University of Science and Technology at Zewail City
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Bachelor Thesis

Spherically symmetric solutions in modified Teleparallel gravity

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Introduction

1998 marked an outrageous event that changed our comprehension and understanding of the universe. The universe was found to expand at an accelerating rate. Bahamonde [2018] This is of course opposite of what can be expected. Scientists termed the cause of such phenomena like dark energy. However, from a theoretical perspective general relativity does not encode such behavior from the first principle. It turns out that one needs to add a cosmological constant term to effectively describe such behavior. The cosmological constant acts like a perfect fluid that violates strong energy conditions making repulsive gravity. However, there are no theoretical motivations for such modifications. Thus, one must follow an approach to modify gravity. This approach is generally marked as modified theories of gravity. One aims to extend general relativity such that the produced theory could effectively describe current cosmological observations. Additionally, there are other motivations to modify general relativity including explaining dark matter, describing inflation, etc.

1.1 Motivation for modification of general relativity

There are serious issues that motivate the modification of general relativity. A basic modification of general relativity can be gone by changing the right hand of the equation or basically introducing new sources in field equations. Mainly, changing the energy stress tensor part is not considered a part of modified gravity. Modified gravity is usually related to changing of geometries.

One of the main problems is the problem of the cosmological constant. GR needs a cosmological constant to act as a perfect fluid with negative pressure. This is mainly needed to describe the current accelerating behavior of the universe. Equivalently, one can add a scalar field that can produce the same effects. However, the current value of the cosmological constant is drastically different from the one calculated by considering quantum and classical aspects of vacuum energy. Some researchers claim that there is a solution to this problem unless one either considers modifying GR or changing the standard model of particle physics. If one chooses to modify GR, there are some theories that can incorporate such behavior of late time acceleration. Furthermore, there will be some fundamental explanations for such behavior. Nojiri and Odintsov [2006]

Additionally, in the early times of the universe, the universe experienced a very rapid stage of acceleration. This type of acceleration cannot be described by a cosmological constant, since there will be no way of stopping such an expansion. This epoch is usually understood by considering a modification with a scalar field called the inflaton. The problem lies in the origin of such a field. There are no fundamental principles that can predict such modification. Additionally, there are problems with its prediction. They have fine-tuning problems in the parameters. Some examples include slow roll approximations and initial conditions. Modified gravity has enough tools to describe both of those eras. Bahamonde [2018]

Additionally, In general relativity, there are problems related to cosmological singularities. It is impossible to avoid such singularities in GR. However, in some modified theories, some singularities can be avoided. Nojiri and Odintsov [2010]

One other problem is the dark components. It was observed that rotation curves of galaxies exhibit flatness which contradicts our Newtonian prediction. Fortunately, there are some theories of modified gravity that incorporate such effects. They are known as MOND (Modified Newtonian Dynamics). MOND modifies Newtonian gravity to mimic the behavior of the flat rotation curves. Additionally, it can also achieve the luminosity-rotation velocity relation, known as the Tully-Fisher relation. Famaey and McGaugh [2011]

Another problem is the nonrenormalizability of GR. GR cannot be quantized. Thus, it needs to be modified in order to formulate a renormalizable theory.

Other problems can be found at Capozziello and De Laurentis [2011], Sotiriou and Faraoni [2008].

One way to approach this problem is to consider other contractions of the Riemann tensor which may introduce other observational and physical predictions that can describe our universe. However, One faces the Lovelock theorem on modification of gravity.

1.2 Lovelock theorem

In fact considering all contractions of the type $\mathcal{R}(g^{\mu\nu}, \mathcal{R}_{\mu\nu\alpha\beta})$ there are infinite. However, we are seeking contraction that ends up in derivatives less than fourth order. Since in light of stokes theorem, these terms turns out to become negligible surface integral (Gibbons-Hawking-York boundary term). Tian [2016] In general by varying a general contraction of the Riemann tensor one finds fourth order gravitational field equations by the variational derivative,

$$\frac{\delta(\sqrt{-g}\mathcal{R})}{\delta g^{\mu\nu}} = \frac{\delta(\sqrt{-g}\mathcal{R})}{\delta g^{\mu\nu}} - \partial_\alpha \frac{\delta(\sqrt{-g}\mathcal{R})}{\partial\partial_\alpha g^{\mu\nu}} + \partial_\alpha\partial_\beta \frac{\delta(\sqrt{-g}\mathcal{R})}{\partial\partial_\alpha\partial_\beta g^{\mu\nu}} \tag{1.1}$$

However, it was shown that the following Lanczos-Lovelock action is the most general action that generates a second order field equations. Lovelock [1969]

$$\mathcal{I} = \int d^4x\sqrt{-g}\left(aR - 2\Lambda + b\ \delta_{\alpha\beta\gamma\eta}R^{\mu\nu\alpha\beta}R_{\mu\nu}{}^{\gamma\eta} + c\ \mathcal{G} + 16\pi G\mathcal{L}_m\right) \tag{1.2}$$

where a, b, c, Λ are constants and $\mathcal{G}$ is the Gauss bonnet invariant.

$$\mathcal{G} = R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$$

In the Lancoz-Lovelock action, the first two terms are just the usual Hilbert-Einstein action. The third term is generally known as the Chern-Pontryagin density, where $\epsilon_{\alpha\beta\mu\nu} = \sqrt{-g}\delta_{\alpha\beta\mu\nu}$ is the total anti-symmetric Levi-Civita tensor. However, one can easily show that the term $\epsilon_{\alpha\beta\mu\nu}R^{\mu\nu\alpha\beta}R_{\mu\nu}{}^{\gamma\eta}$ is proportional to a divergence of the topological Chern-Simons four-current K^μ where,

$$\epsilon_{\alpha\beta\mu\nu}R^{\mu\nu\alpha\beta}R_{\mu\nu}{}^{\gamma\eta} = -8\partial_\mu K^\mu \tag{1.3}$$

$$K^\mu = \epsilon^{\mu\alpha\beta\gamma}\left(\frac{1}{2}\Gamma^\sigma_{\alpha\tau}\partial_\beta\Gamma^\tau_{\gamma\sigma} + \frac{1}{3}\Gamma^\sigma_{\alpha\tau}\Gamma^\tau_{\beta\eta}\Gamma^\eta_{\gamma\sigma}\right) \tag{1.4}$$

Additionally, the Gauss-Bonnet terms is also equivalent to a divergence of a four current.

$$\sqrt{-g}\mathcal{G} = -\partial_\mu J^\mu \tag{1.5}$$

$$J^\mu = \sqrt{-g}\epsilon^{\mu\alpha\beta\gamma}\epsilon_{\rho\sigma}{}^{\xi\zeta}\Gamma^\rho_{\xi\alpha}\left(\frac{1}{2}R^\sigma_{\zeta\beta\gamma} - \frac{1}{3}\Gamma^\sigma_{\lambda\beta}\Gamma^\lambda_{\zeta\gamma}\right) \tag{1.6}$$

Thus, these two terms reduce to a total surface integral leaving only the Einstein Hilbert action. This is the Lovelock theorem which states the following,

Lovelock Theorem: In four dimensions, the Hilbert-Einstein equation is the only second-order gravitational field equation in pure metric gravities. Lovelock [1969]

Upon this theorem one can approach modifying gravity through either one of these approaches,

1. Consider modifications in terms of general function of the Ricci scalar $f(R)$.

2. Go to higher dimensions. A well known action in literature follows this approach which is the Gauss-Bonnet action in 5 dimensions,

$$\mathcal{I} = \int d^5x\sqrt{-g}(R - 2\Lambda + c\,\mathcal{G} + 16\pi G\mathcal{L}_m)$$

This structure in 5 dimensions still preserves the second order field equations.

3. Go beyond Riemann geometry. Riemann geometry is only equipped with a metric and torsion free Levi-Civita connection. However, more general geometries could be constructed. For example, teleparallel gravity is a pure torsion theory. Other include Einstein Cartan and metric affine gravity.

4. Add extra degrees of freedom like a scalar filed or a vector field.

5. Consider non minimally coupling between curvature and matter for modified gravity.

$$\mathcal{L} = f(R) + 16\pi G\mathcal{L}_m + \tilde{f}(R)\mathcal{L}_m$$

In this retrospect, modifications of general relativity in four dimensions take a general form of $f(R)$ where R is the Ricci scalar. $f(R)$ gravity is extensively discussed in literature where various modifications and predication have been derived. Nojiri and Odintsov [2010] However, in this thesis we follow a different approach to considering the modification of general relativity.

General relativity relies on a connection termed the Levi-Civita connection. The Levi-Civita connection is symmetric and torsion less. One way to extend this is to consider more general cases. This thesis will mainly focus on discussing an alternative way of modified gravity termed teleparallel gravity. General outlines of such theory will be given. Additionally, the theory is totally equivalent to general relativity giving the same observational predictions. However, it has different physical and mathematical interpretations.

On the other hand, if one considers modification of teleparallel gravity, they are no longer equivalent to modified general relativity. We are now facing new predication with new physics coming from the mathematical equations.

2
Teleparallel Gravity

In this chapter, we introduce the gauge theory of translation known as teleparallel gravity. We begin by giving a general introduction to tetrads, a mathematical tool that is heavily used in teleparallel theories. Additionally, we will go through the exercise of gauging the translation group. Furthermore, field equations and equivalence with GR will be established.

