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On Elementary Derivation of Schwarzschild and Kerr Metrics

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Chapter 1

Mathematical Preliminaries

In this section, we will give an introduction to the mathematical tools that we will need to drive the metric. We will begin our discussion with manifold and geodesics equation and end it with a brief introduction about hypersurfaces, especially null hypersurfaces which is of great interest to us.

1.1 Manifolds

Manifolds are objects of great interest in mathematics and physics. We are used to $\mathbf{R}^n$ space, where we define vectors by a n-tuples $(x^1, x^2 \cdots x^n)$. Numerous theories have been developed that analyzed this space (i.e differentiation, integration, etc). However, there are other geometrical spaces that we think of as curved, for example, the surface of a sphere. Additionally, some spaces are topologically complicated. The study of manifolds and differential geometry let us perform analogous operations on these spaces.

The notion of **manifold** was invented to define these surfaces. A Manifold is any curved space that may look topologically complex, but locally it represents $\mathbf{R}^n$. Imagine a sphere, as we zoom in enough a batch on the surface resembles $\mathbf{R}^n$. This is much like us walking on the surface of the earth. We can, thus, construct the whole surface by sewing these tiny little batches together. With this definition, we can analyze functions on a manifold by converting them to local functions in Euclidean space. Now, aided with this definition, everything seems to represent a manifold. However, there are plenty of geometrical objects that are not manifold. Examples include two cones stuck at their vertex. Although the cone surface resembles $\mathbf{R}^2$, there is of course something singular about the vertex. We now need to develop a rigorous mathematical definition with is meant by a manifold. The following discussion follows that of Wald [1] and Carroll[2].

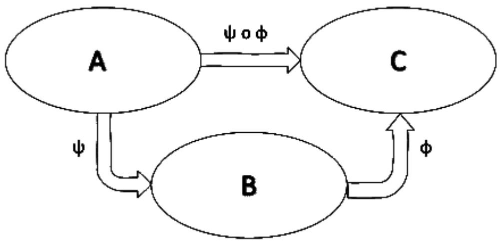

Figure 1.1: Composing ψ and ϕ to move from A to C.

First, we require some elementary notions that are necessary to complete our definition. The first of which is that of a map. A map is a relationship between two sets, say M and N, that takes elements in M and assigns them to exactly one element in N. A map is just a generalization to our notion of functions. Let us say we have two maps, $\phi : A \longmapsto B$ and $\psi : B \longmapsto C$, we can define the composition $\psi \circ \phi : A \longmapsto C$ that maps elements in A to C. This is represented in Fig 1.1. A map $\phi : M \longmapsto N$ is said to be injective (one-to-one) if each element in N has at most one element of M. This means that there may be some elements of M that are not mapped to N. The maps are said to be Onto if each element of N maps at least one element of M. This means that there may exist a subspace $U \in M$ that maps to a single point in N. A map that is both injective and onto is known as invertible. That means that every point on M gets mapped to only a special point on N. This also permits the introduction of an inverse map $\phi^{-1} : N \longmapsto M$, where $(\phi^{-1} \circ \phi)(a) = a$.

continuity notion of maps is suitable to discuss. However, we can think of a map $\phi : \mathbf{R}^m \longmapsto \mathbf{R^n}$. This map can be constructed using n ϕ^i functions,

$$y^1 = \phi^1(x^1, x^2, \cdots, x^m)$$

$$y^2 = \phi^2(x^1, x^2, \cdots, x^m)$$

$$\vdots$$

$$y^n = \phi^n(x^1, x^2, \cdots, x^m)$$

where x and y represent points in $\mathbf{R}^m$ and $\mathbf{R}^n$ respectively. It is referred to one of these functions as C^k if the ks derivative exist and continuous. Through our discussion we will deal with C^∞ maps where all functions are continuous and infinitely differentiable. These maps are called smooth maps. Additionally, the map is called diffeomorphism, if there exist a C^∞ map $\phi : M \longmapsto N$ with a C^∞ inverse $\phi^{-1} : N \longmapsto M$. The two

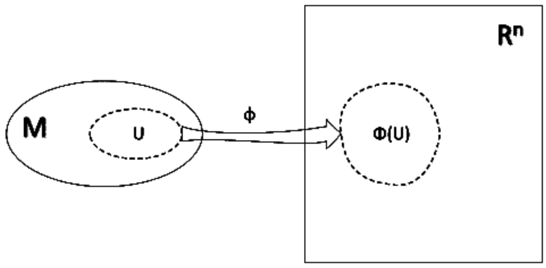

Figure 1.2: A coordinate system covering a subset U of manifold M.

sets then are said to be diffeomorphic. Diffeomorphism is very important in definition of tensors and vectors and moving between coordinate systems.

One can think of an open ball, as the set of all points x in $\mathbf{R}^n : |x-y| < r$ for $y \in \mathbf{R}^n$ and $r \in \mathbf{R}$. An open set in $\mathbf{R}$ is constructed by an arbitrary union of open balls. We can think of open sets as the interior of some $n-1$ dimensional closed surface. The definition of open sets is necessary to define Charts. Charts or coordinate systems are a subset U of setting M with a one-to-one map $\phi : U \longmapsto \mathbf{R}$ such that the image $\phi(U)$ is an open set in $\mathbf{R}^n$. Charts are just coordinate systems on some open set. Fig 1.2 shows a definition of chart. A collection of charts is referred to as an Atlas.

Another important part in the definition is the chain rule. Suppose we have maps $f : \mathbf{R}^m \longmapsto \mathbf{R}^n$ and $g : \mathbf{R}^n \longmapsto \mathbf{R}^l$ where the composition $(g \circ f) : \mathbf{R}^m \longmapsto \mathbf{R}^l$. We label each point in space with x^α, y^β and z^γ components in $\mathbf{R}^m$, $\mathbf{R}^n$ and $\mathbf{R}^l$ respectively. The chain rule that relates the partial derivatives of the composition is as follows,

$$\frac{\partial}{\partial x^\alpha}(g \circ f)^\gamma = \sum_\beta \frac{\partial f^\beta}{\partial x^\alpha}\frac{\partial g^\gamma}{\partial y^\beta}$$

Now we are ready to begin discussing objects that live on manifolds, examples include vectors and tensors. Vectors now have a different notion of what we used to. We used to define vectors in the usual Euclidean space as any object that points from one point to another. We now have to discard this notion and accept that each point has vectors of its own that do not necessarily point to anything in particular. Each point can have an infinite number of associated vectors. We refer to the sets of vectors at a point p on manifold M as the tangent space $T_p(M)$. We now want to construct $T_p(M)$ with objects that are intrinsic to M. Let us define $\mathscr{F}$ to be all C^∞ maps $f : M \longmapsto \mathbf{R}$. We can notice that each curve through p defines an operator that points to a certain direction along the curve. This is the directional derivative that maps $f \longmapsto \frac{df}{d\lambda}$ evaluated at p. We can, thus, claim that the tangent space $T_p(M)$

can be constructed using the space of the directional derivative operator along curves that passes through p. The space of directional derivative is a vectors space. We need, now, to check if it can represent the tangent space. The most obvious answer to this inquiry is to define a basis for the space. Suppose that we have a coordinate system x^μ on manifold M. There exist an obvious set of the basis for this space, which is the set of partial derivative concerning x^μ. To prove this, lets consider an n-Manifold M with coordinates $\phi : M \mapsto \mathbf{R}^n$, a curve $\gamma : \mathbf{R} \mapsto M$ and a function $f : M \mapsto \mathbf{R}$. Now, we need to prove that if λ is the parameter along γ then, it can be represented by partial derivative operators.

$$\begin{aligned}\frac{df}{d\lambda} &= \frac{d}{d\lambda}(f \circ \gamma) \\ &= \frac{d}{d\lambda}((f \circ \phi^{-1}) \circ (\phi \circ \gamma)) \\ &= \frac{\partial(f \circ \phi^{-1})}{\partial x^\mu}\frac{d(\phi \circ \gamma)^\mu}{d\lambda} \\ &= \frac{dx^\mu}{d\lambda}\partial_\mu f\end{aligned}$$

The first line just defines the derivative operations, since we need the derivative to take $\mathbf{R} \mapsto \mathbf{R}$. In the second line, we have done nothing by inserting ϕ and its inverse in between. The third line is the chain rule. Thus, we have

$$\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda}\partial_\mu$$

where the partial derivative represent a suitable basis for the vector space of the directional derivatives and thus the tangent space. Thus, a vector at a point can be thought of as a directional derivative operator along a path that passes through this point. This shows that vectors map smooth functions to smooth functions by taking their derivative.

Another important thing is one-forms that live in the cotangent space T_P^*. One forms can be thought of as a map that takes vectors and produces scalars $\omega : T_p \mapsto \mathbf{R}$. Equivalently, we can think of vectors as linear maps for one forms. One can easily refer these definitions to the usual representation of rows and columns to used to construct. Much like the partial derivative along a curve present a good basis for vectors, the gradients dx^μ represents good basis for the forms. This directly related to the notions of covariant and contravariant vectors we used to [2]. It now natural to define tensors of (k,l) rank as follows,

$$T^{\mu'_1,\mu'_2,\cdots,\mu'_k}{}_{\nu'_1,\nu'_2,\cdots,\nu'_l} = \frac{\partial x^{\mu'_1}}{\partial x^{\mu_1}} \cdots \frac{\partial x^{\mu'_k}}{\partial x^{\mu_k}} \frac{\partial x^{\nu'_1}}{\partial x^{\nu_1}} \cdots \frac{\partial x^{\nu'_l}}{\partial x^{\nu_l}} T^{\mu_1,\mu_2,\cdots,\mu_k}{}_{\nu_1,\nu_2,\cdots,\nu_l}$$

Naturally, we cannot perform this operation of tensor unless we are working on the same manifold. We are doing a coordinate transformation.

Another important notion is that of the metric. The metric is a second rank tensor that is usually represented by $g_{\mu\nu}$. We can think of the metric as an object that helps us calculate distance on the manifold. Manifolds are specifically defined by certain metrics. Association of the metric in the definition is crucial since measuring distance on Riemannian manifolds is different from Lorentzian counterpart, due to their different signatures. Throughout this report, all our will be on Lorentzian manifolds. Let us probably define Lorentzian manifolds. Lorentzian manifolds are necessary to the study of general relativity since the equivalence principle requires geometry to look locally like that of Minkowski no Euclidean as a generalization of special relativity. A Lorentzian manifold M is a differentiable manifold with dimensions $(n+1)$ aided with a Lorentzian metric with a signature $(-,+,\cdots,+)$. Mostly, we will consider the case (3+1) dimensional Lorentzian manifold.

1.2 Geodesics and Geodesic Deviation

Geodesics are basically curves that extremize distances between fixed points. [3] Let λ be an arbitrary parameter that defines the curve between to points a_0 and a_1. We can define the length between the two points as,

$$s = \int_{a_0}^{a_1} \sqrt{g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}d\lambda$$

Using the variational principle $\delta s = 0$, we can note that the Lagrangian is $L = \sqrt{g_{\mu\nu}\dot{x}^\nu\dot{x}^\mu}$, putting that into Euler Lagrange's equation,

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) - \frac{\partial L}{\partial x^\alpha} = 0$$

Substituting term by term we get,

$$\frac{\partial L}{\partial x^\alpha} = \frac{1}{2}\frac{1}{L}\left(\frac{\partial g_{\mu\nu}}{\partial x^\alpha}\dot{x}^\mu\dot{x}^\nu\right)$$

$$\frac{\partial L}{\partial \dot{x}^\alpha} = \frac{1}{L}(g_{\mu\alpha}\dot{x}^\mu)$$

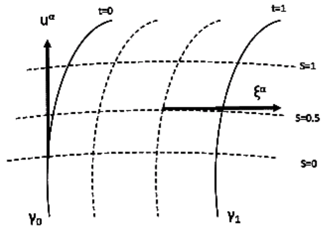

Figure 1.3: A graph showing tangent vector u^α to the geodesics with the geodesics deviation vector ξ^α.

$$\begin{aligned}
\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) &= -\frac{1}{L^2}\frac{\partial L}{\partial \lambda}(g_{\mu\alpha}\dot{x}^\mu) + \frac{1}{L}\frac{d}{d\lambda}\left(\frac{1}{2}g_{\nu\alpha}\dot{x}^\nu + \frac{1}{2}g_{\mu\alpha}\dot{x}^\mu\right) + \frac{1}{L}(g_{\mu\alpha}\ddot{x}^\mu) \\
&= -\frac{1}{L^2}\frac{\partial L}{\partial \lambda}(g_{\mu\alpha}\dot{x}^\mu) + \frac{1}{L}\left(\frac{1}{2}\frac{dg_{\nu\alpha}}{dx^\mu}\dot{x}^\mu\dot{x}^\nu + \frac{1}{2}\frac{dg_{\mu\alpha}}{dx^\nu}\dot{x}^\nu\dot{x}^\mu\right) + \frac{1}{L}(g_{\mu\alpha}\ddot{x}^\mu)
\end{aligned}$$

substituting this result into the equation, we find

$$g_{\mu\alpha}\ddot{x}^\mu + \frac{1}{2}\left(\frac{dg_{\nu\alpha}}{dx^\mu} + \frac{dg_{\mu\alpha}}{dx^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\right)\dot{x}^\mu\dot{x}^\nu = \frac{1}{L}\frac{\partial L}{\partial \lambda}(g_{\mu\alpha}\dot{x}^\mu)$$

Multiplying both sides by $g^{\kappa\alpha}$,

$$\ddot{x}^\kappa + \frac{1}{2}\Gamma^\kappa_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = k(\lambda)\dot{x}^\kappa$$

The geodesic equation can also be written as $u^\alpha_{;\beta}u^\beta = k(\lambda)u^\alpha$, where $u^\alpha = \dot{x}^\alpha$ is the tangent to the geodesic. The choice of the parameterization λ does't affect the equation, however, it may simplifies its structure. For time-like geodesics, we can choose the parameter as the proper time τ. This will give $L =$ constant killing the right hand side.

$$\ddot{x}^\kappa + \frac{1}{2}\Gamma^\kappa_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0$$

An interesting way to illustrate the meaning of the Riemann curvature tensor is through geodesics. Consider geodesics γ_0 and γ_1 as shown in Fig 1.3. Each geodesics is described by a parameter t, $x^\alpha(t)$. Additionally, to build the entire family of geodesics between these two we need another parameter s. Thus, we can collectively describe the entire family of geodesics using relations that involve $x^\alpha(s,t)$. Let the vector $u^\alpha = \partial x^\alpha/\partial t$ be tangent to the geodesics and satisfy the geodesic equation $u^\alpha_\beta u^\beta = 0$. Now, to move between geodesics we need another vector field $\xi^\alpha = \partial x^\alpha/\partial s$. these curves will not be geodesics but will permit us to get a meaningful representation for the Riemannian tensor. For an initial understanding of whats is going on, one can imagine geodesics in flat space (straight lines). Consider connecting two lines together and recording the rate at which these lines move apart. If we desired geodesics to be parallel at the starting point, these will be parallel all the time and we will record no rate of change. On the other hand, imagine yourself with a friend standing on the equator of a sphere at near points. As you move on the surface, your pathways seem to stay parallel, however, you will end by converging at one of the poles. Thus, the rate at which geodesics on a Manifold move or come closer gives an intuitive understating of how much the Manifold is truly curved. Thus, we require to find the acceleration at which move apart from each other,

$$\frac{D^2\xi^\alpha}{dt^2} = (\xi^\alpha_{;\beta}u^\beta)_{;\gamma}u^\gamma$$

Since $u^\alpha = \partial x^\alpha/\partial t$ and $\xi^\alpha = \partial x^\alpha/\partial s$,

$$\frac{\partial \xi^\alpha}{\partial t} = \frac{\partial^2 x^\alpha}{\partial s \partial t} = \frac{\partial^2 x^\alpha}{\partial t \partial s} = \frac{\partial u^\alpha}{\partial s}$$

or

$$\xi^\alpha_{;\beta}u^\beta = u^\alpha_{;\beta}\xi^\beta$$

First, we will use this to prove that $\xi^\alpha u_\alpha$ is a constant along the geodesics.

$$\begin{aligned}
\frac{\partial}{\partial t}(\xi^\alpha u_\alpha) &= (\xi^\alpha u_\alpha)_{;\beta}u^\beta \\
&= \xi^\alpha_{;\beta}u^\beta u_\alpha + \xi^\alpha u_{\alpha;\beta}u^\beta \\
&= u^\alpha_{;\beta}\xi^\beta u_\alpha \\
&= \frac{1}{2}(u^\alpha u_\alpha)_{;\beta}\xi^\beta \\
&= 0
\end{aligned}$$

In the second line, we killed the right term since it is the geodesic equation and commuted the covariant derivative as indicated above. The last equation is zero since $u^\alpha u_\alpha$ is a constant. Thus, we can see that $\xi^\alpha u_\alpha$ is also a constant. Thus, we can rewrite ξ^α in terms of orthogonal and parallel components to u^α.

$$\xi^\alpha = \lambda u^\alpha + \tilde{\xi}^\alpha$$

We can always with proper parameterization and choosing of initial conditions set $\lambda = 0$ with no change of the previous relations. Thus, we can always replace ξ^α with $\tilde{\xi}^\alpha$ making $\xi^\alpha u_\alpha = 0$. This directly implies that the s curves (t is constant) cross the geodesic curves orthogonally. We now can find the acceleration,

$$\begin{aligned}
\frac{D^2\xi^\alpha}{dt^2} &= (\xi^\alpha_{;\beta}u^\beta)_{;\gamma}u^\gamma \\
&= (u^\alpha_{;\beta}\xi^\beta)_{;\gamma}u^\gamma \\
&= u^\alpha_{;\beta;\gamma}\xi^\beta u^\gamma + \xi^\beta_{;\gamma}u^\gamma u^\alpha_{;\beta} \\
&= (u^\alpha_{;\gamma;\beta} - R^\alpha_{\mu\beta\gamma}u^\mu)\xi^\beta u^\gamma + u^\beta_{;\gamma}\xi^\gamma u^\alpha_{;\beta} \\
&= (u^\alpha_{;\gamma}u^\gamma)_{;\beta}\xi^\beta - u^\alpha_{;\gamma}u^\gamma_{;\beta}\xi^\beta - R^\alpha_{\mu\beta\gamma}u^\mu\xi^\beta u^\gamma + u^\alpha_{;\gamma}u^\gamma_{;\beta}\xi^\beta \\
&= -R^\alpha_{\mu\beta\gamma}u^\mu\xi^\beta u^\gamma
\end{aligned}$$

In the fourth line, we used the commutative relation between the covariant derivatives. This relation will be explicitly proved in the symmetries section. Thus, we have shown that geodesics deviation gives a sense of curvature inside space.

