International Center for Theoretical Physics

High Engery and Astroparticle Physics Section

2D Topological Field Theories

Postgraduate Diploma Thesis

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1 Introduction and Physical Motivations

Quantum Field Theory (QFT) is one of the fundamental tools that describe our nature. It is the theoretical framework that describes elementary particles and their interactions. Despite its success in describing most of the interactions observed, there exist a lot of challenges in describing QFTs on a non-perturbative level [2]. This motivates certain digressions to formally define quantum field theories which may reconcile aspects to understand non-perturbative QFT. Herein, we investigate formal aspects of quantum field theories. We focus on Topological Quantum Field Theories (TQFT). Furthermore, we investigate a larger class of theories known as *topological theories* and their construction.

1.1 Motivation

We shall begin our discussion with the basic construction of QFTs. To build a QFT, one needs to have a Riemannian (or pseudo-Riemannian) manifold $\mathcal{M}$ of dimension say n. Usually, for some physical applications, the manifold is endowed with some structure like a metric g [16]. Then, one has to add objects on the manifold which describe the universe. The simplest choice is a scalar field ϕ which defines a map to some field $\mathbb{K}$ (ordinarily $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$),

$$\phi : \mathcal{M} \to \mathbb{K}$$

In general, ϕ could be a map to another manifold $\mathcal{N}$ known as the target space. Let $\mathcal{C}$ denote the configuration space of ϕ on $\mathcal{M}$. Typically, we allow the field to have very small bumps which, by definition, is another configuration. This makes $\mathcal{C}$ an infinite dimensional space. This allows us to specify the action of our theory. It is a function,

$$S : \mathcal{C} \to \mathbb{R}$$

In other words, given the configuration space of some field, the action produces a real number. When writing an action, one usually assumes that it is local, meaning it can be written as,

$$S = \int_{\mathcal{M}} d^n x \sqrt{g} \mathcal{L}(\phi, \partial\phi, \dots)$$

Lastly, the main tool we require to define the quantum theory is the path integral. The path integral defines transition amplitudes for our quantum theory. So, in Euclidean spacetime,

$$\mathcal{Z} = \int_{\mathcal{C}} [\mathcal{D}\phi] e^{-S[\phi]}$$

If the heuristic integral over the field varies from an initial value to a final one, the path integral defines a transition amplitude between these two configurations. The accurate meaning of the path integral resides in the topology of spacetime. Suppose for instance the our spacetime manifold $\mathcal{M}$ has boundaries, say $\partial\mathcal{M} = \cup_i \mathcal{B}_i$. Then to specify the path integral we must put some boundary conditions for the field on each connected component of $\partial\mathcal{M}$. Then for a manifold with a boundary, after specifying the boundary conditions, the path integral defines a map,

$$\otimes_i \mathcal{H}_i \to \mathbb{C}$$

where $\mathcal{H}_i$s are possible configurations of the field on each boundary. Failing to define the boundary conditions of the fields would define a general state on the boundary Hilbert space as illustrated in Fig 1.

The simplest case to consider is a cylinder. Suppose that $\mathcal{M} = \mathcal{N} \times I$ where $\mathcal{N}$ is some $n-1$ dimensional manifold and I is an interval. In that case, the path integral gives a map,

$$U(T) = e^{-HT} : \mathcal{H} \to \mathcal{H}$$

from the ingoing to the outgoing Hilbert space where,

$$\langle \phi_1 | U | \phi_0 \rangle = \int_{\phi|_{\mathcal{N}\times 0} = \phi_0}^{\phi|_{\mathcal{N}\times T} = \phi_1} [\mathcal{D}\phi] e^{-S[\phi]}$$

The Hilbert spaces and the path integral must satisfy certain axioms to accurately describe the quantum theory [8].

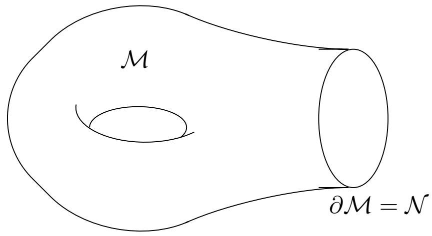


Figure 1: A general state in the Hilbert space $\mathcal{H}(\mathcal{N})$

(1) Functionality Axiom If $f : \mathcal{M}' \to \mathcal{M}$ is a diffeomorphism preserving the extra structure (e.g orientation, metric, etc) with an induced boundary map $\partial f : \partial\mathcal{M}' \to \partial\mathcal{M}$. Then there exists an induced map,

$$f_* : \mathcal{H}(\partial\mathcal{M}') \to \mathcal{H}(\partial\mathcal{M})$$

and

$$(\partial f)_*(\mathcal{Z}_{\mathcal{M}'}) = \mathcal{Z}_{\mathcal{M}}$$

This just states the fact that symmetries of spacetime are reflected in the Hilbert space and that symmetries of spacetimes perceive the path integral.[1]

(2) Orientation Axiom:

$$\mathcal{H}(-\mathcal{N}) = \mathcal{H}(\mathcal{N})^*$$

That is the Hilbert space associated with the boundary with reverse orientation must be the conjugate Hilbert space. This statement is naturally a projection of our interpretation of the time evolution of the state and the previous definition of the path integral.

(3) Multiplicity Axiom: Let $\mathcal{N}$ and $\mathcal{N}'$ be two $n-1$ dimensional manifolds. Then,

$$\mathcal{H}(\mathcal{N}' \cup \mathcal{N}) = \mathcal{H}(\mathcal{N}') \otimes \mathcal{H}(\mathcal{N})$$

(4) Gluing Axiom: Let $\mathcal{N}$ be a codimension one sub-manifold of $\mathcal{M}$. Cutting the manifold along $\mathcal{N}$ would produce a manifold $\mathcal{M}^{\text{cut}}$ with boundary $\partial\mathcal{M}^{\text{cut}} = \partial\mathcal{M} \cup \mathcal{N} \cup -\mathcal{N}$. Then, for the path integral $\mathcal{Z}_{\mathcal{M}} \in \mathcal{H}(\partial\mathcal{M})$ and $\mathcal{Z}_{\mathcal{M}^{cut}} \in \mathcal{H}(\partial\mathcal{M}^{cut}) \cong \mathcal{H}(\partial\mathcal{M}) \otimes \mathcal{H}(\mathcal{N}) \otimes \mathcal{H}(\mathcal{N})^*$, there exist a trace map which sum over all values of the field on $\mathcal{N}$,

$$\mathrm{Tr}_{\mathcal{N}} : \mathcal{H}(\partial\mathcal{M}^{cut}) \to \mathcal{H}(\partial\mathcal{M})$$

Using these general properties one can try to make sense of Quantum Field Theories. A quantum field theory defines a rule which takes a manifold $\mathcal{M}$ as input and produces a Hilbert space defined on the boundaries. From a categorical point of view, a QFT is a functor from the category of all manifolds to the category of Hilbert spaces. However, to make this precise one wishes first to effectively define these two categories. For a brief introduction to category theory basic terminology, one can check [15].

2 Bordisms

In this section, we will be discussing the category of all manifolds. The theory of bordisms establishes a relation between manifolds such that two manifolds are related if their disjoint union is a boundary of some manifold of one dimension higher.

2.1 Manifolds with boundary

We are mainly interested in associating vector spaces to manifolds with boundary. So, we will begin by defining manifolds with boundaries.

[1] This is of course unless the theory has an anomaly

Definition 2.1. A **Manifold $\mathcal{M}$ with boundary** is a Hausdorff second countable topological space which is locally isomorphic to $\mathbb{H}^n$ where,

$$\mathbb{H}^n = \{x \in \mathbb{R}^n | \Lambda(x) \geq 0\}$$

for some linear map $\Lambda : \mathbb{R}^n \to \mathbb{R}$.

A point $p \in \mathcal{M}$ is a boundary point if a chart maps it to a point in $\partial\mathbb{H} = \{x \in \mathbb{R}^n | \Lambda(x) = 0\} \cong \mathbb{R}^{n-1}$ [2]. In an analogy with smooth manifolds, one can define an atlas for manifolds with boundaries. An atlas is a collection of pairs U_α, h_α such that U_α are open sets that cover $\mathcal{M}$ and for each open set h_α is a homomorphism onto an open set of $\mathbb{H}^n$,

$$h_\alpha : U_\alpha \to \mathbb{H}^n$$

One can also define transition maps between charts,

$$h_{\alpha\beta} = h_\alpha \circ h_\beta^{-1}|_{h_\alpha(U_\alpha \cap U_\beta)} : h_\beta(U_\alpha \cap U_\beta) \to h_\alpha(U_\alpha \cap U_\beta)$$

which are homomorphsisms between open sets in $\mathbb{H}^n$. The atlas is called smooth if the transition maps are smooth.

For purposes of this thesis, we will consider orientation as an extra structure on the manifolds. This would help us define in and out boundaries that can be interpreted as the direction of time.

Definition 2.2. An **Orientation** of a real vector space $\mathbb{R}^n$ is a choice of sign $(+, -)$ for each order basis where two bases have the same sign if and only if the transformation matrix has positive determinant.

Using the concept of the tangent space $T_p\mathcal{M}$ at point p on the manifold, we can define an orientation of a manifold.

Definition 2.3. An **Orientation of a topological manifold** is the maximal choice of an orientated atlas. An atlas is called oriented if the coordinate changes maps have positive Jacobin.

A manifold is called orientable if it admits an orientation. Now, we need to define in/out boundaries of a manifold. Suppose that we have chosen a direction say w for time flow at each point on the manifold.

Definition 2.4. Let Σ be a closed-oriented codimension one submanifold of $\mathcal{M}$. At a point $p \in \Sigma$, let $(v_1, \cdots, v_{n-1})$ be positive bases for the tangent space $T_p\Sigma$. Then the vector $w \in T_p\mathcal{M}$ is called **positive normal** if $(v_1, \cdots, v_{n-1}, w)$ are positive bases for $T_p\mathcal{M}$. Similarly, it is negative normal if they are negative bases.

Using this we can classify the connected components of the boundary of the manifold.

Definition 2.5. Let $\mathcal{M}$ be a manifold with a boundary with a specific orientation. A connected component of $\partial\mathcal{M}$ is called an **in-boundary** if its positive normal vector points inward. Similarly, a connected component is called an **out-boundary** if its positive normal points outward.

2.2 Cobordims

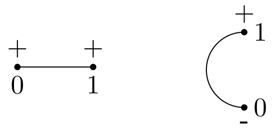
Now, we return to our main focus, bordism or so-called cobordisms. Roughly speaking, a cobordism between Σ_1 and Σ_2 is a manifold of one higher dimension with boundaries $\Sigma_1 \cup \Sigma_2$.

