Tutorial 5: Laplace Equation, Wave Equation & Non-homogenous PDEs

Mohamed Salaheldeen

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Under the supervision of: Dr. Abdallah Aboutahoun

1 Important Points

Consider a steady heat flow in 2 spatial dimensions x and y. Since we are looking for steady state solutions, the solution u is only a function of x and y. Thus, the heat equation have the following form,

$$\begin{aligned} \textbf{PDE: } & \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \\ \textbf{BCs: } & u(0, y) = g_1(y) \\ & u(L, y) = g_2(y) \\ & u(x, 0) = f_1(x) \\ & u(x, L) = f_2(x) \end{aligned} \tag{1}$$

Notice here the the PDEs has non-homogeneous boundary conditions. To solve such equation, one can separate this problem into 4 or 2 problem with homogeneous BC. Then, use the principle of superposition by summing the corresponding solutions. See Problem [][] for examples.

Laplace equation has some qualitative properties which can help us understand theoretically the nature of solutions.

1. **Mean Value Theorem**: If one attempts to utilize the circular symmetry of Laplace equation, one can easily find the following solution,

$$u(r, \theta) = \sum_{n=0}^{\infty} A_n r^n \cos(n\theta) + \sum_{n=1}^{\infty} B_n r^n \sin(n\theta)$$

   Thus, at $r = 0$,

$$u(0, \theta) = a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta$$

   where $f(\theta)$ is a circular boundary condition. Thus, one can consider such integral as an average of the function around a given point. This is called the mean value property. Thus, a steady state heat solution implies that the temperature at any point is the average of the temperature along any circle of radius r_0 centered at that point which is very intuitive.

2. **Maximum principles**: One can use the mean value to prove that in steady state, assuming no sources the temperature cannot attain its maximum or minimum in the interior unless the temperature is constant. Thus a solution of Laplace equation would only attain its maximum and minimum on the boundaries of the domain.

3. **Uniqueness**: A solution to Laplace equation is unique given they satisfy the same boundary conditions. Furthermore, these solutions are well posed meaning that varying the BC slightly would not affect the solution that much which is very physical and intuitive.

Another important PDE that we are interested to solve is the wave equation.

$$\textbf{PDE: } \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{2}$$

with two boundary conditions and two initial conditions. We are interested in bounded and unbounded spatial solutions to the wave equation. Furthermore, a general solution to bounded/unbounded wave equations can be written as,

$$u(x, t) = F(x - ct) + G(x + ct)$$

This is known as D'Alembert principle.

Lastly, we are going to consider solution to non-homogeneous PDEs either time-independent or time-depended. We can introduce non-homogeneity either by adding an explicit term to the form of the PDE or in the boundary conditions. Let us consider the most general case in the heat equation,

$$\begin{aligned} \textbf{PDE:}\ & \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t) \\ \textbf{BCs:}\ & u(0,t) = A(t) \\ & u(L,t) = B(t) \\ \textbf{ICs:}\ & u(x,0) = f(x) \end{aligned} \tag{3}$$

Its not always the case that we can reduce such problem to a similar one with homogeneous boundary conditions. However, for the purposes of this course, consider any reference temperature say $r(x,t)$ with the property that it satisfies the non-homogeneous boundary conditions.

$$r(0,t) = A(t)$$
$$r(L,t) = B(t)$$

There exist infinite number of functions that satisfy such conditions. However, we consider the following distribution,

$$r(x,t) = A(t) + \frac{x}{L}[B(t) - A(t)]$$

Then consider the following function

$$v(x,t) = u(x,t) - r(x,t)$$

It follows that $v(x,t)$ satisfies the homogeneous boundary conditions.

$$v(0,t) = 0$$
$$v(L,t) = 0$$

Please note that the main purpose of this substitution is to homogenize the boundary conditions. Thus, we removed the non-homogeneity from the boundary conditions with the cost of changing the form of the differential equation. One can see this by finding the differential equations satisfied by $v(x,t)$,

$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} + Q(x,t) - \frac{\partial r}{\partial t} + k\frac{\partial^2 r}{\partial x^2} = k\frac{\partial^2 v}{\partial x^2} + q(x,t)$$