1.3 Hypersurfaces

In this section, we will introduce a brief direction of what is a hypersurface and derive some formulas that we will need later in our discussion. The discussion that follows is an excerpt from Poisson [3] and Carroll [2].

In an n-dimensional manifold, a hypersurfaces is an (n-1)-dimensional sub-manifold which can be time-like, space-like or null depending on the nature of its normal vector. A hypersurfaces Σ is specified by putting a restriction on the coordinates thus,

$$\Phi(x^\alpha) = 0$$

or by parameterize the surface using another variables,

$$x^\alpha = x^\alpha(y^\alpha)$$

An intuitive example is that of S^2 sphere in 3-dimensional space. We can clearly define the surface of the sphere using $x^2 + y^2 + z^2 - R^2 = 0$, where R is the radius of the sphere. Additionally, we can parameterize the surface using, $x = R\sin\theta\cos\phi$, $y = R\sin\theta\sin\phi$ and $z = R\cos\theta$. These two types of description give us two usual tools to work on the hypersurfaces, which are normal and tangent vectors.

Clearly, we can find normal vectors of the hypersurfaces by taking the gradient of the function Φ. Thus, $\Phi_{,\alpha}$ is the normal vector to the hypersurfaces. Additionally, we can normalize this normal vector as follows,

$$n^\alpha = \pm \frac{\Phi_{,\alpha}}{\sqrt{g^{\mu\nu}\Phi_{,\mu}\Phi_{,\nu}}}$$

where the $\pm$ is to indicate if the vector is space-like or time-like. Imagining a light cone, one can easily, interpret that space-like hypersurfaces have time-like normal vectors and vice versa.

There is a special kind of hypersurfaces known as null hypersurfaces. Null hypersurfaces are objects of great interest and they are important especially in studying black hole horizons. A Null hypersurface is a surface which has null normal vectors (i.e $n^\alpha n_\alpha = 0$). Such surface is that of the light cone itself where the time component exactly cancels out the spatial components.

Suppose we have a null hypersurfaces Σ. If the tangent vector on the surface is ξ^α, then by definition $\xi^\alpha n_\alpha = 0$. However, since n^α is null, it is tangent to itself. [4].

$$n^\alpha = \frac{\partial x^\alpha}{\partial t}$$

where the surface Σ is parameterized by by parameter t. The nature of the geodesics formed by $x^\alpha(t)$ is null. We can easily prove this. Suppose that $n_\alpha = f\phi_{,\alpha}$ where f is an arbitrary function. We can the geodesic equation as,

$$\begin{aligned}
n_{\alpha;\mu}n^\mu &= n^\mu \partial_\mu f \phi_{,\alpha} + n^\mu f \phi_{,\alpha;\mu} \\
&= n^\mu \partial_\mu f \phi_{,\alpha} + n^\mu f \phi_{,\mu;\alpha} \\
&= n^\mu n_\alpha \frac{\partial_\mu f}{f} + n^\mu f \left(\frac{n_\mu}{f}\right)_{;\alpha} \\
&= n^\mu n_\alpha \partial_\mu \ln(f) + n^\mu n_{\mu;\alpha} - n^\mu n_\mu \partial_\alpha \ln(f) \\
&= n^\mu n_\alpha \partial_\mu \ln(f) + \frac{1}{2}(n^\mu n_\mu)_{;\alpha} - n^\mu n_\mu \partial_\alpha \ln(f) \\
&= k(t) n_\alpha
\end{aligned}$$

In the second line, we used the fact that we can commute derivative on scalar functions since clearly there is no Christoffel symbol. In the last line, we have substituted the definition that n^α is null. Thus, we have the geodesic equation proportional to the vector itself. We can then use appropriate parameterization to make it a null one $n_{\alpha;\mu}n^\mu = 0$.

A convenient application that utilizes hypersurfaces is that of Gaussian normal coordinates. This coordinate system is naturally adapted to some hypersurface Σ. All we need to do is follow certain steps which will gravely reduce the form of the

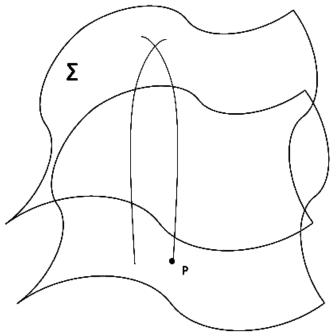

Figure 1.4: Schematic diagram showing the path of geodesics through the submanifolds.

metric. First, we need to specify a hypersurface using the definition we stated before. We choose charting coordinates lets say $\{y^1, y^2, \cdots, y^{n-1}\}$. We, naturally, adapt a normal vector n^α. At each point $p \in \Sigma$, we construct the geodesic for which n^α is a tangent vector and parameterize it with a parameter ,lets say, z. From this we can see that be specifying $\{z, y^1, y^2, \cdots, y^{n-1}\}$ we cover the whole n-dimensional space. Fig 1.4 illustrates the batching of space according to Gaussian normal coordinates. An important condition for the definition of these coordinates is that geodesics must no intersect at a converging point.

with each coordinate function, as mentioned, comes a basis vector, we shall label them as follows,

$$(\partial_z)^\mu = n^\mu$$
$$(\partial_i)^\mu = \mathbf{X}^\mu$$

Let us now construct the metric components, $g_{\mu\nu} = e_\mu \cdot e_\nu$,

$$g_{zz} = (\partial_z) \cdot (\partial_z) = n^\mu n_\mu = \pm 1$$

Now be considering the argument found in the begin of the section and use of Frobenius theorem [1], we can always redefine the parameterization such that,

$$g_{zi} = (\partial_z) \cdot (\partial_i) = n^\mu \mathbf{X}_\mu = 0$$

Thus, now we can write the metric as,

$$ds^2 = \pm dz^2 + \gamma_{ij} dy^i dy^j$$

If the normal vector is not normalized the metric would have the form,

$$ds^2 = C dz^2 + \gamma_{ij} dy^i dy^j$$

where C is an arbitrary constant.

Chapter 2

Symmetries of Einstein Field Equation

Defining symmetries inside the space one works with aid significantly in finding solutions and solving the problem. The mere problem that arises in this context of finding symmetries of a certain manifold is that the mathematical formalism of symmetry arises by establishing an adequate coordinate system. Thus, a legitimate approach should be considered to impose a certain structure on the metric when assuming certain symmetry. In the following, we begin by defining the abstract definition of symmetry of scalars and continue with the discussion of symmetries of the metric. Finally, we use these symmetries to derive a histrionically important solution: the Schwarzschild solution.

A continuous symmetry is imposed for a scalar function by equality upon a certain coordinate transformation. A coordinate transformation $x^\mu \longleftrightarrow {x'}^\mu$ would certainly change the functional form of the function T to T'. However, this change of coordinates should not affect the physical value of the scalar function. Formally speaking, the coordinates react in such a manner that leaves the value invariant. Mathematically, this can be represented through the following,

$$T'(x') = T(x)$$

An elaborate example can be put forward by considering a temperature density map. Points on the density map exist physically accompanied by a certain value of temperature. using different coordinate systems changes the labels we associate to every. However, upon changing labels, the function form of the temperature function changes thus leaving the map invariant.

Nevertheless, if we want to define symmetry inside a scalar map. We need the evaluation of the scalar function to be invariant under certain infinitesimal displacements. Equivalently, the functional form may change while referring to the same point. Thus,

$$T'(x) = T(x)$$

In correspondence with the above argument, we can establish a formula for defining symmetry of spaces. The role of symmetry inside space is to leave the Einstein equation invariant under coordinate transformations. Suppose we are looking for empty space solution where $T_{\mu\nu} = 0$. Thus, the Einstein equation reads,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0$$

To leave this equation invariant,we may consider looking for symmetries of the Ricci tensor where $R'_{\mu\nu}(x) = R_{\mu\nu}(x)$. This will directly leave the scalar invariant. However, this will affect the form of the metric since the Ricci tensor is a function of the metric,

$$R_{\mu\nu} = \partial_k\Gamma^k_{\nu\mu} - \partial_\mu\Gamma^k_{\nu k} + \Gamma^k_{k\sigma}\Gamma^\sigma_{\mu\nu} + \Gamma^k_{\mu\sigma}\Gamma^\sigma_{k\nu}$$

Since Γs are functions of the metric, any other symmetry operation rather than on the metric itself will change the form of the Einstein equation. Thus, we need to find symmetries on the metric itself.

A symmetric space is directly reflected from the construction of the metric, the metric itself should be form-invariant under certain coordinate transformation.

$$g'_{\mu\nu}(x) = g_{\mu\nu}(x) \tag{2.1}$$

unlike scalar symmetry, this a component by component symmetry of the metric. Additionally, we can write the transformed metric as follows,

$$g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}g_{\rho\sigma}(x)$$

We can use (1) to obtain a condition on the metric as follows,

$$g_{\mu\nu}(x) = \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu}g_{\rho\sigma}(x') \tag{2.2}$$

Equation (2) is a very complicated function of x^μ. However, we can reduce such complexity by considering a special case of infinitesimal coordinate transformations,

$$x'^\mu = x^\mu + \epsilon\xi^\mu(x) \qquad \epsilon \ll 1$$

Expanding equation (2) to first order of ϵ,

$$
\begin{aligned}
g_{\mu\nu}(x) &= \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} g_{\rho\sigma}(x') \\
&= \left(\delta^{\rho}_{\mu} + \epsilon \frac{\partial \xi^{\rho}}{\partial x^{\mu}}\right) \left(\delta^{\sigma}_{\nu} + \epsilon \frac{\partial \xi^{\sigma}}{\partial x^{\nu}}\right) g_{\rho\sigma}(x^{k} + \epsilon \xi^{k}(x)) \\
&= \left(\delta^{\rho}_{\mu} + \epsilon \frac{\partial \xi^{\rho}}{\partial x^{\mu}}\right) \left(\delta^{\sigma}_{\nu} + \epsilon \frac{\partial \xi^{\sigma}}{\partial x^{\nu}}\right) \left(g_{\rho\sigma}(x) + \epsilon \xi^{k}(x) \frac{\partial g_{\rho\sigma}}{\partial x^{k}} + O(\epsilon^{2})\right) \\
&= g_{\mu\nu}(x) + \epsilon \xi^{k}(x) \frac{\partial g_{\mu\nu}(x)}{\partial x^{k}} + \epsilon g_{\mu\sigma}(x) \frac{\partial \xi^{\sigma}(x)}{\partial x^{\nu}} + \epsilon g_{\rho\nu}(x) \frac{\partial \xi^{\rho}(x)}{\partial x^{\mu}} + O(\epsilon^{2})
\end{aligned}
$$

Eliminating $g_{\mu\nu}(x)$ from both sides, dividing by ϵ and using symmetry of the metric with renaming $\nu \longleftrightarrow \rho$ and $\mu \longleftrightarrow \sigma$, one can reach the following form,

$$
\xi^{k}(x) \frac{\partial g_{\sigma\rho}(x)}{\partial x^{k}} + g_{\mu\sigma}(x) \frac{\partial \xi^{\mu}(x)}{\partial x^{\rho}} + g_{\rho\nu}(x) \frac{\partial \xi^{\nu}(x)}{\partial x^{\sigma}} = 0 \tag{2.3}
$$

using $\xi_{\sigma} = g_{\mu\sigma} \xi^{\mu}$, we can rewrite the above equation in a covariant derivative form:

$$
\begin{aligned}
0 &= \xi^{k} \frac{\partial g_{\sigma\rho}}{\partial x^{k}} + \left(\frac{\partial \xi_{\sigma}}{\partial x^{\rho}} - \xi^{\mu} \frac{\partial g_{\mu\sigma}}{\partial x^{\rho}}\right) + \left(\frac{\partial \xi_{\rho}}{\partial x^{\sigma}} - \xi^{\nu} \frac{\partial g_{\rho\nu}}{\partial x^{\sigma}}\right) \\
&= \frac{\partial \xi_{\sigma}}{\partial x^{\rho}} + \frac{\partial \xi_{\rho}}{\partial x^{\sigma}} + \xi^{\mu} \left(\frac{\partial g_{\sigma\rho}}{\partial x^{\mu}} - \frac{\partial g_{\mu\sigma}}{\partial x^{\rho}} - \frac{\partial g_{\rho\nu}}{\partial x^{\sigma}}\right) \\
&= \frac{\partial \xi_{\sigma}}{\partial x^{\rho}} + \frac{\partial \xi_{\rho}}{\partial x^{\sigma}} - 2 \xi^{\mu} g_{\mu k} \Gamma^{k}_{\rho\sigma} \\
&= \frac{\partial \xi_{\sigma}}{\partial x^{\rho}} + \frac{\partial \xi_{\rho}}{\partial x^{\sigma}} - 2 \xi_{\mu} \Gamma^{\mu}_{\rho\sigma} \\
&= \xi_{\sigma;\rho} + \xi_{\rho;\sigma}
\end{aligned}
$$

This is what is known as the killing equation after German mathematician Wilhelm Killing. A vector that satisfies this equation is referred to as the killing vector. Now, the problem of determining the isotherm of a given metric is reduced to finding these killing vectors. The killing condition is very restrictive as it allows up to determine the vector from just knowing its value and the value of its covariant derivative at some point p. To see this, we need to find the commutator of the covariant derivative,

$$\begin{aligned}
\xi_{\sigma;\rho;\mu} - \xi_{\sigma;\mu;\rho} &= \left(\partial_\rho \xi_\sigma - \Gamma^\lambda_{\sigma\rho}\xi_\lambda\right)_{;\mu} - \left(\partial_\mu \xi_\sigma - \Gamma^\lambda_{\sigma\mu}\xi_\lambda\right)_{;\rho} \\
&= \partial_\mu\partial_\rho\xi_\sigma - \Gamma^k_{\rho\mu}\partial_k\xi_\sigma - \Gamma^k_{\sigma\mu}\partial_\rho\xi_k - \partial_\mu\Gamma^\lambda_{\sigma\rho}\xi_\lambda + \Gamma^k_{\mu\sigma}\Gamma^\lambda_{k\rho}\xi_\lambda \\
&\quad + \Gamma^k_{\mu\rho}\Gamma^\lambda_{\sigma k}\xi_\lambda - \partial_\rho\partial_\mu\xi_\sigma + \Gamma^k_{\mu\rho}\partial_k\xi_\sigma + \Gamma^k_{\sigma\rho}\partial_\mu\xi_k + \partial_\rho\Gamma^\lambda_{\sigma\mu}\xi_\lambda \\
&\quad - \Gamma^k_{\rho\sigma}\Gamma^\lambda_{k\mu}\xi_\lambda - \Gamma^k_{\mu\rho}\Gamma^\lambda_{\sigma k}\xi_\lambda \\
&= \left(-\partial_\mu\Gamma^\lambda_{\sigma\rho} + \Gamma^k_{\mu\sigma}\Gamma^\lambda_{k\rho} - \Gamma^k_{\rho\sigma}\Gamma^\lambda_{k\mu} + \partial_\rho\Gamma^\lambda_{\sigma\mu}\right)\xi_\lambda \\
&= -R^\lambda_{\sigma\rho\mu}\xi_\lambda
\end{aligned} \tag{2.4}$$

where in the third line we used the fact that Γ is symmetric in the lower indices and eliminated the like terms. With the aid of the cyclic property of the curvature tensor,

$$R^\lambda_{\sigma\rho\mu} + R^\lambda_{\mu\sigma\rho} + R^\lambda_{\rho\mu\sigma} = 0$$