Definition 2.6. Let Σ_1 and Σ_2 be two closed oriented $(n-1)$ manifolds. An **oriented cobordism** $\mathcal{M}$ from Σ_1 to Σ_2 is an oriented n manifold with boundary $\partial\mathcal{M}$ together with two smooth maps $f_1 : \Sigma_1 \to \mathcal{M}$ and $f_2 : \Sigma_2 \to \mathcal{M}$ mapping diffeomorphically Σ_1 to the in-boundaries of $\mathcal{M}$ respectively Σ_2 to the out-boundaries preserving orientation.

$$\begin{tikzcd} & \mathcal{M} & \\ \Sigma_1 \arrow[ur, "f_1"] & & \Sigma_2 \arrow[ul, "f_2"'] \end{tikzcd}$$

[2]It can be shown that if p is a boundary point in some chart map, it is also a boundary point in other maps. This is a consequence of the invariance of domain theorem.

One can consider various examples. The simplest is when $n = 1$ such that the oriented cobordism is an interval $I = [0, 1]$ with standard orientation. When $\{0\}$ and $\{1\}$ both have $(+)$ orientation, I is a cobordism from $\{0\}$ to $\{1\}$. Reversing the orientation of $\{0\}$, results in a cobordism from $\emptyset$ to $\{0\} \cup \{1\}$ as illustrated in the following figure,



Another important notice is that given two diffeomorphic $(n-1)$ dimensional manifolds, one can build a cobordism induced by the diffeomorphism map using the cylinder bordism such that boundaries map to the manifolds.

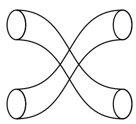
Now, we wish to build a category from these cobordisms, where cobordisms act as morphisms. So, we are left with the task of defining both the identity and composition laws. One may consider the cylinder as an identity that maps Σ to itself. However, if we insist on interpreting the extra dimension as time, different lengths of the cylinder would be evolution through different lengths of time. Thus, the identity cobordism would be a cylinder of length zero, however, this is not an n-dimensional manifold at all. So, basically, we do not have an identity. There are also further subtleties in the composition of bordisms that we will discuss later.

One possible solution to avoid these problems is to consider all diffeomorphic bordism equivalent. So, we pass to equivalence classes of bordisms. Let us make our identification precise,

Definition 2.7. Two oriented cobordisms $\mathcal{M}$ and $\mathcal{M}'$ are *equivalent*, if there exists an orientation preserving diffeomorphism $\phi : \mathcal{M} \to \mathcal{M}'$ that makes the following diagram commute,

$$\begin{tikzcd}
 & \mathcal{M} \arrow[dd, "\phi"] & \\
\Sigma_1 \arrow[ur] \arrow[dr] & & \Sigma_2 \arrow[ul] \arrow[dl] \\
 & \mathcal{M}' &
\end{tikzcd}$$

Before completing our discussion of defining compositions, let us first consider examples of nonequivalent cobordisms. Let Σ and Σ' be two orientable $(n-1)$-dimensional manifolds possibly not connected and consider the following differomorphism $\phi : \Sigma \cup \Sigma' \to \Sigma \cup \Sigma'$ such that $\phi(\Sigma) = \Sigma'$ and $\phi(\Sigma') = \Sigma$. This bordism is different from the identity bordism, it is known as the twist map.



This stems from the fact that $\Sigma' \cup \Sigma$ is not the same as $\Sigma \cup \Sigma'$. This twist map, as we will discuss later, will give the symmetry for the monoidal structure for the category of cobordisms.

2.3 Gluing of cobordisms

Now, we return back to the issue of defining compositions. As discussed, we were able to define an identity by throwing all information on the manifolds and considering only the topology. We will see that this will further supplement defining gluing of cobordisms. To be able to glue manifolds, we need to first define gluing of topologies of the two distinct manifolds, then, check whether this new topological space can be a topological manifold (i.e. we can build an atlas on it). Lastly, we need to check if it admits a smooth structure or not.

Given two topological spaces $\mathcal{M}_0$ and $\mathcal{M}_1$ with common boundary Σ, we can define an equivalence relation $\sim$ on $\mathcal{M}_0 \cup \mathcal{M}_1$: given two points $p_o \in \mathcal{M}_o$ and $p_1 \in \mathcal{M}_1$, $p_o \sim p_1$ if and only if there exists a point $x \in \Sigma$ such that $f_0(x) = p_o$ and $f_1(x) = p_1$. Then,

$$\mathcal{M}_0 \cup_\Sigma \mathcal{M}_1 = \mathcal{M}_0 \cup \mathcal{M}_1 / \sim$$

The topology on $\mathcal{M}_0 \cup_\Sigma \mathcal{M}_1$ is defined by declaring that a subset $U \subset \mathcal{M}_0 \cup_\Sigma \mathcal{M}_1$ is open if and only if its inverse image, in $\mathcal{M}_0$ and $\mathcal{M}_1$, are both open, thus, making the maps continues.

Now, we need to check if we can associate an atlas to this new topological space $\mathcal{M}_0 \cup_\Sigma \mathcal{M}_1$. It is clear that for the points that are not on Σ, we can take the charts of $\mathcal{M}_0$ and $\mathcal{M}_1$. Problems only arise for points on Σ. Given a point $x \in \Sigma$, we need to find a chart $\phi : U \to \mathbb{R}^n$, where U is an open neighborhood of x in $\mathcal{M}_0 \cup_\Sigma \mathcal{M}_1$. Let U be a neighborhood of $x \in \mathcal{M}_0 \cup_\Sigma \mathcal{M}_1$, and let $U_0 = U \cap \mathcal{M}_0$ and $U_1 = U \cap \mathcal{M}_1$. Since U is open, both U_0 and U_1 are open in $\mathcal{M}_0$ and $\mathcal{M}_1$ respectively. One can then choose U to be small enough such that U_0 and U_1 are domain for some charts $\phi_0 : U_0 \to \mathbb{R}^n_-$ and $\phi_1 : U_1 \to \mathbb{R}^n_+$. This gives the following diagram,

$$\begin{tikzcd}
 & \Sigma \arrow[dl] \arrow[dr] & \\
U_0 \arrow[dr] \arrow[dd, "\phi_0"'] & & U_1 \arrow[dl] \arrow[dd, "\phi_1"] \\
 & U \arrow[dd, "\phi"'] & \\
\mathbb{R}^n_- \arrow[dr] & & \mathbb{R}^n_+ \arrow[dl] \\
 & \mathbb{R}^n &
\end{tikzcd}$$

The existence and uniqueness of the map $\phi : U \to \mathbb{R}^n$ is guaranteed by the universal properties of the $\mathbb{R}^n$ and U inside the category.

This gluing of manifolds still does not guarantee a differential structure on the union. Finding a smooth structure on the union is equivalent to finding a smooth manifold S which is diffeomorphic to $\mathcal{M}_0 \cup_\Sigma \mathcal{M}_1$. So, Given such a diffeomorphism we can pull back the atlas from S to $\mathcal{M}_0 \cup_\Sigma \mathcal{M}_1$. Let us first discuss gluing of cylinders. Suppose we have two manifolds that are diffeomorphic to cylinders $\mathcal{M}_0 : \Sigma_0 \to \Sigma_1$ and $\mathcal{M}_1 : \Sigma_1 \to \Sigma_2$. This means that we have these two maps:

$$\phi_0 : \mathcal{M}_0 \to \Sigma_1 \times [0, 1]$$

$$\phi_1 : \mathcal{M}_1 \to \Sigma_1 \times [0, 1]$$

We want to construct a smooth structure on the union $\mathcal{M}_0 \cup_{\Sigma_1} \mathcal{M}_1$. Obviously, S is a cylinder $\Sigma_1 \times [0, 2]$. The homeomorphism $\phi : \mathcal{M}_0 \cup_{\Sigma_1} \mathcal{M}_1 \to S$ is defined by universal property in the category of continuous maps. Since S has a smooth structure, it agrees with ϕ on the two parts. Thus, we have a smooth structure on $\mathcal{M}_0 \cup_{\Sigma_1} \mathcal{M}_1$ using pullback along ϕ. Notice again that the choice for S is arbitrary, meaning we could have chosen any S that would help us define the smooth structure. Now, that we have known how to connect cylinders, we will use the following theorem to help us discuss the connection of general bordisms.

Theorem 2.8. ***Regular Interval Theorem***: *Let $\mathcal{M}$ be a cobordism from Σ_0 to Σ_1 and let $f : \mathcal{M} \to [0, 1]$ be a smooth map with no critical points such that $f^{-1}(0) = \Sigma_0$ and $f^{-1}(1) = \Sigma_1$. Then $\mathcal{M}$ is diffeomorphic to a cylinder. [10]*

Corollary 2.8.1. *Let $\mathcal{M}$ be a cobordism from Σ_0 to Σ_1. There exists a neighborhood $\mathcal{M}_0$ of Σ_0 in $\mathcal{M}$ such that $\mathcal{M}_0$ is diffeomorphic to a cylinder.*

Proof. Let f be a Morse function such that $f : \mathcal{M} \to [0, 1]$. Let $t \in [0, 1]$ be the first critical value. Then, choose $\epsilon < t$ such that the interval $[0, \epsilon]$ is regular. By using theorem 2.8, we proved the corollary. ☐

Now, it is straightforward to generalize this for general cobordisms. All we need to do is to choose an interval through which the junction is regular and connect the parts as we do for cylinders.

