Tutorial 4: Sturm Liouville, Generalized Fourier Series & Heat Equation

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1 Important Points

Consider Sturm Liouville (SL) problem,

$$\frac{d}{dx}(p(x)\frac{d\phi}{dx}) + q(x)\phi(x) + \lambda\sigma(x)\phi(x) = 0$$

with some boundary conditions homogeneous / non-homogeneous and regular/singular. Let us first discuss Sturm Liouville problems with regular homogeneous boundary conditions.

$$\beta_1\phi(a) + \beta_2\phi'(a) = 0$$

$$\beta_1\phi(b) + \beta_2\phi'(b) = 0$$

Problems of this type enjoy a handful of properties.

1. All eigenvalues $\lambda \in \mathbb{R}$
2. There are an infinite number of eigenvalues which is bounded from below,

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$$

3. For each eigenvalue λ_n, there exists a unique eigenfunction $\phi_n(x)$ which has exactly $n-1$ zeros in the interval $a < x < b$. Notice that Sturm Liouville DE is a second-order differential equation. Thus, there is two independent solutions. The previous property shows that only one must survive. As an example, if your differential equation produces $\sin(x)$ are $\cos(x)$ as solutions. Only one must be there in the general solution.
4. The eigenfunctions $\phi_n(x)$ are complete, meaning, we can write any function as a linear combination of the eigenfunctions,

$$f(x) = \sum a_n\phi_n(x)$$

with $f(x)$ being piecewise continuous in the interval.

5. Eigenfunctions are orthogonal with respect to weight function $\sigma(x)$,

$$\int_a^b \phi_n(x)\phi_m(x)\sigma(x)dx = 0$$

if $\lambda_n \neq \lambda_m$.

On the other hand, singular Sturm Liouville requires more care. We have seen 4 cases for singular Sturm Liouville. Each of which needs to be treated in some manner. Before stating the cases, notice that the most important property of regular SL is property 5. Property 5 is highly effective in calculating the coefficients a_n of 4. Thus, we want our eigenfunctions to be orthogonal. Now, we state the cases of singular SL,

1. $p(a) = 0$, and the condition is regular at b.
2. $p(b) = 0$, and the condition is regular at a.
3. $p(a) = p(b) = 0$.
4. $p(b) = p(a)$ and $y(a) = y(b)$, $y'(a) = y'(b)$,

Notice for each of the above cases property 5 of orthogonality of eigenfunctions still holds which is what we want. But why are they that special to study them separately? If you look closely at conditions 1-3 you will notice that the differential ceases to be a DE and becomes an algebraic equation at the boundaries. Thus, you are always in fear whether your proposed solutions can work at the boundaries or not. Sometimes, we get rid of one of the solutions to satisfy these conditions. On the other hand, condition 4 has its problem. For periodic boundary condition (4) property 3 fails. Thus, you expect your eigenspace to have more than one solution. This directly damages property 5 why? Because now for each lambda there exist two independent eigenfunctions which we cannot claim that they would be orthogonal or not. Consequently, we cannot factor out the coefficient in 4.

To get through this problem, we must do some extra work. We need to orthogonalize each eigenspace independently using gram schmit, Furthermore, using property 5, we are confident that the produced linear combinations are orthogonal to all other eigenfunctions. We will see an example of this in problem 6.

Now, that we have discussed two cases of SL. Let us talk more about its eigenfunctions and their orthogonality. As we know eigenfunctions of SL are orthogonal and complete. Thus,

$$\langle \phi_n|\phi_m\rangle_\sigma = \int_a^b \phi_n(x)\phi_m(x)\sigma(x)dx = ||\phi_n||^2\delta_{nm}$$

where δ_{nm} is zero when $n \neq m$ and 1 when $n = m$. $||\phi_n||^2$ is called the norm squared of the eigenfunctions. Its values depend on the given boundary conditions as we have seen in the lectures. Thus, given a piecewise continuous functions $f(x)$ we can write f as a linear combination in ϕ as follows,

$$f(x) = \sum a_n\phi_n$$

multiply the equation with $\sigma(x)\phi_m(x)$ and integrate over x in the interval. One can then obtain,

$$\langle f(x)|\phi_m(x)\rangle_\sigma = \int_a^b \sigma(x)f(x)\phi_m(x)dx = \sum a_n \langle \phi_n|\phi_m\rangle_\sigma = \sum a_n||\phi_n||^2\delta_{nm}$$

since δ_{nm} is zero unless $m = n$,

$$\langle f(x)|\phi_m(x)\rangle_\sigma = a_m||\phi_m||^2$$

Thus, we have extracted the required coefficients in the series as follows,

$$a_m = \frac{\langle f|\phi_m\rangle}{||\phi_m||^2} = \frac{\int_a^b \sigma(x)f(x)\phi_m(x)dx}{\int_a^b \sigma(x)\phi_m^2(x)dx}$$