Thus,

$$\xi_{\sigma;\rho;\mu} - \xi_{\sigma;\mu;\rho} + \xi_{\mu;\sigma;\rho} - \xi_{\mu;\rho;\sigma} + \xi_{\rho;\mu;\sigma} - \xi_{\rho;\sigma;\mu} = -\left(R^\lambda_{\sigma\rho\mu} + R^\lambda_{\mu\sigma\rho} + R^\lambda_{\rho\mu\sigma}\right)\xi_\lambda = 0$$

and using the killing condition,

$$\xi_{\sigma;\rho} + \xi_{\rho;\sigma} = 0$$

we can rewrite the above equation as,

$$\begin{aligned}
0 &= \left(\xi_{\sigma;\rho} - \xi_{\rho;\sigma}\right)_{;\mu} + \left(-\xi_{\sigma;\mu} + \xi_{\mu;\sigma}\right)_{;\rho}\left(-\xi_{\mu;\rho} + \xi_{\rho;\mu}\right)_{;\sigma} \\
&= 2\xi_{\sigma;\rho;\mu} - 2\xi_{\sigma;\mu;\rho} - 2\xi_{\mu;\rho;\sigma}
\end{aligned}$$

From this we can see that,

$$\xi_{\sigma;\rho;\mu} - \xi_{\sigma;\mu;\rho} = \xi_{\mu;\rho;\sigma}$$

which produces the following relation of the second covariant derivative of the killing vectors,

$$\xi_{\mu;\rho;\sigma} = -R^\lambda_{\sigma\rho\mu}\xi_\lambda \tag{2.5}$$

This equation is very powerful in a sense that it permits us to find the all the higher order derivative of ξ at some point p from just knowing the value of the vector and its first covariant derivative at that point. Nevertheless, we can construct the whole vector $\xi_\mu(x)$ from just knowing its value and the of its first covariant derivative at that some point p. We can Taylor expand the vector around point p as follows,

$$\xi_\mu(x) = \xi_\mu(p) + a^\nu\xi_{\mu;\nu}(p) + b^{\nu k}\xi_{\mu;\nu;k}(p) + \cdots$$

However, since all the higher derivatives at point p is known from the previous relations, we can in general write the killing vector as follows,

$$\xi_\mu(x) = A^\lambda_\mu(x)\xi_\lambda(p) + B^{\lambda\nu}_\mu(x)\xi_{\lambda;\nu}(p) \tag{2.6}$$

The explicit values of the A and B functions can be found from the Taylor expansion. Equation 2.6 tells us very important information about the maximal number of killing vectors in a given space of N dimensions. In this space, $\xi_\mu(p)$ has N independent components. However, its covariant derivative has $N(N-1)/2$ components due to the killing equation that asymmetrizes the indices. Thus, we can define the killing vector using,

$$N + \frac{N(N-1)}{2} = \frac{N(N+1)}{2}$$

independent components. From there we can construct a set of killing vectors $\{\xi^{(n)}_\mu\}$, where the initial condition for each vector at point p is specified by the components of $\xi^{(n)}_\mu(p)$ and $\xi^{(n)}_{\mu;\nu}(p)$. Since the vector space of D dimensions has D linearly independent components, the maximum number of killing vectors that can exist in an N-dimensional space is $N(N+1)/2$. A space that admits a maximum number of killing vectors is referred to as a maximally symmetric space.

2.1 Maximally symmetric spaces

The study of maximally symmetric space is particularly important in cosmology and the study of black holes. Most of the studies concentrate on studying homogeneous and isotropic spaces. A homogeneous space is a space that admits killing vectors $\xi^{(n)}_\lambda$ that can take any point p to any point q thus producing an isometry in the metric. Another symmetry of the space is isotropic. Suppose at some point p there exist a time-like vector $\mathbf{u}$ and two orthogonal space-like tangent vectors $\mathbf{s_1}$ and $\mathbf{s_2}$. An isometry that changes $\mathbf{s_1}$ into $\mathbf{s_2}$ leaving $\mathbf{u}$ and p unchanged is an isometry of space. This constraint $\xi_\mu(p)$ to be zero as not to translate the point. However, this leaves $\xi_{\mu;\nu}$ to be arbitrary as to cover all possible rotations.

Our main objective of this discussion is constructing the metric given certain symmetry of space (i.e specify killing vectors). Following same procedures we used in 2.4, we can find the following relation,

$$\xi_{\rho;\nu;\mu;\sigma} - \xi_{\rho;\nu;\sigma;\mu} = -R^\lambda_{\rho\mu\sigma}\xi_{\lambda;\nu} - R^\lambda_{\nu\mu\sigma}\xi_{\rho;\lambda} \tag{2.7}$$

and taking the covariant derivative of 2.5 we obtain,

$$\xi_{\mu;\rho;\sigma;\nu} = -R^\lambda_{\sigma\rho\mu;\nu}\xi_\lambda - R^\lambda_{\sigma\rho\mu}\xi_{\lambda;\nu}$$

substituting in equation 2.7,

$$-R^{\lambda}_{\mu\nu\rho;\sigma}\xi_\lambda - R^{\lambda}_{\mu\nu\rho}\xi_{\lambda;\sigma} + R^{\lambda}_{\sigma\nu\rho;\mu}\xi_\lambda + R^{\lambda}_{\sigma\nu\rho}\xi_{\lambda;\mu} = -R^{\lambda}_{\rho\mu\sigma}\xi_{\lambda;\nu} - R^{\lambda}_{\nu\mu\sigma}\xi_{\rho;\lambda}$$

collecting like terms and using $\xi_{\rho;\lambda} = -\xi_{\lambda;\rho}$,

$$\left(-R^{k}_{\mu\nu\rho;\sigma}\delta^{k}_{\lambda} + R^{k}_{\sigma\nu\rho;\mu}\delta^{k}_{\lambda}\right)\xi_\lambda = \left(R^{\lambda}_{\mu\nu\rho}\delta^{k}_{\sigma} - R^{\lambda}_{\sigma\nu\rho}\delta^{k}_{\mu} - R^{\lambda}_{\rho\mu\sigma}\delta^{k}_{\nu} + R^{\lambda}_{\nu\mu\sigma}\delta^{k}_{\rho}\right)\xi_{\lambda;k}$$

We can immediately see that we can interpret information about the curvature from knowing the form of the killing vectors. To connect this to maximally symmetric spaces, we apply the condition of homogeneity and isotropic. Since, the space is isotropic about every point (i.e maximally symmetric) $\xi_\lambda = 0$ while $\xi_{\lambda;k}$ is arbitrary. This sets the write term on the right-hand side to zero. An elegant way of formalizing this is by requiring the symmetric part in λ and k of the right-hand side to vanish,

$$R^{\lambda}_{\mu\nu\rho}\delta^{k}_{\sigma} - R^{\lambda}_{\sigma\nu\rho}\delta^{k}_{\mu} - R^{\lambda}_{\rho\mu\sigma}\delta^{k}_{\nu} + R^{\lambda}_{\nu\mu\sigma}\delta^{k}_{\rho} = R^{k}_{\mu\nu\rho}\delta^{\lambda}_{\sigma} - R^{k}_{\sigma\nu\rho}\delta^{\lambda}_{\mu} - R^{k}_{\rho\mu\sigma}\delta^{\lambda}_{\nu} + R^{k}_{\nu\mu\sigma}\delta^{\lambda}_{\rho} \quad (2.8)$$

This requirement only considers isotropic spaces. However, if the space is also homogeneous, ξ_λ can have any arbitrary value. Keeping the above requirement, the left hand side is also identically zero,

$$-R^{k}_{\mu\nu\rho;\sigma}\delta^{k}_{\lambda} + R^{k}_{\sigma\nu\rho;\mu}\delta^{k}_{\lambda} = 0 \quad (2.9)$$

contracting equation 2.8 in k and σ,

$$NR^{\lambda}_{\mu\nu\rho} - R^{\lambda}_{\mu\nu\rho} - R^{\lambda}_{\rho\mu\nu} + R^{\lambda}_{\nu\mu\rho} = R^{\lambda}_{\mu\nu\rho} - R^{\sigma}_{\sigma\nu\rho}\delta^{\lambda}_{\mu} - R^{\sigma}_{\rho\mu\sigma}\delta^{\lambda}_{\nu} + R^{\sigma}_{\nu\mu\sigma}\delta^{\lambda}_{\rho}$$

Since $R^{\sigma}_{\sigma\nu\rho} = 0$ and $R^{\sigma}_{\rho\mu\sigma} = R_{\rho\mu}$ is by definition the Ricci tensor. Additionally, with the cyclic property of the curvature tensor,

$$R^{\lambda}_{\mu\nu\rho} + R^{\lambda}_{\rho\mu\nu} + R^{\lambda}_{\nu\rho\mu} = 0$$

and asymmetry in the curvature tensor $R^{\lambda}_{\nu\rho\mu} = -R^{\lambda}_{\nu\mu\rho}$. The equation now reads,

$$(N-1)R^{\lambda}_{\mu\nu\rho} = -R_{\rho\mu}\delta^{\lambda}_{\nu} + R_{\nu\mu}\delta^{\lambda}_{\rho}$$

multiplying both sides by $g_{\lambda k}$, we lower the indices in both sides,

$$(N-1)R_{k\mu\nu\rho} = -R_{\rho\mu}g_{\nu k} + R_{\nu\mu}g_{\rho k} \quad (2.10)$$

Since the left hand side is anti symmetric about k and μ, the right hand side should also satisfy this property,

$$-R_{\rho\mu}g_{\nu k} + R_{\nu\mu}g_{\rho k} = R_{\rho k}g_{\nu\mu} - R_{\nu k}g_{\rho\mu}$$

multiplying by $g^{k\rho}$ and summing,

$$-R_{\rho\mu}\underbrace{g_{\nu k}g^{k\rho}}_{\delta^{\rho}_{\nu}}+R_{\nu\mu}\underbrace{g_{\rho k}g^{k\rho}}_{\delta^{\rho}_{\rho}}=\underbrace{R_{\rho k}g^{k\rho}}_{R^{\rho}_{\rho}}g_{\nu\mu}-R_{\nu k}\underbrace{g_{\rho\mu}g^{k\rho}}_{\delta^{k}_{\mu}}$$

$$-R_{\nu\mu}+NR_{\nu\mu}=Rg_{\nu\mu}-R_{\nu\mu}$$
$$R_{\nu\mu}=\frac{R}{N}g_{\nu\mu} \tag{2.11}$$

Substituting this result in 2.10,

$$R_{k\mu\nu\rho}=\frac{R}{N(N-1)}\left[g_{\nu\mu}g_{\rho k}-g_{\rho\mu}g_{\nu k}\right] \tag{2.12}$$

This is the most restricted form of the remain curvature tensor in a maximally symmetric space. We can even move further to derive a very important results from the form of the Ricci tensor. Using Bianchi identity, we can derive this relation of the Ricci tensor,

$$\left(R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R\right)_{;\mu}=0$$

multiplying a factor of $g_{\nu\rho}$ results in,

$$\left(R^{\mu}_{\rho}-\frac{1}{2}\delta^{\mu}_{\rho}R\right)_{;\mu}=0$$

using equation 2.11,

$$R^{\mu}_{\rho}=\frac{R}{N}\delta^{\mu}_{\rho}$$

Substituting this result in the former equation, we get,

$$\begin{aligned}0&=\left(\frac{R}{N}\delta^{\mu}_{\rho}-\frac{1}{2}\delta^{\mu}_{\rho}R\right)_{;\mu}\\&=\frac{1}{N}R_{;\mu}-\frac{1}{2}R_{;\mu}\\&=\left(\frac{1}{N}-\frac{1}{2}\right)R_{;\mu}\\&=\left(\frac{1}{N}-\frac{1}{2}\right)\frac{\partial R}{\partial x^{\mu}}\end{aligned}$$

Thus, for $N\neq 2$, we have a very astonishing results of the constancy of the Ricci scalar,

$$\frac{\partial R}{\partial x^{\mu}}=0$$

We can now define the curvature in a maximally symmetric space as follows,

$$K = \frac{R}{N(N-1)}$$

where 2.12 now reads,

$$R_{k\mu\nu\rho} = K\left[g_{\nu\mu}g_{\rho k} - g_{\rho\mu}g_{\nu k}\right] \tag{2.13}$$

Weinberg [5] pointed out a very important result which is referred to as the theorem of metric uniqueness. Weinberg states that: given two maximally symmetric metrics with the same K and the same signature, it will always be possible to find a coordinate transformation that carries one metric into another.

The metric is defined, up to a coordinate transformation, by a curvature K and signature.

2.2 Construction of Maximally symmetric spaces

The construction of a general maximally symmetric space may seem a simple task, however, certain assumptions to be taken into consideration. A metric can be easily constructed to fulfill the homogeneous requirement. However, it is rather nonobvious to produce a metric that is additionally invariant under rotations. In the following, we follow the derivation highlighted by Weinberg.

We begin the construction using a general metric of (N+1) dimensions in Gaussian normal coordinates [2] as follows,

$$-d\tau^2 = g_{\mu\nu}dx^\mu dx^\nu = C_{\mu\nu}dx^\mu dx^\nu + K^{-1}dz^2 \tag{2.14}$$

We are aiming to form an isotropic and homogeneous space of N dimensions from this higher dimensional space. The idea behind this is rather a nontrivial one. The problem lies in our construction of isotropic spaces. A procedure could be produced by referring to our intuitive understanding of isotropy. The word isotropy is linked to visualization of a sphere in $\mathbb{R}^3$. From this intuitive thinking of isotropy, we can similarly restrain coordinates to that of the equation of a sphere. Formally, we are embedding an n-dimensional space into this higher space by restricting x^μ and z to a surface of a pseudosphere,

$$KC_{\mu\nu}x^\mu x^\nu + z^2 = 1$$

This will produce a surface in the space of the metric that will produce isotropy. We can use the above formula to find the representation of the metric on that surface,

$$2zdz = -KC_{\mu\nu}\left(x^\mu dx^\nu + x^\nu dx^\mu\right)$$

However, since $C_{\mu\nu}$ is symmetric.

$$2zdz = -2KC_{\mu\nu}x^\mu dx^\nu$$

$$dz^2 = \frac{K^2(C_{\mu\nu}x^\mu dx^\nu)^2}{z^2}$$

substituting this back into equation 2.14,

$$\begin{aligned} g_{\mu\nu}dx^\mu dx^\nu &= C_{\mu\nu}dx^\mu dx^\nu + \frac{K(C_{\mu\nu}x^\mu dx^\nu)^2}{z^2} \\ &= C_{\mu\nu}dx^\mu dx^\nu + \frac{KC_{\mu\lambda}x^\lambda C_{\nu k}x^k}{1 - KC_{\rho\sigma}x^\rho x^\sigma}dx^\mu dx^\nu \\ &= \left(C_{\mu\nu} + \frac{KC_{\mu\lambda}x^\lambda C_{\nu k}x^k}{1 - KC_{\rho\sigma}x^\rho x^\sigma}\right)dx^\mu dx^\nu \end{aligned}$$

Thus, the metric admits the following form,

$$g_{\mu\nu}(x) = C_{\mu\nu} + \frac{K}{1 - KC_{\rho\sigma}x^\rho x^\sigma}C_{\mu\lambda}x^\lambda C_{\nu k}x^k \tag{2.15}$$

Now, it is required that this metric should invariant under rotations and translations in space-time. This will put constraints on $C_{\mu\nu}$ as will be shown later. The metric should be invariant under these transformation,

$$x^\mu \longrightarrow x'^\mu = R^\mu_\nu x^\nu + R^\mu_z z$$

$$z \longrightarrow z' = R^z_\mu x^\mu + R^z_z z$$

where R^μ_ν are constants which produce translations and rotations in space time. We will now substitute this directly into the higher dimension metric.Any second rank tensor will satisfy the following axiom.

