion mind that we are now searching for the nature of the imposed functions F and G:

$$\begin{aligned} \xi_t + \xi_x + \varepsilon\left(F_t + \xi\xi_x\right) + \delta\left(G_t - \tfrac{1}{2}\xi_{xxt}\right) = 0 \\ \xi_x + \xi_t + \varepsilon\left(F_x + 2\xi\xi_x\right) + \delta\left(G_x - \tfrac{1}{6}\xi_{xxx}\right) = 0 \end{aligned} \tag{42}$$

Subtracting both equations, we get:

$$\varepsilon\left(F_x - F_t + \xi\xi_x\right) + \delta\left(G_x - G_t - \frac{1}{6}\xi_{xxx} + \frac{1}{2}\xi_{xxt}\right) = 0 \tag{43}$$

Inserting in eqn.(41), we get:

$$\left.\begin{aligned} 2F_x &= -\xi\xi_x = -\frac{1}{2}\left(\xi^2\right)_x, &&\implies F = -\frac{1}{4}\xi^2, \\ 2G_x &= \frac{2}{3}\xi_{xxx}, &&\implies G = \frac{1}{3}\xi_{xx} \end{aligned}\right. \implies r = \xi - \frac{1}{4}\varepsilon\xi^2 + \frac{1}{3}\delta\xi_{xx} + O^2 \tag{44}$$

With the obtained expression for r, we return to substitute it again in the pair of eqn.(**??**), we obtain two expressions, basically expressing the same nature and behaviour. For historical contexts, we choose to take the second equation that states:

$$\boxed{\xi_t + \xi_x + \tfrac{3}{2}\varepsilon\xi\xi_x + \tfrac{1}{6}\delta\xi_{xxx} = 0} \tag{45}$$

This is essentially the KdV equation that we have been looking for so far.

Whilst it is inconvenient to deal with the perturbation ratios being very small coefficients, we re-scale the equation for the last time, to obtain the "known" version of the equation. We first proceed with eliminating the ξ_x term, which will be achieved by translating ξ using a constant, as follows:

$$\xi_x + \frac{3}{2}\varepsilon\xi\xi_x = \xi_x\left(1 + \frac{3}{2}\varepsilon\xi\right) = \frac{3}{2}\varepsilon\xi_x\left(\xi + \frac{2}{3}/\varepsilon\right) = \frac{3}{2}\varepsilon\left(\xi + \frac{2}{3}/\varepsilon\right)_x\left(\xi + \frac{2}{3}/\varepsilon\right) \equiv \frac{3}{2}\varepsilon\sigma_x\sigma \tag{46}$$

where we define $\sigma \equiv \xi + \frac{2}{3}/\varepsilon$. Hence, the KdV equation is re-stated as:

$$\sigma_t + \frac{3}{2}\varepsilon\sigma\sigma_x = \frac{1}{6}\delta\sigma_{xxx} \tag{47}$$

Let:

$$\left.\begin{aligned} \sigma &= (\delta/\varepsilon)\psi \\ t &= \varrho/\delta, \end{aligned}\right\} \implies \psi_\varrho + \frac{3}{2}\psi\psi_x + \frac{1}{6}\psi_{xxx} = 0 \tag{48}$$

Applying the following re-scales:

$$\left.\begin{aligned} \varrho &\equiv 6t, \\ \psi &\equiv u/9, \end{aligned}\right\} \implies \boxed{\mathrm{u}_t + uu_x + u_{xxx} = 0} \tag{49}$$

The "known" version of the KdV equation that we deduced is categorized to be a Burger's equation with a **dispersive** term, a non-linear equation that is capable of producing (developing) shocks; that in case when the non-linearity balances the dispersion, we obtain a wave of what we defined to be **solitons**. A nice note is to mention a very common misconception that occurs at this point: The obtained *dispersive* equation differs totally from the *dissipative* equation stated as:

$$uu_x + u_t = \varepsilon u_{xx} \tag{50}$$

In nature of their structure, the latter[4] **can** result in equipartitioning of energy whilst the former **can't** lead to this result.

4. also known as *Viscous Burger's equation*

4 Numerical Propagation of the Soliton

We will start the solution analysis by showing the numerical solutions, since this is the best way to analyse the behaviour of the obtained waves. From there, we will present some of the analytic solutions that have been developed for the general case along history.

4.1 Finite Difference Time domain Scheme

We have implemented the Central-Difference in both spatial and temporal derivatives technique (three-layered) in second order scheme to convert the KdV equation into a recursion formula, which reads in its non-conservative form:

$$u_j^{n+1} = u_j^{n-1} - \tfrac{\mu\Delta t}{\Delta x} u_j^n \left(u_{j+1}^n - u_{j-1}^n\right) - \tfrac{\nu\Delta t}{(\Delta x)^3}\left(u_{j+2}^n - 2u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n\right) \tag{51}$$

for which μ is the non-linear coefficient, and ν is the dispersive term.
Whilst to convert it to its discrete conservative form version, using the same CTCS technique, we get:

$$u_j^{n+1} = u_j^{n-1} - \tfrac{\mu\Delta t}{2(\Delta x)}\left(\left(u_{j+1}^n\right)^2 - \left(u_{j-1}^n\right)^2\right) - \tfrac{\nu\Delta t}{\left(\Lambda_j\right)^3}\left(u_{j+2}^n - 2u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n\right) \tag{52}$$

The above expressions come in accordance with the linear stability conditions that have to be satisfied, given by the following equation:

$$\frac{\Delta t}{\Delta x} \leq \frac{2}{\sqrt{3}} * \frac{1}{\left|u_{\max} * \mu - \frac{3\nu}{(\Delta x)^2}\right|} \tag{53}$$

Its worth-mentioning that the assumed technique of solving is correct up to the second order in both Δx and Δt.

4.2 Results Assessment

The above algorithm has already been implemented in MatLAB[5], where we start to show the different factors affecting the behavioural structure of the soliton waves. The following values have been also implemented:

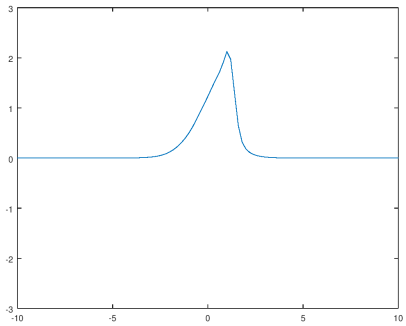
1. $\Delta t = 0.001$
2. $\Delta x = 0.2$
3. Different non-linear terms have been tested, but we present here the graphs when $\mu = 6$.
4. Different dispersive terms have been tested, but we present here the graphs when $\nu = 1$.
5. The KdV equation is subjected to the following initial condition:

$$u_0(j) = n * (n+1) * \operatorname{sech}\big((x(j))\big)^2$$

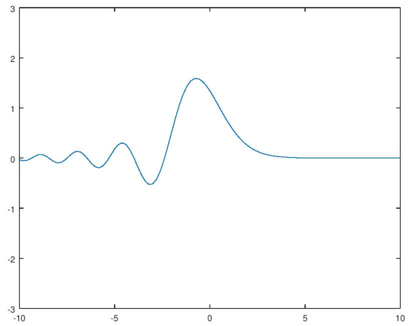
5. The code will be presented at the end of the section