Now, let us see how this works in the generalized Bessel and Legendre series. For the Bessel series, we know that the weight function $\sigma(x) = x$ and $0 \leq x \leq b$. Thus, what is only left is to find the norm squared concerning given boundary conditions. You can consult Zill for the derivation of the following facts,

- Given, $J_n(\alpha b) = 0$,

$$||J_n(\alpha_i x)||^2 = \frac{b^2}{2}J_{n+1}^2(\alpha_i b) \tag{1}$$

  Thus the function is written as follows,

$$f(x) = \sum_{i=1}^{\infty} \frac{2J_n(\alpha_i x)}{b^2J_{n+1}^2(\alpha_i b)}\int_0^b xf(x)J_n(\alpha_i x)dx$$

- Given, $hJ_n(\alpha b) + \alpha bJ'_n(\alpha b) = 0$,

$$||J_n(\alpha_i x)||^2 = \frac{\alpha_i^2b^2 - n^2 + h^2}{2\alpha_i^2}J_n^2(\alpha_i b) \tag{2}$$

Thus the function is written as follows,

$$f(x) = \sum_{i=1}^{\infty} \frac{2\alpha_i^2}{\alpha_i^2 b^2 - n^2 + h^2} \frac{J_n(\alpha_i x)}{J_n^2(\alpha_i b)} \int_0^b x f(x) J_n(\alpha_i x) dx$$

- Given $J_o'(\alpha b) = 0$. (The case of eigenvalue zero) The Fourier series this time has a constant value in addition to other functions due to the existence of $\alpha = 0$ as a solution to the above equation.

$$||J_0(\alpha_i x)||^2 = \frac{b^2}{2} J_0^2(\alpha_i b), \qquad ||1||^2 = b^2/2 \tag{3}$$

Thus the function is written as follows,

$$f(x) = \frac{2}{b^2} \int_0^b x f(x) dx + \sum_{i=1}^{\infty} \frac{2 J_0(\alpha_i x)}{b^2 J_0^2(\alpha_i b)} \int_0^b x f(x) J_0(\alpha_i x) dx$$

The case of Fourier Legendre is trivial since the norm is trivially defined by $\sigma(x) = 1$ and $-1 \leq x \leq 1$,

$$||P_n(x)||^2 = \frac{2}{2n+1} \tag{4}$$

Thus, the function can be written as follows, Fourier

$$f(x) = \sum_{n=0}^{\infty} \frac{2n+1}{2} P_n(x) \int_{-1}^{1} f(x) P_n(x) dx$$

Notice the sum begins from 0, unlike the case of Fourier Bessel.

Important Notice: When we solve SL problem, finding the eigenvalues means that this differential equation has only nontrivial solutions for certain values of λ. However, for other values of λ, only the trivial solution $\phi = 0$ is a solution and no other.

2 Selected Exercises

Problem 1
Given that $J_0(k_n) = 0$, show that,

$$1 - x^2 = 8\sum_{n=1}^{\infty} \frac{J_0(k_n x)}{k_n^3 J_1(k_n)}$$

Solution: This is just case (1) discussed above for Fourier Bessel. We only need to calculate the given integral. First, we read off the value of b. $b = 1$ from the boundary condition $J_0(k_n) = 0$. Now, we calculate the integral,

$$\begin{aligned}\int_0^1 xf(x)J_0(k_n x)dx &= \int_0^1 x(1-x^2)J_0(k_n x)dx \\ &= \int_0^1 xJ_0(k_n x)dx - \int_0^1 x^3 J_0(k_n x)dx\end{aligned}$$

Let us calculate the value of each integral separately,

$$\begin{aligned}\int_0^1 xJ_0(k_n x)dx &= \frac{1}{k_n^2}\int_0^{k_n} uJ_0(u)du && \text{(substitute } u = k_n x\text{)} \\ &= \frac{1}{k_n^2}\int_0^{k_n} \frac{d}{du}[uJ_1(u)]du && \text{(using recursion formula of } J_n\text{)} \\ &= \frac{1}{k_n^2}\left[uJ_1(k_n u)\right]_0^{k_n} \\ &= \frac{J_1(k_n)}{k_n} && \text{(since } J_1(0) = 0\text{)}\end{aligned}$$