Furthermore, the initial condition is also altered to,

$$v(x,0) = f(x) - r(x,0) = f(x) - A(0) - \frac{x}{L}[B(0) - A(0)] = g(x)$$

Thus, we have reduced problem 3 to,

$$\begin{aligned} \textbf{PDE:}\ & \frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} + q(x,t) \\ \textbf{BCs:}\ & v(0,t) = 0 \\ & v(L,t) = 0 \\ \textbf{ICs:}\ & v(x,0) = g(x) \end{aligned} \tag{4}$$

To solve such problem, we employ a method called the method of eigenfunctions expansion. This method works in general either for heat, wave or Laplace equations as long as the boundaries are homogeneous. Here are the steps,

1. Consider the corresponding PDE with no source term $q(x,t)$. Solve the equation with homogeneous boundary conditions and find eigenfunctions $\phi_n(x)$.

2. Expand the solution $v(x,t)$ in terms of the given eigenfunctions,

$$v(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x)$$

Since, $v(x,t)$ has to satisfy the initial conditions. We also have,

$$g(x) = \sum_{n=1}^{\infty} a_n(0)\phi_n(x)$$

Thus, we can use orthogonality of the eigenfunctions to find $a_n(0)$ explicitly.

3. Lastly, our main task is to find $a_n(t)$. To do so, we substitute in the original differential equation.

$$\frac{\partial v}{\partial t} = \sum_n \frac{da_n}{dt}\phi_n$$

$$\frac{\partial^2 v}{\partial x^2} = \sum_n a_n \frac{\partial^2 \phi}{\partial x^2} = -\sum_n a_n(t)\lambda_n\phi_n(x)$$

subsiuting in 4 gives,

$$\sum_{n=1}^{\infty} \left[\frac{da_n}{dt} + \lambda_n k a_n\right]\phi_n(x) = q(x,t)$$

4. Thus, using orthogonality of the eigenfunctions, one can find that,

$$\frac{da_n}{dt} + \lambda_n k a_n = \frac{\int_0^L q(x,t)\pi_n(x)dx}{\int_0^L \phi^2(x)dx} = p(t)$$

This is a first order differential equation which has the solution,

$$a_n(t) = a_n(0)e^{-\lambda_n kt} + e^{-\lambda_n kt}\int_0^t p(\tau)e^{\lambda_n k\tau}d\tau$$

Thus, we found the general solution of the PDE. Finally, we can substitute back to find $u(x,t)$ thus solving 3.
Notice: There is an interesting case when $Q(x,t)$ in 3 is only a function of x with the boundary conditions being constant. This case is known as time-independent PDE with BC. One can directly get rid of the source term by an appropriate function $r(x)$ to kill the term. Thus, we solve

$$k\frac{\partial^2 r}{\partial x^2} + Q(x) = 0$$

This directly gives us back our usual homogeneous PDE which we can solve easily.
Notice: ϕ_n is different for different PDE we only showed the case for the heat equation.

2 Selected Exercises

Problem 1
Solve Laplace equation subject to these boundary conditions,

$$u(0,y) = 10y \qquad \left.\frac{\partial u}{\partial x}\right|_{x=1} = -1$$

$$u(x,0) = 0 \qquad u(x,1) = 0$$

Solution: By seperation of variables, we get,

$$\frac{1}{\phi(x)}\frac{\partial^2 \phi(x)}{\partial x^2} = -\frac{1}{\psi(y)}\frac{\partial^2 \psi(y)}{\partial y^2} = \lambda$$

The boundaries on y are homogeneous thus we use this part to find the eigenvalues. It's clear that the eigenvalues of the separation of variables cannot be zero or negative. Thus, we get,

$$\psi(y) = c_1 \cos(\sqrt{\lambda}y) + c_2 \sin(\sqrt{\lambda}y)$$

By substituting the boundary condition on y. We can find the eigenvalues. Thus,

$$\psi(y) = c_2 \sin(n\pi y)$$

where the eigenvalues $\lambda = n^2\pi^2$. We can then find $\phi(x)$.

$$\phi(x) = c_1 \cosh(n\pi x) + c_2 \sinh(n\pi x)$$

Thus, a general solution would be,

$$u(x,y) = \sum_{n=1}^{\infty} \sin(n\pi y)\left[A_n \cosh(n\pi x) + B_n \sinh(n\pi x)\right]$$