$$g_{\sigma\rho}(x)dx^\mu dx^\nu = g'_{\rho\sigma}(x')dx'^\rho dx'^\sigma$$

As, discussed before a symmetry would be induced if $g'_{\mu\nu}(x') = g_{\mu\nu}(x')$. Thus, in our case of the metric,

$$g_{\sigma\rho}(x)dx^\mu dx^\nu = g_{\rho\sigma}(x')dx'^\rho dx'^\sigma$$

Now substituting this directly in 2.14,

$$\begin{aligned}
g_{\rho\sigma}(x')dx'^{\rho}dx'^{\sigma} &= C_{\rho\sigma}dx'^{\rho}dx'^{\sigma} + K^{-1}dz'^{2} \\
&= C_{\rho\sigma}\left(R^{\rho}_{\nu}dx^{\nu} + R^{\rho}_{z}dz\right)\left(R^{\sigma}_{\mu}dx^{\mu} + R^{\sigma}_{z}dz\right) + K^{-1}\left(R^{z}_{\mu}dx^{\mu} + R^{z}_{z}dz\right)^{2} \\
&= C_{\rho\sigma}\left(R^{\rho}_{\nu}R^{\sigma}_{\mu}dx^{\mu}dx^{\nu} + R^{\rho}_{\nu}R^{\sigma}_{z}dzdx^{\nu} + R^{\rho}_{z}R^{\sigma}_{\mu}dx^{\mu}dz + R^{\rho}_{z}R^{\sigma}_{z}dz^{2}\right) \\
&\quad + K^{-1}\left(R^{z}_{\mu}R^{z}_{\nu}dx^{\mu}dx^{\nu} + 2R^{z}_{\mu}R^{z}_{z}dx^{\mu}dz + (R^{z}_{z})^{2}dz^{2}\right) \\
&= \left[C_{\rho\sigma}R^{\rho}_{\nu}R^{\sigma}_{\mu} + K^{-1}R^{z}_{\mu}R^{z}_{\nu}\right]dx^{\mu}dx^{\nu} \\
&\quad + \left[C_{\rho\sigma}(R^{\rho}_{\mu}R^{\sigma}_{z} + R^{\rho}_{z}R^{\sigma}_{\mu}) + 2K^{-1}R^{z}_{\mu}R^{z}_{z}\right]dx^{\mu}dz \\
&\quad + \left[C_{\rho\sigma}R^{\rho}_{z}R^{\sigma}_{z} + K^{-1}(R^{z}_{z})^{2}\right]dz^{2} \\
&= \left[C_{\rho\sigma}R^{\rho}_{\nu}R^{\sigma}_{\mu} + K^{-1}R^{z}_{\mu}R^{z}_{\nu}\right]dx^{\mu}dx^{\nu} \\
&\quad + \left[2C_{\rho\sigma}R^{\rho}_{\mu}R^{\sigma}_{z} + 2K^{-1}R^{z}_{\mu}R^{z}_{z}\right]dx^{\mu}dz + \left[C_{\rho\sigma}R^{\rho}_{z}R^{\sigma}_{z} + K^{-1}(R^{z}_{z})^{2}\right]dz^{2}
\end{aligned}$$

The last line is a consequence of the fact that $C_{\rho\sigma} = C_{\sigma\rho}$. Now, from the direct equality indicated above we can deduced the following relations,

$$C_{\rho\sigma}R^{\rho}_{\nu}R^{\sigma}_{\mu} + K^{-1}R^{z}_{\mu}R^{z}_{\nu} = C_{\mu\nu}$$

$$C_{\rho\sigma}R^{\rho}_{\mu}R^{\sigma}_{z} + K^{-1}R^{z}_{\mu}R^{z}_{z} = 0$$

$$C_{\rho\sigma}R^{\rho}_{z}R^{\sigma}_{z} + K^{-1}(R^{z}_{z})^{2} = K^{-1}$$

We can distinguish the two types of transformations by following the form of the Matrix R:

$$R^{\mu}_{\nu} = \mathfrak{R}^{\mu}_{\nu} \qquad R^{\rho}_{z} = R^{z}_{\rho} = 0 \qquad R^{z}_{z} = 1$$

Thus we have,

$$C_{\rho\sigma}\mathfrak{R}^{\rho}_{\nu}\mathfrak{R}^{\sigma}_{\mu} = C_{\mu\nu}$$

These are rigid rotations with,

$$x'^{\mu} = \mathfrak{R}^{\mu}_{\nu}x^{\nu}$$

$$z' = z$$

The other case is the case of translations,

$$R^{\mu}_{z} = a^{\mu} \qquad R^{z}_{\mu} = -KC_{\mu\nu}a^{\nu}$$

This will basically induce the following two relation from the equations above,

$$R^{z}_{z} = [1 - KC_{\rho\sigma}a^{\rho}a^{\sigma}]^{1/2} \qquad R^{\mu}_{\nu} = \delta^{\mu}_{\nu} - bKC_{\nu\rho}a^{\rho}a^{\nu}$$

where,

$$KC_{\rho\sigma}a^\rho a^\sigma \leq 1$$

$$b = \frac{1 - (1 - KC_{\sigma\rho}a^\rho a^\sigma)^{1/2}}{KC_{\sigma\rho}a^\rho a^\sigma}$$

These are translations of the form,

$$x'^\mu = x^\mu = a^\mu \left[(1 - KC_{\sigma\rho}a^\rho a^\sigma)^{1/2} - bKC_{\sigma\rho}x^\rho a^\sigma\right]$$

Thus, they take $x^\mu = 0$ into a^μ. We can also preform a brute force calculation of the Riemann christoffel symbol of the metric 2.15. The amazing results that we would find is the K introduced in the form of the metric is the same as that of the curvature in equation 2.13.

After ensuring that our new metric satisfies the maximally symmetric constraints, we return to discussing its form. It's in general well known that we can by similarity transformations change any symmetric matrix to any symmetric matrix we desire as long as we don't change the number of its positive and negative eigenvalues. This is known as Sylvester's law of inertia. In this manner, we consider transformations $C_{\mu\nu}$. If we consider that the metric allows the introduction to a locally Euclidean coordinate system, then we would have all of its eigenvalues to be positive. Thus, we can transform $C_{\mu\nu}$ into $m\delta^\mu_\nu$, where m is an arbitrary constant. Choosing $m = K^{-1}$ and substituting into equation 2.15.

$$\begin{aligned} g_{\mu\nu}(x) &= K^{-1}\delta^\mu_\nu + \frac{K}{1 - Kk^{-1}\delta^\rho_\sigma x^\rho x^\sigma} k^{-2}\delta^\mu_\lambda x^\lambda \delta^\nu_k x^k \\ &= K^{-1}\delta^\mu_\nu + \frac{K^{-1}}{1 - \mathbf{x}^2}\delta^\mu_\lambda x^\lambda \delta^\nu_k x^k \end{aligned}$$

Thus, we can write the metric as,

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = K^{-1}\left[d\mathbf{x}^2 + \frac{(\mathbf{x}\cdot d\mathbf{x})^2}{1 - \mathbf{x}^2}\right]$$

This will be in perfect argument with our intuition of flat spaces (i.e $K = 0$).

$$ds^2 = d\mathbf{x}^2$$

This form of the metric will always produce a maximally symmetric space in Euclidean manifolds of any curvature.

2.3 Tensors in maximally symmetric spaces

Our construction of the form of the metric in a maximally symmetric spaces can be readily extended to any tensor of any rank. Using the same symmetry arguments introduced in section 2.1 and going the infinitesimal transformations of a certain killing vector, one can find the an equivalent form of a tensor T of (0,n) rank.

$$0 = \frac{\partial \xi^{k_1}(x)}{\partial x^{a_1}} T_{k_1 a_2 a_3 \cdots a_n} + \frac{\partial \xi^{k_2}(x)}{\partial x^{a_2}} T_{a_1 k_2 a_3 \cdots a_n} + \cdots + \xi^{\lambda} \frac{\partial}{\partial x^{\lambda}} T_{a_1 a_2 a_3 \cdots a_n} \tag{2.16}$$

A Tensor is said to be maximally symmetric if it satisfies the above relation for all the N(N+1) killing vectors. In a maximally symmetric space, we can choose a vector ξ^{λ} which vanishes at some point s.

$$\xi^{\lambda}(s) = 0$$

as a consequence its covariant derivative is arbitrary,

$$\xi_{\sigma;\mu}(s) = g_{\sigma\lambda}(s) \left(\frac{\partial \xi^{\lambda}(x)}{\partial x^{\mu}} \right)_{x=s}$$

Using this, the last term in equation 2.16 will vanish. Additionally, every partial derivative term can be converted into covariant derivative due to the first condition. Thus, equation 2.16 can be written as follows at point s,

$$0 = \xi_{\sigma;\mu} \left[\delta^{\mu}_{a_1} T^{\sigma}_{a_2 a_3 \cdot a_n} + \delta^{\mu}_{a_2} T^{\sigma}_{a_1 a_3 \cdot a_n} + \cdots + \delta^{\mu}_{a_n} T^{\sigma}_{a_1 a_2 \cdot a_{n-1}} \right]$$

However, $\xi_{\sigma;\mu}$ is anti symmetric. Thus, the symmetric part of the second bracket vanishes.

$$\delta^{\mu}_{a_1} T^{\sigma}_{a_2 a_3 \cdot a_n} + \delta^{\mu}_{a_2} T^{\sigma}_{a_1 a_3 \cdot a_n} + \cdots + \delta^{\mu}_{a_n} T^{\sigma}_{a_1 a_2 \cdot a_{n-1}} = \delta^{\sigma}_{a_1} T^{\mu}_{a_2 a_3 \cdot a_n} + \delta^{\sigma}_{a_2} T^{\mu}_{a_1 a_3 \cdot a_n} + \cdots + \delta^{\sigma}_{a_n} T^{\mu}_{a_1 a_2 \cdot a_{n-1}}$$

This argument will hold for any arbitrary point s. By construction every point in a maximally symmetric space are the same being isotropic. Thus, argument hold everywhere.

An important consequence of this derivation is that it reflects on the form of an arbitrary second rank tensor. For a tensor $B_{\mu\nu}$, the previous equation reads,

$$\delta^{\mu}_{a_1} B^{\sigma}{}_{a_2} + \delta^{\mu}_{a_2} B_{a_1}{}^{\sigma} = \delta^{\sigma}_{a_1} B^{\mu}{}_{a_2} + \delta^{\sigma}_{a_2} B_{a_1}{}^{\mu}$$

contracting a_1 with μ,

$$\delta^{\mu}_{\mu} B^{\sigma}{}_{a_2} + \delta^{\mu}_{a_2} B_{\mu}{}^{\sigma} = \delta^{\sigma}_{\mu} B^{\mu}{}_{a_2} + \delta^{\sigma}_{a_2} B_{\mu}{}^{\mu}$$

$$NB^{\sigma}{}_{a_2} + B_{a_2}{}^{\sigma} = B^{\sigma}{}_{a_2} + \delta^{\sigma}_{a_2} B^{\mu}_{\mu}$$

multiplying by $g_{\sigma\nu}$,

$$NB_{\nu a_2} + B_{a_2 \nu} = B_{\nu a_2} + g_{a_2 \nu} B^{\mu}_{\mu}$$

$$(N-1)B_{\nu a_2} + B_{a_2 \nu} = g_{a_2 \nu} B^{\mu}_{\mu}$$

interchanging ν and a_2,

$$(N-1)B_{a_2 \nu} + B_{\nu a_2} = g_{\nu a_2} B^{\mu}_{\mu} \tag{2.17}$$

subtracting the two previous equations,

$$(N-2)(B_{a_2 \nu} - B_{\nu a_2}) = 0$$

As long as $N \neq 2$,$B_{\nu\mu}$ must be symmetric. Substituting this into equation 2.17,

$$NB_{\mu\nu} = B^k_k g_{\mu\nu}$$

or

$$B_{\mu\nu} = f g_{\mu\nu}$$

where f is some function. We can determine the form of the function by substituting into the killing equation.

$$\frac{\partial \xi^{\rho}}{\partial x^{\mu}} f g_{\rho\nu} + \frac{\partial \xi^{\sigma}}{\partial x^{\nu}} f g_{\mu\sigma} + \xi^{\lambda} \frac{\partial}{\partial x^{\lambda}} (f g_{\mu\nu}) = 0$$

However, $g_{\nu\mu}$ satisfies the equation independently. Thus, we are left with,

$$\xi^{\lambda} \frac{\partial f}{\partial x^{\lambda}} g_{\mu\nu} = 0$$

Since ξ^{λ} can have any value in a maximally symmetric space and $g_{\nu\mu}$ can be no zero, we are left with condition that,

$$\frac{\partial f}{\partial x^{\lambda}} = 0$$

This concludes that any second rank tensor in a maximally symmetric space is just a constant multiplied by the metric.

2.4 Spaces of Maximally symmetric subspace

Our argument thus far has only considered a maximally symmetric metric that looks locally like Euclidean space. However, we are dealing with spaces that look locally like Minkowski. Thus, we wish to extend the argument. Additionally, symmetry arises in numerous problems as batched maximally symmetric spaces of certain coordinates. For example, at r =constant on a sphere a maximally symmetric space can be seen. We, thus, will develop a method to write the metric that contains maximally symmetric subspace.

From the previous definition, it is natural to divide up the coordinate system into two group: one which specifies the location of the batches and the other forms the maximally symmetric subspace in each batch. Thus, if the whole space has N dimensions and each subspace has M dimensions, then we can label each subspace using $N-M$ coordinates ν^a. within each subspace, coordinates u^i span the subspace. Since we assume that the space with constant ν^a is maximally symmetric, it should be invariant under these transformations,

$$u^i \longrightarrow u^{'i} = u^i + \epsilon\xi^i(\nu, u)$$

$$v^a \longrightarrow v^{'a} = v^a$$

where the number of killing vectors are $M(M+1)/2$. Imposing these transformation invariant on the metric will lead to the structure desired. In a similar manner of derivation of equation 2.3, we can find 3 different similar structures where we divided the indices on coordinates u^i and ν^a.
For $\rho = i$, $\sigma = j$ (i.e only u^i coordinates),

$$\xi^k(\nu, u)\frac{\partial g_{ij}(\nu, u)}{\partial u^k} + g_{\mu j}(\nu, u)\frac{\partial \xi^\mu(\nu, u)}{\partial u^i} + g_{i\rho}(\nu, u)\frac{\partial \xi^\rho(\nu, u)}{\partial u^j} = 0$$

For $\rho = i$, $\sigma = a$ (i.e ν^a and u^i coordinates),

$$\xi^k(\nu, u)\frac{\partial g_{ia}(\nu, u)}{\partial u^k} + g_{\mu a}(\nu, u)\frac{\partial \xi^\mu(\nu, u)}{\partial u^i} + g_{i\rho}(\nu, u)\frac{\partial \xi^\rho(\nu, u)}{\partial \nu^a} = 0$$

For $\rho = a$, $\sigma = b$ (i.e only ν^a coordinates),

$$\xi^k(\nu, u)\frac{\partial g_{ab}(\nu, u)}{\partial u^k} + g_{\mu b}(\nu, u)\frac{\partial \xi^\mu(\nu, u)}{\partial v^a} + g_{a\rho}(\nu, u)\frac{\partial \xi^\rho(\nu, u)}{\partial v^b} = 0$$

The first equation simply implies that the part of the metric $g_[ij]$ which deals with the u^i coordinates is independently maximally symmetric for $M(M+1)/2$ ξ^k vectors. However, this is true by construction. Thus, $g_[ij]$ for each ν is both homogeneous and isotropic. The other two equations have information about the elements g_{ai} and g_{ab} which relates to the coordinates ν^a. For a more convenient method to deal with these two equations, we require that g_{ia} vanish. This is a consequence of the fact that we can do some coordinate transformation that leaves the g_{ij} part of the metric with killing vectors independent of ν. However, the killing vector of the whole space will be just a linear combination of these vectors with factors that depend on ν. Thus, it is extremely useful to disentangle the information in the metric that relates ν^a and u^i coordinates. To do this, we need to set g_{ai} to zero. Thus, we do a coordinate transformation where,

$$u^i = U^i(\nu', u')$$
$$v'^a = v^a$$

In the new coordinate system the new metric looks like,

$$\begin{aligned}
g'_{ja}(u', v') &= \frac{\partial u^l}{\partial u'^j}\frac{\partial u^k}{\partial v'^a} g_{lk}(u,\nu) + \frac{\partial u^l}{\partial u'^j} g_{la}(u,\nu) \\
&= \frac{\partial U^l(\nu', u')}{\partial u'^j}\frac{\partial U^k(\nu', u')}{\partial v'^a} g_{lk}(u,\nu) + \frac{\partial U^l(\nu', u')}{\partial u'^j} g_{la}(u,\nu) \\
&= \frac{\partial U^l(\nu', u')}{\partial u'^j}\left\{\frac{\partial U^k(\nu', u')}{\partial v'^a} g_{lk}(u,\nu) + g_{la}(u,\nu)\right\}
\end{aligned}$$

where this represents only the mixed part of the metric in the new basis. If can find a function U that eliminates the term in the brackets,

$$\frac{\partial U^k(\nu', u')}{\partial v'^a} g_{lk}(u,\nu) + g_{la}(u,\nu) = 0$$

This would eliminate the g_{ia} terms from the metric. We can rewrite the equation as,

$$\frac{\partial U^k(\nu', u')}{\partial v'^a} = -F^k_a(U,\nu)$$

where,

$$F^k_a(U,\nu) = g^{lk}(u,\nu) g_{la}(u,\nu)$$

The integrability of such an equation is rather a tedious matter to discuss which is beyond the scope of this text. An interested reader should check Weinberg's book for further reference. Assuming that we can find solutions to the previous equation, we can construct our metric with the new coordinates $(v')^a$ and $(u')^i$ where the mixed terms all vanish. The last two killing equations now reads,

$$\xi^k \frac{\partial g_{ab}}{\partial u^k} = 0$$

$$g_{i\rho}\frac{\partial \xi^{\rho}}{\partial \nu^{a}} = 0$$

Since $g_{i\rho}$ is a non singular matrix,

$$\frac{\partial \xi^{\rho}}{\partial \nu^{a}} = 0$$

where the killing vectors have no dependence on ν^a coordinates. It was also noted that in a maximally symmetric space, we can find vectors ξ^k that can take any arbitrary values which implies,

$$\frac{\partial g_{ab}}{\partial u^{k}} = 0 \tag{2.18}$$

Which shows that the ν^a part of the metric has no dependence on u^i coordinates. Now, we know that at each $\nu = \nu_0$ we have a maximally symmetric space with M(M+1)/2 killing vectors. Each of these killing vectors will satisfy the first killing equation at ν_0 and for general ν,

$$\xi^{k}(u)\frac{\partial g_{ij}(\nu_0,u)}{\partial u^{k}} + g_{\mu j}(\nu_0,u)\frac{\partial \xi^{\mu}(u)}{\partial u^{i}} + g_{i\rho}(\nu_0,u)\frac{\partial \xi^{\rho}(u)}{\partial u^{j}} = 0$$

$$\xi^{k}(u)\frac{\partial g_{ij}(\nu,u)}{\partial u^{k}} + g_{\mu j}(\nu,u)\frac{\partial \xi^{\mu}(u)}{\partial u^{i}} + g_{i\rho}(\nu,u)\frac{\partial \xi^{\rho}(u)}{\partial u^{j}} = 0$$

Since these two equation hold together, we can think of $g_{ij}(\nu,u)$ as a maximally symmetric tensor in the maximally symmetric space of the metric $g_{ij}(\nu_0,u)$. From our discussion of tensor in maximally symmetric spaces, we can see that the only possible form that $g_{ij}(\nu,u)$ can take is a factor independent of u times $g_{ij}(\nu_0,u)$,

$$g_{ij}(\nu,u) = f g_{ij}(\nu_0,u)$$

The only other dependence that f can be just to is ν. Thus, we can write this part of the metric as,

$$g_{ij}(\nu,u) = f(\nu)g_{ij}(u) \tag{2.19}$$

Putting equations 2.18 and 2.19, with the form of the metric, we can write the metric in the following form,

$$-d\tau^{2} = g_{mn}dx^{m}dx^{n} = g_{ab}(\nu)d\nu^{a}d\nu^{b} + f(\nu)g_{ij}(u)du^{i}du^{j} \tag{2.20}$$

where $g_{ab}(\nu)$ and $f(\nu)$ are functions of the ν-coordinates alone, and $g_{ij}(u)$ is a function of the u-coordinates alone that is by itself the metric of an M-dimensional maximally symmetric space.