Theorem 2.9. *Let $\mathcal{M}_0$ and $\mathcal{M}_1$ be two cobordisms such that Σ_1 is an in/out boundary for $\mathcal{M}_0$ /$\mathcal{M}_1$. There always exists a smooth $\mathcal{M}_0\mathcal{M}_1$ which is homeomorphic to $\mathcal{M}_0 \cup_{\Sigma_1} \mathcal{M}_1$.*

Proof. Consider Morse functions $f_0 : \mathcal{M}_0 \to [0, 1]$ and $f_1 : \mathcal{M}_1 \to [1, 2]$. Then, consider the combined topological space $\mathcal{M}_0 \cup_{\Sigma_1} \mathcal{M}_1$ with its induced continues map $\phi : \mathcal{M}_0 \cup_{\Sigma_1} \mathcal{M}_1 \to [0, 2]$. Choose ϵ such that f_0 and f_1 are regular in the intervals $[1 - \epsilon, 1]$ and $[1, 1 + \epsilon]$ respectively. Then, the smooth structure is defined by gluing of cylinders resulting in a gluing of cobordisms. ☐

Note that the way we choose S to find the smooth structure was not unique. However, as it seems, all such ways of choosing S must be diffeomorphic. This is a result of the following theorem,

Theorem 2.10. *Let $\mathcal{M}_0$ and $\mathcal{M}_1$ be smooth n-manifolds. α and β be two smooth structures on $\mathcal{M}_0 \cup_\Sigma \mathcal{M}_1$ such that they both induce original smooth structure on $\mathcal{M}_0$ and $\mathcal{M}_1$. Then, there exist a diffeomorphism $\phi : (\mathcal{M}_0 \cup_\Sigma \mathcal{M}_1, \alpha) \to (\mathcal{M}_0 \cup_\Sigma \mathcal{M}_1, \beta)$ with $\phi_\Sigma = \mathrm{id}_\Sigma$. [13]*

2.4 Category of cobordism classes

Now, that we were able to show how to glue cobordisms, we need to make sure that this procedure is well defined for diffeomorphic bordisms $\mathcal{M}'_0$ and $\mathcal{M}'_1$. This is true because the final connected bordism must also be a result that comes from cutting and gluing along any codimension 1 submanifold. So, consider diffeomorphisms $\psi_0 : \mathcal{M}_0 \to \mathcal{M}'_0$ and $\psi_1 : \mathcal{M}_1 \to \mathcal{M}'_1$ relative to the boundary. Using the above procedure we can construct $\mathcal{M}_0 \cup_\Sigma \mathcal{M}_1$ and $\mathcal{M}'_0 \cup_\Sigma \mathcal{M}'_1$. Similarly, in the category of continuous functions, we glue the diffeomorphisms obtaining $\psi : \mathcal{M}_0 \cup_\Sigma \mathcal{M}_1 \to \mathcal{M}'_0 \cup_\Sigma \mathcal{M}'_1$. Use ψ to define a smooth structure on $\mathcal{M}'_0 \cup_\Sigma \mathcal{M}'_1$. Using theorem 2.10, we have now two smooth structures on $\mathcal{M}'_0 \cup_\Sigma \mathcal{M}'_1$ which must be related. In this way, we were able to define the composition laws of cobordisms. Passing to diffeomorphism classes allowed this structure to be well-defined.

Furthermore, one can check that the composition is associative. Using universal property, there exists a map such that,

$$(\mathcal{M}_0 \cup_{\Sigma_1} \mathcal{M}_1) \cup_{\Sigma_2} \mathcal{M}_2 \to \mathcal{M}_0 \cup_{\Sigma_1} (\mathcal{M}_1 \cup_{\Sigma_2} \mathcal{M}_2)$$

Additionally, the smooth structure on $\mathcal{M}_0 \cup_{\Sigma_1} \mathcal{M}_1 \cup_{\Sigma_2} \mathcal{M}_2$ is made by replacing the charts on the neighborhoods of Σ_1 and Σ_2 with that of the cylinder construction. Since it does not matter which ones we replace first, both structures are diffeomorphic.

Lastly, we return to our identity. It is now clear that an identity map for an object Σ is a cylinder over Σ. This follows from Theorem 2.8.1. Furthermore, composing the cylinder using associativity and gluing of cylinders would give back the usual properties of identity.

Now, we can declare that we have a category of cobordisms.

Definition 2.11. **nCob** is a category given by:

1. **Objects**: smooth oriented $n-1$ manifolds.
2. **Morphisms**: n-cobordism classes, such that in boundaries are domains and out boundaries are co-domains.
3. **Composition**: defined by gluing of cobordisms
4. **Identity**: cylinder construction

The category **nCob** enjoys two properties which are being a symmetric and monoidal category. A monoidal structure on a category is a tensor product operation that satisfies certain compatibility axioms mimicking the tensor product of vector spaces. In particular, there exists a unit element. The tensor product is symmetric if there exists a natural isomorphism exchanging factors, called symmetric braiding. For the bordism category, we know that the disjoint union of two $n-1$ dimensional manifolds is also an $n-1$ dimensional manifold. Similarly, the disjoint union of cobordisms is also another cobordism of the same dimension from the disjoint union of the in boundaries to the disjoint union of the out boundaries. Adding on top of that the empty manifold $\emptyset_{n-1}$ and the empty cobordisms $\emptyset_n$ (i.e a unit), we get a monoidal structure on **nCob**. Lastly, the symmetric structure is defined by the equivalence classes of the twist cobordism denoted τ. One can check that it satisfies the axioms required for symmetric braiding. The symmetry condition is satisfied. Here we will discuss the naturality condition,

Lemma 2.12. *Let $\mathcal{M} : \Sigma_o \to \Sigma_1$ and $\mathcal{M}' : \Sigma'_o \to \Sigma'_1$ be two cobordisms. Then, the following diagram commutes,*

$$
\begin{tikzcd}
\Sigma_0 \cup \Sigma_0' \arrow[r, "\mathcal{M} \cup \mathcal{M}'"] \arrow[d, "\tau_{\Sigma_0 \cup \Sigma_0'}"'] & \Sigma_1 \cup \Sigma_1' \arrow[d, "\tau_{\Sigma_1 \cup \Sigma_1'}"] \\
\Sigma_0' \cup \Sigma_0 \arrow[r, "\mathcal{M}' \cup \mathcal{M}"] & \Sigma_1' \cup \Sigma_1
\end{tikzcd}
$$

Proof. Using the symmetry property of the twist map, the twist map is the inverse of itself. The above is just a description of the following relation,

$$\tau_{\Sigma_0 \cup \Sigma_0'} \circ (\mathcal{M}' \cup \mathcal{M}) \circ \tau_{\Sigma_1' \cup \Sigma_1} = \mathcal{M} \cup \mathcal{M}'$$

This just says that if one reverses the in and out boundaries of a cobordism it is equivalent to the reversed cobordism. To prove this one need to find a diffeomorphism between right and left-hand side manifolds. However, this map is obvious, it is the twist map itself. $\square$

The twist map also satisfies the hexagon axiom which manipulates three $n-1$ dimensional manifolds.

3 Topological Quantum Field Theories

Now, that we were able to construct a category of all manifolds. We need to discuss the target category which is the category of vector spaces. However, this category is trivial. Additionally, it is also equipped with a symmetric monoidal structure. Thus, in correspondence with our intuition about Quantum Field Theories, a QFT should be a functor from the category **nCob** to the category of vector spaces over a field $\mathbb{K}$. This functor should respect the symmetric monoidal structure that both these categories have. These types of QFTs are called Topological quantum field theories. They were first proposed by Atiyah and Segel [1].

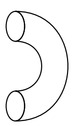
Definition 3.1. A **Topological Quantum Field Theory** of dimension n is a symmetric monoidal functor $\mathcal{Z}$ from the category **nCob** to $\mathbf{Vect}_{\mathbb{K}}$ satisfy the following axioms,

1. $\mathcal{Z}(1_\Sigma) = 1_{\mathcal{Z}(\Sigma)}$
2. $\mathcal{Z}(\mathcal{N} \circ \mathcal{M}) = \mathcal{Z}(\mathcal{N}) \circ \mathcal{Z}(\mathcal{M})$ for any two bordisms such that $\mathcal{N} : \Sigma_0 \to \Sigma_1$ and $\mathcal{M} : \Sigma_1 \to \Sigma_2$
3. $\mathcal{Z}(\Sigma_1 \cup \Sigma_2) = \mathcal{Z}(\Sigma_1) \otimes \mathcal{Z}(\Sigma_2)$ for any two objects in **nCob**
4. $\mathcal{Z}(\mathcal{N} \cup \mathcal{M}) = \mathcal{Z}(\mathcal{N}) \otimes \mathcal{Z}(\mathcal{M})$ for any two bordisms such that $\mathcal{N} : \Sigma_0 \to \Sigma_1$ and $\mathcal{M} : \Sigma_1 \to \Sigma_2$
5. $\mathcal{Z}(\emptyset) = \mathbb{K}$
6. $\mathcal{Z}(\tau_{\Sigma_1,\Sigma_2}) = \sigma_{\Sigma_1,\Sigma_2}$ where σ is permutation symbol.

3.1 General Properties of TQFTs

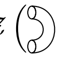
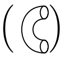
Before studying TQFTs in 1 and 2 dimensions, we state some general properties of TQFTs. For purposes of proving the first proposition, we need the following definition,

Definition 3.2. Let Σ be an object in **nCob**, **U-tubes** are values of the TQFT on following cobordisms,

 $: \Sigma \sqcup -\Sigma \to \emptyset,$  $: \emptyset \to -\Sigma \sqcup \Sigma$

Proposition 3.3. *Let $\mathcal{Z}$ be an n-dimensional TQFTs. Then for any object $\Sigma \in$ **nCob**, $\mathcal{Z}(\Sigma)$ is finite dimensional.*

Proof. This is a consequence of the pairing (U-tube) being non-degenerate. Suppose that $U = \mathcal{Z}(\Sigma)$ and $V = \mathcal{Z}(-\Sigma)$. Then the U-tubes are the following maps,

$\beta = \mathcal{Z}$ () $: U \otimes V \to \mathbb{K}$, and $\gamma = \mathcal{Z}$ () $: \mathbb{K} \to U \otimes V$

Then, using tube diffeomorphism relation, we get the following identity,

$$1_U = (\beta \otimes 1_U) \circ (1_U \otimes \gamma)$$

One is then allowed to choose finitely many basis elements $v_i \in V$ and $u_i \in U$ such that $\gamma(1) = \sum_i v_i \otimes u_i$. Note that any element in $U \otimes V$ can be expressed in this manner [3]. Then,

$$\forall u \in U, u = (\beta \otimes 1_U) \circ ((1_U \otimes \gamma)(u \otimes 1)) = (\beta \otimes 1_U) \circ \left(u \otimes \sum_i v_i \otimes u_i \right) = \sum_i \beta(u, v_i).u_i$$

Since u can be anything one can definitely choose the basis themselves u_j making $\beta(u_j, v_i) = \delta^i_j$. Thus, all bases of U are spanned by a finite sum. $\square$

Furthermore, this also shows that there is a natural isomorphism, $U \to V^*$, the dual of vector space V such that $u \to \beta(u, -)$. Further discussion about non-generate pairings will be found in section 4.1. In general, this shows that,

$$\mathcal{Z}(-\Sigma) = \mathcal{Z}(\Sigma)^*$$

Additionally, TQFTs produce invariants of manifolds with boundary. This can be easily seen since $\mathcal{Z}$ will take all diffeomorphic manifolds to the same vector space. A special case is when we have a closed manifold with $\emptyset$ boundary. This can be viewed as a linear map from $\mathbb{K} \to \mathbb{K}$. An example is the following,