Although being a very successful theory in describing the dynamics of the solar system, GR faces a lot of problems when applied to the entire universe. The problems are mainly summarized as dark energy and dark matter. It turns out that we do not have any fundamental understanding of about 95% of the universe. This is mainly a direct motivation to consider modifying gravity. Wright [2017]

In an attempt to unify electrodynamics and gravity, Einstein introduced an equivalent formulation of general relativity known as teleparallel gravity with tetrads being the dynamical variables instead of the metric. Teleparallel gravity or basically, the Teleparallel equivalent of General relativity (TEGR) is indistinguishable experimentally from general relativity. In TEGR, there is not a geodesic equation as in GR. Instead, similar to electromagnetism, force equations describe the movement of particles under the influence of gravity. The dynamical variable or the tetrad has 16 degrees of freedom while the metric has only 10. This is what motivates Einstein to work with such theory as the other 6 would incorporate electromagnetism. However, he did not succeed in this unification due to the fact the extra degrees of freedom, in the end, were related to the Lorentz invariance of the theory. Khanapurkar [2018]

The key mathematical concept that is heavily used in literature goes back to Weitzenböck, 1923. Aldrovandi and Pereira [2013] He noted that one can choose a connection such that the curvature vanishes everywhere, however, being nontrivial by incorporating torsion. The notion of teleparallelism is coined by the fact that parallelism is global instead of local on flat manifolds. Additionally, due to vanishing of curvature, parallel transportation does not depend on the path chosen. Thus, preserving the angles and the lengths.

Through the following chapter we will be using Hobson et al. [2006], Carroll [2019], Penrose and Rindler [1984] and Frankel [2012].

2.1 Tetrads and Linear Frames

In this section we give an introduction and provide adequate physical motivation for the concept of tetrads as a mathematical object that replaces the metric. Let us consider a observer $\mathcal{O}$ who may be accelerating with respect to some inertial frame $\mathcal{S}$. If the object moves on some worldline with four velocity $\mathbf{u}$, then its 4-acceleration is,

$$\mathbf{a} \equiv \frac{d\mathbf{u}}{d\tau} \tag{2.1}$$

It is also noteworthy to mention that the four acceleration is always perpendicular to the four velocity.

$$\begin{aligned} \mathbf{a} \cdot \mathbf{u} &= \frac{d}{d\tau}\left(\frac{1}{2}\mathbf{u} \cdot \mathbf{u}\right) \\ &= 0 \end{aligned} \tag{2.2}$$

Since the object is accelerating there is no inertial frame in which we can find the particle at rest. However, one can define an instantaneous rest frame $\mathcal{S}'$ such that the observer $\mathcal{O}$ is momentarily at rest. Additionally, since the observer is at rest in $\mathcal{S}'$, the four velocity of the object is parallel to the time-like basis vector denoted as $\mathbf{e'}_0$. Furthermore, at $\mathcal{S}'$ one can build the other orthogonal unit vectors $\mathbf{e'}_i$. Their orthogonality condition will in general depend of the relative velocities of $\mathcal{S}$ and $\mathcal{S}'$ and the relative orientation of their axis. One can then state that any observation made at a point $\mathbf{P}$ on the worldline by observer $\mathcal{O}$ correspond to measurements in the instantaneous rest frame $\mathcal{S}'$ at $\mathbf{P}$.

We can now idealize a notion that uses these orthogonal bases. One can fairly say that any observer whether accelerating or not carries a set of 4 orthogonal unit vectors $\mathbf{e'}_\mu$ that vary from point to point on the manifold and stratify,

$$\mathbf{e'}_\mu \cdot \mathbf{e'}_\nu = \eta_{\mu\nu} \tag{2.3}$$

$$\mathbf{e'}_0 = \hat{\mathbf{u}} \tag{2.4}$$

where $\hat{\mathbf{u}}$ is the normalized four velocity. The tetrads define a time direction with three space directions along the worldline which makes them an adequate basis for the inertial reference frame. Thus, any measurement along the worldline can be given by projections on these tetrad vectors. Lastly, another thing to notice is that these basis vectors must be related to some other vector in some inertial reference frame. In general, any given vector must be a Lorentz transformation of some other vector.

$$\mathbf{e'}_\mu = \Lambda_\mu{}^\nu \mathbf{e}_\nu \tag{2.5}$$

Now one should inquire about the possibility of reformatting general relativity or other theories in terms of these tetrads. Strictly speaking, one needs to inquire about the reason for not spanning the tangent space of the manifold using these bases. We denote the tetrad basis vector by Latin indices $\mathbf{e}_a$. We, thus, again require that the basis are orthogonal to each other. Thus,

$$\mathbf{e}_a \cdot \mathbf{e}_b = g(\mathbf{e}_a, \mathbf{e}_b) = \eta_{ab}$$

where g is the usual metric tensor defined with the manifold. Remembering that the natural basis of the tangent space T_p is $\mathbf{e}_\mu \equiv \partial_\mu$ and that of the cotangent space is $\theta^\mu \equiv dx^\mu$, there is no problem in rewriting these tetrad bases as a linear combination of the natural basis.

$$\mathbf{e}_\mu = \mathbf{e}_\mu{}^a \mathbf{e}_a \tag{2.6}$$

where $\mathbf{e}_\mu{}^a$ form an $n \times n$ matrix. Usually, in literature, the term tetrad is given to this matrix instead of the basis. We will, thus, follow this in the following discussion. The main distinction between these internal frames and the natural frames is the anhonolomy of these bases,

$$[\mathbf{e}_a, \mathbf{e}_b] = [\mathbf{e}^\mu{}_a \partial_\mu, \mathbf{e}^\nu{}_b \partial_\nu] = \mathbf{e}^\mu{}_a \partial_\mu (\mathbf{e}^\nu{}_b \partial_\nu) - \mathbf{e}^\nu{}_b \partial_\nu (\mathbf{e}^\mu{}_a \partial_\mu) \tag{2.7}$$

By taking the derivative of 2.6 one can find the commutator is proportional to the basis through some structure constants,

$$[\mathbf{e}_a, \mathbf{e}_b] = \mathbf{e}^\mu{}_a \mathbf{e}^\mu{}_b (\partial_\mu \mathbf{e}_\nu{}^c - \partial_\nu \mathbf{e}_\mu{}^c) \mathbf{e}_c = f^c_{ab} \mathbf{e}_c \tag{2.8}$$

These frame are called non holonomic frames since the commutator does not commute in disargument for the one of the natural frames. We follow the usual convention in denoting the inverse by swapping the indices which satisfy,

$$\mathbf{e}^\mu{}_a \mathbf{e}_\nu{}^a = \delta^\mu_\nu, \qquad \mathbf{e}_\mu{}^a \mathbf{e}^\mu{}_b = \delta^a_b \tag{2.9}$$

One can now rewrite 2.3 as,

$$g_{\mu\nu} \mathbf{e}^\mu{}_a \mathbf{e}^\nu{}_b = \eta_{ab} \tag{2.10}$$

or equivalently,

$$g_{\mu\nu} = \eta_{ab} \mathbf{e}_\mu{}^a \mathbf{e}_\nu{}^b \tag{2.11}$$

One can then say that the tetrad is the square root of the metric by varying the tetrads and using the above relation.

$$e = \det(\mathbf{e}^a_\mu) = \sqrt{-g} \tag{2.12}$$

Similarly one use tetrads to move from natural basis of the cotangent space to constructed ones. If one have a vector V it can be expressed either in the tetrad basis or the natural basis $V \equiv V^\mu \mathbf{e}_\mu \equiv V^a \mathbf{e}_a$. One can then find the relation between components as,

$$V^a = \mathbf{e}_\mu{}^a V^\mu \tag{2.13}$$

Thus, in general, the tetrads allow us to switch between the Greek and Latin indices. Additionally, we use the Minkowski metric to raise and lower Latin indices and the usual metric for the Greek indices as usual. One can also think of tetrads as the components of (1,1) tensor.

$$e = e_\nu{}^a dx^\nu \otimes \ \mathbf{e}_a \tag{2.14}$$

However, if one looks closely this is just the identity. If this tensor act on a vector it gives back the same vector. Now let us consider coordinate transformations on these basis. We restrict that the basis of tetrads be orthogonal to each other. Thus, using this one can see that proper transformations that keep the orthogonality conditions to hold are Lorentz transformations,

$$\mathbf{e}_a \rightarrow \mathbf{e}_{a'} = \Lambda^a_{a'} \mathbf{e}_a \quad \text{with} \quad \Lambda^a_{a'} \Lambda^b_{b'} \eta_{ab} = \eta_{a'b'} \tag{2.15}$$

Thus, we have the freedom to change Latin indices by Lorentz transformation and the Greek indices with general coordinate transformations. Mixed tensor transform as follows,

$$T^{a'}_{\ \ \mu} = \Lambda^{a'}{}_a \frac{\partial x^\mu}{\partial x^{\mu'}} T^a{}_{\mu'} \tag{2.16}$$

The most important distinction arises when we consider differentiating on a tetrad basis. We should expect that a connection would also arise that will compensate for the action of the usual connection. We label this connection by Spin connection. This fancy naming relates to the fact that it was used to take the covariant derivative of Dirac spinors. Hehl et al. [1976] Thus, each Latin index must get a factor of the spin in the usual way,

$$\nabla_\mu V^a_{\ b} = \partial_\mu V^a_{\ b} + \omega_\mu{}^a{}_c V^c_{\ b} - \omega_\mu{}^c{}_b V^a_{\ c} \tag{2.17}$$

Furthermore, since a tensor is invariant whether written on a tetrad basis or not. This in general will allow us to find a relation between the Christoffel connection and the spin connection.