In this construction, we can now think of the maximally symmetric subspace as spaces which does not depend on time.(look locally like Euclidean spaces). Thus, we can replace last part of the metric with the form we deduced for maximally symmetric spaces,

$$-d\tau^2 = g_{ab}(\nu)d\nu^a d\nu^b + f(\nu)\left[d\mathbf{u}^2 + \frac{(\mathbf{u}\cdot d\mathbf{u})^2}{1-\mathbf{u}^2}\right] \tag{2.21}$$

From this construction, we can directly form metrics of maximally symmetric subspace. An example of such metric will be discussed which is Schwarzschild solutions.

Chapter 3

Schwarzschild Solutions

The most straightforward solution of Einstein equations is that to a spherically symmetric gravitational field. These solutions may describe the field created by a star or planet. Here we are only concerned with the exterior space surrounding a spherical body. [2] This will also lead to the description of solutions of remarkable physical phenomena: Black Holes. Since we are interested in the outside region of a spherical body, we care about Einstein's equation in a vacuum,

$$R_{\mu\nu} = 0$$

We will assume that the source is static (does not change with time) and spherically symmetric. Thus, from our previous discussion, we will divide the space into subspace of constant r and t. We will parameterize each maximally symmetric subspace with,

$$u^1 = \sin\theta\cos\phi$$

$$u^2 = \sin\theta\sin\phi$$

substituting into equation 2.21,

$$-d\tau^2 = g_{tt}(r)dt^2 + 2g_{rt}(r)dtdr + g_{rr}(r)dr^2 + f(r)\left(d\theta^2 + sin^2\theta d\phi^2\right)$$

The general functions of the metric have no explicit time dependence due to the fact that our source is static. Thus, we expect that the space time metric have no explicit dependence on time. We are free to reset our clocks with,

$$t' = t + \Phi(r)$$

with

$$dt = dt' - \frac{\partial\Phi(r)}{\partial r}dr$$

substituting into the previous equation yields,

$$
\begin{aligned}
-d\tau^2 &= g_{tt}\left(dt' - \frac{\partial\Phi(r)}{\partial r}dr\right)^2 + 2g_{rt}\left(dt' - \frac{\partial\Phi(r)}{\partial r}dr\right)dr + g_{rr}dr^2 + f(r)\left(d\theta^2 + sin^2\theta d\phi^2\right) \\
&= g_{tt}(dt')^2 + dt'dr\left(-2g_{tt}\frac{\partial\Phi(r)}{\partial r} + 2g_{rt}\right) + dr^2\left(g_{tt}\left(\frac{\partial\Phi(r)}{\partial r}\right)^2 - 2g_{rt}\frac{\partial\Phi(r)}{\partial r} + g_{rr}\right) \\
&\quad + f(r)\left(d\theta^2 + sin^2\theta d\phi^2\right)
\end{aligned}
$$

Renaming $t' \longrightarrow t$. We can now kill the the cross term between t and r by setting,

$$
\frac{\partial\Phi(r)}{\partial r} = \frac{g_{rt}}{g_{tt}}
$$

Now the equation reads,

$$
-d\tau^2 = g_{tt}dt^2 - K(r)dr^2 + r^2F(r)\left(d\theta^2 + sin^2\theta d\phi^2\right)
$$

Where $K(r)$ can be directly read from the equation. Additionally there is no loss of generality by writing $f(r) = r^2F(r)$. We can further refine the radius by setting $(r')^2 = r^2F(r)$ with $2r'dr' = 2rF(r)dr$. Substituting in the previous equation results in,

$$
-d\tau^2 = g_{tt}(r)dt^2 - K(r)F(r)d(r')^2 + (r')^2\left(d\theta^2 + sin^2\theta d\phi^2\right)
$$

. Now, we can write the metric in the standard form, frankly speaking.

$$
d\tau^2 = B(r)dt^2 - A(r)dr^2 - r^2\left(d\theta^2 + sin^2\theta d\phi^2\right) \tag{3.1}
$$

One can check appendix A for the complete structure of the Christoffel symbol and the Ricci tensor. Using the results, we found in appendix A, one can see that,

$$
B(r) = \frac{1}{A(r)} = 1 - \frac{C}{r}
$$

It's now rather simple to find that constant C by referring to the motion of a slow moving body in a weak filed. If the particle is slowly moving then, $(dt/d\tau^2) \gg (d\vec{x}/d\tau)$. This enables us to write the geodesic equation as follows,

$$
\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\left(\frac{dt}{d\tau}\right)^2 = 0
$$

Since we are very far from the gravitational source, we can linearize the metric as follows,

$$
g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad h_{\mu\nu} \ll 1
$$

Additionally $g_{\mu\nu}$ does not depend on t since our field is static. Finding Christoffel to first order in $h_{\mu\nu}$ yields,

$$\Gamma^{\mu}_{00} = -\frac{1}{2}\eta^{\mu\nu}\frac{\partial h_{00}}{\partial x^{\nu}}$$

Using this form one can break the geodesic equation into two subsequent equations as follows,

$$\frac{d^2t}{d\tau^2} = 0$$

$$\frac{d^2\vec{x}}{d\tau^2} = \frac{1}{2}\left(\frac{dt}{d\tau}\right)^2 \nabla h_{00}$$

This implies that $dt/d\tau$ is a constant. One can change the variable in the second equation to t as follows,

$$\frac{d^2\vec{x}}{d\tau^2} = \frac{d}{d\tau}\left(\frac{dt}{d\tau}\frac{d\vec{x}}{dt}\right) = \cancelto{0}{\frac{d^2t}{d\tau^2}}\frac{d\vec{x}}{dt} + \left(\frac{dt}{d\tau}\right)^2 \frac{d^2\vec{x}}{dt^2}$$

Substituting back into the equation, one finds that,

$$\frac{d^2\vec{x}}{dt^2} = \frac{1}{2}\nabla h_{00}$$

comparing this equation with newton equation for the same body, one finds,

$$\frac{d^2\vec{x}}{dt^2} = \frac{1}{2}\nabla h_{00} = -\nabla\phi$$

where ϕ is the usual gravitational potential $\phi = -Gm/r$. Thus, we can find that the g_{tt} term in the metric takes the following form,

$$g_{tt} = -\left(1 - \frac{2GM}{r}\right)$$

which directly makes $C = 2GM$. Now we can write our metric as,

$$ds^2 = -d\tau^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

This ends our derivation of the Schwarzschild metric for a static spherical object.

Chapter 4

Weak field of a rotating mass

Before attempting to derive the Kerr metric, it's convenient to find solutions of the field at large distances from a rotating object. Furthermore, this will help us determine unknown constants in analogy to the derivation of Schwarzschild. In this section, we will be discussing the weak-field approximation and implement its theory to derive the Lense Thirring metric.

4.1 Linearized Field Equation

It is not suitable to regard the metric at a huge distance from rotating objects like that of Minkowski. One needs to find a suitable representation where the effects of rotation appear even at large distances. On this manner, we seek a linearized version of the field equation, where

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad\qquad |h_{\mu\nu}| \ll 1$$

We expect that the term $h_{\mu\nu}$ will contain all the information we seek in order to describe the rotating object. Here, we will attempt to describe a linearized theory for the nonlinear equation of gravity. All we need now is to find a convenient form the Ricci tensor. We will do all our calculations to a first order of $h_{\mu\nu}$. We can write the Christoffel symbol as,

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}\eta^{\mu\nu}\left(h_{\alpha\nu,\beta} + h_{\beta\nu,\alpha} - h_{\alpha\beta,\nu}\right) \tag{4.1}$$

and the Ricci tensor,

$$
\begin{aligned}
R_{\mu\nu} &= \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} \\
&= \frac{1}{2}\eta^{\alpha k}\left(h_{\mu k,\nu,\alpha} + h_{\nu k,\mu,\alpha} - h_{\mu\nu,k,\alpha} - h_{\mu k,\alpha,\nu} - h_{\alpha k,\mu,\nu} + h_{\mu\alpha,k,\nu}\right) \\
&= \frac{1}{2}\eta^{\alpha k}\left(h_{\mu k,\nu,\alpha} + h_{\nu k,\mu,\alpha} - h_{\mu\nu,k,\alpha} - h_{\alpha k,\mu,\nu}\right) \\
&= \frac{1}{2}\left(h_{\mu}{}^{\alpha}{}_{,\nu,\alpha} + h_{\nu}{}^{\alpha}{}_{,\mu,\alpha} - h_{\mu\nu,\alpha}{}^{,\alpha} - h_{\alpha}{}^{\alpha}{}_{,\mu,\nu}\right)
\end{aligned}
$$

Substituting this result into the field equation, One gets,

$$
h_{\mu}{}^{\alpha}{}_{,\nu,\alpha} + h_{\nu}{}^{\alpha}{}_{,\mu,\alpha} - h_{\mu\nu,\alpha}{}^{,\alpha} - h_{\alpha}{}^{\alpha}{}_{,\mu,\nu} - \eta_{\mu\nu}\left(h_{\alpha\beta}{}^{,\alpha\beta} - h_{,\beta}{}^{,\beta}\right) = 16\pi T_{\mu\nu} \tag{4.2}
$$

where $R \approx \eta^{\mu\nu}R_{\mu\nu}$. Note that upon contracting to find the Ricci scalar, the terms duplicate and we are left with the terms in the brackets. This equation can be reduced by introducing a substitution,

$$
\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h
$$

One can find that the equation reduces to the following form,

$$
-\bar{h}_{\mu\nu,\alpha}{}^{,\alpha} - \eta_{\mu\nu}\bar{h}_{\alpha\beta}{}^{,\alpha,\beta} + \bar{h}_{\mu\alpha}{}^{,\alpha}{}_{,\nu} + \bar{h}_{\mu\alpha}{}^{,\alpha}{}_{,\nu} = 16\pi T_{\mu\nu} \tag{4.3}
$$

One can easily interpret that that the first term is just the D'alembertian and this equation nearly looks like like a waves equation. Thus, one can search for a certain gauge transformation in analogy with the case of electrodynamics that will transform this equation into a wave equation. We can write the components of the Riemannian tensor as,

$$
R_{\alpha\mu\beta\nu} = \frac{1}{2}\left(h_{\alpha\nu,\mu,\beta} + h_{\mu\beta,\nu,\alpha} - h_{\mu\nu,\alpha,\beta} - h_{\alpha\beta,\mu,\nu}\right)
$$

One can see that upon a gauge transformation $h_{\mu\nu} \longrightarrow h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu}$, the Riemannian tensor is left invariant since the normal derivatives will commute in this case. We, thus, regard the Riemannian tensor to be gauge invariant under this transformation. This is analogous to the gauge transformation $A^{\mu}_{,\nu}$ in electrodynamics. Upon this gauge, we can without loss of generality require that $\bar{h}^{\mu\alpha}{}_{,\alpha} = 0$. This will reduce equation 4.3 to the usual form of the wave equation. Additionally, one can further see that the equation is always integrable. Suppose that $\bar{h}^{\mu\alpha}{}_{,\alpha} \neq 0$, we can always find a solution for ξ that would produce $\bar{h}^{\mu\alpha}{}_{,\alpha} = 0$. We can see this as follows,

$$0 = \bar{h}^{\mu\alpha}{}_{,\alpha} = \left(h^{\mu\alpha} - \xi^{\mu,\alpha} - \xi^{\alpha,\mu} - \frac{1}{2}\eta^{\mu\alpha}h \right)_{,\alpha}$$
$$= h^{\mu\alpha}{}_{,\alpha} - \xi^{\mu,\alpha}{}_{,\alpha} - \xi^{\alpha,\mu}{}_{,\alpha} - \frac{1}{2}\eta^{\mu\alpha}h_{,\alpha}$$

where the differential equation can be written as follows,

$$\partial_\alpha\partial^\alpha\xi^\mu + \partial_\alpha\partial^\mu\xi^\alpha = \partial_\alpha h^{\mu\alpha} - \frac{1}{2}\eta^{\mu\alpha}\partial_\alpha h$$

This equation is exactly integrable.[6] Thus, we can always consider that $\bar{h}^{\mu\alpha}{}_{,\alpha} = 0$ without loss of generality. Upon this gauge transformation, one can easily see that equation 4.3 transforms into,

$$-\bar{h}_{\mu\nu,\alpha}{}^{;\alpha} = 16\pi T_{\mu\nu} \tag{4.4}$$

or

$$-\partial_\alpha\partial^\alpha\bar{h}_{\mu\nu} = 16\pi T_{\mu\nu} \tag{4.5}$$

This is exactly the wave equation. We will now use this equation to come up with an approximate solution for the case of the weak field of a rotating object.

4.2 Lense Thirring Metric

One can easily see that the gravitational field of a rotating body should have a different form from its static counterpart. A clear resemblance is of the cases of static and rotating charged spheres [7].