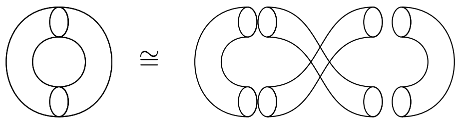
Proposition 3.4. *Let $\mathcal{Z}$ be a TQFT and Σ be a closed manifold. Then,*

$$\mathcal{Z}(\Sigma \times \mathbb{S}^1) = \dim(\mathcal{Z}(\Sigma))$$

Proof. Using the same notation as in the previous proof, let $\gamma(1) = \sum_{i,j} \lambda_{ij} v_i \otimes u_j$. As before, $\forall u \in U, u = \sum_{i,j} \lambda_{ij} \beta(u, v_i).u_j$. If u is a basis element of U. Then,

$$\sum_i \lambda_{i,k} \beta(u_j, v_i) = \delta_{jk}$$

One can then cut, the circle in the middle and reconnect the boundaries using the twist map through the following diffeomorphism,



Which gives the following relation,

$$\mathcal{Z}(\Sigma \times \mathbb{S}^1) = \beta \otimes \sigma_{U,U} \circ \gamma$$

Then,

$$\mathcal{Z}(\Sigma \times \mathbb{S}^1)(1) = \beta \left(\sigma_{U,U} \left(\sum_{i,j} \lambda_{ij} v_i \otimes u_j \right) \right) = \sum_{i,j} \lambda_{ij} \beta(u_j, v_i) = \sum_{i,j} \delta_{jj} = \dim(U)$$

$\square$

One of the most important aspects is to notice that TQFTs form a functor category. That is objects are TQFT functors and morphisms are associated with natural transformations. So, we can write,

$$\mathbf{TQFTs}^{\mathbb{K}}_n = \mathbf{SymMonFun}(\mathbf{nCob}, \mathbf{Vect}_{\mathbb{K}})$$

Since we can exactly classify manifolds in one and two dimensions, we can exactly determine the structure of 1-TQFTs and 2-TQFTS. So, in the following, we will be discussing these two categories with the aim to find an algebra that encodes the structure. With analogy to group theory, we can narrow down our representation of a category to a set of objects, a set of generators for morphisms, and some relations between these generators.

[3] Any element x can be written as $x = \sum_{i,j} c_{ij} v_i \otimes u_j = \sum_j \left(\sum_i c_{ij} v_i\right) \otimes u_j$. Then, one redefines the basis in V as $v'_j = \sum_i c_{ij} v_i$

3.2 1-TQFTS

Let us begin by discussing 1-TQFTs, where objects in **1Cob** are zero-dimensional. In this sense, objects are just points with $(+,-)$ signs to indicate orientations.

$$\underset{-}{\bullet} \qquad \underset{+}{\bullet} \qquad \underset{+}{\bullet}\cup\underset{+}{\bullet} \quad , \cdots$$

For the morphisms, one just needs to see all generators of connected bordisms. Other bordisms which have non-connected pieces would just be a tensor product of connected bordisms. Although this is true for most cases, there is only one disconnected bordism that cannot be constructed from connected parts which is the twist bordism. Twist maps are defined using permutation bordism.

Definition 3.5. Given a permutation $\sigma \in \mathfrak{S}_n$, A **permutation bordism** associated to σ is induced by the diffeomorphism,

$$\Sigma_1 \cup \cdots \cup \Sigma_n \to \Sigma_{\sigma(1)} \cup \cdots \cup \Sigma_{\sigma(n)}$$

Since every permutation can be generated by transpositions, these bordisms are generated by twist bordisms $\Sigma_i \cup \Sigma_j \to \Sigma_j \cup \Sigma_i$. One can then see that the following lemma hold,

Lemma 3.6. *Every bordism can be decomposed into a permutation bordism, a disjoint union of connected bordisms, then another permutation bordism.*

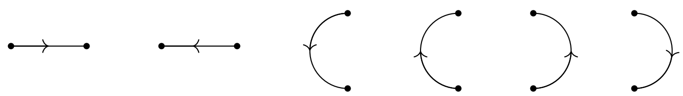
Proof. Let $\mathcal{M}_1, \cdots, \mathcal{M}_n$ be connected components of a generic bordism $\mathcal{M}$ such that $\mathcal{M}_i : \Sigma_{i,1} \sqcup \cdots \sqcup \Sigma_{i,m_i} \to \Sigma'_{i,1} \sqcup \cdots \sqcup \Sigma'_{i,m_i}$. Let $\mathcal{M}'$ be the disjoint union $\mathcal{M}_1 \sqcup \cdots \sqcup \mathcal{M}_n$. Then $\mathcal{M}$ and $\mathcal{M}'$ can only differ in the permutation of the ingoing and outgoing boundaries. This gives,

$$\mathcal{M} = S_{\sigma'} \circ \mathcal{M}' \circ S_\sigma$$

where $S_{\sigma'}$ and S_σ are permutation bordisms. $\square$

By using lemma 3.6 and the fact the morphisms in 1d are just lines or circles, one can reach the following conclusion.

Proposition 3.7. *The monoidal category **1Cob** is generated under composition and disjoint union by the following bordisms.*



Lastly, one needs to find the relations between these generators. There are relations that involve identity morphism, relations associated with the twist map and its naturality condition, snake relation, and most importantly commutativity relations. For more discussion about these relations, one can see [13] [4] Now that we have classified manifolds in **1Cob**, we seek an algebra that resembles 1-TQFTs. We claim that there exists a correspondence between 1-TQFTs and finite-dimensional vector spaces[5].

Theorem 3.8. *There exists a symmetric monoidal equivalence between categories*

$$\mathbf{TQFT}_1^{\mathbb{K}} \cong \mathbf{FinVect}_{\mathbb{K}}^{iso}$$

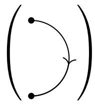
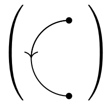
where $\mathbf{FinVect}_{\mathbb{K}}^{iso}$ *is the category of finite dimensional vector spaces with invertible linear maps and morphisms.*

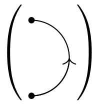
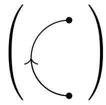
Proof. To prove the above theorem, one needs to show that given a 1-TQFT we can define a finite-dimensional vector space and vice versa. The first statement is trivially defined by $\mathcal{Z} \to \mathcal{Z}(\bullet_+)$. Additionally, by virtue of Proposition 3.3, this vector space is finite-dimensional. Now, we need to show that given a finite-dimensional vector space, we can construct a TQFT. Suppose we are given a TQFT $\mathcal{Z}$, then set $\mathcal{Z}(\bullet_+) = V$ and $\mathcal{Z}(\bullet_-) = V^*$ and more generally,

$$\mathcal{Z}\left(\bullet_+^{\sqcup m} \sqcup \bullet_-^{\sqcup n}\right) = V^{\otimes m} \otimes_{\mathbb{K}} V^{*\otimes n}$$

[4]One can prove that this set of relations is complete meaning that any other relation can be written in terms of them.

One can then pick up a basis v_i for V and define the action on the generators,

$\mathcal{Z} =$  $: V \otimes V^* \to \mathbb{K},$ $\quad \mathcal{Z} =$  $: \mathbb{K} \to V^* \otimes V$ (1)

$\mathcal{Z} =$  $: V^* \otimes V \to \mathbb{K},$ $\quad \mathcal{Z} =$  $: \mathbb{K} \to V \otimes V^*$ (2)

Additionally, the images of the identity must be an identity and twist map also fixed by the symmetric braiding of the functor. One can then check that all relations between the generators hold. One then needs to show that this equivalence holds in a functorial manner (i.e. the correspondence is a functor). To show this, one has to check transformations of morphisms. Let $\alpha : \mathcal{Z} \to \mathcal{Y}$ be a symmetric monoidal natural transformation, $V_{\mathcal{Z}} = \mathcal{Z}(\bullet_+)$ and $V_{\mathcal{Y}} = \mathcal{Y}(\bullet_+)$. In some manner, α acts on the two components of $\mathcal{Z}$ as $\alpha_{\bullet_+} : V_{\mathcal{Z}} \to V_{\mathcal{Y}}$ and $\alpha_{\bullet_-} : V^*_{\mathcal{Z}} \to V^*_{\mathcal{Y}}$. Now, we wish to show that the functor assigns α to $\alpha_{\bullet_+}$. However, for this to be true $\alpha_{\bullet_+}$ must be an invertible transformation. The inverse map is constructed using the following decomposition.

$$V_{\mathcal{Y}} \xrightarrow{\gamma_{\mathcal{Z}} \otimes 1_{V_{\mathcal{Y}}}} V_{\mathcal{Z}} \otimes V^*_{\mathcal{Z}} \otimes V_{\mathcal{Y}} \xrightarrow{1_{V_{\mathcal{Z}}} \otimes \alpha_{\bullet_-} \otimes 1_{\mathcal{Y}}} V_{\mathcal{Z}} \otimes V^*_{\mathcal{Y}} \otimes V_{\mathcal{Y}} \xrightarrow{1_{V_{\mathcal{Z}}} \otimes \beta_{\mathcal{Y}}} V_{\mathcal{Z}}$$

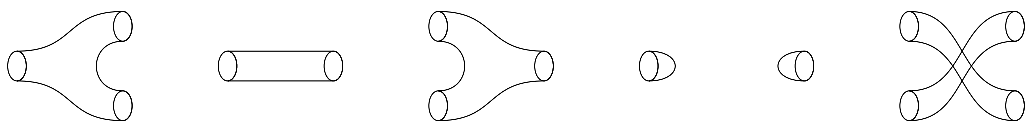
One can then easily show that this composition, denoted $\alpha^{-}_{\bullet_+}$, is indeed the inverse of $\alpha_{\bullet_+}$ using naturality of α. Therefore one can assign $\alpha_{\bullet_+}$ to the correspondence functor. The other direction is also true since given an invertible linear map $f : U \to V$ one can define a symmetric monoidal natural transformation $\alpha_f : \mathcal{Z}_U \to \mathcal{Z}_V$ with two components $\alpha_{f_{\bullet_+}} = f$ and $\alpha_{f_{\bullet_-}} = (f^{-1})^*$. □

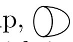
3.3 Generators of 2Cob

In a similar manner to our discussion of the classification of 1-TQFTs, 2D surfaces also are exactly classified by the genus which in turn motivates a search for an algebraic structure that encodes 2-TQFTs. For finding the generators for 2Cob, we will use the following lemma,