Similarly, we evaluate the other integral. As per our last tutorial,

$$\begin{aligned}\int_0^1 x^3 J_0(k_n x)dx &= \frac{1}{k_n^4}\int_0^{k_n} u^3 J_0(u)du \\ &= \frac{1}{k_n^4}\left[(k_n^3 - 4k_m)J_1(k_n) + 2k_n^2 b^2 \cancelto{0}{J_0(k_n)}\right] \\ &= \left[\frac{1}{k_n} - \frac{4}{k_n^3}\right]J_1(k_n)\end{aligned}$$

Thus,

$$\int_0^1 xf(x)J_0(k_n x)dx = \frac{4}{k_n^3}J_1(k_n)$$

Using formula (1),

$$1 - x^2 = \sum_{n=1}^{\infty} \frac{2J_0(k_n x)}{1^2 J_1(k_n)^2}\frac{4}{k_n^3}J_1(k_n) = 8\sum_{n=1}^{\infty}\frac{J_0(k_n x)}{k_n^3 J_1(k_n)}$$

Problem 2
Given that $J_0(k_n) = 0$, show that,

$$\ln(x) = -2\sum_{n=1}^{\infty} \frac{J_0(k_n x)}{[k_n J_1(k_n)]^2}$$

Solution: Again the same boundary as in (1), thus we use the same formula. Again, we need to calculate the integral of c_n. Notice also that $b = 1$ from the boundary condition.

$$\int_0^1 x\ln(x)J_0(k_nx)dx = \frac{1}{k_n^2}\int_0^{k_n} u\ln\left(\frac{u}{k_n}\right)J_0(u) \qquad \text{(substitute } u = k_nx\text{)}$$

$$= \frac{1}{k_n^2}\ln\left(\frac{u}{k_n}\right)J_1(u)\Big|_0^{k_n} - \frac{1}{k_n^2}\int_0^{k_n} J_1(u)du \quad \text{(by parts)}$$

Since $\lim_{x\to 0} x\ln(x) = 0$ by L'Hopital rule and $J_1(0) = 0, \ln(1) = 0$. The boundary term is zero. Thus,

$$\int_0^1 x\ln(x)J_0(k_nx)dx = -\frac{1}{k_n^2}\int_0^{k_n} J_1(u)du$$

$$= \frac{1}{k_n^2}\int_0^{k_n}\frac{d}{du}J_0(u)du$$

$$= \frac{1}{k_n^2}\left[\cancelto{0}{J_0(k_n)} - J_0(0)\right] \qquad \text{(zero by Boundary condition)}$$

$$= -\frac{1}{k_n^2} \qquad (J_0(0) = 1)$$

Thus,

$$\ln(x) = \sum_{n=1}^{\infty}\frac{2J_0(k_nx)}{1^2J_1(k_n)^2}\left[-\frac{1}{k_n^2}\right] = -2\sum_{n=1}^{\infty}\frac{J_0(k_nx)}{[k_nJ_1(k_n)]^2}$$

Problem 3

Given that $J_p'(k_n) = 0$, show that,

$$x^p = 2\sum_{n=1}^{\infty}\frac{k_nJ_{p+1}(k_n)}{(k_n^2 - p^2)J_p^2(k_n)}J_p(k_nx)$$

Solution: Now, we have a different boundary condition. Thus, the norm changes. This boundary condition is similar to (2) with $h = 0$. Again our main task is to evaluate the integral of c_n. Notice also that $b = 1$.

$$\int_0^1 xJ_p(k_nx)x^pdx = \int_0^1 x^{p+1}J_p(k_nx)dx$$

$$= \frac{1}{k_n^{p+2}}\int_0^{k_n}\frac{d}{du}\left[u^{p+1}J_{p+1}(u)du\right] \quad \text{(substitution then recursion formula)}$$

$$= \frac{1}{k_n^{p+2}}(k_n)^{p+1}J_{p+1}(k_n)$$

$$= \frac{1}{k_n}J_{p+1}(k_n)$$

Thus, we can write the function as,

$$x^p = \sum_{n=1}^{\infty}\frac{2k_n^2}{k_n^2 - p^2}\frac{J_p(k_nx)}{J_p^2(k_n)}\frac{1}{k_n}J_{p+1}(k_n) = 2\sum_{n=1}^{\infty}\frac{k_nJ_{p+1}(k_n)}{(k_n^2 - p^2)J_p^2(k_n)}J_p(k_nx)$$

Notice here that this is true for $p \neq 0$. If $p = 0$ then we need to use condition (3). Can you follow the same steps?!