Lastly, we substitute the other two boundary condition to find A_n and B_n. at $x = 0$,

$$u(0,y) = 10y = \sum_{n=1} A_n \sin(n\pi y)$$

This, shows that,

$$A_n = 20\int_0^1 y\sin(n\pi y) = 20\frac{(-1)^{n+1}}{n\pi}$$

Similarly the boundary condition at $x = 1$ gives,

$$-1 = \sum_{n=1}^{\infty} n\pi \sin(n\pi y)\left[A_n \sinh(n\pi) + B_n \cosh(n\pi)\right]$$

Using orthogonality of sin, we have,

$$n\pi\left[A_n \cosh(n\pi) + B_n \sinh(n\pi)\right] = -2\int_0^1 \sin(n\pi y)dy = -\frac{2}{n\pi}[1-(-1^n)]$$

Thus,

$$B_n = -\frac{2}{n^2\pi^2}[1-(-1^n)]\text{sech}(n\pi) - A_n\tanh(n\pi) = -\frac{2}{n^2\pi^2}[1-(-1^n)]\text{sech}(n\pi) + 20\frac{(-1)^n}{n\pi}\tanh(n\pi)$$

Problem 2
Solve Laplace equation subject to these boundary conditions,

$$\begin{aligned} u(0, y) &= 0 \\ u(\pi, y) &= 0 \\ u(x, 0) &= f(x) \end{aligned}$$

and the solution is bounded in the limit $y \to \infty$.

Solution: This is a problem for the semi-infinite plate. Since the boundaries are homogeneous on x we solve this part first to find the eigenvalues. As we did before in the previous question, one can find that,

$$\phi_n(x) = c_1 \sin(ny)$$

with $n > 0$ where $\lambda_n = n^2$ are eigenvalues. Now, we concentrate on the y part. After separation of variables, one gets,

$$\frac{1}{\psi(y)} \frac{\partial^2 \psi}{\partial y^2} = \lambda_n$$

Thus,

$$\psi(y) = c_1 \cosh(ny) + c_2 \sinh(ny)$$

Its better in this cases to use exponential as solutions since the domain is infinite. Thus,

$$\psi(y) = c_1 e^{ny} + c_2 e^{-ny}$$

Now, we substitute the boundary conditions. The full solution is,

$$u(x, t) = \sum_{n=1}^{\infty} \sin(nx) \left\{ A_n e^{ny} + B_n e^{-ny} \right\}$$

Since the solution is bounded as $y \to \infty$, Then, all A_n must be zero. This is because the exponential terms would blow up which contradicts the required property of the solution. Lastly, using $u(x, 0) = f(x)$,

$$u(x, 0) = f(x) = \sum_{n=1}^{\infty} B_n \sin(nx)$$

Thus,

$$B_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx$$

Problem 3
Solve Laplace equation subject to these boundary conditions,

$$u(0, y) = 0 \qquad u(2, y) = y(2 - y)$$

$$u(x, 0) = 0 \qquad u(x, 2) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 \leq x < 2 \end{cases}$$

Solution: Notice here that neither the x nor the y direction have homogeneous boundary conditions. Thus, we must use the principle of superposition to homogenize the boundaries. Thus, we split the problem,

$$\begin{aligned} u(0, y) &= 0 \\ u(2, y) &= 0 \\ u(x, 0) &= 0 \\ u(x, 2) &= \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 \leq x < 2 \end{cases} \end{aligned} \qquad\qquad \begin{aligned} u(0, y) &= 0 \\ u(2, y) &= y(2 - y) \\ u(x, 0) &= 0 \\ u(x, 2) &= 0 \end{aligned}$$

Solving the BC on the left gives,

$$\phi(x) = c_1 \sin(\frac{n\pi}{2}x)$$

$$\psi(y) = c_2 \sinh(\frac{n\pi}{2}y)$$

Thus, the solution would be,

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi}{2}x)\sinh(\frac{n\pi}{2}y)$$

