Tutorial 1: Gamma & Beta Functions

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1 Important Points

Equivalent definitions for the gamma function.

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!n^x}{x(x+1)(x+2)\cdots(x+n)} \tag{1}$$

$$= \int_0^\infty t^{x-1}e^{-t}dt \tag{2}$$

such that $x > 0$. Useful properties,

$$\Gamma(x) = \frac{\Gamma(x+k)}{x(x+1)(x+2)\cdots(x+k-1)} \tag{3}$$

such that $k = 1, 2, \cdots$ and $x > -k$.

$$\Gamma(1) = 1 \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \tag{4}$$

Useful Substitutions,

1. $t = u^2$
$$\Gamma(x) = 2\int_0^\infty e^{-u^2}u^{2x-1}du$$
2. $t = \ln(1/u)$
$$\Gamma(x) = \int_0^1 \left(\ln\left(\frac{1}{u}\right)\right)^{x-1} du$$

Important Relations,

$$\int_0^{\pi/2} \sin^{2x-1}(\theta)\cos^{2y-1}(\theta)d\theta = \frac{\Gamma(x)\Gamma(y)}{2\Gamma(x+y)} \tag{5}$$

Legendre duplication formula,

$$2^{2x-1}\Gamma(x)\Gamma\left(x+\frac{1}{2}\right) = \sqrt{\pi}\Gamma(2x) \tag{6}$$

Weierstrass infinite product,

$$\frac{1}{\Gamma(x)} = xe^{\gamma x}\prod_{n=1}^{\infty}\left(1+\frac{x}{n}\right)e^{-x/n} \tag{7}$$

which produces the following relation,

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)} \tag{8}$$

Lastly, the beta function with $x, y > 0$,

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt \tag{9}$$

$$= \int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}}du \tag{10}$$

$$= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{11}$$

2 Selected Exercises

Problem 1
Show that

$$\Gamma(x) = (\ln(b))^x \int_o^\infty t^{x-1} b^{-t}$$

with $x > 0$ and $b > 0$.

Solution: using formula 2 and since $b = e^{\ln(b)}$, one can use substitution

$$t = u \ln(b) \implies dt = \ln(b) du$$

. Since $b > 0$, the limits won't change and the logarithm function is well-defined.

$$\begin{aligned} \Gamma(x) &= \int_0^\infty t^{x-1} e^{-t} dt \\ &= \int_0^\infty (u \ln(b))^{x-1} e^{-u \ln(b)} \ln(b) du \\ &= (\ln(b))^x \int_o^\infty t^{x-1} b^{-t} \end{aligned}$$

Problem 2
Show that

$$\int_a^\infty e^{2ax - x^2} dx = \frac{1}{2}\sqrt{\pi} e^{a^2}$$

Solution: First, we need to complete the square.

$$2ax - x^2 = -(x^2 - 2ax + a^2 - a^2) = -(x-a)^2 + a^2$$

Thus, we can rewrite the integral as,

$$\begin{aligned} \int_a^\infty e^{2ax - x^2} dx &= e^{a^2} \int_a^\infty e^{-(x-a)^2} dx \\ &= e^{a^2} \int_0^\infty e^{-x^2} dx && (x \to x + a) \\ &= \frac{1}{2} e^{a^2} \int_0^\infty u^{-1/2} e^{-u} du && (\text{substitute } u = x^2) \\ &= \frac{1}{2} e^{a^2} \Gamma(1/2) = \frac{1}{2}\sqrt{\pi} e^{a^2} \end{aligned}$$

Problem 3
Show that,

$$\int_0^1 x^k (\ln(x))^n dx = \frac{(-1)^n n!}{(k+1)^{n+1}}$$

with $k > -1$ and $n \in \mathbb{Z}^+$.