Lemma 3.9. *Let $\mathcal{M}$ be a compact oriented manifold with $f : \mathcal{M} \to [0,1]$ a Morse function. If there exists a unique critical point x such that x has an index 1. Then is diffeomorphic to either pair of pants or reverse pair of pants. [10]*

Proposition 3.10. *The monoidal category **2Cob** is generated under the composition and disjoint union of the following bordisms.*



Proof. Suppose that we have a bordism $\mathcal{M} : \Sigma_0 \to \Sigma_1$. Let f be a Morse function such that $f : \mathcal{M} \to [0,1]$ with $f^{-1}(0) = \Sigma_0$ and $f^{-1}(1) = \Sigma_1$. Then consider the following sequence $[x_0, \cdots, x_n]$ where $[x_i, x_{i+1}]$ contain at most one critical point. The part $\mathcal{M}_{[x_i, x_{i+1}]}$ may be composed of several connected pieces where only one of them contains the critical point if it exists. By lemma 3.6, all other similar bordisms are equivalent up to permutations. Without loss of generality we can assume $\mathcal{M}_{[x_i, x_{i+1}]}$ is connected. So, now if x has an index 0 then we have a local minimum for which we associate the following bordism, . If x has index 2, we associate the cup, . Finally, by Lemma 3.9, we associate a pair of pants or reverse pair of pants for a critical point with index 1. □

4 Frobenius algebras and structure of 2-dimensional TQFTs

In this section, we are aiming to prove a correspondence between Frobenius algebras and 2-TQFTS. This correspondence was first investigated by Dijkgraff [7]. We want to construct pieces in the algebra that are mapped to the generators to prove the correspondence. Let us first give a small introduction about Frobenius Algebras to help us see the correspondence.

4.1 Basic Definitions

Definition 4.1. A $\mathbb{K}$**-algebra** is a $\mathbb{K}$ vector space A equipped with two linear maps $\mu : A \otimes A \to A$ and $\eta : \mathbb{K} \to A$ such that the following diagrams commute,

$$\begin{tikzcd}
 & A \otimes A \otimes A \arrow[dl, "\mu \oplus 1_A"'] \arrow[dr, "1_A \oplus \mu"] & \\
A \otimes A \arrow[dr, "\mu"'] & & A \otimes A \arrow[dl, "\mu"] \\
 & A &
\end{tikzcd}$$

$$\begin{tikzcd}
 & A \otimes A \arrow[dr, "\mu"] & \\
\mathbb{K} \otimes A \arrow[ur, "\eta \oplus 1_A"] \arrow[rr, equal] & & A
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \otimes A \arrow[dr, "\mu"] & \\
A \otimes \mathbb{K} \arrow[ur, "1_A \oplus \eta"] \arrow[rr, equal] & & A
\end{tikzcd}$$

One notices that the A is a ring where the addition is inherited from the definition of the vector spaces and the multiplication is such that,

$$\begin{aligned} . : & A \otimes A \to A \\ & (a, b) \to \mu(a \otimes b) \end{aligned}$$

Now, we give the definition of a Frobenius algebra,

Definition 4.2. A **Frobenius Algebra** is a finite-dimensional $\mathbb{K}$-algebra A with a linear map $\epsilon : A \to \mathbb{K}$ whose null space is,

$$\text{null}(\epsilon) = \{a \in A | \epsilon(a) = 0\}$$

has no nontrivial left ideals.

Notice that since a nontrivial left ideal contains a nontrivial principal ideal, the previous definition is equivalent to requiring that the null space has no nontrivial principal ideals. First, we want to show that a Frobenius algebra admits a paring that is non-degenerate to make correspondence with tube bordisms. Let us first give discuss the notions of pairing and co-pairing.

Definition 4.3. A **pairing** between two $\mathbb{K}$-vector spaces V and W is a linear map $\beta : V \otimes W \to \mathbb{K}$. Similarly, a co-pairing is linear map $\gamma : \mathbb{K} \to V \otimes W$.

Definition 4.4. A pairing is $\beta : V \otimes W \to \mathbb{K}$ **non-degenerate** in V if there exist $\gamma_V : \mathbb{K} \to V \otimes W$ such that $\beta \otimes \gamma_V = \text{id}_V$. Similarly for W.

Lemma 4.5. *If β is a non-degenerate pairing then, $\gamma_W = \gamma_V$.*

Proof. One sees this equivalence by considering the following map,

$$\mathbb{K} \xrightarrow{\gamma_W \otimes \gamma_V} W \otimes V \otimes W \otimes V \xrightarrow{1_W \otimes \beta \otimes 1_V} W \otimes V$$

This map can be factorized either by applying γ_V or γ_W first such that $\gamma_W \otimes \gamma_V = \gamma_W \circ (1_W \otimes 1_V \otimes \gamma_V) = \gamma_V \circ (\gamma_W \otimes 1_W \otimes 1_V)$. In the first case, the map evaluates to γ_V, and in the second case it evaluates to γ_W due to non-degeneracy of β. This proves equality. $\square$

Definition 4.6. Let A be a $\mathbb{K}$-algebra. A pairing $\beta : A \otimes A \to \mathbb{K}$ is **associative** if the following diagram commutes,

$$
\begin{tikzcd}
 & A \otimes A \otimes A \arrow[dl, "\mu \oplus 1_A"'] \arrow[dr, "1_A \oplus \mu"] & \\
A \otimes A \arrow[dr, "\beta"'] & & A \otimes A \arrow[dl, "\beta"] \\
 & \mathbb{K} &
\end{tikzcd}
$$

Now, we want to show that a Frobenius algebra admits a non-degenerate pairing. For this, we first prove the following two lemmas,

Lemma 4.7. *Let V and W be two vector spaces of the same finite dimension and $\beta : V \otimes W \to \mathbb{K}$ a pairing. Then, the following are equivalent,*

1. *β is non-degenerate*
2. *if $\beta(v, w) = 0\ \forall v \in V \implies w = 0$*
3. *if $\beta(v, w) = 0\ \forall w \in W \implies v = 0$*

Proof. We just need to prove that 1 implies 2 and vice versa for β being non-degenerate in W and the rest follows in the same manner. The direction 1 implies 2 is straightforward. Due to the non-degeneracy of β, this implies the existence of a co-pairing γ. We can choose the γ such that $\gamma(1) = \sum_i v_i \otimes w_i$ for some $v_i \in V$ and $w_i \in W$. This implies,

$$w \xrightarrow{\gamma \otimes 1_W} \sum_i w_i \otimes v_i \otimes w \xrightarrow{1_W \otimes \beta} \sum_i \beta(v_i, w) w_i = w$$

This clearly shows that if $\beta(v_i, w) = 0$, $\forall v_i$ then $w = 0$. To prove the other direction, let $\{w_1, \cdots, w_n\}$ be basis for W. Since the kernel of β is only zero by virtue of 2, then the map $W \to V^*$ is injective. This can be seen as follows: suppose that the kernel of the previous map is non-trivial in V^*.

$$\sum_i \lambda_i \beta(-, w_i) = 0 \implies \forall v \in V \sum_i \lambda_i \beta(v, w_i) = 0$$

Since β is linear, this implies,

$$\forall v \in V, \quad \beta(v, \sum_i \lambda_i w_i) = 0 \implies \lambda_i = 0$$

Thus, one chooses a basis in the dual space such that $\beta(v_i, w_j) = \delta_{ij}$. We, then, define the co-pairing $\gamma : \mathbb{K} \to W \otimes V$ as $\gamma(1) = \sum_i v_i \otimes w_i$. To prove the non-degeneracy of β, we check the action on a general vector.

$$\sum_j \lambda_j w_j \xrightarrow{\gamma \otimes 1_W} \sum_{i,j} \lambda_j w_i \otimes v_i \otimes w_j \xrightarrow{1_W \otimes \beta} \sum_{i,j} \lambda_j \beta(v_i, w_j) w_i = \sum_{i,j} \lambda_j \delta_{ij} w_i = \sum_j \lambda_j w_j$$

So β is clearly non-degenerate. $\square$

Lemma 4.8. *Let A be a $\mathbb{K}$-algebra. There exists 1-to-1 correspondence between linear form $\epsilon : A \to \mathbb{K}$ and associate pairing $\beta : A \otimes A \to \mathbb{K}$.*

Proof. Given the form ϵ, one can construct the pairing β as follows,

$$\begin{aligned} \beta :& A \otimes A \to \mathbb{K} \\ & x \otimes y \to \epsilon(\mu(x, y)) \end{aligned}$$

where μ is the multiplication map. Similarly, given a pairing β, we can construct a linear form,

$$\begin{aligned} \epsilon :& A \to \mathbb{K} \\ & x \to \beta(1, x) = \beta(x, 1) \end{aligned}$$

clearly, in both cases the pairing is associative. $\square$

Theorem 4.9. *A Frobenius algebra A is equipped with a non-degenerate pairing $\beta : A \otimes A \to \mathbb{K}$.*

Proof. Using lemma 4.9, we are only left with the task of proving that null(ϵ) contains no nontrivial left ideals if and only if the corresponding pairing is non-degenerate. Using lemma 4.8, β is non degenerate if and only if $\beta(A, x) = 0 \implies x = 0$. This is equivalent to $\epsilon(\mu(A, x)) = 0 \implies x = 0$ showing that ϵ has no nontrivial ideals. □

We have now seen that Frobenius algebra has structures that would correspond to the pairing, co-pairing, unit, co-unit, and pair of pants. We need to check whether it also admits a structure that corresponds to the reverse pair of pants. We can define $\delta : A \to A \otimes A$ which would correspond to the reverse pair of pants as a composition of the following two maps,

$$A \xrightarrow{\gamma \otimes 1_A} A \otimes A \otimes A \xrightarrow{1_A \otimes \mu} A \otimes A$$

We need to show that is map is unique.

Theorem 4.10. *Given a Frobenius algebra (A, ϵ). Then, there exists a unique co-multiplication map $\delta : A \to A \otimes A$.*

Proof. Suppose there exists w satisfying the same properties of co-multiplication. Consider the following diagram,

$$\begin{tikzcd}
& & A \otimes A \otimes A \arrow[dr, "1_A \oplus \mu"] \arrow[ddrr, bend left, "1_A \oplus \beta"] & & \\
& A \otimes A \arrow[ur, "w \oplus 1_A"] \arrow[dr, "\mu"] & & A \otimes A \arrow[dr, "1_A \oplus \epsilon"] & \\
A \arrow[ur, "\eta \oplus 1_A"] \arrow[rr, "1_A"'] \arrow[uurr, bend left, "\gamma \oplus 1_A"] & & A \arrow[ur, "w"] \arrow[rr, "1_A"'] & & A
\end{tikzcd}$$

From the diagram, one can see that $(\gamma \otimes 1_A) \circ (1_A \otimes \mu) = w$, but that is exactly the definition of δ which proves uniqueness. □

One can also prove that (A, ϵ, δ) is a $\mathbb{K}$-co-algebra[5] by diagram tracing. From this fact, one can rewrite the definition of Frobenius algebra in the following manner.