$$\begin{aligned}
\nabla V &= (\nabla_\mu V^\nu) dx^\mu \otimes \partial_\nu \\
&= (\partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda) dx^\mu \otimes \partial_\nu \\
&= (\nabla_\mu V^a) dx^\mu \otimes \mathbf{e}_a \\
&= (\partial_\mu V^a + \omega_\mu{}^a{}_b V^b) dx^\mu \otimes \mathbf{e}_a \\
&= (\partial_\mu (\mathbf{e}_\nu{}^a V^\nu) + \omega_\mu{}^a{}_b \mathbf{e}_\lambda{}^b V^\lambda) dx^\mu \otimes (\mathbf{e}^\sigma{}_a \partial_\sigma) \\
&= \mathbf{e}^\sigma{}_a (V^\nu \partial_\mu (\mathbf{e}_\nu{}^a) + \mathbf{e}_\nu{}^a \partial_\mu (V^\nu) + \omega_\mu{}^a{}_b \mathbf{e}_\lambda{}^b V^\lambda) dx^\mu \otimes \partial_\sigma \\
&= (\delta^\sigma_\nu \partial_\mu V^\nu + V^\nu \mathbf{e}^\sigma{}_a \partial_\mu (\mathbf{e}_\nu{}^a) + \omega_\mu{}^a{}_b \mathbf{e}^\sigma{}_a \mathbf{e}_\lambda{}^b V^\lambda) dx^\mu \otimes \partial_\sigma \\
&= (\partial_\mu V^\nu + V^\lambda \mathbf{e}^\nu{}_a \partial_\mu (\mathbf{e}_\lambda{}^a) + \omega_\mu{}^a{}_b \mathbf{e}^\nu{}_a \mathbf{e}_\lambda{}^b V^\lambda) dx^\mu \otimes \partial_\nu
\end{aligned} \tag{2.18}$$

Now by equating the second line with the last line, one can find the following relation between the spin connection and the Christoffel symbol,

$$\Gamma^\nu_{\mu\lambda} = \mathbf{e}^\nu{}_a \partial_\mu (\mathbf{e}_\lambda{}^a) + \omega_\mu{}^a{}_b \mathbf{e}^\nu{}_a \mathbf{e}_\lambda{}^b \tag{2.19}$$

or

$$\omega_\mu{}^a{}_b = \mathbf{e}_\nu{}^a \mathbf{e}^\lambda{}_b \Gamma^\nu_{\mu\lambda} - \mathbf{e}^\lambda{}_b \partial_\mu \mathbf{e}_\lambda{}^a \tag{2.20}$$

One can then easily use these relation to derive what is known as the tetrad postulate.

$$\begin{aligned}
\nabla_\mu \mathbf{e}_\nu{}^a &= \partial_\mu \mathbf{e}_\nu{}^a - \Gamma^\lambda_{\mu\nu} \mathbf{e}_\lambda{}^a + \omega_\mu{}^a{}_b \mathbf{e}_\nu{}^b \\
&= 0
\end{aligned} \tag{2.21}$$

It is also noteworthy to point out that this is true for any geometry since we did not assume anything about the connection. In the same manner, as the usual connection, the spin connection must also transform in-homogeneously to fix up the derivative term. Thus, it is convenient to expect that the spin connection transforms as follows,

$$\omega_\mu{}^{a'}{}_{b'} = \Lambda^{a'}{}_a \Lambda^{b'}{}_b \omega_\mu{}^a{}_b - \Lambda^c{}_{b'} \partial_\mu \Lambda^{a'}{}_c \tag{2.22}$$

2.2 Gravitational gauge theory

Let now introduce the theory of teleparallel gravity. We claim in this section that teleparallel gravity is the gauge theory of translation. Thus, upon localizing the global translation invariance, teleparallel gravity should pop up as an adequate theory. We being this section by reviewing some necessary formulation of symmetries that will be utilized. Most of the following discussion is adapted from Freedman and Van Proeyen [2012], Hehl et al. [2013] and Bennett [2021].

Any symmetry transformation is determined by a parameter ϵ^A and an operation $\delta(\epsilon)$. The operation depends linearly on the parameter and acts on the fields of the dynamical system. Thus, one can write

$$\delta(\epsilon) = \epsilon^A T_A \tag{2.23}$$

where T_A is an operator on the space of the fields which many symbolize translation generator, rotation generator, etc. Now let t_A be matrix generators of the lie algebra of the operator with the following commutation relations.

$$[t_A, t_B] = f_{AB}^{\ \ C} t_C \tag{2.24}$$

Thus the action of the generator T_A on some field ϕ^i can be expanded as follows,

$$T_A \phi^i = -(t_A)^i_j \phi^j \tag{2.25}$$

Now are seeking the product operation of two infinitesimal transformations of the same operator. This will be come in hand later.

$$\delta(\epsilon_1)\delta(\epsilon_2)\phi^i = \epsilon_1^A T_A(\epsilon_2^B T_B \phi^i) \tag{2.26}$$

Since the generator matrices t_A are constant matrices the operator will act on them but rather on the field itself. Thus, we have,

$$\begin{aligned} \delta(\epsilon_1)\delta(\epsilon_2)\phi^i &= \epsilon_1^A T_A(-\epsilon_2^B (t_B)^i_j \phi^j) \\ &= \epsilon_1^A \epsilon_2^B T_A(-(t_B)^i_j \phi^j) \\ &= \epsilon_1^A \epsilon_2^B (t_A)^i_k (t_B)^k_j \phi^j \end{aligned} \tag{2.27}$$

We can now find the commutator as follows,

$$\begin{aligned}[\delta(\epsilon_1), \delta(\epsilon_2)] &= \epsilon_1^A \epsilon_2^B [T_A, T_B] \\ &= \epsilon_1^A \epsilon_2^B f_{AB}^{\ \ C} T_C\end{aligned} \tag{2.28}$$

Following the same procedure of localizing the transformation parameter, one will need to properly transform the derivative. Thus, for $\epsilon^A \equiv \epsilon^A(x)$,

$$\begin{aligned}\delta(\epsilon)\partial_\mu \phi &= \delta(\epsilon) \lim_{h\to 0} \frac{1}{h} \left(\phi(x+h) - \phi(x)\right) \\ &= \lim_{h\to 0} \frac{1}{h} \left(\phi'(x'+h) - \phi(x+h) - \phi'(x') + \phi(x)\right) \\ &= \lim_{h\to 0} \frac{1}{h} \left(\delta(\epsilon)\phi(x+h) - \delta(\epsilon)\phi(x)\right) \\ &= \partial_\mu(\delta(\epsilon)\phi)\end{aligned} \tag{2.29}$$

Thus, we have

$$\begin{aligned}\delta(\epsilon)\partial_\mu \phi &= \partial_\mu(\delta(\epsilon)\phi) \\ &= \partial_\mu(\epsilon^A T_A \phi) \\ &= \partial_\mu(\epsilon^A) T_A \phi + \epsilon^A \partial_\mu (T_A \phi)\end{aligned} \tag{2.30}$$

We require that the derivative transform as the field upon which we define a covariant derivative as follows,

$$\begin{aligned}D_\mu &= \partial_\mu - \delta(B_\mu) \\ &= \partial_\mu - B_\mu^{\ A} T_A\end{aligned} \tag{2.31}$$

One should note that $\delta(B_\mu)$ is not the variation of the gauge field transformation, but rather we are using the gauge field as a parameter for symmetry transformation. Since the generator will act only on the field, relation 2.28 will be true even when localizing the parameter. Now, if one replaces ϵ_1 with B_μ and ϵ_2 with ϵ, one gets,

$$\begin{aligned}[\delta(B_\mu), \delta(\epsilon)]\phi &= \delta(B_\mu)\delta(\epsilon)\phi - \delta(\epsilon)\delta(B_\mu)\phi \\ &= \delta(B_\mu)(\epsilon^A T_A \phi) - \delta(\epsilon)(B_\mu^{\ A} T_A \phi) \\ &= \epsilon^A \delta(B_\mu)(T_A \phi) - B_\mu^{\ A} \delta(\epsilon)(T_A \phi) \\ &= B_\mu^{\ A} \epsilon^B f_{AB}^{\ \ C} T_C \phi\end{aligned} \tag{2.32}$$

We are now seeking a transformation law for the field or more precisely we want to know the value of $\delta(\epsilon)B_\mu{}^A$. By requiring that the derivative to transforms like the field, one can find,

$$\begin{aligned}\delta(\epsilon)D_\mu\phi &= \delta(\epsilon)(\partial_\mu - \delta(B_\mu))\phi \\ &= \delta(\epsilon)\partial_\mu\phi - \delta(\epsilon)(B_\mu{}^A T_A\phi) \\ &= \partial_\mu(\epsilon^A)T_A\phi + \epsilon^A\partial_\mu(T_A\phi) - \delta(\epsilon)(B_\mu{}^A)T_A\phi - B_\mu{}^A\delta(\epsilon)(T_A\phi)\end{aligned} \tag{2.33}$$

Now, equating this with $\epsilon^A D_\mu T_A\phi$ (i.e the required transformation law). One can find the following relation,

$$\begin{aligned}-\epsilon^A\delta(B_\mu)T_A\phi &= \partial_\mu(\epsilon^A)T_A\phi - \delta(\epsilon)(B_\mu{}^A)T_A\phi - B_\mu{}^A\delta(\epsilon)(T_A\phi) \\ &= \partial_\mu(\epsilon^A)T_A\phi - \delta(\epsilon)(B_\mu{}^A)T_A\phi + B_\mu{}^A\epsilon^B f_{AB}^{\ \ C}T_C\phi - \epsilon^A\delta(B_\mu)(T_A\phi)\end{aligned} \tag{2.34}$$

or,

$$\delta(\epsilon)(B_\mu{}^A)T_A\phi = \partial_\mu(\epsilon^A)T_A\phi + B_\mu{}^A\epsilon^B f_{AB}^{\ \ C}T_C\phi \tag{2.35}$$