6. Different time steps have been evaluated, as will be shown respectively within the coming graphs[6].

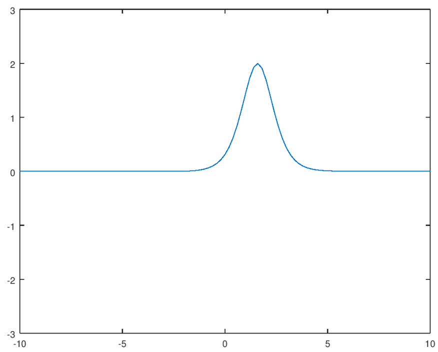
We first show the sole effect of the presence of the non-linear term within the wave solution, as shown in figure (1a). Then, we tend to present the sole effect of the dispersive term within the wave solution, as shown in figure (1b). At last, we tend to show the effect of combining both of the effects together, and anticipate the behaviour of the resulting wave, as shown in figure (1c). A soliton wave is characterized by having the effect of the non-linear term neutralized by the presence of the dispersive term, which possesses then weird behavioural characteristics.



(a) Non-linear Term Sole Effect with time step = 100



(b) Dispersive Term Sole Effect with time step = 300

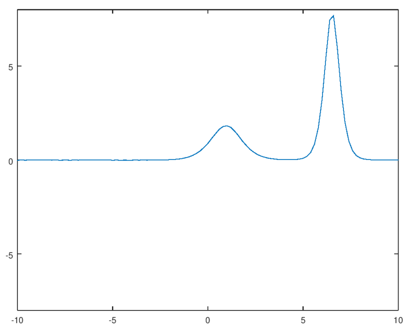


(c) Combined neutral effect, resulting in a1- solition wave, with time step = 400

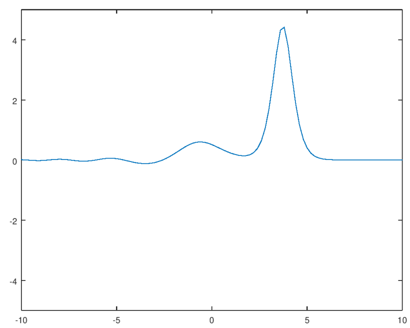
Figure 1: Different effects of the characteristics of the wave solution of the KdV equation.

We now look into the behaviour when 2-solitons interact with each other[7] as shown per figure (2a). The most apparent characteristic, is that the superposition of both effects tends to result in defective amplitude function, where the height of the total soliton tends to be **less** than the summation of the actual amplitudes of the original sole effects. Another interesting wave solution to look at, is when a half integer solitons interact with each other. The lower amplitude soliton appears to act as a dispersive term , that fades away totally as the time passes by. This is shown as per figure (2b).

6. Using different time steps does not alter the resulting wave solution, but rather refines the attained grpahs for the inserted values.
7. same conditions apply as described before



(a) The resulted 2-soliton behaviour at time step = 400



(b) Interaction of 1.5 soliton at time step = 400

Figure 2: N-solitons superpositionin with each other.

4.3 MatLAB Code

The code is as follows:

```
l = 20; mu = 6;  nu = 1;  dx = 0.2;  dt = 0.001;  t_end = 400 ;

x = -l:dx:l;  uf = zeros(length(x),1);
u = zeros(length(x),1);  ub = zeros(length(x),1);
u_0 = zeros(length(x),1);

n=1.5;
for j=1:length(x)
u_0(j) = n*(n+1)*sech((x(j)))^2;
%u_0(j) = n*(n+1)*cos(pi*(x(j)));
end

uf=u_0; u = u_0; ub = u_0;

for n = 1:t_end
plot(x,uf);
axis([-10 10 -5 5]);

uf(1) = ub(1) - ((dt*mu)/(2*dx))*(u(2)^2-u(length(x))^2) - ((dt*nu)/(dx^3))*(u(3)-2*u(2)
+2*u(length(x))-u(length(x)-1));
uf(2) = ub(2) - ((dt*mu)/(2*dx))*(u(3)^2-u(1)^2) - ((dt*nu)/(dx^3))*(u(4)-2*u(3)
+2*u(1)-u(length(x)));
for i= 3:length(x)-2
```

```
%Non-conservative form
%uf(i) = ub(i)  - ((dt*mu)/(dx))*u(i)*(u(i+1)-u(i-1)); %nonlinear term
%uf(i) = ub(i)  - ((dt*nu)/(dx^3))*(u(i+2)-2*u(i+1)+2*u(i-1)-u(i-2));  %dispersive term
%uf(i) = ub(i) - ((dt*mu)/(dx))*u(i)*(u(i+1)-u(i-1)) - ((dt*nu)/(dx^3))*(u(i+2)-2*u(i+1)
+2*u(i-1)-u(i-2));

%conservative form
%uf(i) = ub(i)  - ((dt*mu)/(2*dx))*(u(i+1)^2-u(i-1)^2); %nonlinear term
%uf(i) = ub(i)  - ((dt*nu)/(dx^3))*(u(i+2)-2*u(i+1)+2*u(i-1)-u(i-2));  %dispersive term
uf(i) = ub(i) - ((dt*mu)/(2*dx))*(u(i+1)^2-u(i-1)^2) - ((dt*nu)/(dx^3))*(u(i+2)-2*u(i+1)
+2*u(i-1)-u(i-2));
end
uf(length(x)-1) = ub(length(x)-1) - ((dt*mu)/(2*dx))*(u(length(x))^2-u(length(x)-2)^2)
-((dt*nu)/(dx^3))*(u(1)-2*u(length(x))+2*u(length(x)-2)-u(length(x)-3));
uf(length(x)) = ub(length(x)) - ((dt*mu)/(2*dx))*(u(1)^2-u(length(x)-1)^2)
- ((dt*nu)/(dx^3))*(u(2)-2*u(1)+2*u(length(x)-1)-u(length(x)-2));

ub=u;
u = uf;
refreshdata
drawnow;
%pause(0.05);
end
```

5 Methods for Analytical Solutions

5.1 Preliminary Approach

The KdV equation gives rise to many solutions of different identities. The common thing about these solutions is that they propagate at some speed for example c with keeping its identity. There is an interesting approach to implement here. We can actually transform ourselves to the frame where the wave propagates, thus only retaining its structure as time passes. Thus, we introduce the new variable $\xi = x - ct$, where c is the speed of the wave. This approach will transform the above partial differential equation to an ordinary differential equation with no loss of generality. This transforms the equation as follows:

$$\frac{\partial u}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial u}{\partial \xi} = -c\frac{\partial u}{\partial \xi} \tag{54}$$