We will now begin inspecting the form of the metric very far from a rotating object or near an object that posses a non relativistic angular momentum. We assume that the rotation is steady and time independent. From which, we suspect the field produced will also be time independent. There we can write equation 4.5 as,

$$-\nabla^2\bar{h}^{\mu\nu}(\vec{x}) = T^{\mu\nu}$$

One can then see that this equation has a general solution as [7],

$$\bar{h}^{\mu\nu}(\vec{x}) = -\frac{1}{4\pi}\int \frac{T^{\mu\nu}(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|}d^3\vec{x}\,'$$

We can approximate the solution be expanding the integral function in the previous equation since we are only interested in a weak field case. [8] Using Taylor expansion of multivariable functions one can easily expand $1/|\vec{x} - \vec{x}\,'|$ as,

$$\frac{1}{|\vec{x} - \vec{x}\,'|} \approx \frac{1}{r} + \sum_k \frac{x'^k x^k}{r^3} + O\left(\frac{1}{r^2}\right)$$

We can now write the approximate solution as,

$$\bar{h}^{\mu\nu}(\vec{x}) = -\frac{1}{4\pi r}\int T^{\mu\nu}(\vec{x}\,')d^3\vec{x}\,' - \sum_k \frac{x^k}{4\pi r^3}\int x'^k T^{\mu\nu} d^3\vec{x}\,' \tag{4.6}$$

We will attempt to write the integrals of the energy stress tensor more elaborately. We begin by placing our coordinate system on the rotating body where the origin coincides with the center of mass. Since the origin of our coordinate system coincides with the center of mass, we have

$$\int x'^k T^{00}(\vec{x}\,')d^3\vec{x}\,' = 0 \tag{4.7}$$

Additionally, we believe that the $T^{\mu\nu}$ is symmetric and obeys the continuity equation. We can see that,

$$\int T^{00}(\vec{x}\,')d^3\vec{x}\,' = M \tag{4.8}$$

and from the continuity equation $\nabla_\mu T^{\mu\nu} = 0$ or,

$$\frac{\partial T^{0\nu}}{\partial x'^0} + \frac{\partial T^{a\nu}}{\partial x'^a} = 0$$

where $a = 1, 2, 3$. Since the matter distribution is time independent by assumption, then,

$$\frac{\partial T^{a\nu}}{\partial x'^a} = 0 \tag{4.9}$$

We use this equation to derive two important relations. Multiplying equation 4.2 by x'^k and integrating by parts, one gets,

$$\int x'^k \frac{\partial T^{a\nu}}{\partial x'^a} d^3\vec{x}\,' = -\int T^{k\nu}(\vec{x}\,')d^3\vec{x}\,' = 0 \tag{4.10}$$

where k runs from 1 to 3. Where the boundary term vanish since we can integrate in a volume bounded by a surface where $T_\mu = 0$. Additionally, one can multiply equation 4.2 by $x'^n x'^k$ integrating the equation by parts, one obtains,

$$-\int x'^k T^{n\mu} d^3\vec{x}\,' - \int x'^n T^{k\mu} d^3\vec{x}\,' = 0 \tag{4.11}$$

In an obvious analogy with its dynamical counterpart, the spin angular momentum J is defined to be the integral of the cross product of the position vector and the density of momentum. Thus,

$$J_n = \int \epsilon_{nka} x'^k T^{a0} d^3\vec{x}\,'$$

Regarding equation 4.11, The spin angular momentum can be rewritten in terms of one of the contributions from the cross product since they have the same value. Thus, for example we can write the x component of the spin angular momentum as,

$$J^1 = \int x'^2 T^{30} d^3\vec{x}\,' - \int x'^3 T^{20} d^3\vec{x}\,' = 2\int x'^2 T^{30} d^3\vec{x}\,'$$

Similarly for the other components. Thus, we can evaluate this integral as,

$$\int x'^k T^{n0} d^3\vec{x}\,' = \frac{1}{2}\epsilon^{nkl} J^l \tag{4.12}$$

Lastly, due to the continuity equation, we can see that we can write the energy momentum tensor as,

$$T^{pq} = \frac{1}{2}\frac{\partial}{\partial x'^m}\left[x'^q T^{pm} + x'^p T^{mq}\right]$$

Thus, now we can evaluate this integral,

$$\begin{aligned}
\int x'^k T^{pq} d^3\vec{x}\,' &= \frac{1}{2}\int x'^k \frac{\partial}{\partial x'^m}\left[x'^q T^{pm} + x'^p T^{mq}\right] d^3\vec{x}\,' \\
&= -\frac{1}{2}\int \delta^k_m \left[x'^q T^{pm} + x'^p T^{mq}\right] d^3\vec{x}\,' \\
&= -\frac{1}{2}\int \left[x'^q T^{pk} + x'^p T^{kq}\right] d^3\vec{x}\,' \\
&= 0
\end{aligned} \tag{4.13}$$

where the last line is zero due to equation 4.11. By combining our results from equations, 4.2, 4.7, 4.10, 4.12 and 4.13, and substituting into equation 4.6, we obtain,

$$\bar{h}^{00}(\vec{x}) = -\frac{M}{4\pi r}$$

$$\bar{h}^{a0}(\vec{x}) = -\frac{1}{8\pi r^3}\epsilon^{akn} x^k J^n$$

$$\bar{h}^{an}(\vec{x}) = 0$$

We can now write the corresponding values for $h_{\mu\nu}$ as follows,

$$h_{00}(\vec{x}) = -\frac{2M}{r}$$

$$h_{a0}(\vec{x}) = \frac{4}{r^3}\epsilon_{kan}x^k J^n$$

$$h_{an}(\vec{x}) = -\frac{2M}{r}\delta^a_n$$

where we can write the corresponding metric as,

$$\begin{aligned} ds^2 = &-\left[1 - \frac{2M}{r} + O\left(\frac{1}{r^2}\right)\right] dt^2 - \left[\frac{4}{r^3}\epsilon_{kan}x^k J^n + O\left(\frac{1}{r^2}\right)\right] dt dx^a \\ &+ \left[\left(1 + \frac{2M}{r}\right)\delta^a_b + O\left(\frac{1}{r^2}\right)\right] dx^a dx^b \end{aligned} \tag{4.14}$$

Chapter 5

The Kerr metric

Schwarzschild solution is a description of the exact geometry produced by non rotating point particles. It was clearly shown by Oppenheimer and Snyder that the inward analytic extension of Schwarzschild solution represents a non rotating black hole.[9] However, astronomically, most objects (star and planets) are observed to rotate in space. From the weak field approximation of Einstein equations, we can get the form of the metric of an isolated rotating body with mass M and momentum J. [10]

$$ds^2 = -\left[1 - \frac{2GM}{r} + O(\frac{1}{r^2})\right] dt^2 - \left[\frac{4J\sin^2\theta}{r} + O(\frac{1}{r^2})\right] d\phi dt$$
$$+ \left[1 + \frac{2GM}{r} + O(\frac{1}{r^2})\right] \left(dr^2 + r^2 d\Omega^2\right) \tag{5.1}$$

Additionally, there is no general reason for us to think of black holes as non-rotating bodies. This can be interpreted by considering two huge stars that combine to end up forming a black hole (Binary system). The final object that will result from their collapse is expected to have a net angular momentum from their rotational collapse. Thus, in this section, we will present the exact solution of rotating static black holes (the Kerr metric) as discussed in Chandrasekhar book [11].

5.1 Axial symmetric metric

In Newtonian physics the axial symmetry of a gravitational field produces a gravitational field that is independent of ϕ coordinate while using cylindrical coordinates. [12] In correspondence with this intuition, we can free that the metric of the axial symmetric rotating body of ϕ dependence in analogy with the Newtonian potential. Additionally, we will further assume that the field is static (i.e has no time dependence). Thus, we will think of the metric as a function of x^2 and x^3 and independent of ϕ and t. Additionally, We will again define space into batches of maximally symmetric subspace of coordinates x^1 and x^2. Using same argument as found in section 2.4, we can kill the terms that couple ϕ and t with x^2 and x^3, leaving a form that is nearly similar to that produced by subspace symmetry,

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}d\phi dt + g_{\phi\phi}d\phi^2 + g_{22}d(x^2)^2 + 2g_{23}d(x^2)d(x^3) + g_{33}d(x^3)^2$$

A further reduction of the metric can be achieved by reducing the last three terms to a diagonal form. Now suppose we have the contravariant metric of the following form,

$$ds^2 = g^{22}d(x_2)^2 + 2g^{23}d(x_2)d(x_3) + g^{33}d(x_3)^2$$

We can make a coordinate transformation such that,

$$\xi^1 = \phi(x^2, x^3), \qquad \xi^2 = \psi(x^2, x^3)$$

and

$$(g')^{\mu\nu} = \frac{\partial \xi^\mu}{\partial x^\sigma}\frac{\partial \xi^\nu}{\partial x^\rho} g^{\sigma\rho}$$

writing the components of the transformed contravariant metric explicitly yields,

$$(g')^{23} = \left(\frac{\partial \phi}{\partial x^2}\frac{\partial \psi}{\partial x^2}\right) g^{22} + \left(\frac{\partial \phi}{\partial x^2}\frac{\partial \psi}{\partial x^3} + \frac{\partial \phi}{\partial x^3}\frac{\partial \psi}{\partial x^1}\right) g^{23} + \left(\frac{\partial \phi}{\partial x^3}\frac{\partial \psi}{\partial x^3}\right) g^{33} \tag{5.2}$$

To get a diagonal form of the metric we need this part end up being zero. This will happen if the functions ϕ and ψ satisfy the following relation,

$$\frac{\partial \phi}{\partial x^2} = k\left(g^{32}\frac{\partial \psi}{\partial x^2} + g^{33}\frac{\partial \psi}{\partial x^3}\right)$$

$$\frac{\partial \phi}{\partial x^3} = -k\left(g^{22}\frac{\partial \psi}{\partial x^2} + g^{23}\frac{\partial \psi}{\partial x^3}\right)$$

Substituting these into equation 5.2,

$$\begin{aligned}(g')^{23} &= \left(k\left(g^{32}\frac{\partial \psi}{\partial x^2} + g^{33}\frac{\partial \psi}{\partial x^3}\right)\frac{\partial \psi}{\partial x^2}\right) g^{22} + g^{23}k\left(g^{32}\frac{\partial \psi}{\partial x^2} + g^{33}\frac{\partial \psi}{\partial x^3}\right)\frac{\partial \psi}{\partial x^3} \\ &\quad - kg^{23}\left(g^{22}\frac{\partial \psi}{\partial x^2} + g^{23}\frac{\partial \psi}{\partial x^3}\right)\frac{\partial \psi}{\partial x^2} + \left(\frac{\partial \phi}{\partial x^3}\frac{\partial \psi}{\partial x^3}\right) g^{33} \\ &= 0\end{aligned}$$

Satisfying these conditions, the new form of the metric will be,

$$ds^2 = (g')^{22}(dx_2')^2 + (g')^{33}(dx_3')^2$$

a further reduction can be achieved by requiring a full diagonal form thus setting,

$$(g')^{22} = (g')^{33} \tag{5.3}$$

using the same transformations as above, we can find a condition of the form of k to satisfy both requirements,

$$(g')^{22} = g^{22}\frac{\partial\phi}{\partial x^2}\frac{\partial\phi}{\partial x^2} + 2g^{23}\frac{\partial\phi}{\partial x^2}\frac{\partial\phi}{\partial x^3} + g^{33}\frac{\partial\phi}{\partial x^3}\frac{\partial\phi}{\partial x^3}$$

$$(g')^{33} = g^{22}\frac{\partial\psi}{\partial x^2}\frac{\partial\psi}{\partial x^2} + 2g^{23}\frac{\partial\psi}{\partial x^2}\frac{\partial\psi}{\partial x^3} + g^{33}\frac{\partial\psi}{\partial x^3}\frac{\partial\psi}{\partial x^3}$$

Requiring 5.3,

$$\begin{aligned}
0 &= (g')^{22} - (g')^{33} \\
&= g^{22}\left(\frac{\partial\phi}{\partial x^2}\frac{\partial\phi}{\partial x^2} - \frac{\partial\psi}{\partial x^2}\frac{\partial\psi}{\partial x^2}\right) + 2g^{23}\left(\frac{\partial\phi}{\partial x^2}\frac{\partial\phi}{\partial x^3} - \frac{\partial\psi}{\partial x^2}\frac{\partial\psi}{\partial x^3}\right) + g33\left(\frac{\partial\phi}{\partial x^3}\frac{\partial\phi}{\partial x^3} - \frac{\partial\psi}{\partial x^3}\frac{\partial\psi}{\partial x^3}\right) \\
&= \left(k^2(g^{22}g^{33} - (g^{23})^2) - 1\right)\left(g^{22}\left(\frac{\partial\psi}{\partial x^2}\right)^2 + 2g^{23}\frac{\partial\psi}{\partial x^2}\frac{\partial\psi}{\partial x^3} + g^{33}\left(\frac{\partial\psi}{\partial x^3}\right)^2\right)
\end{aligned}$$

One can easily see that the second bracket cannot vanish on a given manifold. Thus to satisfy the second requirement, we need,

$$k^2 = \frac{1}{g^{22}g^{33} - (g^{23})^2}$$

One can easily notice that the denominator is just the determinant of the contravariant metric. Thus, we can easily infer that k is just the Jacobian of the subspace,

$$k = \sqrt{det(g)} = \mathcal{J}$$

reducing the subspace into a diagonal form is very effective in computations of the Kerr metric. The final form of the subspace is,

$$ds^2 = e^{\mu}(d(x_2')^2 + d(x_3')^2)$$

where e^{μ} is an arbitrary function. The full line element can now be written as,

$$ds^2 = e^{2\nu}dt^2 - e^{2\psi}(d\phi - \omega dt)^2 + e^{\mu}(d(x^2)^2 + d(x^3)^2)$$

There is no loss of generality be writing the line the line element as follow, however, it will greatly simplify the computations. This line element is derived by Papapetrou for implementation of axial symmetry. Although, this form will severely reduce the computation, we will not use it. We will be using a more general form of the metric as pointed by Chandrasekhar,

$$ds^2 = e^{2\nu}dt^2 - e^{2\psi}(d\phi - \omega dt)^2 - e^{\mu_2}d(x^2)^2 - e^{\mu_3}d(x^3)^2 \tag{5.4}$$

5.2 The vacuum solution

As seen from appendix B, the equations seems to have very complex structure. However, by using reductions suggested by Chandrasekhar we can even reduce its form readily. We will label the derivative with respect to x and y as $_{,2}$ and $_{,3}$ respectively. Letting $\beta = \psi + \nu$, we can rewrite equations B.7, B.8.

$$\left(e^{\beta+\mu_3-\mu_2}\nu_{,2}\right)_{,2} + \left(e^{\beta+\mu_2-\mu_3}\nu_{,3}\right)_{,3} = \frac{1}{2}e^{3\psi-\nu}\left(e^{\mu_3-\mu_2}(\omega_{,2})^2 + e^{\mu_2-\mu_3}(\omega_{,3})^2\right) \tag{5.5}$$

$$\left(e^{\beta+\mu_3-\mu_2}\psi_{,2}\right)_{,2} + \left(e^{\beta+\mu_2-\mu_3}\psi_{,3}\right)_{,3} = -\frac{1}{2}e^{3\psi-\nu}\left(e^{\mu_3-\mu_2}(\omega_{,2})^2 + e^{\mu_2-\mu_3}(\omega_{,3})^2\right) \tag{5.6}$$

adding and subtracting equations 5.5 and 5.6,

$$\left(e^{\mu_3-\mu_2}(e^{\beta})_{,2}\right)_{,2} + \left(e^{\mu_2-\mu_3}(e^{\beta})_{,3}\right)_{,3} = 0 \tag{5.7}$$

$$\left(e^{\beta+\mu_3-\mu_2}(\psi-\nu)_{,2}\right)_{,2} + \left(e^{\beta+\mu_2-\mu_3}(\psi-\nu)_{,3}\right)_{,3} = -e^{3\psi-\nu}\left(e^{\mu_3-\mu_2}(\omega_{,2})^2 + e^{\mu_2-\mu_3}(\omega_{,3})^2\right) \tag{5.8}$$

subtracting equations B.11 and B.12 yields,

$$\begin{aligned} &4e^{\mu_3-\nu_2}\left(\beta_{,2}\mu_{3,2} + \psi_{,2}\nu_{,2}\right) - 4e^{\mu_2-\nu_3}\left(\beta_{,3}\mu_{2,3} + \psi_{,3}\nu_{,3}\right) \\ &= 2e^{-\beta}\left(\left(e^{\mu_3-\mu_2}(e^{\beta})_{,2}\right)_{,2} - \left(e^{\mu_2-\mu_3}(e^{\beta})_{,3}\right)_{,3}\right) - e^{3\psi-\nu}\left(e^{\mu_3-\mu_2}(\omega_{,2})^2 + e^{\mu_2-\mu_3}(\omega_{,3})^2\right) \end{aligned} \tag{5.9}$$

As mentioned before, the calculations are very tedious, but end up in a form that is very convenient to use. We can without loss of generality write the axial symmetric metric in the following form,

$$ds^2 = e^{\beta}\left[\chi(dt)^2 - \frac{1}{\chi}(d\phi - \omega dt)^2\right] - \frac{e^{\mu_2+\mu_3}}{\sqrt{\Delta}}\left(d(x^2)^2 + \Delta d(x^3)^2\right) \tag{5.10}$$

where,

$$\Delta = e^{\mu_3-\mu_2}, \qquad \beta = \psi + \nu, \qquad \chi = e^{\nu-\psi}$$

Direct substitution will end with the same form in equation 5.4. An important feature of this form of the metric is that we can realize an invariance under transformation,

$$t \longrightarrow i\phi, \qquad \phi \longrightarrow -it$$

The only affected part in the term in the first bracket. Performing the transformation, one can yield,

$$
\begin{aligned}
\left[\chi(dt')^2 - \frac{1}{\chi}(d\phi' - \omega dt')^2\right] &= -\chi d\phi^2 + \frac{1}{\chi}(dt + \omega d\phi)^2 \\
&= d\phi^2\left[-\chi + \frac{\omega^2}{\chi}\right] + \frac{2\omega}{\chi}d\phi dt + \frac{1}{\chi}dt^2 \\
&= \left[\frac{\omega^2 - \chi^2}{\chi}\right]\left(d\phi^2 + \frac{2\omega}{\omega^2 - \chi^2}d\phi dt\right) + \frac{1}{\chi}dt^2 \\
&= \left[\frac{\omega^2 - \chi^2}{\chi}\right]\left(d\phi + \frac{\omega}{\omega^2 - \chi^2}dt\right)^2 \\
&\quad + \frac{1}{\chi}dt^2 - \left[\frac{\omega^2 - \chi^2}{\chi}\right]\left[\frac{\omega^2}{(\omega^2 - \chi^2)^2}\right]dt^2 \\
&= \left[\frac{\omega^2 - \chi^2}{\chi}\right]\left(d\phi + \frac{\omega}{\omega^2 - \chi^2}dt\right)^2 + \left[\frac{1}{\chi} - \frac{\omega^2}{\chi(\omega^2 - \chi^2)}\right]dt^2 \\
&= \left[\frac{\omega^2 - \chi^2}{\chi}\right]\left(d\phi + \frac{\omega}{\omega^2 - \chi^2}dt\right)^2 - \left[\frac{\chi}{(\omega^2 - \chi^2)}\right]dt^2
\end{aligned}
\tag{5.11}
$$

redefining,

$$
\left[\frac{\chi^2 - \omega^2}{\chi}\right] = \frac{1}{\tilde{\chi}}, \qquad \tilde{\omega} = \frac{\omega}{\chi^2 - \omega^2}
$$

one can reach the same form of the metric,

$$
\left[\tilde{\chi}dt^2 - \frac{1}{\tilde{\chi}}(d\phi - \tilde{\omega}dt)^2\right]
$$

This, shows that both (ω, χ) and $(\tilde{\omega}, \tilde{\chi})$ solve for the same metric. The second metric is refereed to as the conjugate metric.