By renaming the last factor $C \to A, A \to B, B \to C$

$$\delta(\epsilon)(B_\mu{}^A)T_A\phi = \partial_\mu(\epsilon^A)T_A\phi + B_\mu{}^B\epsilon^C f_{BC}^{\ \ A}T_A\phi \tag{2.36}$$

Thus, the transformation law of the derivative is found to be,

$$\delta(\epsilon)(B_\mu{}^A) = \partial_\mu(\epsilon^A) + f^A_{\ BC}B_\mu{}^B\epsilon^C \tag{2.37}$$

One can then build the field strength from the covariant derivatives as follows,

$$\begin{aligned}\delta(R_{\mu\nu}) &= -[D_\mu, D_\nu] \\ &= -\left[\partial_\mu - \delta(B_\mu), \partial_\nu - \delta(B_\nu)\right] \\ &= -[\partial_\mu, \partial_\nu] + [\partial_\mu, \delta(B_\nu)] + [\delta(B_\mu), \partial_\nu] - [\delta(B_\mu), \delta(B_\nu)] \\ &= \delta(\partial_\mu B_\nu) - \delta(\partial_\nu B_\mu) + \delta(B_\mu{}^B B_\nu{}^C f_{BC}^{\ \ A})\end{aligned} \tag{2.38}$$

That makes,

$$R_{\mu\nu}{}^A = \partial_\mu B_\nu{}^A - \partial_\nu B_\mu{}^A + B_\mu{}^B B_\nu{}^C f_{BC}^{\ \ A} \tag{2.39}$$

Let us now return to the main purpose of this section which is deriving the teleparallel equations. Teleparallel gravity is based on the notion of global parallelism. In the framework of GR, one cannot have global parallelism unless one works with flat space described by Minkowski spacetime. However, general manifolds can incorporate torsion where the covariant derivatives do not commute. This allows the possibility of the construction of nontrivial geometries in addition to being globally flat. It was found that teleparallel gravity can be written as a gauge theory of translation. For mathematical consistency, we will gauge the full Poincare transformations and then limit them to the case of translations. A Poincare transformation transforms spacetime coordinates as,

$$x^\mu \to x^{\mu'} = (\Lambda^{-1})^\mu{}_\nu (x^\nu - a^\nu) \tag{2.40}$$

The Poincare transformations include both generators of translations P_μ and the anti-symmetric $M_{\mu\nu}$ of Lorentz generators. The generators have the following algebra,

$$[P_\mu, P_\nu] = 0 \tag{2.41}$$

$$\begin{aligned}[M_{\mu\nu}, P_\rho] &= \eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu \\ &= 2\eta_{\rho[\nu} P_{\mu]}\end{aligned} \tag{2.42}$$

$$\begin{aligned}[M_{\mu\nu}, M_{\rho\sigma}] &= \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\mu\rho} \\ &= 4\eta_{[\mu[\rho} M_{\sigma]\nu]}\end{aligned} \tag{2.43}$$

An element of the Lorentz and translation subgroups can be constructed as,

$$U(\Lambda) = e^{-\frac{1}{2}\lambda^{\mu\nu} M_{\mu\nu}} \quad U(a) = e^{a^\mu P_\mu}$$

One can then write infinitesimal transformations for both translation and Lorentz transformations as follows,

$$\begin{aligned}\delta(\lambda)\phi &= -\frac{1}{2}\lambda^{\mu\nu} M_{\mu\nu}\phi \\ &= -\frac{1}{2}\lambda^{\mu\nu}(x_\mu \partial_\nu - x_\nu \partial_\mu)\phi \\ &= \lambda^{\mu\nu} x_\nu \partial_\mu \phi\end{aligned} \tag{2.44}$$

$$\begin{aligned}\delta(a)\phi &= a^\mu P_\mu \phi \\ &= a^\mu \partial_\mu \phi\end{aligned} \tag{2.45}$$

where $\lambda^{\mu\nu}$ is anti-symmetric. One can rewrite equations 2.42 and 2.43 in a form similar to that of the generator with structure coefficients, Freedman and Van Proeyen [2012]

$$f_{[ab],[cd]}{}^{[ef]} = 8\eta_{[c[b}\delta^{[e}_{a]}\delta^{f]}_{d]} \tag{2.46}$$

$$f_{a,[bc]}{}^{d} = 2\eta_{a[b}\delta^{d}_{c]} \tag{2.47}$$

One can now attach a gauge field $e_\mu{}^a$ to translations and a gauge field $\omega_\mu{}^{ab}$ to local Lorentz transformation M_{ab}. Now, instead of working with the two gauge fields, we can consider only those rotations. However, one has to add the field for the translations. In this context, we now have two fields the gauge field of LLT $\omega_\mu{}^{ab}$ and the nongauge field $e_\mu{}^a$ of translations. Since a nongauge field is also present equation 2.37 is not correct. The most general form of equation 2.37 is,

$$\delta(\epsilon)(B_\mu{}^A) = \partial_\mu(\epsilon^A) + f^A_{\ BC}B_\mu{}^B\epsilon^C + \epsilon^B\mathcal{M}_{\mu B}{}^A \tag{2.48}$$

where the last term is proportional to the field and its structure factors. Upon this identification, if we use $B_\mu{}^a = \omega_\mu{}^{ab}$ and $\mathcal{M}_{\mu B}^{\ A} = e_\mu{}^a f_{aB}{}^A$ where the structure factors came from the commutation of generators of translations and local Lorentz transformations. Using these relations one can find the following relation between field strength and transformation of the gauge field,

$$\delta(\lambda)\omega_\mu{}^{ab} = \partial_\mu\lambda^{ab} + \lambda^{ca}\omega_\mu{}^{bc} - \lambda^{cb}\omega_\mu{}^{ac} \tag{2.49}$$

$$R_{\mu\nu}{}^{ab}(\omega) = \partial_\mu\omega_\nu{}^{ab} - \partial_\nu\omega_\mu{}^{ab} - \omega_\mu{}^a{}_c\omega_\nu{}^{cb} + \omega_\nu{}^a{}_c\omega_\mu{}^{cb} \tag{2.50}$$

On the other hand if one considers the gauge field to the tetrad field $e_\mu{}^a$ of translations and the nongauge field to that of local Lorentz transformations. One can find the following transformation rules for the translation field and its field strength,

$$\delta(\epsilon)e_\mu{}^a = \partial_\mu\epsilon^a - \epsilon^b\omega_\mu{}^{ab} + \lambda^{ab}e_\mu{}^b \tag{2.51}$$

$$R_{\mu\nu}{}^a = \partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a - \omega_\mu{}^a{}_b e_\nu{}^b + \omega_\nu{}^a{}_b e_\mu{}^b \tag{2.52}$$

We define the last strength tensor as the torsion tensor. Its equivalence to geometric torsion will be established in the next section.

2.3 Geometric Interpenetration of torsion

Now we will consider a general manifold. In general the covariant derivative need not commute or $D_\mu D_\nu A^{\beta\cdots}_{\alpha\cdots} \neq D_\nu D_\mu A^{\beta\cdots}_{\alpha\cdots}$. We will first consider working with scalars.

$$\begin{aligned}[D_\mu, D_\nu]f &= -\Gamma^\lambda_{\mu\nu}\partial_\lambda f + \Gamma^\lambda_{\nu\mu}\partial_\lambda f \\ &\equiv -T^\lambda_{\ \mu\nu}\partial_\lambda f\end{aligned} \tag{2.53}$$

where we have defined the torsion tensor as,

$$T^\lambda_{\ \mu\nu} \equiv \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} \tag{2.54}$$

GR is a torsion-free theory where the connection is symmetric. However, if one considers theories like teleparallel gravity it will depend heavily on torsion. Since the torsion is skew symmetric in two indices it contains $n^2(n-1)/2$ independent parameters. Additionally, one can easily check that torsion is indeed a tensor by writing the transformation laws for the connection. One will then notice that the offensive part will cancel leaving only the homogeneous parts. Nevertheless, one should inquire about what torsion does for the field or tensors on the manifold. This can be seen by considering a 2-dimensional geometry. Consider a point P_0 that is located at x^μ and P_1 and P_2 are located at $x^\mu + dx_1^\mu$ and $x^\mu + dx_2^\mu$ respectively. Now, let us parallel transport vectors dx_1^μ and dx_2^μ along one another. Define the transported vectors as dx_3^μ and dx_4^μ. One can then use a parallel transport equation to find the coefficients,

$$dx_3^\mu = dx_2^\mu - \Gamma^\mu_{\nu\rho}dx_1^\rho dx_2^\nu \tag{2.55}$$

and

$$dx_4^\mu = dx_1^\mu - \Gamma^\mu_{\nu\rho}dx_1^\nu dx_2^\rho \tag{2.56}$$

One can use these vectors to define points P_3 and P_4 with coordinates,

$$x^\mu(P_3) = x^\mu(P_1) + dx_3^\mu, \quad x^\mu(P_4) = x^\mu(P_2) + dx_4^\mu$$

Since we are closing the parallelogram one should it expect that must close up or that the two coordinate of P_3 and P_4 are the same. However, this is not the case if the manifold has torsion. Subtracting the value of the two points one can get,

$$\begin{aligned}x^\mu(P_3) - x^\mu(P_4) &= (\Gamma^\mu_{\nu\rho} - \Gamma^\mu_{\rho\nu})dx_1^\nu dx_2^\rho \\ &= T^\mu_{\ \nu\rho}\ dx_1^\nu\ dx_2^\rho\end{aligned} \tag{2.57}$$

Thus, torsion measures the failure to close an infinitesimal parallelogram. In Teleparallel theories of gravity when the torsion tensor is different from zero, this geometrical effect plays a very important role.