Similarly, the x derivatives can be stated as follows:

$$\frac{\partial u}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial \xi} \tag{55}$$

$$\frac{\partial^3 u}{\partial x^3} = \frac{\partial^3 u}{\partial \xi^3} \tag{56}$$

Thus, the equation can be written as:

$$-cu' + 6uu' + u''' = 0 \tag{57}$$

The derivatives are taken over ξ. From this stage, we can easily integrate this equation to reduce its order, stated as below:

$$\begin{aligned} -c\int u' d\xi + 6\int uu' d\xi + \int u''' d\xi &= A \\ -cu + 3u^2 + u'' &= A \end{aligned} \tag{58}$$

Where A is an integration constant. However, we demand that the waves goes to zero as $t \longrightarrow \infty$. So, does its derivatives. Hence, A can be easily set to zero.

Thus, now we have:

$$-cu + 3u^2 + u'' = 0 \tag{59}$$

The above expression can be put into an integral form by multiplying by $2u'$ and then integrating.

$$\begin{aligned} &- cu + 3u^2 + u'' = 0 \\ &- 2cuu' + 6u'u^2 + 2u'u'' = 0 \\ &- 2c\int uu' d\xi + 6\int u'u^2 d\xi 2\int u'u'' d\xi = B \\ &- cu^2 + 2u^3 + 2(u')^2 = B \end{aligned} \tag{60}$$

We can also set the $B = 0$ using the aforementioned boundary conditions. Now the equation reads:

$$(u')^2 = cu^2 - 2u^3 \quad \text{or} \quad u' = \sqrt{cu^2 - 2u^3} \tag{61}$$

This is a separable ODE which can be integrated as follows:

$$\begin{aligned} \int \frac{du}{\sqrt{cu^2 - 2u^3}} &= \int d\xi \\ \frac{1}{\sqrt{2}} \int \frac{du}{|u|\sqrt{\frac{c}{2} - u}} &= \xi - \xi_0 \end{aligned}$$

Now, we anticipate that u is positive and perform the following substitution, $u = \frac{c}{2}\operatorname{sech}^2(\theta)$, $du = -c\operatorname{sech}^2(\theta)\tanh(\theta)d\theta$. This yields:

$$\begin{aligned} \frac{1}{\sqrt{2}} \int \frac{-c\operatorname{sech}^2(\theta)\tanh(\theta)d\theta}{\frac{c}{2}\operatorname{sech}^2(\theta)\sqrt{\frac{c}{2} - \frac{c}{2}\operatorname{sech}^2(\theta)}} &= -\frac{2}{\sqrt{c}} \int \frac{\operatorname{sech}^2(\theta)\tanh(\theta)d\theta}{\operatorname{sech}^2(\theta)\tanh(\theta)} \\ &= -\frac{2}{\sqrt{c}}\theta \end{aligned}$$

From the substitution, we see that:

$$\begin{aligned} \frac{2u}{c} &= \operatorname{sech}^2(\theta) \\ \ln\left(\frac{2u}{c}\right) &= 2\ln\big(\operatorname{sech}(\theta)\big) \\ \frac{1}{2}\ln\left(\frac{2u}{c}\right) &= \ln\big(\operatorname{sech}(\theta)\big) \\ e^{\ln\left(\frac{2u}{c}\right)/2} &= \operatorname{sech}(\theta) \\ \theta &= \operatorname{sech}^{-1}(e^{\ln\left(\frac{2u}{c}\right)/2}) \end{aligned} \tag{62}$$

Thus, we can easily see that one of the solutions is:

$$\boxed{\mathrm{u}(\xi) = \frac{c}{2}\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}\xi\right)} \tag{63}$$

since $\operatorname{sech}(\theta) = \operatorname{sech}(-\theta)$. If we implement a substitution of $u = \frac{c}{2}\tanh^2(\theta)$ instead, we can find the other solution given as:

$$\boxed{\mathrm{u}(\xi) = -\frac{c}{2}\operatorname{csch}^2\left(\frac{\sqrt{c}}{2}\xi\right)} \tag{64}$$

5.2 Bäcklund transformation

The Bäcklund transformation deals with rigorous mathematics in its nature. However, we will only stick to the method of the potential function. We have to introduce a potential $\phi_x = u$. This substituting this results in the KdV equation gives the following.

$$\phi_{xt} + 6\phi_x\phi_{xx} + \phi_{xxxx} = 0 \tag{65}$$

Now, we can easily integrate this with respect to x to get,:

$$\phi_t + 3(\phi_x)^2 + \phi_{xxx} = g(t)$$

where $g(t)$ is a general function of time. We can get rid of this function without loss of generality by introducing a new substitution,

$$\eta = \phi + \int_{t_0}^{t} g(t)$$

substituting this into the equation will give the homogeneous equation.

$$\eta_t + 3(\eta_x)^2 + \eta_{xxx} = 0 \tag{66}$$

Moving to the frame of the wave by the usual substitution $\xi = x - ct$ and doing the usual integration by hyperbolic identities as before, we find that:

$$\begin{aligned} \eta &= \sqrt{c}\tanh\left(\frac{\sqrt{c}}{2}(x - ct)\right) \\ \eta &= \sqrt{c}\coth\left(\frac{\sqrt{c}}{2}(x - ct)\right) \end{aligned} \tag{67}$$

Taking the derivative of these potential functions gives the same previous answers we found for u.

5.3 Adomian's Decomposition Method

5.3.1 Introduction

Adomian's Decomposition is a method for finding a solving for general operator equations regardless of the linearity (differential operator,algebraic operator, non-linear terms).[8]
In form:

$$\mathscr{L}[u] = \mathscr{B}[u]$$

such that:

- u is a function representing a solution to the operator equation.
- $\mathscr{L}$ is an invertible operator.
- $\mathscr{B}$ is any of the above mentioned operators of a mixture of them.

8. G Adomian, *Nonlinear Stochastic Operator Equations.*, OCLC: 1041863658 (Kent: Elsevier Science, 2015), ISBN: 978-1-4832-5909-3, accessed January 12, 2021, http://qut.eblib.com.au/patron/FullRecord.aspx?p=1901494.