5.3 Gauge choice and Initial conditions

In this section, we will begin by finding the exact forms of the metric functions by defining initial conditions and specifying the gauge freedom. Our first attempt is forecasting initial conditions that will generally produce horizons of black in clear analogy to that of Schwarzschild. The horizon can be defined by a null hypersurface $\Phi = 0$ in which light-like geodesics going in and out coincide. [13] We can find the conditions for the black hole horizon by finding the limit for which a photon can travel radially. [14] [15]

$$\frac{dr}{dt} = 0$$

A light ray follows a null geodesic where $ds^2 = 0$. It is also convenient to regard x^2 and x^3 as the radial distance r and the polar angle θ.

$$\begin{aligned} 0 &= ds^2 \\ &= e^{\beta}\left[\chi(dt)^2 - \frac{1}{\chi}(d\phi - \omega dt)^2\right] - \frac{e^{\mu_2+\mu_3}}{\sqrt{\Delta}}\left(dr^2 + \Delta d\theta^2\right) \end{aligned}$$

Rewriting the above equation to have a term for dr/dt,

$$\begin{aligned} \left(\frac{dr}{dt}\right)^2 &= -\Delta\left(\frac{d\theta}{dt}\right)^2 + \frac{\sqrt{\Delta}}{e^{\mu_2+\mu_3}}e^{\beta}\left[\chi - \frac{1}{\chi}(\frac{d\phi}{dt} - \omega)^2\right] \\ &= 0 \end{aligned}$$

Thus, the condition for the event horizon is,

$$\Delta(r) = 0$$

where we have exercised the gauge freedom to restrict Δ to be only a function of r. This can be done by finding some coordinate transformation that will make Δ a function of r only. Additionally, another requirement for event horizon as indicated by Chandrasekhar [11] and Hooft [13] is that the function g_{tt} which is multiplied by the dt^2 term in the metric tend to zero as one approaches the event Horizon. This can be seen from the definition of the event horizon since it separates two region which are causally disconnected and by requiring that the manifold remain smooth and differentiable, this function must tend to zero since it flips its sign. This directly implies that the 2-dimensional null hypersurface of (t , ϕ) have a vanishing determinant.

$$\begin{vmatrix} e^{\beta}\left[\chi - \frac{\omega^2}{\chi}\right] & \frac{e^{\beta}\omega}{\chi} \\ \frac{e^{\beta}\omega}{\chi} & -\frac{e^{\beta}}{\chi} \end{vmatrix} = e^{2\beta}\left(-1 + \frac{\omega^2}{\chi^2} - \frac{\omega^2}{\chi^2}\right) = -e^{2\beta} = 0$$

In this manner we can think of $e^{2\beta}$ as general function times Δ. We can even put the function into a more restricted form by removing the dependence on r. Thus, we have,

$$e^{\beta} = \Delta^{1/2} f(\theta)$$

substituting this into equation 5.7, one can find a partial differential equation for Δ and f as follows,

$$\left(\Delta^{1/2}(\Delta^{1/2})_{,r}\right)_{,r} + \frac{f_{,\theta,\theta}}{f} = 0$$

One can easily solve the two equations by separation of variables. we will find the following,

$$f(\theta) = C_1 \sin(\alpha\theta) + C_2 \cos(\alpha\theta)$$
$$\Delta = \alpha^2 r^2 - 2Mr + a^2$$

To specify the constant, We require initial conditions. On this line, We have to assume the full structure of the functions and compare it with a known metric. An example of this is the Lense Thrilling metric in the weak field approximation. After the full derivation, we compare the coefficients. There we will find that $C_2 = 0$, $\alpha = 1$, $C_1 = 1$ with M and a being general constants that represent mass and the angular momentum per unit mass of the black hole.

5.4 Completing the Derivation

First we need to make a transformation of coordinates,

$$\theta \longrightarrow \xi = cos(\theta)$$

Equation 5.8 can be written as,

$$(\Delta(\psi - \nu)_{,r})_{,r} + (\delta(\psi - \nu)_{,\xi})_{,\xi} = -e^{2\psi - 2\nu}\left(\Delta(\omega_{,r})^2 + \delta(\omega_{,\xi})^2\right) \tag{5.12}$$

and equation B.9

$$\left(\Delta e^{2\psi-2\nu}\omega_{,r}\right)_{,r} + \left(\delta e^{2\psi-2\nu}\omega_{,\xi}\right)_{,\xi} = 0 \tag{5.13}$$

where,

$$\delta = \sin^2(\theta)$$

with $\chi = e^{-\psi+\nu}$, can write the above equations as follows,

$$\left(\frac{\Delta}{\chi}\chi_{,r}\right)_{,r} + \left(\frac{\delta}{\chi}\chi_{,\xi}\right)_{,\xi} = \frac{1}{\chi^2}\left(\Delta(\omega_{,r})^2 + \delta(\omega_{,\xi})^2\right) \tag{5.14}$$

$$\left(\frac{\Delta}{\chi^2}\omega_{,r}\right)_{,r} + \left(\frac{\delta}{\chi^2}\omega_{,\xi}\right)_{,\xi} = 0 \tag{5.15}$$

Now, we have reached the equation we desire the most. Now, the computations is a matter of substitutions to solve these two differential equations. The most straight forward approach is by using Ernest equation. [16] One can realize that we can write equation 5.15 in terms of a potential Φ as,

$$\Phi_{,r} = \frac{\delta}{\chi^2}\omega_{,\xi}, \qquad \Phi_{,\xi} = -\frac{\Delta}{\chi^2}\omega_{,r}$$

Using this potential equations 5.15 is clearly satisfied. Also, expressing equation 5.14 in the same potential,

$$\left(\frac{\Delta}{\chi}\chi_{,r}\right)_{,r} + \left(\frac{\delta}{\chi}\chi_{,\xi}\right)_{,\xi} = \frac{\chi^2}{\delta}(\Phi_{,r})^2 + \frac{\chi^2}{\Delta}(\Phi_{,\xi})^2$$

Defining

$$\Psi^2 = \frac{\Delta\delta}{\chi^2}$$

We can reduce the two equations into the following form,

$$\Psi\left[(\Delta\Psi_{,r})_{,r} + (\delta\Psi_{,\xi})_{,\xi}\right] = \Delta\left[(\Psi_{,r})^2 - (\Phi_{,r})^2\right] + \delta\left[(\Psi_{,\xi})^2 - (\Phi_{,\xi})^2\right]$$

$$\Psi\left[(\Delta\Phi_{,r})_{,r} + (\delta\Phi_{,\xi})_{,\xi}\right] = 2\Delta\Phi_{,r}\Psi_{,r} + 2\Delta\Phi_{,\xi}\Psi_{,\xi}$$

These two equations can be combined by letting $Z = \Psi + i\Phi$. Combining the two equations yields,

$$\Re(Z)\left[(\Delta Z_{,r})_{,r} + (\delta Z_{,\xi})_{,\xi}\right] = \Delta(Z_{,r})^2 + \delta(Z_{,\xi})^2$$

We can now reach Ernest equations by substituting

$$Z = -\frac{1+E}{1-E}$$

After substituting we obtain,

$$(EE^* - 1)\left[(\Delta E_{,r})_{,r} + (\delta E_{,\xi})_{,\xi}\right] = 2E^*\left[\Delta(E_{,r})^2 + \delta(E_{,\xi})^2\right] \tag{5.16}$$

For a more adequate representation, we can perform the transformation,

$$\eta = \frac{r-M}{\sqrt{M^2-a^2}}$$

where,

$$\begin{aligned}\Delta &= r^2 - 2Mr + a^2\\ &= \left(\eta\sqrt{M^2-a^2}+M\right)^2 - 2M\left(\eta\sqrt{M^2-a^2}+M\right) + a^2\\ &= (M^2-a^2)\eta^2 + M^2 - 2M^2 + a^2\\ &= (M^2-a^2)(\eta^2-1)\end{aligned}$$

and $\delta = sin^{\theta} = 1-\xi^2$, we can 5.16 to have this form,

$$(EE^* - 1)\left[((\eta^2-1)E_{,\eta})_{,\eta} + ((1-\xi^2)E_{,\xi})_{,\xi}\right] = 2E^*\left[(\eta^2-1)(E_{,\eta})^2 + (1-\xi^2)(E_{,\xi})^2\right] \tag{5.17}$$

In his paper, Ernst pointed out a solution to a similar form of this equation. [17] The two equations have a slight difference in structure. The factors of $(\eta^2 - 1)$ and $(1 - \xi^2)$ do not exist in his original equation. Despite this fact, equation 5.16 still satisfy the elementary solution,

$$E = -p\eta - iq\xi$$

Upon direct substituting in equation 5.17, we can obtain a condition on p and q such that, $p^2 + q^2 = 1$ to satisfy the equation. All we need now is to substitute back to find the form of the original function. We can now find Z,

$$Z = -\frac{1+E}{1-E} = \frac{p\eta + iq\xi - 1}{p\eta + iq\xi + 1}$$

reverting back to the variable r,

$$Z = \frac{p(r-M) + iq(\sqrt{M^2+a^2})\xi - (\sqrt{M^2+a^2})}{p(r-M) + iq(\sqrt{M^2+a^2})\xi + (\sqrt{M^2+a^2})}$$

We can simplify this equation be proper choice of p and q. A adequate choice is,

$$p = \frac{\sqrt{M^2-a^2}}{M} \qquad q = \frac{a}{M}$$

This choice clearly satisfies the condition on p and q and it reduces the above equation to

$$Z = \frac{r + ia\xi - 2M}{r + ia\xi} = \frac{r^2 + a^2\xi^2 - 2Mr + 2iaM\xi}{r^2 + a^2\xi^2}$$

We can now find Ψ and Φ as follows,

$$\Psi = \Re(Z) = \frac{r^2 + a^2(1-\delta) - 2Mr}{r^2 + a^2\xi^2} = \frac{\Delta - a^2\delta}{\rho^2}$$

$$\Phi = \Im(Z) = \frac{2aM\xi}{r^2 + a^2\xi^2} = \frac{2aM\xi}{\rho^2}$$

where we have defined $\rho^2 = r^2 + a^2\xi^2$. We can now find two differential equations for ω,

$$\Phi_{,r} = \frac{\delta}{\chi^2}\omega_{,\xi} = \frac{\Psi^2}{\Delta}\omega_{,\xi} = \frac{(\Delta - a^2\delta)^2}{\rho^4\Delta}\omega_{,\xi}$$

$$\Phi_{,\xi} = -\frac{\Delta}{\chi^2}\omega_{,r} = -\frac{\Psi^2}{\delta}\omega_{,r} = -\frac{(\Delta - a^2\delta)^2}{\rho^4\delta}\omega_{,r}$$

Thus,

$$\omega_{,r} = -\frac{2aM\delta(r^2 - a^2\xi^2)}{(\Delta - a^2\delta)^2}. \qquad \omega_{,\xi} = -\frac{4aM\Delta r\xi}{(\Delta - a^2\delta)^2}$$

We can easily see by integrating $\omega_{,\xi}$, that a suitable solution for these equations is,

$$\omega = \frac{2aMr\delta}{\Delta - a^2\delta}$$

Additionally χ can be found to the following form,

$$\chi = \frac{\sqrt{\delta\Delta}}{\Psi} = \frac{\rho^2\sqrt{\delta\Delta}}{\Delta - a^2\delta}$$

The only renaming unknown function is $e^{\mu_2+\mu_3}$. An important identity that will simply our calculations is derived as follows,

$$\begin{aligned}\rho^2\sqrt{\Delta} \pm 2aMr\sqrt{\delta} &= \rho^2\sqrt{\Delta} \pm 2aMr\sqrt{\delta} \mp (r^2 - r^2 + a^2 - a^2)a\sqrt{\delta} \\ &= \rho^2\sqrt{\Delta} \mp (r^2 - 2Mr + a^2)a\sqrt{\delta} \pm (r^2 + a^2)a\sqrt{\delta} \\ &= (r^2 + a^2 - a^2\delta)\sqrt{\Delta} \mp a\Delta\sqrt{\delta} \pm (r^2 + a^2)a\sqrt{\delta} \\ &= \left[r^2 + a^2 \mp a\sqrt{\Delta\delta}\right]\left(\sqrt{\Delta} \pm a\sqrt{\delta}\right)\end{aligned} \tag{5.18}$$

We will use that last two equation we have not used up til now which are equations 5.9 and B.10. Using elementary reduction, doing the ξ transformation and defining $x = \chi + \omega$ and $y = \chi - \omega$, we can rewrite these two equations in the following form,

$$-\frac{\xi}{\delta}(\mu_3 + \mu_2)_{,r} + \frac{r - M}{\Delta}(\mu_3 + \mu_2)_{,\xi} = \frac{2}{(x + y)^2}(x_{,r}y_{,\xi} + x_{,\xi}y_{,r}) \tag{5.19}$$

$$2(r-M)(\mu_3+\mu_2)_{,r}+2\xi(\mu_3+\mu_2)_{,\xi} = \frac{4}{(x + y)^2}(\Delta x_{,r}y_{,r} - x_{,\xi}y_{,\xi}\delta) - 3\frac{(r - M)^2 - \Delta}{\Delta} + \frac{\xi^2 + \delta}{\delta} \tag{5.20}$$

From the structure of these two equation, they clearly are two equations constructed to find $\mu_3 + \mu_2$ function. Using 5.18, We can now find that,

$$\chi \pm \omega = \frac{\sqrt{\delta}(\rho^2\sqrt{\Delta} \pm 2aMr\sqrt{\delta})}{\Delta - a^2\delta} = \frac{\sqrt{\delta}\left[r^2 + a^2 \mp a\sqrt{\Delta\delta}\right]\left(\sqrt{\Delta} \pm a\sqrt{\delta}\right)}{\left(\sqrt{\Delta} + a\sqrt{\delta}\right)\left(\sqrt{\Delta} - a\sqrt{\delta}\right)}$$

We can use the above equation to find the form of x and y explicitly. We can, then, find their derivatives respectively,

$$x_{,r} = -\frac{\sqrt{\delta}}{\sqrt{\Delta}(\sqrt{\Delta} - a\sqrt{\delta})^2}\left[\rho^2(r - M) - 2r(\sqrt{\Delta} - a\sqrt{\delta})\sqrt{\Delta}\right]$$

$$x_{,\xi} = -\frac{\xi\sqrt{\Delta}}{\sqrt{\delta}(\sqrt{\Delta}-\sqrt{\delta})^2}\left[r^2+a^2+a^2\delta-2a\sqrt{\delta\Delta}\right]$$

$$y_{,r} = -\frac{\sqrt{\delta}}{\sqrt{\Delta}(\sqrt{\Delta}+a\sqrt{\delta})^2}\left[\rho^2(r-M)-2r(\sqrt{\Delta}+a\sqrt{\delta})\sqrt{\Delta}\right]$$

$$y_{,\xi} = -\frac{\xi\sqrt{\Delta}}{\sqrt{\delta}(\sqrt{\Delta}+\sqrt{\delta})^2}\left[r^2+a^2+a^2\delta+2a\sqrt{\delta\Delta}\right]$$

Substituting these into equations 5.20 and 5.19, one obtains the following equations,

$$-\frac{\xi}{\delta}(\mu_3+\mu_2)_{,r}+\frac{r-M}{\Delta}(\mu_3+\mu_2)_{,\xi}=\frac{\xi}{\rho^2\Delta\delta}\Big((r-M)(\rho^2+2a^2\delta)-2r\Delta\Big)$$

$$2(r-M)(\mu_3+\mu_2)_{,r}+2\xi(\mu_3+\mu_2)_{,\xi}=4-3\frac{(r-M)^2-\Delta}{\Delta}+\frac{\xi^2+\delta}{\delta}$$

Solving these two equations is rather complicated. However, one can readily check that a adequate solutions is,

$$\mu_3+\mu_2=\ln(\frac{\rho^2}{\sqrt{\Delta}})$$

We have now derived all the functions. The line of the Kerr have the following form,

$$ds^2=\Delta\sin^2\theta\left[\frac{\rho^2\sqrt{\delta\Delta}}{\Delta-a^2\delta}dt^2-\frac{\Delta-a^2\delta}{\rho^2\sqrt{\delta\Delta}}\left(d\phi-\frac{2aMr\delta}{\Delta-a^2\delta}dt\right)^2\right]+\frac{\rho^2}{\Delta}dr^2-\rho^2d\theta^2$$

This is, however, not a famous form. We can find the form that is commonly used in literature by using the conjugate solution $(\tilde{\chi},\tilde{\omega})$.

$$\tilde{\omega}=\frac{2Mar}{\Sigma^2}$$

$$\tilde{\chi}=\frac{\rho^2\sqrt{\Delta}}{\Sigma^2\sqrt{\delta}}$$

where

$$\Sigma^2=\frac{\rho^4\Delta-4a^2M^2r^2\delta}{\Delta-a^2\delta}=(r^2+a^2)^2-a^2\delta\Delta$$

substituting back into the line element, one can easily get the form of the Kerr metric.

$$ds^2=\rho^2\frac{\Delta}{\Sigma^2}dt^2-\frac{\Sigma^2}{\rho^2}\sin^2\theta\left(d\phi-\frac{2Mar}{\Sigma^2}dt\right)^2+\frac{\rho^2}{\Delta}dr^2-\rho^2d\theta^2 \tag{5.21}$$

Lastly, the remaining task is to make sure that indeed M and a represent the mass and the angular momentum per unit mass of the object. All we need to do is expand the previous equations at $r = \infty$. One can easily implement Mathematica to effectively expand the previous metric. One can, then, see that the components correctly matches that of the Lense Thirring metric, equation 5, with M the mass of the rotating object and a is J/M or the angular momentum per unit mass.