It is also very useful in our discussion to split the connection into a symmetric part and an anti-symmetric part. This in general can be done to any object. To drive such a formula, one begins with the metric compatibility condition,

$$\begin{aligned} D_\sigma g_{\mu\nu} &= \partial_\sigma g_{\mu\nu} - \Gamma^\lambda_{\mu\sigma} g_{\lambda\nu} - \Gamma^\lambda_{\nu\sigma} g_{\mu\lambda} \\ &= 0 \end{aligned} \tag{2.58}$$

One can then cycle the indices by forming three equivalent equations of the above equation. Adding two terms and subtracting the third with some algebra and substitution of the value of the torsion tensor one can find the following relation for the connection.

$$\Gamma^\lambda_{\mu\nu} = \overset{\circ}{\Gamma}{}^\lambda_{\mu\nu} + K_\mu{}^\lambda{}_\nu \tag{2.59}$$

where $\overset{\circ}{\Gamma}{}^\lambda_{\mu\nu}$ is the symmetric Levi-Civita connection found in GR and $K_\mu{}^\lambda{}_\nu$ is known as the contorsion tensor which is equivalent to,

$$K_\mu{}^\lambda{}_\nu = \frac{1}{2}\left(T^\lambda_{\mu\nu} - T_{\nu\mu}{}^\lambda + T_\mu{}^\lambda{}_\nu\right)$$

This is valid for any spacetime unless it contains nonmetricity. Since the Christoffel split into two parts one contain torsion and the other does not, we should expect the same for the curvature term. That is indeed the case.

$$R^\lambda{}_{\mu\sigma\nu} = \overset{\circ}{R}{}^\lambda{}_{\mu\sigma\nu} + \overset{\circ}{\nabla}_\sigma K_\nu{}^\lambda{}_\mu - \overset{\circ}{\nabla}_\nu K_\sigma{}^\lambda{}_\mu + K_\sigma{}^\lambda{}_\rho K_\nu{}^\rho{}_\mu - K_\nu{}^\lambda{}_\rho K_\sigma{}^\rho{}_\mu \tag{2.60}$$

where any quantity with a circle on top signifies that it is with respect to the Levi-Civita connection. We can directly see that if torsion is zero we return to the usual GR. Now we return to equation 2.52 of the tetrad gauge field if we define this strength tensor as,

$$R_{\mu\nu}{}^a = T^a{}_{\mu\nu}$$

One can easily find that it is equivalent to equation 2.19 and $e_\mu{}^a$ is indeed the tetrad as we guessed.

2.4 Field equations

Now, we move on to drive the teleparallel field equations and prove their equivalence to general relativity. One can then move on to construct the Ricci tensor from equation 2.60. If one then contracts the Ricci tensor with the metric one obtains the Ricci scalar which has the following relation,

$$\begin{aligned} R &= \mathring{R} + T - \frac{2}{e}\partial_\mu(eT^\mu) \\ &= \mathring{R} + T - B \end{aligned} \tag{2.61}$$

where B is called the boundary term,

$$B = \frac{2}{e}\partial_\mu(eT^\mu)$$

and T is called the torsion scalar which is given by,

$$T = \frac{1}{4}T^{\mu\nu\lambda}T_{\mu\nu\lambda} + \frac{1}{2}T^{\mu\nu\lambda}T_{\nu\mu\lambda} - T^\mu T_\mu \tag{2.62}$$

and T_μ is known as the torsion scalar $T_\mu = T^\lambda{}_{\lambda\mu}$. For convenience, it is better to introduce the superpotential,

$$\begin{aligned} S_\sigma{}^{\mu\nu} &= \frac{1}{4}(T_\sigma{}^{\mu\nu} - T^\nu{}_\sigma{}^\mu - T^\mu{}_\sigma{}^\nu) + \frac{1}{2}(\delta^\nu_\sigma T^\mu - \delta^\mu_\sigma T^\nu) \\ &= \frac{1}{2}(K_\sigma{}^{\mu\nu} - \delta^\nu_\sigma T^\mu + \delta^\mu_\sigma T^\nu) \end{aligned} \tag{2.63}$$

One can then easily check that the torsion scalar can be written as follows,

$$T = S_\lambda{}^{\mu\nu}T^\lambda{}_{\mu\nu}$$

In teleparallel gravity, one applies the notion of global parallelism. The Weitzenböck connection provides a globally flat spacetime with zero curvature and then with zero scalar curvature equation 2.50. Thus, making the Ricci scalar zero. Thus, we obtain,

$$\begin{aligned} R &= \mathring{R} + T - B \\ &= 0 \end{aligned} \tag{2.64}$$

or

$$\overset{\circ}{R} = -T + B \tag{2.65}$$

It should be emphasized that T and B are computed with the Weitzenböck connection ω. One can follow the most convenient method of constructing the Lagrangian from the torsion scalar. A convenient Lagrangian will be,

$$\mathcal{S}_{\text{TEGR}} = \frac{1}{2\kappa^2}\int d^4xeT + \mathcal{S}_m \tag{2.66}$$

One then notices a very important relationship. Since the Ricci scalar and torsion differ only by a boundary term B. One can notice that $\mathcal{L}_{\text{GR}} = \sqrt{-g}\overset{\circ}{R} = e\overset{\circ}{R}$. The difference between Lagrangian of teleparallel gravity and general relativity is,

$$\begin{aligned} \mathcal{L}_{\text{GR}} - \mathcal{L}_{\text{TEGR}} &= e(\overset{\circ}{R} + T) \\ &= eB \\ &= \partial_\mu(eT^\mu) \end{aligned} \tag{2.67}$$

which will be converted to a negligible surface integral. Thus, if one takes the variation of any of these Lagrangians, the same equations would pop up. Thus, we have proven the equivalence between general relativity and the teleparallel equivalent of general relativity. Since teleparallel gravity has the same equations as GR, teleparallel gravity is experimentally confirmed as well. One can conclude that it is a matter of interpretation whether gravity is described by a zero torsion theory and non-zero curvature (GR) or a theory with zero curvature and non-zero torsion (TEGR). Although being equivalent, the two theories have different physical interpretations.

Lastly, let us drive the equations of motion of teleparallel gravity. Throughout the whole thesis, as pointed out before, we will be using the pure tetrad formalism where the spin connection vanishes. There is a problem with this formulation. The torsion is no longer covariant under Lorentz transformations. However, due to the fact that the teleparallel action is invariant under local Lorentz transformations by a boundary term, the theory with a vanishing spin connection is a quasi-local Lorentz covariant. Thus, it is not a huge problem. We will discuss more of this problem in the next chapter. Taking the variation of the action with respect to the tetrad field one gets,

$$\delta\mathcal{S}_{\text{TEGR}} = \frac{1}{2\kappa^2}\int d^4x(e\delta T + T\delta e) \tag{2.68}$$

By using $g^{\mu\nu} = \eta^{ab} e^{\mu}{}_{a} e^{\nu}{}_{b}$ with the relation between tetrads and its inverse, it is straight forward to show that,

$$\delta g^{\mu\nu} = -\left(g^{\nu\lambda} e^{\mu}{}_{a} + g^{\mu\lambda} e^{\nu}{}_{a}\right) \delta e_{\lambda}{}^{a} \tag{2.69}$$

$$\begin{aligned} \delta e &= \frac{1}{2} e g^{\mu\nu} \delta g_{\mu\nu} \\ &= e e^{\lambda}{}_{a} \delta e_{\lambda}{}^{a} \end{aligned} \tag{2.70}$$

Then the second term in 2.68 is straight forward. However, variation of the torsion needs some work. Since the spin connection vanish, one can write the connection as,

$$\Gamma^{\nu}_{\mu\lambda} = e^{\nu}{}_{a} \partial_{\mu}(e_{\lambda}{}^{a})$$

using this with variation of the tetrad field,

$$\delta e^{\sigma}{}_{a} = -e^{\sigma}{}_{b} e^{\mu}{}_{a} \delta e_{\mu}{}^{b}$$

One can find the following relations for the variation of the torsion tensor and torsion vector,

$$\delta T^{\lambda}{}_{\mu\nu} = -e^{\lambda}{}_{a} T^{\beta}{}_{\mu\nu} \delta e_{\beta}{}^{a} + e^{\lambda}{}_{a} (\partial_{\mu} \delta e_{\nu}{}^{a} - \partial_{\nu} \delta e_{\mu}{}^{a}) \tag{2.71}$$

$$\delta T^{\mu} = -\left(e^{\mu}{}_{a} T^{\lambda} + g^{\mu\lambda} T_{a} + T^{\lambda}{}_{a}{}^{\mu}\right) \delta e_{\lambda}{}^{a} + g^{\mu\nu} e^{\lambda}{}_{a} \left(\partial_{\lambda} \delta e_{\nu}{}^{a} - \partial_{\nu} \delta e_{\lambda}{}^{a}\right) \tag{2.72}$$

Now its a matter of substitution to show the following,

$$\delta(T_{\mu} T^{\mu}) = -2\left(T^{\beta} T^{\alpha}{}_{\beta\mu} + T^{\alpha} T_{\mu}\right) e^{\mu}{}_{a} \delta e_{\alpha}{}^{a} + 2(T^{\alpha} e^{\mu}{}_{a} - T^{\mu} e^{\alpha}{}_{a}) \partial_{\alpha} \delta e_{\mu}{}^{a} \tag{2.73}$$

$$\delta(T_{\alpha\mu\nu} T^{\mu\alpha\nu}) = 2\left(T^{\beta\mu\alpha} - T^{\alpha\mu\beta}\right) T_{\mu\alpha\nu} e^{\nu}{}_{a} \delta e_{\beta}{}^{a} + \left(T^{\alpha}{}_{\mu}{}^{\beta} - T^{\beta}{}_{\mu}{}^{\alpha}\right) e^{\mu}{}_{a} \partial_{\alpha} \delta e_{\beta}{}^{a} \tag{2.74}$$

$$\delta(T_{\alpha\mu\nu} T^{\alpha\mu\nu}) = -4 T^{\alpha\mu\nu} T_{\alpha\mu\beta} e^{\beta}{}_{a} \delta e_{\nu}{}^{a} + 4 T^{\mu\nu}_{\alpha} e^{\alpha}{}_{a} \partial_{\mu} \delta e_{\nu}{}^{a} \tag{2.75}$$