Problem 4
Develop Legendre series for,

$$f(x) = \begin{cases} -1 & -1 \leq x < 0, \\ 1 & 0 < x \leq 0 \end{cases}$$

Solution: The Legendre series is trivial to solve since we don't have all of these boundary conditions. We just need to evaluate the integral,

$$\int_{-1}^{1} f(x)P_n(x)dx = -\int_{-1}^{0} P_n(x)dx + \int_{0}^{1} P_n(x)dx$$

Now, we need to find a method to evaluate these types of integrals. The idea here is to replace P_n with something containing the derivative. However, this is direct using the recursion formula,

$$P_n(x) = \frac{1}{2n+1}\left[P'_{n+1}(x) - P'_{n-1}(x)\right]$$

Thus, integral,

$$\int_{a}^{b} P_n(x) = \frac{1}{2n+1}\left[P_{n+1} - P_{n-1}\right]_a^b$$

Thus, what is left is a substitution. Let us do them one by one,

$$\int_{-1}^{0} P_n(x) = \frac{1}{2n+1}\left[P_{n+1} - P_{n-1}\right]_{-1}^{0} = \frac{1}{2n+1}\left[P_{n+1}(0) - P_{n-1}(0) - P_{n+1}(-1) + P_{n-1}(-1)\right]$$

Since, $P_n(-1) = (-1)^n P_n(1) = (-1)^n$. Thus, $P_{n+1}(-1) - P_{n-1}(-1) = (-1)^{n+1} - (-1)^{n-1} = (-1)^n(-1+1) = 0$. Thus,

$$\int_{-1}^{0} P_n(x) = \frac{1}{2n+1}\left[P_{n+1} - P_{n-1}\right]_{-1}^{0} = \frac{1}{2n+1}\left[P_{n+1}(0) - P_{n-1}(0)\right]$$

Similarly,

$$\int_{0}^{1} P_n(x) = -\frac{1}{2n+1}\left[P_{n+1}(0) - P_{n-1}(0)\right]$$

Thus,

$$\int_{-1}^{1} f(x)P_n(x)dx = \frac{2}{2n+1}\left[P_{n-1}(0) - P_{n+1}(0)\right]$$

What is left to evaluate the Legendre polynomials at $x = 0$. However, this is trivial from the generating function as we did before in past tutorials. We have seen that,

$$P_{2k}(0) = \frac{(-1)^k(2k)!}{2^{2k}(k!)^2} \qquad P_{2k+1}(0) = 0$$

Thus, the integral is zero unless $n = 2k+1$ meaning n is odd. Now, we can simplify the integral as follows,

$$\begin{aligned}
\int_{-1}^{1} f(x)P_{2k+1}(x)dx &= -\frac{2}{4k+3}\left[P_{2k+2}(0) - P_{2k}(0)\right] \\
&= -\frac{2}{4k+3}\left[\frac{(-1)^{k+1}(2k+2)!}{2^{2k+2}((k+1)!)^2} - \frac{(-1)^k(2k)!}{2^{2k}(k!)^2}\right] \\
&= -\frac{2(-1)^{k+1}}{4k+3}\frac{(2k)!}{2^{2k}(k!)^2}\left[1 + \frac{(2k+2)(2k+1)}{2^2(k+1)^2}\right] \\
&= -\frac{2(-1)^{k+1}}{4k+3}\frac{(2k)!}{2^{2k}(k!)^2}\left[\frac{2(k+1)}{2(k+1)} + \frac{(2k+1)}{2(k+1)}\right] \\
&= -\frac{2(-1)^{k+1}}{4k+3}\frac{(2k)!}{2^{2k}(k!)^2}\left[\frac{4k+3}{2(k+1)}\right] \\
&= \frac{(-1)^k(2k)!}{2^{2k}(k!)^2(k+1)}
\end{aligned}$$

Thus, the function can be written as,

$$f(x) = \sum_{k=0}^{\infty}\frac{4k+3}{2}P_{2k+1}(x)\frac{(-1)^k(2k)!}{2^{2k}(k!)^2(k+1)} = \sum_{k=0}^{\infty}\frac{(-1)^k(2k)!(4k+3)}{2^{2k+1}(k!)^2(k+1)}P_{2k+1}(x)$$

Problem 5

Find a general solution and steady-state solution to the following problem,

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

$$\textbf{BC: } \frac{\partial u}{\partial x}(0,t) = 0, \qquad \frac{\partial u}{\partial x}(L,t) = 0$$

$$\textbf{IC: } u(x,0) = f(x)$$

where $0 < x < L$ and $t > 0$.