Appendix A

Schwarzschild Vacuum Riemannian tensor

In this appendix we will present the vacuum solutions for a spherically symmetric static object. We begin by finding the Christoffel symbol of the proposed form of the metric (3.1) as follows,

$$\Gamma^t_{tr} = \Gamma^t_{rt} = \frac{\partial_r B(r)}{2B(r)} \qquad \Gamma^r_{tt} = \frac{\partial_r B(r)}{2A(r)} \tag{A.1}$$

$$\Gamma^r_{rr} = \frac{\partial_r A(r)}{2A(r)} \qquad \Gamma^r_{\theta\theta} = -\frac{r}{A(r)} \tag{A.2}$$

$$\Gamma^r_{\phi\phi} = -\frac{r\sin^2\theta}{A(r)} \qquad \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{r} \tag{A.3}$$

$$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \qquad \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta \tag{A.4}$$

and Ricci tensor,

$$R_{tt} = \frac{1}{2}\frac{B_{,r,r}}{A} - \frac{1}{4}\left(\frac{B_{,r}}{A}\right)\left(\frac{B_{,r}}{B} + \frac{A_{,r}}{A}\right) + \frac{1}{r}\frac{B_{,r}}{A} \tag{A.5}$$

$$R_{rr} = -\frac{1}{2}\frac{B_{,r,r}}{B} + \frac{1}{4}\left(\frac{B_{,r}}{B}\right)\left(\frac{B_{,r}}{B} + \frac{A_{,r}}{A}\right) - \frac{1}{r}\frac{A_{,r}}{A} \tag{A.6}$$

$$R_{\theta\theta} = 1 - \frac{r}{2A}\left(-\frac{A_{,r}}{A} + \frac{B_{,r}}{B}\right) - \frac{1}{A} \tag{A.7}$$

$$R_{\phi\phi} = \sin^2\theta R_{\theta\theta} \tag{A.8}$$

$$\tag{A.9}$$

and $R_{\mu\nu} = 0$ for $\mu \neq \nu$. Now, solving for $R_{\mu\nu} = 0$ will give,

$$\frac{1}{2}\frac{B_{,r,r}}{A} - \frac{1}{4}\left(\frac{B_{,r}}{A}\right)\left(\frac{B_{,r}}{B} + \frac{A_{,r}}{A}\right) + \frac{1}{r}\frac{B_{,r}}{A} = 0$$

$$-\frac{1}{2}\frac{B_{,r,r}}{B}+\frac{1}{4}\left(\frac{B_{,r}}{B}\right)\left(\frac{B_{,r}}{B}+\frac{A_{,r}}{A}\right)-\frac{1}{r}\frac{A_{,r}}{A}=0$$
$$1-\frac{r}{2A}\left(-\frac{A_{,r}}{A}+\frac{B_{,r}}{B}\right)-\frac{1}{A}=0$$

We now need to solve these equations together to find the relation between A(r) and B(r). One can see that using the first two equation give,

$$\frac{R_{tt}}{B}+\frac{R_{r}}{A}=0$$

or

$$\frac{1}{rA}\left(-\frac{A_{,r}}{A}+\frac{B_{,r}}{B}\right)=0$$

Or,

$$\frac{A_{,r}}{A}=\frac{B_{,r}}{B}$$

integrating both sides with respect to r, one can easily find,

$$A(r)B(r)=\text{constant}$$

Another restriction on our metric is that when $r\longrightarrow\infty$ it return back to Minkowski. This results in this two conditions,

$$\lim_{r\longrightarrow\infty}A(r)=1 \qquad \lim_{r\longrightarrow\infty}B(r)=1$$

Thus, a convenient way to write the relation between A and B is,

$$A(r)=\frac{1}{B(r)}$$

Now, substituting in the third equation,

$$1-rB_{,r}-B=0$$

or

$$(rB)_{,r}=1$$

Integrating with respect to r,

$$B(r)=1-\frac{C}{r}$$

Appendix B

Kerr Vacuum Riemannian tensor

From this point, the computations will be very tedious and hard to follow. Thus, we will be using the Mathematica OGRe package to aid in the computation of the Ricci and Einstein tensors. One can find vacuum solutions either by,

$$G_{\mu\nu} = 0$$

or

$$R_{\mu\nu} = 0$$

For our current proposes, following Chandrasekhar, we would need only R_{tt}, $R_{\phi\phi}$, $R_{t\phi}$, R_{23}, G_{22} and G_{33} for the full derivation. Where 2 and 3 sub indexes refer to x^2 and x^3 respectively. For our current derivation, we will use the following functions to reduce the equations,

$$K = \nu + \psi - \mu_2 + \mu_3$$

$$M = \nu - 3\psi + \mu_2 - \mu_3$$

$$P = \nu + \psi + \mu_2 - \mu_3$$

$$N = \nu - 3\psi - \mu_2 + \mu_3$$

Now, one can find the components of Einstein and Ricci tensor after setting them to zero and using the Fullsimplfy package in Mathematica. With renaming $x^2 = x$ and $x^3 = y$,

$\mathbf{R_{tt}} = 0$:

$$
\begin{aligned}
&e^{2(\nu+\psi)}\left(-e^{2\mu_3}\left(2\omega^2\left(\frac{\partial\psi}{\partial x^2}+\frac{\partial\psi}{\partial x}\frac{\partial K}{\partial x}\right)+2\omega\left(\frac{\partial\omega}{\partial x^2}-\frac{\partial\omega}{\partial x}\frac{\partial M}{\partial x}\right)+\left(\frac{\partial\omega}{\partial x}\right)^2\right)\right.\\
&\quad\left.-e^{2\mu_2}\left(2\omega^2\left(\frac{\partial\psi}{\partial y^2}+\frac{\partial\psi}{\partial y}\frac{\partial P}{\partial y}\right)+2\omega\left(\frac{\partial\omega}{\partial y^2}-\frac{\partial\omega}{\partial y}\frac{\partial N}{\partial y}\right)+\left(\frac{\partial\omega}{\partial y}\right)^2\right)\right)\\
&\quad+2e^{4\nu}\left(e^{2\mu_3}\left(\frac{\partial\nu}{\partial x^2}+\frac{\partial\nu}{\partial x}\frac{\partial K}{\partial x}\right)\right.\\
&\qquad\left.+e^{2\mu_2}\left(\frac{\partial\nu}{\partial y^2}+\frac{\partial\nu}{\partial y}\frac{\partial P}{\partial y}\right)\right)=e^{4\psi}\omega(x,y)^2\left(e^{2\mu_3}\left(\frac{\partial\omega}{\partial x}\right)^2+e^{2\mu_2}\left(\frac{\partial\omega}{\partial y}\right)^2\right)
\end{aligned}
\tag{B.1}
$$

$\mathbf{R}_{\phi\phi} = 0$:

$$
\begin{aligned}
&e^{2\mu_3}\left(-2e^{2\nu}\left(\frac{\partial\psi}{\partial x^2}+\frac{\partial\psi}{\partial x}\frac{\partial K}{\partial x}\right)-e^{2\psi}\left(\frac{\partial\omega}{\partial x}\right)^2\right)\\
&\quad+e^{2\mu_2}\left(-2e^{2\nu}\left(\frac{\partial\psi}{\partial y^2}+\frac{\partial\psi}{\partial y}\frac{\partial P}{\partial x}\right)-e^{2\psi}\left(\frac{\partial\omega}{\partial y}\right)^2\right)=0
\end{aligned}
\tag{B.2}
$$

$\mathbf{R}_{t\phi} = 0$:

$$
\begin{aligned}
&e^{2\mu_3}\left(e^{2\nu}\left(2\omega\left(\frac{\partial\psi}{\partial x^2}+\frac{\partial\psi}{\partial x}\frac{\partial K}{\partial x}\right)+\frac{\partial\omega}{\partial x^2}-\frac{\partial\omega}{\partial x}\frac{\partial M}{\partial x}\right)+e^{2\psi}\omega\left(\frac{\partial\omega}{\partial x}\right)^2\right)\\
&\quad+e^{2\mu_2}\left(e^{2\nu}\left(2\omega\left(\frac{\partial\psi}{\partial y^2}+\frac{\partial\psi}{\partial y}\frac{\partial P}{\partial x}\right)+\frac{\partial\omega}{\partial y^2}-\frac{\partial\omega}{\partial y}\frac{\partial N}{\partial x}\right)+e^{2\psi}\omega\left(\frac{\partial\omega}{\partial y}\right)^2\right)=0
\end{aligned}
\tag{B.3}
$$

$\mathbf{R}_{23} = 0$:

$$
\begin{aligned}
&-\frac{\partial^2\nu}{\partial x\partial y}-\frac{\partial^2\psi}{\partial x\partial y}+\frac{\partial\mu_2}{\partial y}\left(\frac{\partial\nu}{\partial x}+\frac{\partial\psi}{\partial x}\right)+\frac{\partial\nu}{\partial y}\left(\frac{\partial\mu_3}{\partial x}-\frac{\partial\nu}{\partial x}\right)\\
&\quad+\frac{\partial\psi}{\partial y}\left(\frac{\partial\mu_3}{\partial x}-\frac{\partial\psi}{\partial x}\right)+\frac{1}{2}e^{2\psi-2\nu}\frac{\partial\omega}{\partial x}\frac{\partial\omega}{\partial y}=0
\end{aligned}
\tag{B.4}
$$

$\mathbf{G}_{22} = 0$:

$$
\begin{aligned}
&e^{2\mu_3}\left(4e^{2\nu}\left(\frac{\partial\mu_3}{\partial x}\left(\frac{\partial\nu}{\partial x}+\frac{\partial\psi}{\partial x}\right)+\frac{\partial\nu}{\partial x}\frac{\partial\psi}{\partial x}\right)+e^{2\psi}\left(\frac{\partial\omega}{\partial x}\right)^2\right)\\
&\quad+e^{2\mu_2}\left(4e^{2\nu}\left(\frac{\partial\nu}{\partial y^2}+\frac{\partial\psi}{\partial y^2}-\frac{\partial\mu_3}{\partial y}\left(\frac{\partial\nu}{\partial y}+\frac{\partial\psi}{\partial y}\right)+\frac{\partial\nu}{\partial y}\frac{\partial\psi}{\partial y}+\left(\frac{\partial\nu}{\partial y}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2\right)\right.\\
&\quad\left.-e^{2\psi}\left(\frac{\partial\omega}{\partial y}\right)^2\right)=0
\end{aligned}
\tag{B.5}
$$

$\mathbf{G}_{33} = 0$:

$$
\begin{aligned}
&e^{2\mu_3}\left(4e^{2\nu}\left(\frac{\partial\nu}{\partial x^2}+\frac{\partial\psi}{\partial x^2}-\frac{\partial\mu_2}{\partial x}\left(\frac{\partial\nu}{\partial x}+\frac{\partial\psi}{\partial x}\right)+\frac{\partial\nu}{\partial x}\frac{\partial\psi}{\partial x}+\left(\frac{\partial\nu}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial x}\right)^2\right)\right.\\
&\quad\left.-e^{2\psi}\left(\frac{\partial\omega}{\partial x}\right)^2\right)+e^{2\mu_2}\left(4e^{2\nu}\left(\frac{\partial\mu_2}{\partial y}\left(\frac{\partial\nu}{\partial y}+\frac{\partial\psi}{\partial y}\right)+\frac{\partial\nu}{\partial y}\frac{\partial\psi}{\partial y}\right)+e^{2\psi}\left(\frac{\partial\omega}{\partial y}\right)^2\right)\\
&\quad=0
\end{aligned}
\tag{B.6}
$$

It now a matter of simple substitution and rearrangement to bring the above equations to this desired form,

$\mathbf{R_{tt}} = 0$:

$$
e^{-\mu_2}\left(\frac{\partial^2\nu}{\partial x^2}+\frac{\partial\nu}{\partial x}\frac{\partial K}{\partial x}\right)+e^{-\mu_3}\left(\frac{\partial^2\nu}{\partial y^2}+\frac{\partial\nu}{\partial y}\frac{\partial P}{\partial y}\right)=\frac{1}{2}e^{2\psi-2\nu}\left(e^{-\mu_2}\left(\frac{\partial\omega}{\partial x}\right)^2+e^{-\mu_3}\left(\frac{\partial\omega}{\partial y}\right)^2\right)
\tag{B.7}
$$

$\mathbf{R}_{\phi\phi} = 0$:

$$
\begin{aligned}
&e^{-\mu_2}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial\psi}{\partial x}\frac{\partial K}{\partial x}\right)+e^{-\mu_3}\left(\frac{\partial^2\psi}{\partial y^2}+\frac{\partial\psi}{\partial y}\frac{\partial P}{\partial y}\right)\\
&\quad=-\frac{1}{2}e^{2\psi-2\nu}\left(e^{-\mu_2}\left(\frac{\partial\omega}{\partial x}\right)^2+e^{-\mu_3}\left(\frac{\partial\omega}{\partial y}\right)^2\right)
\end{aligned}
\tag{B.8}
$$

$\mathbf{R}_{t\phi} = 0$:

$$
\frac{\partial}{\partial x}\left(e^{-M}\frac{\partial\omega}{\partial x}\right)+\frac{\partial}{\partial y}\left(e^{-N}\frac{\partial\omega}{\partial y}\right)=0
\tag{B.9}
$$

$\mathbf{R}_{23} = 0$:

$$
\frac{\partial^2(\psi+\nu)}{\partial x\partial y}-\frac{\partial(\psi+\nu)}{\partial x}\frac{\partial\mu_2}{\partial y}-\frac{\partial(\psi+\nu)}{\partial y}\frac{\partial\mu_3}{\partial x}+\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial y}+\frac{\partial\nu}{\partial x}\frac{\partial\nu}{\partial y}=\frac{1}{2}e^{2\psi-2\nu}\frac{\partial\omega}{\partial x}\frac{\partial\omega}{\partial y}
\tag{B.10}
$$

$\mathbf{G}_{22} = 0$:

$$
\begin{aligned}
&e^{-2\mu_3}\left(\frac{\partial^2(\psi+\nu)}{\partial y^2}+\frac{\partial(\nu-\mu_3)}{\partial y}\frac{\partial(\psi+\nu)}{\partial y}+\left(\frac{\partial\psi}{\partial y}\right)^2\right)\\
&+e^{-2\mu_2}\left(\frac{\partial\nu}{\partial x}\frac{\partial(\psi+\mu_3)}{\partial x}+\frac{\partial\psi}{\partial x}\frac{\partial\mu_3}{\partial x}\right)=\\
&-\frac{1}{4}e^{2\psi-2\nu}\left(e^{-2\mu_2}\left(\frac{\partial\omega}{\partial x}\right)^2-e^{-2\mu_3}\left(\frac{\partial\omega}{\partial y}\right)^2\right)
\end{aligned}
\tag{B.11}
$$

$\mathbf{G}_{33} = 0$:

$$
\begin{aligned}
&e^{-2\mu_2}\left(\frac{\partial^2(\psi+\nu)}{\partial x^2}+\frac{\partial(\nu-\mu_2)}{\partial x}\frac{\partial(\psi+\nu)}{\partial x}+\left(\frac{\partial\psi}{\partial x}\right)^2\right)+e^{-2\mu_3}\left(\frac{\partial\nu}{\partial y}\frac{\partial(\psi+\mu_2)}{\partial y}\right.\\
&\left.+\frac{\partial\psi}{\partial y}\frac{\partial\mu_2}{\partial y}\right)=\frac{1}{4}e^{2\psi-2\nu}\left(e^{-2\mu_2}\left(\frac{\partial\omega}{\partial x}\right)^2-e^{-2\mu_3}\left(\frac{\partial\omega}{\partial y}\right)^2\right)
\end{aligned}
\tag{B.12}
$$

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