5.3.2 Solution Steps

Strategy of implementing the solution goes as follows:

1. The solution u is assumed to in form $u = \sum_{n=0}^{\infty} u_n$ such that u_n is certain polynomial to be found.
2. We get $\mathscr{L}^{-1}\mathscr{L}u = \mathscr{L}^{-1}\mathscr{B}[u] \implies u = f + \mathscr{L}^{-1}\mathscr{B}[u]$ by applying the inverse of $\mathscr{L}$ in both sides of the operator equation (Note: $\mathscr{L}\mathscr{L}^{-1}$ doesn't necessarily $= I$).
3. In case of non-linear operator, we let u_0 be the solution to the linear part and Adomian polynomials A_n which are to be defined as follows:
$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{i=0}^{n}\lambda^i u_i\right)\right]_{\lambda=0} \quad , \quad n = 0, 1, 2, \cdots$$
where F is the non-linear operator.
4. By plugging in $u = \sum_{n=0}^{\infty} u_n$ we may compare terms to obtain our desired solution by recursion.
$$u_0 = f \ , \ u_1 = \mathscr{L}^{-1}\mathscr{B}[u_0] \ , \ u_2 = \mathscr{L}^{-1}\mathscr{B}[u_1] \ldots$$

5.3.3 Examples

E1: Linear First Order Homogeneous ODE

$$\frac{d}{dx}u(x) = u(x) \ , \ u(0) = A$$

Identifying $\mathscr{L} = \frac{d}{dx}$ an appropriate inverse is $\int_0^x [.]dx$

$$\implies \mathscr{L}^{-1}\frac{d}{dx}u(x) = \mathscr{L}^{-1}u(x)$$

$$u(x) - u(0) = \int_0^x u(x)dx$$

Letting $u = \sum_{n=0}^{\infty} u_n$, $u(0) = A$ we get,

$$(u_0 + u_1 + u_2......) = A + \int_0^x (u_0 + u_1 + u_2....)dx$$

By letting $u_0 = A$, $u_{k+1} = \int_0^x u_k dx$ We get the following:

$$u_0 = A \ , \ u_1 = Ax \ , \ u_2 = \frac{Ax^2}{2} \ , \ \ldots u_k = \frac{Ax^k}{k!}$$

$$\implies u(x) = A + Ax + A\frac{Ax^2}{x!}....$$

$$= A(\sum_{k=0}^{\infty}\frac{x^k}{k!}) = Ae^x$$

E2: Second Order Linear ODE

$$\frac{d^2}{dx^2}u + (1 + x + x^2)u = 0\ ,\ u(0) = A\ ,\ \frac{d}{dx}u]_{x=0} = B$$

Identifying $\mathscr{L} = \frac{d^2}{dx^2}$, an appropriate inverse is $\mathscr{L}^{-1} = \int_0^x \int_0^x [.]dxdx$, Which yields:

$$\mathscr{L}^{-1}[\frac{d^2}{dx^2}u] = u - A - Bx$$

$$\implies u = A + Bx - \mathscr{L}^{-1}[(1 + x + x^2)u]$$

Letting $u = \sum_{k=0}^{\infty} u_n$ we get

$$u_0 = A + Bx$$
$$u_1 = -\int_0^x \int_0^x (1 + x + x^2)(A + Bx)dxdx$$
$$= -\int_0^x \int_0^x A(1 + x + x^2) + B(x + x^2 + x^3)dxdx$$
$$= -A(\frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{12}) - B(\frac{x^3}{6} + \frac{x^4}{12} + \frac{x^5}{20})$$

Doing the same for u_2, u_3 and so on we get

$$u = \sum_{n=0}^{\infty} u_n = A(1 - \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{12} + ...) + B(x - \frac{x^3}{6} + \frac{x^4}{12} + \frac{x^5}{20} + ...)$$

E3: Nonlinear ODE Consider the following ODE

$$\frac{d}{dx}u + u^2 = -1, u(0) = 0$$

Identifying $\mathscr{L} = \frac{d}{dx}$ with an appropriate inverse is $\mathscr{L}^{-1} = \int_0^x [.]dx$, $\mathscr{B}(u) = u^2$

$$\implies u = u(0) - \mathscr{L}^{-1}[1] - \mathscr{L}^{-1}\mathscr{B}[u] = -1 - \mathscr{L}^{-1}\mathscr{B}[u]$$

By letting $u = \sum_{n=0}^{\infty} u_n$ along with $\mathscr{L}^{-1}\mathscr{B}[u] = \mathscr{L}^{-1}[\sum_{n=0}^{\infty} A_n]$
$= \mathscr{L}^{-1}[A_0(u_0) + A_1(u_0, u_1) + A_2(u_0, u_1, u_2)....]$
For convenient each term in A_n must include terms with a sum of coefficients = n, so we get the following:

$$A_0 = (u_0)^2\ ,\ A_1 = 2u_0u_1\ ,\ A_2 = u_1^2 + 2u_0u_2\ ,\ A_3 = 2u_0u_3 + 2u_1u_2\ ,\ ......$$

$$u_0 = -x, A_0 = x^2$$
$$u_1 = -\mathscr{L}^{-1}(A_0) = -\int_0^x (x)^2 dx = -\frac{x^3}{3}, A_1 = \frac{2x^4}{3}$$
$$u_2 = -\mathscr{L}^{-1}(A_1) = -\int_0^x \frac{2x^4}{3}dx = -\frac{2x^5}{15}\ ,\ A_2 = \frac{x^6}{9} + \frac{2x^6}{15} = \frac{17x^6}{15}$$

After a few iterations, we get:

$$u = -(x + \frac{x^3}{3} + \frac{2x^5}{15} + ...)$$

5.4 Adomian's solution of KdV equation

5.4.1 Solution Setup

The Korteweg de-Vries (KdV) equation is given by:

$$u_t + auu_x + u_{xxx} = 0$$

Standard Operator form[9]

Which can be written in the following operator form:

$$\mathscr{L}_t[u] = -a\mathscr{B}[u] - \mathscr{L}_{xxx}[u]$$

Such that $\mathscr{L}_t[u] = u_t$, $\mathscr{B}[u]$, $\mathscr{L}_{xxx}[u] = u_{xxx}$
A proper inverse the L.H.S is $\mathscr{L}^{-1} = \int_0^t [.]dt$ which yields the following:

$$u(x,t) = u(x,0) - a\mathscr{L}^{-1}[\mathscr{B}[u]] - \mathscr{L}^{-1}[\mathscr{L}_{xxx}[u]]$$

After letting $u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$ we need to determine Adomian's polynomials which are given by the nonlinear term $\mathscr{B}[u]$ as follows:

$$\begin{aligned}\sum_{n=0}^{\infty} A_n = \mathscr{B}[u] &= (u_0 + u_1 + u_2 + u_3 \ldots.)(u_{0x} + u_{1x} + u_{2x} \ldots.) \\ &= u_0u_{0x} + u_1u_{0x} + u_0u_{1x} + u_2u_{0x} + u_1u_{1x} + u_0u_{2x} \ldots.\end{aligned}$$