Solution: We proceed to solve this problem using separation of variables,

$$u(x,t) = \phi(x)G(t)$$

Thus, the PDE gets the following form,

$$\frac{1}{kG(t)}\frac{dG(t)}{dt} = \frac{1}{\phi(x)}\frac{d^2\phi(x)}{dx^2} = -\lambda$$

since the separated parts are functions of different variables, they must be constants. The time-dependent part is trivial to solve,

$$\frac{1}{kG(t)}\frac{dG(t)}{dt} = -\lambda$$

Thus,

$$G(t) = ce^{-k\lambda t}$$

Now, we solve the time-independent part.

$$\frac{1}{\phi(x)}\frac{d^2\phi(x)}{dx^2} = -\lambda$$

The solution of this ODE can be written in terms of exponential or sin and cos. Which one to choose to represent our answer? We will follow a certain convection to write the answer that would help us find the eigenvalues. If x is bounded use trigonometric or other related functions. If x is unbounded use exponential representation.
Let us now solve this for the cases of λ.

1. $\lambda < 0$. This case is not accepted physically. why? Since the time-dependent part would grow exponentially in time. Thus, producing a non-physical solution.

2. $\lambda = 0$. The ODE now have the following form,

$$\frac{d^2\phi(x)}{dx^2} = 0$$

Thus,

$$\phi(x) = c_1 x + c_2$$

Now, we use the boundary conditions. Since the boundary conditions are defined for all values of t. These conditions apply directly to the function $\phi(x)$. Thus,

$$\frac{\partial\phi}{\partial x}(0) = 0, \qquad \frac{\partial\phi}{\partial x}(L) = 0$$

Both of these boundary conditions give $c_1 = 0$. However, c_2 is undetermined. Thus, we get

$$\phi(x) = c_2$$

for $\lambda = 0$.

3. $\lambda > 0$. The differential equation has the form,

$$\frac{1}{\phi(x)}\frac{d^2\phi(x)}{dx^2} = -\lambda$$

with boundary conditions,

$$\frac{\partial\phi}{\partial x}(0) = 0, \qquad \frac{\partial\phi}{\partial x}(L) = 0$$

since x is bounded we choose the trigonometric representation of the solution. Thus,

$$\phi(x) = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)$$

Now, we substitute for the boundary conditions. The first boundary condition gives,

$$c_2\sqrt{\lambda} = 0$$

Since $\lambda > 0$ by assumption. c_2 must be zero. substituting the second boundary condition produces,

$$-c_1\sqrt{\lambda}\sin(\sqrt{\lambda}L) = 0$$

Since we are searching for a non-trivial solution $c_1 \neq 0$. Thus, the argument of sin must be a zero of the function. Thus,

$$\sqrt{\lambda}L = n\pi$$

Now, we have found the eigenvalues $\lambda = \frac{n^2\pi^2}{L^2}$ with eigenfunctions $\phi_n(x) = c_1\cos(\frac{n\pi}{L}x)$.Notice that the eigenspace contains a single function of cos as we have expected from solving a regular Sturm Liouville problem.

Now we can write the general solution as,

$$u(x,t) = \sum_{\lambda} \phi_\lambda(x) G_\lambda(t) = A_0 e^{-(0)kt} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) e^{-\frac{n^2\pi^2}{L^2}kt} = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) e^{-\frac{n^2\pi^2}{L^2}kt}$$

What is left is for you to substitute for the initial condition $f(x)$. If we look again at the time-independent equation. This is an SL problem that produces eigenfunctions that are orthogonal with respect to weight function $\sigma(x) = 1$. Thus, at $t = 0$,

$$u(x,0) = f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right)$$

and we can find A_n easily as,

$$A_n = \frac{\int_0^L f(x) \cos\left(\frac{n\pi}{L}x\right)}{\int_0^L \cos^2\left(\frac{n\pi}{L}x\right)}$$

Lastly, we want to find the steady state solution. The steady-state solution is defined as the limit of the solution as t goes to infinity. Thus,

$$u_{\text{steady}} = \lim_{t\to\infty} u(x,t) = \lim_{t\to\infty} \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) e^{-\frac{n^2\pi^2}{L^2}kt} = A_0$$

Notice that you can also find the steady-state solution by making taking the partial derivative of u to zero with respect to t and solving the equation.