By substituting these values in the action, integrating by parts and neglecting the boundary terms, one can reach the following variation of the torsion,

$$e\delta T = 4\left(e T^{\sigma}{}_{\mu a} S_{\sigma}{}^{\lambda\mu} - \partial_{\mu}(e S_{a}{}^{\mu\lambda})\right) \delta e_{\lambda}{}^{a} \tag{2.76}$$

One can then obtain the following teleparallel field equations,

$$\frac{2}{e}\partial_\mu(eS_a{}^{\mu\lambda}) - 2eT^\sigma{}_{\mu a}S_\sigma{}^{\lambda\mu} - Te^\lambda{}_a = \kappa^2 T^\lambda_a \tag{2.77}$$

where T^λ_a is the energy momentum tensor found by varying the matter Lagrangian with respect to the tetrad field. One can also perform some more algebraic manipulation and find that,

$$\mathring{G}^\lambda_\nu = \frac{2}{e}\partial_\mu(eS_a{}^{\mu\lambda}) - 2eT^\sigma{}_{\mu a}S_\sigma{}^{\lambda\mu} - Te^\lambda{}_a \tag{2.78}$$

which is expected since the two theories have equivalent actions.

2.5 Force equations

Let us now prove the equivalence of the trajectories of massive particles. Using $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, one can find that the action of a spinless massive particles as follows,

$$\begin{aligned}\mathcal{S} &= m\int_a^b ds \\ &= m\int_a^b g_{\mu\nu}\frac{dx^\mu}{ds}dx^\nu \\ &= m\int_a^b g_{\mu\nu}u^\mu dx^\nu \\ &= m\int_a^b u_a e^a\end{aligned} \tag{2.79}$$

where $u_a = u_\mu e^\mu{}_a$. One can also write e^a in terms of the tetrad field by expanding in the basis of the natural coordinates as,

$$e^a = dx^a + B_\mu{}^a dx^\mu \tag{2.80}$$

where in this frame the spin connection vanish. One can then vary the above equation with respect to x^μ. Using $\delta x^a = (\partial_\mu x^a)\delta x^\mu$ and $\delta B_\mu{}^a = \partial_\nu(B_\mu{}^a)\delta x^\nu$. Doing some simplifications, one can obtain,

$$\delta\mathcal{S} = m\int_a^b \left(e_\mu{}^a\frac{d^2x_a}{ds^2} - (\partial_\mu B_\nu{}^a - \partial_\nu B_\mu{}^a)u_a u^n u\right)\delta x^\mu ds \tag{2.81}$$

and noting that the second term is just the torsion, one can obtain,

$$e_\mu{}^a \frac{d^2 x_a}{ds^2} = T^\lambda{}_{\mu\nu} u_\lambda u^\nu \tag{2.82}$$

By using the tetrad postulate and doing some simplification one could reach the following form of the particle trajectory,

$$\frac{d^2 x_\mu}{ds^2} - \Gamma^\lambda_{\mu\nu} u_\lambda u^\nu = -K^\lambda{}_{\mu\nu} u_\lambda u^\nu \tag{2.83}$$

where we can see that contorsion acts like a force on the particles in spacetime. Thus, this equation is termed the teleparallel force equation where we interpret gravity as a force. Thus, instead of following geodesics as in GR, particles follow the force equation. However, the two equations are exactly the same by noting that $K^\lambda{}_{\mu\nu} - \Gamma^\lambda_{\mu\nu} = \overset{\circ}{\Gamma}{}^\lambda_{\mu\nu}$. This indicates that TEGR is consistent with the equivalence principle.

3

Modified Teleparallel Gravity

After proving equivalence with general relativity, one now seeks a modification that we hope to incorporate the current cosmological behavior. The natural extension to teleparallel gravity is the so-called modified teleparallel gravity or $f(T)$. However, this formulation has some issues related to the breaking of Lorentz covariance. In this chapter, we will discuss this issue and give an introduction to the field equations.

3.1 Covariant Formulation

Teleparallel gravity is a theory that is invariant under general coordinate transformation since it is based on contractions of the torsion tensor. However, what is the problem with Lorentz covariance? A Lorentz transformation on the tangent space can be written as follows,

$$x^a \longrightarrow \Lambda^{a'}{}_{a} x^a$$

Special relativity and general relativity are both Lorentz invariant theories. Since Lorentz is experimentally proven to be a fundamental transformation of nature, one should expect that a theory that is not Lorentz invariant will not be adequate. Let us now perform Lorentz transformations on the tetrads and spin connection.

$$\omega_{\mu}{}^{a'}{}_{b'} = \Lambda^{a'}{}_{a}\Lambda^{b'}{}_{b}\omega_{\mu}{}^{a}{}_{b} - \Lambda^{c}{}_{b'}\partial_{\mu}\Lambda^{a'}{}_{c} \tag{3.1}$$

$$e_{\mu}{}^{a'} = \Lambda^{a'}{}_{a}e_{\mu}{}^{a} \tag{3.2}$$

It is now clear that the torsion tensor will transform homogeneously under local Lorentz transformations.

$$T^{a'}_{\ \mu\nu} = \Lambda^{a'}{}_{a}T^{a}{}_{\mu\nu} \tag{3.3}$$

However, this is only the case when the spin connection is nonvanishing. This proves that teleparallel gravity is invariant under local Lorentz transformations. However, teleparallel gravity is formulated in the so-called pure tetrad formalism for some historical reasons. In this formalism, one chooses a frame from the beginning. When one considers the frame where the spin connection vanishes, one obtains a weird transformation law for the torsion tensor,

$$T^{a'}_{\ \mu\nu} = \Lambda^{a'}{}_{a}T^{a}{}_{\mu\nu} + \Lambda^{a'}{}_{a}(e_{\nu}{}^{b}\partial_{\mu}\Lambda_{b}{}^{a} - e_{\mu}{}^{b}\partial_{\nu}\Lambda_{b}{}^{a}) \tag{3.4}$$

Thus, it is not Lorentz covariant. However, according to Krššák and Saridakis [2015], one can prove that Lagrangian is still invariant under local Lorentz transformations even though the torsion tensor does not transform covariantly. One can rewrite the torsion as follows,

$$T(e_{\mu}{}^{a}, \omega^{a}{}_{b\mu}) = T(e_{\mu}{}^{a}, 0) + \frac{4}{e}\partial_{\mu}(e\omega^{\mu}) \tag{3.5}$$

where the last term is just a boundary term that is integrated out of the equation. Thus, the theory is said to be quasi-Lorentz invariant up to a boundary term. Thus, any linear combination in T will still admit Lorentz covariance. This makes it clear that whether using the covariant formulation or the pure tetrad formulation one must get the same equations of motion and the same physics.

This does not seem to be a problem when working with teleparallel gravity. However, will that be true when one considers modifications of the equation?

Generally speaking, since we consider theories of the form $f(T)$ they will break the Lorentz covariance and we will not get a general boundary term as in teleparallel gravity. So now, how do we avoid this issue? There are two general prescription which we could follow they are developed by Golovnev et al. [2017], Tseng [2018].

The first of which is to work within the covariant formalism when the spin connection is different from zero. However, the spin connection is not uniquely determined it depends on the observer. Additionally, it contributes to the equations of motion. Krššák and Saridakis [2015] developed this formulation where one can obtain the spin connection by using a reference tetrad and setting the gravitational constant to zero. However, this approach is still under development and there is a huge debate on how to find the spin connection.

Another approach that we will utilize in the following discussion, is to work within the pure tetrad formulation where the spin connection vanishes. However, one should now incorporate a notion called good tetrad which we will discuss below.

3.2 $f(T)$ gravity

Now, we will be driving the field equation when considering a general modification to teleparallel gravity. Additionally, we will discuss the notion of good tetrads. This section is very important for discussing the spherically symmetric solution in teleparallel gravity.

3.2.1 Action

A well-known and a well-studied modification of general relativity is $f(\mathring{R})$ gravity with the following action,

$$\mathcal{S} = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}f(\mathring{R}) + \mathcal{S}_m \tag{3.6}$$

This action is a direct modification of general relativity. One just replaced $\mathring{R}$ with a general function $f(\mathring{R})$. One could also obtain the usual general relativity by taking the limit of $f(\mathring{R}) = \mathring{R}$. If one considers this modification, it can be directly seen that it will be a fourth-order theory since one must invoke integration by parts twice. Again as discussed above only GR yields second-order field equations.