Adomian's Polynomials A_n
A convenient choice for A_n is by having sum of indices in each term be equal n as the following:

$$\begin{aligned}A_0 &= u_0u_{0x} \\ A_1 &= u_1u_{0x} + u_0u_{1x} \\ A_2 &= u_2u_{0x} + u_1u_{1x} + u_0u_{2x} \\ &\vdots\end{aligned}$$

that would yield the following:

$$(u_0 + u_1 + u_2 \cdots) = u(x,0) - a\int_0^t (A_0 + A_1 \cdots)dt - \int_0^t (u_{0xxx} + u_{1xxx} + u_{2xxx} \cdots) \tag{68}$$

Form of Solution
By comparing both sides we get the following:

$$\begin{aligned}u_0 = u(x,0) \ &, \ A_0 = u(x,0)u_x(x,0) \\ u_1 = -a\int_0^t (A_0)dt - \int_0^t u_{0xxx}dt \ &, \ A_1 = u_1u_{0x} + u_0u_{1x} \\ u_2 = -a\int_0^t (A_1)dt - \int_0^t u_{1xxx}dt \ &, \ A_2 = u_2u_{0x} + u_1u_{1x} + u_0u_{2x}\end{aligned} \tag{69}$$

And the iteration shall continue until an acceptable solution is obtained.

9. Abdul-Majid Wazwaz, *Partial differential equations and solitary waves theory*, Nonlinear physical science, OCLC: 310400928 (Beijing: Higher Education Press, 2009), ISBN: 978-3-642-00251-9 978-3-642-00250-2 978-7-04-025480-8, https://books.google.com.eg/books?id=ZnrjZZbrsgUC.

5.4.2 Examples

Case 1 Consider the following Kdv equation subject to the given IC.

$$u_t - uu_x + u_{xxx} = 0 \, , u(x,0) = 6x$$

Having a=-6, the form of solution is given by:

$$u_0 = 6x \ , \ A_0 = (6x) * (6) = 36x$$
$$u_1 = 6\int_0^t (36x)dt = 6^3xt \ , \ A_1 = (6^3xt)(6) + (6x)(6^3t) = 2 * 6^4xt$$
$$u_2 = 6\int_0^t (2 * 6^4xt)dt = 6^5xt^2$$
$$A_2 = (6^5xt^2)(6) + (6^3xt)(6^3t) + (6)(6^5xt^2) = 3 * 6^6xt^3$$
$$u_3 = 6\int_0^t (36^6xt^2)dt - \int_0^t 0dt = 6^7xt^3.......$$
$$\implies u(x,t) = 6x(1 + 36t + (36t)^2 + (36t)^3...) = \frac{6x}{1-36t} \ , \ |36t| < 1$$

Case 2 Consider the following KdV equation subject to the given IC:

$$u_t - uu_x + u_{xxx} = 0 \, , u(x,0) = \frac{k^2}{2}\operatorname{sech}^2(\frac{k}{2}x)$$

The exact solution to this case is well known and it is:[10]

$$u(x,t) = \frac{-k^2}{2}\operatorname{sech}^2\left[\frac{k}{2}\left(x - k^2t\right)\right]$$

Using ADM we can obtain an approximated solution up to our desired order of accuracy as illustrated. The first terms of $\sum_{n=0}^{\infty} u_n$ are given by:

$$u_0(x,t) = \frac{-k^2}{2}\operatorname{sech}^2\left[\frac{k}{2}x\right]$$
$$u_1(x,t) = \frac{-k^5}{2}\operatorname{sech}^2\left[\frac{k}{2}x\right]\tanh\left[\frac{k}{2}x\right]t$$
$$u_2(x,t) = \frac{-k^8}{8}\operatorname{sech}^4\left[\frac{k}{2}x\right](-2 + \cosh[kx])t^2)$$
$$u_3(x,t) = \frac{-k^{11}}{48}\operatorname{sech}^5\left[\frac{k}{2}x\right]\left(-11\sinh\left[\frac{k}{2}x\right] + \sinh\left[\frac{3k}{2}x\right]\right)t^3$$
$$u_4(x,t) = \frac{-k^{14}}{384}\operatorname{sech}^6\left[\frac{k}{2}x\right](33 - 26\cosh[kx] + \cosh[2kx])t^4$$
$$u_5(x,t) = \frac{-k^{17}}{3840}\operatorname{sech}^7\left[\frac{k}{2}x\right]\left(302\sinh\left[\frac{k}{2}x\right] - 57\sinh\left[\frac{3k}{2}x\right] + \sinh\left[\frac{5k}{2}x\right]\right)t^5$$

Those terms were calculated systematically using Mathematica software due to the complexity of the calculations.

10. Tamer A. Abassy, Magdy A. El-Tawil, and Hassan K. Saleh, "The Solution of KdV and mKdV Equations Using Adomian Pade Approximation," *International Journal of Nonlinear Sciences and Numerical Simulation* 5, no. 4 (January 2004), ISSN: 2191-0294, 1565-1339, accessed January 12, 2021, https://doi.org/10.1515/IJNSNS.2004.5.4.327, https://www.degruyter.com/view/j/ijnsns.2004.5.4/ijnsns.2004.5.4.327/ijnsns.2004.5.4.327.xml.

5.5 Hirota's Bi-linear Method

Herota method is an alternative to the Bäcklund transform and is somtimes avaialve when the baclund transform is not cabale of the solving the problem. It was established buy hirto in 1971 to find N-Soliton solution of the KdV equation.[11]

5.5.1 Motivation

Following the same procedures we have done in the Bäcklund transform, we have reached a solution of 1-Soliton of the KdV equation as:

$$u(x,t) = \frac{c}{2}\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(x - x_0 - ct)\right)$$

Now for simplifying the equation we substitute $\mu = \frac{\sqrt{c}}{2}$. The equation becomes,

$$u(x,t) = 2\mu^2 \operatorname{sech}^2\left(\mu(x - x_0 - 4\mu^2 t)\right)$$

and of course we can write this solution in term of the potential as η

$$u = \eta_x = 2\mu \tanh\left(\mu(x - x_0 - 4\mu^2 t)\right)$$

We can go further by integrating the previous equation we get,

$$u = \eta_x = 2\frac{\partial^2}{\partial x^2}\left(\ln\left(\cosh\left(\mu(x - x_0 - 4\mu^2 t)\right)\right)\right)$$

We can substitute $\xi = x - x_0 - 4\mu^2 t$ and $\cosh(x) = \frac{e^x + e^{-x}}{2}$, we get

$$\begin{aligned}
u &= 2\frac{\partial^2}{\partial x^2}\left(\ln\left(\frac{e^{-\mu\xi}(1 + e^{2\mu\xi}}{2}\right)\right) \\
&= 2\frac{\partial^2}{\partial x^2}\left[\ln\left(1 + e^{2\mu\xi}\right) - \ln(2) - \mu\xi\right] \\
&= 2\frac{\partial^2}{\partial \xi^2}\left[\ln\left(1 + e^{2\mu\xi}\right) - \ln(2) - \mu\xi\right] \\
&= 2\frac{\partial^2}{\partial \xi^2}\ln\left(1 + e^{2\mu\xi}\right)
\end{aligned}$$

Which can be written as,

$$u(x,t) = 2\frac{\partial^2}{\partial x^2}\ln\left(1 + e^{2\mu(x - x_0 - 4\mu^2 t)}\right) \tag{70}$$

11. P. G Drazin and R. S Johnson, *Solitons: an introduction*, OCLC: 818173318 (Cambridge [England]; New York: Cambridge University Press, 1989), ISBN: 978-1-139-17205-9, accessed January 9, 2021, https://doi.org/10.1017/CBO9781139172059.

