Tutorial 2: Legendre Polynomials

Mohamed Salaheldeen

October 2023

Under the supervision of: Dr. Abdallah Aboutahoun

1 Important Points

Generating function of Legendre Polynomials,

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n \tag{1}$$

with $|x| \leq 1$ and $t < 1$. You can use equation 1 to show that $P_n(1) = 1$ and $P_n(-1) = (-1)^n$. Note that differentiating the generating function with respect to x would introduce the derivative instead.

Most of the questions that you will encounter while solving for identities of Legendre polynomials require the use of recursion formulas.

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0 \tag{2}$$
$$P'_{n+1}(x) - 2xP'_n(x) + P'_{n-1}(x) - P_n(x) = 0 \tag{3}$$
$$xP'_n(x) - P'_{n-1}(x) - nP_n(x) = 0 \tag{4}$$
$$P'_{n+1}(x) - xP'_n(x) = (n+1)P_n(x) \tag{5}$$
$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x) \tag{6}$$
$$\frac{d}{dx}((1-x^2)P'_n(x)) + n(n+1)P_n(x) = 0 \tag{7}$$

In most cases, you will be faced with some problems and utilize one or more of these equations.The use of these equations varies depending on the context. Here are some examples,

- Formula 1 is mostly used to replace a product of $xP_n(x)$ with a sum of Legendre Polynomials.
- Formula 4 and 5 are used to replace $xP'_n(x)$ with a primed and unprimed polynomial. Notice the difference between the two formulas where x lies beside the highest order of n in equation 4 and the least order of n in equation 5.

Furthermore, some questions may require deriving an integral that has Legendre polynomials. The main task, you need to do, is trying to simplify the integral to just some product of Legendre polynomials P_nP_m. (Why?) So we can use the orthogonality relation of Legendre polynomials.

$$\int_{-1}^{1} P_n(x)P_m(x) = \begin{cases} 0 & m \neq n \\ \frac{2}{2n+1} & m = n \end{cases} \tag{8}$$

Thus, the integral can be easily evaluated. Other representations of Legendre Polynomials,

$$P_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k(2n-2k)!x^{n-2k}}{2^n k!(n-k)!(n-2k)!} \tag{9}$$
$$= \frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^n] \qquad \text{(Rodriguez formula)} \tag{10}$$

Equations 9 and 10 are not usually used to solve a lot of problems unless explicitly clear. Nevertheless, any representation can prove any identity with a certain amount of mathematical manipulations.

2 Selected Exercises

Problem 1
Find
$$P'_{2n}(0) \text{ and } P'_{2n+1}(0)$$

Solution: The easiest method to evaluate specific values for Legendre polynomials like this one is to compare coefficients in the generating function equation 1. We want the derivative, thus, we need to take the derivative of equation 1 with respect to x. This gives the following equation.

$$\frac{t}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} P'_n(x)t^n$$

We just now need to substitute $x = 0$,

$$\frac{t}{(1+t^2)^{3/2}} = \sum_{n=0}^{\infty} P'_n(0)t^n$$

To find the answer, we need to compare coefficients in powers of t. The most direct way to do so is to expand the left-hand side,

$$(1+t^2)^{-3/2} = \sum_{n=0}^{\infty} \binom{-3/2}{n} (1)^{-3/2-n}(t^2)^n$$

Thus, we have,

$$\sum_{n=0}^{\infty} \binom{-3/2}{n} t^{2n+1} = \sum_{n=0}^{\infty} P'_n(0)t^n$$

Now, we can easily compare the coefficients. It is clear that the left-hand side has no even powers of t since the exponent is $2n+1$. Thus, we can conclude that,

$$P'_{2n+1} = \binom{-3/2}{n} \qquad P'_{2n} = 0$$

Problem 2
Show that
$$\sum_{k=0}^{n}(2k+1)P_k(x) = P'_{n+1}(x) + P'_n(x)$$

Solution: We can see that equation 6 resembles most of the left hand without a summation sign. Thus, we sum from $k = 1$ to $k = n$ since there exist no P'_{-1}.

$$\begin{aligned}
\sum_{k=1}^{n}(2k+1)P_k(x) &= \sum_{k=1}^{n} P'_{k+1} - P'_{k-1} \\
&= -P'_0(x) - P'_1(x) + P'_n(x) + P'_{n+1}(x) && \text{(Cancel like terms)} \\
&= -P_0(x) + P'_n(x) + P'_{n+1}(x) && (P_0(x) = 1 \& P_1(x) = x)
\end{aligned}$$

Thus, moving $P_0(x)$ to the other side,

$$P_0(x) + \sum_{k=1}^{n}(2k+1)P_k(x) = P'_n(x) + P'_{n+1}(x) = \sum_{k=0}^{n}(2k+1)P_k(x)$$

Problem 3
Show that,

$$t^2(1-2xt+t^2)^{-3/2} = \sum_{n=0}^{\infty} \left[xP_n'(x) - nP_n(x)\right] t^n$$

.

Solution: Notice that we can get the right-hand side just by differentiating the generating function with respect to x and multiplying by t. Thus,

$$\begin{aligned}
t^2(1-2xt+t^2)^{-3/2} &= t\frac{d}{dx}\left[(1-2xt+t^2)^{-1/2}\right] \\
&= t\frac{d}{dx}\left[\sum_{n=0}^{\infty} P_n(x)t^n\right] && \text{(using eqn 1)} \\
&= \sum_{n=0}^{\infty} P_n'(x)t^{n+1} && (n \mapsto n-1) \\
&= \sum_{n=1}^{\infty} \left[xP_n'(x) - nP_n(x)\right] t^n && \text{(using eqn 5)}
\end{aligned}$$

Problem 4
Prove that

$$\int_{-1}^{1} P_n(x)dx = 0$$

for $n = 1, 2, \cdots$

Solution: The idea to solve this problem is to notice that $P_0(x) = 1$, which mean,

$$\int_{-1}^{1} P_n(x)dx = \int_{-1}^{1} P_0(x)P_n(x)dx$$

However, this is exactly zero by the orthogonality relation since $n \neq 0$. Other solutions may utilize the Rodriguez formula.