Definition 4.11. A **Frobenius $\mathbb{K}$-algebra** is a quintuple $(A, \mu, \eta, \delta, \epsilon)$ such that,

1. (A, μ, η) is a $\mathbb{K}$-algebra.
2. (A, δ, ϵ) is a $\mathbb{K}$-co-algebra.
3. The following diagram commutes (Frobenius relation),

$$\begin{tikzcd}
& A \otimes A \otimes A \arrow[ddr, "1_A \oplus \mu"] & \\
& A \arrow[dr, "\delta"'] & \\
A \otimes A \arrow[uur, "\delta \oplus 1_A"] \arrow[ur, "\mu"'] \arrow[dr, "1_A \oplus \delta"'] & & A \otimes A \\
& A \otimes A \otimes A \arrow[ur, "\mu \oplus 1_A"'] &
\end{tikzcd}$$

The last piece we want to add is the twist map. This is defined using the symmetric braiding $\sigma_{A,A}$. The existence of a symmetric braiding makes a Frobenius algebra a commutative Frobenius algebra.

[5] Same as the definition of $\mathbb{K}$-algebra with arrows reversed.

Definition 4.12. A $\mathbb{K}$-algebra / $\mathbb{K}$-co-algebra is **commutative/co-commutative** is the following diagrams commute,

$$\begin{tikzcd} & A\otimes A \arrow[dl, "\delta"'] \arrow[dr, "\mu"] & \\ A\otimes A \arrow[rr, "\mu"'] & & A\otimes A \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} & A\otimes A \arrow[dl, "\delta"'] \arrow[dr, "\sigma"] & \\ A\otimes A \arrow[rr, "\delta"'] & & A\otimes A \end{tikzcd}$$

Definition 4.13. A Frobenius $\mathbb{K}$-algebra is **commutative** if (A, μ, η) is a commutative $\mathbb{K}$-algebra.

4.2 The category of Frobenius algebras

Now that we have identified the required pieces for the correspondence, we need to build a category of Frobenius algebra. Thus, we have to define morphisms to construct the category. First, we need to describe the morphisms of algebras and co-algebras,

Definition 4.14. A $\mathbb{K}$**-algebra homomorphism** $f : A_1 \to A_2$ is a linear map such that the following diagram commutes,

$$\begin{tikzcd} A_1\otimes A_1 \arrow[r, "\mu_1"] \arrow[d, "f\otimes f"'] & A_1 \arrow[d, "f"] \\ A_2\otimes A_2 \arrow[r, "\mu_2"'] & A_2 \end{tikzcd} \qquad \begin{tikzcd} & A_1 \arrow[dd, "f"] \\ \mathbb{K} \arrow[ur, "\eta_1"] \arrow[dr, "\eta_2"'] & \\ & A_2 \end{tikzcd}$$

In a similar fashion, the definition for $\mathbb{K}$-co-algebra homomorphism with all arrows reversed and μ and η replaced with δ and ϵ respectively. Using these, we can define a Frobenius algebra homomorphism as follows,

Definition 4.15. A **Frobenius algebra homomorphism** f, is a linear map that is both a $\mathbb{K}$-algebra homomorphism and $\mathbb{K}$-co-algebra homomorphism.

Thus, one can form the category of Frobenius algebras $\mathbf{FA}_{\mathbb{K}}$. Furthermore, one can show that this category enjoys a symmetric monoidal structure by constructing a Frobenius algebra from the tensor product. If we require that the objects be commutative Frobenius algebra, the category is then $\mathbf{cFA}_{\mathbb{K}}$.

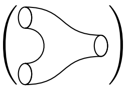
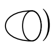
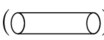
4.3 Equivalence of categories

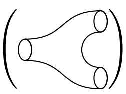
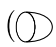
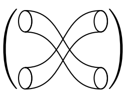
In this section, we prove the equivalence between the category of $\mathbf{TQFTs}_2^{\mathbb{K}}$ and $\mathbf{cFA}_{\mathbb{K}}$ and give a basic example.

Theorem 4.16. *There exists a symmetric monoidal equivalence between*

$$\boldsymbol{TQFTs}_2^{\mathbb{K}} \cong \boldsymbol{cFA}_{\mathbb{K}}$$

Proof. In the same manner that we proved the equivalence in 1-TQFTs, we want to associate a map given every generator and vice versa. Consider a symmetric monoidal functor $F : \mathbf{TQFTs}_2^{\mathbb{K}} \to \mathbf{cFA}_{\mathbb{K}}$ such that $\mathcal{Z}(\bigcirc) = A$,where we define,

$\mu = \mathcal{Z}\Big($  $\Big) : A\otimes A \to A$ $\qquad$ $\eta = \mathcal{Z}\big($  $\big) : \mathbb{K} \to A$ $\qquad$ $1_A = \mathcal{Z}\big($  $\big) : A \to A$

$\delta = \mathcal{Z}\Big($  $\Big) : A \to A\otimes A$ $\qquad$ $\epsilon = \mathcal{Z}\big($  $\big) : A \to \mathbb{K}$ $\qquad$ $\sigma_{A,A} = \mathcal{Z}\Big($  $\Big) : A\otimes A \to A\otimes A$

Again, using these definitions one can check the validity for all the relations as in definition 4.11. This produces a commutative Frobenius algebra. Conversely, given a Frobenius algebra A, we assign the same bordisms for the elements of the vector space using a functor G. Furthermore, all relations in the category of **2Cob** are satisfied since the functor is symmetric monoidal. Thus, one can think of these two functors as inverses of each other. Lastly, one should check the transformations on morphisms. Let $\alpha : \mathcal{Z} \to \mathcal{Y}$ be a natural transformation. Now, we want to see how this functor would transform α. It is straightforward to show that $F(\alpha) = A_{\mathcal{Z}} \to A_{\mathcal{Y}}$ is the right assignment. Furthermore, any map on the disjoint union of circles is just the tensor product of $F(\alpha)$. On the other hand, if one is given a morphism $f : A_1 \to A_2$ between two Frobenius algebras, the G functor assigns a natural transformation $\alpha_f : \mathcal{Z}_{A_1} \to \mathcal{Z}_{A_2}$ such that $\alpha_{f_n} = f^{\otimes n}$. □

Example 4.1. Now, we give an example of a TQFT defined by Frobenius algebra $\mathbb{K}[x]/\{x^2 - 1\}$ and Frobenius form that takes $1 \to 1$ and $x \to 0$. Thus, our space is finite-dimensional with basis elements $\{1, x\}$. Using multiplication defined on the algebra of polynomials, we can find the tensor components of μ_{ij}^k. The relations are:

$$1 \otimes 1 = 1, \;\; 1 \otimes x = x, \;\; x \otimes x = 1$$

So, we have $\mu_{11}^2 = \mu_{12}^1 = \mu_{21}^1 = \mu_{22}^2 = 0$ and $\mu_{11}^1 = \mu_{12}^2 = \mu_{21}^2 = \mu_{22}^1 = 1$. One can then find tube bordism β and γ, as follows,

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Lastly, we wish to find the action on a closed surface with genus g. For the construction of holes, we need, first, to find the delta map. The delta map have coefficients $\delta_i^{jk} = \gamma^{mj}\mu_{im}^k = \mu_{ij}^k$. Thus, $\delta_1^{11} = \delta_1^{22} = \delta_2^{21} = \delta_2^{12} = 1$ and $\delta_1^{12} = \delta_1^{21} = \delta_2^{11} = \delta_2^{22} = 0$. Thus, we can evaluate the handle element which is $\delta_m^{ij}\mu_{ji}^n$. It just multiplies any basis element with 2. Lastly, η can be evaluated using ϵ and γ. It takes 1 and gives 1. Thus, a surface of genus g would evaluate to 2^g.

5 Universal Construction of Topological Theories

We have seen that we could build a QFT that takes objects from **nCob** and produces a vector space. This functor turns out to be symmetric and monoidal to respect the structures in **nCob** and $\mathbf{Vect}_{\mathbb{K}}$. Now, one can inquire whether this is the only possible way to define this functor (i.e. can we drop the requirement of the functor to symmetric and monoidal?). In this section, we return back to our original task of assigning a vector space to objects in **nCob**. We describe another method to build the functor $\mathcal{Z}$ called universal construction.

5.1 Universal construction theorem

We will begin by giving the setup to build topological theories as described in [6] [4] [11]. The main ingredient that would build these topological theories is an invariant α of a diffeomorphic manifold. The invariant α takes closed manifolds and assigns to them a scalar quantity. Additionally, we require that this invariant to multiplicative meaning if $\mathcal{M}_1$ and $\mathcal{M}_2$ are two closed manifolds, then, the map of the disjoint union $\alpha(\mathcal{M}_1 \sqcup \mathcal{M}_2) = \alpha(\mathcal{M}_1)\alpha(\mathcal{M}_2)$. This in turn implies that $\alpha(\emptyset) = 1$ by replacing $\mathcal{M}_2 = \emptyset$. Using this map, one can define a bilinear form on manifolds with boundaries in the following manner. A bilinear form takes two manifolds with the same boundary N,

$$(\mathcal{M}_1, \mathcal{M}_2) = \alpha(-\mathcal{M}_1 \cup_N \mathcal{M}_2)$$

That is, it flips the orientation of one of the manifolds to build a closed manifold and then acts with α. As expected the form must be symmetric. Now, to each closed $(n-1)$ manifold N, we want to associate a vector space. This is done through the following theorem,

Theorem 5.1. *Let α be a multiplicative invariant of n-dimensional manifolds, such that $\alpha : \mathrm{Hom}(\emptyset, \emptyset) \to \mathbb{K}$. Then, there exists a cobordism-generated functor $\mathcal{Z}$: **nCob** $\to$ $\mathbf{Vect}_{\mathbb{K}}$ which is restricted by the invariant.*