5.5.2 KdV equation in quadratic form

This form of the solution will help write the KdV equation in a so called bi-linear form. We will first attempt to write the KdV equation in a quadratic form and observe its structure. Inspired by the previous form the solution of the 1-soliton, we can write,

$$u = 2\frac{\partial^2}{\partial x^2} f \qquad \eta = 2\frac{\partial}{\partial x} f = 2\frac{f_x}{f}$$

We now can substitute this form in the KdV, but first we need to find the derivatives of η

$$\frac{1}{2}\eta_t = \frac{f_{xt} f - f_x f_t}{f^2} \qquad \frac{1}{2}\eta_x = \frac{f_{xx} f - f_x^2}{f^2}$$

Finally, we need η_{xxx} the answer can be found as,

$$\frac{1}{2}\eta_{xxx} = \frac{f_{xxxx}}{f} - 4\frac{f_{xxx} f_x}{f^2} - 3\frac{f_{xx}^2}{f^2} + 12\frac{f_{xx} f_x^2}{f^3} - 6\frac{f_x^4}{f^4}$$

Now we can substitute in eqn.(66), and we find;

$$\begin{aligned}
\eta_t + 3(\eta_x)^2 + \eta_{xxx} &= 2\left(\frac{f_{xt} f - f_x f_t}{f^2}\right) + 3\left[2\frac{f_{xx} f - f_x^2}{f^2}\right]^2 + 2\left(\frac{f_{xxxx}}{f} - 4\frac{f_{xxx} f_x}{f^2} - 3\frac{f_{xx}^2}{f^2} + 12\frac{f_{xx} f_x^2}{f^3} - 6\frac{f_x^4}{f^4}\right) = 0 \\
&= \frac{f_{xt} f}{f^2} - \frac{f_x f_t}{f^2} + 6\left[\frac{f_{xx}^2 f^2}{f^4} - \frac{2 f_{xx} f_x^2 f}{f^4} + \frac{f_x^4}{f^4}\right] + \frac{f_{xxxx}}{f} - 4\frac{f_{xxx} f_x}{f^2} - 3\frac{f_{xx}^2}{f^2} + 12\frac{f_{xx} f_x^2}{f^3} - 6\frac{f_x^4}{f^4} \\
&= \frac{f_{xt} f}{f^2} - \frac{f_x f_t}{f^2} + 6\frac{f_{xx}^2 f^2}{f^4} - \cancel{\frac{12 f_{xx} f_x^2 f}{f^4}} + 6\cancel{\frac{f_x^4}{f^4}} + \frac{f_{xxxx}}{f} - 4\frac{f_{xxx} f_x}{f^2} - 3\frac{f_{xx}^2}{f^2} + 12\cancel{\frac{f_{xx} f_x^2}{f^3}} - 6\cancel{\frac{f_x^4}{f^4}} \\
&= \frac{f_{xt}}{f} - \frac{f_x f_t}{f^2} + 3\frac{f_{xx}^2}{f^2} - 4\frac{f_{xxx} f_x}{f^2} + \frac{f_{xxxx}}{f} = 0
\end{aligned}$$

Now, if we multiply by f^2 throughout we find that,

$$f f_{xt} - f_x f_t + 3 f_{xx}^2 - 4 f_{xxx} f_x + f f_{xxxx} = 0 \tag{71}$$

This equation is quadratic in f and is known as the quadratic form of th KdV equation.[12]

At first site, equation (71) does not seem to be much of a progress. However, it has an a advantage that it is quadratic in f and its derivatives can be written in a neat way. For this, we need to defines a new operation performed by the Hirota's bilinear operator.

5.5.3 Hirota's Bi-linear Operator

They are bi-linear differential operator $\mathbf{D}$ that map a pair of functions (f, g) into a single function $\mathbf{D}(f, g)$. This operator is unlike the usual operators we dealt with before like n-deravtive with respect to x, that map a single function into a single function.

$$\mathbf{D} : C^\infty \times C^\infty \mapsto C^\infty$$

12. Alan C. Newell, *Solitons in mathematics and physics*, CBMS-NSF regional conference series in applied mathematics 48 (Philadelphia, Pa: Society for Industrial / Applied Mathematics, 1985), ISBN: 978-0-89871-196-7.

Now the differential operator is defined for $m, n \geq 0$ as,

$$\left[\mathbf{D}_t^m \mathbf{D}_x^n (f,g)\right](x,t) := \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n f(x,t)g(x',t') \mid_{x'=x,t'=t}$$

For better illustrations we proceed though some examples that will helps us factorize equation (71).

$$\begin{aligned}
[\mathbf{D}_t(f.g)](x,t) &= \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right) f(x,t)g(x',t') \mid_{x'=x,t'=t} \\
&= f_t(x,t)g(x',t') - f(x,t)g_{t'}(x',t') \mid_{x'=x,t'=t} \\
&= f_t(x,t)g(x,t) - f(x,t)g_t(x,t)
\end{aligned}$$

Which implies that $\mathbf{D}_t(f,f) = 0$. Now lets, look at another example,

$$\begin{aligned}
[\mathbf{D}_t\mathbf{D}_x(f.g)](x,t) &= \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right) f(x,t)g(x',t') \mid_{x'=x,t'=t} \\
&= \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)(f_x(x,t)g(x',t') - f(x,t)g_{x'}(x',t')) \mid_{x'=x,t'=t} \\
&= f_{xt}(x,t)g(x,t) - f_t(x,t)g_x(x,t) - f_x(x,t)g_t(x,t) + f(x,t)g_{xt}(x,t)
\end{aligned}$$