Mostly in literature, this theory is heavily studied when discussing inflation and studying the cosmological history of the universe. Capozziello and De Laurentis [2011] Sotiriou and Faraoni [2008]

Similarly as $f(\mathring{R})$, one can invoke a modification of teleparallel gravity by considering an arbitrary function of torsion scalar $f(T)$ which can be described using the following action,

$$\mathcal{S} = \frac{1}{2\kappa^2}\int d^4xef(T) + \mathcal{S}_m \tag{3.7}$$

However, this time torsion depends only on the first derivative of tetrads which makes this a theory that produces field equations that are second order only. Additionally, as noted above, $f(T)$ is no longer a covariant theory under local Lorentz transformations. One can also think of this as a trade-off. One gets a second-order theory while being a non-Lorentz covariant. One can also see that $f(\mathring{R})$ and $f(T)$ will no longer differ by a boundary term. Thus, in general, these theories are no longer equivalent. Let us now vary the action 3.7,

$$\begin{aligned}\delta\mathcal{S} &= \frac{1}{2\kappa^2}\int d^4x\delta(e)f(T) + e\delta(f(T)) + \delta(e\mathcal{L}) \\ &= \delta\mathcal{S} \\ &= \frac{1}{2\kappa^2}\int d^4x\delta(e)f(T) + ef_T\delta(T) + \delta(e\mathcal{L})\end{aligned} \tag{3.8}$$

where f_T is the derivative of the function with respect to the torsion. We have previously seen the variation of torsion and the determinant of the tetrad. However, now when integrating by parts one will get some derivative that will act of the function f_T. After doing some algebra, one could easily find,

$$\begin{aligned}ef_T\delta T &= 4\left(ef_T T^{\sigma}{}_{\mu a}S_{\sigma}{}^{\lambda\mu} - e(\partial_\mu f_T)S_a{}^{\mu\lambda} - f_T\partial_\mu(eS_a{}^{\mu\lambda})\right)\delta e_\lambda{}^a \\ &= 4\left(ef_T T^{\sigma}{}_{\mu a}S_{\sigma}{}^{\lambda\mu} - ef_{TT}(\partial_\mu T)S_a{}^{\mu\lambda} - f_T\partial_\mu(eS_a{}^{\mu\lambda})\right)\delta e_\lambda{}^a\end{aligned} \tag{3.9}$$

where only one term was added that contains the partial derivative of f_T. One can then arrive at the field equation of teleparallel gravity,

$$4ef_{TT}(\partial_\mu T)S_\nu{}^{\mu\lambda} + 4f_Te_\nu{}^a\partial_\mu(eS_a{}^{\mu\lambda}) - 4ef_TT^{\sigma}{}_{\mu a}S_\sigma{}^{\lambda\mu} - ef\delta_\nu^\lambda = 2\kappa^2e\mathcal{T}_\nu^\lambda \tag{3.10}$$

Clearly, if takes the limit $f(T) = T$ one recovers 2.77. Additionally, Ferraro and Guzmán [2018] it was found that $f(T)$ has an extra degree of freedom other than GR or TEGR. Mostly, one can interpret this extra degree as a scalar field.

3.2.2 Choice of Tetrads

Now we return to discuss the choice of tetrads which we marked as important in our previous discussion. We will follow the discussion about good and bad tetrads found in Tamanini and Boehmer [2012]. This approach presents a method to work probably when one considers modification of teleparallel gravity. A noted different choice of tetrads gives rise to distinct equations of motion. This approach in some sense will give a solution of Lorentz symmetry breaking of $f(T)$.

A good tetrad is defined to be the tetrad that does not impose any conditions on the structure and form of f. Thus, the structure and formalism can be extended to any teleparallel theory of gravity. One can see that Lorentz's transformation will not change the metric under discussion since they are just rotation in 4-dimensional space. Lorentz transformations will only change the equations of motion. Lorentz transformations will not affect the spherical symmetry of spacetime since they only operate on the tangent space. Due to its importance for the last chapter, we now will discuss the case of a spherically symmetric metric in spherical coordinates (t, r, θ, ϕ). The metric generally be written in the form,

$$ds^2 = A(r,t)^2 dt^2 - B(r,t)^2 dr^2 - r^2 d\Omega \tag{3.11}$$

Now what tetrads can be used to form this metric according to $g_{\mu\nu} = \eta_{ab} e_\mu{}^a e_\nu{}^b$. There are infinitely many choices. One possible choice which is the most straightforward is a diagonal tetrad given by,

$$e_\mu{}^a = \begin{pmatrix} A(r,t) & 0 & 0 & 0 \\ 0 & B(r,t) & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin\theta \end{pmatrix} \tag{3.12}$$

However, according to Bahamonde [2018], an $f(T)$ that utilizes this tetrad satisfies the Birkhoff theorem but the Schwarzschild solution is not a solution to the equations. Schwarzschild metrics are very important when discussing space with spherical symmetry. So, one way to solve such a problem is to make a Lorentz transformation on the tetrad. This will change the equations of motion which may solve a problem. One way to do this is to consider rotation in the 3-dimensional space parameterized by three Euler angles (ϕ, θ, ψ). One can then write the Lorentz transformation matrix as follow,

$$\Lambda^{a'}_{\ \ a} = \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{R}(\phi,\theta,\psi) \end{pmatrix} \tag{3.13}$$

where $\mathcal{R}(\theta,\phi,\psi) = \mathcal{R}_z(\psi)\mathcal{R}_y(\theta)\mathcal{R}_x(\phi)$For our purpose, we will chose this specific chose of angles,

$$\phi = \gamma, \quad \theta = \theta - \frac{\pi}{2}, \quad \psi = \phi \tag{3.14}$$

One would get the following transformed tetrad,

$$e_\mu^{\ a'} \tag{3.15}$$

$$= \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & B\cos\phi\sin\theta & -r(\cos\theta\cos\phi\sin\gamma + \cos\gamma\sin\phi) & r\sin\theta(\cos\gamma\cos\theta\cos\phi - \sin\gamma\sin\phi) \\ 0 & B\sin\phi\sin\theta & r(\cos\theta\cos\gamma - \cos\theta\sin\gamma\sin\phi) & -r\sin\theta(\cos\gamma\cos\theta\sin\phi - \sin\gamma\cos\phi) \\ 0 & B\cos\theta & r\sin\gamma\sin\theta & r\cos\gamma\sin^2\theta \end{pmatrix}$$

Now one should substitute this tetrad in the field equation of $f(T)$. After substitution one can easily reach these two constraints,

$$f_{TT}\frac{\partial T}{\partial r}\cos\gamma = 0, \quad f_T\frac{\partial B}{\partial t} = 0 \tag{3.16}$$

Now, the whole idea lies in eliminating the constraints on f. Thus, we assume $f_{TT} \neq 0$ and $f_T \neq 0$. One can directly see that $B(r,t) \equiv B(r)$ only a function of r. Additionally, $\partial_r T$ is a choice where T is constant which is very trivial. Thus, one chooses $\cos\gamma = 0$ or $\gamma = -\pi/2$. One could then reach the following form,

$$e_\mu^{\ a'} = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & B\cos\phi\sin\theta & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ 0 & B\sin\theta\sin\phi & r\cos\theta\sin\phi & r\cos\phi\sin\theta \\ 0 & B\cos\theta & -r sin\theta & 0 \end{pmatrix} \tag{3.17}$$

One can also see that this give a scalar torsion of the form,

$$T = \frac{2(B-1)(A(B-1) - 2rA')}{r^2AB^2} \tag{3.18}$$

One can use packages like OGRe for the calculation of those tensors using this tetrad with zero spin connection. This tetrad is indeed a good tetrad since it did not impose any constraints on f. Additionally, one also have teleparallel gravity in background if $f = T$. Furthermore, the Birkhoff theorem is valid.

4

Spherically symmetric solutions

In this section, we will be reviewing recent research findings that are concerned with spherical symmetry in modified theories of teleparallel gravity. We will mainly follow discussion outlines in Golovnev and Guzman [2021]. Additionally, we will give general comment on Bahamonde et al. [2021] and DeBenedictis et al. [2022].

4.1 Equations of Motion

For symbolic manipulation throughout this discussion, one can either find tensors by hand or use simple symbolic manipulation packages like OGRe in Mathematica. Golovnev and Guzman [2021] found out that upon using a specific choice of the tetrad, one can get equations of motion that are not written in terms of f. It was also assumed that if there exists an adequate $f(T)$ theory that can describe current cosmological observations, it has to respect spherical symmetry. Thus, they utilize different approaches to solve the equations of motion given to be the tetrad 3.17. They have reduced the problem of finding the spherically symmetric solution to only two equations, one radial and the other independent of f. Since one of the equations is independent of f this can allow them to discuss the validity of certain solutions. Using tetrad 3.17, one can easily calculate nonvanishing components of the connection as,

$$\Gamma^t_{rt} = \frac{A'}{A} \qquad \Gamma^r_{rr} = \frac{B'}{B} \tag{4.1}$$

$$\Gamma^r_{\theta\theta} = -\frac{r}{B} \qquad \Gamma^r_{\phi\phi} = -\frac{\sin^2\theta}{B} \tag{4.2}$$

$$\Gamma^\theta_{r\theta} = \Gamma^\phi_{r\phi} = \frac{1}{r} \qquad \Gamma^\theta_{\theta r} = \Gamma^\phi_{\phi r} = \frac{1}{r} \tag{4.3}$$

$$\Gamma^\theta_{\phi\phi} = -\cos\theta\sin\theta \qquad \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta \tag{4.4}$$

Additionally, the non vanishing components of the torsion are,

$$T^t{}_{tr} = -T^t{}_{rt} = -\frac{A'}{A} \tag{4.5}$$

$$T^\theta{}_{r\theta} = -T^\theta{}_{\theta r} = T^\phi{}_{r\phi} = -T^\phi{}_{r\phi} = \frac{1-B}{r} \tag{4.6}$$

Lastly, the superpotential would have non-vanishing components of,

$$S_t{}^{tr} = -S_t{}^{rt} = \frac{1-B}{rB^2} \tag{4.7}$$

$$S_\theta{}^{r\theta} = S_\phi{}^{r\phi} = -S_\theta{}^{\theta r} = -S_\phi{}^{\phi r} = \frac{A - AB + rA'}{2rAB^2} \tag{4.8}$$