Lastly, we need to substitute the last boundary condition to find A_n,

$$u(x,2) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi}{2}x)\sinh(n\pi)$$

and by using orthogonality of the sine function,

$$A_n = \frac{1}{\sinh(n\pi)}\left[\int_0^1 x\sin(\frac{n\pi}{2}x)dx + \int_1^2 (2-x)\sin(\frac{n\pi}{2}x)dx\right] = \frac{8\sin(\frac{n\pi}{2})}{n^2\pi^2\sinh(n\pi)} = \frac{4[1-(-1)^n]}{n^2\pi^2\sinh(n\pi)}$$

Similarly, we do the same steps for the BC on the right. This time y is homogeneous, thus, one can easily obtain,

$$\psi(y) = c_1 \sin(\frac{n\pi}{2}y)$$

$$\phi(x) = c_2 \sinh(\frac{n\pi}{2}x)$$

Giving the solution,

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi}{2}y)\sinh(\frac{n\pi}{2}x)$$

Similarly, we find B_n by using the orthogonality of sine.

$$B_n = \frac{1}{\sinh(n\pi)}\left[\int_0^2 y(2-y)\sin(\frac{n\pi}{2}y)dy\right] = \frac{16[1-(-1)^n]}{n^3\pi^3\sinh(n\pi)}$$

Thus, the full solution is,

$$u(x,y) = \sum_{n=1}^{\infty}\left[\frac{4[1-(-1)^n]}{n^2\pi^2\sinh(n\pi)}\right]\sin(\frac{n\pi}{2}x)\sinh(\frac{n\pi}{2}y) + \left[\frac{16[1-(-1)^n]}{n^3\pi^3\sinh(n\pi)}\right]\sin(\frac{n\pi}{2}y)\sinh(\frac{n\pi}{2}x)$$

Problem 4
Solve the wave equation subject to the following conditions,

$$u(0,t) = 0 \qquad u(\pi,t) = 0$$

$$u(x,0) = \frac{1}{6}x(\pi^2 - x^2) \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$$

Solution: After separation of variables one gets,

$$\frac{1}{c^2 G(t)}\frac{\partial^2 G(t)}{\partial t^2} = \frac{1}{\phi(x)}\frac{\partial^2 \phi(x)}{\partial x^2} = -\lambda$$

Its clear that the x direction is homogeneous. Thus, we can easily solve the corresponding DE giving,

$$\phi(x) = c_1 \sin(nx)$$

where $n > 0$. Similarly, the equation for $G(t)$ is,

$$G(t) = c_1 \cos(cnt) + c_2 \sin(cnt)$$

Thus, giving rise to the general solution,

$$u(x,t) = \sum_{n=1}^{\infty} \sin(nx)\left[A_n \cos(cnt) + B_n \sin(cnt)\right]$$

Now, we substitute by the last two initial conditions for time. Since $\frac{\partial u}{\partial t}\big|_{t=0} = 0$,

$$\sum_{n=1}^{\infty} cnB_n \sin(nx) = 0$$

Thus,

$$B_n = 0$$

The last conditions gives,

$$\sum_{n=1}^{\infty} A_n \sin(nx) = \frac{1}{6}x(\pi^2 - x^2)$$

Thus,

$$A_n = \frac{1}{3\pi}\int_0^{\pi} x(\pi^2 - x^2)\sin(nx)dx = \frac{2(-1)^{n+1}}{n^3}$$

Thus, the full solution is,

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n^3} \sin(nx)\cos(cnt)$$

Problem 5

Solve the wave equation subject to the following conditions,

$$u(0,t) = 0 \qquad u(1,t) = 0$$

$$u(x,0) = x(1-x) \qquad \frac{\partial u}{\partial t}\bigg|_{t=0} = x(1-x)$$

Solution: Similar to the previous problem the x direction is homogeneous thus,

$$\phi(x) = c_1 \sin(n\pi x)$$

$$G(t) = c_1 \cos(cn\pi t) + c_2 \sin(cn\pi t)$$

$$u(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x)\left[A_n \cos(cn\pi t) + B_n \sin(cn\pi t)\right]$$

We just need to solve for coefficients using the initial conditions given. at $t = 0$,