Particularly, if we take $f = g$, we find

$$\mathbf{D}_t\mathbf{D}_x(f,f) = 2(ff_{xt} - f_t f_x)$$

which looks like the first two terms of equation (71). For finding the last three terms we proceed by finding $\mathbf{D}_x^4(f,g)$,

$$\begin{aligned}
\mathbf{D}_x^4(f,g) &= \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^4 f(x,t)g(x',t') \mid_{x'=x,t'=t} \\
&= \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^3 (f_x(x,t)g(x',t') - f(x,t)g_{x'}(x',t')) \mid_{x'=x,t'=t} \\
&= \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^2 (f_{xx}(x,t)g(x',t') - 2f_x(x,t)g_{x'}(x',t') + f(x,t)g_{x'x'}(x',t')) \mid_{x'=x,t'=t} \\
&= \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)(f_{xxx}(x,t)g(x',t') - 3f_{xx}(x,t)g_{x'}(x',t') + 3f_x(x,t)g_{x'x'}(x',t') + f(x,t)g_{x'x'x'}(x',t')) \mid_{x'=x,t'=t} \\
&= f_{xxxx}g - 4f_{xxx}g_x + 6f_{xx}g_{xx} - 4f_x g_{xxx} + fg_{xxxx}
\end{aligned}$$

If $f = g$,

$$\mathbf{D}_x^4(f,f) = 2(ff_{xxxx} - 4f_x f_{xxx} + 3f_{xx}^2)$$

These are the remaining 3 terms we need for equation (71). Now, we are able to write the KdV equation in its quadratic form as:

$$\left(\mathbf{D}_t\mathbf{D}_x + \mathbf{D}^4\right)(f,f) = 0 \tag{72}$$

This is called the Bi-linear form of the KdV equation.[13]

13. Wazwaz, *Partial differential equations and solitary waves theory*.

5.5.4 Solutions of the Bilinear form

We will first introduce some ideas that Hirtota himself through of while finding the solutions of the KdV equation.[14] We have got some intuition that the KdV equation has a solution of the shape:

$$u = 2\frac{\partial^2}{\partial x^2}\ln(f) \tag{73}$$

where if $f = 1 + e^{2\mu(x-x_0-5\mu^t)}$, the results will be a 1-soliton solution. The bi-linear form of the KdV equation alongside this intuition of the solution of first solution suggests that the multi-soliton solution might be of the form of a sum of exponential that are linear function of x and t.

Let's check his idea for the 1-soliton soliton solution. Thus, we propose the solution would have a f of the form $1 + e^{\theta}$, where $\theta = ax + bt + c$. It rather seems ubiquitous to substitute this directly and find the solution from the bi-linear form. Thus, we will basically derive up some lemmas that will us even further in our discussion.

Lemma 5.1. *If* $\theta_i = a_i x + b_i t + c_i$, *where* $(i = 1, 2)$ *then,*

$$\begin{aligned}
\mathbf{D}_t^m\mathbf{D}_x^n(e^{\theta_1}, e^{\theta_2}) &= \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n e^{\theta_1+\theta_2} \\
&= \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^{n-1} (a_1 - a_2)e^{\theta_1+\theta_2} \\
&= \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^{n-2} (a_1 - a_2)^2 e^{\theta_1+\theta_2} \\
&= \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m (a_1 - a_2)^n e^{\theta_1+\theta_2} \\
&= (a_1 - a_2)^n \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^{m-1} (b_1 - b_2)e^{\theta_1+\theta_2} \\
&= (b_1 - b_2)^m (a_1 - a_2)^n e^{\theta_1+\theta_2}
\end{aligned}$$

In particular, $\mathbf{D}_t^m\mathbf{D}_x^n(e^{\theta}, e^{\theta}) = 0$ *unless* $m = n = 0$. *and,*

$$\begin{aligned}
\mathbf{D}_t^m\mathbf{D}_x^n(e^{\theta}, 1) &= \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n e^{\theta} \\
&= b^m a^n e^{\theta}
\end{aligned}$$

while,

$$\begin{aligned}
\mathbf{D}_t^m\mathbf{D}_x^n(1, e^{\theta}) &= \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n e^{\theta} \\
&= (-b)^m(-a)^n e^{\theta} \\
&= (-1)^{n+m} b^m a^n e^{\theta} \\
&= (-1)^{n+m}\mathbf{D}_t^m\mathbf{D}_x^n(e^{\theta}, 1)
\end{aligned}$$

14. Drazin and Johnson, *Solitons*.

Now we can write the bi-linear form of the KdV equation for $f = 1 + e^\theta$ as,

$$\begin{aligned}
0 &= \left(\mathbf{D}_t\mathbf{D}_x + \mathbf{D}^4\right)(1+e^\theta, 1+e^\theta) \\
&= \left(\mathbf{D}_t\mathbf{D}_x + \mathbf{D}^4\right)\left(\cancelto{0}{(1,1)} + (1,e^\theta) + (e^\theta,1) + \cancelto{0}{(e^\theta,e^\theta)}\right) \\
&= 2\left(\mathbf{D}_t\mathbf{D}_x + \mathbf{D}^4\right)(e^\theta, 1) \\
&= 2(ba + a^4)e^\theta = 2a(b + a^3)e^\theta
\end{aligned}$$

Thus, by plugging the answer we obtained an algebraic equation. Since the exponential cannot vanish, then either $a = 0$ or $b + a^3 = 0$. The first one give u that is independent of x which is just the trivial solution. The second factor gives the one soliton solution.

5.5.5 N-Soliton solutions

Hirota thought of a usful idea for finding other solton solutions for the KdV equation by looking for power series solutions in a auxillary parameter ϵ.[15] [16]

$$f(x,t) = \sum_{n=0}^{\infty} \epsilon^n f_n(x,t)$$

with $f_0 = 1$. In this assumption we hope that the series terminates for a finite value of N. After finding the solution we can set this parameter $\epsilon = 1$, thus making it finite. Now we plug this form of f in the bi-linear form of the KdV equation. We first define $\mathbf{B}(f,f) = \left(\mathbf{D}_t\mathbf{D}_x + \mathbf{D}^4\right)$. After substitution we get:

$$\begin{aligned}
0 &= \mathbf{B}\Big(\sum_{n_1=0}^{\infty} \epsilon^{n_1} f_{n_1}(x,t), \sum_{n_2=0}^{\infty} \epsilon^{n_2} f_{n_2}(x,t)\Big) \\
&= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \epsilon^{n_1+n_2}\mathbf{B}(f_{n_1}, f_{n_2})
\end{aligned}$$

Next we gather terms of the same degree in ϵ by preforming the substitution $n = n_1 + n_2$ and $n_2 = m$,

$$0 = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \epsilon^n \mathbf{B}(f_{n-m}, f_m) = \sum_{n=1}^{\infty}\epsilon^n \sum_{m=0}^{n} \mathbf{B}(f_{n-m}, f_m)$$

Since $\mathbf{B}(1,1) = 0$. For the last equation to be true each coefficient of ϵ must vanish for every power. Thus,

$$\sum_{m=0}^{n} \mathbf{B}(f_{n-m}, f_m) = 0 \qquad \forall \ n = 1,2,3,\cdots \tag{74}$$

We can rewrite equation (74) as follows,

$$\mathbf{B}(f_n, 1) + \mathbf{B}(1, f_n) = -\sum_{m=1}^{n-1} \mathbf{B}(f_{n-m}, f_m) = \textbf{Experssion only in } f_1, f_2, \cdots f_{n-1} \tag{75}$$