One can then easily find the torsion scalar which is given by equation 3.18. We will solve the equation with no matter present. One can then substitute into equation 3.10 to find the temporal ($\lambda = \nu = 0$), radial ($\lambda = \nu = 1$), and angular ($\lambda = \nu = 2$) equations of motion respectively as follows,

$$\begin{aligned}
&-\frac{1}{2}f - \frac{2}{r^2AB^3}f_T\left(r(B-1)BA' + A(B(B-1) + rB')\right) \\
&+ \frac{8(B-1)}{r^4A^2B^5}f_{TT}\Big((B-1)\Big(A^2(B(B-1) - rB') - r^2BA'^2\Big) \\
&+ rA\left(2rA'B' + B(A'(1-rB') - rA'') + B^2(-A' + rA'')\right)\Big) = 0
\end{aligned} \tag{4.9}$$

$$-\frac{1}{2}f - \frac{2}{r^2AB^2}f_T\left(A(B-1) + rA'(B-2)\right) = 0 \tag{4.10}$$

$$\begin{aligned}
&-\frac{1}{2}f + \frac{1}{r^2AB^3}f_T\left(A(B^3 + B - 2B^2 - rB')\right.\\
&\left.+ r(-2B^2A' - rA'B' + B(3A' + rA''))\right)\\
&- \frac{4(A - AB + rA')}{r^4A^3B^5}f_{TT}\Big((B-1)\Big(A^2(B(B-1) - rB') - r^2BA'^2\Big)\\
&+ rA\left(2rA'B' + B(A'(1 - rB') - rA'') + B^2(-A' + rA'')\right)\Big) = 0
\end{aligned} \tag{4.11}$$

It was found that only two of those equations are independent. One has to keep the radial equation since it has the simplest form. The temporal and angular equations could be simplified together since they have the same factor of f_{TT}. Doing so, one can find,

$$\begin{aligned}
&f\left(A^2B^2r^2(B-1) + r^3AA'B^2\right)\\
&+ f_T\Big(4A^2B(B-1) - 4A^2B^2(B-1) + 4rAA' - 8rAA'B\\
&+ 4rAA'B^2 + 4r^2A'^2(B-1) + 4r^2AA'B' - 4r^2AA''(B-1)\Big) = 0
\end{aligned} \tag{4.12}$$

One can now use this equation with the radial equation to eliminate the unknown function. Thus, leaving,

$$-A^2(B+1)(B-1)^2 + r^2A'^2 + r^2A(A'B' - A''(B-1)) = 0 \tag{4.13}$$

One can then try to assume some certain behavior on the form of A and find out how B behaves. This has been discussed in their paper.

4.2 Perturbative solutions

We now will focus on finding solutions 4.13 by requiring certain behavior on the functions A and B. After which one can substitute into the temporal equation and find a perturbative expansion of the function f in terms of the torsion scalar. One requirement that was suggested by Golovnev and Guzman [2021] is that the solution is asymptotically flat. Meaning that the function must be constants at spatial infinities. This allows power expansion in terms of $1/r$. One can find different results in their paper. One can now assume the following perturbative expansion of A and B,

$$\begin{aligned} A(r) &= a_0 + \frac{a_1}{r} + \frac{a_2}{r^2} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{a_k}{r^k} \end{aligned} \tag{4.14}$$

$$\begin{aligned} B(r) &= b_0 + \frac{b_1}{r} + \frac{b_2}{r^2} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{b_k}{r^k} \end{aligned} \tag{4.15}$$

One can now substitute this into 4.13 and solve the equation term by term. We will here present the other branch that was discussed in the paper. We have some simple Mathematica code that can collect the coefficient and find their solutions. Here are the results. The coefficients of the A function are,

$$\begin{aligned} a_0 &= 1 & a_1 &= \text{undetermined} \\ a_2 &= -\frac{a_1^2}{2} & a_3 &= \frac{a_1^3}{2} \\ a_4 &= -\frac{1}{8}\left(5a_1^4\right) & a_5 &= \frac{7a_1^5}{8} \\ a_6 &= -\frac{1}{16}\left(21a_1^6\right) & a_7 &= \frac{33a_1^7}{16} \\ a_8 &= -\frac{1}{128}\left(429a_1^8\right) & a_9 &= \frac{715a_1^9}{128} \end{aligned}$$

which seems like that of Schwarzschild solutions. Here are the solutions of the B function coefficients,

$$b_0 = -1$$

$$b_1 = a_1$$

$$b_2 = 3a_2$$

$$b_3 = \frac{1}{2}\left(a_1^3 + 4a_2a_1 + 12a_3\right)$$

$$b_4 = \frac{1}{8}\left(5a_1^4 + 48a_2a_1^2 + 46a_3a_1 + 44a_2^2 + 80a_4\right)$$

$$b_5 = \frac{1}{16}\left(21a_1^5 + 208a_2a_1^3 + 246a_3a_1^2 + 392a_2^2a_1 + 192a_4a_1 + 408a_2a_3 + 240a_5\right)$$

$$b_6 = \frac{1}{64}\big(205a_1^6 + 2188a_2a_1^4 + 2334a_3a_1^3 + 5632a_2^2a_1^2 + 2064a_4a_1^2 + 7568a_2a_3a_1 \\ + 1376a_5a_1 + 2016a_2^3 + 1776a_3^2 + 3008a_2a_4 + 1344a_6\big)$$

$$b_7 = \frac{1}{64}\big(513a_1^7 + 6308a_2a_1^5 + 6446a_3a_1^4 + 20248a_2^2a_1^3 + 5296a_4a_1^3 + 29088a_2a_3a_1^2 \\ + 3840a_5a_1^2 + 15312a_2^3a_1 + 8704a_3^2a_1 + 14976a_2a_4a_1 + 2240a_6a_1 + 13696a_2^2a_3 \\ + 6272a_3a_4 + 4928a_2a_5 + 1792a_7\big)$$

$$b_8 = \frac{1}{128}\big(2941a_1^8 + 38489a_2a_1^6 + 38508a_3a_1^5 + 148652a_2^2a_1^4 + 30912a_4a_1^4 \\ + 220222a_2a_3a_1^3 + 21248a_5a_1^3 + 171240a_2^3a_1^2 + 71316a_3^2a_1^2 + 123216a_2a_4a_1^2 \\ + 13152a_6a_1^2 + 222072a_2^2a_3a_1 + 66240a_3a_4a_1 + 52928a_2a_5a_1 + 6816a_7a_1 \\ + 28816a_2^4 + 59616a_2a_3^2 + 10752a_4^2 + 51328a_2^2a_4 + 19872a_3a_5 + 14976a_2a_6 \\ + 4608a_8\big)$$

which does look like the other Schwarzschild part. It's an adequate solution according to the perturbative search.

One can also demand other behaviors on the functions of A and B. DeBenedictis et al. [2022] has analyzed the solutions for covariant $f(T)$ gravity. In a space with spherical symmetry, if demands that the solutions be regular at the center ($r = 0$), one would get Minkowski spacetime in GR. However, that may not be the case in $f(T)$ there may be nontrivial solutions that possess this property. Especially, if something like this exists, one can then begin analyzing singularities and other stuff of the same kind. They did so by analyzing the force equation of a spinless massive particle. They assumed a radially falling particle that crosses $r == 0$. Expanded the function in a Taylor expansion (since they are assumed to be regular) and solved the equation perturbatively in r. They found out after calculating some orders that all the derivatives of A and B are zeros whereupon they conjugated that demanding a regularity as such also gives back Minkowski spacetime.

Thus, the regularity is not an interesting case. However, they also assumed that a horizon must form somewhere ($r = r_H$ with $A(r_H) = 0$) in the manifold. Upon which, they studied the properties the solutions must have in order for the horizon to exist. They also did assume that $B(r_H)$ is not zero and not infinite. One can then substitute these assumptions into the equations of motion where one requires the numerator of the equations of motion to vanish. They found out that for all the conditions hold probably, one requires that,

$$A'(r)B'(r)|_{r=r_H} = 0 \tag{4.16}$$

They have shown that by requiring that $A'(r_H)$ vanish if one requires a regular horizon solution. Additionally, the solution will be non singular by requiring $A''(r_H) = 0$. Furthermore, if one requires the solutions to analytic, they found out the $B'(r_H)$ should vanish. However, the produced functions will not solve the equations of motion. Most of this analysis was done by examining the components of the Riemann-Christoffel symbol. Other demands that one may consider may include that the product of A and B be flat at spatial infinity. $\lim_{r\to}\infty(A(r)B(r)) = 1 + \epsilon h$ where ϵ is a small perturbation around the flat metric. Bahamonde et al. [2021].

5
Conclusion

In this thesis, we dealt with a manifold that incorporated torsion and being globally flat. This theory is termed teleparallel gravity. Its equivalence to GR has been established. Additionally, we considered modification and extensions to this theory in terms of an arbitrary function of the torsion scalar. Additionally, we have discussed a possible spherically symmetric solution that any f theory could have by requiring various demands on their behavior.

One of the possible advantages of working with $f(T)$ theories, is the produced equations are second order rather than fourth order as in general relativity. Thus, if one adds non-minimally couplings with scalar fields or matter, the equations are mathematically simpler in modified Teleparallel gravity than in modified GR. It is also interesting that teleparallel gravity is written in terms of gauge theory much like other forces of nature. This allows different aspects of unification in comparison with general relativity which cannot be written as a gauge theory. Another advantage is related to the equivalence principle. GR is mainly based on the equivalence principle. Thus, if at some scale the equivalence principle fails, one can rule out GR. However, TEGR is consistent with universality but it is not needed to construct the theory. Thus, its violation will not affect the theory.

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