Quiz 1: Gamma & beta functions

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Problem 1
Evaluate,

$$\int_a^\infty e^{2ax-x^2}dx$$

Solution: First, we need to complete the square.

$$2ax - x^2 = -(x^2 - 2ax + a^2 - a^2) = -(x-a)^2 + a^2$$

Thus, we can rewrite the integral as,

$$\begin{aligned}
\int_a^\infty e^{2ax-x^2}dx &= e^{a^2}\int_a^\infty e^{-(x-a)^2}dx \\
&= e^{a^2}\int_0^\infty e^{-x^2}dx && (x \to x+a) \\
&= \frac{1}{2}e^{a^2}\int_0^\infty u^{-1/2}e^{-u}du && (\text{substitute } u = x^2) \\
&= \frac{1}{2}e^{a^2}\Gamma(1/2) = \frac{1}{2}\sqrt{\pi}e^{a^2}
\end{aligned}$$

Problem 2
Evaluate,

$$\int_0^1 x^k(\ln(x))^n dx$$

Solution:

$$\begin{aligned}
\int_0^1 x^k(\ln(x))^n dx &= \int_0^\infty (-t)^n e^{-kt}e^{-t}dt && (\text{substitute } x = e^{-t}) \\
&= (-1)^n\int_0^\infty t^n e^{-(k+1)t}dt \\
&= \frac{(-1)^n}{(k+1)^{n+1}}\int_0^\infty u^n e^{-u}du && (\text{substitute } u = (k+1)t) \\
&= \frac{(-1)^n\Gamma(n+1)}{(k+1)^{n+1}} = \frac{(-1)^n n!}{(k+1)^{n+1}}
\end{aligned}$$

Note that in the third step, the limits won't change if $k > -1$. The integral diverges otherwise (check).

Problem 3

$$\int_0^a x^4\sqrt{a^2-x^2}dx$$

Solution: This integral is fundamentally the beta integral upon substitution $x^2 = a^2u$. Notice also that,

$$dx = \pm\frac{u^{-1/2}}{2|a|}a^2du = \pm|a|\frac{u^{-1/2}}{2}du$$

We will take either the positive or the negative branch depending on the sign of a (since a positive a means all values of x are essentially positive and vice versa).

$$\begin{aligned}
\int_0^a x^4\sqrt{a^2-x^2}dx &= \pm\int_0^1 (a^2u)^2\sqrt{(a^2)(1-u)}|a|\frac{u^{-1/2}}{2}du = \pm\frac{|a|^6}{2}\int_0^1 u^{3/2}(1-u)^{1/2}du \\
&= \pm\frac{a^6}{2}B(5/2, 3/2)
\end{aligned}$$

positive if $a > 0$ and negative otherwise.

Problem 4
Show that
$$\int_0^\infty e^{-st}(1-e^{-t})^n dt = \frac{n!\Gamma(s)}{\Gamma(s+n+1)}$$
where $s > 0$ and $n = 0, 1, 2, \cdots$

Solution: This form looks like the beta function. Thus, a motivated substitution is $u = e^{-t}$. Since $e^{-st} = (e^{-t})^s$. Check that the substitution gives the following formula,

$$\int_0^\infty e^{-st}(1-e^{-t})^n dt = \int_0^1 u^{s-1}(1-u)^n du = B(s, n+1) = \frac{\Gamma(s)\Gamma(n+1)}{\Gamma(s+n+1)} = \frac{n!\Gamma(s)}{\Gamma(s+n+1)}$$

Problem 5
Show that
$$\int_{-1}^1 (1-x^2)^n dx = 2^{2n+1}\frac{(n!)^2}{(2n+1)!}$$
where $n = 0, 1, 2, \cdots$

Solution: A quick guess to solve this integral would be a substitution $u = x^2$ to have a beta function. However, this would send both limits (1,-1) to 1. However, notice the function is symmetric thus we can write integral as,

$$\int_{-1}^1 (1-x^2)^n dx = 2\int_0^1 (1-x^2)^n dx$$

Now substitute your clever guess $u = x^2$. Thus, one can get,

$$2\int_0^1 (1-x^2)^n dx = \int_0^1 (1-u)^n u^{-1/2} du = B(1/2, n+1) = \frac{\Gamma(1/2)\Gamma(n+1)}{\Gamma(3/2+n)}$$

Now you can easily use the Legendre duplication formula with $x \mapsto n+1$ to get the desired result since,
$$\Gamma(1/2)\Gamma(2n+2) = 2^{2n+1}\Gamma(n+1)\Gamma(n+3/2)$$

Thus,
$$\int_{-1}^1 (1-x^2)^n dx = 2^{2n+1}\frac{\Gamma(n+1)^2}{\Gamma(2n+2)} = 2^{2n+1}\frac{(n!)^2}{(2n+1)!}$$