15. Drazin and Johnson, *Solitons*.
16. Wazwaz, *Partial differential equations and solitary waves theory*.

This is a recurrence relation that shows that we can solve for the coefficients f_n from the previous found coefficients. We can use second outcome of lemma 3.1 to simplify equation (74) to,

$$\mathbf{B}(f_n, 1) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} + \frac{\partial^3}{\partial x^3}\right) f_n = -\frac{1}{2}\sum_{m=1}^{n-1} \mathbf{B}(f_{n-m}, f_m) \tag{76}$$

For example, we can solve equation 9 for $n = 1$. The summation vanishes and we are left with this partial differential equation.

$$\left(\frac{\partial}{\partial t} + \frac{\partial^3}{\partial x^3}\right) f_1 = 0$$

For this equation we can propose a simple solution, that is related to the boundary conditions. We propose that,

$$f_1 = \sum_{i=1}^{N} e^{a_i x + a_i^3 t + c_i} = \sum_{i=1}^{N} e^{\theta_i}$$

Since we have a simple solution for f_1, we can find the other solution recursively using equation (76). When we do the same for the other f_ns, we are left with N as being the highest order power f in the proposed solution. Thus, this turn out to be the N-soliton solution of the KdV which is given by,

$$f(x,t) = \sum_{n=0}^{N} \epsilon^n f_n(x,t) \tag{77}$$

Then we easily set $\epsilon = 1$ or absorb it in the constant e^{c_i}. We have seen before that when $N = 1$ we get the 1-Soliton solution. Thus, with $f_1 = e^{\theta} = e^{ax - a^3 t + c}$ we write the second equation for f_2 as follows,

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} + \frac{\partial^3}{\partial x^3}\right) f_2 = -\frac{1}{2}\mathbf{B}(e^{\theta}, e^{\theta}) = 0$$

This equation admits the trivial solution thus make $f_2 = 0$. Similarly, $f_3 = f_4 = \cdots = 0$. From this we can find the general f as follows,

$$f = 1 + e^{\theta}$$

which is just the 1-soliton solution. Now we look for $N = 2$.

$$f_1 = e^{\theta_1} + e^{\theta_2}$$

We can find the second f_2 as follows,

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} + \frac{\partial^3}{\partial x^3}\right) f_2 = -\frac{1}{2}\mathbf{B}(e^{\theta_1} + e^{\theta_2}, e^{\theta_1} + e^{\theta_2}) = -\mathbf{B}(e^{\theta_1}, e^{\theta_2})$$

Using lemma 3.1, this is equal to

$$-\mathbf{B}(e^{\theta_1}, e^{\theta_2}) = -(a_1 - a_2)(-a_1^3 + a_2^3 + (a_1 - a_2)^3)e^{\theta_1 + \theta_2}$$

Now, we finally have

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} + \frac{\partial^3}{\partial x^3}\right) f_2 = 3a_1 a_2 (a_1 - a_2)^2 e^{\theta_1 + \theta_2}$$

Now let's suppose that $f_2 = Ae^{\theta_1+\theta_2}$ where A is a constant that have to be determined. After substation, we can easily interpret that

$$A = \left(\frac{a_1 - a_2}{a_1 + a_2}\right)^2$$

Now let's find the equation for f_3. For this, we need to compute $\mathbf{f_2}, \mathbf{f_1}$. We can easily see that any $\mathbf{B}$ that act on f_1 and f_2 will be zero. Thus, making $f_3 = 0$ and similarity for other fs.

Now then our finial f can be written as,

$$f = 1 + \epsilon e^{\theta_1} + \epsilon e^{\theta_2} + \epsilon^2 \left(\frac{a_1 - a_2}{a_1 + a_2}\right)^2 e^{\theta_1+\theta_2}$$

This equation is just the 2-soliton solution of the KdV equation. One can check these relations by studying the asymptotic relations of the generated function.

Now, we would to put a simple geerating methods for the General N-term or the N-soliton. We will not go into any details for deriving this, however we will some a pattern in the 2-Soliton and complete upon it. The complete derivation is found at Newell's book.[17] We can neatly write the 2-soliton solution as follows,

$$\begin{aligned} f &= 1 + \epsilon e^{\theta_1} + \epsilon e^{\theta_2} + \epsilon^2 \left(\frac{a_1 - a_2}{a_1 + a_2}\right)^2 e^{\theta_1+\theta_2} \\ &= (1 + \epsilon e^{\theta_1})(1 + \epsilon e^{\theta_2}) - \epsilon^2 e^{\theta_1+\theta_2} + \epsilon^2 \left(\frac{a_1 - a_2}{a_1 + a_2}\right)^2 e^{\theta_1+\theta_2} \\ &= (1 + \epsilon e^{\theta_1})(1 + \epsilon e^{\theta_2}) - \epsilon^2 \frac{4a_1a_2}{(a_1 + a_2)^2} e^{\theta_1+\theta_2} \end{aligned}$$

This form can be interpreted as a matrix of dimension 2×2 as follows,

$$f = \begin{vmatrix} 1 + \epsilon e^{\theta_1} & \epsilon \frac{2a_1}{a_1+a_2} e^{\theta_2} \\ \epsilon \frac{2a_2}{a_1+a_2} e^{\theta_1} & 1 + \epsilon e^{\theta_2} \end{vmatrix}$$

Where the matrix elements S_{ij} are given by the following expression,

$$S_{ij} = \delta_{ij} + \epsilon \frac{2a_i}{a_i + a_j} e^j \tag{78}$$

Actually this way of writing the elements is the way that can be generalized to higher N with S an $N \times N$ matrix of the element form as in equation (78). This gives the N soliton solution of the KdV equation.

17. Newell, *Solitons in mathematics and physics*.

6 Conclusion

We have introduced the KdV equation, that is considered to be the baseline for explaining the modern water wave theory, starting from modelling the shallow water waves. We constructed the geometry of the problem, obtained a scenario where perturbations have been ignored at the very beginning, and obtaining the linearised version of the equation, followed by relaxing the conditions to obtain the general version. Our scope of study was restricted only towards the dispersive coefficients and not the dissipation coefficients, and based on this our solutions were developed. Numerical solutions were discussed based on implementing the CTCS (central-difference) second order scheme, whilst other explicit higher order schemes could have been used, such as Zabusky and Kruskal explicit scheme, Hopscotch Method for implicit and explicit schemes, the Proposed Scheme etc. The latter scheme is based on the inverse scattering method, a technique that is used to attain analytic solutions for our equation of interest. Rather, we presented analytic solutions by a preliminary approach at the beginning, coming across Bäcklund transformation, Adomian's decomposition method, and finally the Hirota's Bi-linear method for several cases like the 1-soliton till discussing the N-soliton case.

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