Problem 6

Heat conduction in an insulted ring,

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

$$\textbf{BC: } u(-L,t) = u(L,t), \qquad \frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(L,t)$$

$$\textbf{IC: } u(x,0) = f(x)$$

where $0 < x < L$ and $t > 0$.

Solution: After the usual separation of variables. We get the same factor for the exponential part and again we are left with the bounded ODE,

$$\frac{1}{\phi(x)}\frac{d^2\phi(x)}{dx^2} = -\lambda$$

$$\phi(-L) = \phi(L) \qquad \phi'(-L) = \phi'(L)$$

Let us do this for the cases of λ as before,

1. $\lambda < 0$: is unphysical.
2. $\lambda = 0$. Thus, as before we get the solution,

   $$\phi(x) = c_1 x + c_2$$

   Now, we substitute the boundary conditions. The first boundary condition gives $c_1 = 0$. However, there are no constraints on c_2. Thus, for $\lambda = 0$. We get one solution which is the constant value.

3. $\lambda > 0$, Since x is bounded, we use the representation,

$$\phi(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$$

Since cos is even and sin is odd. By substituting the first boundary condition, We get

$$c_1 \cos(\sqrt{\lambda}L) + c_2 \sin(\sqrt{\lambda}L) = \cos(\sqrt{-\lambda}L) + c_2 \sin(\sqrt{-\lambda}L)$$

Thus,

$$2c_2 \sin(\sqrt{\lambda}L) = 0$$

Before concluding that c_2 is zero, let us check the other condition,

$$\phi'(x) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}x)$$

By substituting the value at L and -L and equating the two-equation, we get,

$$2c_1\sqrt{\lambda}\sin(\sqrt{\lambda}L) = 0$$

The two conditions are the same and if we choose that $\sin(\sqrt{\lambda}L) \neq 0$ we get only the trivial solution. Thus, we must have $\sin(\sqrt{\lambda}L) = 0$ producing the eigenvalues $\lambda = \frac{n^2\pi^2}{L^2}$. However, the eigenfunctions are,

$$\phi_n(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$$

Notice here that the space is spanned by two functions instead of one like regular SL. Furthermore, we are very lucky the two functions turn out to be orthogonal in this case. Thus, we don't need to do any other thing.

Thus, the general solution can be written as,

$$u(x,t) = A_0 + \sum_{n=1}^{\infty}\left[A_n \cos\left(\frac{n\pi}{L}x\right) + B_n \sin\left(\frac{n\pi}{L}x\right)\right] e^{-\frac{n^2\pi^2}{L^2}kt}$$

Then as usual we substitute with the initial condition and use the orthogonality of sin and cos to extract the coefficients. Again if the eigenspace turns out to have more than one function, we must make sure the functions are orthogonal to extract the coefficients.

Problem 7
Heated conduction in an insulted ring with lateral dissipation,

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - hu$$

$$\textbf{BC:}\ \frac{\partial u}{\partial x}(0,t) = 0, \qquad \frac{\partial u}{\partial x}(L,t) = 0$$

$$\textbf{IC:}\ u(x,0) = f(x)$$

where h is a constant, $0 < x < L$, and $t > 0$.

Solution: Again for this problem, we use the method of separation of variables. So, $u(x,t) = \phi(x)G(t)$. The differential equation now takes the form,

$$\phi(x)\frac{\partial G(t)}{\partial t} = kG(t)\frac{\partial^2 \phi(x)}{\partial x^2} - h\phi(x)G(t)$$

Dividing by $kG(t)\phi(x)$,

$$\frac{1}{kG(t)}\frac{\partial G(t)}{\partial t} = \frac{1}{\phi(x)}\frac{\partial^2 \phi(x)}{\partial x^2} - \frac{h}{k}$$

Since the time ODE is easier to solve, we separate as follows,

$$\frac{1}{kG(t)}\frac{\partial G(t)}{\partial t} + \frac{h}{k} = \frac{1}{\phi(x)}\frac{\partial^2 \phi(x)}{\partial x^2} = -\lambda$$

The spatial part can be solved as in question 5. What is left is the time-dependent part.

$$\frac{1}{kG(t)}\frac{\partial G(t)}{\partial t} + \frac{h}{k} = -\lambda$$

However, the solution of this DE is easy,

$$G(t) = ce^{-(h+\lambda k)t}$$

Thus, the general solution is now trivial to write.