Problem 5
Show that,

$$\int_{-1}^{1} P_n(x)P_{n+1}'(x)dx = 2$$

Solution: Notice here that we want to substitute something instead of a primed Legendre polynomial. However, using any of the formulas 3-6 would introduce at least another primed polynomial. The idea here is to reduce the order of the primed Legendre function. Thus, one needs to use equation 6 which would not introduce any factors of x.

$$\begin{aligned}
\int_{-1}^{1} P_n(x)P_{n+1}'(x)dx &= \int_{-1}^{1} P_n(x)\left[(2n+1)P_n(x) + P_{n-1}'(x)\right]dx && \text{(using eqn 6)} \\
&= \int_{-1}^{1} P_n(x)^2 dx + \int_{-1}^{1} P_n(x)P_{n-1}'(x)dx \\
&= 2 + \int_{-1}^{1} P_n(x)P_{n-1}'(x)dx && \text{(using eqn 8)}
\end{aligned}$$

Notice that we ended up with the same integral but with two orders less in the primed polynomial. Thus, naturally, one would repeat the previous step to reduce until zero for n odd or 1 for n even. However, in both cases, the extra integral would evaluate exactly to zero (Check!!).

Problem 6
Show that,

$$\int_{-1}^{1} xP_n'(x)P_n(x)dx = \frac{2n}{2n+1}$$

Solution: The main idea to evaluate such an integral is trying to replace all derivatives and xs with Legendre polynomials using equations 2-6. As it seems, we have to introduce at least another primed factor. The most convenient is formulae 4. Thus,

$$\begin{aligned}
\int_{-1}^{1} xP_n'(x)P_n(x)dx &= \int_{-1}^{1} \left[P_{n-1}'(x) + nP_n(x)\right] P_n(x)dx && \text{(using eqn 4)} \\
&= n\int_{-1}^{1} P_n(x)^2 dx && \text{(same as problem 5)} \\
&= \frac{2n}{2n+1} && \text{(using eqn 8)}
\end{aligned}$$

Problem 7
Show that

$$\int_{-1}^{1} x^2 P_{n+1}(x)P_{n-1}(x)dx = \frac{2n(n+1)}{(4n^2-1)(2n+3)}$$

Solution: Here we have two factors of x that we have to get rid of. Thus, we need to substitute for both of the Legendre functions in this case. Again, we are searching for the recursion formula that would introduce the least factors of xs and derivatives. Formula 2 is ideal since it replaces $xP_n(x)$ with a sum of Legendre polynomials. Writing two versions of this equation,

$$\begin{aligned}
xP_{n-1}(x) &= \frac{n}{2n-1}P_n + \frac{n-1}{2n-1}P_{n-2} \\
xP_{n+1}(x) &= \frac{n+2}{2n+3}P_{n+2} + \frac{n+1}{2n+3}P_n
\end{aligned}$$

Multiplying these two equations together and using equation 8, we can obtain,

$$\int_{-1}^{1} x^2 P_{n+1}(x)P_{n-1}(x)dx = \frac{n}{2n-1}\frac{n+1}{2n+3}\int_{-1}^{1} P_n(x)^2 dx = \frac{2n(n+1)}{(4n^2-1)(2n+3)}$$

Problem 8
Show that

$$\int_{-1}^{1} f(x)P_n(x)dx = \frac{(-1)^n}{2^n n!}\int_{-1}^{1} f(x)^{(n)}(x^2-1)^n dx$$

Solution: It's clear from the form that we have to use the Rodriguez formula as a representation for Legendre polynomials. Thus,

$$\int_{-1}^{1} f(x)P_n(x)dx = \frac{1}{2^n n!}\int_{-1}^{1} f(x)\frac{d^n}{dx^n}[(x^2-1)^n]$$

It's clear that we need to increase the order of derivatives on $f(x)$ and reduce that on $(x^2-1)^n$. This can be done using integration by parts with the cost of a total derivative and a minus sign. Thus, doing integration by parts once gives,

$$\frac{1}{2^n n!}\int_{-1}^{1} f(x)\frac{d^n}{dx^n}[(x^2-1)^n] = \frac{1}{2^n n!}f(x)\frac{d^{n-1}}{dx^{n-1}}[(x^2-1)^n]\Big|_{-1}^{1} - \frac{1}{2^n n!}\int_{-1}^{1}\frac{d}{dx}[f(x)]\frac{d^{n-1}}{dx^{n-1}}[(x^2-1)^n]$$

Notice that $\frac{d^r}{dx^r}[(x^2-1)^n]\Big|_{-1}^{1} = 0$ when $r < n$ since each term in the expansion has at least one factor of (x^2-1). Thus, the total derivative finishes exactly. Thus, we can swap the derivative sign with a cost of the minus sign only. Doing integration by parts $n-1$ more times gives,

$$\frac{1}{2^n n!}\int_{-1}^{1} f(x)\frac{d^n}{dx^n}[(x^2-1)^n] = \frac{(-1)^n}{2^n n!}\int_{-1}^{1} f(x)^{(n)}(x^2-1)^n dx$$