Solution:

$$\begin{aligned} \int_0^1 x^k (\ln(x))^n dx &= \int_0^\infty (-t)^n e^{-kt} e^{-t} dt && (\text{substitute } x = e^{-t}) \\ &= (-1)^n \int_0^\infty t^n e^{-(k+1)t} dt \\ &= \frac{(-1)^n}{(k+1)^{n+1}} \int_0^\infty t^n e^{-t} dt && (\text{substitute } u = (k+1)t) \\ &= \frac{(-1)^n \Gamma(n+1)}{(k+1)^{n+1}} = \frac{(-1)^n n!}{(k+1)^{n+1}} \end{aligned}$$

Note that in the third step, the limits won't change since $k > -1$.

Problem 4
Prove that
$$B(x+1,y) + B(x,y+1) = B(x,y)$$
with $x, y > 0$.

Solution: Using definition 10 for the beta function, we can rewrite the left-hand side as,

$$\begin{aligned} B(x+1,y) + B(x,y+1) &= \int_0^\infty \frac{u^x}{(1+u)^{x+y+1}}du + \int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y+1}}du \\ &= \int_0^\infty \frac{u^x + u^{x-1}}{(1+u)^{x+y+1}}du \\ &= \int_0^\infty \frac{u^{x-1}(u+1)}{(1+u)^{x+y+1}}du \\ &= \int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}}du = B(x,y) \end{aligned}$$

Problem 5
Show that,
$$\Gamma(x) = \int_0^\infty e^{-t}(t-x)t^{x-1}\ln(t)dt$$

Solution: Notice that,

$$\frac{d}{dx}t^x = \ln(t)t^x \tag{12}$$

Then, we can rewrite the above integral as,

$$\begin{aligned} \int_0^\infty e^{-t}(t-x)t^{x-1}\ln(t)dt &= \int_0^\infty e^{-t}\frac{d}{dx}t^x dt - x\int_0^\infty e^{-t}\frac{d}{dx}t^{x-1}dt \\ &= \frac{d}{dx}\int_0^\infty e^{-t}t^x dt - x\frac{d}{dx}\int_0^\infty e^{-t}t^{x-1}dt \\ &= \frac{d}{dx}\Gamma(x+1) - x\frac{d}{dx}\Gamma(x) \\ &= \frac{d}{dx}(x\Gamma(x)) - x\frac{d}{dx}\Gamma(x) = \Gamma(x) \end{aligned}$$

Problem 6
Show that,
$$\Gamma(x)\cos(ax) = b^x\int_0^\infty t^{x-1}e^{-bt\cos(a)}\cos(bt\sin(a))dt$$
such that $x, b > 0$ and $-\pi/2 \le a \le \pi/2$.

Solution: Let us begin by simplifying the integral by absorbing b using substitution $u = bt$ which gives,
$$\int_0^\infty u^{x-1}e^{-u\cos(a)}\cos(u\sin(a))du$$

Using Euler formula,

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \implies \cos(\theta) = \Re(e^{i\theta}) \tag{13}$$

we can rewrite the integral as,

$$\begin{aligned}
\int_0^\infty u^{x-1}e^{-u\cos(a)}\Re(e^{iu\sin(a)})du &= \Re\left\{\int_0^\infty u^{x-1}e^{-u(\cos(a)-i\sin(a))}du\right\} \\
&= \Re\left\{\int_0^\infty u^{x-1}e^{-ue^{-ia}}du\right\} && \text{(using eq 13 with } -\theta) \\
&= \Re\left\{\int_0^\infty e^{iax}t^{x-1}e^{-t}dt\right\} && \text{(substitute } t = ue^{-ia}) \\
&= \Re\left\{e^{iax}\right\}\int_0^\infty t^{x-1}e^{-t}dt \\
&= \Gamma(x)\cos(ax)
\end{aligned}$$

Note that in the third line, the limits did not change since a is restricted such that $\cos(a)$ is always positive.