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) = x(1-x)$$

Thus,

$$A_n = 2\int_0^1 x(1-x)\sin(n\pi x)dx = \frac{4(1-(-1)^n)}{n^3\pi^3}$$

similarly, the other initial condition gives,

$$\sum_{n=1}^{\infty} cn\pi B_n \sin(n\pi x) = x(1-x)$$

Thus,

$$B_n = \frac{2}{cn\pi}\int_0^1 x(1-x)\sin(n\pi x) = \frac{4(1-(-1)^n)}{cn^4\pi^4}$$

Problem 6
Solve the following PDE,

$$\begin{aligned}
\textbf{PDE:}\ & \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + 2\beta \frac{\partial u}{\partial t} \\
\textbf{BCs:}\ & u(0,t) = 0 \\
& u(\pi, t) = 0 \\
\textbf{ICs:}\ & u(x,0) = f(x) \\
& \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0
\end{aligned}$$

with $0 < \beta < 1$.

Solution: Using separation of variables and dividing by $G(t)\phi(x)$, one can get

$$\frac{1}{G(t)} \frac{\partial^2 G(t)}{\partial t^2} - \frac{2\beta}{G(t)} \frac{\partial G(t)}{\partial t} = \frac{1}{\phi(x)} \frac{\partial^2 \phi(x)}{\partial x^2} = -\lambda$$

since BC are homogeneous in x we solve for eigenvalues,

$$\phi(x) = c_1 \sin(nx)$$

All we need to do now is to solve the equation for the time variable,

$$\frac{\partial^2 G(t)}{\partial t^2} - 2\beta \frac{\partial G(t)}{\partial t} + nG(t) = 0$$

To solve this DE, we solve the indicial equation $k^2 - 2\beta k + n = 0$. Thus, $k = \beta \pm \sqrt{\beta^2 - n}$ giving the solution,

$$G(t) = c_1 e^{\left[\beta + \sqrt{\beta^2 - n}\right]t} + c_2 e^{\left[\beta - \sqrt{\beta^2 - n}\right]t}$$

since $\beta < 1$, the square root produces a complex number. Thus, we can simplify $G(t)$ even more,

$$G(t) = e^{\beta t}\left\{c_1 e^{i\sqrt{n-\beta^2}t} + c_2 e^{-i\sqrt{n-\beta^2}t}\right\} = e^{\beta}\left\{m_1 \cos\left(\sqrt{n-\beta^2}t\right) + m_2 \sin\left(\sqrt{n-\beta^2}t\right)\right\}$$

Thus, the general solution is,

$$u(x,t) = \sum_{n=1}^{\infty} \sin(nx) e^{\beta t}\left\{A_n \cos\left(\sqrt{n-\beta^2}t\right) + B_n \sin\left(\sqrt{n-\beta^2}t\right)\right\}$$

Notice that we used cos and sine instead of the exponential to simplify the terms. Lastly, we use the initial conditions to find A_n and B_n. Using the last condition, one can find

$$A_n \beta + B_n \sqrt{n - \beta^2} = 0$$

Thus,

$$B_n = -A_n \frac{\beta}{\sqrt{n-\beta^2}}$$

Thus, we can simplify the solution as,

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(nx) e^{\beta t}\left\{\cos\left(\sqrt{n-\beta^2}t\right) - \frac{\beta}{\sqrt{n-\beta^2}} \sin\left(\sqrt{n-\beta^2}t\right)\right\}$$

Now, we use the third condition to find A_n. At $t = 0$,

$$f(x) = \sum_{n=1}^{\infty} A_n \sin(nx)$$

Thus,

$$A_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx$$

Problem 7
Solve the following PDE,

$$\begin{aligned} \textbf{PDE:}\ & \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + e^{-t} \sin(3x) \\ \textbf{BCs:}\ & u(0,t) = 0 \\ & u(\pi, t) = 1 \\ \textbf{ICs:}\ & u(x,0) = f(x) \end{aligned}$$

Solution: Its clear that there is no homogeneous conditions to solve this PDE. Thus, we need to make the boundary conditions homogeneous by finding $r(x,t)$. As discussed above,

$$r(x) = 0 + \frac{x}{\pi}[1 - 0] = \frac{x}{\pi}$$

Thus, we change u with,

$$v(x,t) = u(x,t) - \frac{x}{\pi}$$

This directly transforms the problem into,

$$\begin{aligned} \textbf{PDE:}\ & \frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + e^{-t} \sin(3x) \\ \textbf{BCs:}\ & v(0,t) = 0 \\ & v(\pi, t) = 0 \\ \textbf{ICs:}\ & v(x,0) = f(x) - \frac{x}{\pi} \end{aligned}$$

Since we successfully homogenized the x direction we can now apply the method of eigenfunciton expansion.