Proof. Consider all diffeomorphism classes with boundary N, $\mathrm{Hom}(\emptyset, N)$. Denote $F(N) = \mathrm{Span}\{\mathrm{Hom}(\emptyset, N)\}$ the freely generated set of all these cobordism. Furthermore, let $F'(N) = \mathrm{Span}\{\mathrm{Hom}(N, \emptyset)\}$ be the set with opposite orientation. Then, consider the action of bilinear one $F'(N) \otimes F(N)$. We define a functor $\mathcal{Z} : \mathbf{nCob} \to \mathbf{Vect}_{\mathbb{K}}$ by fixing $\mathcal{Z}$ on objects as,

$$\mathcal{Z}_N = F(N)/\mathrm{Ker}(F(N), F(N)')$$

where $\mathrm{Ker}(F(N), F(N')) = \{x \in F(N) | \forall y \in F'(N), (x, y) = 0\}$ and similarly we define the functor $\mathcal{Z}' : \mathbf{nCob}^{\mathrm{op}} \to \mathbf{Vect}_{\mathbb{K}}$

$$\mathcal{Z}'_N = F'(N)/\mathrm{Ker}(F(N)', F(N))$$

Note here that the pairing is non-degenerate by construction. Now, we need to check the action of these functors on morphisms. Let $\mathcal{M} \in \mathrm{Hom}(N_1, N_2)$ be a morphism in $\mathbf{nCob}$ and $[\mathcal{N}] \in \mathrm{Hom}(\emptyset, N_1)$ be a basis element in $\mathcal{Z}_{N_1}$.Then,

$$\mathcal{Z}(\mathcal{M})[N] = [\mathcal{M} \cup_{N_1} \mathcal{N}]$$

Thus, it takes a basis element to another basis element. Consequently, we can define compositions. Let $[\mathcal{M}'] \in \mathrm{Hom}(N_2, N_3)$, then,

$$\mathcal{Z}(\mathcal{M} \cup_{N_2} \mathcal{M}')[\mathcal{N}] = \mathcal{Z}(\mathcal{M}) \circ \mathcal{Z}(\mathcal{M}')[\mathcal{N}]$$

Basically, This is the usual multiplication of matrices. We can define the same laws for Z'. One can then observe that these two functors are determined by each other. This can be seen since,

$$(\mathcal{Z}'(\mathcal{M})[\mathcal{N}_1], [\mathcal{N}_2]) = [\mathcal{N}_1 \cup \mathcal{M} \cup \mathcal{N}_2] = ([\mathcal{N}_1], \mathcal{Z}(\mathcal{M})[\mathcal{N}_2])$$

One can see then that $\mathcal{Z}(\emptyset_{n-1}) = F(\emptyset_n) = \mathrm{Hom}(\emptyset_{n-1}, \emptyset_{n-1})$ since $\alpha(\emptyset_n) = 1$. Since the map is linear in both arguments, $\mathcal{Z}(\emptyset_{n-1}) \cong \mathbb{K}$. Using this one can easily show that $\mathcal{Z}$ is cobordism generated by action on $[\emptyset_n]$. $\square$

It is very important to notice that this functor may not be monoidal. For instance, consider $\mathcal{Z}(\Sigma) = F(\Sigma)/\sim$ and $\mathcal{Z}(\Sigma') = F(\Sigma')/\sim$. $\mathcal{Z}(\Sigma \cup \Sigma') = F(\Sigma \cup \Sigma')/\sim$ may include additional bases that connect the two boundaries, like a U-tube. It is also straightforward to see that there exists an inclusion map from $\mathcal{Z}(\Sigma) \otimes \mathcal{Z}(\Sigma')$ to $\mathcal{Z}(\Sigma \cup \Sigma')$.

5.2 Universal Construction in two dimensions

In this section, we aim to classify all topological field theories of dimension 2 guided by our motivation that all surfaces of 2 dimensions are classified by the genus. Thus, the invariant is determined by a sequence $(\alpha_0, \alpha_1, \cdots)$, where $\alpha_i = \alpha(S_i)$ (invariant on the surface with genus i). Then, consider the following generating function,

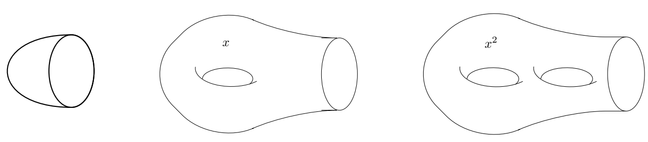
$$Z(T) = \sum_{g \geq 0} \alpha_g T^g$$

This generating function encodes all information about our topological theory. Consequently, one can think of this function as a definition of the theory itself. One, then, wishes to extract information from this function. Our main task would be determining the state space of the disjoint union of circles,

$$A(k) = \alpha \left(\bigsqcup_k \mathbb{S}^1 \right), k \geq 0$$

This state space is clearly spanned by all 2-manifolds with these boundary circles. $A(k)$ is a vector space quotient of the freely generated vector spaces modulo the kernel of the bilinear which is determined by $Z(T)$.

Let us first discuss the generators of this state space. We want to show that we have an algebra encoded in these state spaces induced by the pants cobordism. To see this, consider the state space $A(1)$, where its spanned by the following cobordism,

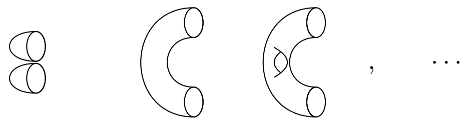


, $\cdots$

Thus, the algebra A is spanned by elements $x^n = [\mathbb{S}^1_n]$. Obviously, one can generate x^2 by gluing two x cobordisms together using the pants cobordism. One then expects that there exists a natural map between the ring polynomial generated by x and state space of $A(1)$,

$$\rho : R[x] \to A = \alpha(\mathbb{S}^1)$$

One may think that this map is an isomorphism. However, that generally depends on the theory $Z(T)$ that we are working with. $Z(T)$, in general, may quotient the space by a nontrivial ideal. Next, we have $A(2)$. As noted above, $A(2)$ may have bordisms that connect the two circles like tube bordisms which makes $A(2)$, in general, have more basis elements than the tensor product. The following figure shows some bordisms in $A(2)$.



If one considers the pants and cap cobordisms, one may for instance think that there is a possibility that they can make a Frobenius algebra. However, this is not true, in general. That is because a basis like a tube bordism may not have an image in $A(1)$ at all making the diagrams of either the algebra or the co-algebra not commute. However, it is important to notice the following, since our ring is a field, the algebra A is Frobenius if and only if it is finite-dimensional over $\mathbb{K}$. This is due to the simple reason that any quotient of $\mathbb{K}[x]$ by a nontrivial ideal is a Frobenius algebra. In a similar fashion, we want to build generators for k disjoint circles.

Proposition 5.2. *For any k, commutative monoid $\boldsymbol{Cob}_{2,k}$ has generator $x_i, 1 \leq i \leq k$ and $y_{i,j}, 1 \leq i \leq j \leq k$ such that*

$$y_{i,j}^2 = y_{i,j}x_i = y_{i,j}x_j$$

$$y_{i,j}y_{j,k} = y_{i,j}y_{i,k} = y_{i,k}y_{j,k}, i \leq j \leq k$$

where x_i is a cup cobordism in $k-1$ circles with only one holed cobordism with boundary circles i and $y_{i,j}$ is $k-2$ cup with a tube between circle i and j.

Proof. First notice that multiplication in this algebra is defined using the pants bordism. Then, notice that any holed connected bordism can be constructed by multiplication of the connected part and a holed bordism x_i. This proves the first relation. Additionally, one can see that there are only the required generators. Thus, we only need to construct the connected components of genus 0 of k circles. However, this set is parameterized by the decomposition of k elements into nonempty subsets,

$$\{1, 2, \cdots, k\} = I_1 \cup \cdots \cup I_n$$

Bordisms that include subsets of elements greater than one can be constructed again by joining tubes together (i.e. having one similar index). $\square$

Now, we return back to the task of classifying these theories. Notice that our definition of $A(1)$ is just a quotient $\mathbb{K}[x]$ by a nontrivial ideal induced by the kernel of the bilinear.

$$(x^n, x^m) = \alpha(\mathbb{S}_{m+n}) = \alpha_{m+n} \in \mathbb{K}$$

$$A(1) = \alpha(\mathbb{S}_1) \cong \mathbb{K}/\mathrm{Ker}(,)$$

So, in this case, the bilinear acts as a dot product on the state space of $A(1)$. Thus, one naturally would construct a matrix formed by the product to check the independence of the corresponding vectors. This matrix is called the Hankel matrix.

$$H = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \cdots \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots \\ \alpha_2 & \alpha_3 & \alpha_4 & \cdots \\ \alpha_3 & \alpha_4 & \alpha_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

This directly leads to the following proposition.

Proposition 5.3. *Elements* $\{x^k, x^{k+1}, \cdots, x^{k+N-1}\}$ *are* $\mathbb{K}$ *linearly independent in* $A(1)$ *if and only if* $\det(H_{k,k+N-1}) \neq 0$.

Now, we shall focus on the case where $A(1)$ is finite-dimensional. Thus clearly we have a Frobenius algebra. It turns out that these theories are exactly classified with correspondence to rational functions through the following theorem.

Theorem 5.4. *Let* $Z(T) = \sum_g \alpha_g T^g \in \mathbb{K}[[T]]$*, where* $\mathbb{K}$ *is any field. For* $m, s \geq 0$*.* $Z(T)$ *is a quotient of two polynomials,*

$$Z(T) = \frac{P(T)}{Q(T)}$$

if and only if $\exists\ m \geq 0$ *and* l *such that* $\det(H_{s,m}) = 0$ *for any* $s \geq l$*.[12]*

Proof. Suppose that $Z(T)$ is a quotient of polynomials. Let $P(T) = \sum_{i=0}^{k} p_i T^i$ and $Q(T) = \sum_{i=0}^{n} q_i T^i$. then,

$$P(T) = Z(T)Q(T) = \sum_{g=0}^{\infty}\sum_{i=0}^{n} \alpha_g q_i T^{i+g} = \sum_{i=0}^{n}\sum_{g=i-n}^{\infty} \alpha_{g-n+i} q_{n-i} T^g$$

Now, if the power g is greater than the degree of $P(T)$ all its coefficients must be zero. This for $g > \max(k, n)$,

$$\sum_{i=0}^{n} \alpha_{g-n+i} q_{n-i} = 0 \tag{3}$$

It is now clear that we can construct a set of equations with zero on the right-hand side. So, let $l = \max(k - n + 1, 1)$ and let $m = n$ (i.e. we begin with g that would give zero). For $s \geq l$, one would get the following set of equations for $g = s + n, \cdots, s + 2n$:

$$\begin{gathered}
\alpha_s q_n + \alpha_{s+1} q_{n-1} + \cdots + \alpha_{s+n} q_0 = 0 \\
\alpha_{s+1} q_n + \alpha_{s+2} q_{n-1} + \cdots + \alpha_{s+n+1} q_0 = 0 \\
\vdots \\
\alpha_{s+n} q_n + \alpha_{s+1+n} q_{n-1} + \cdots + \alpha_{s+2n} q_0 = 0
\end{gathered}$$

Thus, the coefficients matrix of q_i's, which are non-zero by definition, is singular and has a zero determinant proving the first direction.