Problem 7
Show that,

$$B(x, y+1) = \frac{y}{x}B(x+1, y) = \frac{y}{x+y}B(x, y)$$

Solution: Using identity 11,

$$\begin{aligned}
B(x, y+1) &= \frac{\Gamma(x)\Gamma(y+1)}{\Gamma(x+y+1)} \\
&= \frac{y}{x}\frac{\Gamma(x+1)\Gamma(y)}{\Gamma(x+y+1)} = \frac{y}{x}B(x+1, y) && \text{(Using eq 3 with k=1)} \\
&= \frac{y}{x+y}\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \frac{y}{x+y}B(x+1, y)
\end{aligned}$$

Problem 8
Evaluate,

$$\int_a^b (b-x)^{m-1}(x-a)^{n-1}dx$$

Solution: This integral looks similar to the beta integral in equation 9 upon finding a map such that $a \mapsto 0$ and $b \mapsto 1$. This map turns out to be

$$x = (b-a)t + a$$

(Check this gives the correct limits for t). Using this substitution, the integral transforms into,

$$\int_0^1 (b-a)^{m-1}(1-t)^{m-1}(b-a)^{n-1}(t)^{n-1}(b-a)dt = (b-a)^{m+n-1}B(n, m)$$

Problem 9
Show that
$$\int_0^\infty \frac{x^{p-1}\ln(x)}{1+x}dx = -\pi^2\csc(p\pi)\cot(p\pi)$$

Solution: Using observation 12, we can rewrite the integral as,

$$\begin{aligned}
\int_0^\infty \frac{x^{p-1}\ln(x)}{1+x}dx &= \int_0^\infty \frac{\frac{d}{dp}x^{p-1}}{1+x}dx \\
&= \frac{d}{dp}\left\{\int_0^\infty \frac{x^{p-1}}{1+x}dx\right\} \\
&= \frac{d}{dp}\left\{B(p,1-p)\right\} && \text{(eq 10)} \\
&= \frac{d}{dp}\left\{\frac{\Gamma(p)\Gamma(1-p)}{\Gamma(1)}\right\} && \text{(eq 11)} \\
&= \frac{d}{dp}\left\{\pi\csc(p\pi)\right\} = -\pi^2\csc(p\pi)\cot(p\pi) && \text{(eq 8)}
\end{aligned}$$

Problem 10
Show that
$$B(x,y) = \int_0^1 \frac{t^{x-1}+t^{y-1}}{(t+1)^{x+y}}dt$$

Solution: Note that this integral has a very similar form to equation 10. However, the limits are totally different. Thus, this form would be of no use to us since any transformation that would restore the limits would miss up the integral (Check). Let us first try to simplify the integral using substitution $t=\tan^2(u)$ to get rid of the denominator.

$$\int_0^1 \frac{t^{x-1}+t^{y-1}}{(t+1)^{x+y}}dt = 2\left[\int_0^{\pi/4}\sin^{2x-1}(u)\cos^{2y-1}(u)du + \int_0^{\pi/4}\sin^{2y-1}(u)\cos^{2x-1}(u)du\right]$$

Check that this substitution indeed produces the above form. This integral has a similar form to equation 5, however, these limits are different. Let us now try to combine the integral into a single integral by transforming the sine into a cosine function using the identities.

$$\cos(x) = \sin\left(\frac{\pi}{2}-x\right)$$
$$\sin(x) = \cos\left(\frac{\pi}{2}-x\right)$$

This gives the following form for the first integral,

$$\begin{aligned}
\int_0^{\pi/4}\sin^{2x-1}(u)\cos^{2y-1}(u)du &= \int_0^{\pi/4}\cos^{2x-1}(\pi/2-u)\sin^{2y-1}(\pi/2-u)du \\
&= \int_{\pi/4}^{\pi/2}\cos^{2x-1}(t)\sin^{2y-1}(t)dt && \text{(substitute } t=\pi/2-u\text{)}
\end{aligned}$$

We can now combine the two integrals into a single one as follows,

$$\int_0^1 \frac{t^{x-1}+t^{y-1}}{(t+1)^{x+y}}dt = 2\int_0^{\pi/2}\sin^{2x-1}(u)\cos^{2y-1}(u)du = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = B(x,y)$$