1. First we need to find eigenfunctions of the homogeneous problem

$$\begin{aligned} \textbf{PDE:}\ & \frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} \\ \textbf{BCs:}\ & v(0,t) = 0 \\ & v(\pi, t) = 0 \end{aligned}$$

The eigenfucntions and eigenvalues in the x direction are easy to derive,

$$\phi(x) = c_1 \sin(nx)$$

2. Now, we expand out $v(x,t)$ in terms of the eigenfunctions,

$$v(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin(nx)$$

with

$$v(x,0) = f(x) - \frac{x}{\pi} = \sum_{n=1}^{\infty} a_n(0)\sin(nx)$$

Thus,

$$a_n(0) = \frac{2}{\pi}\int_0^{\pi}\left[f(x) - \frac{x}{\pi}\right]\sin(nx)dx$$

3. Now, we substitute back our expansion into the original PDE. Thus, we get,

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + e^{-t}\sin(3x) \implies \sum_{n=1}^{\infty}\left[\frac{da_n(t)}{dt} + n^2 a_n(t)\right]\sin(nx) = e^{-t}\sin(3x)$$

4. Lastly, we solve for a_n. Its to our advantage in this problem that we can easily compare coefficients,

$$\frac{da_n(t)}{dt} + n^2 a_n(t) = \begin{cases} 0, & n \neq 3 \\ e^{-t}, & n = 3 \end{cases}$$

Thus, when $n \neq 3$, we can solve for a_n as,

$$a_n = a_n(0)e^{-n^2 t}$$

similarly for $n = 3$,

$$a_3 = \frac{1}{8}e^{-t} + \left\{a_3(0) - \frac{1}{8}\right\}e^{-9t}$$

Now, we can write $v(x,t)$ explicitly and correspondingly find $u(x,t)$.

Problem 8
Solve the following PDE,

$$\begin{aligned} \textbf{PDE: } & \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Ae^{-\beta x} \\ \textbf{BCs: } & u(0,t) = 0 \\ & u(1,t) = 0 \\ \textbf{ICs: } & u(x,0) = f(x) \end{aligned}$$

Solution: Although we can directly solve this problem by eigenfunction expansion directly since the x direction is homogeneous. We can further reduce the problem by choosing an appropriate $r(x)$ to absorb the source term since the source term is only a function of x. So, again make the substitution $u(x,t) = v(x,t) + r(x)$ this gives the following PDE,

$$\begin{aligned} \textbf{PDE: } & \frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} + k\frac{\partial^2 r}{\partial x^2} + Ae^{-\beta x} \\ \textbf{BCs: } & v(0,t) + r(0) = 0 \\ & v(1,t) + r(1) = 0 \\ \textbf{ICs: } & v(x,0) = f(x) - r(x) \end{aligned}$$

Thus, we choose $r(x)$ such that,

$$k\frac{\partial^2 r}{\partial x^2} + Ae^{-\beta x} = 0$$

with $r(0) = r(1) = 0$. We can easily solve for $r(x)$ as follows,

$$r(x) = \frac{A}{k\beta^2}\left[1 - e^{-\beta x}\right] + \frac{A}{k\beta^2}\left[e^{-\beta} - 1\right]x$$

Since we would find $r(x)$ with these properties, our original problem transforms into a homogeneous problem,

$$\begin{aligned}
\textbf{PDE: } & \frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} \\
\textbf{BCs: } & v(0,t) = 0 \\
& v(1,t) = 0 \\
\textbf{ICs: } & v(x,0) = f(x) - r(x)
\end{aligned}$$

Now, you can easily solve for $v(x,t)$,

$$v(x,t) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) e^{-n^2 k\pi^2 t}$$

where

$$A_n = 2\int_0^1 [f(x) - r(x)] \sin(n\pi x) dx$$