Conversely, suppose that $\forall s \geq l, \det(H_{s,m}) = 0$ such that m is minimal. First note that to prove that the multiplication is a polynomial, we need to show that q_i's are solutions to the set of equations as in 3. Since this must be for all g, these q's must also be a solution for all g's greater than some number. One way to show this is by showing that the last row of the matrix can always be written as a linear combination with non-zero coefficients by the other rows. However, this may not be the case in many instances. However, this would happen if $\det(H_{s,m}) = 0$ and $\det(H_{s,m-1}) \neq 0\ \forall s > l$ (all the past rows are independent).

The first statement is already our premise thus we only need to show that $\det(H_{s,m}) = 0$ and $\det(H_{s,m-1}) \neq 0\ \forall s > l$. We prove this by contradiction. Suppose $\exists s \geq l$ s.t $\det(H_{s,m-1}) = 0$. Then one of the rows in the matrix of $H_{s,m-1}$ can be put to zero using a linear combination from other rows. There are two cases here: one when the row is the first row and the other when it is not. The following two matrices show the two cases:

$$\begin{pmatrix}
\alpha_s & \alpha_{s+1} & \cdots & \alpha_{s+m} \\
\alpha_{s+1} & \alpha_{s+2} & \cdots & \alpha_{s+m+1} \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \beta \\
\vdots & \vdots & \vdots & \vdots \\
\alpha_{s+m} & \alpha_{s+1+m} & \cdots & \alpha_{s+2m}
\end{pmatrix}
\qquad
\begin{pmatrix}
0 & 0 & 0 & \beta \\
\alpha_{s+1} & \alpha_{s+2} & \cdots & \alpha_{s+m+1} \\
\vdots & \vdots & \vdots & \vdots \\
\alpha_{s+m-1} & \alpha_{s+m} & \cdots & \alpha_{s+2m-1} \\
\alpha_{s+m} & \alpha_{s+1+m} & \cdots & \alpha_{s+2m}
\end{pmatrix}$$

If the row index is not zero then the determinant of the matrix $H_{s+1,m-1}$ is zero. If the index is not zero, since the determinant of the whole matrix $H_{s,m}$ is zero, either $\beta = 0$ or $\det(H_{s+1,m-1}) = 0$. In both cases, these two lead to the same conclusion $\det(H_{s+1,m-1}) = 0$ since the following two matrices are exactly the same (the dashed box and solid box),

$$\begin{pmatrix} 0 & 0 & \cdots & \beta \\ \alpha_{s+1} & \alpha_{s+2} & \cdots & \alpha_{s+m+1} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{s+m-1} & \alpha_{s+m} & \cdots & \alpha_{s+2m-1} \\ \alpha_{s+m} & \alpha_{s+1+m} & \cdots & \alpha_{s+2m} \end{pmatrix}$$

One can then conclude the same fact by induction $\forall s' > s$. Now we have found a matrix $\det(H_{s',m-1}) = 0$, $\forall s' > l$. This is a contradiction to the minimality of m. Thus, $\det(H_{s,m-1}) \neq 0$. This shows that all previous rows are linearly independent and the only dependent vector is the last row which can be written as a linear combination of all the others. This statement is true by induction for all $s > l$ making q_i's be a solution for all g's. Thus, their product must be a polynomial of degree less than $l + m$. $\square$

This consequently implies the finite dimensionality of $A(1)$ since the determinant of the Hankel matrix will evaluate to zero. Thus, the maximum number of linearly independent vectors must be the maximum degree to which we get equation 3. However, this is just the maximum of $k + 1$ or n. This leads to the following proposition.

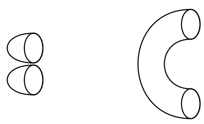
Proposition 5.5. *$A(1)$ is finite-dimensional if and only if $Z(T) = P_n(T)/Q_m(T)$ is a rational function such that elements $\{1, x, x^2, \cdots, x^{k-1}\}$ span $A(1)$ where $k = \max(n + 1, m)$.*

Now that we have understood how this works let us look at specific examples.

Example 5.1. Consider the following generating function

$$Z(T) = \beta$$

we want to construct the state space of $A(k)$. As per theorem 5.5, we know that span$\{A(1)\} = \{$  $\}$. For $A(k)$, we know that adding any holes in any connected or disconnected parts will give a zero row in the Hankel matrix. Thus, one can only consider bordisms with genus 0. For example $A(2)$ has only two cobordisms,



Thus, one may be tempted to conclude that the state space of $A(k)$ is just the Bell numbers. However, this is not true. Notice that we can only determine independence using proposition 5.3. Thus, there may exist certain relations between the vectors that would reduce the number of independent vectors. For more discussion about these relations refer to [11]. Since this may seem tedious to check by hand, we made a simple Mathematica code that would calculate the Hankel matrix and produce the number of independent vectors. Here are some results that were produced for polynomial $Z(T)$,

$Z(T)$	$A(1)$	$A(2)$	$A(3)$	$A(4)$
β	1	2	5	14
$\beta + \alpha T$	2	5	13	41

Other cases can also be computed for general polynomials. The code can also be modified to include the general rational case. In general, as we have seen in an example, $A(k)$ is also finitely generated as long as $A(1)$ is finitely generated. One can also prove this in general [11].

5.3 Summing over Topolgies

Now, we turn our attention to a specific calculation that is important in the theory of quantum gravity. It has been noted by various literature [9][14] that when one considers the Euclidean path integral in quantum gravity, one needs to sum overall topologies with the integration over all metrics. It seems tedious to calculate such a sum even for theories that are metric-independent. Thus, here, we will consider this sum over topologies for TQFTs and give some aspects regarding topological theories.

The sum over topologies in 2 dimensions has been investigated in recent literature [3] [14] as follows: suppose that we want to calculate the sum over topologies from n_i ingoing boundaries to n_o outgoing boundaries. The sum can be written as,

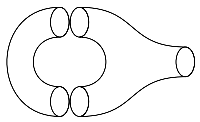
$$\mathfrak{A}(n_i, n_o) = \sum_{Y: n_i \to n_o} \frac{1}{|\mathrm{Aut}(Y)|} \mathcal{Z}(n_i, n_0)$$

The first thing to notice is that the bordisms that connect these two boundaries may have various connected components. Furthermore, there always exists a sum over closed manifolds each with a certain genus. The latter was discussed in [14]. They pointed out that to add the contribution from these closed manifolds, one just needs to compute the sum over all closed connected surfaces and call it λ. Then, by denoting $\mathfrak{A}(n_i, 0)$ as the sum over all bordism with n_i are incoming boundaries and zero outgoing boundaries, one sees that,

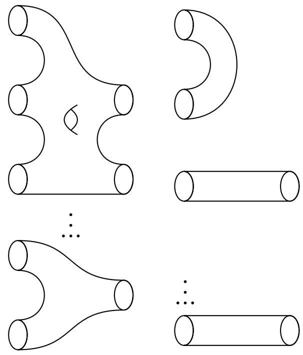
$$\mathfrak{A}(n, 0) = e^{\lambda} B_n(\lambda)\mathbf{v}$$

where B_n is the Bell polynomial of order n and $\mathbf{v}$ is some vector. If one was able to construct $\mathfrak{A}(n, 0)$, then action by tube cobordism would produce $\mathfrak{A}(n_i, n_o)$ such that $n_i + n_o = n$. Various examples are discussed in [3].

One now wishes to perform such calculations for universal construction. However, one faces much more issues. First of all, one does not have an analytic form that encodes the basis for $A(k)$. Suppose for instance that we found them by hand, and let us consider a single bordism from n_i to n_o. One can represent all basis by standard representation. Since all basis are cobordism generated, attaching a basis from the left produces a cobordism that can be written as a linear combination of basis elements in $A(n_o)$ as illustrated.

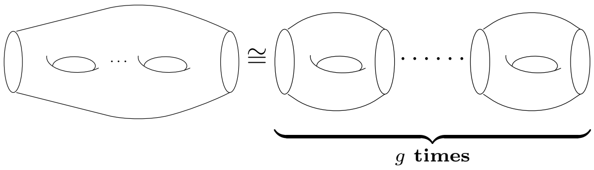

$\in A(1)$

Thus, given basis elements and relations in $A(k)$, one needs first to construct all bordism from $A(k)$ to $A(k)$. Thus, we have constructed the matrix $\mathfrak{A}(k, k)$. One can then multiply this matrix from the right by the matrix constructed from n_i tubes and n_o cylinders (note that to construct this matrix one needs to know all bases in $A(k + n_0)$). This is illustrated in the following figure



Now, we have a map from $A(k + n)$ to $A(k - n)$ which is the total sum $\mathfrak{A}(k + n, k - n)$. Thus, given n_0 and n_i, one needs to construct $\mathfrak{A}(k, k)$ from the basis elements where $k = (n_0 + n_i)/2$ and multiply by the tube bordisms as described above. This may reduce the computation, however, it requires knowledge of basis elements of $A(n_i)$, $A(n_o)$, and $A(k)$. Furthermore, it requires the sum to be even. If the sum is odd, one needs to construct $\mathfrak{A}(\lfloor k \rfloor, \lceil k \rceil)$ and again apply tube bordism. Then one would add all closed manifold constrictions by multiplying by $e^{Z(1)}$. The subtle point here is the method of constructing $\mathfrak{A}(k, k)$ or $\mathfrak{A}(\lfloor k \rfloor, \lceil k \rceil)$. As far as I am aware, in literature, there were no methods of constructing such sums of topologies for universal construction.

Example 5.2. Lastly, we will give an example illustrating the method of finding the matrices. We wish to calculate the matrix for the surface connecting $A(1)$ to itself with genus g for polynomial $Z(T)$.



This case is simple since we already know, be proposition 5.5, that the maximum number of basis we can have is $k = n + 1$. One can then choose standard bases as a representation for span$\{A(1)\}$. As shown, in the figure, one needs to find the handle element matrix M. Then, the matrix for the general surface with genus g is just M^g. However, this matrix is obvious, it just adds one hole to the generator bordism. Thus, it must have the following form,

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

which is nilpotent. Then, one can easily do the sum.

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